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Full counting statistics of a charge pump in the Coulomb blockade regime

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We study full charge counting statistics (FCCS) of a charge pump based on a nearly open single electron transistor. The problem is mapped onto an exactly soluble problem of a nonequilibrium $g=1/2$ Luttinger liquid with an impurity. We obtain an analytic expression for the generating function of the transmitted charge for an arbitrary pumping strength. Although this model contains fractionally charged excitations only *integer* transmitted charges can be observed. In the weak pumping limit FCCS correspond to a Poissonian transmission of particles with charge $e^*=e/2$ from which all events with odd numbers of transferred particles are excluded.

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Charge pumping has attracted considerable theoretical and experimental interest. It occurs when the Hamiltonian of the system changes with the time. At the end of the pumping cycle, when the Hamiltonian returns to its initial value, a finite charge may be transmitted through the system. The amount of the transferred charge depends on the details of the pumping cycle. Thouless[1] showed that in certain one-dimensional systems the transmitted charge is quantized in the adiabatic limit. Most of research efforts have focussed on charge pumping through mesoscopic devices.[2–13]

Motivated by the efforts to build an accurate standard of electric current most experiments concentrated on single electron pumps in which the charge pumped during one cycle is quantized due to the Coulomb blockade effects.[4,9] Such devices are already used in metrological applications to produce an accurate capacitance standard.[9]

Understanding of noise properties of the pumped current and of the accuracy of quantization of the pumped charge are very important for metrological applications. In this case it is desirable to know not only the average pumping current and its second moment (noise power) but the whole distribution function of the pumped charge. Such full charge counting statistics (FCCS) were first considered in Refs. 14,15 for systems with noninteracting electrons.

In the present paper FCCS for a charge pump based on a single electron transistor are considered. More precisely, the device in question consists of a quantum dot connected to the left and right leads by single channel quantum point contacts labeled by the index $\alpha=\pm 1$, see Fig. 1. Such devices can be fabricated in semiconductor heterostructures[16] where the electrons in the two-dimensional electron gas (2DEG) in a heterostructure are electrostatically confined to the area of the dot by a negative voltage which is applied to the metallic gates located on top of the 2DEG. The reflection amplitudes r_α in the contacts are controlled by the voltages on gates α and are assumed to be small throughout the pumping cycle $r_\alpha \ll 1$. The Coulomb interaction of electrons in the dot can be treated within the constant interaction model

$$H_C=E_C[\hat{N}-N(t)]^2, \tag{1}$$

where $\hat{N}$ is the number of electrons in the dot, E_C is the charging energy, and $N(t)$ is the dimensionless parameter proportional to the voltage on the central gate G.

At low temperatures $T\ll E_c$, the electron transport across the device is dominated by cotunneling processes. The quantum dot is assumed to be sufficiently large so that elastic cotunneling effects[17] can be neglected and the transport of electrons across the device is dominated by the inelastic cotunneling.[18,19] In addition the electrons are assumed to be spin polarized. This can be realized experimentally by applying a strong magnetic field parallel to the plane of the 2DEG.

Sufficiently strong pumping can lead to a nonequilibrium distribution of electrons in the dot. Below we assume that the deviations from equilibrium may be neglected. This requirement imposes a limitation on the number of pumping cycles in the absence of energy relaxation in the dot. Indeed, inelastic cotunneling can be thought of as a coherent process in which an electron, say from the left lead, enters a certain quantum state in the dot and an electron from a different state in the dot leaves into the right lead. As a result of such a process an electron is transferred across the device and an electron-hole pair is created in the dot. Upon completion of N pumping cycles roughly N electron-hole pairs will be created. The number of electron-hole pairs in equilibrium may

