Anomalous Skin Effect in a Magnetic Field*

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(Received October 14, 1957)

A classical and quantum mechanical derivation of cyclotron resonance in metals is given. The classical result differs slightly from that obtained by Azbel and Kaner. The quantum derivation yields the same result as the classical calculation except that in the limit of low quantum numbers or high magnetic fields a de Haas-van Alphen type of variation of the surface impedance occurs rather than the resonance behavior.

I. INTRODUCTION

PIPPARD[1] has indicated the importance of microwave surface impedance measurements for obtaining information concerning the electronic energy band structure in metals. When the electron mean free path is much greater than the skin depth, the collision term can be neglected in the transport calculation, and measurements can be interpreted directly in terms of the anisotropy of the Fermi surface. By measuring anisotropies in the surface impedance in the anomalous skin effect region, Pippard[1] has given a detailed picture of the Fermi surface in copper. Azbel and Kaner[2] suggested that the application of a dc magnetic field parallel to the surface of the metal and to the ac electric field should yield a periodic variation of the surface impedance, $Z(0)$, from which one can determine an average effective mass for electrons at the Fermi surface. The derivation of the Azbel and Kaner result in terms of Pippard's "ineffectiveness" concept has been given by Heine.[3] An effect like that which Azbel and Kaner predicted was first observed by Fawcett[4] in samples of tin and copper, and later in tin by Kip *et al.*[5] with better resolution of the resonance lines.

Past theoretical treatments of the skin effect in a magnetic field are not quite satisfactory for the following reasons. One might suspect that a periodicity of $Z(0)$ should occur in any metal showing a de Haas-van Alphen effect, as electronic transport processes like the Hall effect and the magnetoresistance are affected by the quantization of the electronic levels and show de Haas-van Alphen periods in high magnetic fields. The quantum transport treatment enables one to show that at high magnetic fields, i.e., at fields where de Haas-van Alphen periods are observed in the susceptibility, the cyclotron resonance data are not easily interpretable, as the de Haas-van Alphen periods dominate. Secondly, Azbel and Kaner solve a Boltzmann equation, the validity of which is questionable at high magnetic fields, i.e., fields such that $\omega_c\tau>1$ where $\omega_c=eH_0/mc$ and τ if the relaxation time. Argyres[6] has shown that one predicts appreciably different results for the magnetoresistance in the region $\omega_c\tau>1$ by a quantum-mechanical calculation as opposed to a solution of the Boltzmann transport equation.

Section II of this paper contains the solution to the Boltzmann equation for a simple parabolic energy band and the assumption that a relaxation time exists. This case can be solved exactly. The result differs slightly from that obtained by Azbel and Kaner. In Sec. III the quantum mechanical problem is formulated and in Sec. IV application is made to anomalous skin effect problems. The result for the skin effect without applied magnetic fields is precisely the same as that obtained by the solution of the Boltzmann equation. A detailed quantum mechanical treatment of anomalous skin effects with applications to superconductors will be given in a paper by Mattis and Bardeen.[7] Section IV also makes the application of the results obtained in Sec. III to the longitudinal and transverse cyclotron resonance problem. The resonance result is the same as that obtained in Sec. II for the longitudinal case, provided $\hbar\omega\sim\hbar\omega_c\ll\mathcal{E}_F$, where $\mathcal{E}_F$ is the Fermi energy of the metal and ω the applied rf field angular velocity. The case, $\hbar\omega\ll\hbar\omega_c\sim\mathcal{E}_F$ shows a de Haas-van Alphen type of periodicity. The case $\hbar\omega\sim\hbar\omega_c\sim\mathcal{E}_F$ is quite complex, and in this region one must be quite cautious in interpreting the resonance data.

II. CLASSICAL TREATMENT OF LONGITUDINAL CYCLOTRON RESONANCE

The problem is solved first by means of the Boltzmann equation, which serves to introduce the means of handling the specular reflection boundary condition and of outlining the form which the later quantum

* This research was sponsored by the Office of Ordnance Research, United States Army. It is based in part upon material in a dissertation submitted by D.M. in partial fulfillment of the requirements for the Ph.D. degree at the University of Illinois.

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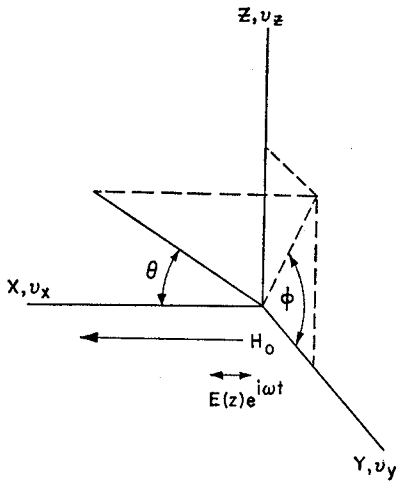

FIG. 1. Coordinate system and configuration of electric and magnetic fields for longitudinal cyclotron resonance. The xy plane is the surface of the metal which occupies the space for $z>0$.

calculation will assume. The diffuse reflection case, though favored by experimental results, does not seem to give appreciably different results from the case treated here, and the mathematical simplicity obtained for specular reflection is considerable.

