

# Modes of growth in dynamic systems

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Regardless of a system's complexity or scale, its growth can be considered to be a spontaneous thermodynamic response to a local convergence of down-gradient material flows. Here it is shown how system growth can be constrained to a few distinct modes that depend on the time integral of past flows and the current availability of material and energetic resources. These modes include a law of diminishing returns, logistic behavior and, if resources are expanding very rapidly, super-exponential growth. For a case where a system has a resolved sink as well as a source, growth and decay can be characterized in terms of a slightly modified form of the predator-prey equations commonly employed in ecology, where the perturbation formulation of these equations is equivalent to a damped simple harmonic oscillator. Thus, the framework presented here suggests a common theoretical under-pinning for emergent behaviors in the physical and life sciences. Specific examples are described for phenomena as seemingly dissimilar as the development of rain and the evolution of fish stocks.

## 1. Introduction

Very generally, the physical universe can be considered as a locally continuous distribution of energy and matter in the three dimensions of space. Conservation laws dictate that total energy and matter are conserved. The Second Law of Thermodynamics requires that a positive direction for time is characterized by a net material flow from high to low energy density. The rate of flow depends on the precise physical forces at hand. Spatial variability in flows allows for a local convergence in the density field. [Onsager(1931), de Groot and Mazur(1984)].

Most often though, we categorize our world in terms of discrete, identifiable "things", species, systems, or particles that require that we artificially invoke some local discontinuity that distinguishes the system of interest from its surroundings. Local variability within the system is ignored, not necessarily because it doesn't exist, but rather because we lack the ability or interest to resolve any finer structure, at least in anything other than a purely statistical sense. The system evolves according to flows to and from its surroundings as determined by interactions across the predefined system boundaries.

General formulations have been developed for characterizing rates of potential energy dissipation within heterogeneous systems [de Groot and Mazur(1984), Kjelstrup and Bedeaux(2008)]. However, these do not explicitly express rates of growth for a discrete system itself, nor how these system growth rates change with time. What this paper explores is a unifying framework for expressing the emergent growth of discrete systems, and discusses a few simple expressions for the types of evolutionary

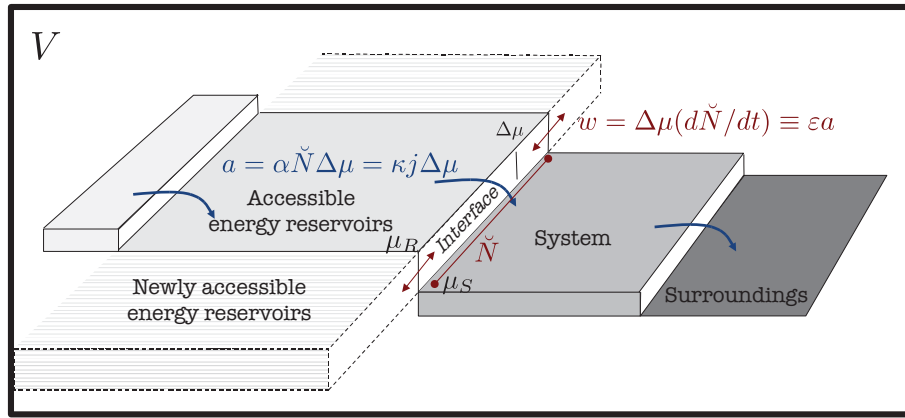


Figure 1. A schematic for the thermodynamic evolution of a system at potential  $\mu_S$  in response to flows from a reservoir at higher potential  $\mu_R$ . The size of an interface  $\tilde{N}$  along the gradient  $\Delta\mu$  between the surfaces determines the speed of downhill material flow  $j$  and rate of energy dissipation  $a$ . The interface itself grows or shrinks by doing work at rate  $w$ , whose sign and efficiency  $\varepsilon$  depends on whether there is material convergence or divergence at  $\mu_S$ . Growth expands flows by bootstrapping the interface into previously inaccessible reservoirs.

phenomena that are thermodynamically possible. Some, such as a law of diminishing returns, explosive or super-exponential growth, and non-linear oscillatory behavior, have been identified in a very broad range of scientific disciplines, ranging from cloud physics [Koren and Feingold(2011)] to ecology [Berryman(1992)] to energy economics [Höök et al.(2010)]. These are shown here to have common physical roots.

## 2. Growth and decay of flows

Fig. 1 is an illustration of flows between discrete systems. A closed system of volume  $V$  contains a locally resolved fluctuation in pressure, density, or potential energy per unit matter, that is represented as a discrete potential “step”  $\mu_S$ . Defined as a system at local thermodynamic equilibrium,  $\mu_S$  is resolved only as a surface of uniform potential energy per unit matter.

Thus, the nature of the step can be of arbitrary internal complexity because, defined as a whole, the internal details are unresolved. For example,  $\mu_S$  might represent the sum of the specific energies associated with any choice of force fields. As an example, in the atmospheric sciences, the mass specific moist static energy  $h_m$  of an air parcel is often employed as a simple, conserved tracer, even if it is physically derived from the more complex sum of the potentials from gravitational forces, molecular motions, and molecular bonds.

Because the step is itself open system, there are flows to and from it. Flows from some higher potential  $\mu_R$  that characterizes a “reservoir” for the system are down a small jump  $\Delta\mu = \mu_R - \mu_S \ll \mu_S$  that separates the two steps. There is an interface between the system and the reservoir that is defined by a quantity of matter  $\tilde{N}$ , such that the total magnitude of the potential difference along the interface is  $\Delta G = \tilde{N}\Delta\mu$ . Because

$\Delta\mu/\mu_S$  is small, the potential difference as defined is never “far from equilibrium” [Nicolis(2007)], even if the magnitude of  $\Delta G$  is large.