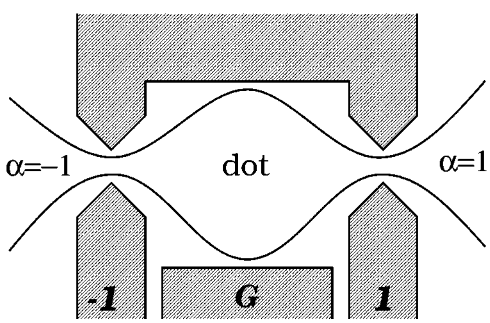

FIG. 1. Schematic drawing of a single electron transistor electrostatically defined on a surface of a two-dimensional electron gas. The quantum dot is connected to two leads by single channel quantum point contacts labeled by α. The voltages on the gates G and ± 1 determine, respectively, the average electron number in the dot $N(t)$ and the reflection amplitudes $r_{\pm 1}(t)$ in QPC's.

be estimated as T/δ_1, where δ_1 is the single particle mean level spacing in the dot. Therefore, for the deviations from equilibrium to be small we assume that the dot is sufficiently large so that the mean level spacing $\delta_1 \ll T/N$. In the presence of energy relaxation in the dot this condition may be relaxed.

At frequencies below the charging energy E_C the pumping cycle at hand is described by a single complex variable[19,8] $z(t)=r_1(t)\exp[i\pi N(t)]+r_{-1}(t)\exp[-i\pi N(t)]$. The average pumping current for this cycle was obtained in Ref. 8.

Here we study FCCS for this pump. The probability distribution function $P_N(Q)$ for the charge Q transmitted through the dot upon completion of N pumping cycles is determined by the generating function

$$F_N(\lambda)=\sum_Q \exp(i\lambda Q)P_N(Q), \tag{2}$$

where the charge Q is measured in units of the absolute value of the electron charge e and the sum goes over all its possible values. The nth cumulant $\langle\langle Q^n\rangle\rangle$ of the transmitted charge may be determined from $F_N(\lambda)$ through the relation

$$\langle\langle Q^n\rangle\rangle=\left.\frac{d^n \ln F_N(\lambda)}{i^n d\lambda^n}\right|_{\lambda=0}. \tag{3}$$

Below we concentrate on the pumping cycle in which $z(t)=z_0\exp(-i\omega t)$. In this case we obtain the following generating function:

$$\begin{aligned}\ln F_N(\lambda)=\frac{1}{2}\sum_{l=-\infty}^{\infty}\ln\{1+\cos^2\theta(\varepsilon_l)[n_-(\varepsilon_l)[1-n_+(\varepsilon_l)]\\ \times(e^{i\lambda}-1)+n_+(\varepsilon_l)[1-n_-(\varepsilon_l)](e^{-i\lambda}-1)]\}\\ -iN\lambda.\end{aligned} \tag{4}$$

Here $\varepsilon_l=\omega(2l+1)/(2N)$, with l being an integer, denotes the discrete fermionic frequency, $n_\pm(\epsilon)=n_0(\epsilon\pm\omega)=(e^{(\epsilon+\omega)/T}+1)^{-1}$ is the Fermi distribution function shifted by $\pm\omega$, and $\exp[i\theta(\epsilon)]=(\epsilon+i\Gamma)/\sqrt{\epsilon^2+\Gamma^2}$, where $\Gamma=2\gamma|z_0|^2E_C/\pi^2$, $\gamma=\exp C$, with $\ln\gamma=C\approx0.5772\cdots$ being the Euler constant. Note that since the generating function $F_N(\lambda)$ in Eq. (4) is periodic in λ with the period 2π, $F_N(\lambda+2\pi)=F_N(\lambda)$, only *integer* values of charge Q can be transmitted.

The Eq. (4) acquires a particularly simple form at low temperatures $T\ll\omega$. Approximating the Fermi functions by the step functions $n_\pm(\epsilon_l)=\Theta(-\epsilon_l\mp\omega)$ we observe from Eq. (4) that only the energy interval $-\omega<\epsilon_l<\omega$ contributes to the pumped charge. Using the Poisson summation formula we can write Eq. (4) as

$$\begin{aligned}\ln F_N(\lambda)=-iN\lambda+\Upsilon N\sum_{n=-\infty}^{\infty}\int_{-1/\Upsilon}^{1/\Upsilon}dx\, e^{in(x\Gamma\tau+\pi)}\\ \times\ln(x^2e^{i\lambda/2}+1)-\ln(x^2+1),\end{aligned} \tag{5}$$

