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Quasi-linear series in three-dimensional electromagnetic modeling

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Abstract. We have recently introduced a quasi-linear (QL) approximation for the solution of the three-dimensional (3-D) electromagnetic modeling problem. In this paper we discuss an approach to improving its accuracy by considering the QL approximations of the higher-order. This approach can be considered the natural generalization of the Born series. We use the modified Green's operator with the norm less than 1 to ensure the convergence of the higher orders QL approximations to the true solution. This new approach produces the converged QL series, which makes it possible to estimate the accuracy of the original QL approximation without direct comparison with the rigorous full integral equation solution. It also opens principally new possibilities for fast and accurate 3-D EM modeling and inversion.

1. Introduction

In our recent publications [*Zhdanov and Fang*, 1996a, b] we developed a novel approach to three-dimensional (3-D) electromagnetic (EM) modeling based on linearization of the integral equations for scattered EM fields. We called this approach a quasi-linear (QL) approximation. It is based on the assumption that the anomalous field $\mathbf{E}^a$ is linearly related to the background (normal) field $\mathbf{E}^b$ in the inhomogeneous domain $\mathbf{E}^a = \hat{\lambda}\mathbf{E}^b$, where $\hat{\lambda}$ is an electrical reflectivity tensor. The reflectivity tensor inside inhomogeneities is approximated by slowly varying functions which are determined numerically by a simple optimization technique.

The results of numerical calculations have demonstrated that QL approximation gives an accurate estimate of the 3-D EM response for a much stronger conductivity contrast (up to 100 times) than conventional Born approximation and for a wide range of frequencies. It has also been shown that this method is much faster than the computer codes based on the full integral equation (IE) solution.

There is a possibility, however, of increasing the accuracy of the QL approximation by constructing QL approximations of a higher order. It is well known, for example, that conventional Born approximation can be applied iteratively, generating the Nth order Born approximations. This approximation can be treated as the sum of N terms of Born (or Neumann) series. However, the convergence of the Born series is questionable and depends on the norm of integral equation (Green's) operator.

It seems to be attractive to construct similar series on the basis of QL approximation. In this paper we present a solution of this problem. It is based on a new method of constructing the converged Born series developed recently by *Pankratov et al.* [1995]. This method can be considered as the generalization of iterative dissipative method (IDM) developed by *Singer and Fainberg* [1995]. It transforms the conventional EM Green's integral operator $\mathbf{G}_b$ of forward modeling in inhomogeneous media into modified Green's operator $\mathbf{G}_b^m$ with the norm smaller than 1: $\| \mathbf{G}_b^m \| \leq 1$. Using this method, *Pankratov et al.* [1995] have constructed new Born series which are almost always converged.

We use this method to generate QL series and calculate the accuracy of QL approximation. The important theoretical result is that the developed QL series are always converged for any lossy background medium. A new method of QL series opens the possibility for fast and accurate solution of 3-D EM scattering problems.

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To make the presentation clearer, we feel it is necessary to begin our paper with a short review of the methods of constructing the classical and always converged Born series. In the subsequent sections we introduce QL series based on QL approximation. The QL series make it possible to estimate the accuracy of the original QL approximation [*Zhdanov and Fang*, 1996a] without direct comparison with the rigorous full integral equation (IE) solution and to construct a fast and accurate iterative method for 3-D forward modeling. The last section presents numerical results illustrating the efficiency of the new forward modeling approach.

2. Born Series in Forward Modeling

Consider a 3-D geoelectric model with the background (normal) complex conductivity $\tilde{\sigma}_b$ and local inhomogeneity D with an arbitrarily varying complex conductivity $\tilde{\sigma} = \tilde{\sigma}_b + \Delta\tilde{\sigma}$, which can be, in general case, frequency dependent. We assume that $\mu = \mu_0 = 4\pi \times 10^{-7}$H/m, where μ_0 is the free-space magnetic permeability. The model is excited by an electromagnetic field generated by an arbitrary source. This field is time harmonic as $e^{-i\omega t}$. Complex conductivity includes the effect of displacement currents $\tilde{\sigma} = \sigma - i\omega\varepsilon$, where σ and ε are electrical conductivity and dielectric permittivity, respectively. The electromagnetic fields in this model can be presented as a sum of background (normal) and anomalous fields:

$$\mathbf{E} = \mathbf{E}^b + \mathbf{E}^a, \quad \mathbf{H} = \mathbf{H}^b + \mathbf{H}^a,$$

where the background field is a field generated by the given sources in the model with the background distribution of conductivity $\tilde{\sigma}_b$, and the anomalous field is produced by the anomalous conductivity distribution $\Delta\tilde{\sigma}$.

The EM field in this model satisfies Maxwell's equations, which can be written separately for background field $\mathbf{E}^b$, $\mathbf{H}^b$:

$$\nabla \times \mathbf{H}^b = \tilde{\sigma}_b \mathbf{E}^b + \mathbf{j}^e$$

$$\nabla \times \mathbf{E}^b = i\omega\mu\mathbf{H}^b,$$

(where $\mathbf{j}^e$ is the density of extraneous electric currents) and for anomalous field $\mathbf{E}^a$, $\mathbf{H}^a$:

$$\nabla \times \mathbf{H}^a = \tilde{\sigma}_b \mathbf{E}^a + \mathbf{j}^a$$

$$\nabla \times \mathbf{E}^a = i\omega\mu\mathbf{H}^a, \tag{1}$$

where

$$\mathbf{j}^a = \Delta\tilde{\sigma}\mathbf{E} = \Delta\tilde{\sigma}\left(\mathbf{E}^b + \mathbf{E}^a\right) \tag{2}$$

is the density of excess electric currents within inhomogeneity D .

It is well known that the anomalous field can be presented as an integral over the excess currents in the inhomogeneous domain D [*Hohmann*, 1975]; [*Weidelt*, 1975]:

$$\mathbf{E}^a(\mathbf{r}_j) = \int\int\int_D \widehat{\mathbf{G}}^b(\mathbf{r}_j \mid \mathbf{r})\, \mathbf{j}^a dv = \mathbf{G}_b(\mathbf{j}^a), \tag{3}$$

where $\widehat{\mathbf{G}}^b(\mathbf{r}_j \mid \mathbf{r})$ is the electromagnetic Green's tensor defined for an unbounded conductive medium with the background conductivity $\tilde{\sigma}_b$, $\mathbf{G}_b$ is a corresponding Green's linear operator, and excess currents $\mathbf{j}^a$ are determined by equation (2).

We can combine equations (3) and (2) in another operator form:

$$\mathbf{E}^a = \mathbf{A}\left[\mathbf{E}^a\right], \tag{4}$$

where operator $\mathbf{A}$ is determined by the formula

$$\mathbf{A}\left[\mathbf{E}^a\right] = \mathbf{G}_b\left[\Delta\tilde{\sigma}\mathbf{E}^b\right] + \mathbf{G}_b\left[\Delta\tilde{\sigma}\mathbf{E}^a\right] \tag{5}$$

The operator equation (4) can be solved by the method of successive iterations:

$$\mathbf{E}^{a(N)} = \mathbf{A}\left[\mathbf{E}^{a(N-1)}\right], \; N = 1, 2, 3... \tag{6}$$

It is well known that successive iterations converge if operator $\mathbf{A}$ is a contraction operator (Banach theorem), that is,

$$\left\|\mathbf{A}\left[\mathbf{E}^{a(1)} - \mathbf{E}^{a(2)}\right]\right\| \leq k\left\|\mathbf{E}^{a(1)} - \mathbf{E}^{a(2)}\right\|, \tag{7}$$

where $\|...\|$ is L_2 norm, $k < 1$, and $\mathbf{E}^{a(1)}$ and $\mathbf{E}^{a(2)}$ are any two different solutions. Substituting (5) into (7), we obtain

$$\left\|\mathbf{A}\left[\mathbf{E}^{a(1)} - \mathbf{E}^{a(2)}\right]\right\| \leq \|\mathbf{G}_b\| \, \|\Delta\tilde{\sigma}\| \left\|\mathbf{E}^{a(1)} - \mathbf{E}^{a(2)}\right\|. \tag{8}$$

So condition (7) holds if

$$\|\mathbf{G}_b\| \, \|\Delta\tilde{\sigma}\| < 1. \tag{9}$$

Under this condition

$$\mathbf{E}^{a(N)} \to \mathbf{E}^a, \text{ when } N \to \infty. \tag{10}$$

The conventional Born approximation $\mathbf{E}^B$ arises if one takes the initial approximation (zero-order iteration) to be equal to zero:

$$\mathbf{E}^{a(0)} = 0$$

In this case

$$\mathbf{E}^{B} = \mathbf{E}^{a(1)} = \mathbf{A}\left[0\right] = \mathbf{G}_b\left[\Delta\widetilde{\sigma}\mathbf{E}^b\right]. \qquad (11)$$

The second iteration is equal to

$$\mathbf{E}^{a(2)} = \mathbf{A}\left[\mathbf{E}^{a(1)}\right] = \mathbf{G}_b\left[\Delta\widetilde{\sigma}\mathbf{E}^b\right] + \mathbf{G}_b\left[\Delta\widetilde{\sigma}\mathbf{E}^B\right] =$$

$$\left(\mathbf{G}_b\mathbf{M}_{\Delta\widetilde{\sigma}}\right)\left[\mathbf{E}^b\right] + \left(\mathbf{G}_b\mathbf{M}_{\Delta\widetilde{\sigma}}\right)^2\left[\mathbf{E}^b\right],$$

where we use an operator $\mathbf{M}_{\Delta\widetilde{\sigma}}$ of multiplication by the function $\Delta\widetilde{\sigma}$. The Nth iteration is represented as the sum of N terms of the Born series:

$$\mathbf{E}^{a(N)} = \left(\mathbf{G}_b\mathbf{M}_{\Delta\widetilde{\sigma}}\right)\left[\mathbf{E}^b\right] + \left(\mathbf{G}_b\mathbf{M}_{\Delta\widetilde{\sigma}}\right)^2\left[\mathbf{E}^b\right] + ... + \left(\mathbf{G}_b\mathbf{M}_{\Delta\widetilde{\sigma}}\right)^N\left[\mathbf{E}^b\right]. \qquad (12)$$

The Born series could be a powerful tool for EM modeling if they would converge [*Torres-Verdin and Habashy*, 1994]. However, in practice, the condition (9) does not hold, because in a general case the L_2 norm of the Green's operator is bigger than 1. That is why the Born series did not find a wide application in EM modeling.

3. Modified Born Series

Following *Pankratov et al.* [1995], we apply some linear transformations to the Green's operator to obtain a modified Green's operator $\mathbf{G}_b^m$ with the norm less than 1. The specific form of this linear transformation is motivated by energy inequality (A6), introduced by *Singer* [1995] and *Pankratov et al.* [1995], and described in Appendix A. Actually, we construct a new linear operator $\mathbf{G}_b^m$, which transforms the integrand from the right part of energy inequality (A6) into its left part:

$$\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\left(\mathbf{E}^a + \frac{\mathbf{j}^a}{2\mathrm{Re}\widetilde{\sigma}_b}\right) =$$

$$\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{G}_b\left[2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\left(\frac{\mathbf{j}^a}{2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}}\right)\right] +$$

$$\frac{\mathbf{j}^a}{2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}} = \mathbf{G}_b^m\left(\frac{\mathbf{j}^a}{2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}}\right). \qquad (13)$$

Operator $\mathbf{G}_b^m$ can be applied to any vector function

$$\mathbf{G}_b^m(\mathbf{x}) = \sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{G}_b\left(2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{x}\right) + \mathbf{x}$$

$\mathbf{x}(r) \in L_2(D)$, where $L_2(D)$ is the Hilbert space of the vector functions determined in the domain D and integrable in D with the norm

$$\|\mathbf{x}\| = \sqrt{\int\int\int_D |\mathbf{x}(r)|^2\, dv}. \qquad (14)$$

The remarkable property of this operator, established by *Singer* [1995] and *Pankratov et al.* [1995], is that according to inequality (A6),

$$\|\mathbf{G}_b^m(\mathbf{x})\| \preceq \|\mathbf{x}\|$$

for any $\mathbf{x}(r) \in L_2(D)$. In other words, the L_2 norm of modified Green's operator is always less than or equal to one:

$$\|\mathbf{G}_b^m\| \preceq 1. \qquad (15)$$

Equation (13) can be simplified by taking into account (2):

$$a\mathbf{E}^a + b\mathbf{E}^b = \mathbf{G}_b^m\left[b\left(\mathbf{E}^a + \mathbf{E}^b\right)\right], \qquad (16)$$

where

$$a = \frac{2\mathrm{Re}\widetilde{\sigma}_b + \Delta\widetilde{\sigma}}{2\sqrt{\mathrm{Re}\widetilde{\sigma}_b}},\ b = \frac{\Delta\widetilde{\sigma}}{2\sqrt{Re\widetilde{\sigma}_b}}. \qquad (17)$$