While total matter and energy within the total volume  $V$  are conserved, a continuous flow redistributes potential energy downward. This downhill dissipation of potential energy across  $\Delta\mu$  manifests itself as an energetic “heating” of the system occupying  $\mu_S$  at rate

$$a = \alpha \check{N} \Delta\mu \quad (2.1)$$

where  $\alpha$  is a constant rate coefficient that can be related to the speed of flow across the interface. The energetic heating is tied to a material flow  $j$  through a coefficient  $\kappa = (da/dj) / \Delta\mu$ , such that

$$a = \kappa j \Delta\mu \quad (2.2)$$

For example, matter falls down a gravitational potential gradient, and the radiative dispersion of light can be expressed in terms of a flow of photons from high to low energy density. The energetic and material convergence into the fixed potential  $\mu_S$  causes an orthogonal “stretching” along  $\mu_S$ , allowing thermodynamic work to be done at rate

$$w = \left( \frac{\partial \check{N}}{\partial t} \right)_{\mu_S, \mu_R} \Delta\mu \quad (2.3)$$

Work here is a linear expansion of the interface at constant density. Depending on whether or not there is net convergence or divergence of flows at  $\mu_S$ , work can be either positive or negative, in which case the interface  $\check{N}$  either grows or shrinks. From Eqs. 2.1 to 2.3, the dimensionless efficiency  $\varepsilon$  with which the dissipative heating  $a$  is converted to work  $w$  is

$$\varepsilon = \frac{w}{a} = \frac{d\check{N}/dt}{j} = \frac{1}{\alpha} \frac{d \ln \check{N}}{dt} \quad (2.4)$$

Thus, the sign of  $\varepsilon$  dictates whether there is exponential growth or decay in  $\check{N}$ . Combining Eqs. 2.1 to 2.4 leads to

$$\frac{d \ln j}{dt} = \frac{d \ln \check{N}}{dt} = \alpha \varepsilon = \eta \quad (2.5)$$

where  $\eta$  is the instantaneous rate at which flows into the system either grow or decay (i.e.  $j = j_0 \exp(\eta t)$ ).

Integrating Eq. 2.5 yields the integral equation  $j(t) = j(0) + \int_0^t \eta j dt'$ . The implication is that current flows are a function of past flows, or systems have memory. Nucleation, or emergence, requires *past* efficiencies  $\varepsilon$ , growth rates  $\eta = \alpha \varepsilon$ , and rates of doing work  $w = \varepsilon a$ , to be positive. Equations of this form are commonly used to describe transport phenomena, such as radiative emission from planetary atmospheres [Liou(2002)]. The solution requires finding a time-dependent formulation for  $\eta = \alpha \varepsilon$ .

### 3. Definition of the interface driving flows

The first step towards describing the evolution of growth rates  $\eta = d \ln \check{N} / dt$  is to parameterize the system interface  $\check{N}$ . Supposing that in Fig. 1 the only resolved flows

are those into the system from a higher potential, it follows from Eq. 2.2 that the material flow  $j$  across the interface  $\check{N}$  results in an increase in the amount of matter (or energy) in the system  $N_S$  at the expense of the reservoir  $N_R$

$$j = \alpha \check{N} = \left( \frac{\partial N_S}{\partial t} \right)_{\mu_S} = - \left( \frac{\partial N_R}{\partial t} \right)_{\mu_R} \quad (3.1)$$

The relationship to energy dissipation is given by Eq. 2.2. Eq. 3.1 is proportional to an increase in the system volume  $V_S = N_S/n_S$ , assuming no resolved internal variations in the system density  $n_S$ .

A first guess might be that the size of the interface  $\check{N}$  that is enabling flows is determined by the product the reservoir and system sizes  $N_S N_R$ . This is the approach that is most commonly taken when modeling ecological populations [Berryman(1992)] and in the application of the logistic equation to long-range modeling of national energy reserve consumption [Bardi and Lavacchi(2009), Höök et al.(2010)].

However, perfect multiplication is only suitable when  $N_S$  and  $N_R$  can be treated as being perfectly well-mixed. But it is not possible to resolve flows between two components of a perfect mixture. Rather, if  $N_S$  and  $N_R$  can be distinguished, then they must interact through physical flows across some sort of interface between the two. Because fluid flows are always down a potential gradient, the interface driving the flux  $j$  from  $N_R$  to  $N_S$  is most appropriately defined as a concentration gradient normal to a surface. It is the exterior surface of the system, and a density gradient away from the surface, that provides the resolvable contrast allowing for a net flow.

Perhaps the simplest possible example of this physics is the diffusional growth of a particulate sphere of radius  $r$  within a larger volume  $V$ . Fick's Law dictates that a concentration gradient  $n$  drives a diffusive flux across the sphere surface at rate

$$j = 4\pi r^2 \mathcal{D} n_{\mu_S} \left. \frac{\partial \ln n}{\partial x} \right]_{x=r} \quad (3.2)$$

where  $\mathcal{D}$  is a diffusivity (units area per time) that expresses the speed of material transfer across a surface with radius  $r$  along radial coordinate  $x$ . If the gradient is approximated as a small discretized concentration jump between two potential surfaces  $\Delta n = n_{\mu_R} - n_{\mu_S}$ , and the particle volume  $V_S$  is small compared to the total volume  $V$ , then the flux of matter down the gradient is

$$j \simeq 4\pi r \mathcal{D} \Delta n = \frac{4\pi r \mathcal{D}}{V} N_R \quad (3.3)$$

where  $N_R = \Delta n V$  is the amount of matter in the higher potential reservoir that is available to flow to the particle volume  $V_S$  and contribute to the number of material elements in the particle  $N_S = n_S V_S$ , where  $n_S$  is the particle material density.