where $\Upsilon=\Gamma/\omega$ is the relative pumping strength. For long observation times $\tau\Gamma=2\pi\Upsilon N\gg1$ the terms $n\neq0$ become small due to the presence of quickly oscillating factors in their integrands. Explicit evaluation of the main, $n=0$, term gives

$$\begin{aligned}\ln F_N(\lambda)=N\ln\frac{\Upsilon^2e^{-i\lambda}+1}{\Upsilon^2+1}-2\Upsilon N\arctan(\Upsilon^{-1})\\ +2\Upsilon Ne^{-i\lambda/2}\arctan(\Upsilon^{-1}e^{i\lambda/2}).\end{aligned} \tag{6}$$

With the aid of Eq. (3) we obtain for the average pumping current

$$I=\frac{e\langle Q\rangle}{\tau}=-\frac{e\Gamma}{2\pi}\arctan(\Upsilon^{-1}). \tag{7}$$

The initial growth of the current with the pumping frequency ω saturates at $I=-e\Gamma/4$ for large ω.

In the strong pumping limit, $\Upsilon\gg1$, Eq. (6) yields

$$\ln F_N(\lambda)=-iN\lambda+\frac{N}{3\Upsilon^2}(e^{i\lambda}-1), \tag{8}$$

where the first term contributes only to the average current, and the second term describes a Poisson process for particles with an integer charge $e^*=e$ and transmission frequency $\omega/(6\pi\Upsilon^2)$. In the limit of weak pumping, $\Upsilon\ll1$, we can perform the integration over x in Eq. (5) from $-\infty$ to ∞. Retaining the terms with $n\neq0$ we obtain

$$F_N(\lambda)=\frac{\cosh[\pi\Upsilon Ne^{-i\lambda/2}]}{\cosh[\pi\Upsilon N]}. \tag{9}$$

To evaluate the cumulants (3) it suffices to know $F_N(\lambda)$ at $\lambda\to0$. For long observation times $\tau\Gamma\gg1$ we can neglect the exponentially small terms in Eq. (9) and write the logarithm of the generating function as

$$\ln F_N(\lambda)=\pi\Upsilon N(e^{-i\lambda/2}-1). \tag{10}$$

This formula can also be derived directly from Eq. (6). It corresponds to a Poisson process which describes independent transmission of quasiparticles with the average transmission frequency $\Gamma/2$ and *fractional* charge $e^*=e/2$. The true limiting expression for weak pumping, Eq. (9) is periodic in λ with the period 2π, allowing transmission of only integer charges. It is easy to check that Eq. (9) describes a Poisson process for charge $e/2$ particles from which all transmission events with odd numbers of transferred particles have been excluded. The corrections to Eq. (9) are small $\sim\Upsilon^2$.

One may define the effective charge e^* of the carriers through the ratio of the variance of the transmitted charge to its average value for intermediate pumping strengths as well. However Eqs. (10),(8) show it can be interpreted as a charge of independently transmitted particles only in the limits of weak, $\Upsilon\ll1$, and strong, $\Upsilon\gg1$, pumping. The coefficients in the Taylor expansion of Eq. (6) in powers $1/\Upsilon^n$ (for strong fpumping) or in powers Υ^n (for weak pumping) represent Poissonian transmission processes of multiple charge ne^*, in agreement with Ref. 21.

Below we present the derivation of the above results. At $T,\omega \ll E_C$ the pumping cycle is described by the Hamiltonian[18,8]

$$H=\int_{-\infty}^{\infty} dk\left[\frac{\epsilon_k}{2}\Psi_k^\dagger\sigma_3\Psi_k+\frac{\kappa\zeta\Psi_k^\dagger}{\sqrt{2\pi}}\begin{pmatrix}-z^*(t)\\ z(t)\end{pmatrix}\right], \quad (11)$$

where $\kappa=\sqrt{\gamma v E_C/\pi^2}$. In Eq. (11) Ψ_k is a vector fermion operator in Gorkov-Nambu notations and is expressed through the creation and annihilation operators c_k and $c_k^\dagger$ as $\Psi_k^\dagger=(c_k^\dagger,c_{-k})$, and σ_3 is the Pauli matrix. In this model electrons have a linear spectrum $\epsilon_k=vk$ and are coupled to a resonant state described by a Majorana fermion ζ, $\zeta^2=1$. The current through the pump is given by

$$I=-\frac{ev_F}{2}\int_{-\infty}^{\infty} dk \quad \Psi_k^\dagger\Psi_k. \quad (12)$$

