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Emergent exclusion statistics of quasiparticles in two-dimensional topological phases

Yuting Hu,[1,*] Spencer D. Stirling,[1,2,†] and Yong-Shi Wu[1,3,‡]

[1]*Department of Physics and Astronomy, University of Utah, Salt Lake City, Utah 84112, USA*

[2]*Department of Mathematics, University of Utah, Salt Lake City, Utah 84112, USA*

[3]*Key State Laboratory of Surface Physics, Department of Physics and Center for Field Theory and Particle Physics, Fudan University, Shanghai 200433, China*

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We demonstrate how the generalized Pauli exclusion principle emerges for quasiparticle excitations in 2D topological phases. As an example, we examine the Levin-Wen model with the Fibonacci data (specified in the text), and construct the number operator for fluxons living on plaquettes. By numerically counting the many-body states with fluxon number fixed, the matrix of exclusion statistics parameters is identified and is shown to depend on the spatial topology (sphere or torus) of the system. Our work reveals the structure of the (many-body) Hilbert space and some general features of thermodynamics for quasiparticle excitations in topological matter.

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I. INTRODUCTION

By now it is well known that (quasi)particles in strongly entangled many-body systems may exhibit exotic quantum statistics, other than the familiar Bose-Einstein and Fermi-Dirac ones. In addition to the anyonic or exchange statistics [1] in two dimensional systems, statistical weight of many-body quantum states may also obey new combinatoric counting rules [2] following a generalized Pauli exclusion principle [3], in which the number of available single-particle states, when adding one more quasiparticle into the system, linearly depends on the number of existing quasiparticles. A typical new feature is mutual exclusion between different species, resulting in a matrix of statistical parameters [3] and leading to unusual thermodynamics for ideal gases with only statistical interactions [2,4]. (For a review see, e.g., Ref. [5].)

More precisely, following Ref. [2], in the case with only one species of quasiparticles, the number of N-particle states is assumed to be given by the binomial coefficient:

$$W_{G,N} = \binom{G_{\text{eff}} + (N-1)}{N}, \tag{1}$$

with $G_{\text{eff}} = G - \alpha(N-1)$ being the number of available single-particle states, while G is the number of single-particle states when $N = 1$. Then $\alpha = 0$ corresponds to bosons and $\alpha = 1$ fermions; other values of α gives rise to exotic exclusion statistics. Similarly, in the multispecies case, the number of many-particle states is assumed to be given by ($a,b = 1,\ldots,m$ labeling species)

$$W_{\{G_a,N_a\}} = \prod_a \binom{G_a + N_a - 1 - \sum_{b=1}^{m} \alpha_{ab}\,(N_b - \delta_{ab})}{N_a}. \tag{2}$$

Here coefficients α_{ab} form the (mutual) statistics matrix.

It has been shown [6] that the thermodynamic ansatz [7] for one-dimensional solvable many-particle models is actually a special case of the exotic exclusion statistics. (See also Refs. [8] and [9].) It has also been numerically verified that quasiparticle excitations in the fractional quantum Hall (FQH) systems indeed obey [10] Eq. (1), or Eq. (2) allowing mutual exclusion between different species [11]. Moreover either the Haldane or Jain hierarchy in the FQH effect can be theoretically understood from the exclusion statistics of quasiparticles [5,12].

Recently there has been revived interest in the study of quasiparticle statistics in 2D topological states of matter (including FQH systems), because of the possibility of using their braiding to do (fault tolerant) topological quantum computation (TQC) [13,14]. In order to know better about the error of TQC at finite temperature, it is needed to understand better how exclusion statistics of quasiparticles emerges in 2D topological matter, which governs the thermodynamics of the system.

In this paper, we carry out the many-body state counting in an exactly solvable discrete model, i.e., the Levin-Wen model [15] (with a special set of data), that describes a 2D topological quantum fluid [16] of Fibonacci anyons [17], with doubled Fibonacci anyons as fluxon excitations living on plaquettes. The Fibonacci anyons are the simplest non-Abelian anyons. They occur as quasiparticles in the $k = 3$ Read-Rezayi state [18] in an FQH state with filling fraction $\nu = \frac{12}{5}$, and can be used for universal topological quantum computation [14]. (Recently, it is proposed [19] that the physics of interacting Fibonacci anyons may be studied in a Rydberg lattice gas.)