Equation (16) can be treated as an integral equation with respect to the product $a\mathbf{E}^a$:

$$a\mathbf{E}^a = \mathbf{C}\left(a\mathbf{E}^a\right), \qquad (18)$$

where $\mathbf{C}(a\mathbf{E}^a)$ is an integral operator of the anomalous field

$$\mathbf{C}\left(a\mathbf{E}^a\right) = \mathbf{G}_b^m\left(\beta a\mathbf{E}^a\right) + \mathbf{G}_b^m\left(\beta a\mathbf{E}^b\right) - \beta a\mathbf{E}^b, \qquad (19)$$

$$\beta = \frac{b}{a}.$$

The solution of this integral equation is similar to equation (4) and also can be obtained using the method of successive iterations which is governed by the equations

$$a\mathbf{E}^{a(N)} = \mathbf{C}\left[a\mathbf{E}^{a(N-1)}\right],\ N = 1, 2, 3... \qquad (20)$$

These iterations are always converged, because the operator $\mathbf{C}$ is a contraction operator. To prove this result, consider the following inequality:

$$\left\|\mathbf{C}\left[a\mathbf{E}^{a(1)}-a\mathbf{E}^{a(2)}\right]\right\|=\left\|\mathbf{G}_b^m\left[\beta\left(a\mathbf{E}^{a(1)}-a\mathbf{E}^{a(2)}\right)\right]\right\|$$

$$\leq\|\beta\|_\infty\left\|\mathbf{G}_b^m\right\|\left\|a\mathbf{E}^{a(1)}-a\mathbf{E}^{a(2)}\right\|,$$

where

$$\|\beta\|_\infty=\max_{r\in D}\left|\frac{b(r)}{a(r)}\right|=\max_{r\in D}\frac{|\Delta\tilde{\sigma}|}{|2\mathrm{Re}\tilde{\sigma}_b+\Delta\tilde{\sigma}|}.$$

Taking into account (15), we can conclude that $\mathbf{C}$ is a contraction operator if

$$\|\beta\|_\infty<1. \tag{21}$$

Simple calculations show that

$$\beta^2=\frac{|\Delta\tilde{\sigma}|^2}{|2\mathrm{Re}\tilde{\sigma}_b+\Delta\tilde{\sigma}|^2}=1-\frac{4\mathrm{Re}\tilde{\sigma}\mathrm{Re}\tilde{\sigma}_b}{|\tilde{\sigma}-\tilde{\sigma}_b|^2+4\mathrm{Re}\tilde{\sigma}\mathrm{Re}\tilde{\sigma}_b}<1, \tag{22}$$

under the natural condition that

$$0<Re\tilde{\sigma}_{\min}\leq\mathrm{Re}\tilde{\sigma}\leq\mathrm{Re}\tilde{\sigma}_{\max}<\infty,$$

$$0<\mathrm{Re}\tilde{\sigma}_{b\,\min}\leq\mathrm{Re}\tilde{\sigma}_b\leq\mathrm{Re}\tilde{\sigma}_{b\,\max}<\infty.$$

Thus we have proved that $\mathbf{C}$ is a contraction operator for any lossy background medium (where $\mathrm{Re}\tilde{\sigma}_{b\,\min}>0$).

Therefore the Nth iteration approaches the actual anomalous field

$$\mathbf{E}^{a(N)}\to\mathbf{E}^a$$

when $N\to\infty$.

We obtain the first iteration in the solution of operator equation (18), assuming that the initial approximation $\mathbf{E}^{a(0)}$ (zero-order iteration) is selected to be equal to zero ($\mathbf{E}^{a(0)}=0$):

$$a\mathbf{E}^{a(1)}=\mathbf{C}[0]=\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^b\right)-\beta a\mathbf{E}^b, \tag{23}$$

where we use an operator $\mathbf{M}_\beta$ of multiplication by the function β. By analogy with the classical Born approximation, we will call expression (23) a modified Born approximation $\mathbf{E}^{Bm}$:

$$\mathbf{E}^{a(1)}=\mathbf{E}^{Bm}=\frac{1}{a}\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^b\right)-\beta\mathbf{E}^b. \tag{24}$$

The second iteration is equal to

$$a\mathbf{E}^{a(2)}=\mathbf{C}\left(a\mathbf{E}^{a(1)}\right)=\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^{Bm}\right)+a\mathbf{E}^{Bm}.$$

The third iteration is equal to

$$a\mathbf{E}^{a(3)}=\mathbf{C}\left(a\mathbf{E}^{a(2)}\right)=\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^{a(2)}\right)+a\mathbf{E}^{Bm}=$$

$$\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^2\left(a\mathbf{E}^{Bm}\right)+\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^{Bm}\right)+a\mathbf{E}^{Bm}.$$

Finally, the Nth iteration can be treated as the sum of N terms of the Born (or Neumann) series:

$$a\mathbf{E}^{a(N)}=a\mathbf{E}^{Bm}+\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^{Bm}\right)+$$

$$+\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^2\left(a\mathbf{E}^{Bm}\right)...+\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^{N-1}\left(a\mathbf{E}^{Bm}\right). \tag{25}$$

The remarkable result is that these series are always converged for any lossy medium with $\mathrm{Re}\tilde{\sigma}_{b\,\min}>0$!

Note that converged Born series have been obtained by *Pankratov et al.* [1995]. for auxiliary vector fields. Our result differs from *Pankratov et al.* [1995] in the way that we construct the always converged Born series directly for the anomalous field.

Let us analyze more carefully the modified Born approximation introduced by formula (24). Simple calculations show that

$$\mathbf{E}^{Bm}=\frac{1}{a}\mathbf{G}_b^m\mathbf{M}_\beta\left(a\mathbf{E}^b\right)-\beta\mathbf{E}^b=\frac{1}{a}\sqrt{\mathrm{Re}\tilde{\sigma}_b}\mathbf{E}^B, \tag{26}$$

where $\mathbf{E}^B=G_b\left[\Delta\tilde{\sigma}\mathbf{E}^b\right]$ is a conventional Born approximation [*Born*, 1933]. Therefore we obtain

$$\mathbf{E}^{Bm}=\frac{2\mathrm{Re}\tilde{\sigma}_b}{2\mathrm{Re}\tilde{\sigma}_b+\Delta\tilde{\sigma}}\mathbf{E}^B. \tag{27}$$

From equation (27) we can see that the modified Born approximation is equal to the conventional Born approximation outside inhomogeneous domain D:

$$\mathbf{E}^{Bm}=\mathbf{E}^B,\ \text{if}\ r\notin D.$$

However, inside D they are different. Actually, this difference makes the new Born series to be converged.

Taking into account formula (26), we can obtain the following expression for the modified Nth order Born approximation as the sum of N terms of the Born (or Neumann) series, which helps to understand better the internal structure of new series:

$$a\mathbf{E}^{a(N)}=\sqrt{\mathrm{Re}\tilde{\sigma}_b}\mathbf{E}^B+\mathbf{G}_b^m\mathbf{M}_\beta\left(\sqrt{\mathrm{Re}\tilde{\sigma}_b}\mathbf{E}^B\right)+$$

$$\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^2\left(\sqrt{\mathrm{Re}\tilde{\sigma}_b}\mathbf{E}^B\right)...+\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^{N-1}\left(\sqrt{\mathrm{Re}\tilde{\sigma}_b}\mathbf{E}^B\right).$$

These series unconditionally converge to the anomalous field

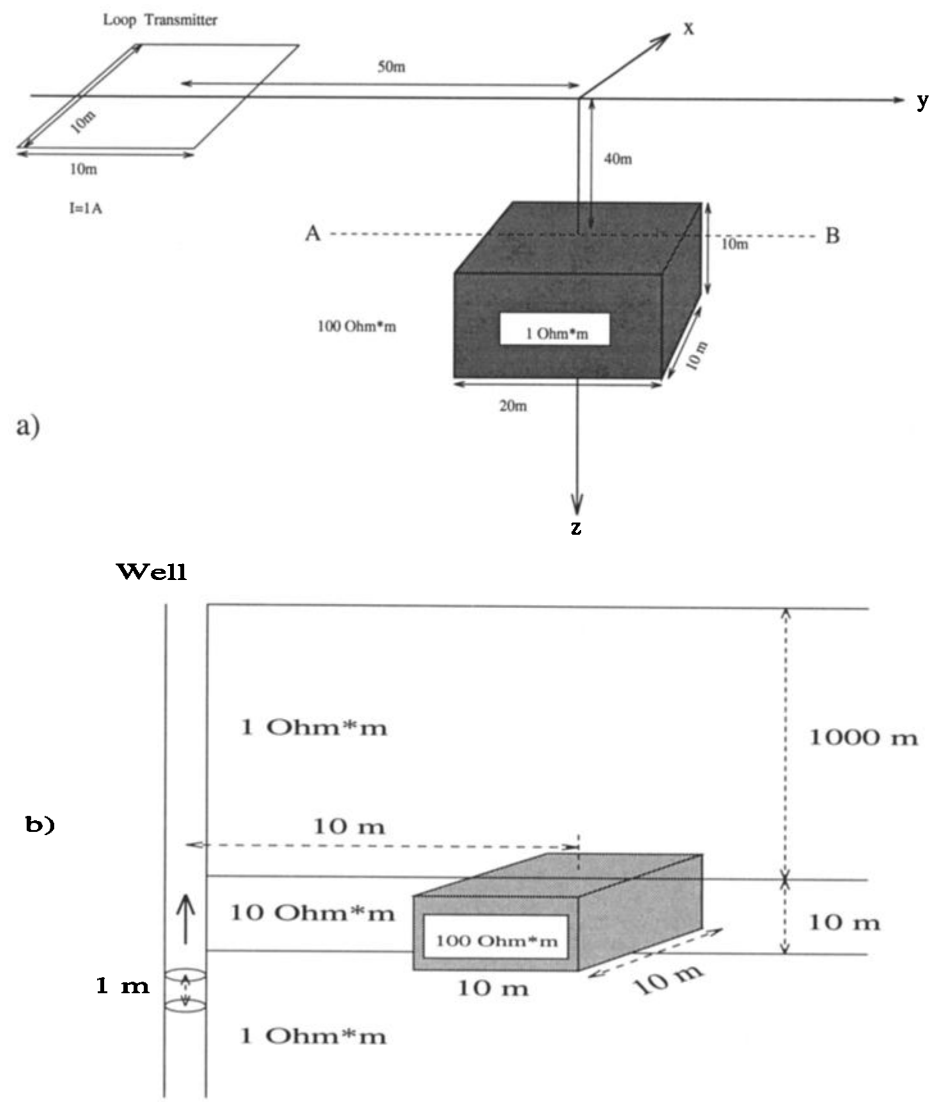

Figure 1. (a) Three-dimensional (3-D) model of rectangular conductive structure in a homogeneous half space, excited by a rectangular loop, and (b) 3-D borehole model of cube resistive structure in a three-layer background model, excited by a vertical magnetic dipole.

$$\mathbf{E}^a = \frac{1}{a}\sum_{k=0}^{\infty}\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^k\left(\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{E}^B\right),$$

because operator $(\mathbf{G}_b^m\mathbf{M}_\beta)$ is a contraction operator:

$$\left\|\mathbf{G}_b^m\mathbf{M}_\beta\right\| \le \left\|\mathbf{G}_b^m\right\|\left\|\beta\right\|_\infty \prec 1.$$

We can estimate the accuracy δ_B of the Nth iteration:

$$\delta_B = \left\|a\mathbf{E}^a - a\mathbf{E}^{a(N)}\right\| =$$

$$\left\|\sum_{k=N}^{\infty}\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^k\left(\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{E}^B\right)\right\| =$$

$$\left\|\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^N\sum_{k=0}^{\infty}\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^k\left(\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\mathbf{E}^B\right)\right\| =$$

$$\left\|\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^N a\mathbf{E}^a\right\| \le$$

$$\left\|\left(\mathbf{G}_b^m\mathbf{M}_\beta\right)^N\right\|\left\|a\mathbf{E}^a\right\| \le \left\|\beta\right\|_\infty^N\left\|a\mathbf{E}^a\right\|. \qquad (28)$$