Note, however, that it is a length dimension of  $4\pi r$  that drives flows rather than the whole particle volume or its surface area. In this respect, the electrostatic analogy for flows is that they are proportional to a capacitance, which in cgs units has dimensions of length. For shapes more complex than spheres, such as snow crystals, cows, or any other species [Wood et al.(2001), Maia et al.(2005), Kooijman(2010)], the length dimension can be retained but generalized such that the flux equation given by Eq. 3.3 becomes

$$j = \lambda \mathcal{D} N_R \quad (3.4)$$

where  $\lambda$  is the system's effective length or capacitance density within the the total volume  $V$ . The flux of  $N_R$  to  $N_S$  has a time constant  $1/(\lambda\mathcal{D})$ .

However, interactions between particles or species are not always referenced with respect to space. For example, thermal heating requires a radiation pressure contrast, but the distance between the source and receiver is not normally considered because light is so fast.

So a more convenient expression for Eq. 3.4 might be to represent the diffusivity through a rate coefficient  $\alpha$  that has dimensions of inverse time, and to represent the length density or capacitance density  $\lambda$  through  $kN_S^{1/3}$  where  $k$  is a dimensionless coefficient that depends on the system geometry. The proportionality of  $N_S^{1/3}$  to a length dimension  $L$  comes from  $(N_S/n_S)^{1/3} = V_S^{1/3} \propto (L^3)^{1/3} = L$ .

Accordingly, the rate coefficient  $\lambda\mathcal{D}$  can be generalized to the geometry-independent expression  $\alpha k N_S^{1/3}$ , in which case Eq. 3.4 becomes

$$j = \alpha \check{N} = \alpha k N_S^{1/3} N_R \quad (3.5)$$

For the diffusional growth of a particle, as given by Eqs. 3.3 or 3.4, then Eq. 3.5 is based on the relations

$$\alpha = \frac{\mathcal{D}}{n_S^{1/3} V} \quad (3.6)$$

$$k = \lambda \frac{V}{V_S^{1/3}} \quad (3.7)$$

If the particle is a sphere, then  $\lambda = 4\pi r/V$ ,  $V_S = 4\pi r^3/3$  and the dimensionless geometric coefficient  $k$  is given by  $(48\pi^2)^{1/3}$ .

The point here is that the material interface  $\check{N}$  is best treated as being proportional to two quantities. The first is a length density  $\lambda$  for the system within the total volume  $V$ , which is related to the system's total number of material elements to a one third power  $N_S^{1/3}$ . The second is the material availability in the energy reservoir  $N_R$ . It is the product that enables material flows at rate  $j = \alpha \check{N}$  and dissipates energy at rate  $a = \alpha \check{N} \Delta\mu$ .

Thus, flows are proportional to a surface area *and* a local concentration gradient (i.e.  $N_S^{1/3} N_R$ ), rather than the system volume (i.e.  $N_S N_R$ ) or its surface area alone (i.e.  $N_S^{2/3} N_R$ ). The one third exponent arises because flows are through a cross-section proportional to  $N_S^{2/3}$ , and they are down a linear concentration gradient that is proportional to  $N_R/N_S^{1/3}$ .

#### 4. Diminishing returns

The sub-unity exponent for  $N_S$  lends itself to widely-observed mathematical behaviors. Systems as seemingly disparate as droplets [Pruppacher and Klett(1997)], boundary layers [Turner(1979)], animals [Kooijman(2010)] and plants [Montieth(2000)] show growth behavior that is initially rapid but slows with time, in what might be termed a "law of diminishing returns".

To see why, consider that flows evolve at rate  $\eta = d \ln j / dt$  (Eq. 2.5) where  $j \propto N_S^{1/3} N_R$ . Thus, from Eq. 3.5

$$\eta = \frac{1}{3} \left( \frac{\partial \ln N_S}{\partial t} \right)_{N_R} + \left( \frac{\partial \ln N_R}{\partial t} \right)_{N_S} \quad (4.1)$$

$$= \frac{1}{3} \eta_S - \eta_R \quad (4.2)$$

Here,  $\eta_S$  and  $\eta_R$  represent the respective growth rates of the system and the reservoir, assuming the other is held fixed. The rate  $\eta_S$  represents the positive feedback that comes from system expansion. Growth lengthens the interface with respect to previously inaccessible reservoirs, allowing for increasing flows (Fig. 1). The rate  $\eta_R$  is a negative feedback since reservoirs are simultaneously being depleted.

A system with two potentials  $\mu_S$  and  $\mu_R$ , and no external sources to the volume  $V$ , is characterized by  $N_S + N_R = N_T$  and  $dN_T/dt = 0$ ; total (but not potential) energy is conserved. Then, from Eq. 3.1, Eq. 4.1 can be rewritten as:

$$\eta = \frac{d \ln j}{dt} = j \left( \frac{1}{3N_S} - \frac{1}{N_R} \right) \quad (4.3)$$

which suggests a dimensionless ‘‘Adjustment number’’ expressing whether the evolution of flows is dominated by negative or positive feedbacks:

$$A = \frac{\eta_S}{3\eta_R} = \frac{N_R}{3N_S} \quad (4.4)$$

Flows are in a mode of either emergent growth or decay depending on whether  $A$  is greater or less than unity, respectively.

