Type-IIB-string–M-theory duality and longitudinal membranes in M(atrix) theory

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In this paper we study duality properties of the M(atrix) theory compactified on a circle. We present evidence for the equivalence of this theory to the strong coupling limit of type-IIB string theory compactified on a circle. In the M(atrix) theory context, our evidence for this duality consists of showing the appearance (upon compactification) of a topological term recently discovered in the D-string action, identifying the BPS states of type-IIB strings in the spectrum and finding the remnant symmetry of SL(2,Z) and the associated τ moduli. By this type-IIB-string–M-theory duality, a number of insights are gained into the physics of longitudinal membranes in the infinite momentum frame. [S0556-2821(97)05622-1]

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I. INTRODUCTION

The recent revelation of various string dualities indicates clearly that all the previously known five consistent perturbative superstring theories in ten dimensions are closely related to each other; they appear to represent five corners of the moduli space for vacuua of the one and same theory, which yet in another corner of the moduli space is most conveniently viewed as a theory in 11 dimensions, dubbed M theory. (For recent reviews see, e.g., [1–6].) Banks *et al.* [7] have proposed a definition of M theory in the infinite momentum frame (IMF) as a large N limit of maximally supersymmetric quantum mechanics of $N\times N$ matrices describing $D0$-branes. This proposal has passed a number of consistency tests [7–21]. Many of the tests consist of verifying that M(atrix) theory indeed reproduces the right properties and interactions of Dp-branes (with even p) expected from type-IIA string theory. And others examine the way certain string dualities are realized upon compactification on tori or more complicated spaces, and/or address some other issues.

The IMF description is essential to this formulation of M(atrix) theory, which lacks manifest Lorentz invariance in 11 dimensions, and may give rise to some technical complications: For example, T duality is not manifest. An issue related to this is how to construct a membrane wrapped in the longitudinal direction that defines the IMF. Another important issue is how strings emerge in M(atrix) theory as collective states of $D0$-branes. In this paper, we try to address the two issues by studying M(atrix) theory compactified on a circle, and the manifestation of the type-IIB-string–M-theory duality [22,23] in this theory. We expect to see how D-strings emerge upon compactification, and to infer some properties of a longitudinal membrane from those of a D-string in accordance to type-IIB-string–M-theory duality.

The basic idea is the following. Uncompactified M(atrix) theory is supposed to be equivalent to the strong coupling limit of type-II string theory [24,25]. Therefore, combining this with the well-known type-IIA–type-IIB duality [26,27], M(atrix) theory *compactified on a circle* should be equivalent to the *strong coupling limit* of type-IIB superstring theory compactified on a circle. The usual type-IIB-string–M-theory duality [22,23] asserts that M theory compactified on a torus is dual to type-IIB string theory compactified on a circle. One way to test this is to compactify M(atrix) theory on a transverse torus [7]. But in this paper we are going to compactify the M(atrix) theory on a transverse circle, and to exploit the fact that in the IMF formulation of the M(atrix) theory, the longitudinal 11th direction has to be compactified as an infrared regularization. Thus, the type-IIB-string–M-theory duality we are studying here is a limiting situation of the usual one, in which the torus being compactified involves the longitudinal direction, with radius R_{11} approaching infinity. It is natural to expect that the type-IIB-string–M-theory duality in such limit should allow us to infer some properties of longitudinal membranes.

In the context of M(atrix) theory, we will provide the following evidence for the above mentioned type-IIB-string–M-theory duality. First, we will show that upon compactification on a circle, the M(atrix) theory action can be interpreted as describing many interacting D-strings (or $D1$-branes). In particular, we show the appearance in M(atrix) theory of a topological term recently discovered [28] in the action for a single $D1$-brane. Next we identify, in the IMF spectrum of M(atrix) theory, excitations corresponding to the Bogoniol'nyi-Prasad-Sommerfield (BPS) states of type-IIB strings. They include oscillation modes on the D-string resulting from compactification which, according to type-IIB-string–M-theory duality, can be identified as excitations of a longitudinal membrane. Then, we will show that relations among the parameters of equivalent type-IIA, type-IIB, and M theories (all compactified on a circle) dictated by duality of each pair are satisfied in M(atrix) theory. Moreover, it is shown that the remnant symmetry of the celebrated S duality SL(2,Z) for type-IIB strings in the IMF is a subgroup isomorphic to the group of all integers; also the τ moduli of the strongly coupled type-IIB string theory is identified with the geometric tilt of the circle on which M(atrix) theory is compactified. In the last section, we summarize a number of insights we gain into the IMF description of the properties of longitudinal membranes in M(atrix) theory, which are inferred from the properties of type-IIB D-strings.

At the end of the paper, we present an Appendix devoted to the description of moduli-dependent aspects of compactification on a slanted transverse torus.

II. TYPE-IIB-STRING–M-THEORY DUALITY REVISITED

Let us first consider what we would expect to happen in M theory, in accordance with the type-IIB-string–M-theory duality [22,23], when compactified on a transverse circle in the IMF.

