Algebraic and geometric aspects of generalized quantum dynamics

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We briefly discuss some algebraic and geometric aspects of the generalized Poisson bracket and noncommutative phase space for generalized quantum dynamics, which are analogous to properties of the classical Poisson bracket and ordinary symplectic structure.

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Recently, one of us (S.L.A.) proposed a generalization of Heisenberg picture quantum mechanics, termed *generalized quantum dynamics*, which gives a Hamiltonian dynamics for general noncommutative degrees of freedom [1,2]. The formalism permits the direct derivation of equations of motion for field operators, without first proceeding through the intermediate step of "quantizing" a classical theory. In a complex Hilbert space, generalized quantum dynamics gives results compatible with standard canonical quantization. It is also applicable to the construction of quantum field theories in quaternionic Hilbert spaces, where canonical methods fail, basically because the *matrix elements* of operators are themselves elements of the noncommutative quaternion algebra. It is hoped that the methods of generalized quantum dynamics will facilitate answering the question of whether quantum field theories in quaternionic Hilbert space are relevant to the unification of the standard model forces with gravitation at energies above the grand unified theory (GUT) scale.

As applied to quantum theory, generalized quantum dynamics is formulated by defining a Hilbert space V_H (based either on complex number or quaternionic scalars) which is the direct sum of a bosonic space V_H^+ and a fermionic space V_H^-. Next, following Witten [3], one defines an operator $(-1)^F$ with eigenvalue $+1$ for states in V_H^+ and -1 for states in V_H^-. Finally, one needs a trace operation $\mathbf{Tr}\,\mathcal{O}$ for a general operator $\mathcal{O}$, defined by

$$\mathbf{Tr}\,\mathcal{O} = \mathrm{Re}\,Tr\,(-1)^F\mathcal{O} = \mathrm{Re}\sum_n \langle n|(-1)^F\mathcal{O}|n\rangle. \quad (1)$$

It is easy to show that the trace $\mathbf{Tr}$ vanishes for operators $\mathcal{O}$ which anticommute with $(-1)^F$, and so $\mathbf{Tr}\,\mathcal{O}$ acts nontrivially only on the part of $\mathcal{O}$ which commutes with $(-1)^F$.

Let $\{q_r(t)\}$ be a finite set of time-dependent quantum variables, which act as operators on the underlying Hilbert space, with each individual q_r of either bosonic or fermionic type, defined respectively as commuting or anticommuting with $(-1)^F$. No other *a priori* assumptions about commutativity of the q_r are made. The Lagrangian $\mathbf{L}[\{q_r\},\{\dot{q}_r\}]$ is then defined as the trace of a polynomial function of $\{q_r(t)\}$ and its time derivative $\{\dot{q}_r(t)\}$, or as a suitable limit of such functions. The action $\mathbf{S}$ is defined as the time integral of $\mathbf{L}$, and generalizations of the Euler-Lagrange equations follow from the requirement that $\delta\mathbf{S}=0$ for arbitrary (same-type) variations of the operators. Derivatives of $\mathbf{L}$ with respect to q_r and $\dot{q}_r$ are defined by writing the variation of $\mathbf{L}$, for infinitesimal variations in the $\{q_r\}$, in the form

$$\delta\mathbf{L} = \mathbf{Tr}\sum_r\left(\frac{\delta\mathbf{L}}{\delta q_r}\,\delta q_r + \frac{\delta\mathbf{L}}{\delta\dot{q}_r}\,\delta\dot{q}_r\right), \quad (2)$$

where cyclic permutations of operators inside $\mathbf{Tr}$ have been used to order δq_r and $\delta\dot{q}_r$ to the right. The momentum p_r conjugate to q_r is defined by

