Computing Hulls in Positive Definite Space

P. Thomas Fletcher, John Moeller, Jeff M. Phillips, Suresh Venkatasubramanian

THE UNIVERSITY OF UTAH

P(n): a *Riemannian manifold*

Definition: symmetric positive-definite $n \times n$ matrices
Applications:

- Diffusion Tensor MRI (DT-MRI)
  Flow through voxel modeled in P(3)
- Elasticity Tensors
  Modeled by elements of P(6)
- Machine Learning
  Used in kernels

Convex Hulls

Data on P(n):

- Want to analyze this data
- Centerpoints, clustering, shape

Convex hull (CH) is a useful data analysis tool

- Describes shape of the data
- Can use max CH peeling depth to find a centerpoint

Goal: Compute hulls in P(n)

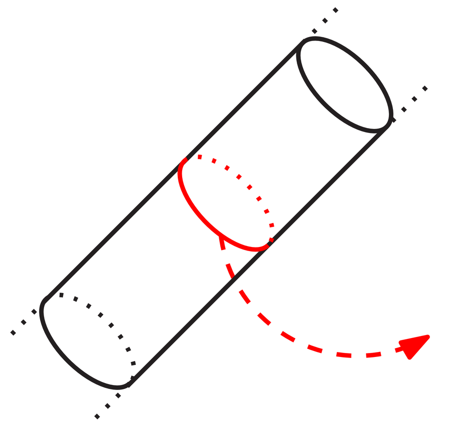
Problem!

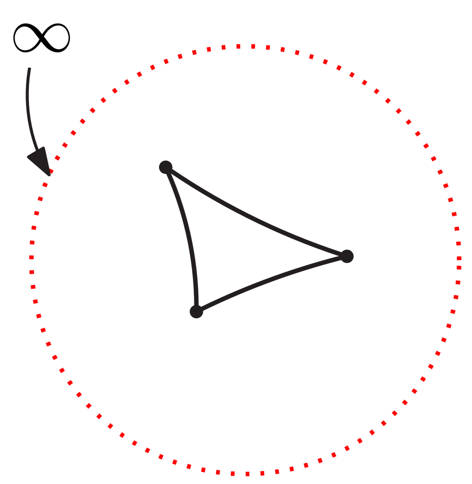
Convex hull of 3 points:

In $\mathbb{R}^n$	In general manifolds
2-dimensional	Might be full-dimensional (measure > 0)
Finite representation	Might not have a finite representation
Closed	Might not be closed (standing conjecture: no)

Makes computation of convex hulls or testing membership in convex hulls in general manifolds difficult

Modeling P(2)


P(2) as a 3-d space

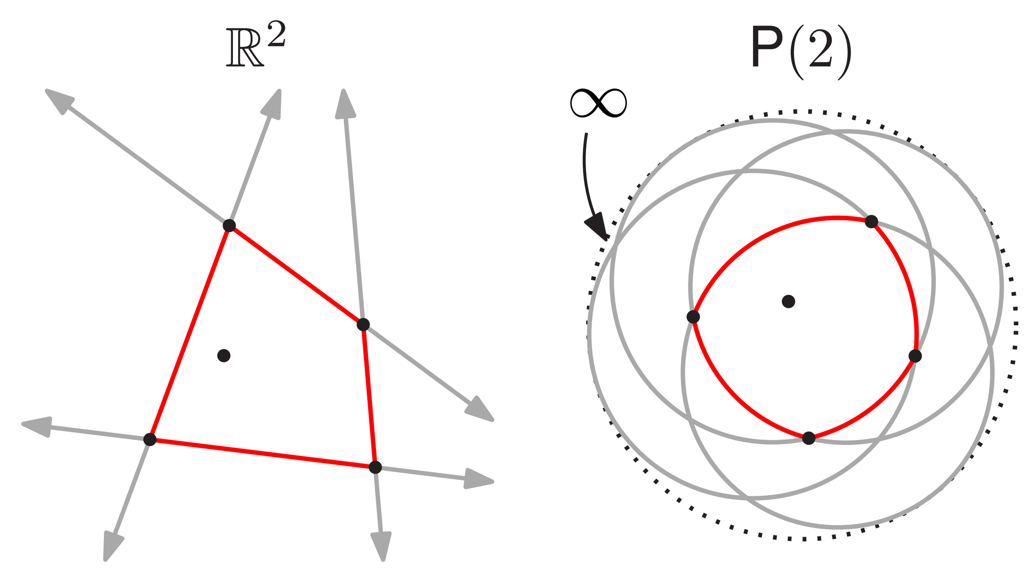


2-d cross-section of P(2)

