A Review of Shear Stress Sign Convention in Interpreting Mohr's Circle

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Abstract:

The pole of planes method is a popular technique for interpreting Mohr's circle to determine the stresses (normal and shear) on planes of differing rotations in 2-D space. A survey of undergraduate textbooks on soil mechanics shows differing viewpoints on the sign convention for interpreting the shear stresses. This paper makes a rigorous evaluation of the consequences of using a clockwise (CW) or counterclockwise (CCW) positive sign convention for the proper interpretation of Mohr's circle of stresses. In either case, the shear stress axis is considered positive upwards while the normal stress axis is considered positive to the right. It was found that if the shear stresses acting on an element are considered CW positive, the resulting stresses found on a rotated plane do not satisfy static equilibrium. However, the stresses determined using a CCW sign convention do satisfy static equilibrium and hence, if the pole method is to be used to interpret Mohr's circle for stresses, a CCW positive sign convention for shear stresses must be used.

CE Database keywords: equilibrium, shear stress, normal stress, Mohr's circle, sign convention

Introduction:

Mohr's circle, which graphically describes the state of stress at a point in either 2-D or 3-D space, is a fundamental concept introduced in all undergraduate mechanics curriculum. While Mohr's circle may be used in either 2-D or 3-D space, this note will be limited to2-D only; specifically, the plane that contains the major and minor principal stresses, σ_1 and σ_3 only. In undergraduate soil mechanics courses, the pole (or origin) of planes is generally introduced as an interpretation tool to estimate the magnitude and orientation of the normal and shear stresses that act on planes of any desired rotation. In determining the sign convention for shear stresses, some undergraduate textbooks use a clockwise or CW (also known as a left-hand rule) positive sign convention while others use a counterclockwise or CCW (also known as a right-hand rule) positive sign convention. Table 1 presents the results of a survey of undergraduate soil mechanics texts that have been published over the past several decades. It is observed that while the right hand rule has been the most popular, each sign convention has been adopted by several texts through history. To illustrate the general confusion regarding the sign convention, Holtz and Kovacs (1981) states that the use of a compression-positive convention, typically used for soil mechanics, requires that a CCW positive shear convention be used but does not provide justification for this requirement. However, Sowers and Sowers (1970) state that the sign convention on the shear stress is of no consequence at all and that shear stresses may be plotted either as positive or negative. Because the use of Mohr's circle is of a fundamental nature and is of the greatest importance in soil mechanics, (Lambe and Whitman 1969), this paper seeks to investigate the consequences of interpreting the state of stress at a point when using either a CW or CCW positive sign convention for the shear stresses.

Fig. 1 shows a set of stresses that act on an element. The magnitudes of the stresses were chosen randomly and are only for illustrative purposes. For the purposes of summing the forces in different directions, the element is assumed to be a cube with dimensions (including normal thickness) of 0.1 m.

Clockwise Positive Sign Convention:

For the first sign convention, we will denote a positive shear stress as one that tends to rotate the element in a clockwise fashion. Drawing the Mohr's circle for this case, we develop the following diagram shown in Fig. 2. The pole is denoted as **P** and the major principal plane, drawn at an angle of 22.5° CW to the horizontal.

The principal plane and the major principal (normal) stress acting on it as drawn above can be superimposed on the original element. When this is done, the resulting element is shown in Fig. 3.

From this element, the forces may be summed in both the vertical (Z) and horizontal (X) direction. The horizontal plane is arbitrarily left at its original length, 0.1 m. From trigonometry, the length of the vertical plane is found to be 0.0414 m and the length of the principal (rotated) plane is found to be 0.1082 m long. All planes remain with a normal thickness of 0.1 m. The sign of each force is obtained from the general orientation shown in Fig. 3 (positive upward and to the right).