$$\frac{\delta\mathbf{L}}{\delta\dot{q}_r} = p_r, \quad (3)$$

and the Hamiltonian $\mathbf{H}$ is given by

$$\mathbf{H} = \mathbf{Tr}\sum_r p_r\dot{q}_r - \mathbf{L}. \quad (4)$$

In complete analogy with the Lagrangian derivatives defined in Eq. (2), for a general trace functional $\mathbf{A}$, constructed as the trace $\mathbf{Tr}$ of a (bosonic) polynomial function of operator arguments, one can define unique derivative $\delta\mathbf{A}/\delta q_r$ with respect to the operator q_r (and of the same bosonic or fermionic type as q_r) by the relation

$$\delta\mathbf{A} = \mathbf{Tr}\,\frac{\delta\mathbf{A}}{\delta q_r}\,\delta q_r. \quad (5)$$

Again, cyclic invariance of the trace has been used to reorder all δq_r factors to the right in the respective terms in which they occur. Using this derivative, one can then define generalized Poisson brackets, as follows. Let $\{q_r\}$, $\{p_r\}$ be the set of operator phase space variables introduced above, which for each r are either both bosonic or both fermionic, in the sense that they commute or anticommute with $(-1)^F$. Again, no further *a priori* assumptions are made about their commutativity. If we now let $\mathbf{A}[\{q_r\},\ \{p_r\}]$ and $\mathbf{B}[\{q_r\},\ \{p_r\}]$ be two trace functionals of their arguments, then the generalized Poisson bracket $\{\mathbf{A},\ \mathbf{B}\}$ is defined by

$$\{\mathbf{A},\ \mathbf{B}\} = \mathbf{Tr}\left[\sum_r \varepsilon_r\left(\frac{\delta\mathbf{A}}{\delta q_r}\,\frac{\delta\mathbf{B}}{\delta p_r} - \frac{\delta\mathbf{B}}{\delta q_r}\,\frac{\delta\mathbf{A}}{\delta p_r}\right)\right], \quad (6)$$

with $\varepsilon_r = +1(-1)$ according to whether q_r and p_r are bosonic (fermionic). Using the generalized bracket, the time development of a general trace functional $\mathbf{A}[\{q_r\},\ \{p_r\},\ t]$ takes the form [1,2]

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \{\mathbf{A},\ \mathbf{H}\}, \tag{7}$$

with $\mathbf{H}$ the total trace Hamiltonian. It was conjectured in Refs. [1] and [2] that the generalized bracket obeys the Jacobi identity,

$$0 = \{\mathbf{A},\ \{\mathbf{B},\mathbf{C}\}\} + \{\mathbf{C},\ \{\mathbf{A},\ \mathbf{B}\}\} + \{\mathbf{B},\ \{\mathbf{C},\ \mathbf{A}\}\}, \tag{8}$$

and this conjecture has recently been proved by Adler, Bhanot, and Weckel [4]. The key observation is that despite the absence of both commutativity and the product rule, and the lack of a definition for the double derivative, pairwise cancellations still occur in the right-hand side of Eq. (8) because of cyclic permutability inside the trace $\mathbf{Tr}$. The proof of Eq. (8) is, in fact, independent of the Hilbert space arena on which the operators $\{q_r\}$ act. All that is used are the definition of derivative of Eq. (5), and the assumptions that operator multiplication is associative, and that there exists a graded trace $\mathbf{Tr}$ permitting cyclic permutation of noncommuting operator variables, according to the formula

$$\mathbf{Tr}\ \mathcal{O}_{(1)}\mathcal{O}_{(2)} = \pm\mathbf{Tr}\ \mathcal{O}_{(2)}\mathcal{O}_{(1)}, \tag{9}$$

with the $+$ $(-)$ sign holding when $\mathcal{O}_{(1)}$ and $\mathcal{O}_{(2)}$ are both bosonic (fermionic).

Evidently the generalized bracket of Eq. (6) can be viewed as an extension of the classical Poisson bracket, which permits the introduction of noncommuting phase space variables $\{q_r\}$, $\{p_r\}$. Our aim in this note is to document a number of further algebraic and geometric properties of noncommutative phase space, which closely relate to the existence of the generalized Poisson bracket that satisfies the Jacobi identity of Eq. (8), but which do not enter into the proof given in Ref. [4].