In this paper, we first construct the number operator for fluxons in the model, which helps us identify the states with localized excitations. Then we numerically count the (many-body) states with fluxon-number N fixed, from $N = 1$ up to $N = 7$, for the system on a sphere and torus, respectively. The results exhibit a pattern closely related to the Fibonacci numbers, which in turn is put in the form of Eq. (2), thus determining a topology-dependent statistics parameter matrix. Our work reveals that exotic exclusion emerges among quasiparticles due to interplay between various "hidden" degrees of freedom (d.o.f.) in addition to fluxon locations. These "hidden" d.o.f. are very similar to the pseudospecies, previously introduced in the literature on conformal field

*yuting@physics.utah.edu
†stirling@physics.utah.edu
‡wu@physics.utah.edu

theory [20], which do not contribute to energy but contribute to state counting in accordance with an exclusion statistics parameter matrix. Finally, we briefly discuss the thermodynamics of the system.

II. THE MODEL

We consider a discrete model for a "spin" system on a trivalent graph on a closed surface, e.g., a sphere or torus. We adopt a simplified formulation of the Levin-Wen model [15], with the Fibonacci data (e.g., see Ref. [14]) given as follows: Each link is assigned a "spin" type labeled by j, and configurations of the labels on all links form an orthonormal basis in the Hilbert space. The key input of the Fibonacci data is that the "spin" -type index j takes only two values $j = 0,1$, and they satisfy an algebra (called the Fibonacci algebra), which describes how to fuse two "spin" types through the branching rules:

$$0 \otimes j = j \otimes 0 = j, \quad 1 \otimes 1 = 0 \oplus 1. \tag{3}$$

These rules are similar to those for the (direct sum) decomposition of (tensor) products of irreducible representations of a group, with $j = 0$ playing the role of the unit element for the (tensor) product. [It is conjectured that the Levin-Wen model describes a class of doubled (time-reversal invariant) topological phases [21].]

The Hamiltonian of the model is of the form

$$\hat{H} = U \sum_v (1 - \hat{Q}_v) + \epsilon \sum_p (1 - \hat{B}_p), \quad \hat{B}_p = \frac{1}{D} \sum_{s=0,1} d_s \hat{B}_p^s. \tag{4}$$

The two summations here run over all vertices v and plaquettes p, respectively. For $\hat{B}_p$, the summation runs over the "spin" -type $s = 0,1$, and $d_0 = 1, d_1 = \phi \equiv (\sqrt{5}+1)/2$, and $D = 1 + \phi^2$. U and ϵ are positive constants. The explicit form of the operators $\hat{Q}_v$ and $\hat{B}_p^s$ are given in the supplemental material [22]. (By adding more competing interactions, Ref. [23] has used this model to discuss topological phase transitions in the Fibonacci anyon liquid. Here we restrict to the original Levin-Wen model and discuss emergent exclusion statistics for quasiexcitations.)

A notable property of the model is that by construction, $\hat{Q}_v$ and $\hat{B}_p$ are mutually commuting projection operators: $[\hat{Q}_v, \hat{B}_p] = 0$, $\hat{Q}_v \hat{Q}_{v'} = \delta_{vv'} \hat{Q}_v$ and $\hat{B}_p \hat{B}_{p'} = \delta_{pp'} \hat{B}_p$. Thus the Hamiltonian is exactly solvable. The energy eigenstates are the simultaneous eigenvectors of these projections $\hat{Q}_v$ and $\hat{B}_p$. The ground states are those $|\Phi\rangle$ that satisfy $\hat{Q}_v|\Phi\rangle = |\Phi\rangle = \hat{B}_p|\Phi\rangle$, for all v and p. Using the method developed in Ref. [25], one can compute ground state degeneracy: GSD $= 1$ on sphere and GSD $= 4$ on torus.

The quasiparticle excitations are the states with zero eigenvalue of $\hat{Q}_{v'}$ for some v' and/or of $\hat{B}_{p'}$ for some p'. In this paper we restrict ourselves to study the so-called fluxons, satisfying $\hat{Q}_v = 1$ for all v, $\hat{B}_{p'} = 0$ for a specified set of p' (where fluxons live), and $\hat{B}_p = 1$ for all other p. (See Fig. 1.)