The spectrum of type-IIB theory compactified on a circle of radius R_B for a string of NS-NS and RR charges (q_1,q_2), with q_1,q_2 mutually prime, is

$$M_B^2=\left(\frac{n}{R_B}\right)^2+(2\pi R_B T_{(q_1,q_2)}l)^2+4\pi T_{(q_1,q_2)}(N_R+N_L). \tag{1}$$

The first term is the contribution from the Kaluza-Klein excitations, the second term from the winding modes, and the third term from the string oscillation modes. The tension of the (q_1,q_2) string is

$$T_{(q_1,q_2)}=[(q_1-\chi_0 q_2)^2+g_B^{-2}q_2^2]^{1/2}T_s, \tag{2}$$

where χ is the axion field and the subscript 0 stands for its vacuum expectation value; T_s is the fundamental string tension in the string metric (with l_s the string length scale):

$$T_s=\frac{1}{2\pi\alpha'}=\frac{1}{2\pi l_s^2}. \tag{3}$$

Using the level-matching condition $N_R-N_L=nl$ and the BPS condition $N_R=0$ or $N_L=0$, one finds the spectrum of type-IIB BPS states:

$$M_B=\frac{n}{R_B}+2\pi R_B T_{(q_1,q_2)}l. \tag{4}$$

The spectrum of M theory compactified on a torus of modular parameter $\tau=\tau_1+i\tau_2$ with radii R_1 and R_{11} $(\tau_2=R_1/R_{11})$ is [22,29]

$$M_M^2=\left(\frac{N}{R_{11}}\right)^2+\left(\frac{m-\tau_1 N}{R_1}\right)^2+(A_M T_2^M n)^2+\cdots, \tag{5}$$

where $A_M=(2\pi R_{11})(2\pi R_1)$ is the area of the torus. The first two terms come from the Kaluza-Klein modes, the third from the winding modes, and the contribution of membrane excitations are not written down because the quantum theory of the membrane is what we are going after. Comparing the spectra (1) and (5), one finds that the Kaluza-Klein (winding) modes in type-IIB theory match the winding (Kaluza-Klein) modes in M theory [22,23], if we make the identification $m=q_1 l$, $N=q_2 l$, and[1]

$$R_B=1/(A_M T_2^M), \tag{6}$$

$$R_B=l_s^2/R_1, \tag{7}$$

$$\chi_0=\tau_1, \tag{8}$$

$$g_B=R_{11}/R_1(=1/\tau_2). \tag{9}$$

Therefore the modular parameter of the torus is identified with the vacuum expectation value of the complex field $\chi+ie^{-\phi}$, where ϕ is the dilaton field and $g_B=e^{\phi_0}$.

Here the consistency of Eqs. (6) and (7) requires the relation

$$T_s=2\pi R_{11}T_2^M, \tag{10}$$

implying that a fundamental type-IIB string is identified with a membrane wrapped on R_{11}. Therefore, given the paramteters of M theory, the paramenters of type-IIB theory are determined by Eqs. (6) and (8)–(10). Conversely, given the parameters of type-IIB theory, the parameters of M theory are determined by

$$R_1=\frac{l_s^2}{R_B},\quad R_{11}=g_B\frac{l_s^2}{R_B},\quad T_2^M=g_B^{-1}R_B T_s^2. \tag{11}$$

This duality also matches a membrane wrapped around a cycle of the torus in M theory with a string in type-IIB theory: A cycle on the torus is specified by two mutually prime integers (q_1,q_2) with a minimal length

$$L_{(q_1,q_2)}=2\pi R_{11}[(q_1-\tau_1 q_2)^2+\tau_2^2 q_2^2]^{1/2}. \tag{12}$$

Hence the tension of a type-IIB string of charge (q_1,q_2) is $T_{(q_1,q_2)}=L_{(q_1,q_2)}T_2^M$, in agreement with the above relations.

Once the type-IIB-string–M-theory duality is justified, the spectrum for M theory can be completed by looking at the type-IIB spectrum (4) [22]:

$$M_M=\left[\left(\frac{N}{R_{11}}\right)^2+\left(\frac{m-\tau_1 N}{R_1}\right)^2\right]^{1/2}+A_M T_2^M n, \tag{13}$$

for arbitrary integers N, m, and n. Comparing this with Eq. (5) and its type-IIB analogues (1) and (4), we see that the second term is a mixture of the contributions from winding modes and excitations on the membrane. To compare this spectrum to that of M(atrix) theory, we need to go to the IMF by boosting in the direction of R_{11} so that $P_{11}\equiv N/R_{11}\gg m/R_1$. [This Lorentz transformation corresponds to a change in the Ramond-Ramond (RR) charge q_2 in type-IIB theory to a large number, which can be achieved by an SL(2,**Z**) symmetry transformation.] Thus, in the IMF, the spectrum in M theory appears to be

$$M_M=\frac{N}{R'_{11}}+\frac{R'_{11}}{2N}\left(\frac{m}{R'_1}\right)^2-\frac{\tau_1 m}{\tau_2 R'_1}+A_M T_2^M n+\cdots, \tag{14}$$

where (see Fig. 1)

$$R'_{11}=\frac{\tau_2}{|\tau|}R_{11},\quad R'_1=\frac{|\tau|}{\tau_2}R_1. \tag{15}$$

We note that the fourth term in Eq. (14) is finite in the limit $N\to\infty$, without a factor of $1/P_{11}$ in front of it, showing that it is dominated by the energy of excitations on a longitudinal membrane which scales as P_{11} under a boost in the longitudinal direction [14].

[1]Throughout this paper all quantities appearing in the same equation are given with respect to the same metric.

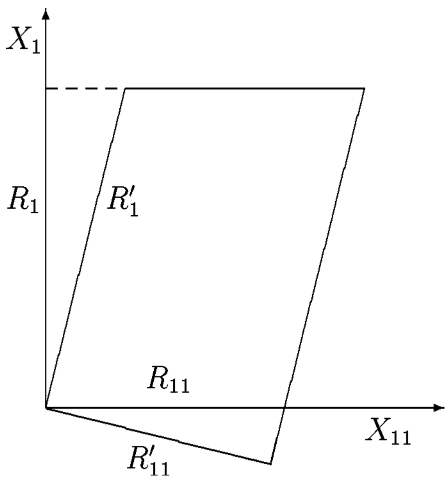

FIG. 1. Longitudinal torus.