The first of these involves the algebraic structure of the trace functionals, under the product operation used to construct the antisymmetric bracket of Eq. (6). Letting $\mathbf{A}$ and $\mathbf{B}$ be any two trace functionals defined on phase space, a product $\mathbf{A} \circ \mathbf{B}$ that remains a trace functional can be defined by

$$\mathbf{A} \circ \mathbf{B} \equiv \mathbf{Tr}\left[\sum_r \varepsilon_r \frac{\delta \mathbf{A}}{\delta q_r}\ \frac{\delta \mathbf{B}}{\delta p_r}\right], \tag{10}$$

in terms of which the generalized Poisson bracket takes the form of a commutator:

$$\{\mathbf{A},\mathbf{B}\} = \mathbf{A} \circ \mathbf{B} - \mathbf{B} \circ \mathbf{A}. \tag{11}$$

The algebra $\mathcal{A}_\circ$ of trace functionals under the product $\circ$ can now be characterized in terms of the standard classification [5] of nonassociative algebras. It is *associative* if the associator $(\mathbf{A},\mathbf{B},\mathbf{C})$ defined by

$$(\mathbf{A},\mathbf{B},\mathbf{C}) \equiv (\mathbf{A} \circ \mathbf{B}) \circ \mathbf{C} - \mathbf{A} \circ (\mathbf{B} \circ \mathbf{C}) \tag{12}$$

vanishes. It is *flexible* if the associator obeys

$$(\mathbf{A},\mathbf{B},\mathbf{C}) = -(\mathbf{C},\mathbf{B},\mathbf{A}), \tag{13}$$

and it is *Lie admissible* if the associator obeys

$$\begin{aligned} 0 = {} & (\mathbf{A},\mathbf{B},\mathbf{C}) - (\mathbf{A},\mathbf{C},\mathbf{B}) + (\mathbf{B},\mathbf{C},\mathbf{A}) \\ & -(\mathbf{B},\mathbf{A},\mathbf{C}) + (\mathbf{C},\mathbf{A},\mathbf{B}) - (\mathbf{C},\mathbf{B},\mathbf{A}). \end{aligned} \tag{14}$$

Evidently, any associative algebra is Lie admissible, but the converse is of course not true. Now by substituting Eq. (12) into Eq. (14) and rearranging using Eq. (11), we find that Eq. (14) is equivalent to

$$0 = \{\mathbf{A},\{\mathbf{B},\mathbf{C}\}\} + \{\mathbf{C},\{\mathbf{A},\mathbf{B}\}\} + \{\mathbf{B},\{\mathbf{C},\mathbf{A}\}\}, \tag{15}$$

which is true by virtue of the Jacobi identity for the generalized Poisson bracket. To see that Eq. (12) does not vanish and that Eq. (13) does not hold, it suffices to consider the special case in which the variables $\{q_r\}$ and $\{p_r\}$ are commuting (bosonic) c numbers. This is just the classical case in which $\{\mathbf{A},\mathbf{B}\}$ is proportional to the standard Poisson bracket, and a simple calculation of multiple derivatives (see, e.g., Ref. [5], Sec. 7.3) shows that both the vanishing of Eq. (12) and the identity of Eq. (13) are false for the product defined by Eq. (10). Hence the algebra $\mathcal{A}_\circ$ is neither associative nor flexible, and therefore is only of secondary interest. But as in the case of its classical analog, $\mathcal{A}_\circ$ is Lie admissible by virtue of the Jacobi identity, and hence the resulting Lie structure defined by Eq. (11) is of primary importance. Thus, the trace functionals form a Lie algebra under the generalized Poisson bracket of Eq. (11) and, in particular, the total trace conserved symmetry generators that commute with the total trace Hamiltonian form a Lie subalgebra [4].