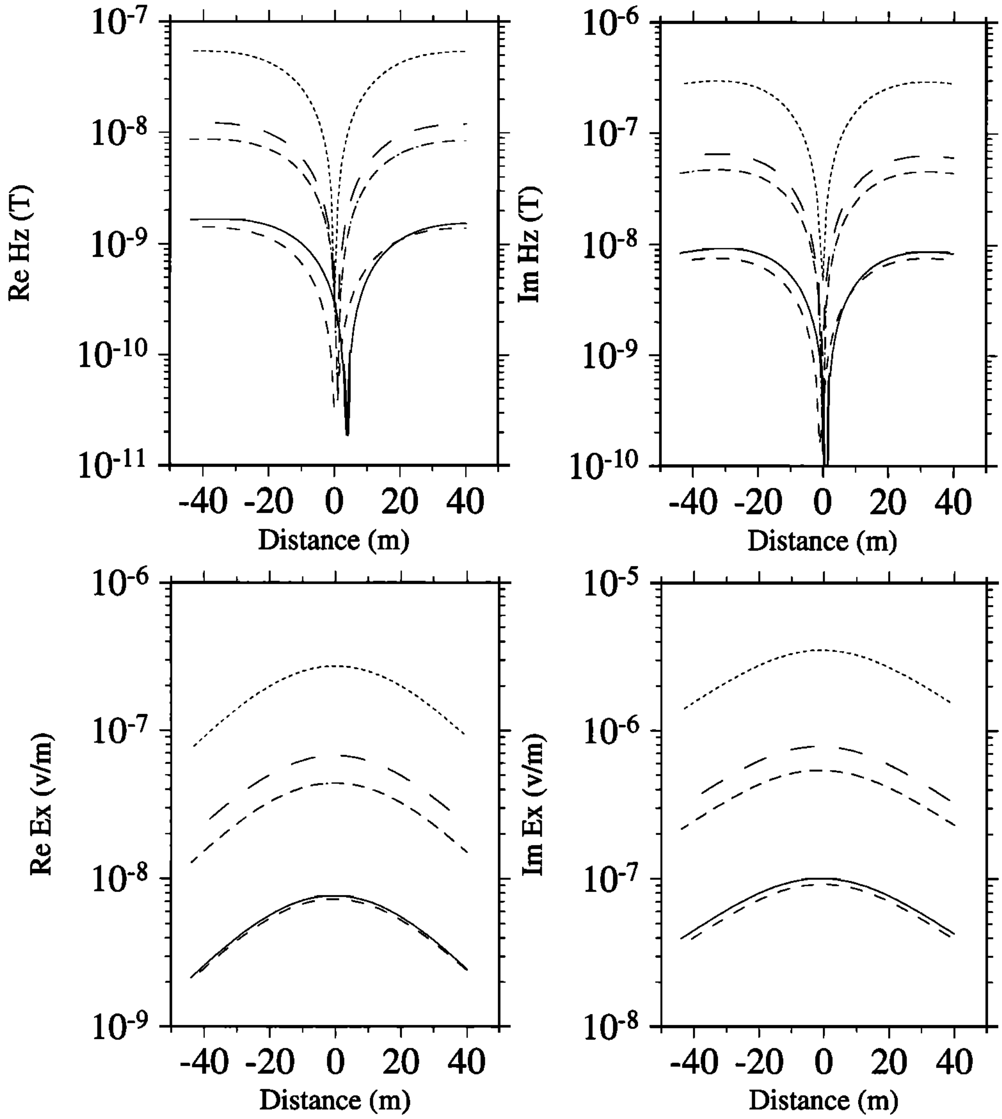

Figure 2. Numerical comparison of full integral equation (IE) solution (solid line), QL approximation (short-dashed line), Born approximation (dotted line), second-order modified Born approximation (long-dashed line), and third-order modified Born approximation (dash-dotted line) computed for model 1 (Figure 1a) at frequency 1000 Hz. Calculations are done for receivers located along the Y axes on the surface.

Taking into account that $\|\beta\|_\infty < 1$, we conclude that δ progressively goes to zero with the number of iterations N.

4. Quasi-linear Approximation of the Modified Green's Operator

We will obtain a more accurate approximation even on the first step if we assume that the anomalous field $\mathbf{E}^a$ inside the inhomogeneous domain is not equal to zero, as it was supposed in the previous section, but is linearly related to the background field $\mathbf{E}^b$ by some tensor $\widehat{\lambda}$ [*Zhdanov and Fang*, 1996a]:

$$\mathbf{E}^a(\mathbf{r}) \approx \widehat{\lambda}(\mathbf{r})\,\mathbf{E}^b(\mathbf{r}). \tag{29}$$

Subsequently, we use expression (29) as the zero-order approximation for the scattered field inside the inhomogeneity

$$\mathbf{E}^{a(0)} = \widehat{\lambda}\mathbf{E}^b$$

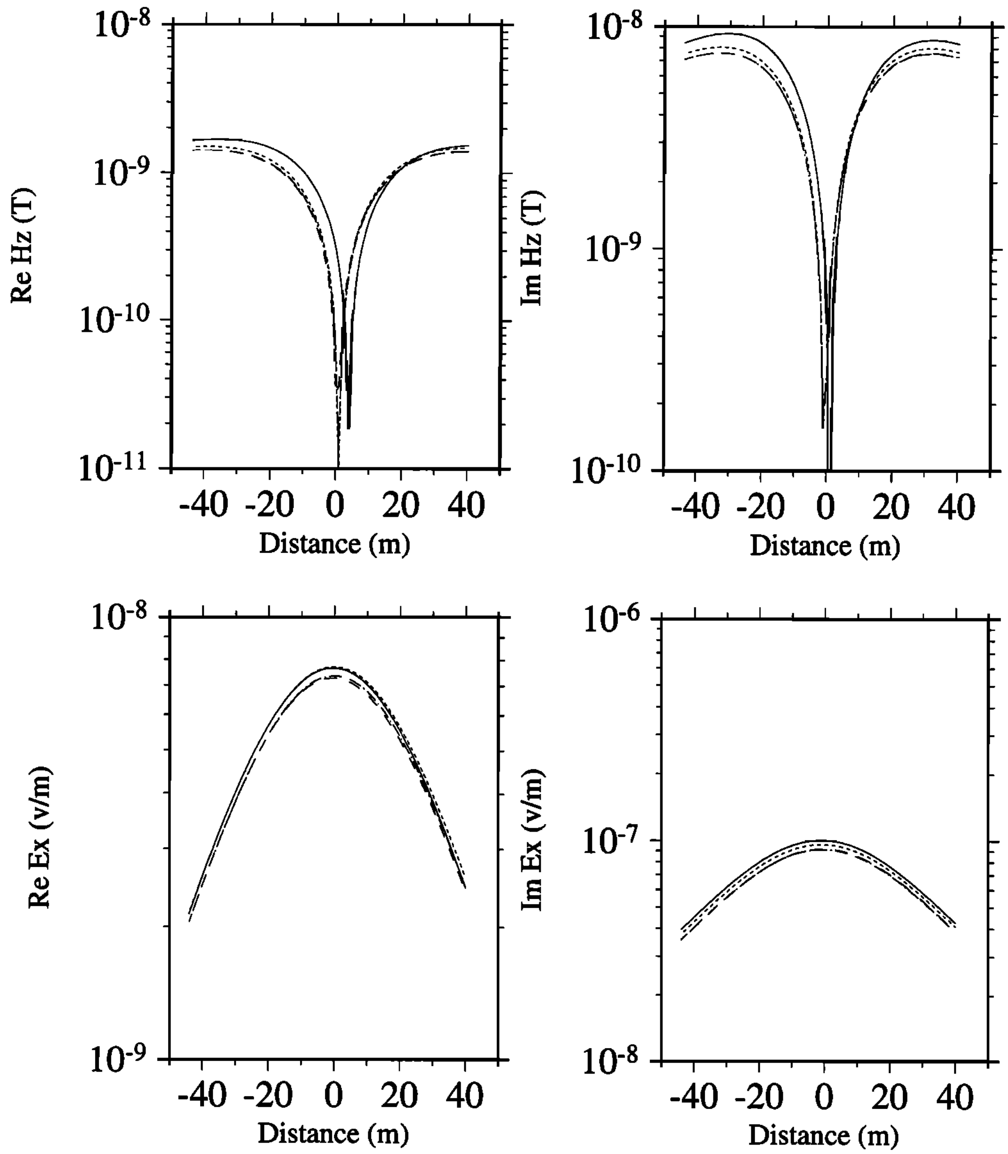

Figure 3. Numerical comparison of full IE solution (solid line), QL approximation (dashed line), second-order QL approximation (dash-dotted line), and third-order QL approximation (dotted line) computed for model 1 (Figure 1a) at frequency 1000 Hz. Calculations are done for receivers located along the Y axes on the surface.

and calculate the first approximation as follows:

$$a\mathbf{E}^{a(1)} = \mathbf{C}\left[a\widehat{\lambda}\mathbf{E}^b\right] =$$

$$\mathbf{G}_b^m\beta\left(a\widehat{\lambda}\mathbf{E}^b\right) + \mathbf{G}_b^m\beta\left(a\mathbf{E}^b\right) - \beta a\mathbf{E}^b =$$

$$\mathbf{G}_b^m\beta\left(a\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right) - \beta a\mathbf{E}^b. \tag{30}$$

We call this approximation a quasi-linear approximation of the first order $\mathbf{E}_{ql}^{a(1)}$ for anomalous field

$$\mathbf{E}_{ql}^{a(1)} = \frac{1}{a}\mathbf{G}_b^m\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right) - \beta\mathbf{E}^b. \tag{31}$$

Note that original QL approximation is given by the formula [*Zhdanov and Fang*, 1996a]

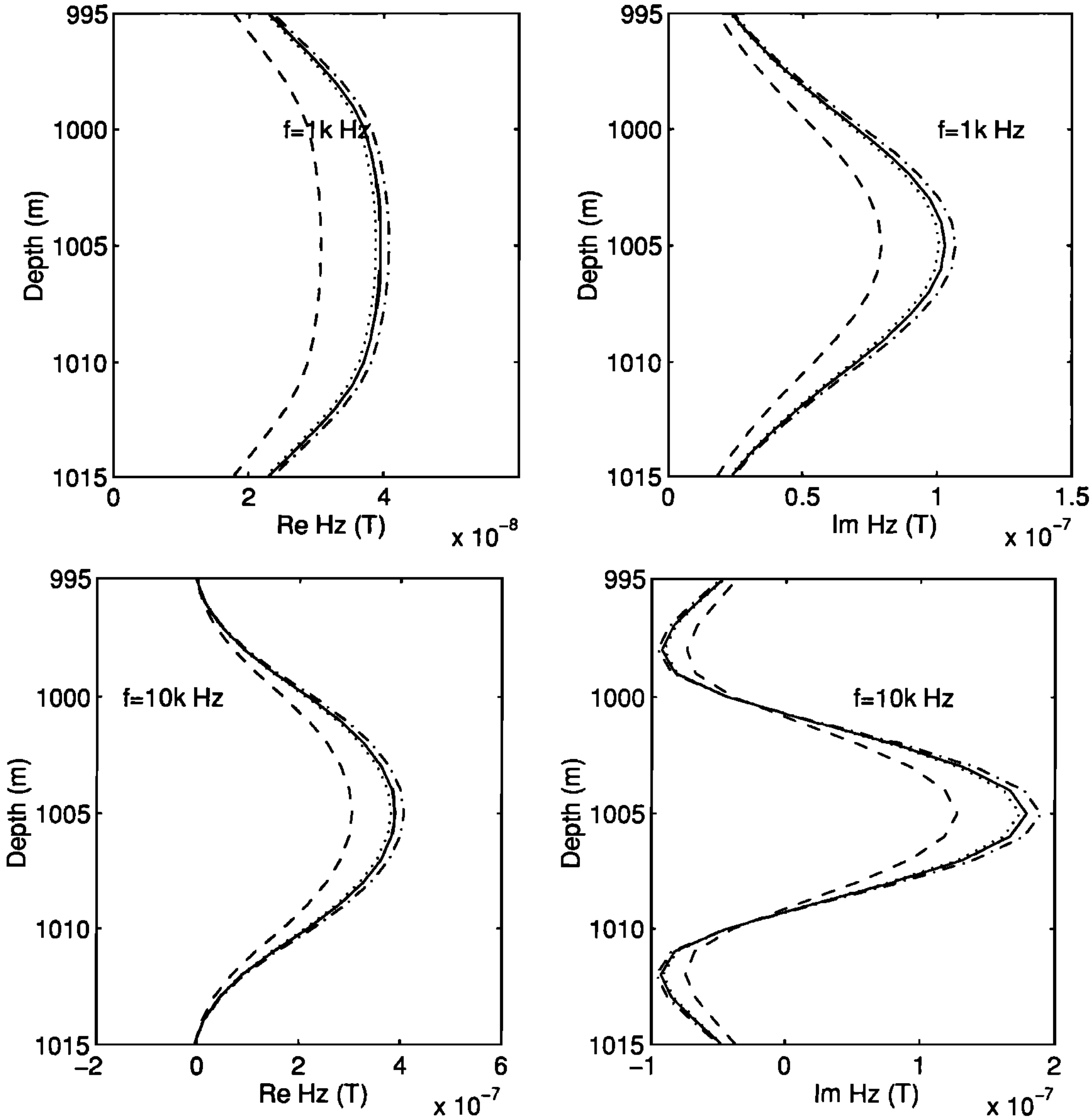

Figure 4. Numerical comparison of full IE solution (solid line), Born approximation (dashed line), second-order modified Born approximation (dash-dotted line), and third-order modified Born approximation (dotted line) computed for model 2 (Figure 1b) at frequencies 1 and 10 kHz. Calculations are done for receivers moved along the borehole.