Substituting Eq. 4.4 into Eq. 4.3, the expression for the evolution of flows becomes

$$\eta(t) = \alpha k N_S^{1/3} (A - 1) \quad (4.5)$$

Decay dominates when  $A < 1$ , in which case

$$\eta \simeq -\eta_R = -\alpha k N_S^{1/3} \quad (4.6)$$

Emergent growth requires that  $A \gg 1$ , in which case  $\eta \simeq \eta_S/3$ , where

$$\eta_S = \alpha k \frac{N_R}{N_S^{2/3}} \quad (4.7)$$

Note that if it had been assumed that  $j \propto N_S N_R$  rather than  $N_S^{1/3} N_R$ , then emergent growth rates would have depended only on the reservoir size  $N_R$ , and not on  $N_S = \int_0^t j dt'$ , and therefore on past flows. Rather, as shown by Eq. 4.7, growth rates have a power-law relationship given by  $N_S^{-2/3}$ , or the ratio of system length and volume.

The reason that growth in flows stagnates is that current flows are proportional to the system length  $N_S^{1/3}$  (Eq. 3.4), while length grows one third as fast as volume. Thus, current flows become progressively diluted in the volume accumulation of past flows  $N_S = \int_0^t j dt'$ . Large systems tend to grow at a slower rate (Eq. 4.7) and with lower thermodynamic efficiency  $\varepsilon = \eta/\alpha$  (Eq. 2.4) than small systems.

Mathematically, if a system is in its emergent growth stage, such that  $A \gg 1$ , then its rate of growth evolves at rate

$$\frac{d \ln \eta_S}{dt} \simeq -\frac{2}{3} \eta_S \quad (4.8)$$

While the system growth rate  $\eta_S$  stays positive, its own rate of change  $d \ln \eta_S / dt$  is negative. The solution to Eq. 4.8 is

$$\eta_S(t) = \frac{\eta_{S0}}{1 + 2\eta_{S0}t/3} \quad (4.9)$$

where,  $\eta_{S0}$  is the initial value of  $\eta_S$  at time  $t = 0$ . Provided the system is initially small (i.e.  $N_S \ll N_R$ ), its growth rate has a half life of  $3/(2\eta_S)$ . Eq. 4.9 accounts for the phenomenon of a “law of diminishing returns” where a system is growing in response to conserved flows from a potential energy reservoir. Relative growth rates start quickly, but they asymptote to zero over time. Current flows become diluted in past flows <sup>1</sup>.

### 5. Logistic and explosive growth

Two phenomena often seen in physical, biological and social systems are sigmoidal growth [Cohen(1995), Tsoularis and Wallace(2002)] and super-exponential (sometimes termed “faster than exponential”), or “explosive” growth [Bettencourt et al.(2007), Garrett(2011)]. Sigmoidal behavior, as described by the logistic equation, starts exponentially but saturates. By contrast, explosive instabilities exhibit rates of change that grow super-exponentially with time, such that  $\eta_S$  and  $d \ln \eta_S / dt$  are both greater than zero. One immediately recognizable example is the historically explosive growth of the world population [Pollock(1988), Johansen and Sornette(2001)].

Explosive growth requires that the reservoir  $N_R$  be open to some external source. Then, from Eq. 4.7 the system growth rate  $\eta_S$  evolves at rate

$$\frac{d \ln \eta_S}{dt} = -\frac{2}{3} \eta_S + \eta_R^{net} \quad (5.1)$$

where  $\eta_R^{net} = d \ln N_R / dt = \eta_D - \eta_R$  represents a balance between rates of reservoir discovery  $\eta_D$  due to flows into the reservoir, and depletion  $\eta_R$  due to flows out of the reservoir and into the system. This suggests a “Growth Number”

$$G = \frac{3 \eta_R^{net}}{2 \eta_S} \quad (5.2)$$

Explosive growth with  $d \ln \eta_S / dt > 0$  is possible provided that  $G > 1$ , in which case the reservoir is growing at least two-thirds as fast as the system is growing. Steady-state growth occurs when  $G = 1$  and  $\eta_R^{net} = 2\eta_S/3$ .

Eq. 5.1 is expressible as a logistic equation for rates of growth

$$\frac{d\eta_S}{dt} = \eta_R^{net} \eta_S - \frac{2}{3} \eta_S^2 \quad (5.3)$$

<sup>1</sup> As shown in the Appendix,  $\eta_S$  is equivalent to the local rate of entropy production.