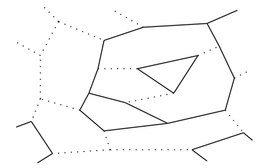

FIG. 1. A configuration in the subspace with only fluxons allowed, with solid lines for $j = 1$ and dotted lines $j = 0$. Open strings (with only single solid line at some vertex) are forbidden by $\delta_{001} = 0$.

III. NUMBER OPERATOR OF FLUXONS

A crucial property of the model is that $\hat{B}_p^s$ defined above forms an Abelian algebra [15]:

$$\hat{B}_p^r \hat{B}_p^s = \sum_{t=0,1} \delta_{rst} \hat{B}_p^t. \tag{5}$$

Here $\delta_{ijk} = \delta_{jki} = \delta_{jik}$ is given by $\delta_{000} = \delta_{011} = \delta_{111} = 1$, arising from the branching rules [Eq. (3)]. From the generators we construct the operators ($i = 0,1$):

$$\hat{n}_p^i = \sum_{j=0,1} S_{i0} S_{ij} \hat{B}_p^j, \tag{6}$$

where S is the modular matrix given by

$$S = \frac{1}{\sqrt{D}} \begin{pmatrix} 1 & \phi \\ \phi & -1 \end{pmatrix}, \tag{7}$$

with S_{ij} for fixed i being a one-dimensional representation of the algebra (5). One can check that $\hat{n}_p^i$ ($i = 0,1$) form a complete set of orthonormal projections:

$$\hat{n}_p^i \hat{n}_p^k = \delta_{ik} \hat{n}_p^k, \quad \hat{n}_p^0 + \hat{n}_p^1 = \mathbf{1}. \tag{8}$$

There is a fluxon at p in a state $|\Psi\rangle$, if $\hat{n}_p^1|\Psi\rangle = |\Psi\rangle$. A ground state $|\Phi\rangle$ contains no fluxon because $\hat{n}_p^0 = \hat{B}_p$. Hence the model has only one type of fluxons, and there is no state with two fluxons living at the same plaquette p. This seems to indicate that the flux in this model should be a fermion (or a hard boson). In the following we will present a study of many-fluxon state counting in this model, which reveals that actually the fluxons in this model obey instead exotic exclusion statistics, which was proposed in Refs. [2] and [3].[1]

IV. EXCLUSION STATISTICS ON A SPHERE

Let us count the N-fluxon states in the model with P plaquettes on a sphere. Pick up a set of N fixed plaquettes and denote it by $\mathcal{C} = \{p_1, p_2, \ldots, p_N\}$ ($N < P$). The states with exactly N fluxons occupying the selected plaquettes are those $|\psi\rangle$ satisfying

$$\hat{n}_p^j|\Psi\rangle = \delta_{j1}|\Psi\rangle, \quad \text{for } p \in \mathcal{C},$$
$$\hat{n}_{p'}^j|\Psi\rangle = \delta_{j0}|\Psi\rangle, \quad \text{for } p' \notin \mathcal{C}. \tag{9}$$

[1]Different state-counting formulas appeared in Ref. [23] for the particular case with Fibonacci data, but no connection to exclusion statistics was made. See also Ref. [24] which, among other things, discusses counting of Fibonacci nets analytically.

TABLE I. State counting on sphere.

Fluxon number N	0	1	2	3	4	5	6	7
State counting $w_{P,N,\mathcal{C}}$	1	0	1	1	4	9	25	64

Thus $(\prod_{p\in\mathcal{C}} \hat{n}^1_p \prod_{p'\notin\mathcal{C}} \hat{n}^0_{p'})$ is the projector onto the subspace of such states. Tracing this projection computes the total number of the N-fluxon states in the configuration $\mathcal{C}$:

$$w_{P,N,\mathcal{C}} = \mathrm{tr}\left(\prod_{p\in\mathcal{C}} \hat{n}^1_p \prod_{p'\notin\mathcal{C}} \hat{n}^0_{p'}\right). \qquad (10)$$

We numerically compute Eq. (10) on random graphs on a sphere with $P(\geqslant 7)$ plaquettes, with the stable result presented in Table I.