$$\mathbf{E}^a_{ql} = \mathbf{G}_b\left(\Delta\widetilde{\sigma}\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right). \qquad (32)$$

Obviously, outside inhomogeneities, where $\Delta\widetilde{\sigma} = 0$, the QL approximation of the first order is identically equal to the original QL approximation:

$$\mathbf{E}^{a(1)}_{ql} = \mathbf{E}^a_{ql}, \text{ if } r \notin D. \qquad (33)$$

However, inside inhomogeneities ($\Delta\widetilde{\sigma} \neq 0$) they are different. In particular, inside domain D we obtain the following approximate formula for $\mathbf{E}^{a(1)}_{ql}$:

$$\mathbf{E}^{a(1)}_{ql} \approx \widehat{\lambda}\mathbf{E}^b \approx \frac{1}{a}\mathbf{G}^m_b\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right) - \beta\mathbf{E}^b =$$

$$\frac{1}{a}\sqrt{\mathrm{Re}\widetilde{\sigma}_b}\,\mathbf{G}_b\left(\Delta\widetilde{\sigma}\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right) + \beta\widehat{\lambda}\mathbf{E}^b.$$

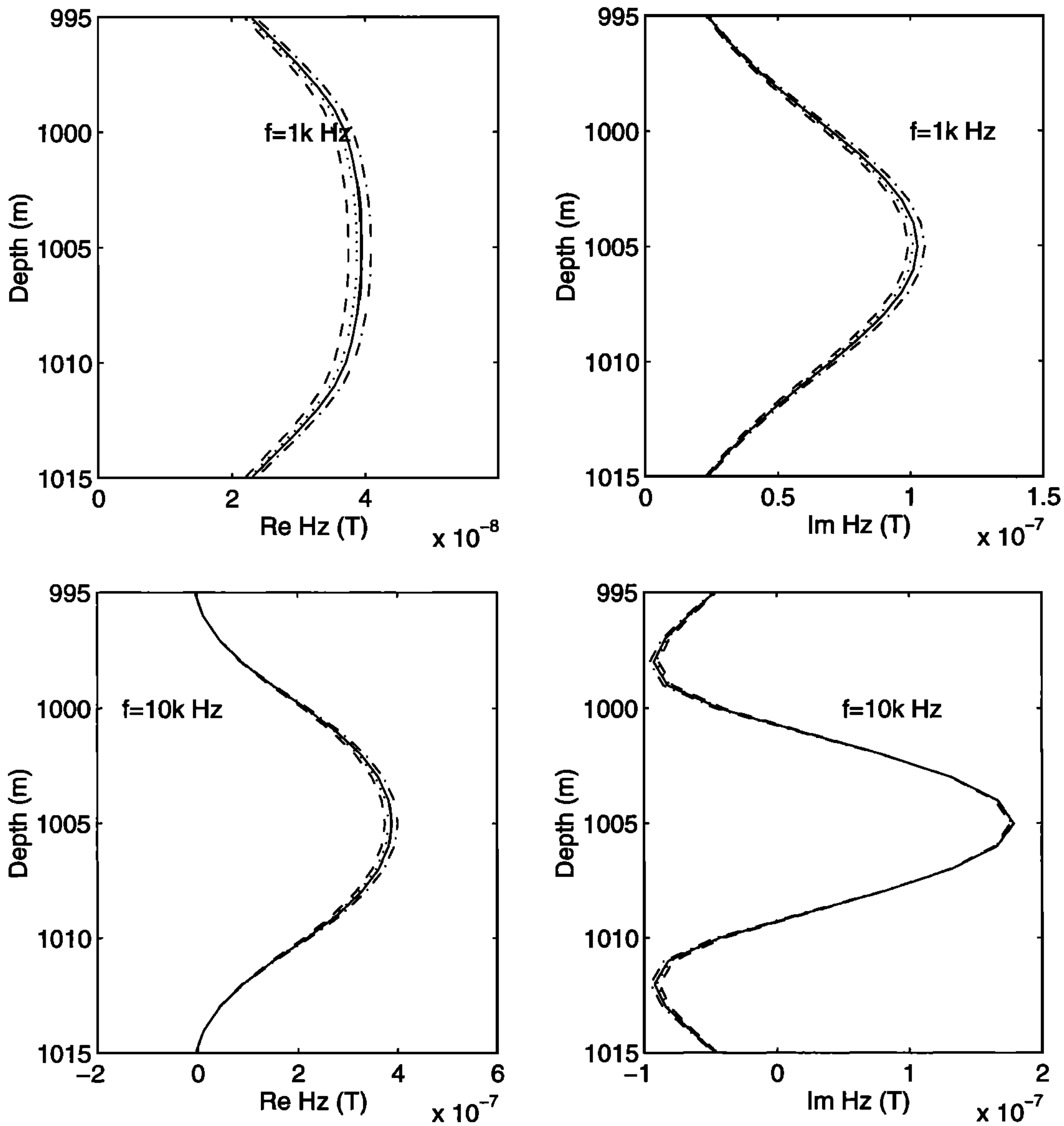

Figure 5. Numerical comparison of full IE solution (solid line), QL approximation (dashed line), second-order QL approximation (dash-dotted line), and third-order QL approximation (dotted line) computed for model 2 (Figure 1b) at frequencies 1 and 10 kHz. Calculations are done for receivers moved along the borehole.

We estimate the accuracy of the first-order QL approximation using the following formula derived in Appendix B:

$$\left\| a\left(\mathbf{E}^a - \mathbf{E}^a_{ql}\right)\right\| \leq \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty}\varphi\left(\widehat{\lambda}\right), \qquad (34)$$

where $\varphi\left(\widehat{\lambda}\right)$ is determined by formula

$$\varphi\left(\widehat{\lambda}\right) = \left\| a\mathbf{E}^a_{ql} - a\widehat{\lambda}\mathbf{E}^b\right\| =$$

$$\left\| \mathbf{G}^m_b\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right) - b\mathbf{E}^b - a\widehat{\lambda}\mathbf{E}^b\right\|.$$

Formula (34) shows that the minimum of $\varphi\left(\widehat{\lambda}\right)$ determines the accuracy of the QL approximation. This criterion is used to find the electrical reflectivity

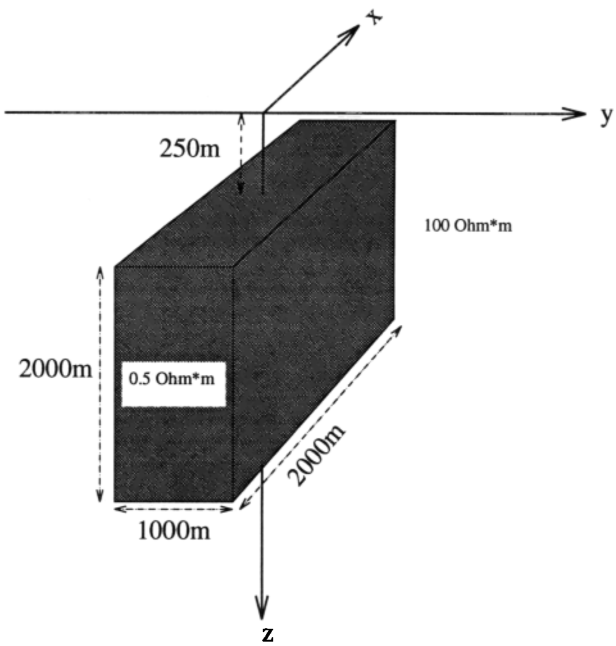

Figure 6. Three-dimensional geoelectric model containing one conductive body in a homogeneous half space with a plane wave excitation (model 3).

tensor

$$\left\|\mathbf{G}_b^m\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^b\right)-b\mathbf{E}^b-a\widehat{\lambda}\mathbf{E}^b\right\|=\varphi\left(\widehat{\lambda}\right)=\min. \quad (35)$$

Note that the minimization problem (35) is equivalent to the minimization problem which we have used in determining electrical reflectivity tensor for original QL approximation [*Zhdanov and Fang*, 1996a]. This means that we can use exactly the same reflectivity tensor in both cases.

5. QL Series

We have noticed that the background of the modified Born approximation and the new first-order QL approximation is the same. The main difference is that in the case of the Born approximation, the starting point (zero-order approximation) for the iteration process is the zero anomalous field, while in QL approach we start with the anomalous field proportional to the background field:

$$\mathbf{E}_{ql}^{a(0)}=\widehat{\lambda}\mathbf{E}^b. \quad (36)$$

In principle, we can extend our approach to computing all iterations by (30). In this case we will obtain a complete analog of Born series. For example, the second-order QL approximation is equal to

$$a\mathbf{E}_{ql}^{a(2)}=\mathbf{C}\left(a\mathbf{E}_{ql}^{a(1)}\right)=$$

$$=\left(\mathbf{G}_b^m\beta\right)^2\left(a\widehat{\lambda}\mathbf{E}^b\right)+\mathbf{G}_b^m\left(a\mathbf{E}^{Bm}\right)+a\mathbf{E}^{Bm}. \quad (37)$$

The third-order QL approximation is given by the formula

$$a\mathbf{E}_{ql}^{a(3)}=\mathbf{C}\left(a\mathbf{E}^{a(2)}\right)=\left(\mathbf{G}_b^m\beta\right)^3\left(a\widehat{\lambda}\mathbf{E}^b\right)+$$

$$+\left(\mathbf{G}_b^m\beta\right)^2\left(a\mathbf{E}^{Bm}\right)+\mathbf{G}_b^m\left(a\mathbf{E}^{Bm}\right)+a\mathbf{E}^{Bm}.$$

Finally, the Nth order QL approximation can be treated as the sum of N terms of the QL series

$$a\mathbf{E}_{ql}^{a(N)} = \sum_{k=0}^{N-1}\left(\mathbf{G}_b^m\beta\right)^k\left(a\mathbf{E}^{Bm}\right)+\left(\mathbf{G}_b^m\beta\right)^N\left(a\widehat{\lambda}\mathbf{E}^b\right)$$

$$= a\mathbf{E}^{a(N)}+\left(\mathbf{G}_b^m\beta\right)^N\left(a\widehat{\lambda}\mathbf{E}^b\right), \quad (38)$$

where according to (25), $\mathbf{E}^{a(N)}$ is Nth- order modified Born approximation. Let us compare the Nth-order Born approximation with the Nth-order QL approximation:

$$\mathbf{E}_{ql}^{a(N)}-\mathbf{E}^{a(N)}=\frac{1}{a}\left(\mathbf{G}_b^m\beta\right)^N\left(a\left(\widehat{\lambda}\mathbf{E}^b\right)\right). \quad (39)$$

From the last formula we can see that QL approximations of the higher orders take into account an additional term in the series in comparison with the modified Born approximation of the higher orders, which makes them more accurate. We will illustrate this theoretical fact by the results of numerical modeling presented in the next section.

The accuracy of the QL approximation of the Nth-order is estimated in Appendix B and can be determined by the formula

$$\left\|a\mathbf{E}^a-a\mathbf{E}_{ql}^{a(N)}\right\|\leq\frac{\|\beta\|_\infty^N}{1-\|\beta\|_\infty}\varphi\left(\widehat{\lambda}\right). \quad (40)$$

Note that in the framework of the QL method the reflectivity tensor λ is computed by minimization $\varphi\left(\widehat{\lambda}\right)$. In turn, the $\varphi\left(\widehat{\lambda}\right)$ minimum value according to (40) determines the accuracy of QL approximations of any order N.