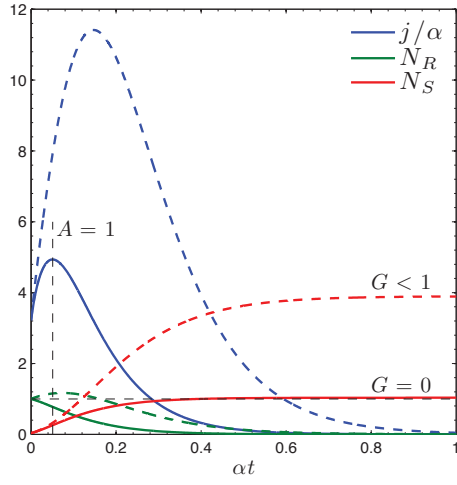


Figure 2. Numerical solution for the evolution of fluxes  $j$  and the reservoir size  $N_R$  and the system size  $N_S$  for two cases. Solid lines:  $N_S + N_R$  is a constant and the initial value of  $G = 0$  (Eq. 5.2); peak flows occur where  $A = N_R / (3N_S) = 1$  (Eq. 4.4). Dashed lines: there is “discovery” of new reservoirs at rate  $\eta_D > 0$ , but at a rate that is smaller than what is required for super-exponential growth, so  $G < 1$ . In both cases reservoirs ultimately give way to net depletion (i.e.  $\eta_R^{net} < 0$ ).

The prognostic solution for Eq. 5.3, with initial conditions given by  $G = 3\eta_{S0}/2\eta_{R0}^{net}$ , is of standard sigmoidal form

$$\eta_S(t) = \frac{G\eta_{S0}}{1 + (G - 1)e^{-\eta_{R0}^{net}t}} \quad (5.4)$$

The growth rate  $\eta_S$  adjusts sigmoidally to  $G\eta_{S0}$ , or 50% faster than the net energy reservoir expansion rate  $\eta_{R0}^{net}$ . Illustrations of the logistic nature of emergent growth rates, and how they ultimately give way to reservoir depletion, are shown in Figs. 2 and 3.

## 6. Rapid production of cloud droplets and rain

One example of how instability can lead to runaway explosive growth is in the formation of embryonic raindrops. The growth of the droplet radius through vapor diffusion is constrained by a law of diminishing returns. Production of embryonic raindrops requires a rapid transition of cloud droplet size from about  $10 \mu\text{m}$  to  $50 \mu\text{m}$  radius through interdroplet collision and coalescence [Langmuir(1948), Pruppacher and Klett(1997)]. What remains poorly explained is how this “autoconversion” process can happen as rapidly as has been observed [Wang et al.(2006)].

Within the context of the discussion above, consider a droplet population with a number density  $n_d$  in a total volume  $V$ , and with each droplet having a molecular density  $n_l = N_S/V_S$  within the droplet volume  $V_S$ . Eq. 4.6 becomes the relaxation rate of the available vapor supply in reponse to condensational flows  $\eta_R = 4\pi r\mathcal{D}n_d$  [Squires(1952), Kostinski(2009)]. Eq. 4.7 for system (or droplet volume) growth



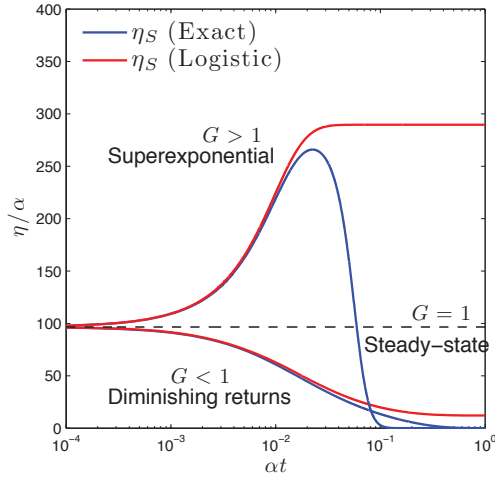


Figure 3. The evolution of the system growth rate  $\eta_S = d \ln N_S / dt$  as a function of time for three regimes of the growth number  $G$  (Eq. 5.2). Red lines show analytical solutions for emergent growth given by the logistic expression in Eq. 5.4. Exact numerical solutions given by the blue lines account for how flow rates eventually decay as  $N_R$  is depleted.

becomes

$$\eta_S = \frac{3D\Delta n_v}{n_l r^2} \quad (6.1)$$

where  $\Delta n_v = n_v - n_v^{sat} = N_R / V$  is the local vapor density surplus relative to the saturation value  $n_v^{sat}$  at the droplet surface [Baker et al.(1980)]. Note how growth rates slow as  $r$  grows.

A droplet can overcome this law of diminishing returns by “discovering” new mass reservoirs through the droplet collision-coalescence process. If droplets are generally uniformly distributed and efficiently collected, with a dimensionless mass mixing ratio in air of  $q_l$ , then a larger, falling, collector droplet with mass  $m$  will grow through collisions at rate

$$\eta_D = \frac{d \ln m}{dt} = C q_l r \quad (6.2)$$

where  $C \sim 10^5 m^{-1} s^{-1}$  (Details in Appendix). If the depletion of droplets through this process  $\eta_R$  remains small, then  $\eta_R^{net} = \eta_D - \eta_R \simeq \eta_D$  and the collision-coalescence leads to explosive growth provided that Eq. 5.2 satisfies

$$G = \left( \frac{C n_l}{2D \Delta n_v} \right) q_l r^3 > 1 \quad (6.3)$$

For example, conditions characteristic of a small cumulus cloud might have a liquid mixing ratio  $q_l$  of  $0.5 \text{ g kg}^{-1}$  and a supersaturation  $S = \Delta n_v / n_v^{sat}$  of 0.5%. In this case, diminishing returns can be overcome, leading to explosive droplet growth, provided that a fraction of the droplet population exceeds a radius of about  $20 \mu\text{m}$ . This is in fact the threshold radius that is commonly observed as being necessary for warm rain production [Rangno and Hobbs(2005)].