The field configuration and coordinate system are as shown in Fig. 1. The Boltzmann equation is

$$(1+i\omega\tau)\delta f+v\tau\sin\theta\sin\varphi\frac{\partial}{\partial z}\delta f+\omega_c\tau\frac{\partial}{\partial\varphi}\delta f=-ef_0'\tau E(z)v\cos\theta, \quad (1)$$

where v, θ, φ are the polar coordinates in velocity space, the distribution function $f=f_0+\delta fe^{i\omega t}$, $f_0'=\partial f_0/\partial\mathcal{E}$, τ is the relaxation time, $E(z)e^{i\omega t}$ is the rf electric field which damps out in the metal, and $\omega_c=eH_0/mc$. In terms of the Fourier transforms

$$\delta f_q=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{iqz}\delta f dz,$$
$$E_q=\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty e^{iqz}E(z)dz, \quad (2)$$

Eq. (1) becomes

$$\left[1-iqv\bar{\tau}\sin\theta\sin\varphi+\omega_c\bar{\tau}\frac{\partial}{\partial\varphi}\right]\delta f_q=-ef_0'E_qv\bar{\tau}\cos\theta, \quad (3)$$

where

$$\bar{\tau}=\tau/(1+i\omega\tau).$$

Equation (3) has the solution

$$\delta f_q=ef_0'v\cos\theta E_q\omega_c^{-1}\int_\varphi^\infty d\varphi'\exp\{(\omega_c\bar{\tau})^{-1}\times[(\varphi'-\varphi)+iv\bar{\tau}q\sin\theta(\cos\varphi'-\cos\varphi)]\}. \quad (4)$$

The constant of integration has been determined by the boundary condition $\delta f_q(\varphi+2\pi)=\delta f_q(\varphi)$. The Fourier transform of the x component of the current is given by

$$I_q=2em^3h^{-3}\int\delta f_q\, v^3dv\cos\theta\sin\theta d\theta d\varphi. \quad (5)$$

The integration over the velocity is accomplished by use of the relation

$$\int_0^\infty\frac{\partial f_0}{\partial v}v^3g(v)dv=-\frac{3}{8\pi}\left(\frac{2\pi h}{m}\right)^3Ng(v_F), \quad (6)$$

where v_F is the velocity at the Fermi surface and N is the electron concentration. Substituting (4) into (5) and making use of (6), one obtains

$$I_q=-\frac{3}{4\pi}\frac{Ne^2E_q}{m\omega_c}\left[1-\exp\left(\frac{2\pi}{\omega_c\bar{\tau}}\right)\right]^{-1}\int_0^\pi\cos^2\theta\sin\theta d\theta \times\int_0^{2\pi}d\varphi\int^{\varphi+2\pi}d\varphi'\exp\{(\omega_c\bar{\tau})^{-1}[(\varphi'-\varphi)+iv_F\bar{\tau}q\sin\theta(\cos\varphi'-\cos\varphi)]\}. \quad (7)$$

The change of variable

$$\alpha=\tfrac{1}{2}(\varphi'-\varphi), \quad \beta=\tfrac{1}{2}(\varphi'+\varphi)$$

enables one to write (7) as

$$I_q=\frac{3}{2\pi}\frac{Ne^2E_q}{m\omega_c}\left[1-\exp\left(\frac{2\pi}{\omega_c\bar{\tau}}\right)\right]^{-1}\int_0^\pi\cos^2\theta\sin\theta d\theta\int_0^\pi d\alpha\int_\alpha^{\alpha+2\pi}d\beta\exp\{2(\omega_c\bar{\tau})^{-1}[\alpha-iv_F\bar{\tau}q\sin\theta\sin\beta\sin\alpha]\}, \quad (8)$$

which upon integration gives

$$I_q=\tfrac{3}{4}\frac{\sigma_0E_q}{1+i\omega\tau}K_b(iv_F\bar{\tau}q), \quad (9)$$

where $\sigma_0=Ne^2\tau/m$, $b=\omega_c\bar{\tau}$, and

$$K_b(s)=4\sum_{n=0}^\infty\frac{s^{2n}}{(2n+3)(2n+1)}\prod_{\nu=0}^n\frac{1}{1+\nu^2b^2}. \quad (10)$$

In the limit of high magnetic fields, $\omega_c\tau \gg \omega\tau$ and $\omega_c\tau \gg 1$, only the $n=0$ term of the series is important and one obtains Ohm's law

$$I_q = [\sigma_0/(1+i\omega\tau)]E_q. \tag{11}$$

The case of zero magnetic field can also be handled easily. In this case the summation yields

$$K_0(s) = \frac{1}{s^3}\left\{2s - (1-s^2)\ln\left(\frac{1+s}{1-s}\right)\right\}, \tag{12}$$

which gives, for small s ($v_F q\tau \ll |1+i\omega\tau|$ or $\lambda/\delta \ll |1+i\omega\tau|$; λ is the mean free path and δ the skin depth),

$$K_0(0) = \tfrac{4}{3} \tag{13}$$

resulting once again in Ohm's law for the current. In the extreme anomalous limit ($v_F q\tau \gg |1+i\omega\tau|$), one has the asymptotic expansion

$$K_0(s) \simeq -i\pi/s, \tag{14}$$

which yields the current

$$I_q = \tfrac{3}{4}\pi \frac{Ne^2 E_q}{m v_F q} \tag{15}$$

independent of the relaxation time. The asymptotic expansion of (10) for large sb^{-1} ($v_F q/\omega_c \gg 1$ or $r_c/\delta \gg 1$; r_c is the cyclotron radius) is

$$K_b(s) \simeq -i(\pi/s)\coth(\pi/b), \tag{16}$$

giving the current

$$I_q = \tfrac{3}{4}\pi \frac{Ne^2 E_q}{m v_F q}\coth\left(\pi \frac{1+i\omega\tau}{\omega_c\tau}\right), \tag{17}$$

which for $\omega\tau \gg 1$ shows periodic oscillations.