Summation of forces in horizontal is as follows:

$$\sum F_X = \text{shear force on horiz plane} - \text{normal force on vert. plane} + \text{horiz component force from } \sigma_1$$

$$\sum F_X = \left(100\,\text{kPa}\cdot 0.1\,\text{m} - 300\,\text{kPa}\cdot 0.0414\,\text{m} + 541.4\,\text{kPa}\cdot 0.1082\,\text{m}\cdot\cos(22.5°)\right)\cdot 0.1\,\text{m} =$$

$$\sum F_X = 1.0\,\text{kN} - 1.243\,\text{kN} + 2.243\,\text{kN} = 2.0\,\text{kN} \neq 0.0$$

Summation of forces in vertical is as follows:

$$\sum F_Z = \text{shear force on vert plane} - \text{normal force on horiz. plane} + \text{vert component of } \sigma_1$$

$$\sum F_Z = (100\text{kPa}\cdot 0.0414\text{m} - 500\text{kPa}\cdot 0.1\text{m} + 541.4\text{kPa}\cdot 0.1082\text{m}\cdot \sin(22.5°))\cdot 0.1\text{m} =$$

$$\sum F_Z = 0.414\text{kN} - 5.0\text{kN} + 5.414\text{kN} = 0.828\text{kN} \neq 0.0$$

The summation of forces are not equal to zero and hence, the element is not in equilibrium.

Counterclockwise Positive Sign Convention:

The second analysis will use a sign convention with a positive shear stress as one that tends to rotate the element in a counterclockwise (CCW) fashion. Drawing the Mohr's circle for this sign convention, we develop the construction shown in Fig. 4. Similar to Fig. 2, the pole is denoted as **P** and the major principal plane, drawn at an angle of 22.5° CCW to the horizontal.

The principal plane is now seen to be oriented at an angle of 22.5° CCW from the horizontal instead of 22.5° CW from the horizontal. The principal plane and stress drawn above can be superimposed on the original element. When this is done, the resulting element is shown in Fig. 5.

As performed previously, the forces may be summed in both the vertical and horizontal direction. As before, the horizontal plane is arbitrarily left at its original length, 0.1 m, the length of the vertical plane is found to be 0.0414 m, and the length of the principal (rotated) plane is found to be 0.1082 m long. Again, all planes remain with a normal thickness of 0.1 m. The sign of each force is obtained from the general orientation shown in Fig. 5.

Summation of forces in horizontal:

$$\sum F_X = \text{normal force on vert. plane} + \text{shear force on horiz plane} - \text{horiz component of force from } \sigma_1$$

$$\sum F_X = (300\text{kPa}\cdot 0.0414\text{m} + 100\text{kPa}\cdot 0.1\text{m} - 541.4\text{kPa}\cdot 0.1082\text{m}\cdot \cos(22.5°))\cdot 0.1\text{m} =$$

$$\sum F_X = 1.243\text{kN} + 1.0\text{kN} - 2.243\text{kN} = 0.0$$

Summation of forces in vertical:

$$\sum F_Z = \text{vert component of } \sigma_1 - \text{shear force on vert plane} - \text{normal force on horiz. plane}$$

$$\sum F_Z = (541.4\text{kPa}\cdot 0.1082\text{m}\cdot \sin(22.5°) - 100\text{kPa}\cdot 0.0414\text{m} - 500\text{kPa}\cdot 0.1\text{m})\cdot 0.1\text{m} =$$

$$\sum F_Z = 5.414\text{kN} - 5.0\text{kN} - 0.414\text{kN} = 0.0$$

The summations of forces are equal to zero and hence, the element is in equilibrium.

Stresses on Rotated Element:

What if the stresses acting on surfaces rotated 30° CW and 30° CCW from the principal plane are to be evaluated? Although it can be shown that the resulting elements are not in static equilibrium, the pole of planes solution using a CW positive (left-hand rule) sign convention is graphically shown in Fig. 6.

If a CCW positive sign convention is used, the pole of planes solution for these two orientations is graphically shown in Figs. 7a and b, respectively. On the rotated planes, the magnitudes of the stresses (normal and shear) are found to be consistent with those in Fig. 6, although again, the orientations of the planes and the resulting stresses are very different from that shown earlier.