The pattern of the N dependence is obvious:

$$w_{P,N,\mathcal{C}} = F^2_{N-1}, \qquad (11)$$

where F_n is the Fibonacci number that satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. Both numerically and analytically we have checked that Eq. (11) is independent of the graph, of the total number P of plaquettes, as well as the locations of the N fluxons. The appearance of the squared in Eq. (11) is consistent with the conjecture that the LW model describes a *doubled* topological phase [21,25].

Summing over configurations $\mathcal{C}$ (i.e., over possible distributions of N plaquettes in a fixed graph), we get the total number of N-fluxon states:

$$W^{\mathrm{sphere}}_{P,N} = \sum_{\mathcal{C}} w_{P,N,\mathcal{C}} = \binom{P}{N} F^2_{N-1}. \qquad (12)$$

The first factor counts the ways to distribute N fluxons over P plaquettes. The second factor counts the states of the link d.o.f., which are not unique, given N and $\mathcal{C}$. The independence of $w_{P,N,\mathcal{C}}$ on P and $\mathcal{C}$ implies the degeneracy of the excited states is topological in the sense that it does not depend on the detailed structure of the underlying graph, and not on the relative positions between the fluxons as well. We have numerically checked particularly this property (see the supplemental material [22]). The origin of this property lies in the topological symmetry of the model under mutations of the underlying graph [25].

To find the exclusion statistics, we rewrite (12):

$$W^{\mathrm{sphere}}_{P,N} = \binom{P}{N} \sum_{N_1,N_2=0}^{[\frac{1}{2}(N-2)]} \binom{N-N_1-2}{N_1}\binom{N-N_2-2}{N_2}, \qquad (13)$$

where $[x]$ is the greatest integer less than or equal to x. Now Eq. (13) is of the form of Eq. (2), by introducing two additional pseudospecies $a = 1,2$, which do not contribute to the total energy but are helpful for state counting. This is similar to what was suggested for state counting in some conformal field theories [20]. Including the original fluxon species labeled by $a = 0$, from Eq. (13) we read the exclusion statistics parameters α_{ab} $(a,b, = 0,1,2)$:

$$\alpha^{\mathrm{sphere}} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}. \qquad (14)$$

The diagonal α_{aa} is the self-exclusion statistics for species a. The $\alpha_{00} = 1$ implies the hard-core boson behavior that takes care of the first combinatoric factor in $\binom{P}{N}$ in Eqs. (12) and (13). This can be understood with Eq. (8).

The pseudospecies provides a way to count states, in the presence of fluxons, of link d.o.f., which are not uniquely determined by the constraints (9). The value $\alpha_{11} = \alpha_{22} = 2$ implies that one pseudoparticle makes two single-particle states (or "seats") unavailable to an additional pseudoparticle. The negative mutual statistics $\alpha_{20} = \alpha_{30} = -1$ tells us that each fluxon present creates one vacant "seat" for each pseudospecies. So the maximum particle number of each pseudospecies is naturally $[(N-1)/2]$. These results help us understand the structure of the (many-body) Hilbert space for excited states of the system, and help derive analytically the state counting formula (13). (A sketch of such a derivation is presented in the supplemental material [22].)

We note that the many-body counting formula Eq. (2), proposed in Ref. [2], with the statistical matrix (14), exactly reproduces the result, Table I, of numerical counting for fluxon numbers from $N = 0$ to $N = 7$. It is remarkable that the counting formula Eq. (2) is valid even for very small values of the fluxon number, so we believe it is an exact result, true for all values of N, including the thermodynamical limit.

V. EXCLUSION STATISTICS ON A TORUS

We proceed to consider the model on a torus. The ground state degeneracy [25] is 4. Thus the system exhibits the global topological d.o.f., and we can study their effects on excited states by counting the pseudoparticle states.

Pick up N plaquettes $(N < P)$. The number of states with N fluxons on these plaquettes is computed numerically as in Table II. The pattern of its dependence on N is

$$W^{\mathrm{torus}}_{P,N} = \binom{P}{N} L^2_N, \qquad (15)$$

with L_n the Lucas number, a modified version of the Fibonacci number, satisfying the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with $L_1 = 1, L_2 = 3$.