In next section we will try to identify the spectrum (14) in M(atrix) theory compactified on a circle.

III. M(ATRIX) THEORY AND COMPACTIFICATION

In this section, we first review the uncompactified M(atrix) theory, with an eye on the guiding role of the type-IIA-string–M-theory duality and on its IMF nature. Then, we will describe compactification of the theory on a circle, *not necessarily perpendicular* to the longitudinal direction that defines the IMF.

A. Type-IIA-string–M-theory duality and uncompactified M(atrix) theory

The Banks-Fischler-Shenker-Susskind (BFSS) action for M theory in the IMF is a large N limit of the supersymmetric matrix quantum mechanics obtained by dimensionally reducing the supersymmetric U(N) Yang-Mills action from $9+1$ dimensions to $0+1$ dimension [7]:

$$S=\frac{1}{g}\int dt\,\mathrm{tr}\left(-\frac{1}{2}\nabla_0 A_i\nabla_0 A_i+\frac{1}{4}[A_i,A_j][A_i,A_j]+\frac{i}{2}\bar{\Psi}(\gamma^0\nabla_0\Psi+\gamma^i[A_i,\Psi])\right), \tag{16}$$

where $i,j=1,2,\ldots,9$ and $\nabla_0=\partial_t+[A_0,\cdot\,]$. A_μ and Ψ are anti-Hermitian $N\times N$ matrices; A^a_μ are real and Ψ^a are Majorana-Weyl spinors in ten dimensions.

This matrix action was originally suggested [30] for a regularized supermembrane in 11 dimensions, since the U(N) gauge symmetry

$$A_0\to UA_0U^\dagger+U\partial_tU^\dagger,\quad A_i\to UA_iU^\dagger,\quad \Psi\to U\Psi U^\dagger \tag{17}$$

can be interpreted as the area-preserving diffeomorphism group of a membrane in the large N limit.

In the context of string theory, this action describes dynamics of N $D0$-branes in type-IIA theory [31]. In the temporal gauge $A_0=0$, using the Hermitian matrices $X_i=-iA_i$ ($i=1,2,\ldots,9$), the Hamiltonian is

$$H=\mathrm{tr}\left(\frac{g}{2}\Pi_i^2-\frac{1}{4g}[X_i,X_j]^2+\frac{1}{2g}\bar{\Psi}\gamma^i[X_i,\Psi]\right). \tag{18}$$

A minimum of the potential term in the Hamiltonian is reached when $\Psi=0$ and all the X_i's can be simultaneously diagonalized. Introducing

$$X_i=T_s x_i, \tag{19}$$

then $(x_i)_{\alpha\alpha}$ ($\alpha=1,2,\ldots,N$) is interpreted as the ith coordinate of the αth $D0$-brane. An off-diagonal entry $(x_i)_{\alpha\beta}$ ($\alpha\neq\beta$) represents the effects due to open strings stretched between the αth and the βth $D0$-brane. Hence the energy of a stretched string, given by the string tension times the distance between the $D0$-branes, should equal the mass of the field $(x_i)_{\alpha\beta}$ in the action [31]. The coefficient of the action (16) is thus $1/g=T_0/T_s^2$, where T_0 is the $D0$-brane tension (the D-particle mass).

In accordance with the well-known type-IIA-string–M-theory duality between M theory compactified on S^1 (with radius R_{11}) and type-IIA theory [24,25], membranes wrapped around S^1 are identified with fundamental type-IIA strings, and unwrapped membranes with $D2$-branes. The string tension and $D2$-brane tension are therefore related to the membrane tension T_2^M by $T_s=2\pi R_{11}T_2^M$ and $T_2=T_2^M$. Recall that the Dp-brane tension in type-II string theory is given by [32,33]

$$T_p=1/[(2\pi)^p g_s l_s^{(p+1)}], \tag{20}$$

where g_s is the string coupling g_A (g_B) in type-IIA (-IIB) theory for p even (odd). Therefore the compactification radius R_{11} and the membrane tension T_2^M can be given in terms of the type-IIA parameters (g_A,l_s) as [24,25]

$$R_{11}=g_A l_s,\quad T_2^M=\frac{1}{(2\pi)^2 g_A l_s^3} \tag{21}$$

or, conversely,

$$g_A=2\pi R_{11}^{3/2}(T_2^M)^{1/2},\quad l_s=\frac{1}{2\pi R_{11}^{1/2}(T_2^M)^{1/2}}. \tag{22}$$

[The Planck length l_p in M theory is defined by $T_2^M=1/(2\pi)^2 l_p^3$, implying that $l_p=g_A^{1/3}l_s$.] It follows that

$$\frac{T_s^2}{g}=T_0=\frac{1}{R_{11}}. \tag{23}$$

According to Eq. (18), the center-of-mass kinetic energy of N $D0$-branes is

$$\frac{R_{11}}{2N}\sum_i (p_i^{\mathrm{com}})^2, \tag{24}$$

where p_i^{com} is the conjugate momentum of the center-of-mass position $x_i^{\mathrm{com}}=\mathrm{tr}(x_i)/N$. The prefactor has an interpretation in M theory as the momentum of N partons ($D0$-branes) in the compactified direction: $P_{11}=N/R_{11}$. So the Hamiltonian (18) is understood as the light-cone energy with P_{11} very large, implying the 11-dimensional Lorentz invariance with the IMF expansion of kinetic energy: $K=(P_{11}^2+\Sigma P_i^2)^{1/2}=P_{11}+(\Sigma_i P_i^2)/2P_{11}+\cdots$.