Following a method outlined by Serber,[8] one can use expressions (11), (15), and (17) to obtain the surface impedance for the case of specular reflection in a rather simple manner. Maxwell's equations give

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}E = \frac{4\pi i\omega}{c^2}I, \tag{18}$$

the Fourier transform of which is

$$[-q^2 + (\omega^2/c^2)]E_q = (4\pi i\omega/c^2)I_q. \tag{19}$$

An electron which is specularly reflected from the surface will follow a trajectory after reflection which is just the mirror image of the trajectory which it would have followed if it had been allowed to cross over into the other half-plane. When magnetic fields are present the dc magnetic field must be reversed in the upper half-plane, as a magnetic field is an axial vector which reverses sign upon reflection. The problem may now be considered in an infinite medium if the following extensions are made:

$$E(-z) = E(z), \quad E_{-q} = E_q, \quad I_{-q}(-\omega_c) = I_q(\omega_c). \tag{20}$$

The latter is true in (17) by virtue of the reversal of sign of ω_c for plus and minus q. In addition, the solution of E *vs* z must show a discontinuity in the first derivative at $z=0$. This is accomplished by adding the term $2E'(0)\delta(z)$ to the right-hand side of (18) or $(2/\pi)^{\frac{1}{2}}E'(0)$ to the right-hand side of (19), where $E'(0) = (dE/dz)_{z=0}$. The equation relating the Fourier coefficients in which the boundary conditions are already contained is

$$\left[-q^2 + \frac{\omega^2}{c^2}\right]E_q = \frac{4\pi i\omega}{c^2}I_q + \left(\frac{2}{\pi}\right)^{\frac{1}{2}}E'(0). \tag{21}$$

This procedure is equivalent to introducing a current sheet on the $z=0$ plane of the infinite medium. The medium has been made infinite to account for the specular reflection of the electrons from the surfaces, after which a current sheet must be introduced at $z=0$ to produce the correct boundary conditions for the electric field.

The surface impedance is defined as

$$Z(0) = R + iX = \frac{4\pi}{c}\frac{E_x(0)}{H_y(0)} = -\frac{4\pi i\omega}{c^2}\frac{E(0)}{E'(0)}. \tag{22}$$

The quantity $E(z)$ is given by the inverse Fourier transform

$$E(z) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} E_q e^{-iqz}dq, \tag{23}$$

where E_q is obtained from (21) and the expression for I_q obtained by a solution of the transport equation. The high field limit, $r_c \ll \delta$, gives, using (11), (21), and (23),

$$\frac{E(0)}{E'(0)} = \frac{2}{\pi}\int_0^{\infty} dq\left[-q^2 + \frac{\omega^2}{c^2} - \frac{4\pi i\omega}{c^2}\left(\frac{\sigma_0}{1+i\omega\tau}\right)\right]^{-1} = \frac{-1}{q_c}, \tag{24}$$

where

$$q_c = \left[-\frac{\omega^2}{c^2} + \frac{4\pi i\omega\sigma_0}{c^2(1+i\omega\tau)}\right]^{\frac{1}{2}}, \tag{25}$$

and the sign of the square root is such that the real part of q_c is positive. This is identical to the skin effect problem

[8] R. Serber (unpublished). The authors are indebted to M. H. Cohen for calling this work to their attention.

in the absence of a magnetic field with $E(z)$ following the exponential law $e^{-q_0 z}$. The low-field limit, $r_c \gg \delta$, is obtained in a similar manner using the current expression (17). The ratio $E(0)/E'(0)$ is

$$\frac{E(0)}{E'(0)}=-\frac{2}{\pi}\int_0^\infty q\,dq\left[q^3-\frac{\omega^2}{c^2}q+\frac{3\pi^2\omega\sigma_0}{c^2 v_F\tau}i\coth\left(\frac{\pi}{\omega_c\bar{\tau}}\right)\right]^{-1}, \tag{26}$$

which integrates to yield (neglecting the displacement current)

$$\frac{E(0)}{E'(0)}=-\tfrac{2}{3}\left(1-\frac{i}{\sqrt{3}}\right)\left(\frac{3\pi^2\sigma_0\omega}{c^2 v_F\tau}\right)^{-\frac{1}{3}}\tanh^{\frac{1}{3}}\left[\frac{\pi}{\omega_c\tau}(1+i\omega\tau)\right]. \tag{27}$$

The power absorption is proportional to the real part of the surface impedance, R, which is given by

$$R=\frac{16\pi\omega}{3^{\frac{3}{2}}c^2}\frac{\cos[\frac{1}{3}(\alpha+\pi)]}{(3\pi^2\sigma_0\omega/c^2v_F\tau)^{\frac{1}{3}}}\frac{[\sinh^2(\pi/\omega_c\tau)\cosh^2(\pi/\omega_c\tau)+\sin^2(\pi\omega/\omega_c)\cos^2(\pi\omega/\omega_c)]^{1/6}}{[\cos^2(\pi\omega/\omega_c)\cosh^2(\pi/\omega_c\tau)+\sin^2(\pi\omega/\omega_c)\sinh^2(\pi/\omega_c\tau)]^{\frac{1}{3}}}, \tag{28}$$

where $\tan\alpha=\sin(2\pi\omega/\omega_c)[\sinh(2\pi/\omega_c\tau)]^{-1}$. A plot of R for several values of the relaxation time is shown in Fig. 2. Harmonic absorption occurs and leads to decreases in the value of R whenever ω/ω_c is an integer.

III. QUANTUM TRANSPORT

The use of density matrix techniques in transport calculations has recently been reviewed by Nakajima.[9]

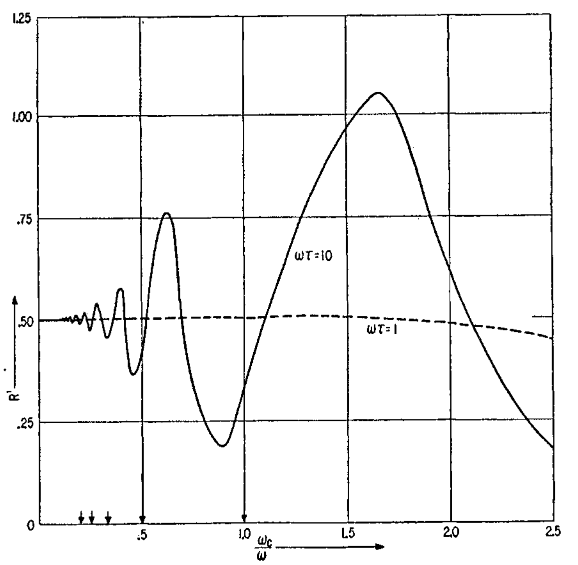

FIG. 2. $R'=R(16\pi\omega/3^{\frac{3}{2}}c^2)^{-1}(3\pi^2\sigma_0\omega/c^2v_F\tau)^{\frac{1}{3}}$ *vs* ω_c/ω for $\omega\tau=1$ and 10. The first five harmonics are indicated by arrows. The fundamental and first harmonic are appreciably shifted toward lower magnetic fields. This shift remains even for longer relaxation times.