For the pumping cycle considered here the gauge transformation

$$\Psi_k\to\exp(i\sigma_3\omega t)\Psi_k, \quad (13)$$

removes the time dependence of the Hamiltonian $z(t)\to z_0$. As a result, the chemical potentials of electrons and holes shift by $\pm\omega$. The current operator in this gauge acquires an additional anomalous term and takes the form

$$I=-\frac{ev_F}{2}\int_{-\infty}^{\infty} dk \quad \Psi_k^\dagger\Psi_k+\frac{e\omega}{2\pi}. \quad (14)$$

The stationary Hamiltonian (11) [with $z(t)\to z_0$] was diagonalized by Matveev[18] in terms of the linear combinations of particle and hole operators

$$\tilde{C}_k=\frac{c_k+c_{-k}^\dagger}{\sqrt{2}}, \quad (15a)$$

$$C_k=\frac{\epsilon_k\pm i\Gamma}{\sqrt{\epsilon_k^2+\Gamma^2}}\frac{c_k-c_{-k}^\dagger}{\sqrt{2}}-\zeta\sqrt{\frac{v_F\Gamma}{2\pi(\epsilon_k^2+\Gamma^2)}}+\frac{\Gamma}{\pi\sqrt{\epsilon_k^2+\Gamma^2}}\int\frac{d\epsilon_{k'}}{\epsilon_k-\epsilon_{k'}\pm i0}\frac{c_{k'}-c_{-k'}^\dagger}{\sqrt{2}}. \quad (15b)$$

Both signs in Eq. (15b) give equivalent expressions. For the upper/lower sign the last term in Eq. (15b) gives a vanishing contribution to $\Psi(x)$ at $x\to\pm\infty$ after a Fourier transformation to the real space. [The second term in Eq. (15b) corresponds to the resonant state vanishing for $x\to\pm\infty$.]

Having observed these asymptotic properties of operators (15) we can readily build scattering states corresponding to the scattering of an electron

$$\frac{1}{\sqrt{2}}(\tilde{C}_k+C_k e^{i\theta(\epsilon_k)})=\begin{cases}c_k, & x\to-\infty,\\ e^{i\theta(\epsilon_k)}[c_k\cos\theta(\epsilon_k)-ic_{-k}^\dagger\sin\theta(\epsilon_k)], & x\to+\infty,\end{cases} \quad (16)$$

and a hole

$$\frac{1}{\sqrt{2}}(\tilde{C}_k-C_k e^{i\theta(\epsilon_k)})=\begin{cases}c_{-k}^\dagger, & x\to-\infty,\\ e^{i\theta(\epsilon_k)}[-ic_k\sin\theta(\epsilon_k)+c_k^\dagger\cos\theta(\epsilon_k)], & x\to+\infty,\end{cases} \quad (17)$$

where $\cos\theta(\epsilon_k)=\epsilon_k/\sqrt{\epsilon_k^2+\Gamma^2}$ and $\sin\theta(\epsilon_k)=\Gamma/\sqrt{\epsilon_k^2+\Gamma^2}$. We can now write the scattering matrix

$$\hat{S}(\epsilon_k)=e^{i\theta(\epsilon_k)}\begin{pmatrix}\cos\theta(\epsilon_k) & -i\sin\theta(\epsilon_k)\\ -i\sin\theta(\epsilon_k) & \cos\theta(\epsilon_k)\end{pmatrix} \quad (18)$$

for the scattering between electron (positive current) and hole (negative current) states.