As before, these planes may be superimposed on the original element (Fig. 8) and the forces in both the vertical and horizontal are summed. Using the Law of Sines, the lengths of the other two sides, beginning with the upper-most side, are found to be 0.070 and 0.114 m, respectively.

Summation of forces in horizontal:

$$\sum F_X = \text{normal force} + \text{horiz component of upper shear force} - \text{horiz component of} \ldots$$

$$\ldots \text{upper normal force} + \text{horiz component of lower shear force} - \text{horiz component..}$$

$$\ldots \text{of lower normal force}$$

$$\sum F_X = [300\,kPa \cdot 0.1\,m + 122.5\,kPa \cdot 0.07\,m \cdot \cos(7.5°) - 470.7\,kPa \cdot 0.07\,m \cdot \sin(7.5°) \ldots$$

$$\ldots + 122.5\,kPa \cdot 0.114\,m \cdot \sin(37.5°) - 470.7\,kPa \cdot 0.114\,m \cdot \cos(37.5°)\ ]$$

$$\sum F_X = 3.00\,kN + 0.854\,kN - 0.432\,kN + 0.854\,kN - 4.275\,kN = 0.0$$

Summation of forces in vertical:

$$\sum F_Z = -\text{shear force on vert. plane} - \text{vert component of upper shear force} - \text{vert component of} \ldots$$

$$\ldots \text{upper normal force} + \text{vert component of lower shear force} + \text{vert component..}$$

$$\ldots \text{of lower normal force}$$

$$\sum F_Z = [-100\,kPa \cdot 0.1\,m - 122.5\,kPa \cdot 0.07\,m \cdot \sin(7.5°) - 470.7\,kPa \cdot 0.07\,m \cdot \cos(7.5°) \ldots$$

$$\ldots + 122.5\,kPa \cdot 0.114\,m \cdot \cos(37.5°) + 470.7\,kPa \cdot 0.114\,m \cdot \sin(37.5°)\ ]$$

$$\sum F_Z = -1.00\,kN - 0.1125\,kN - 3.280\,kN + 1.1125\,kN - 3.280\,kN = 0.0$$

The summations of forces are equal to zero, at least to the extent that significant digits are used and hence, the element is verified to be in equilibrium.

Conclusion:

While the majority of undergraduate texts on soil mechanics have used a CCW (or right hand rule) sign convention for the interpretation of shear stresses, none have stated why the CCW direction is required, particularly in demonstrating that this is necessary for the element to remain in equilibrium. Moreover, two recent texts have used a CW (or left hand rule) sign convention, thus creating a point of confusion.

The pole is a unique point on Mohr's circle in which all planes of real orientations pass through and hence, the sign convention used for the pole method is not arbitrary. While a CW (left-hand rule) positive sign convention on the shear stress does give the proper magnitude of stress on rotated planes, it does not provide the proper orientation of that shear stress. As the orientation is incorrect, static equilibrium of the rotated element will not be established. The proper interpretation of Mohr's circle has particular consequences in slope stability and lateral earth pressure problems where the rotation of principal stresses is important. Hence, when interpreting Mohr's circle with the Pole method, a CCW (right-hand rule) sign convention on axis with the normal stress positive to the left and shear stress positive upwards should always be used.

Acknowledgments:

The author would like to thank Dr. Evert C. Lawton (University of Utah), Dr. Jonathan D. Bray (University of California), and Dr. Muniram Budhu (The University of Arizona) for their technical assistance in preparation of this technical note.

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Table 1: Results of survey of undergraduate soil mechanics texts on sign convention for Mohr's circle using the pole method.