The general considerations are given briefly in this section followed by an application to the anomalous skin effect problem in Sec. IV.

The Hamiltonian for the system will be written

$$\mathcal{H}=\mathcal{H}_0+\epsilon\mathcal{H}_1, \tag{29}$$

where $\mathcal{H}_0$ is time independent with eigenfunctions, φ_n, and eigenvalues, $\mathcal{E}_n=\hbar\omega_n$, and $\epsilon\mathcal{H}_1$ is a time-dependent perturbation. The solution to the time-dependent problem,

$$\mathcal{H}\Psi_\alpha=i\hbar\partial\Psi_\alpha/\partial t, \tag{30}$$

is then expanded in terms of the eigenfunctions of $\mathcal{H}_0$, namely

$$\Psi_\alpha=\sum_n a_\alpha(n,t)\varphi_n e^{-i\omega_n t}. \tag{31}$$

The a_α's satisfy the differential equation

$$\dot{a}_\alpha(n,t)=(i\hbar)^{-1}\sum_{n'}a_\alpha(n',t)(n|\epsilon\mathcal{H}_1|n')e^{-it(\omega_{n'}-\omega_n)} \tag{32}$$

or the integral equation

$$a_\alpha(n,t)=a_\alpha(n,-\infty)+(i\hbar)^{-1}\sum_{n'}\int_{-\infty}^t a_\alpha(n',t')g_{nn'}(t')dt', \tag{33}$$

where

$$g_{nn'}(t)=(n|\epsilon\mathcal{H}_1(t)|n')\exp[it(\omega_n-\omega_{n'})]. \tag{34}$$

By iteration of (33), one obtains

$$a_\alpha(n,t)=a_\alpha(n,-\infty)+(i\hbar)^{-1}\sum_{n'}a_\alpha(n',-\infty)\int_{-\infty}^t g_{nn'}(t')dt' +(i\hbar)^{-2}\sum_{n',n''}a_\alpha(n'',-\infty)\int_{-\infty}^t g_{nn'}(t')dt'\int_{-\infty}^{t'}g_{n'n''}(t'')dt''+\cdots. \tag{35}$$

The matrix element of the current operator is given by

$$\mathbf{i}_{mn}=\frac{e}{2m}\left\{\varphi_m^*\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\varphi_n\exp\{it(\omega_m-\omega_n)\}+\text{c.c.}\right\}. \tag{36}$$

[9] S. Nakajima, in *Advances In Physics*, edited by N. F. Mott (Taylor and Francis Ltd., London, 1955), Vol. 4, p. 363.

The total current being given by

$$\mathbf{I}=\mathrm{Tr}(\rho\mathbf{i})=\sum_{n,m}\rho_{nm}\mathbf{i}_{mn},\tag{37}$$

where ρ is the density matrix whose matrix elements are

$$\rho_{nm}(t)=\frac{1}{N}\sum_{\alpha}a_{\alpha}(n,t)a_{\alpha}^{*}(m,t).\tag{38}$$

Substitution of the perturbation expansion (35) into (38) enables one to express $\rho_{nm}(t)$ as

$$\rho_{nm}(t)=\rho_{nm}(-\infty)+(i\hbar)^{-1}\sum_{n'}\rho_{n'm}(-\infty)\int_{-\infty}^{t}g_{nn'}(t')dt'-(i\hbar)^{-1}\sum_{n'}\rho_{nn'}(-\infty)\int_{-\infty}^{t}g_{n'm}(t')dt'+\cdots.\tag{39}$$

If one assumes that at $t=-\infty$ the perturbation is turned on adiabatically and that the density matrix is diagonal, then

$$\rho_{nm}(-\infty)=\delta_{nm}f_0(\mathcal{E}_m),\tag{40}$$

where $f_0(\mathcal{E}_m)$ is the Fermi function. The matrix elements of the density matrix at time t are

$$\rho_{nm}(t)=\delta_{nm}f_0(\mathcal{E}_m)+(i\hbar)^{-1}[f_0(\mathcal{E}_m)-f_0(\mathcal{E}_n)]\int_{-\infty}^{t}g_{nm}(t')dt'+\cdots.\tag{41}$$

The evaluation of the trace in (37) results in

$$\mathbf{I}(\mathbf{r})=\frac{e}{2m}\bigg\{\sum_{n}f_0(\mathcal{E}_n)\varphi_n^{*}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\varphi_n+(i\hbar)^{-1}\sum_{n,m}[f_0(\mathcal{E}_n)-f_0(\mathcal{E}_m)]$$
$$\times\int_{-\infty}^{t}g_{nm}(t')dt'\varphi_m^{*}\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)\varphi_n\exp[it(\omega_m-\omega_n)]+\cdots+\text{c.c.}\bigg\}.\tag{42}$$