Formula (40) also shows that QL series converge with the rate proportional to the power function of

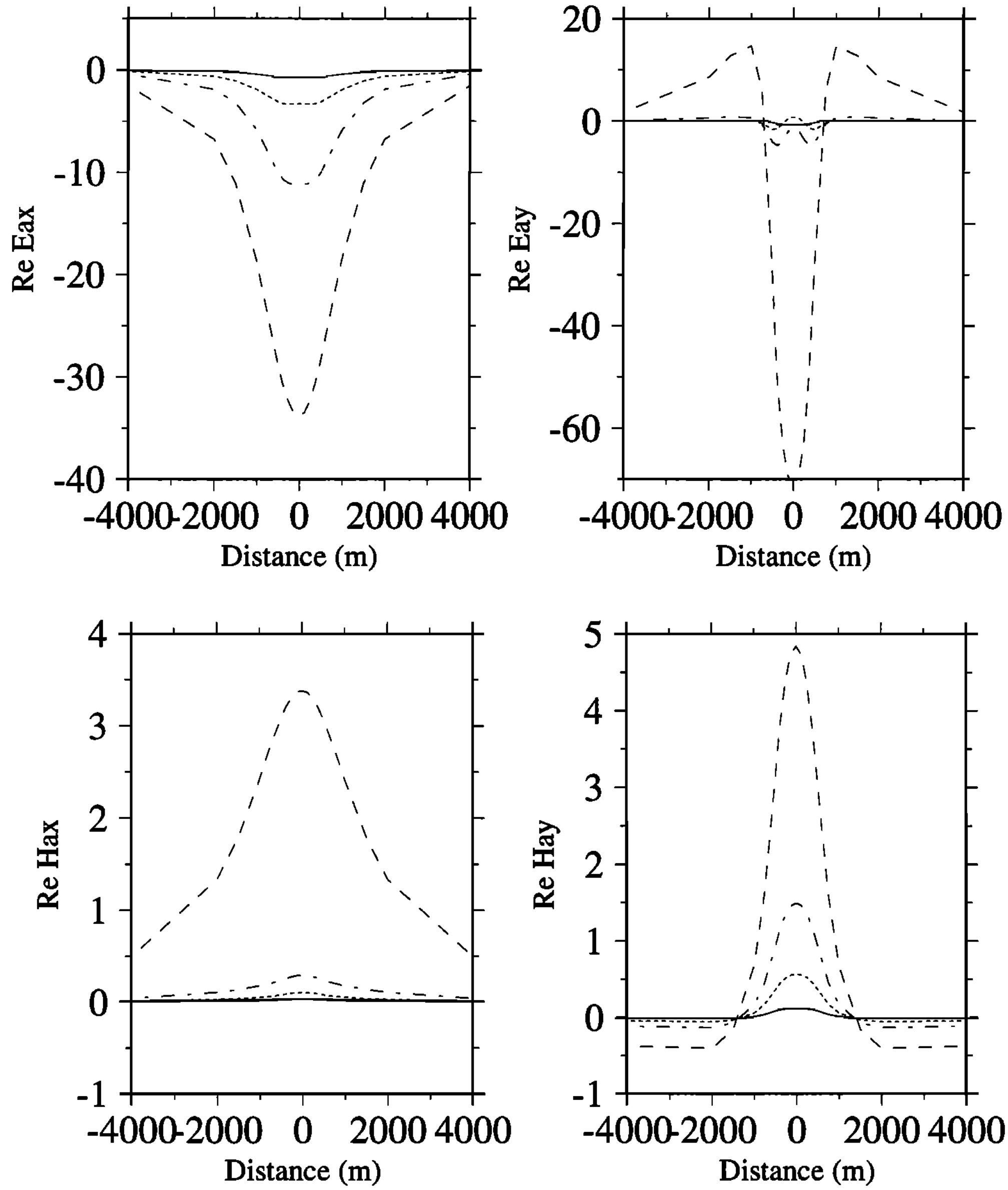

Figure 7. Numerical comparison of full IE solution (solid line), the modified Born approximations of the first order (dashed line), the third order (dash-dotted line), and the fifth order (dotted line) computed for model 3 (Figure 6) at frequency 1 Hz. Calculations are done for receivers located along the Y axes on the surface.

β. So the convergence rate of QL series directly depends on the value of parameter β: If it is small, the convergence rate is fast, and if it is close to one, the QL series converges slowly to exact solution.

Equation (22) demonstrates that parameter β can be close to one in two cases: (1) for a lossless background medium when $\mathrm{Re}\tilde{\sigma}_{b\,\mathrm{min}} \approx 0$ and (2) for a model with a strong conductivity anomalies when $|\Delta\tilde{\sigma}| \gg \mathrm{Re}\tilde{\sigma}_b$. These two extreme models can be considered the difficult cases for QL series application.

Comparison of the inequalities (28) and (40) clearly demonstrates that the accuracy of the Born approximation depends only on the order N, while the ac-

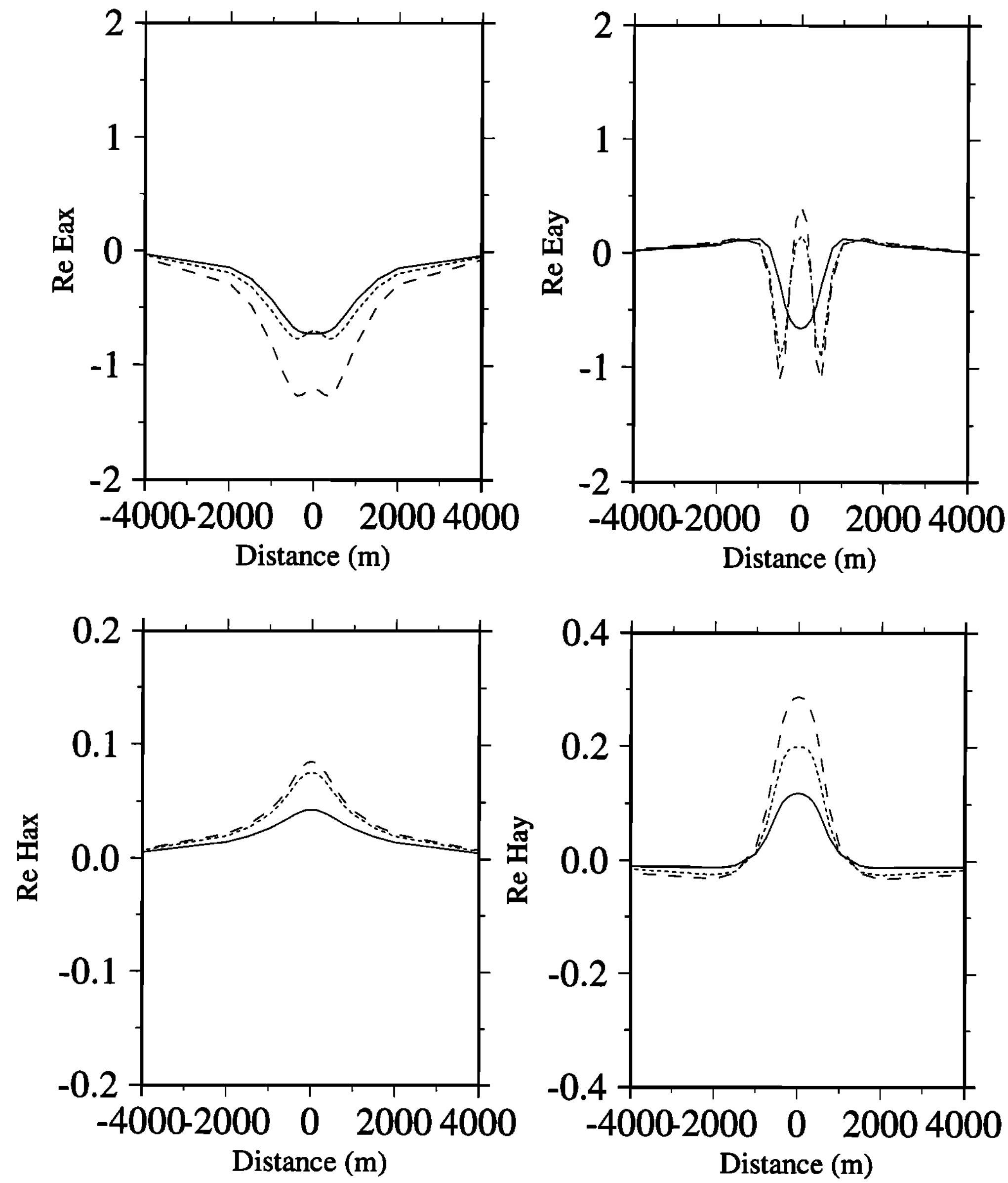

Figure 8. Numerical comparison of full IE solution (solid line), the modified Born approximations of the seventh order (dashed line), and the ninth order (dotted line) computed for model 3 (Figure 6) at frequency 1 Hz. Calculations are done for receivers located along the Y axes on the surface.

curacy of QL approximation of the same order can be increased by a proper selection of $\widehat{\lambda}$. This circumstance makes the QL approximation a more efficient tool for EM modeling than conventional or modified Born series.

At the same time, we can use the inequality (40) to prove that QL series determined by these equations are always converged. This result comes from the fact that $\|\beta\|_\infty < 1$ for any geoelectrical model with the lossy background medium $Re\widetilde{\sigma}_{b\,\min} > 0$. Therefore

$$\left\| a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)} \right\| \to 0, \text{ when } N \to \infty.$$

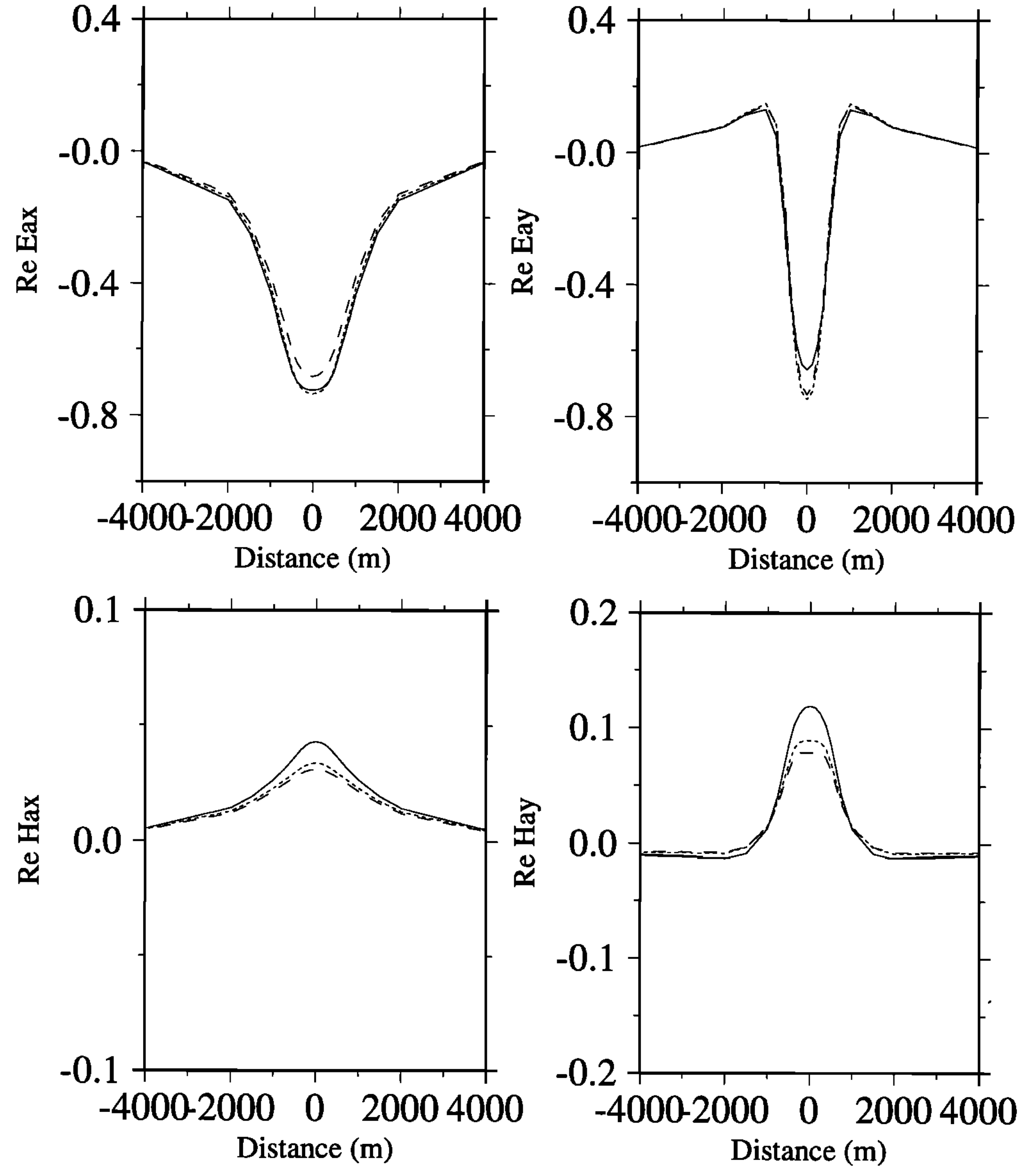

Figure 9. Numerical comparison of full IE solution (solid line), the QL approximations of the first order (dashed line), and the third order (dotted line), computed for model 3 (Figure 6) at frequency 1 Hz. Calculations are done for receivers located along the Y axes on the surface.