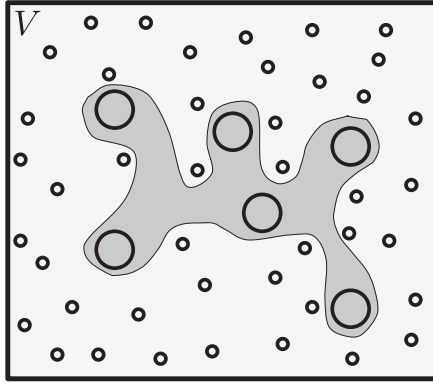


Figure 4. A predator population (large circles) acts as a system that interacts with a reservoir of prey (small circles) through an arbitrarily shaped linear interface within a total volume  $V$ .

### 7. Thermodynamics of predator-prey relationships

If, in addition to a source, a sink for a system is explicitly resolved in Fig. 1, then the logistic expressions for  $N_S$  and  $N_R$  can be expressed in terms of predators and prey, as is commonly considered in the ecological sciences and more recently for physical representations of stratocumulus cloud dynamics [Feingold et al.(2010), Koren and Feingold(2011)]. A fall in predators is followed by a rise in prey. The response is renewed predation at the sacrifice of the prey. This oscillatory behavior is canonically represented by the Lotka-Volterra equations [Lotka(1925)], which represent the one-way fluxes of populations of prey to predators in terms of the product of the biomass densities of each, i.e,  $N_S N_R$ . Many improvements to this model have been made over the past century in order to more faithfully reproduce observed behavior, but not necessarily by appealing to physical conservation laws [Berryman(1992)].

The physical framework discussed here can be interpreted as a one-way material flow of “prey” biomass  $N_R$  to “predator” biomass  $N_S$ . As discussed above, representing species interactions as a product of predator and prey populations, e.g.  $N_S N_R$ , would seem to require the unphysical condition that predators and prey interact in the absence of a local gradient. Physically, this is best addressed by introducing an arbitrarily shaped interface (Fig. 4), requiring that species interactions be proportional to  $N_S^{1/3} N_R$ . In this case, the modified predator-prey relationships are

$$\begin{aligned} \frac{dN_R}{dt} &= \beta N_R - \gamma N_S^{1/3} N_R \\ \frac{dN_S}{dt} &= \gamma N_S^{1/3} N_R - \delta N_S \end{aligned} \quad (7.1)$$

where  $\beta$ ,  $\gamma$  and  $\delta$  are constant coefficients. The coefficient  $\beta$  is equivalent to the discovery rate  $\eta_D$  discussed previously,  $\gamma = \alpha k$  (Eq. 3.5), and  $\delta$  represents the sink rate of  $N_S$  to its surroundings, as shown in Fig. 1.

So, while the Lotka-Volterra equations lead to non-dissipative limit cycles, the simple addition of a one third exponent to the predators allows populations to converge on

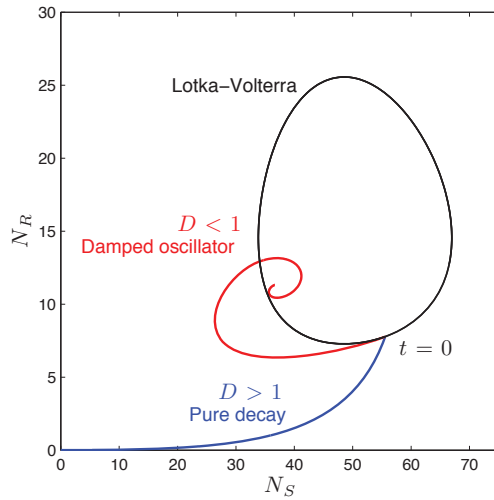


Figure 5. Phase plot for  $N_S$  and  $N_R$  for the predator-prey equations given by Eq. 7.1 (blue and red), depending on the value of the damping number  $D = \delta / (3\beta)$ . The limit cycle behavior given by the canonical Lotka-Volterra Equations (black) is shown for the same set of initial conditions.

an equilibrium state given by  $N_R = \delta\beta^2/\gamma^3$  and  $N_S = (\beta/\gamma)^3$ , or  $N_R/N_S = \delta/\beta$ . As shown in Appendix and in Fig. 5, the nature of the convergence depends on a damping number  $D = \delta / (3\beta)$ . For the under-damped case that  $D < 1$ , then  $N_S$  behaves as a damped simple harmonic oscillator with angular frequency  $\omega_1 = \omega_0\sqrt{1 - D}$ , where  $\omega_0 = (\delta\beta/3)^{1/2}$ . For the over-damped case that  $D \geq 1$ , then equilibrium is approached in monotonic decay.

The key point here is that the sub-unity exponent for  $N_S$  allows for inter-species interactions to evolve more slowly than the respective populations themselves, introducing a damped or “buffered” [Koren and Feingold(2011)] response. The general perturbation solution for the damping of  $N_S$  is  $a \exp(-\delta t) \exp[i(\omega_1 t - \phi)]$  where  $a$  and  $\phi$  are determined by the initial conditions.

Damped simple harmonic oscillators are ubiquitous in physics, for example in the interactions of light with matter [Liou(2002)], so it is particularly noteworthy that it requires only a very small modification to the Lotka-Volterra predator-prey framework in order to arrive at an expression of this form. In fact, even in ecology, damped oscillatory behavior is being observed in the response of forage fish populations to a collapse of predatory cod stocks from overfishing [Frank et al.(2011)]. From the above, a possible interpretation is that forage fish biomass densities ( $N_S$ ) initially thrived when predator cod stocks collapsed (a drop in  $\delta$ ), but then they overshot and declined themselves as a consequence of excessive plankton ( $N_R$ ) depletion at rate  $\gamma N_S^{1/3} N_R$ . As forage fish populations fell, plankton recovered at rate  $\beta N_R$ , and the forage fish soon followed. Equilibrium is being restored, but in the manner of damped oscillations.