It is convenient to deal with the Fourier transform of (42) which is given by

$$\mathbf{I}(\mathbf{q})=(2\pi)^{-\frac{3}{2}}\int_{-\infty}^{\infty}e^{i\mathbf{q}\cdot\mathbf{r}}\mathbf{I}(\mathbf{r})d\mathbf{r}$$
$$=(2\pi)^{-\frac{3}{2}}\frac{e}{2m}\bigg\{\sum_{n,n'}f_0(\mathcal{E}_n)(n|e^{i\mathbf{q}\cdot\mathbf{r}}|n')\left(n'\left|\mathbf{p}-\frac{e}{c}\mathbf{A}\right|n\right)$$
$$+(i\hbar)^{-1}\sum_{n,n',n''}[f_0(\mathcal{E}_{n'})-f_0(\mathcal{E}_n)]\int_{-\infty}^{t}g_{n'n}(t')dt'(n|e^{i\mathbf{q}\cdot\mathbf{r}}|n'')$$
$$\times\left(n''\left|\mathbf{p}-\frac{e}{c}\mathbf{A}\right|n'\right)\exp[it(\omega_n-\omega_{n'})]+\cdots+\text{c.c.}\bigg\}.\tag{43}$$

IV. QUANTUM MECHANICAL TREATMENT OF ANOMALOUS SKIN EFFECTS

As a first application of the quantum mechanical treatment of a skin effect problem, the anomalous skin effect in the absence of a magnetic field is now given. The result is identical to that obtained by Reuter and Sondheimer[10] for the classical case provided the skin depth δ is much greater than the de Broglie wavelength of an electron at the Fermi surface, i.e., $\delta k_F\gg1$, a condition which is satisfied for all metals in the skin effect region.

The one-electron Hamiltonian for a free-electron gas is

$$\mathcal{H}_0=p^2/2m.\tag{44}$$

An rf electric field in the x direction is represented by the vector potential

$$\mathbf{A}_1=(ie^{at}c/2\omega)(e^{i\omega t}-e^{-i\omega t})E(z)\mathbf{i},\tag{45}$$

where $\mathbf{i}$ is a unit vector in the x direction; this vector potential builds up exponentially from $t=-\infty$ and gives rise to the perturbation

$$\epsilon\mathcal{H}_1=(-ei/2\omega m)e^{at}(e^{i\omega t}-e^{-i\omega t})E(z)p_x,\tag{46}$$

where the xy plane is the surface of the metal and the electric field is dependent on z, the depth into the metal.

[10] G. Reuter and E. Sondheimer, Proc. Roy. Soc. (London) **A195**, 336 (1949).

The unperturbed problem has plane wave eigenfunctions

$$\varphi_{\mathbf{k}}=(2\pi)^{-\frac{3}{2}}e^{i\mathbf{k}\cdot\mathbf{r}}, \tag{47}$$

and energy eigenvalues

$$\mathcal{E}_{\mathbf{k}}=\hbar\omega_{\mathbf{k}}=\hbar^2k^2/2m, \quad \mathcal{E}_{k_z}=\hbar\omega_{k_z}=\hbar^2k_z^2/2m. \tag{48}$$

The matrix elements required to perform the summation (43) are

$$(\mathbf{k}|e^{i\mathbf{q}\cdot\mathbf{r}}|\mathbf{k}'')=\delta(-\mathbf{k}+\mathbf{q}+\mathbf{k}''), \tag{49}$$

$$\left(\mathbf{k}''\left|\mathbf{p}-\frac{e}{c}\mathbf{A}\right|\mathbf{k}'\right)=\hbar\mathbf{k}'\delta(\mathbf{k}'-\mathbf{k}'')-\mathbf{i}\frac{ei}{2\omega(2\pi)^{\frac{1}{2}}}e^{at}(e^{i\omega t}-e^{-i\omega t})\delta(k_x'-k_x'')\delta(k_y'-k_y'')E(k_z'-k_z''), \tag{50}$$

$$g_{\mathbf{k}'\mathbf{k}}(t)=\frac{e}{m\omega(2\pi)^{\frac{1}{2}}}\exp\{it(\omega_{\mathbf{k}'}-\omega_{\mathbf{k}}-ia)\}\frac{1}{2i}(e^{i\omega t}-e^{-i\omega t})\hbar k_x\delta(k_x-k_x')\delta(k_y-k_y')E(k_z-k_z'). \tag{51}$$

Substitution into (43) yields

$$I_y(\mathbf{q})=I_z(\mathbf{q})=0,$$

$$I_x(\mathbf{q})=-2\left(\frac{1}{2\pi}\right)^2\frac{e^2}{4m\omega}\delta(q_x)\delta(q_y)E(q_z)e^{at}\Big\{i(e^{i\omega t}-e^{-i\omega t})\sum_{\mathbf{k}}f_0(\mathcal{E}_{\mathbf{k}})+\frac{i\hbar}{m}\sum_{\mathbf{k}}k_x^2[f_0(\mathcal{E}_{k_z-q_z})-f_0(\mathcal{E}_{\mathbf{k}})]$$
$$\times[e^{i\omega t}(\omega+\hbar^{-1}\mathcal{E}_{k_z-q_z}-\hbar^{-1}\mathcal{E}_{\mathbf{k}}-ia)^{-1}-e^{-i\omega t}(-\omega+\hbar^{-1}\mathcal{E}_{k_z-q_z}-\hbar^{-1}\mathcal{E}_{\mathbf{k}}-ia)^{-1}]+\cdots+\text{c.c.}\Big\}, \tag{52}$$

where a factor of two is inserted to account for the sum over electron spin. If one assumes $q_z\ll k_F$, i.e., $\delta\gg$de Broglie wavelength of an electron at the Fermi surface, then the integration over k results in

$$I_x(\mathbf{q})=2\pi\delta(q_x)\delta(q_y)\tfrac{3}{4}i\frac{Ne^2}{m\omega}E(q_z)e^{at}\tfrac{1}{2}\left\{e^{i\omega t}K_0\left(\frac{v_Fq_z}{\omega-ia}\right)-e^{-i\omega t}K_0\left(\frac{v_Fq_z}{\omega+ia}\right)\right\}, \tag{53}$$

where N is the electron concentration, v_F is the Fermi velocity, and $K_0(s)$ is given by Eq. (12). To obtain the conductivity, recall that the electric field is given by

$$E_x=-\frac{1}{c}\frac{\partial A_1}{\partial t}=\tfrac{1}{2}e^{at}E(z)\left[\left(\frac{\omega-ia}{\omega}\right)e^{i\omega t}+\left(\frac{\omega+ia}{\omega}\right)e^{-i\omega t}\right]. \tag{54}$$