Moreover, based on inequality (B12) developed in Appendix B, we can estimate the accuracy of the QL approximation of the Nth-order by comparing it with the QL approximation of the $(N-1)$th-order:

$$\varepsilon_N = \frac{\left\| a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)} \right\|}{\left\| a\mathbf{E}_{ql}^{a(N)} \right\|} \le \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty} r_N, \tag{41}$$

where $\mathbf{E}_{ql}^{a(0)} = \widehat{\lambda}\mathbf{E}^b$, and r_N is the relative convergence rate of the QL approximations:

$$r_N = \frac{\left\| a\mathbf{E}_{ql}^{a(N)} - a\mathbf{E}_{ql}^{a(N-1)} \right\|}{\left\| a\mathbf{E}_{ql}^{a(N)} \right\|} \tag{42}$$

Table 1. Comparison of CPU Time for Electromagnetic Modeling Using Different Methods (Model 3)

Cells' Number in Anomalous Domain	250 Cells	400 Cells	800 Cells
Full integral equation solution	1064.6	4785.3	18370.3
QL approximation	221.6	504.5	778.4
Second-order QL approximation	243.4	556.4	1016.3
Third-order QL approximation	284.7	666.7	1348.1
Born approximation	191.5	356.5	393.0
Fifth-order Born approximation	285.8	632.7	1249.1
Ninth-order Born approximation	361.4	840.4	2210.8

Values are given in seconds. QL, quasi-linear.

In particular, the accuracy of the original QL approximation $\mathbf{E}^a_{ql}$ can be estimated by computing using formula

$$\varepsilon_1 = \frac{\left\| a\mathbf{E}^a - a\mathbf{E}^a_{ql} \right\|}{\left\| a\mathbf{E}^a_{ql} \right\|} \leq \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty} r_1 =$$

$$\frac{\|\beta\|_\infty}{1-\|\beta\|_\infty} \cdot \frac{\left\| a\mathbf{E}^a_{ql} - a\widehat{\lambda}\mathbf{E}^n \right\|}{\left\| a\mathbf{E}^a_{ql} \right\|}, \tag{43}$$

which can be also obtained from equation (40).

The important result is that formulae (41) and (43) make it possible to obtain a quantitative estimation of the QL approximation accuracy without direct comparison with the rigorous full IE forward modeling solution.

6. Numerical Modeling Results

In this section we present the numerical results of comparison between the Born approximation, QL approximation, modified Born series, and QL series for simple 3-D geoelectrical models, presented in Figures 1 and 6.

Model 1 shown in Figure 1a consists of a homogeneous half space (with resistivity 100 Ohm-m) and a conductive rectangular inclusion with the resistivity 1 Ohm-m. The electromagnetic field in the model is excited by a horizontal rectangular loop, located 50 m to the left of the model, with the loop 10 m on a side and the current at 1 A. We have used the integral equation program SYSEM for computing the frequency domain response of the complex conductivity structure [*Xiong*, 1992]. Figure 2 shows the comparison of the different solutions for real and imaginary parts of the anomalous electrical field E^a_x and the anomalous magnetic field H^a_z computed for the model 1 at the frequency 1 kHz. Calculations are done for receivers located along the axes Y on the surface. In the case of QL approximation we have used the simplest scalar reflectivity tensor. One can see that the full integral equation solution (solid line) and the QL approximations (short-dashed line) produce very similar results, while the conventional Born approximation (dotted line) produces the cor-

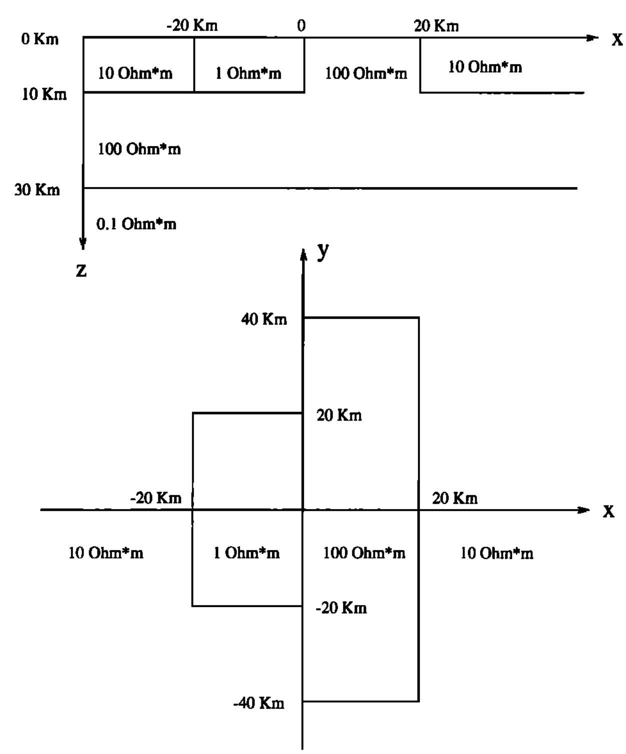

Figure 10. Three-dimensional geoelectric model containing one conductive body and one resistive body in a three-layer background resistivity cross section with resistivities 10, 100, and 0.1 Ohm-m and with a plane wave excitation (model 4).

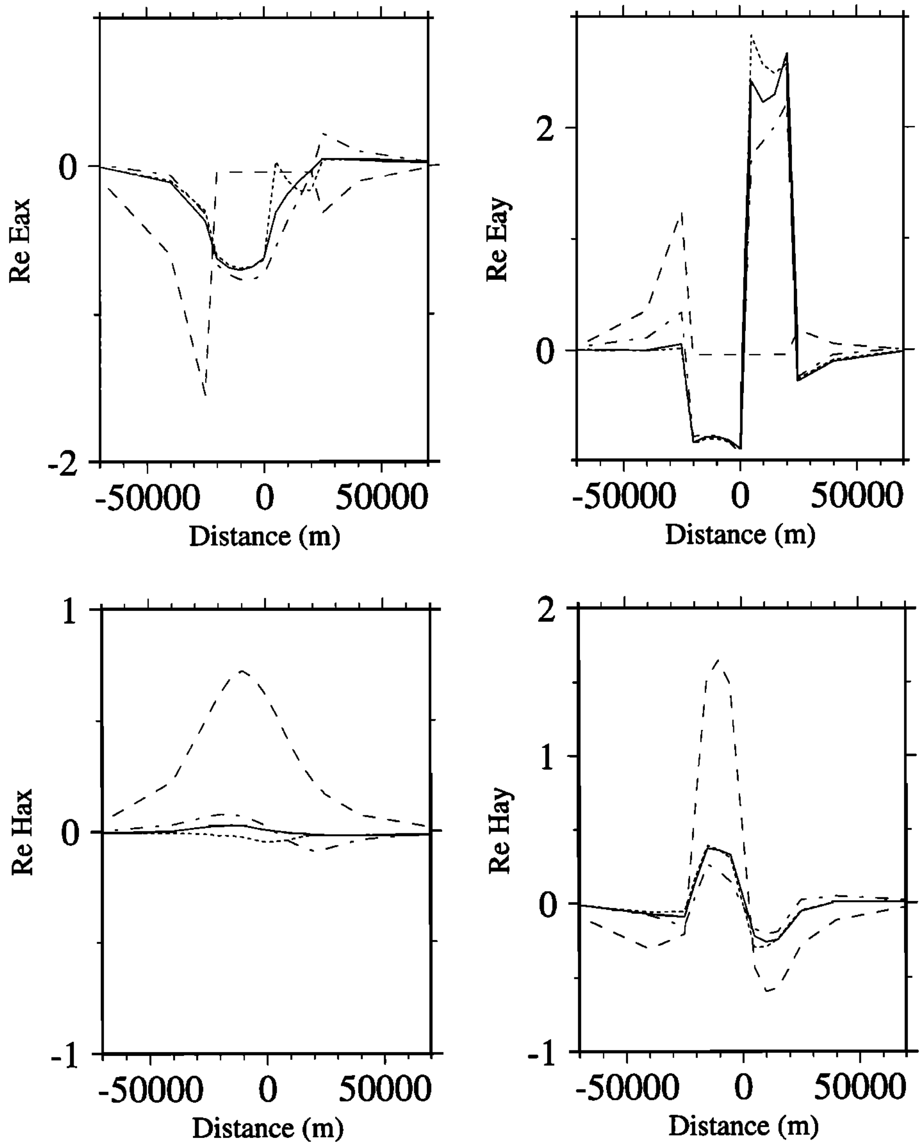

Figure 11. Numerical comparison of full IE solution (solid line), the modified Born approximations of the first order (dashed line), the third order (dash-dotted line), and the fifth order (dotted line) computed for model 4 (Figure 10) at frequency 0.01 Hz. Calculations are done for receivers located along the Y axes on the surface.

rect shape but incorrect magnitude. The second-order (long-dashed line) and the third-order (dash-dotted line) modified Born approximations go closer to the true solution (solid line) but still have the wrong magnitude. Figure 3 presents the QL approximations of the different orders. One can see that the QL approximation of the third order lies close to the integral equation solution. There is a very small difference only for the ReH_z component, which has the smallest magnitude among all analyzed curves.

Figure 1b shows a 3-D borehole model which consists of a three-layer background resistivity cross section (with resistivities 1, 10, and 1 Ohm-m) and a resistive rectangular inclusion with the resistivity 100 Ohm-m in the middle layer. We have numerically simulated the borehole EM observations in this

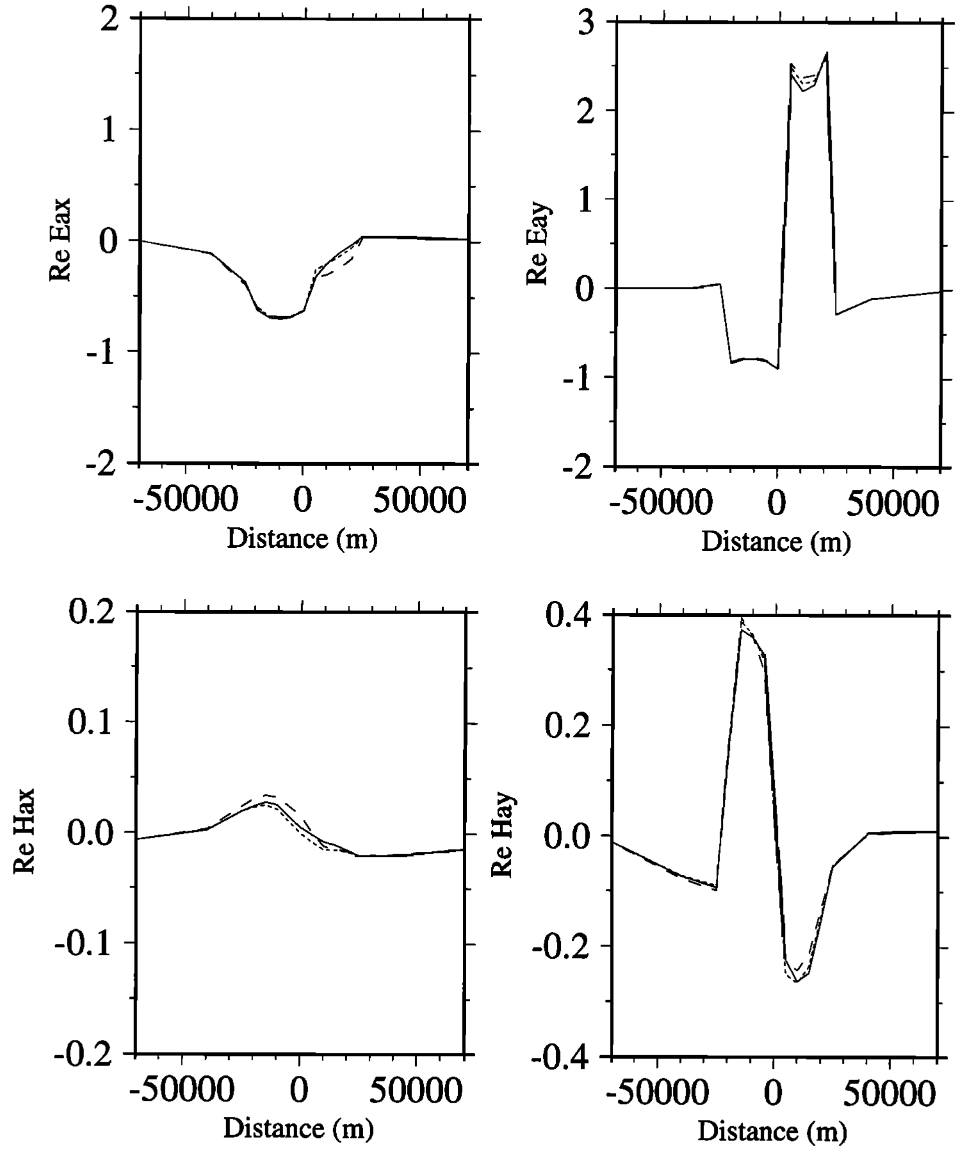

Figure 12. Numerical comparison of full IE solution (solid line), the modified Born approximations of the seventh order (dashed line), and the ninth order (dotted line) computed for model 4 (Figure 10) at frequency 0.01 Hz. Calculations are done for receivers located along the Y axes on the surface.

model with the moving dipole-dipole array consisting of the vertical magnetic dipole transmitter and receiver vertically separated by 1 m.