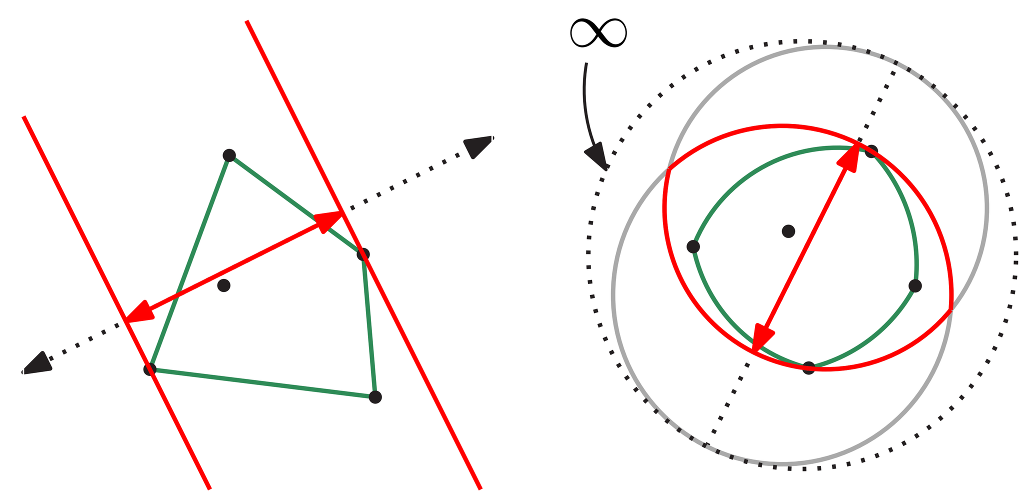
Meeting Convex Hulls Halfway

- In $\mathbb{R}^n$ **CH** is intersection of **halfspaces**
- Draw a ball, Fix surface at one point, Send center away to infinity along a geodesic c: In the limit, called a *horoball*
- Compute by taking sublevel set of *Busemann function*: $b_c(p) = \lim_{t\to\infty}(d(p, c(t)) - t)$
- Horoball stands in for halfspace (in $\mathbb{R}^n$ they are identical)
- **Ball Hull (BH)** is an intersection of **horoballs**

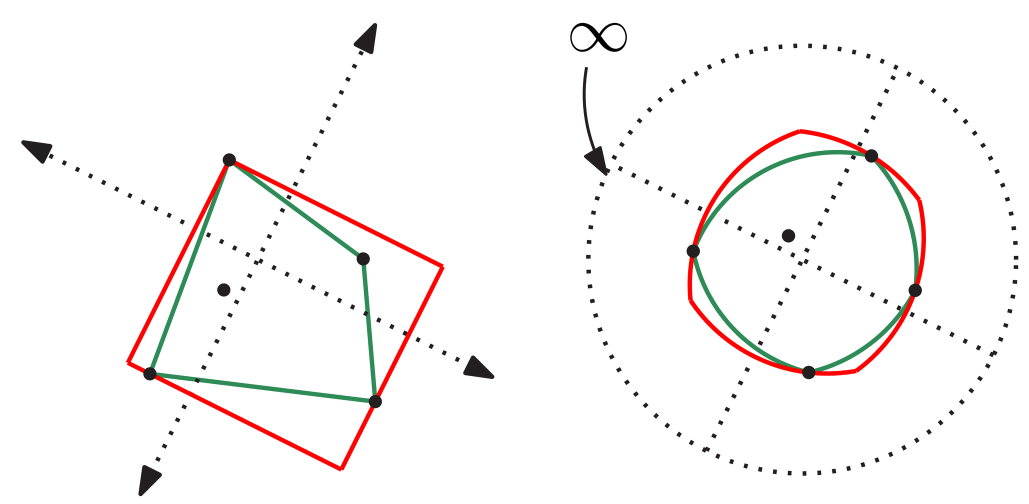
ε-Ball Hull



- $\mathcal{B}(X) = \bigcap_{b_c, r} B(b_c, r), \quad X \subset B(b_c, r)$
- Problem: How many horoballs are needed for BH? Could be ∞



- Extent (horoextent) of X — width of lens containing X w.r.t. a direction c: $E_c(X) = |\max_{x\in X} b_{c^+}(x) + \max_{x\in X} b_{c^-}(x)|$
- Extent of approximate shape should be close in every direction