If one assumes a time dependence $e^{i\omega t}$, then (53) and (54) imply a complex conductivity

$$\sigma=\tfrac{3}{4}\frac{Ne^2}{mi(\omega-ia)}K_0\left(\frac{v_Fq_z}{\omega-ia}\right), \tag{55}$$

which is identical to (9) for zero magnetic field where the relaxation time τ is identified with a^{-1}.

Thus, provided $\delta k_F\gg1$, the quantum calculation gives precisely the same result as the solution of the Boltzmann equation for the skin effect in the absence of a magnetic field.

The skin effect problem for a dc magnetic field, H_0, parallel to the surface which has been treated classically in Sec. II can be solved quantum mechanically by using the expressions developed in Sec. III. The quantum problem gives the same results as the classical calculation; however, one now sees clearly the limits of validity for the expressions developed and hence under what conditions the interpretation of the experimental results by means of the Boltzmann equation is in question.

The vector potential for the dc magnetic field is chosen as

$$\mathbf{A}_0=yH_0(0,0,1), \tag{56}$$

and the unperturbed Hamiltonian

$$\mathcal{H}_0=\left(\mathbf{p}-\frac{e}{c}\mathbf{A}_0\right)^2\Big/2m, \tag{57}$$

which has eigenvalues

$$\mathcal{E}_{n,k_x}=\hbar\omega_c(n+\tfrac{1}{2})+\hbar^2k_x^2/2m, \tag{58}$$

and eigenfunctions

$$\phi_{n,k_x,k_z}=(2\pi)^{-1}\exp\{i(k_xx+k_zz)\}\phi_n(y+\lambda^2k_z), \tag{59}$$

where

$$\lambda^2 = h/m\omega_c$$

and $\phi_n(y)$ is the normalized harmonic oscillator wave function. The perturbation caused by the rf field is

$$\epsilon\mathcal{H}_1 = -(ei/2\omega m)e^{at}(e^{i\omega t}-e^{-i\omega t})E(z)p_\alpha, \quad (60)$$

where $\alpha = x$ or y for longitudinal or transverse cyclotron resonance, respectively. An evaluation of matrix elements in the expression for the Fourier transform of the x component of the current in the longitudinal cyclotron resonance case, gives

$$I_y(\mathbf{q}) = I_z(\mathbf{q}) = 0,$$

$$I_x(\mathbf{q}) = -\frac{e^2}{8\pi^2 m\omega}\delta(q_x)E(q_z)e^{at}\Big\{i(e^{i\omega t}-e^{-i\omega t})\sum_{n,n',k_x,k_z} J_{nn'}(q_y,k_z,k_z-q_z)J_{n'n}(0,k_z-q_z,k_z)f_0(\mathcal{E}_{nk_x})$$
$$+\frac{\hbar}{im}\sum_{n,n',k_x',k_z} k_x^2[f_0(\mathcal{E}_{nk_x})-f_0(\mathcal{E}_{n'k_x})]J_{nn'}(q_y,k_z,k_z-q_z)J_{n'n}(0,k_z-q_z,k_z)$$
$$\times\{e^{i\omega t}[\omega-\omega_c(n'-n)-ia]^{-1}-e^{-i\omega t}[-\omega+\omega_c(n'-n)-ia]^{-1}\}+\cdots+\text{c.c.}\Big\}, \quad (61)$$

where

$$J_{nn'}(q_y,k_z,k_z') = \int_{-\infty}^{\infty} e^{iq_y y}\phi_n^*(y+\lambda^2 k_z)\phi_{n'}(y+\lambda^2 k_z')dy,$$

and

$$J_{nn'}(q_y,k_z,k_z-q_z) = \exp(-iq_y\lambda^2 k_z)J_{nn'}(q_y,0,-q_z). \quad (62)$$

The sum over k_z results in a delta function in q_y; hence

$$I_x(\mathbf{q}) = \frac{e^2}{4\pi m\omega\lambda^2}e^{at}\delta(q_x)\delta(q_y)E(q_z)\Big\{i(e^{i\omega t}-e^{-i\omega t})\sum_{n,n',k_x} f_0(\mathcal{E}_{nk_x})J_{nn'}(0,0,-q_z)J_{n'n}(0,-q_z,0)$$
$$+\frac{\hbar}{im}\sum_{n,n',k_x}[f_0(\mathcal{E}_{nk_x})-f_0(\mathcal{E}_{n'k_x})]k_x^2 J_{nn'}(0,0,-q_z)J_{n'n}(0,-q_z,0)$$
$$\times\{e^{i\omega t}[\omega-ia+\omega_c(n'-n)]^{-1}-e^{-i\omega t}[-\omega-ia+\omega_c(n'-n)]^{-1}\}+\cdots+\text{c.c.}\Big\}. \quad (63)$$