In the limit $R_{11}\to\infty$ with l_p kept fixed, type-IIA theory goes to the strong coupling limit while its dual theory—M theory—becomes uncompactified in 11 dimensions and is dominated by massless $D0$-branes (supergravitons). *Guided by the type-IIA-string–M-theory duality*, the BFSS formulation of M(atrix) theory [7] just postulates that in the limit both R_{11} and N/R_{11} going to infinity, the Hamiltonian (18) describes the uncompactified M theory in the IMF.

Here R_{11} is used as an infrared cutoff for the uncompactified theory: All winding modes are supposed to be thrown away, except those which wind R_{11} at most once corresponding to longitudinal branes.

To test the type-IIB-string–M-theory duality we revisited in last section, normally one would consider compactification of M(atrix) theory on a transverse torus [7,34,9,10,19], which needs to examine a $(2+1)$-dimensional quantum gauge theory. However, as we are going to demonstrate, it is more direct to test type-IIB-string–M-theory duality in M(atrix) theory compactified only on a *circle* which, by combining the usual type-IIA–type-IIB and type-IIA-string–M-theory dualities, is expected to be the strong coupling limit of type-IIB string theory. One expects to gain interesting insights into physics in the IMF formulation from this study, because it will involve longitudinal membranes that wrap around R_{11}.

B. M(atrix) theory compactified on an oblique circle

Now let us consider M(atrix) theory compactified on a circle, which is normally [7,34,10] taken to be in a direction, say, X_1, perpendicular to the longitudinal X_{11}, with radius R_1. For our purpose, it is necessary to incorporate moduli parameters for equivalent type-IIB string theory, which requires us to consider the compactification on an *oblique* circle, in a tilted X_1'-direction in the X_1-X_{11} plane, with radius R_1'. This makes sense in M(atrix) theory. From the target-space point of view, at least in low energy supergravity, what is relevant is the Kaluza-Klein metric [22,23] which contains the moduli parameters of type-IIB strings. From the world-volume point of view the M(atrix) theory action, which results from compactification and describes $D1$-branes, will be defined on the dual circle, whose radius depends on the tilt modular parameter and is essential for the quantized value of the momentum of the $D1$-brane.

First recall the usual case with S^1 in the X_1 direction. By gauging a discrete subgroup of U(N) representing periodic translations in X_1, the action for $D0$-branes in the compactified space can be written as an action for $D1$-branes on the dual circle [7,34,10]. In accordance with the type-IIA–type-IIB T duality, the winding modes around the circle for open strings stretched between $D0$-branes become the discretized momentum modes for $D1$-branes on the dual circle, while the compactified coordinate X^1 of the target space turns into the covariant derivative with a gauge field A_1 on the world sheet, leading to the action (in the temporal gauge) [34]

$$S=\int dt\int_0^{2\pi R_B}\frac{dx}{2\pi R_B}\frac{1}{2g}\,\mathrm{tr}\Big(\dot X_i^2-\dot A_1{}^2-(\nabla_1X_i)^2+\frac{1}{2}[X_i,X_j]^2+i\bar\Psi(\gamma^0\dot\Psi+\gamma^1\nabla_1\Psi+\gamma^i[iX_i,\Psi])\Big),\tag{25}$$

where $i=2,3,\ldots,9$, $\nabla_1=\partial_x+[A_1,\cdot]$, and $R_B=l_s^2/R_1$ is the dual radius of R_1, exactly as required by the type-IIA–type-IIB duality.

Now let us consider the case of an oblique S^1 in a tilted X_1' direction. In low energy supergravity, the Kaluza-Klein metric in such coordinates will contain a gauge field in the X_1' direction. This gauge field can be gauged away except its Wilson line (or holonomy) degree of freedom. Motivated by this observation, in the dual description, we expect the appearance of a constant background electric field E for the gauge field on the $D1$-brane world sheet so that the field strength F_{01} is shifted to $(F_{01}'-E)$, which is $(\dot A_1'-E)$ in the temporal gauge. Now we denote the gauge field as A_1', because it corresponds to the tilted X_1' coordinate and has a period R_1'. Indeed we can see how this modification comes about by considering compactification on a slanted torus. This leads to (see the Appendix) the following modified action:

$$S=\int dt\int_0^{2\pi R_B}\frac{dx}{2\pi R_B}\frac{1}{2g'}\,\mathrm{tr}\Big(\dot X_i^2-(\dot A_1'-E)^2-(\nabla_1X_i)^2+\frac{1}{2}[X_i,X_j]^2+i\bar\Psi(\gamma^0\dot\Psi+\gamma^1\nabla_1\Psi+\gamma^i[iX_i,\Psi])\Big),\tag{26}$$

where $\nabla_1=\partial_x+[A_1',\cdot]$; $R_B=l_s^2/R_1$ is unchanged, while

$$E=-i\lambda T_s\equiv-i\frac{\tau_1}{\tau_2}T_s,\qquad\frac{T_s^2}{g'}=\frac{1}{R_{11}'}.\tag{27}$$

Here $\tau=\tau_1+i\tau_2$ is the modular parameter of the slanted longitudinal torus with radii R_1 and R_{11}; the relations between (R_1',R_{11}') and (R_1,R_{11}) are exactly those of Eq. (15). Note that the coefficient g' is defined as Eq. (23) with R_{11} replaced by R_{11}', while the change of (R_1,R_{11}) to (R_1',R_{11}') does not affect the area A_M. Moreover, here the values of the field A_1' range between 0 and $2\pi R_1'T_s^2$, with R_1' just the radius of the (0,1) cycle on the slanted torus [see Eq. (12)]. Using Eqs. (7) and (19), one finds that if Eq. (9) is used to define g_B, then the prefactor of the action (26) is precisely the tension $T_{(0,1)}=|\tau|T_s$, appropriate for a type-IIB string of charge (0,1).