Reference:	Sign Convention	Specific Notes:
Terzaghi (1943)	CCW	One of the first introductions of the pole method. No explicit discussion of the sign convention is made, but the right hand rule is used through examples and equations.
Taylor (1948)	CCW	States "… counterclockwise shears (on the element) are positive".
Hough (1957)	--	Pole method introduced. The sign convention is not explicitly discussed. Additionally, stresses are not shown on a rotated plane and hence, the implied sign convention cannot be determined..
Terzaghi and Peck (1967)	--	No discussion of pole method is made.
Lambe and Whitman (1969)	CCW	Pole method is introduced; the sign convention is illustrated through a number of examples.
Sowers and Sowers (1970)	--	States that the sign convention does not matter, that shear stresses may be arbitrarily plotted (chosen) as positive or negative.
Sowers (1979)	CCW	This is the 4th edition of Sowers and Sowers (1970). Here it is stated "Shear stresses are plotted upward positive if they produce a counterclockwise couple, downward negative if clockwise". Hence, the RH rule is used.
Dunn et al. (1980)	CCW	States "A shear stress that produces a counterclockwise torque about the center of the free body is considered to be positive". This is the right hand rule.
Holtz and Kovacs (1981)	CCW	Sign convention is explicitly discussed. It is stated "… our sign convention has compressive forces and stresses positive … This sign convention then requires that a positive shear stress produce counterclockwise couples on our element". However, no justification (for instance, with respect to equilibrium) is provided.
Spangler and Handy (197, 1982)	CCW	Pole method is introduced, right hand rule is used in discussion.
Das (1983)	CCW	States "The shear stresses on a given plane is positive if it tends to produce a clockwise rotation about a point outside the soil element, it is negative if it tends to produce a counterclockwise rotation".
Bowles (1988)	CCW	Pole method introduced. While the sign convention is not explicitly discussed, the right hand rule is implied by example.
French (1989)	--	Mohr's circle is introduced. Discussions of associated sign convention or the pole method are not explicitly made. However, Figs 3-7 and 3-8 may imply that a left hand sign convention is used.
Craig (1992)	--	Mohr's circle is introduced, but the pole method is not.
Das (1994)	CCW	Pole method is introduced and the sign convention is explicitly discussed. States "… shear stresses are considered positive if … they tend to produce a counterclockwise rotation (of the element). .
Cernica (1995)	CCW	Pole method introduced. While the sign convention is not explicitly discussed, the right hand rule is implied by example.
Liu and Evett (1992, 1998, 2004)	CCW	Pole method introduced. While the sign convention is not explicitly discussed, the right hand rule is implied by example.
McCarthy (1998)	CW	Sign convention explicitly addressed. Stated that a shear stress acting clockwise on the element is considered positive.
Coduto (1999)	CW	Pole method introduced. While the sign convention is not discussed, the left hand rule is directly provided by example.
Budhu (2000)	CW	Sign convention explicitly addressed. States that a "… clockwise shear is positive… ".
Das (2000)	--	No discussion is made of Mohr's circle for the purpose of finding stresses on rotated planes.
Aysen (2002)	CCW	The pole method is not discussed, but transformation equations are used. The right hand rule is used in determining positive shear stresses.

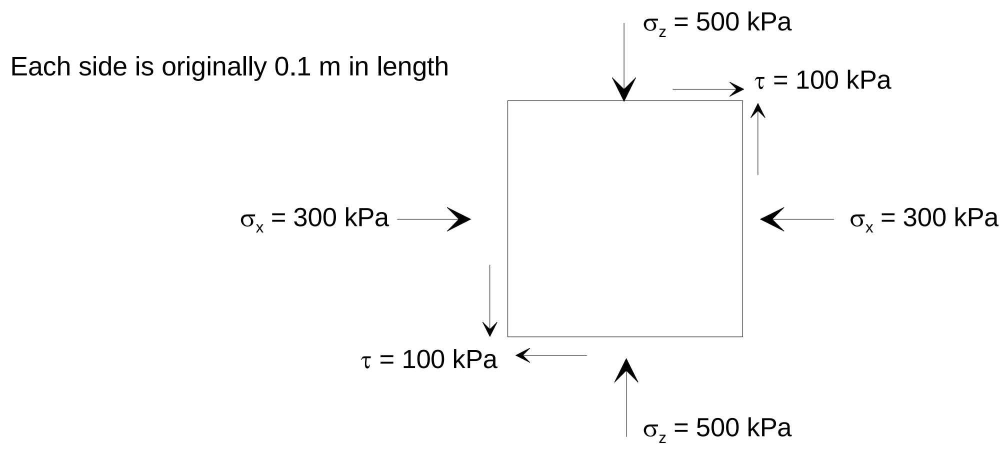

Fig. 1: Stresses acting on an element in the X-Z plane.