For transverse cyclotron resonance, one obtains

$$I_y(\mathbf{q}) = \frac{e^2}{4\pi m\omega\lambda^2}e^{at}\delta(q_x)\delta(q_y)E(q_z)\Big\{i(e^{i\omega t}-e^{-i\omega t})\sum_{n,n',k_x} f_0(\mathcal{E}_{nk_x})J_{nn'}(0,0,-q_z)$$
$$\times J_{n'n}(0,-q_z,0)+(i\hbar m)^{-1}\sum_{n,n',n'',n''',k_x}[f_0(\mathcal{E}_{nk_x})-f_0(\mathcal{E}_{n'k_x})]J_{n'n''}(0,-q_z,0)(p_y)_{n''n}$$
$$\times J_{nn'''}(0,0,-q_z)(p_y)_{n'''n'}\{e^{i\omega t}[\omega-ia+\omega_c(n'-n)]^{-1}-e^{-i\omega t}[-\omega-ia+\omega_c(n'-n)]^{-1}\}+\cdots+\text{c.c.}\Big\}, \quad (64)$$

where the momentum matrix element between the harmonic oscillator states is

$$(p_y)_{n''n} = \frac{\hbar}{i\lambda}\left[\left(\frac{n}{2}\right)^{\frac{1}{2}}\delta_{n'',n-1}-\left(\frac{n+1}{2}\right)^{\frac{1}{2}}\delta_{n'',n+1}\right]. \quad (65)$$

The integrals (62) are orthogonal for the special case

$$\sum\nolimits_{n''} J_{n'n''}(0,0,q)J_{n''n}(0,q,0) = \delta_{n'n}, \quad (66)$$

and are related by

$$J_{nn'}^*(q_y,k_z,k_z') = J_{n'n}(q_y,k_z',k_z). \quad (67)$$

One can once again use Eq. (54) for the electric field and obtain the complex conductivity for longitudinal cyclotron resonance:

$$\sigma_{\text{long}} = \frac{e^2 i}{2\pi^2\lambda^2 m(\omega-ia)}\Big\{\sum_{n,k_x} f_0(\mathcal{E}_{nk_x})+\frac{\hbar\omega_c}{m}\sum_{n,n',k_x}[f_0(\mathcal{E}_{nk_x})-f_0(\mathcal{E}_{n'k_x})]$$
$$\times k_x^2(n'-n)|J_{n'n}(0,-q_z,0)|^2[(\omega-ia)^2-\omega_c^2(n'-n)^2]^{-1}\Big\}, \quad (68)$$

and for transverse cyclotron resonance:

$$\sigma_{\text{tran}}=\frac{e^2i}{2\pi^2\lambda^2m(\omega-ia)}\left\{\sum_{n,k_x} f_0(\mathcal{E}_{nk_x})+\frac{\omega_c}{\hbar m}\sum_{n,n',k_x}[f_0(\mathcal{E}_{nk_x})-f_0(\mathcal{E}_{n'k_x})]\right.$$
$$\left.\times(n'-n)|(J(0,-q_z,0)p_y)_{n'n}|^2[(\omega-ia)^2-\omega_c^2(n'-n)^2]^{-1}\right\}, \quad (69)$$

where $J(0,-q_z,0)p_y$ is a matrix product and the operator $J(0,0,q)=J(0,-q,0)=\exp[i\hbar^{-1}\lambda^2qp_y]$.

A Taylor's series expansion of the Fermi function and integration by parts simplifies (68) to

$$\sigma_{\text{long}}=\frac{\sqrt{2}e^2i(\omega-ia)}{m\omega_c^2\pi^2\lambda^3}\sum_{n=0}^{n_F}(n_F-n)^{\frac{1}{2}}\sum_{n'=0}^{\infty}\frac{|J_{n'n}(0,-q_z,0)|^2}{[(\omega-ia)/\omega_c]^2-(n'-n)^2}, \quad (70)$$

where

$$n_F=\mathcal{E}_F(\hbar\omega_c)^{-1}-\tfrac{1}{2}.$$

After integration over k_z the conductivity for the transverse resonance can be written

$$\sigma_{\text{tran}}=\frac{2\sqrt{2}e^2i}{\pi^2m(\omega-ia)\lambda\hbar^2}\sum_{n=0}^{n_F}(n_F-n)^{\frac{1}{2}}\sum_{n'=0}^{\infty}\left[\frac{|(p_y)_{n'n}|^2}{n'-n}+\frac{(n'-n)}{[(\omega-ia)/\omega_c]^2-(n'-n)^2}|[J(0,-q_z,0)p_y]_{n'n}|^2\right], \quad (71)$$

where the f-sum rule has been used in the manipulation of the first term in the bracket.

The two-center harmonic oscillator integrals (62) can be evaluated[11] to give

$$J_{nn'}(0,q,0)=\exp[-(\lambda q/2)^2](n!/n'!)^{\frac{1}{2}}(-\lambda q/\sqrt{2})^{n'-n}L_n^{n'-n}(\lambda^2q^2/2), \quad n'\geq n, \quad (72)$$

where $L_n^\alpha(x)$ is the associated Laguerre polynonial

$$L_n^\alpha(x)=(n!)^{-1}e^xx^{-\alpha}\frac{d^n}{dx^n}(e^{-x}x^{n+\alpha}). \quad (73)$$

For large values of n one can use the asymptotic expansion

$$L_n^\alpha(x)=e^{x/2}x^{-\alpha/2}\frac{\Gamma(n+\alpha+1)}{n![(2n+\alpha+1)/2]^{\alpha/2}}J_\alpha([2x(2n+\alpha+1)]^{\frac{1}{2}})+O(n^{\alpha/2-\frac{3}{4}}), \quad (74)$$

where $J_\alpha(x)$ is the Bessel function of order α. Thus for large n and $n<n'$ one obtains from (72), (74), and the asymptotic expansion for the Γ function

$$J_{nn'}(0,q_z,0)=(-1)^{n'-n}J_{n'-n}(\lambda q_z[n+n'+1]^{\frac{1}{2}}). \quad (75)$$

The limit of infinite skin depth, $\lambda q_z\rightarrow 0$, reduces the two-center integral (62) to

$$J_{nn'}(0,q,0)=\delta_{nn'},$$

which when substituted into (70) and (71) yield the classical result[12]

$$\sigma_{\text{long}}=\frac{\sigma_0}{1+i\omega\tau}, \quad (76)$$

and

$$\sigma_{\text{tran}}=\sigma_0\left[\frac{1+i\omega\tau}{(1+i\omega\tau)^2+\omega_c^2\tau^2}\right], \quad (77)$$

where the relaxation time τ is identified with a^{-1}.