The comparisons of the magnetic field component H_z for model 2 are shown in Figures 4 and 5. Figure 4 presents the plots of the vertical magnetic field component H_z (real and imaginary parts) computed for model 2 at frequencies 1 and 10 kHz using full IE solution (solid line), conventional Born approximation (dashed line), the second-order modified Born approximation (dash-dotted line), and the third-order modified Born approximation (dotted line). Calculations are done for vertical magnetic dipole-dipole array distributed at the different depths along the borehole. Figure 5 shows similar plots computed using original QL approximation (dashed line), the second-order QL approximation (dash-dotted line), and the third-order QL approximation (dotted line).

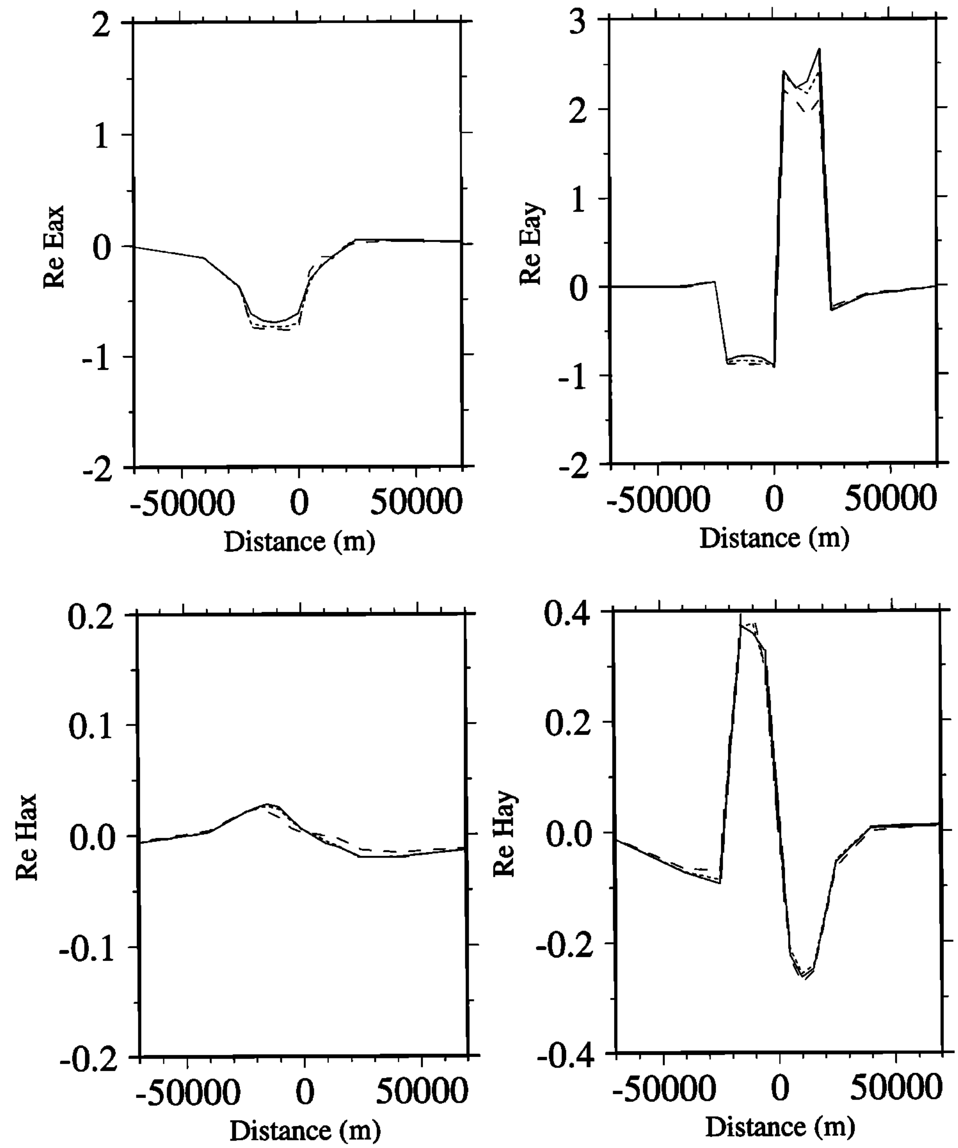

Figure 13. Numerical comparison of full IE solution (solid line), the QL approximations of the first order (dashed line), and the third order (dotted line), computed for model 4 (Figure 10) at frequency 0.01 Hz. Calculations are done for receivers located along the Y axes on the surface.

One can see that the QL approximations converge more rapidly to the full IE solution than modified Born approximations.

Next, model 3 is shown in Figure 6. This model presents one conductive body in a homogeneous half space excited by the vertically propagating plane wave. Figure 7 compares the modified Born approximations of the first, third, and fifth orders for the normalized anomalous fields ReH_{ax}, ReH_{ay} and ReE_{ax}, ReE_{ay} at a frequency of 1 Hz. Figure 8 shows the modified Born approximations of the seventh and ninth orders. We can see that these approximations, in full compliance with the theory, converge to the full IE solution (solid line), but this convergence is rather slow. One can notice a significant 'overshooting' of ReE_{ax} at distance zero, which shows that even a modified Born approximation produces too strong an anomaly. Figure 9 demonstrates

Table 2. Comparison of the Relative Convergence Rate Computed by Equation (42) for Models 1 - 4

	QL	Second-Order QL	Third-Order QL
Model 1 (1000 Hz)	0.0071	0.0036	0.0023
Model 2 (1000 Hz)	0.746	0.301	0.209
Model 3 (1 Hz)	0.0037	0.0021	0.0015
Model 4 (0.01 Hz)	0.094	0.050	0.034

the results of QL series calculation for the same model. The QL approximation of the first-order produces a more accurate result than the modified Born of the ninth-order. The third-order QL approximation has an accuracy of 1% in the extremum of the anomalous curves.

It is important to compare the time of numerical modeling needed for full IE solution, modified Born series, and QL series. The results of a comparison obtained for model 3 are presented in Table 1.

One can see from this table that CPU time increases exponentially with the number of cells for full IE solution, while it increases only linearly for QL approximation and Born approximation. In the models with 800 cells in anomalous domain, the QL approximation is an order of magnitude faster than the full IE solution. It takes only about twice as much time to compute the QL approximation than the Born approximation. The third-order QL approximation requires 15 times less CPU time than full IE modeling but produces practically the same result. The modified Born approximation of the ninth order requires more CPU time and still cannot reach the same accuracy.

Figure 10 presents the geoelectrical model from the set developed in the framework of the International COMMEMI (Comparison of the Methods of Electromagnetic Modeling International) project on the comparison of the different numerical modeling methods [*Zhdanov et al.*, 1990]. This model contains prismatic conductive (with resistivity 1 Ohm-m) and resistive (with resistivity 100 Ohm-m) inserts in the first layer of the three-layer background resistivity cross section (with resistivities 10, 100, and 0.1 Ohm-m). Figure 10 shows its vertical section in the plane $y = 0$ (upper panel) and a plan view of the model (lower panel).

Figure 11 presents the results of the forward modeling based on the full IE solution (solid lines) and the modified Born approximations of the first, third, and fifth orders for the normalized anomalous fields $\mathrm{Re}H_{ax}$, $\mathrm{Re}H_{ay}$ and $\mathrm{Re}E_{ax}$, $\mathrm{Re}E_{ay}$ at a frequency of 0.01 Hz. The modified Born approximations of the seventh and ninth orders are shown in Figure 12. Once again we observe a successive convergence to the full IE solution (solid line). The ninth-order modified Born approximation produces a reasonable estimation of the IE solution. Figure 13 demonstrates the results of computing the QL approximations of the first and third orders. The third-order QL approximation practically coincides with the IE solution. However, for this model the CPU time for computing the third-order QL approximation is equal to 1569.4 s, while the CPU time for computing the ninth-order modified Born approximation is equal to 2725.4 s (with 1500 cells in anomalous domain).

Table 2 shows the relative convergence rate r_N for all four models. This parameter, multiplied by coefficient $\|\beta\|_\infty / (1 - \|\beta\|_\infty)$, determines the upper bound of the accuracy ε_N of the QL series.

Table 3 presents the results of the accuracy estimation ε_N computed for models 1, 3, and 4 using formula (41). Note that for models 1 and 3, parameter a is a constant within the domain D, so the estimation (41) is simplified to

$$\varepsilon_N = \frac{\left\|\mathbf{E}^a - \mathbf{E}_{ql}^{a(N)}\right\|}{\left\|\mathbf{E}_{ql}^{a(N)}\right\|} \leq \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty} \cdot \frac{\left\|\mathbf{E}_{ql}^{a(N)} - \mathbf{E}_{ql}^{a(N-1)}\right\|}{\left\|\mathbf{E}_{ql}^{a(N)}\right\|}. \tag{44}$$

Table 3. Comparison of the Accuracy Estimation Computed by Equation (41) for Models 1, 3, and 4

	QL	Second-Order QL	Third-Order QL
Model 1 (1000 Hz)	0.35	0.18	0.11
Model 3 (1 Hz)	0.09	0.05	0.04
Model 4 (0.01 Hz)	0.43	0.23	0.16

Table 4. Comparison of the Accuracy Estimation Computed by Equation (41) for Model 2

	Fifth-Order QL	Seventh-Order QL	Ninth-Order QL
Convergence rate	0.119	0.067	0.039
Epsilon	0.54	0.31	0.18

We can see from this table that expression (41) produces rather conservative estimation of the accuracy, which practically can be much higher, as we could see by direct comparison with the full IE solution. In this connection we also should notice that formulae (41) and (44) actually give the accuracy estimation of the integral equation (18) solution within the inhomogeneous domain D. The approximation errors of the anomalous field computing outside domain D are usually much smaller due to the contraction properties of the integral operator G_b^m.

At the same time, expression (41) can be used as a tool to control the order of the QL approximations which would satisfy to the required accuracy. For example, calculations show that it takes nine iterations for Model 2 to reach the accuracy estimation $\varepsilon_N = 0.18$ (see Table 4). Thus we can guarantee that the ninth-order QL approximation for model 2 produces the forward modeling solution with the given accuracy. In practice, however, the accurate modeling result can be reached even by the QL approximations of the lower orders, as we could see in Figures 4 and 5.

7. Conclusion

In this paper we have developed a new approach to fast EM modeling by considering the QL approximations of the higher orders. We have proved that the corresponding QL series are always converged for the models with the lossy background medium. On the basis of the analytical and numerical comparisons between modified Born series and QL series we can conclude the following:

1. Both the modified Born series and QL series are converged.

2. The modified Born series work reasonably well for resistive bodies.

3. The QL series provide accurate approximation for both conductive and resistive structures.

4. The QL series have superior convergence for all models.

5. It is possible to calculate a quantitative estimation of the QL approximation accuracy without direct comparison with the rigorous full IE forward modeling solution.

6. Calculation of the individual modified Born approximations of the different orders requires less time than the computing QL series of the same orders. However, to reach the same accuracy, one typically should generate the modified Born approximations of the higher order than in the case of the QL approximation.

These facts make the QL series a more efficient tool for EM modeling than full IE solution or modified Born series. We believe that always converged QL series open principally new possibilities for fast and accurate 3-D EM modeling and inversion.

Appendix A: EM Energy Inequality

Fundamental energy inequality for the anomalous EM field has been derived by *Singer* [1995] and *Pankratov et al.* [1995]. We demonstrate this inequality here for completeness.

One can calculate the average per period energy flow of anomalous EM field through the surface of the Earth Σ as

$$Q = \mathrm{Re} \int\int_{\Sigma} \mathbf{P} \cdot \mathbf{n} ds = \frac{1}{2} \mathrm{Re} \int\int_{\Sigma} (\mathbf{E}^a \times \mathbf{H}^{a*}) \cdot \mathbf{n} ds, \tag{A1}$$

where $\mathbf{n}$ is the unit vector of normal to the surface Σ, directed to the upper half space (assuming that the sources of the anomalous field are located in the lower half space) and $\mathbf{P}$ is the Poynting vector [*Stratton*, 1941], introduced by the following formula:

$$\mathbf{P} = \frac{1}{2} \mathbf{E}^a \times \mathbf{H}^{a*},$$

where the asterisk indicates complex conjugate value.