A more physical parameter space for the phase diagram shown in Fig. 5 could be either the orthogonal basis of  $j = \gamma N_S^{1/3} N_R$  and  $dN_S/dt$ , or alternatively  $N_S^{1/3}$  and  $N_R$ . For the latter, the area carved out by the curves in a phase diagram would be proportional to the energy dissipated by the trophic cascade.

## 8. Summary

Regardless of scale or complexity, anything that can be defined requires some local contrast to be observable. Contrasts require a gradient and therefore a local exchange of material and energetic flows. Physically, flows are across an interface that is related to the magnitude of the local gradient, normal to the surface of the system. Dimensional reasoning requires that flows must be proportional to a length dimension, or a one third exponent with respect to the system volume or mass.

Table 1. Evolutionary modes for emergent system growth rates.

	$d \ln \eta_S / dt$	$\eta_R^{net}$	$G$
explosive growth	$> 0$	$> 2\eta_S/3$	$> 1$
steady state	$0$	$= 2\eta_S/3$	$1$
diminishing returns	$< 0$	$< 2\eta_S/3$	$< 1$

The consequence of the one third exponent is that the time evolution of flow rates follows mathematical behaviors that can be partitioned into a limited set of regimes, depending on a dimensionless parameter  $G$  (Table 1). In general, spontaneous emergence is governed by the logistic equation, exhibiting a sigmoidal curve for system growth rates  $\eta_S$ . The one third exponent requires that current flows become increasingly diluted in an accumulation of past flows, so spontaneously emergent systems have a natural propensity to exhibit a law of diminishing returns. Explosive, faster-than-exponential growth occurs if energy reservoirs are expanding at a rate  $\eta_R^{net}$  that is at least two-thirds as fast as the rate of system growth. However, even explosive growth ultimately lends itself towards decay in flow rates. The faster a system grows, the faster it depletes its potential energy reservoirs.

Where a system is open to downhill flows to and from it, the system size itself can either grow or decay, depending on the sign of net convergence in flows. In this case, the growth equations are very similar to the canonical Lotka-Volterra predator-prey equations used to model ecological systems, differing only in a one third exponent for the “predators”. This subtle but important difference leads to the perturbation equations for a damped simple harmonic oscillator that are ubiquitous in the physical sciences and have also been identified in ecological systems. Whether the oscillator is under- or over-damped depends on the ratio of the natural growth rates for the “predators” and “prey”.

The mathematical expressions described here are independent of complexity or scale, and any physics more specific than thermodynamic laws. They offer a simple framework for expressing how a redistribution of matter and energy evolves through a cascading flow between distinguishable systems.

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## Appendix

### *Entropy production*

The equation for the growth rate of  $N_S$  is

$$\eta_S = \frac{d \ln N_S}{dt} = \frac{j}{N_S} \quad (\text{A.1})$$

$N_S$  is an accumulation of past flows, so Eq. A.1 can be rewritten as

$$\eta_S = \frac{j}{\int_0^t j dt'} \quad (\text{A.2})$$

The current growth rate of flows  $\eta_S$  is tied to the integrated history of past flows  $\int_0^t j dt'$ .

From Eq. 2.2, the potential energy dissipation rate  $a$  along the gradient  $\Delta\mu$  is proportional to the material flow  $j$  down the gradient. Thus

$$\eta_S = \frac{a}{\int_0^t a dt'} \quad (\text{A.3})$$

The expression  $\int_0^t a dt'$  is the total time-integrated heating that has been applied to the constant potential surface  $\mu_S$ . The local rate of production of entropy  $\mathcal{S}$  can be written as the energy dissipation rate relative to the local potential, i.e.  $\sigma = d\mathcal{S}/dt = a/\mu_S$ . The well-known Clausius expression is obtained for the restrictive case that  $\mu_S$  is proportional only to temperature  $T$ .

It follows that the accumulation of added entropy within a volume  $V$  that contains fixed potentials  $\mu_R$  and  $\mu_S$  is  $\mathcal{S} = \int_0^t a dt' / \mu_S$ . Thus,  $\eta_S$  has the following thermodynamic expressions

$$\eta_S = \frac{d \ln N_S}{dt} = \frac{\sigma}{\mathcal{S}} = \frac{d \ln \mathcal{S}}{dt} = \frac{a}{\int_0^t a dt'} \quad (\text{A.4})$$

Energy dissipation at rate  $a$  drives conservative material flows at rate  $j$  from a high potential  $\mu_R$  to a lower potential  $\mu_S$ . The growth rate  $\eta_S$  of the amount of material  $N_S$  in the lower potential is proportional to the rate at which entropy is increasing locally through  $\sigma = \mathcal{S} d \ln N_S / dt$ .

Total entropy in the volume  $V$  always increases because flows are downhill. The net redistribution of matter is always towards lower values of  $\mu$  and higher values of  $\mathcal{S}$ . Thus, for an arbitrarily large number of potential steps, total entropy production is

$$\sum_i \sigma_i = \sum_i \mathcal{S}_i \frac{d \ln N_{S_i}}{dt} > 0 \quad (\text{A.5})$$

where the summation is across potential surfaces  $\mu_i$ .