Comparing this to the action (25), we see that the term $\dot A_1^2$ has been replaced by $(\dot A_1'-E)^2$, with the background gauge field E given by the above relation. In addition to a constant term, this modification adds a topological term of the form $-i\lambda\int\dot A_1'$ to the action for $D1$-branes. This is an analogue of the θ term in two-dimensional gauge theory [35].

By imposing periodic boundary conditions in time, a change in λ by b results in a change in the action by

$-i2\pi|\tau|^2 b/\tau_2$ times an integer. So the period for λ in the path integral measure e^{iS} is $\tau_2/|\tau|^2$.

We note that this moduli-dependent topological term recently also appears in Ref. [28] for the action of a $D1$-brane. It was used there to recover the fundamental string action by the electric-magnetic duality. Here in our treatment this term appears automatically, and is just right (see next section) to reproduce the moduli-dependent term, the third term in Eq. (14), in the IMF spectrum of M theory. We also note that the action (26) has an additional constant term proportional to E^2, which is just right for the Hamiltonian to have a minimum of zero energy, which is consistent with unbroken supersymmetry.

IV. TYPE-IIB-STRING–M-THEORY DUALITY IN M(ATRIX) THEORY

We propose to interpret the M(atrix) theory action (26) as describing strongly coupled type-IIB theory compactified on a circle in terms of D-strings.

A. Spectrum of type-IIB BPS states

As evidence for this equivalence, we now show that the BPS spectrum (14) expected from the type-IIB-string–M-theory duality can be found in M(atrix) theory.

First, the kinetic energy of the U(1) factor of A_1', $\mathrm{tr}(A_1')/N$, gives the second term in Eq. (14), which is simply part of the matching (24) mentioned in the last section. In addition, the contribution of the topological term (due to the Wilson line) to the Hamiltonian just matches the third term in Eq. (14) as well, because of the relation (27) between E and τ_1/τ_2.

In the following we will take the limit $R_1/l_p \to 0$ so that the $[X_i,X_j]^2$ term in the Hamiltonian indicates that low energy states have a vanishing commutator $[X_i,X_j]=0$. Now consider the particular configurations which satisfy $[X_i,X_j]=0$ and $\Psi=0$, for which one can simultaneously diagonalize the X_i's. Then the action (26) becomes proportional to the free action, $\Sigma_\alpha[(\dot{X}_i)^2_{\alpha\alpha}-(\partial_x X_i)^2_{\alpha\alpha}]$, for a system of closed $D1$-branes ($D1$-strings). By Fourier expansion $X_i=\Sigma_k X_{ik}e^{ikx/R_B}$, one finds that the excitation (or oscillation) modes of X_i on the closed $D1$-brane have the spectrum $M=n/R_B=2\pi T_s R_1 n$ upon quantization. The operator n is defined by $n=\Sigma_{ik\alpha}kN_{ik\alpha}$, where $N_{ik\alpha}$ is the number operator for the mode $(X_{ik})_{\alpha\alpha}$. Obviously this part of energy should be identified with the fourth term in Eq. (14) or, equivalently, with the first term in (4).

We observe that in the present IMF formulation, this part of the energy, which involves the excitations of the D-string, does *not* have a prefactor of $1/(2P_{11})$, in contrast to the usual IMF energies for purely transverse excitations. This implies that the proper interpretation of this part of energy in M theory should be attributed to excitations on a (longitudinal) membrane wrapped on the 11th direction, just as expected from the type-IIB-string–M-theory duality. (See the comment at the end of Sec. II.) Note that in this argument we have taken advantage of the IMF formulation, without the need of constructing a semiclassical longitudinal membrane.

One can check that both the topological configuration of A_1' and the oscillation modes of X_i are indeed BPS states in M(atrix) theory.[2] The 11-dimensional supersymmetry transformations in the IMF consist of the dynamical part

$$\delta A_\mu = \frac{i}{2}\bar{\epsilon}\gamma_\mu\Psi, \quad \delta\Psi = -\frac{1}{4}F_{\mu\nu}\gamma^{\mu\nu}\epsilon, \tag{28}$$

where $F^{0i}=\dot{X}^i$, $F^{1i}=\nabla_1 X^i$, and $F^{ij}=[X^i,X^j]$, and the kinematical part

$$\delta A_\mu = 0, \quad \delta\Psi = i\tilde{\epsilon}, \tag{29}$$

where ϵ and $\tilde{\epsilon}$ are both Majorana-Weyl spinors times a unit matrix. Each part of the supersymmetry has 16 generators and the total supersymmetry has 32 generators. It turns out that the supersymmetry algebra involves central terms, which can be interpreted as various RR charges [14]. The topologically nontrivial configuration of A_1' preserves one half of the total supersymmetry (SUSY) as a linear combination of the dynamical part and the kinematical part. The associated charge is simply the Kaluza-Klein momentum P_1 proportional to m. The purely left-moving (or right-moving) oscillation modes preserve one-quarter of the total SUSY (half of the dynamical part) and give one nonzero RR charge Z^1 (defined in Ref. [14] as a central term in the SUSY algebra) proportional to k, corresponding to branes wrapped around R_{11}.