The second aspect to be discussed relates to the tangent vector fields associated with the generalized dynamics. Let $X_{\mathbf{A}}$ be the tangent vector field associated with a trace functional $\mathbf{A}$, defined as a formal derivative operator by

$$X_{\mathbf{A}} \equiv \mathbf{Tr}\left[\sum_r \left(\varepsilon_r \frac{\delta \mathbf{A}}{\delta q_r}\ \frac{\delta}{\delta p_r} - \frac{\delta \mathbf{A}}{\delta p_r}\ \frac{\delta}{\delta q_r}\right)\right], \tag{16}$$

and defined operationally by its action on any trace functional $\mathbf{B}$,

$$X_{\mathbf{A}}\mathbf{B} = \mathbf{B}X_{\mathbf{A}} + (X_{\mathbf{A}}\mathbf{B}), \tag{17}$$

with $(X_{\mathbf{A}}\mathbf{B})$ given by

$$\begin{aligned} (X_{\mathbf{A}}\mathbf{B}) &= \mathbf{Tr}\left[\sum_r \left(\varepsilon_r \frac{\delta \mathbf{A}}{\delta q_r}\ \frac{\delta \mathbf{B}}{\delta p_r} - \frac{\delta \mathbf{A}}{\delta p_r}\ \frac{\delta \mathbf{B}}{\delta q_r}\right)\right] \\ &= \mathbf{Tr}\left[\sum_r \varepsilon_r \left(\frac{\delta \mathbf{A}}{\delta q_r}\ \frac{\delta \mathbf{B}}{\delta p_r} - \frac{\delta \mathbf{B}}{\delta q_r}\ \frac{\delta \mathbf{A}}{\delta p_r}\right)\right] = \{\mathbf{A},\mathbf{B}\}. \end{aligned} \tag{18}$$

In terms of this operator, the time development of a general trace functional $\mathbf{B}[\{q_r\},\ \{p_r\}]$, under the dynamics governed by $\mathbf{A}$ as total trace Hamiltonian, can be rewritten as [cf. Eq. (7)]

$$\frac{d\mathbf{B}}{dt} = -(X_{\mathbf{A}}\mathbf{B}). \tag{19}$$

Thus the tangent vector field $X_{\mathbf{A}}$ can be viewed as (minus) the directional derivative along the time evolution orbit (called the phase flow in Ref. [6]) of the point $(\{q_r\}, \{p_r\})$ in phase space, which is determined by the Hamiltonian equations of motion [1]

$$\frac{dq_r}{dt} = \varepsilon_r \frac{\delta \mathbf{A}}{\delta p_r}, \qquad \frac{dp_r}{dt} = -\frac{\delta \mathbf{A}}{\delta q_r}, \tag{20}$$

with $\mathbf{A}$ acting as the total trace Hamiltonian. Following Ref. [6], we call a tangent vector field of the form of Eq. (16) a Hamiltonian vector field, the same name as for its classical counterpart.

We note that with respect to the product defined by Eq. (10), the directional derivative $X_{\mathbf{A}}$ does not obey the Leibniz product rule:

$$(X_{\mathbf{A}}(\mathbf{B} \circ \mathbf{C})) \neq (X_{\mathbf{A}}\mathbf{B}) \circ \mathbf{C} + \mathbf{B} \circ (X_{\mathbf{A}}\mathbf{C}). \tag{21}$$

(It is easy to verify that the same is true in the classical case.) However, it does obey the Leibniz product rule for the generalized Poisson bracket or the commutator defined by Eq. (11),

$$(X_{\mathbf{A}}\{\mathbf{B},\mathbf{C}\}) = \{(X_{\mathbf{A}}\mathbf{B}),\mathbf{C}\} + \{\mathbf{B},(X_{\mathbf{A}}\mathbf{C})\}, \tag{22}$$

because, in view of Eq. (14), this equation is equivalent to the Jacobi identity of Eq. (8).