The argument of the Bessel function in (75) is approximately r_c/δ, which is large in metals at the magnetic fields in question. Hence one can use the asymptotic expansion for the Bessel function and obtain

$$J_{nn'}(0,q_z,0)=2^{\frac{1}{2}}[\pi|\lambda q_z|(n+n'+1)^{\frac{1}{2}}]^{-\frac{1}{2}}\cos(\lambda q_z[n+n'+1]^{\frac{1}{2}}), \quad (78)$$

[11] See A. Erdelyi *et al.*, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 292.
[12] See Kittel, *Introduction to Solid State Physics*, (John Wiley and Sons, Inc., New York, 1956), second edition, p. 372.

and

$$[J(0,q_z,0)p_y]_{nn'}=\frac{\hbar}{i\lambda}\left[\frac{2(n+n'+1)^{\frac{1}{2}}}{\pi|\lambda q_z|}\right]^{\frac{1}{2}}\sin(\lambda q_z[n+n'+1]^{\frac{1}{2}}). \tag{79}$$

Substitution of (78) into (70) and (79) into (71) and replacing the square of the sine and cosine by $\frac{1}{2}$ gives for the longitudinal conductivity

$$\sigma_{\text{long}}=\frac{\sqrt{2}e^2ip}{m\omega_c\pi^3\lambda^4q_z}\sum_{n=0}^{n_F}(n_F-n)^{\frac{1}{2}}\sum_{\alpha=-n}^{\infty}\frac{1}{p^2-\alpha^2}[2n+\alpha+1]^{-\frac{1}{2}}, \tag{80}$$

and for the transverse conductivity

$$\sigma_{\text{tran}}=\frac{\sqrt{2}e^2i}{m(\omega-ia)\pi^3\lambda^4q_z}\sum_{n=0}^{n_F}(n_F-n)^{\frac{1}{2}}\sum_{\alpha=-n}^{\infty}\frac{1}{p^2-\alpha^2}[p^2+\alpha^2+2\alpha(2n+1)][2n+\alpha+1]^{-\frac{1}{2}}, \tag{18}$$

where $p=(\omega-ia)/\omega_c$.

The summations in (80) and (81) may be evaluated with the aid of the integral

$$\frac{1}{2\pi i}\int_C\frac{f(z)}{p^2-z^2}\pi\cot\pi z \tag{82}$$

over the contour shown in Fig. 3. The integral over the contour vanishes provided $f(z)$ diverges slower than z^2. If $f(z)$ has no singularities inside the contour, one obtains

$$\sum_{\alpha=-n}^{\infty}\frac{f(\alpha)}{p^2-\alpha^2}=\frac{\pi}{2p}(\cot\pi p)[f(p)+f(-p)]. \tag{83}$$

The summation is carried out under the assumption that $p\ll n_F$; i.e., a Taylor's series expansion is used for $f(p)$. The final summation over n is replaced by an integration, an approximation which is very good for large n_F. With these approximations both (80) and (81) reduce to the classical expression (17).

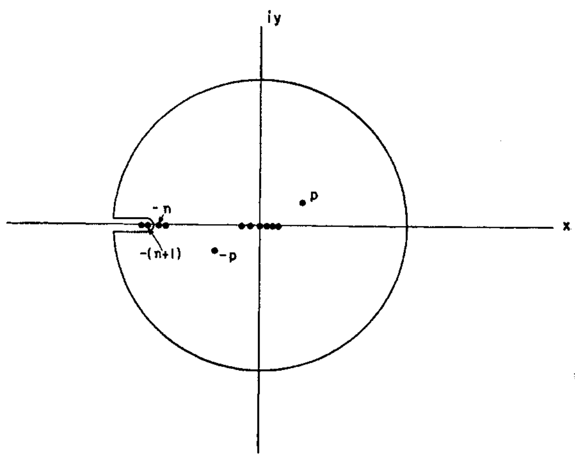

FIG. 3. The contour used to evaluate summation (83).

In very high magnetic fields where $\hbar\omega\ll\hbar\omega_c\sim\mathcal{E}_F$, the asymptotic expansion for the associated Laguerre polynomial is no longer valid, and one must use the exact expression for the two-center harmonic oscillator integral (72). This limit gives a de Haas-van Alphen type of oscillations of the surface impedance, which results from the fact that as the magnetic field decreases from infinite fields where only the $n=0$ state is occupied, additional quantum states become occupied and these quantum states for small values of n have appreciably different q_z dependence from the asymptotic expansion used for large n. Hence the surface impedance changes in a discontinuous fashion as each new oscillator state begins to be occupied. New terms enter into the summation (70) and (71) when

$$\mathcal{E}_F/\hbar\omega_c=(2m+1)/2, \tag{84}$$

where m is an integer, and thus a discontinuous change in $Z(0)$ will be observed when (84) is satisfied.‡

ACKNOWLEDGMENTS

This work is based in part on a section of the thesis by one of us (D.M.) which was composed under the direction and inspiration of Professor J. Bardeen. The authors owe much to friendly discussions with their colleagues in the Departments of Physics at the University of Illinois and Cornell University, respectively. In addition, one of us (G.D.) would like to thank the General Electric Research Laboratory for making available its stimulating environment during the summer of 1957 when much of this work was accomplished.

‡ *Note added in proof.*—M. Ia Azbel [J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 969 (1958)] has also discussed this possibility.