It is instructive to compare the above identification of the spectrum for a longitudinal membrane with that for a transverse membrane [7]. The configuration in M(atrix) theory that represents a membrane wrapped n times around a transverse torus with radii R_1 and R_2 is given by $x_1=R_1 p$, $x_2=R_2 q$, with p and q satisfying $[p,q]=2\pi i n/N$. [While the representation of the canonical commutation relation can only be realized in infinite dimensional representations, the commutation relation above makes sense with the understanding that x_i ($i=1,2$) resides on a circle of radius R_i and that the right-hand side is appropriately normalized by the dimension of the representation N.[3]] As is easy to verify, it is the potential term $\frac{1}{4}[x_i,x_j]^2$ in the light-cone Hamiltonian (18) that correctly reproduces the membrane spectrum $A_M T_2^M n$, where $A_M=(2\pi R_1)(2\pi R_2)$ is the area of the torus, for this configuration.

In this discussion, to match the type-IIB spectrum, it is necessary to have the membrane wound around the R_{11} only once. This is in agreement with the use of R_{11} in M(atrix) theory as a cutoff, so that longitudinal branes are allowed to wind it only once. In addition, this is exactly what has been conjectured by Schwarz in his discussions on the type-IIB-string–M-theory duality [1], namely, that in M theory the membrane wrapped on a torus should select a preferred cycle in which it is wrapped many times, and this preferred direction must be the one defined by the type-IIB-theory Kaluza-Klein excitations, which is nothing but the R_1 direction. This can be argued as follows. The type-IIB Kaluza-Klein modes by T duality are type-IIA string winding modes around R_1. The type-IIA strings are membranes wound around R_{11} only once by the type-IIA-string–M-theory duality. Hence the

[2]We thank M. Li for discussions on this matter.
[3]We thank B. Zumino for pointing this out to us.

type-IIB Kaluza-Klein modes are membranes wound around R_{11} once and R_1 an arbitrary number of times.

B. Relations among type-IIA–type-IIB-M-theory parameters

The above spectrum matching can also be understood heuristically from type-IIA–type-IIB [26,27] and type-IIA-string–M-theory dualities [24,25]. Let us recall that the quantity $(X_{ik})_{\alpha\alpha}$ has a dual interpretation in type-IIA or -IIB language. In type-IIA language, when compactifying $D0$-branes on a circle of radius R_1, an open string wound k times around the circle with both ends on the same $D0$-brane labeled α is known to be represented by $(X_{ik})_{\alpha\alpha}$ [34]. On the other hand, in the dual (type-IIB) language, $(X_{ik})_{\alpha\alpha}$ represents an oscillation mode on the D-string. Because strings in type-IIA theory are thought of, by the type-IIA-string–M-theory duality, as membranes wound around R_{11}, we are again led to the identification in the last subsection of the D-string oscillation modes with excitations on the longitudinal membrane winding around R_1 and R_{11}.

Thus, what we have here is type-IIA–type-IIB–M-theory triality; i.e., M(atrix) theory compactified on a circle is equivalent to either the strong coupling limit of type-IIA or type-IIB theory, each compactified on a circle too. The type-IIA-string–M-theory duality and type-IIB-string–M-theory duality we considered separately in the above are linked by the usual type-IIA–type-IIB duality.

If we consider the type-IIA-string–M-theory duality for M theory compactified on a torus and type IIA on a circle, the type-IIA-string–M-theory relations for this case are simply Eqs. (21) and (22) together with

$$R_A = R_1, \tag{30}$$

where R_A is the radius of the circle in type-IIA theory. The type-IIA–type-IIB duality identifies Dp-branes in type-IIA with $D(p\pm 1)$-branes in type-IIB theory by wrapping or unwrapping [27]. Therefore we have Eq. (7) and [26]

$$g_B = g_A \frac{l_s}{R_A}. \tag{31}$$

It can be easily checked that these relations, required by dualities of each pair of type-IIA, type-IIB, and M theories, are satisfied in M(atrix) theory compactified on a circle. Also using any two of the dualities, one can derive uniquely the third one. If one uses the dualities to go cyclicly from one theory through the other two to return to the original theory, one finds that the parameters are unchanged after the journey. If it were not the case it would mean that there exist new self-dualities in these theories.

Because R_{11} is to be treated as a cutoff, it should be much larger than any other length scale in the theory. Hence in the limit $R_{11}\to\infty$, the matrix model gives S^1-compactified M theory dual to type-IIB theory compactified on the dual circle in the strong coupling limit according to Eq. (9).

C. SL(2,Z) duality of type-IIB theory

The SL(2,**Z**) duality of type-IIB theory [22] transforms a (q_1,q_2) string to a string with charge (q_1',q_2') given by

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{32}$$

where $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ is an SL(2,**Z**) matrix. The coupling and string tension transform as

$$g_B \to g_B' = |c\tau + d|^2 g_B, \tag{33}$$

$$T_s \to T_s' = |c\tau + d| T_s. \tag{34}$$

These agree, by the the type-IIB-string–M-theory duality relations (6)–(11), with the modular transformation

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d} \tag{35}$$

of the torus in M theory, on which the theory is compactified. For the modular transformation to be a geometric symmetry of M theory, the area A_M of the torus is to be fixed and thus the radii transform as

$$R_1 \to \frac{R_1}{|c\tau + d|}, \quad R_{11} \to |c\tau + d| R_{11}. \tag{36}$$

The spectrum in M theory, $\{(N/R_{11})^2 + [(m - \tau_1 N)/R_1]^2\}^{1/2}$, is invariant under the transformation (35) and (36) and

$$\begin{pmatrix} m \\ N \end{pmatrix} \to \begin{pmatrix} m' \\ N' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} m \\ N \end{pmatrix}. \tag{37}$$

In the IMF description of M(atrix) theory compactified on a circle, we do not expect to have the full SL(2,Z) symmetry, since the longitudinal direction is preferred, and the limit of R_{11} may be different from that of R_1. However, the theory may be invariant under a subgroup. We have checked that if we make the transformation (35)–(37), then the invariance of the IMF spectrum (14) requires that $a=1$, $b=0$, and $d=1$. So the remnant symmetry is

$$\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \tag{38}$$

where c is an integer.