Discrete, resolved systems along any given potential surface  $\mu_i$  can experience either positive or negative entropy production as they grow or shrink, but still satisfy the Second Law requirement that global entropy production is positive.

### *Collision-coalescence*

The growth equation for the mass  $m = 4\pi\rho_l r^3/3$  of a collector drop with radius  $r$  and density  $\rho_l$ , that falls with terminal velocity  $v_T$  through a cloud of droplets with mass

liquid water mixing ratio in air  $q_l$  is

$$\frac{dm}{dt} \simeq \pi r^2 v_T \rho_{air} q_l \quad (\text{A.6})$$

where  $\rho_{air}$  is the air density, and it is assumed that the collector drop has a relatively large cross-section and the collection efficiency is near unity. In the initial stages of growth, when the collector drop is smaller than about  $35 \mu\text{m}$ , the drop terminal velocity is determined by a balance between Stokes drag and the gravitational force  $mg$ , such that

$$v_T = \frac{2\rho_l g}{9\rho_{air}\nu} r^2 \quad (\text{A.7})$$

where  $\nu$  is the kinematic viscosity of air. Thus,

$$\eta_D = \frac{d \ln m}{dt} = \frac{g}{6\nu} q_l r \simeq C q_l r \quad (\text{A.8})$$

where  $C \simeq 10^5 \text{ m}^{-1} \text{ s}^{-1}$ .

#### *Perturbation solutions for the predator-prey equations*

The original set of predator-prey equations is

$$\begin{aligned} \frac{dN_R}{dt} &= \beta N_R - \gamma N_S^{1/3} N_R \\ \frac{dN_S}{dt} &= \gamma N_S^{1/3} N_R - \delta N_S \end{aligned} \quad (\text{A.9})$$

which can be re-written in a more amenable mathematical form as

$$\begin{aligned} \frac{dx}{dt} &= \beta x - \gamma xy \\ 3y^2 \frac{dy}{dt} &= \gamma xy - \delta y^3 \end{aligned} \quad (\text{A.10})$$

where  $x = N_R$  and  $y = N_S^{1/3}$ . The equilibrium solutions for  $x$  and  $y$  are  $x_{eq} = \delta\beta^2/\gamma^3$  and  $y_{eq} = \beta/\gamma$ .

Supposing a perturbation solution

$$\begin{aligned} x &= \frac{\delta\beta^2}{\gamma^3} + x' \\ y &= \frac{\beta}{\gamma} + y' \end{aligned} \quad (\text{A.11})$$

and noting that  $dx'/dt = dx/dt$  and  $dy'/dt = dy/dt$ , Eqs. A.10 are transformed to

$$\begin{aligned} \frac{dx'}{dt} &= -\left(\frac{\delta\beta^2}{\gamma^2}\right) y' \\ \frac{dy'}{dt} &= \left(\frac{\gamma^2}{3\beta}\right) x' - \left(\frac{2\delta}{3}\right) y' \end{aligned} \quad (\text{A.12})$$

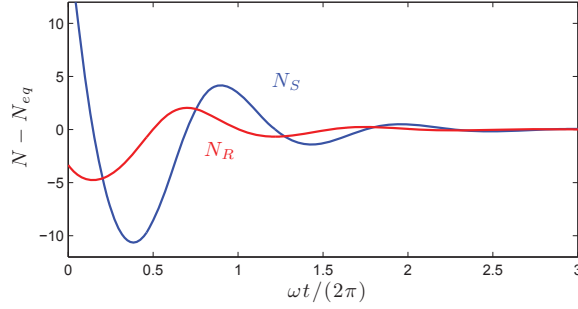


Figure A.6. Time series for perturbation from equilibrium in  $N_S$  and  $N_R$  for the predator-prey equations given by Eq. A.9, for the special case that  $D < 1$  and the perturbations behave as damped simple harmonic oscillators.

where second order perturbation terms have been neglected. Taking the second derivative leads to the equation for a damped simple harmonic oscillator

$$\frac{d^2 y'}{dt^2} + \frac{2\delta}{3} \frac{dy'}{dt} + \frac{\delta\beta}{3} y' = 0 \quad (\text{A.13})$$

The natural oscillator angular frequency is  $\omega_0 = (\delta\beta/3)^{1/2}$ . Eq. A.13 has the general solution  $y' = a \exp^{\eta_{y1}t} + b \exp^{\eta_{y2}t}$  where  $\eta_{y1}$  and  $\eta_{y2}$  are the quadratic roots

$$\eta_y = \frac{\delta}{3} \left[ -1 \pm \sqrt{1 - 3\frac{\beta}{\delta}} \right] \quad (\text{A.14})$$

Since the real part of  $\eta_y$  is always negative,  $y'$  always decays. The nature of the decay depends on a damping ratio

$$D = \frac{\delta}{3\beta} \quad (\text{A.15})$$

The value of  $\eta_y$  is complex if  $D < 1$ , in which case decay is oscillatory with frequency

$$\omega_1 = \omega_0 \sqrt{1 - D} \quad (\text{A.16})$$

In terms of  $N_S$ , for the real component  $\eta_S = 3\eta_y$  since  $d \ln y / dt = (d \ln N_S / dt) / 3$ . Thus, the solution for  $N_S$  is

$$N_S = a \exp(-\delta t) \exp[i(\omega_1 t - \phi)] \quad (\text{A.17})$$

where  $a$  and  $\phi$  are determined by the initial conditions.

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