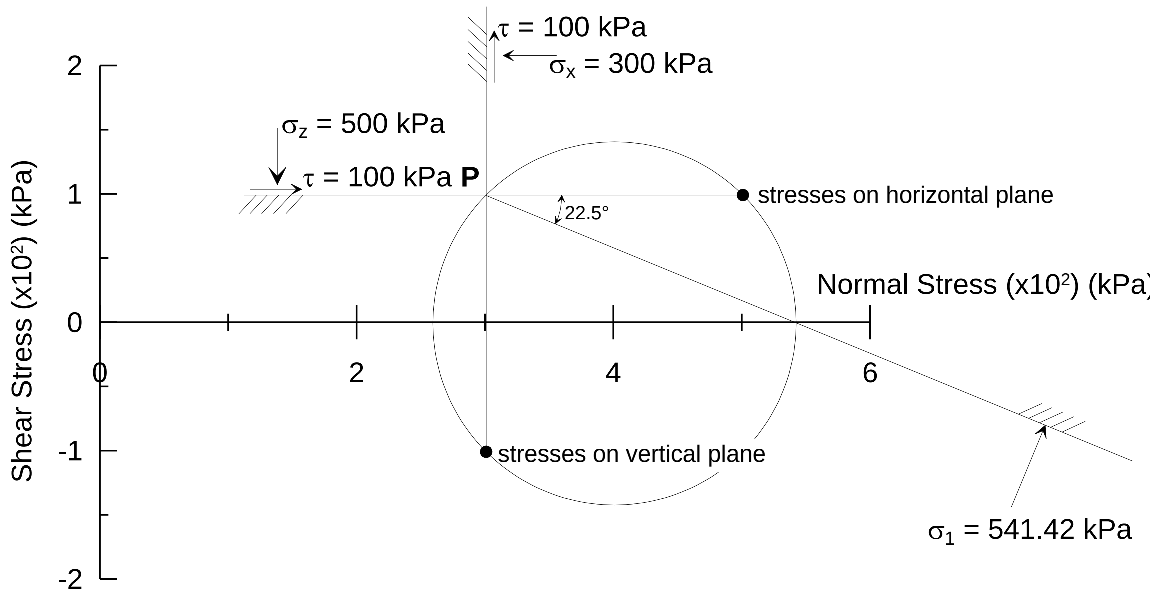

Fig. 2: Development of Mohr's circle given CW positive shear stresses.

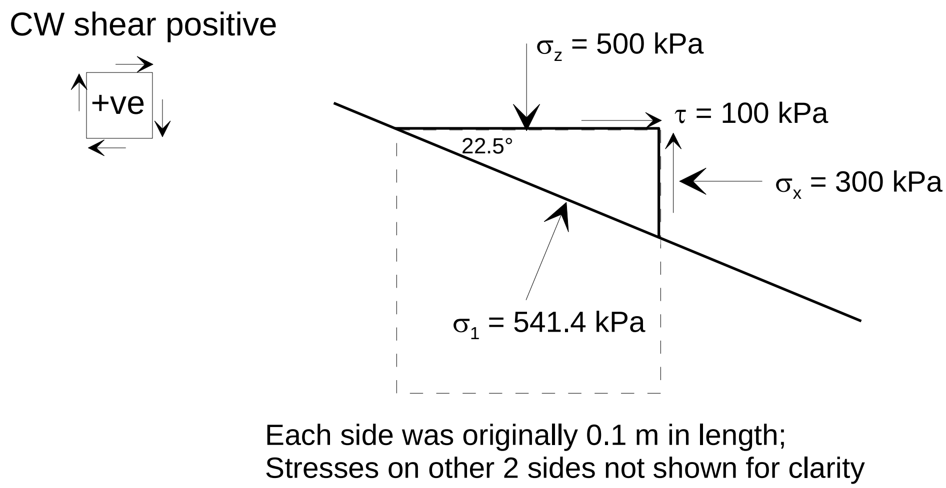

Fig. 3: Principal Plane shown superimposed on original element with clockwise positive shear convention.