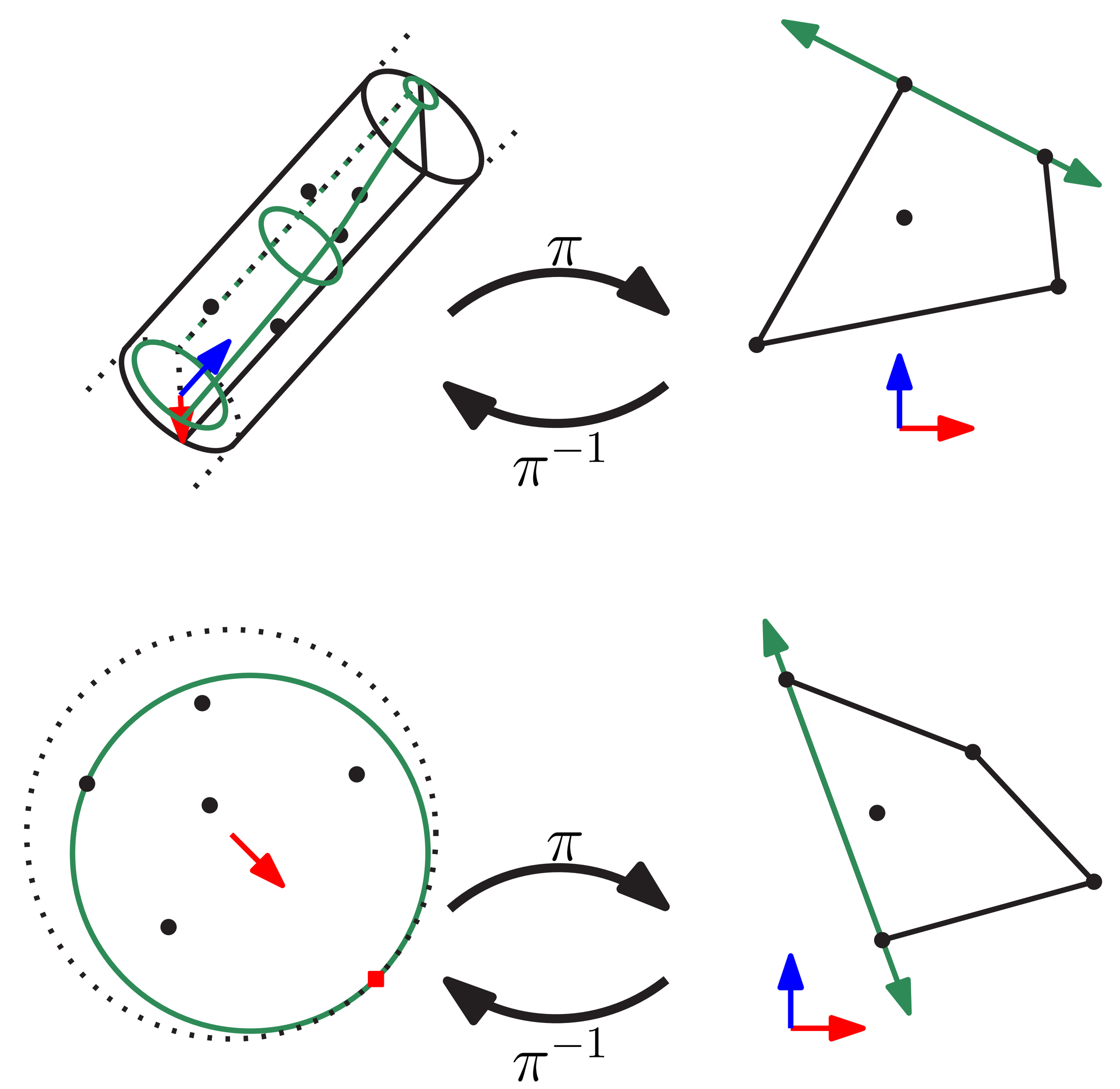


- **ε-ball hull of X ($\mathcal{B}_\varepsilon(X)$):** A finite collection of horoballs that approximate the ball hull w.r.t. extent: $|E_c(\mathcal{B}_\varepsilon(X)) - E_c(\mathcal{B}(X))| \le \varepsilon, \ \forall c$

Ideal Approach

- Each Busemann function corresponds to a direction
- For a discrete set of directions, find best horoball for each (like an ε-kernel)
- Problem: we don't know how to analyze this yet — but there's another approach

Alternate Approach



- P(n) has Euclidean subspaces (flats)
- Compute CH in flat
- Map faces back to horoballs
- We only need to discretize along directions modulo flats

Result

Theorem:
For a set $X \subset$ P(2) of size N, we can construct an $\mathcal{B}_\varepsilon(X)$ of size $O(\sinh(g_X)N/\varepsilon)$ in time $O(\sinh(g_X)(N \log N)/\varepsilon)$.

Conclusion

- A framework for analyzing shape in spaces where CH is difficult to work with (ball hull)
- An approximation to the ball hull (ε-ball hull)
- A way to measure width as a side benefit (extent)
- Horofunctions provide a good way to analyze manifolds like this