D. Finiteness of the spectrum and limits of parameters

For the spectrum M_M, Eq. (14), we obtained in Sec. IV A for M(atrix) theory to make sense in the limit $R_{11}\to\infty$, other parameters in the theory have to take appropriate limits accordingly. To be more precise about these limits, one should consider only dimensionless quantities. For instance,

$$r_B = R_B / l_s, \quad r_B^c = g_B^{-1/4} R_B / l_s \tag{39}$$

are the type-IIB radius measured in the type-IIB string metric and the canonical metric, where $l_s = 1$ and $g_B^{1/4} l_s = 1$, respectively, and

$$r_1 = R_1 / l_p, \quad r_{11} = R_{11} / l_p \tag{40}$$

are the values of R_1 and R_{11} measured in the 11-dimensional Planck units. Similarly, the finiteness of the spectrum is to be considered with respect to a certain system of units.

In M theory it is natural to measure everything in terms of the Planck scale l_P, and so the dimensionless spectrum is $m_P = M_M l_P$. In type-IIB theory one can choose to use the string metric or the canonical metric, where the spectrum appears to be $m_s = M_M l_s$ and $m_s^c = g_B^{1/4} M_M l_s$, respectively. The results in the type-IIB canonical metric are the same as in the Planck units. In the following we discuss separately in the Planck units and string units the appropriate limits of various parameters in the theory for the spectrum to be finite.

In the Planck units, for both the third and fourth terms in Eq. (14) to have finite limits, we need

$$r_B^c \sim \text{finite}, \quad r_1 \sim r_{11}^{-1}, \quad \tau_1 \sim r_{11}^{-3}, \quad \tau_2 \sim r_{11}^{-2}. \quad (41)$$

As a consequence, the parameter $\lambda \equiv \tau_1/\tau_2$ does not have a finite period since its period $\tau_2/|\tau|^2$ goes to infinity. It is also easy to check that the modular parameter τ_1/τ_2 of our longitudinal torus, scaled by $(l_p/l_s)^2$ so that it has a finite limit, is invariant under the remnant subgroup of SL(2,**Z**) mentioned in the previous subsection. In this case one has $r_1 \to 0$. This gives a version of M theory dual to type-IIB theory compactified on a finite circle in the strong coupling limit. The part of the spectrum considered above has a finite limit in both the Planck units and the type-IIB canonical metric. The opposite extremum $r_1 \to \infty$ is just the original case considered in Ref. [7].

In the string metric we need

$$r_B \sim \text{finite}, \quad \tau_2 \sim r_{11}^{-3/2}, \quad (42)$$

for arbitrary τ_1. In this case it is possible to choose $\tau_1 \sim \tau_2^{1/2}$ so that the parameter λ has a finite period. The modular parameter τ_1/τ_2 can have an arbitrary limit by choosing the limit of τ_1. But in any case τ_1/τ_2 scaled by an appropriate power of l_p/l_s is invariant under the remnant subgroup. Here we get a strong coupling limit of type-IIB theory compactified on a circle with a finite radius in string units but zero radius in the canonical metric.

Note that in both cases considered above $r_1 = R_1/l_P$ approaches zero in the limit, which is consistent with our assumption made in the begining of Sec. IV A.

It is certainly possible to take other limits. For instance, one can take the units such that R_1 is finite, and then one needs $T_2^M \to 0$. This is a special limit in the type-IIB-string–M-theory duality which is the strong coupling limit of type-IIB theory (or, by T duality, type-IIA theory) compactified on a circle as well as the weak tension limit in M theory compactified on the dual circle.

V. DISCUSSION: INSIGHTS INTO LONGITUDINAL MEMBRANES

Previously the type-IIB-string–M-theory duality refers to the equivalence between M theory compactified on a torus and type-IIB superstring theory compactified on a circle. The recently proposed nonperturbative IMF formulation of M(atrix) theory makes it possible to discuss the equivalence between M(atrix) theory compactified on a circle and (the strong coupling limit of) type-IIB theory also compactified on a circle, as a special limit of the generic case in which one cycle on the torus is in the longitudinal direction and is taken to infinity. In this paper we establish this the type-IIB-string–M-theory duality in M(atrix) theory context. Several pieces of evidence we provide are described in Sec. IV, and summarized in the abstract and Introduction.

Here we would like to concentrate on the insights we have gained into the IMF description of M(atrix) theory for the behavior of a longitudinal membrane.

No one has succeeded in constructing a longitudinal membrane, in the way a transverse membrane is constructed [7], and no one doubts the existence of a longitudinal membrane in M(atrix) theory. From our study we have seen indeed that the longitudinal membranes are hiding in the theory. By the the type-IIB-string–M-theory duality established above, properties of a membrane that wraps the longitudinal direction once can be extracted from those of the D-string obtained from compactification on a circle. These properties are the following.

The excitations of a longitudinal membrane are identified with the quantum oscillation modes on the D-string, by energy matching and noting that the light-cone energy of these modes is finite. This is in sharp contrast to the energy of a purely transverse membrane, which comes from the commutator potential term.