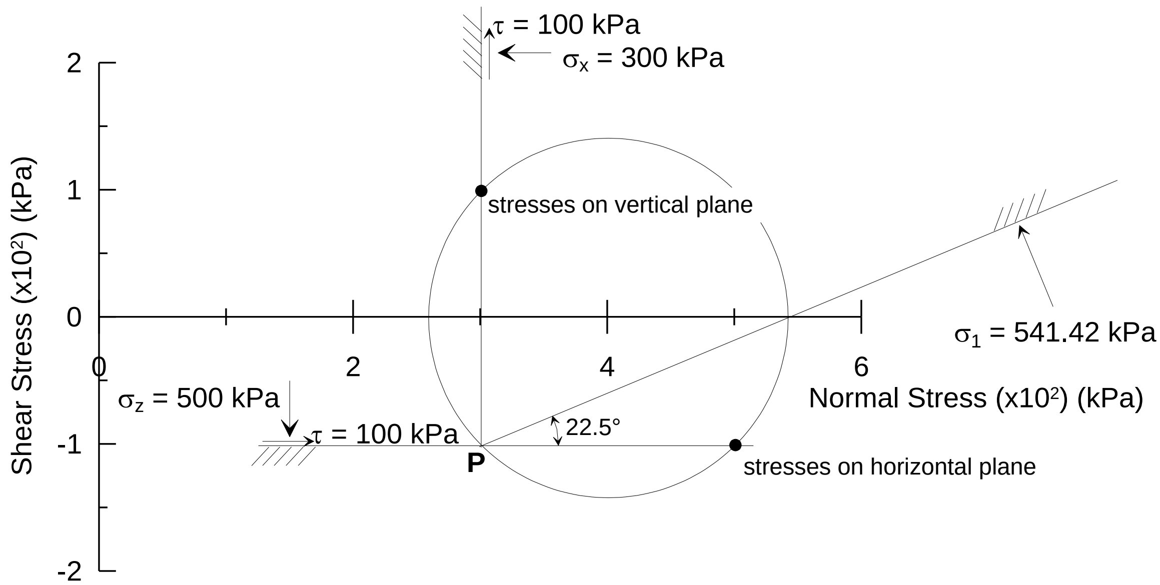

Fig. 4: Development of Mohr's circle given CW positive shear stresses.

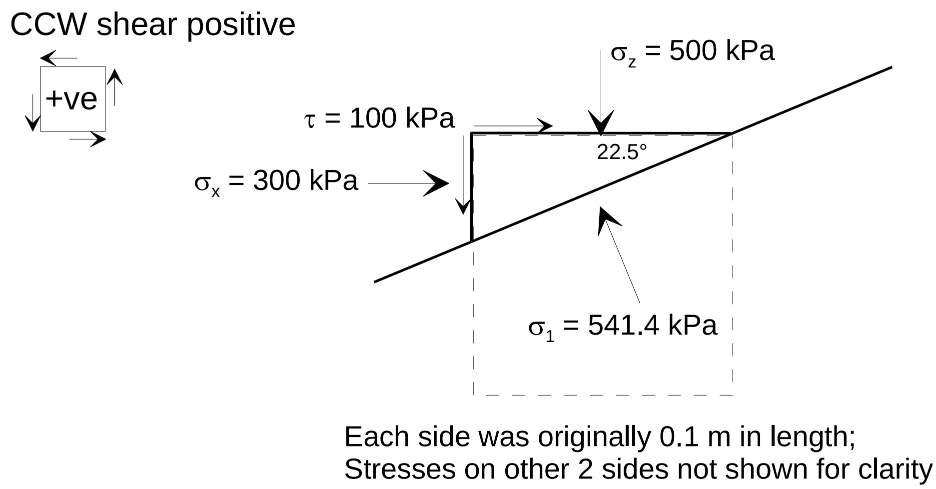

Fig. 5: Principal plane and stress shown superimposed on original element with counterclockwise positive shear convention.