What is the algebraic structure of the Hamiltonian vector fields? Let us compute the action of the commutator of two tangent vector fields $X_{\mathbf{A}}$ and $X_{\mathbf{B}}$ on a third trace functional $\mathbf{C}$:

$$\begin{aligned}([X_{\mathbf{A}},\ X_{\mathbf{B}}]\mathbf{C}) &= (X_{\mathbf{A}}(X_{\mathbf{B}}\mathbf{C})) - (X_{\mathbf{B}}(X_{\mathbf{A}}\mathbf{C})) \\ &= \{\mathbf{A},\{\mathbf{B},\mathbf{C}\}\} - \{\mathbf{B},\{\mathbf{A},\mathbf{C}\}\} \\ &= \{\mathbf{A},\{\mathbf{B},\mathbf{C}\}\} + \{\mathbf{B},\{\mathbf{C},\mathbf{A}\}\}. \end{aligned} \tag{23}$$

Using Eq. (14) with $\mathbf{A}$ replaced by $\{\mathbf{A},\mathbf{B}\}$ and $\mathbf{B}$ replaced by $\mathbf{C}$, we also get

$$(X_{\{\mathbf{A},\mathbf{B}\}}\mathbf{C}) = \{\{\mathbf{A},\mathbf{B}\},\mathbf{C}\}, \tag{24}$$

and subtracting Eq. (24) from Eq. (23) gives finally

$$\begin{aligned}&(([X_{\mathbf{A}},X_{\mathbf{B}}] - X_{\{\mathbf{A},\mathbf{B}\}})\mathbf{C}) \\ &\quad = \{\mathbf{A},\{\mathbf{B},\mathbf{C}\}\} + \{\mathbf{B},\{\mathbf{C},\mathbf{A}\}\} + \{\mathbf{C},\{\mathbf{A},\mathbf{B}\}\} \\ &\quad = 0. \end{aligned} \tag{25}$$

Hence validity of the Jacobi identity for the generalized Poisson bracket implies that the Hamiltonian vector fields $X_{\mathbf{A}}$ defined by Eqs. (16)–(18) obey the commutator algebra

$$[X_{\mathbf{A}},X_{\mathbf{B}}] = X_{\{\mathbf{A},\mathbf{B}\}}, \tag{26}$$

and, therefore, form a Lie algebra that is isomorphic to the Lie algebra of trace functionals under the generalized Poisson bracket, which is the generalized quantum dynamics analogue of a standard result [6] in classical mechanics.

Finally, we address the geometric structure underlying generalized quantum dynamics. As is well known, there is a geometry which underlies classical Hamiltonian dynamics, namely the symplectic geometry of ordinary phase space. Can we generalize symplectic geometry to noncommutative phase space? If a generalized symplectic structure exists, is it preserved by phase space flows (or Hamiltonian time evolutions) as in classical mechanics [6]? In the following we present a discussion of these questions with affirmative answers, which is readable to physicists who are not familiar with differential forms [7].

Ordinary symplectic geometry is defined by a standard (constant) antisymmetric metric in the tangent or cotangent spaces of a phase space. (By way of contrast, Riemannian geometry, which is perhaps more familiar to physicists, is defined by a *symmetric* metric in the tangent or cotangent spaces of a manifold.) To avoid differential forms, let us consider the cotangent space, which is known to be spanned by covariant vectors whose components form the gradient (or differential) of a function on phase space. The standard (antisymmetric) symplectic metric, or the inner product, between two covariant vectors that are the gradients of two classical functions $A(q_r,p_r)$ and $B(q_r,p_r)$ on phase space, is provided by the classical Poisson brackets $\{A,B\}$. In a noncommutative phase space, the analogues of functions are trace functionals, and the analogues of the differentials of functions are the differentials of trace functionals, i.e., Eq. (5) adapted to phase space:

$$\delta\mathbf{A} = \mathbf{Tr}\left[\sum_r \left(\frac{\delta\mathbf{A}}{\delta q_r}\,\delta q_r + \frac{\delta\mathbf{A}}{\delta p_r}\,\delta p_r\right)\right]. \tag{27}$$