We rewrite Eq. (15) in terms of binomial coefficients:

$$W^{\mathrm{torus}}_{P,N} = \binom{P}{N} \sum_{N_1,N_2=0,1} \binom{1}{N_1}\binom{1}{N_2} \times \sum_{N_3,N_4=0}^{[\frac{1}{2}(N-2)]} \binom{N-2N_1-N_3}{N_3}\binom{N-2N_2-N_4}{N_4}, \qquad (16)$$

TABLE II. State counting on torus.

Fluxon number N	0	1	2	3	4	5	6
State counting	2^2	1	3^2	4^2	7^2	11^2	18^2

and get the exclusion statistics parameters α_{ab} $(a,b = 0,1,2,3,4)$:

$$\alpha^{\text{torus}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 2 & 0 \\ -1 & 0 & 2 & 0 & 2 \end{pmatrix}, \quad (17)$$

where we denote by $a = 0$ the fluxon species.

Equation (16) shows that one needs to introduce four pseudospecies $a = 1,2,3,4$. The pseudospecies $a = 1,2$ are interpreted as the topological d.o.f. on the torus, for the following reasons. The allowed "particle number" $N_1,N_2 = 0,1$ of these pseudospecies are independent of the number N of fluxons. Particularly when there is no fluxon present, the configurations $N_1,N_2 = 0,1$ characterize the four degenerate ground states. Then the pseudospecies $a = 3,4$ provide a way to count the states of link d.o.f. given a ground state and fluxon number.

The state counting of excitations on a torus is shown differently from that on a sphere. (A state counting formula of different form from ours, which also exhibits the dependence on the spatial topology, is reported in Ref. [23], without making connection to exclusion statistics.) Indeed the mutual statistics parameters $\alpha_{31} = \alpha_{42} = 2$ imply that the number of states of link d.o.f. $a = 3$ $(a = 4)$ are affected by the topological d.o.f. $a = 1$ $(a = 2)$, respectively. On the other hand, the topological d.o.f. are not affected by the fluxons present and the link d.o.f. So the degenerate ground states can be used to label the sectors of excitations. We note that in the sector with $N_1 = N_2 = 1$, the state counting for fluxons is exactly the same as that on a sphere.

VI. STATISTICAL THERMODYNAMICS

Now we assume that only fluxons can be thermally excited; this is the case when $U \gg \epsilon,kT$ in Eq. (4). In the thermodynamic limit, the Hilbert space dimension of N-fluxon states (occupying N fixed plaquettes) is asymptotically

$$\text{on sphere:} \quad \lim_{N\to\infty} F_{N-1}^2 \sim \phi^{2N-2}/5,$$
$$\text{on torus:} \quad \lim_{N\to\infty} L_N^2 \sim \phi^{2N}. \quad (18)$$

(ϕ^2 is called the quantum dimension of the fluxon.) On a torus, for example, the canonical partition function is

$$Z^{\text{torus}} = \sum_{N=0}^{P} \binom{P}{N} L_N^2 e^{-N\epsilon/kT} \sim (\phi^2 e^{-\epsilon/kT} + 1)^P. \quad (19)$$

It can be interpreted as the grand canonical partition function of the many-fluxon system, which behaves like a fermionic system with a *temperature-independent fugacity* z given by the quantum dimension:

$$z = \phi^2. \quad (20)$$

The fugacity z counts the effective number of states per fluxon located at a plaquette. Note that z is irrational rather than integer. This is a manifestation that the many-fluxon states are highly entangled ones with long-range entanglement. They are superpositions of highly constrained j configurations on the links, obviously not of the form of a direct product of localized fluxon states.

The statistical distribution of the average occupation number of fluxons is obtained from Eq. (19):

$$\langle n\rangle = \langle N\rangle/P = \frac{1}{e^{\epsilon/kT}\phi^{-2} + 1}. \quad (21)$$

Many useful thermodynamic observables are then computable. The probability for thermal excitations of fluxons that cause errors in topological quantum computation, which uses the code based on this model, can then be estimated more accurately than before.

Though the model considered here is very simple, in the sense that the defining branching rules [Eq. (3)] for the Levin-Wen model look simple, we believe the features revealed in this paper should be quite general for emergent exotic exclusion statistics and thermodynamics for quasiparticle excitations in a wide class of 2D topological phases. Moreover, the knowledge and insights gained in this model for the Hilbert space structure of many-fluxon states may be useful in the future for fault-tolerant quantum computation codes and algorithms that explore systems in topological phases.

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