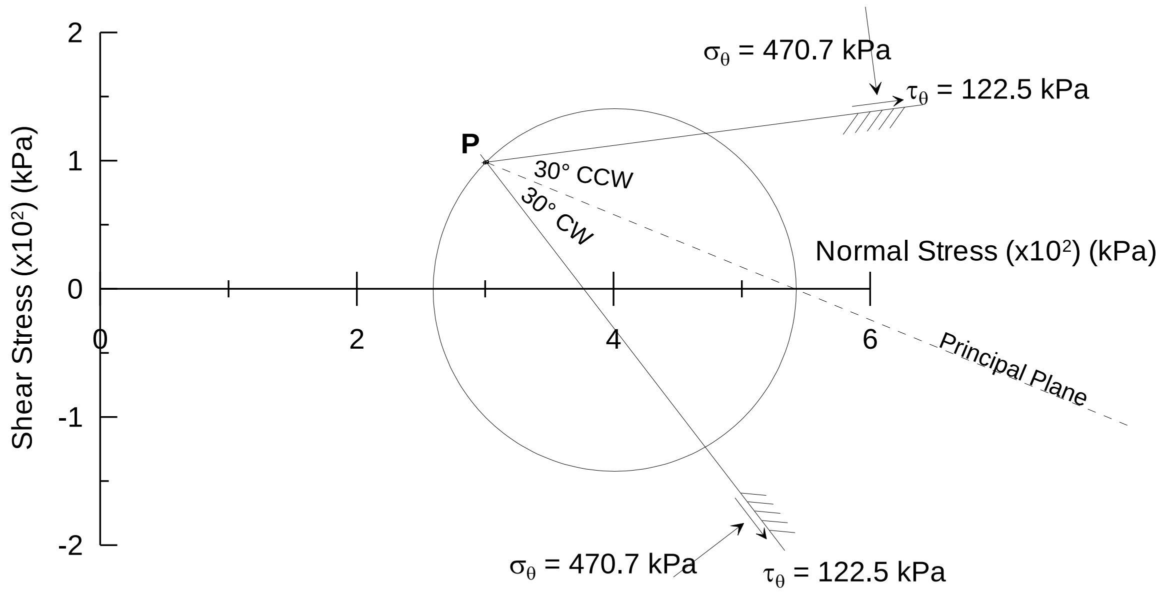

Fig. 6: Stresses acting on planes rotated at angles of 30° CW and 30° CCW using a CW positive sign convention.