Expression (A1) can be rewritten using the Gauss formula

$$Q = \mathrm{Re} \int\int\int_{O^+} \nabla \cdot \mathbf{P} dv = \frac{1}{2} \mathrm{Re} \int\int_{\Sigma} (\mathbf{E}^a \times \mathbf{H}^{a*}) \cdot \mathbf{n} ds, \tag{A2}$$

where O^+ is the lower half space.

Following a conventional approach [*Stratton*, 1941] we can obtain the Poynting's theorem for anomalous field by scalar multiplication of the first equation in formula (1) by $\mathbf{E}^{a*}$ and the complex conjugate second equation in formula (1) by $\mathbf{H}^{a}$ and subtracting one from another:

$$\begin{aligned} 2\nabla\cdot\mathbf{P} &= \nabla\cdot\left(\mathbf{E}^{\mathbf{a}}\times\mathbf{H}^{\mathbf{a}*}\right) \\ &= \mathbf{H}^{a}\cdot\nabla\times\mathbf{E}^{a*}-\mathbf{E}^{a*}\cdot\nabla\times\mathbf{H}^{a} \\ &= -\widetilde{\sigma}_b\left|\mathbf{E}^{a}\right|^2-\mathbf{E}^{a*}\cdot\mathbf{j}^{a}-i\omega\mu\left|\mathbf{H}^{a}\right|^2. \end{aligned}$$

Thus the total energy flow Q of the anomalous field through the surface of observation Σ can be calculated using the formula

$$Q=-\frac{1}{2}\mathrm{Re}\int\int\int_{O+}\left\{\widetilde{\sigma}_b\left|\mathbf{E}^{a}\right|^2+\mathbf{E}^{a*}\cdot\mathbf{j}^{a}\right\}dv. \tag{A3}$$

Pankratov et al. [1995] have proved an important theorem, according to which the energy flow Q of the anomalous field is nonnegative:

$$Q\geq 0. \tag{A4}$$

This result can be obtained from the equations (A2) and (A3) applied to the upper half space O^-

$$Q=\frac{1}{2}\mathrm{Re}\int\int_{\Sigma}\left(\mathbf{E}^{a}\times\mathbf{H}^{a*}\right)\cdot\mathbf{n}ds=$$

$$\frac{1}{2}\int\int\int_{O-}\mathrm{Re}\widetilde{\sigma}_b\left|\mathbf{E}^{a}\right|^2dv\geq 0.$$

On the basis of this theorem, they have derived an energy inequality, which can be obtained after some algebraic transformations from (A4) and (A3):

$$\mathrm{Re}\int\int\int_{O+}\left\{\widetilde{\sigma}_b\left|\mathbf{E}^{a}\right|^2+\mathbf{E}^{a*}\cdot\mathbf{j}^{a}\right\}dv=$$

$$\int\int\int_{O+}\left\{\mathrm{Re}\widetilde{\sigma}_b\left|\mathbf{E}^{a}+\frac{\mathbf{j}^{a}}{2\mathrm{Re}\widetilde{\sigma}_b}\right|^2-\frac{\left|\mathbf{j}^{a}\right|^2}{4\mathrm{Re}\widetilde{\sigma}_b}\right\}dv\preceq 0. \tag{A5}$$

From the last formula we have

$$\int\int\int_{O+}\mathrm{Re}\widetilde{\sigma}_b\left|\mathbf{E}^{a}+\frac{\mathbf{j}^{a}}{2\mathrm{Re}\widetilde{\sigma}_b}\right|^2dv\preceq\int\int\int_{O+}\frac{\left|\mathbf{j}^{a}\right|^2}{4\mathrm{Re}\widetilde{\sigma}_b}dv. \tag{A6}$$

Appendix B: Accuracy Estimation of QL Approximation of the First and Higher Orders

Let us estimate the accuracy of the QL approximation. According to (19) and (18), the actual anomalous field $\mathbf{E}^{a}$ can be determined by the equation

$$a\mathbf{E}^{a}=\mathbf{G}_b^m\left(b\left(\mathbf{E}^{a}+\mathbf{E}^{b}\right)\right)-b\mathbf{E}^{b}. \tag{B1}$$

Comparing $a\mathbf{E}^{a}$ with the corresponding formula for modified QL approximation (31), we can obtain accuracy criteria for a QL solution

$$\left\|a\mathbf{E}^{a}-a\mathbf{E}^{a}_{ql}\right\|=\left\|\mathbf{G}_b^m\left(\beta\left(a\mathbf{E}^{a}-a\widehat{\lambda}\mathbf{E}^{b}\right)\right)\right\|\preceq$$

$$\preceq\|\beta\|_\infty\left\|\mathbf{G}_b^m\right\|\left\|a\mathbf{E}^{a}-a\widehat{\lambda}\mathbf{E}^{b}\right\|, \tag{B2}$$

where

$$\|\beta\|_\infty=\max_{r\in D}\left|\frac{b(r)}{a(r)}\right|=\max_{r\in D}\frac{|\Delta\widetilde{\sigma}|}{|2\mathrm{Re}\widetilde{\sigma}_b+\Delta\widetilde{\sigma}|}. \tag{B3}$$

Using equation (31), we can express $\left\|a\mathbf{E}^{a}-a\widehat{\lambda}\mathbf{E}^{b}\right\|$ as follows:

$$\left\|a\mathbf{E}^{a}-a\widehat{\lambda}\mathbf{E}^{b}\right\|=$$

$$\left\|a\mathbf{E}^{a}-a\mathbf{E}^{a}_{ql}+\mathbf{G}_b^m\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^{b}\right)-b\mathbf{E}^{b}-a\widehat{\lambda}\mathbf{E}^{b}\right\|.$$

$$\leq\left\|a\mathbf{E}^{a}-a\mathbf{E}^{a}_{ql}\right\|+\varphi\left(\widehat{\lambda}\right), \tag{B4}$$

where

$$\varphi\left(\widehat{\lambda}\right)=\left\|a\mathbf{E}^{a}_{ql}-a\widehat{\lambda}\mathbf{E}^{b}\right\|$$

$$=\left\|\mathbf{G}_b^m\left(b\left(1+\widehat{\lambda}\right)\mathbf{E}^{b}\right)-b\mathbf{E}^{b}-a\widehat{\lambda}\mathbf{E}^{b}\right\|. \tag{B5}$$

From inequalities (B2) and (B4) we have

$$\left\|a\mathbf{E}^{a}-a\mathbf{E}^{a}_{ql}\right\|\leq\|\beta\|_\infty\cdot\left\|\mathbf{G}_b^m\right\|\cdot$$

$$\cdot\left\{\left\|a\mathbf{E}^{a}-a\mathbf{E}^{a}_{ql}\right\|+\varphi\left(\widehat{\lambda}\right)\right\}. \tag{B6}$$

According to inequalities (15) and (21),

$$\|\beta\|_\infty\left\|\mathbf{G}_b^m\right\|<1.$$

Under this condition we can rewrite inequality (B6) as

$$\left\|a\left(\mathbf{E}^{a}-\mathbf{E}^{a}_{ql}\right)\right\|\leq\frac{\|\beta\|_\infty}{1-\|\beta\|_\infty}\varphi\left(\widehat{\lambda}\right) \tag{B7}$$

Now let us estimate the accuracy of the QL approximation of the Nth-order:

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)}\right\| =$$

$$\left\|\sum_{k=N}^{\infty}\left(\mathbf{G}_b^m\beta\right)^k\left(a\mathbf{E}^{Bm}\right) - \left(\mathbf{G}_b^m\beta\right)^N\left(a\widehat{\lambda}\mathbf{E}^b\right)\right\| =$$

$$\left\|\left(\mathbf{G}_b^m\beta\right)^{N-1}\left[\sum_{k=0}^{\infty}\left(\mathbf{G}_b^m\beta\right)^k\left(a\mathbf{E}^{Bm}\right) - a\mathbf{E}^{Bm}\right]\right.$$

$$\left. - \left(\mathbf{G}_b^m\beta\right)^N\left(a\widehat{\lambda}\mathbf{E}^b\right)\right\| =$$

$$\left\|\left(\mathbf{G}_b^m\beta\right)^{N-1}\left[a\mathbf{E}^a - a\mathbf{E}^{Bm} - \mathbf{G}_b^m\beta\left(a\widehat{\lambda}\mathbf{E}^b\right)\right]\right\|$$

$$\leq \left\|\left(\mathbf{G}_b^m\beta\right)^{N-1}\right\|\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^a\right\|.$$

Thus we have

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)}\right\| \leq \|\beta\|_\infty^{N-1}\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^a\right\|$$

$$\leq \frac{\|\beta\|_\infty^N}{1-\|\beta\|_\infty}\varphi\left(\widehat{\lambda}\right). \tag{B8}$$

Comparison of inequalities (28) and (B8) clearly demonstrates that the accuracy of the Born approximation depends only on the order N, while the accuracy of QL approximation of the same order can be increased by a proper selection of $\widehat{\lambda}$.

Note that it is possible to obtain a more accurate estimation of the accuracy of the higher-order QL approximations by comparison the $(N+1)$th and Nth QL iterations. According to equations (18), (20), and (19), we can obtain the following inequalities for the difference between exact solution and QL approximation of the Nth-order:

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)}\right\| \leq \left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N+1)}\right\| +$$

$$+\left\|a\mathbf{E}_{ql}^{a(N+1)} - a\mathbf{E}_{ql}^{a(N)}\right\|, \tag{B9}$$

and

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N+1)}\right\| = \left\|\mathbf{C}\left(a\mathbf{E}^a\right) - \mathbf{C}\left(a\mathbf{E}_{ql}^{a(N+1)}\right)\right\| =$$

$$\left\|\mathbf{G}_b^m\beta\left(a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N+1)}\right)\right\| \leq \|\beta\|_\infty\cdot\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)}\right\|. \tag{B10}$$

Substituting (B10) into (B9), we can write

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N)}\right\| \leq \frac{1}{1-\|\beta\|_\infty}\cdot\left\|a\mathbf{E}_{ql}^{a(N+1)} - a\mathbf{E}_{ql}^{a(N)}\right\|. \tag{B11}$$

The last inequality provides an important practical tool to estimate the accuracy of the QL approximations of the different orders without direct comparison with the rigorous full IE solution. However, to estimate the accuracy of the Nth-order QL approximation, we have to compute the $(N+1)$-th order QL approximation. It would be useful to find the accuracy estimation of the highest-order QL approximation. We can solve this problem by substituting expression (B11) back in the inequality (B10):

$$\left\|a\mathbf{E}^a - a\mathbf{E}_{ql}^{a(N+1)}\right\| \leq \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty}\cdot\left\|a\mathbf{E}_{ql}^{a(N+1)} - a\mathbf{E}_{ql}^{a(N)}\right\|. \tag{B12}$$

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References

Born, M., *Optics*, Springer-Verlag, New York, 1933.

Hohmann., G. W. , Three-dimensional induced polarization and EM modeling, *Geophysics, 40,* 309-324, 1975.

Pankratov, O. V., D.B. Avdeev, and A. V. Kuvshinov, Scattering of electromagnetic field in inhomogeneous earth: Forward problem solution, *Izv. Akad. Nauk. SSSR Fiz. Zemli, 3,* 17-25, 1995.

Singer, B. Sh., Method for solution of Maxwell's equation in non-uniform media, *Geophys. J. Int., 120,* 590-598, 1995.

Singer, B. Sh., and E. B. Fainberg, Generalization of the iterative dissipative method for modeling electromagnetic fields in nonuniform media with displacement currents, *Journal of Applied Geophysics, 34,* 41-46, 1995.

Stratton, J.A., *Electromagnetic Theory*, 615 pp., McGraw-Hill, New-York, 1941.

Torres-Verdin, C., and T. M. Habashy, Rapid 2.5-dimensional forward modeling and inversion via a new nonlinear scattering approximation, *Radio Sci., 29(4)*, 1051-1079, 1994.

Weidelt, P., EM induction in three-dimensional structures, *J. Geophys., 41*, 85-109, 1975.

Xiong, Z., EM modeling of three-dimensional structures by the method of system iteration using integral equations, *Geophysics, 57,* 1556-1561, 1992.

Zhdanov, M. S., and S. Fang, Quasi-linear approximation in 3D EM modeling, *Geophysics, 61*, 646-665, 1996a.

Zhdanov, M. S., and S. Fang, Three-dimensional quasi-linear electromagnetic inversion, *Radio Science, 31(4)*, 741-754, 1996b.

Zhdanov, M. S., N. G. Golubev, I. M. Varentsov, and V. A. Krylov, *The Methods for Electromagnetic Fields Modeling*, Nauka Moscow, 1990.

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