With the generalized Poisson brackets of Eq. (6) available, we can use it to define a generalized symplectic structure Ω on the noncommutative phase space, through defining the inner product between two cotangent vectors $\delta\mathbf{A}$ and $\delta\mathbf{B}$ as follows:

$$\begin{aligned}\Omega(\delta\mathbf{A},\delta\mathbf{B}) &= \{\mathbf{A},\ \mathbf{B}\} \\ &\equiv \mathbf{Tr}\left[\sum_r \varepsilon_r\left(\frac{\delta\mathbf{A}}{\delta q_r}\,\frac{\delta\mathbf{B}}{\delta p_r} - \frac{\delta\mathbf{B}}{\delta q_r}\,\frac{\delta\mathbf{A}}{\delta p_r}\right)\right]. \end{aligned} \tag{28}$$

To see that such a symplectic structure is preserved by any Hamiltonian phase flow of Eq. (20), we observe that the time derivative of the inner product along the phase-flow orbit is

$$\frac{d}{dt}\,\Omega(\delta\mathbf{B},\delta\mathbf{C}) = \frac{d}{dt}\,\{\mathbf{B},\ \mathbf{C}\} = \{\{\mathbf{B},\ \mathbf{C}\},\ \mathbf{A}\}, \tag{29}$$

while that of the differential $\delta\mathbf{B}$ along the same flow is

$$\frac{d}{dt}\,\delta\mathbf{B} \equiv \delta\dot{\mathbf{B}} \tag{30}$$

where the dot abbreviates the time derivative. Therefore, we have

$$\Omega(\delta\dot{\mathbf{B}},\ \delta\mathbf{C}) + \Omega(\delta\mathbf{B},\ \delta\dot{\mathbf{C}}) = \{\dot{\mathbf{B}},\ \mathbf{C}\} + \{\mathbf{B},\ \dot{\mathbf{C}}\}$$
$$= \{\{\mathbf{B},\ \mathbf{A}\},\mathbf{C}\} + \{\mathbf{B},\ \{\mathbf{C},\mathbf{A}\}\}\ . \quad (31)$$

Therefore the Jacobi identity of Eq. (8) implies

$$\frac{d}{dt}\ \Omega(\delta\mathbf{B},\delta\mathbf{C}) = \Omega(\delta\dot{\mathbf{B}},\ \delta\mathbf{C}) + \Omega(\delta\mathbf{B},\ \delta\dot{\mathbf{C}}); \quad (32)$$

that is, the symplectic structure is invariant under Hamiltonian phase flow. This statement can be viewed as a (dual) form of the generalized quantum dynamics analogue of the Liouville theorem.

Thus, generalized quantum dynamics, albeit with noncommuting operator phase space variables, has an underlying generalized symplectic geometry which is preserved by the time evolution generated by any total trace Hamiltonian. Basically this is due to the existence of a (graded) trace **Tr** that permits cyclic permutation of noncommuting operator variables, which implies the validity of the Jacobi identity for the generalized Poisson bracket. As in classical mechanics, we expect that the basic concepts and theorems of generalized quantum dynamics will be invariant under the group of symplectic transformations, i.e., under transformations which preserve the generalized symplectic structure.

To conclude, we have seen that in many algebraic and geometric aspects, the generalized quantum dynamics proposed in Refs. [1] and [2] is analogous to classical mechanics. It is really surprising that with the help of a cyclically permutable (graded) trace alone, so many features of classical mechanics can be generalized to a noncommutative phase space. (We remind readers once more that in Ref. [1] and in our present discussion, *no phase space variable commutation relations such as commutativity, anticommutativity, or q commutators are assumed.*) Further developments in generalized quantum dynamics, paralleling to some extent aspects of existing quantization schemes, are expected.

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[7] A more mathematical treatment, using the notions of generalized differential forms in noncommutative phase space, may be possible. This is left for a future publication.