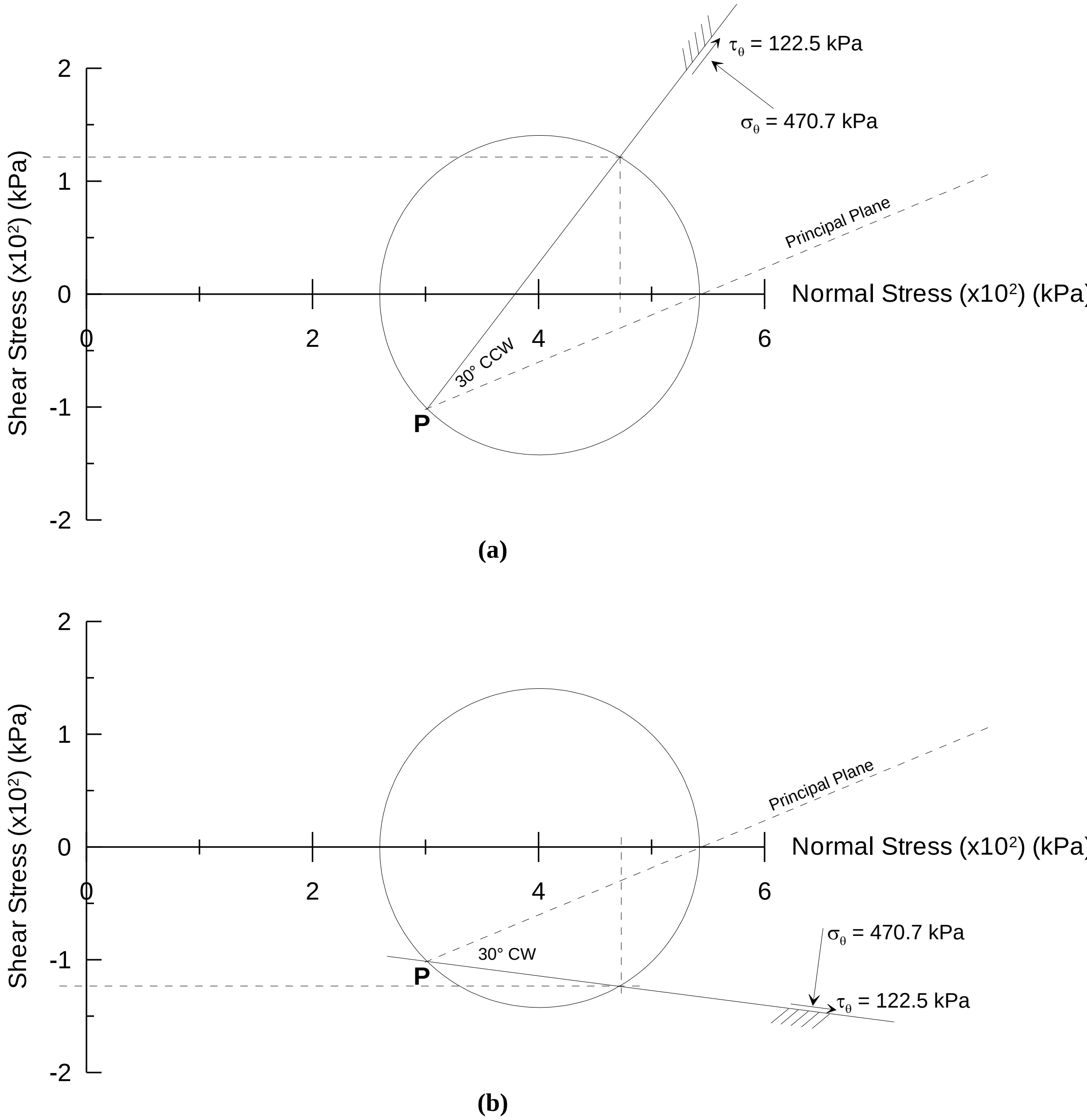

Fig. 7: Stresses acting on a plane rotated at an angle of (a) 30° CW, and (b) 30° CCW using a CCW positive sign convention.

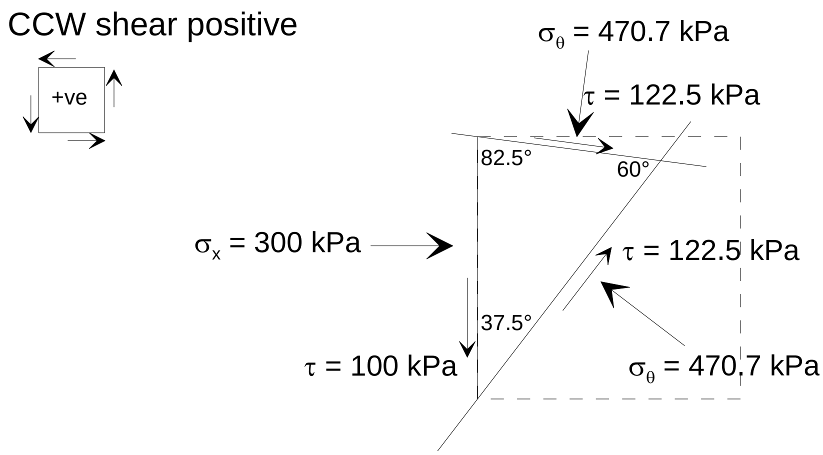

Fig. 8: Rotated elements with determined stresses acting on rotated planes superimposed over original element.