For a longitudinal membrane wrapped on the longitudinal torus with modular parameter τ, its light-cone energy has a term dependent on the ratio τ_1/τ_2, and independent of P_{11}, as expected from general grounds [see Eq. (14)]. The mechanism in M(atrix) theory responsible for this energy is similar to that in a θ vacuum on the D-string, since the latter contains a topological term with τ_1/τ_2 as coefficient, analogous to the θ vacuum parameter.

The properties of a longitudinal membrane we extract through studying $D1$-branes are always such that they can be viewed as the limit of a membrane wrapped on a transverse torus with one of the two cycles going to infinity. This fact provides us one more piece of evidence for the 11-dimensional Lorentz invariance of M(atrix) theory.

In summary, we conclude that the properties of a membrane which wraps once in the longitudinal direction can be extracted from those of a D-string obtained by compactifying M(atrix) theory on a circle.

Another interesting way to look at the model considered in this paper is to interchange the roles of R_1 and R_{11}. Namely, previously we had $R_A = R_1$ and $g_A = R_{11}/l_s$, but now we set $R_A' = R_{11}$ and $g_A' = R_1/l_s$. The ten dimensions of type-IIA theory are thus the (0,2,3, . . . ,9,11)th dimensions. If $\tau_1 = 0$, the first two terms in the spectrum (14) are the large P_{11} expansion of the relativistic kinetic energy $\sqrt{P_{11}^2 + (mT_0)^2}$ where mT_0 is the mass of m $D0$-branes. The fourth term in Eq. (14) is the winding energy of n type-IIA string wound around R_{11} once. With $R_{11} \to \infty$ and $R_1 \to 0$, this model should be equivalent to uncompactified type-IIA theory in the weak coupling limit (or its dual type-IIB theory).

Note added. When after submission of the writing of this paper a paper by T. Banks and N. Seiberg, Nucl. Phys. **B497**, 41 (1997), appeared, which, among other things, has some overlap with part of what we address here.

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APPENDIX: MODULAR PARAMETER AND COMPACTIFICATION ON A TORUS

The general idea of the compactification of M(atrix) theory on a compact manifold [7,34,10] is to consider the compact manifold as the quotient of a covering space over a discrete group. The matrix model on the compact space is then obtained by taking the quotient of the U(N) matrices over the discrete group, which is to be embedded in U(N) as a subgroup.

For toroidal compactifications [34,10], one needs to choose unitary matrices $\{U_i,\ i=1,\dots,d\}$, where d is the dimension of the torus, commuting with each other so that the discrete group generated by them is isomorphic to the fundamental group of the torus $\mathbf{Z}^d$. The action of U_i on the coordinates is

$$U_i X^\mu U_i^\dagger = X^\mu + e_i^\mu, \tag{A1}$$

where e_i^μ is the basis of the lattice whose unit cell is the torus. Obviously U_i is the operator translating all fields a whole cycle along e_i.

The matrix theory is then restricted to be invariant under the action of the U_i's. This is most easily realized by viewing $e^i_{\ \mu} X^\mu$ as the covariant derivative $\partial/\partial x^i + A_i$ in the direction of e^i, dual to e_i, and the U_i as $e^{2\pi i x^i}$, where x^i is a coordinate on the dual torus and so we have interchanged the role of coordinates and momenta. Naturally the matrix model becomes a $d+1$ dimensional gauge field theory.

For a slanted two-torus with modular parameter $\tau=\tau_1+i\tau_2$ and radii $R_1, R_2=\tau_2 R_1$, it is convenient to introduce slanted coordinates (X_1', X_2') by

$$\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} 1 & -\dfrac{\tau_1}{\tau_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{A2}$$

out of the orthonormal coordinate system (X_1, X_2). The nice thing about the X_i''s is that they reside on circles of radii R_i. The discrete group generated by U_1, U_2 with $U_1U_2=U_2U_1$ acts on them simply as $U_i X_j' U_i^\dagger = X_j' + 2\pi\delta_{ij} R_j$. From the action $\frac{1}{2}(\dot{X}_1^2+\dot{X}_2^2)$ one defines conjugate variables and finds

$$(P_1' P_2') = (P_1 P_2) \begin{pmatrix} 1 & \dfrac{\tau_1}{\tau_2} \\ 0 & 1 \end{pmatrix}. \tag{A3}$$

It follows that the kinetic energy is $\frac{1}{2}(P_1^2+P_2^2)=\frac{1}{2}\{P_1'^2 + [P_2' - (\tau_1/\tau_2)P_1']^2\}$, giving the spectrum of $(m_1/R_1)^2 + [(m_2-\tau_1 m_1)/R_2]^2$.

The case of a longitudial membrane can be inferred from this result. To do so, note that R_1 and R_2 here correspond to R_{11} and R_1, respectively, in this paper for the longitudinal torus. Therefore the first two terms in Eq. (5) follow. Further, the action $\frac{1}{2}(\dot{X}_1^2+\dot{X}_2^2)$ can be written as $\frac{1}{2}[\dot{Y}_1^2 + (\dot{Y}_2 + \tau_1 \dot{Y}_1 \tau_2)^2]$, where $Y_1=\tau_2 X_1'/|\tau|$ and $Y_2=|\tau| X_2'/\tau_2$. Applying this result to the longitudinal case in the infinite momentum frame ($\dot{Y}_1=1$), we see that the prescription is to replace R_{11} by R_{11}' and $\dot{A}_1$ by $[\dot{A}_1' + i(\tau_1/\tau_2)T_s]$ where A_1' (corresponding to $iT_s Y_2$) is valued in the range $iT_s[0, 2\pi R_1')$. This gives what we found in Sec. IV A.

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