Thus, the problem reduces to the problem of nonequilibrium chiral fermions scattering off a resonant state at zero energy. The electrons and holes here are characterized by nonequilibrium distribution functions $n_\pm(\epsilon)$ defined below Eq. (4) and may be represented by the diagonal matrix

$$\hat{n}(\epsilon)=\begin{pmatrix}n_-(\epsilon) & 0\\ 0 & n_+(\epsilon)\end{pmatrix}. \quad (19)$$

The full counting statistics for non-equilibrium non-interacting fermions have been extensively studied.[14] The generating function of the transmitted charge is given by

$$F_N(\lambda)=\exp(-iN\lambda)\exp\left\{\mathrm{Tr}\sum_{k=0}^{\infty}\ln\{\mathbb{1}+\hat{n}(\epsilon_k)\times[\hat{S}_{-\lambda}^\dagger(\epsilon_k)\hat{S}_\lambda(\epsilon_k)-\mathbb{1}]\}\right\}, \quad (20)$$

where $\hat{S}_{\pm\lambda}(\epsilon_k)=\exp[\pm(i/4)\sigma_3\lambda]\hat{S}(\epsilon_k)\exp[\pm(i/4)\sigma_3\lambda]$ with $\hat{S}(\epsilon_k)$ defined in Eq. (18). The first term in this equation arises from the anomalous term in the current operator, Eq. (14). Since electron and hole operators describe the same physical states, the sum over energies is restricted to positive frequencies in order to avoid double counting of degrees of freedom. Substituting Eqs. (18) and (19) into Eq. (20) we obtain the generating function (4).

In conclusion, we have obtained full counting statistics for a charge pump based on a nearly open single electron transistor. In the spin-polarized case the problem is mapped onto an exactly soluble chiral fermion model, Eq. (11). In the weak pumping regime $\Gamma\ll\omega$ the generating function given by Eq. (9) corresponds to a Poisson process of charge e^*

$=e/2$ particles with transmission rate $\Gamma/2$ from which all events with odd numbers of transferred particles are excluded. Although all the moments of the transferred charge obtained from Eq. (9) are practically indistinguishable from those of a simple Poisson process for charge $e^* = e/2$ particles only integer transferred charges may be observed in a pumping experiment. Since the Hamiltonian (11) of this model describes a $g=\frac{1}{2}$ Luttinger liquid with an impurity one may expect that similar conclusion hold for other coupling strengths $g \neq \frac{1}{2}$ and other problems with fractionally charged excitations realized for example in Quantum Hall experiments.[22,23,11]

Equation (9) differs from the weak pumping result in Ref. 13. The reason for this discrepancy and for charge fractionalization lies in the failure of perturbation theory in the reflection amplitudes $r_{\pm 1}$ for our model at sufficiently low energies $\epsilon < \Gamma$. The effective reflection coefficient $\Gamma^2/(\epsilon^2 + \Gamma^2)$ that determines the strength of pumping approaches unity in this energy range. Thus, the true expansion parameter at weak pumping is not the reflection amplitude $r_{\pm 1}$ but the ratio of energy scales $\Upsilon = \Gamma/\omega$. Perturbation theory in the reflection amplitude fails for Luttinger models with an impurity at other interaction strengths $g \neq \frac{1}{2}$ as well which leads to the appearance of an energy scale analogous to Γ below which the system is in the strong coupling limit.[20,21]

Although we have focused on the low temperature case $T \ll \omega$ the validity of the result (4) is restricted only by the condition $T < E_C$. Using Eq. (4) we obtain the general expression for the average pumping current

$$I = -\frac{e\omega}{2\pi} + \frac{e}{4\pi}\int_{-\infty}^{\infty} \frac{\epsilon^2 d\epsilon}{\epsilon^2 + \Gamma^2} \frac{\sinh\frac{\omega}{T}}{\cosh\frac{\epsilon}{T} + \cosh\frac{\omega}{T}}. \tag{21}$$

This formula reduces to the result of Ref. 8 in the linear response regime $\omega \ll T$.

The case of zero pumping and finite external bias V can be obtained from the above expressions by substituting $\omega \to eV$ and omitting the anomalous term in the current. For example the nonlinear I-V characteristic is obtained in this way from the second term of Eq. (21).

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