# COMPLEXITY OF COMPUTING TOPOLOGICAL DEGREE OF LIPSCHITZ FUNCTIONS IN N DIMENSIONS

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**Abstract.** We find lower and upper bounds on the complexity, comp(deg), of computing the topological degree of functions defined on the $n$-dimensional unit cube $C^n, f : C^n \rightarrow R^n, n \geq 2$, which satisfy a Lipschitz condition with constant $K$ and whose infinity norm at each point on the boundary of $C^n$ is at least $d$, $d > 0$, and such that $\frac{K}{8d} \geq 1$.

A lower bound, $\text{comp}_{\text{low}} \simeq 2n(\frac{K}{8d})^{n-1}(c+n)$ is obtained for comp(deg), assuming that each function evaluation costs $c$ and elementary arithmetic operations and comparisons cost unity.

We prove that the topological degree can be computed using $A = (\lfloor \frac{K}{2d} + 1 \rfloor + 1)^n - (\lfloor \frac{K}{2d} + 1 \rfloor - 1)^n$ function evaluations. It can be done by an algorithm $\varphi^*$ due to Kearfott, with cost given by $\text{comp}(\varphi^*) \cong A(c + \frac{n^2}{2}(n-1)!)$. Thus for small $n$, say $n \leq 5$, and small $\frac{K}{2d}$, say $\frac{K}{2d} \leq 9$, the degree can be computed in time at most $10^5(c+300)$. For large $n$ and/or large $\frac{K}{2d}$ the problem is intractable.

## 1. INTRODUCTION.

The problem of computing the topological degree of a function has been studied in many recent papers, see Kearfott (1977,1979), Stenger (1975), and Stynes (1979a,1979b,1981). From the topological degree one may ascertain whether there exists a zero of a function inside a domain. Namely, Kronecker's theorem, see Ortega and Rheinboldt (1970), states that if the degree is not zero, then there exists at least one zero of a function inside the domain. By computing a sequence of domains with nonzero degrees and decreasing diameters one can obtain a region with arbitrarily small diameter which contains at least one zero of the function, see Kearfott (1977,1979) and Stynes (1981). Algorithms proposed in these papers were tested by their authors on relatively easy examples. They concluded that the degree of an arbitrary continuous function could be computed. It was observed, however, see Kearfott (1977,1979) and Stynes (1981), that the algorithms may require an unbounded number of function evaluations.

In this paper we restrict the class of functions, which enables us to compute the degree for every element in the restricted class using an a priori bounded number of function evaluations. We consider the class $F$ of Lipschitz functions with constant $K$, defined on the unit cube $C^n \subset R^n$, $f: C^n \to R^n$, such that $\|f(x)\|_\infty \geq d > 0$, for every $x \in \partial C^n$, the boundary of $C^n$, and $\frac{K}{8d} \geq 1$. Note that if $\frac{K}{2d} < 1$ then the functions in $F$ do not have zeros and therefore the degree is zero for every $f \in F$. The case $1 \leq \frac{K}{2d} < 4$ is open. The information on $f$, $N_m(f)$, consists of $m$ values of $f$ on $\partial C^n$ which may be computed adaptively. This form of information is assumed since the topological degree is defined by the values of $f$ on $\partial C^n$, see Ortega and Rheinboldt (1970). The topological degree is computed by means of an algorithm $\varphi$ which is a mapping depending on the information, $\varphi : N_m(F) \to I$, where $I$ denotes the set of all integers.

In this paper we solve the following problems:

(1.1) We exhibit information $N_m^*$ which uniquely determines the degree of $f$ for every $f \in F$.

This information consists of

$$A = \left(\left\lfloor \frac{K}{2d} + 1 \right\rfloor + 1\right)^n - \left(\left\lfloor \frac{K}{2d} + 1 \right\rfloor - 1\right)^n$$

function evaluations, see Sect. 3.

(1.2) We exhibit an algorithm $\varphi^*$ due to Kearfott (1979) which uses $N_m^*$ to compute the degree, see Sect. 4.

(1.3) We find a lower bound $m^*$, roughly equal to $2n(\lfloor \frac{K}{8d} \rfloor)^{n-1}$, on the number of function evaluations necessary to find the degree of $f$ for every $f$ in $F$ using arbitrary information $N_m$, see Sect. 5.

We remark that information $N_m^*$ is *parallel (nonadaptive)*, i.e., the evaluation points are given a priori. Thus $N_m^*$ can be efficiently computed in parallel yielding an almost optimal speed-up, see Traub and Woźniakowski (1984) for further discussion.

Assuming that each function evaluation costs $c$ and elementary operations cost unity, (1.1) yields a lower bound $\text{comp}_{\text{low}}$ on the complexity, comp(deg), of the problem

$$\text{comp}_{\text{low}} \simeq 2n \left(\frac{K}{8d}\right)^{n-1} (c+n) \,.$$

If $\frac{K}{8d}$ is large and/or $n$ is large then the lower bound is so huge that the problem is intractable. For example take $\frac{K}{8d} = 10^3$ and $n = 10$ then the $\text{comp}_{\text{low}} \simeq 2 \cdot 10^{28}(c+10)$.

The cost of algorithm $\varphi^*$ is roughly $A(c + \frac{n^2}{2}(n-1)!)$. Thus for small $n$, say $n \leq 5$ and small $\frac{K}{2d}$, say $\frac{K}{2d} < 10$, $\varphi^*$ computes the degree in time at most roughly $10^5 \cdot (c+300)$.

We remark that in Boult and Sikorski (1985a) (see also Boult (1986)) we find the complexity $\text{comp}_2(\text{deg})$ for the two dimensional case,

$$\text{comp}_2(\text{deg}) = 4 \left\lfloor \frac{K}{4d} \right\rfloor (c+a) - 1 \tag{1.4}$$

where a $\in$ [2,24].

In Boult and Sikorski (1985a) we exhibit an algorithm with cost as (1.4) with $a$=24. This algorithm ($n = 2$) as well as the $n$-dimensional algorithm $\varphi^*$ (for small $n$, $n \geq 3$) exhibited here are implemented in Boult and Sikorski (1985b), see also Boult (1986).

## 2. Basic Definitions

Let $C^n = [0,1]^n$ be the unit cube in $R^n$, $n \geq 2$, $I$ the set of all integers, $||\cdot|| = ||\cdot||_\infty$ the infinity norm in $R^n$ and $\theta = (0,\ldots,0) \in R^n$. For a given positive $d$ and $K$ define

$$(2.1)\qquad \begin{aligned} F = \{f : C^n \to R^n, f = (f_1,\ldots,f_n), ||f(x) - f(y)|| \leq K||x-y||,\\ ,\forall x, y \in C^n \text{ and } ||f(x)|| \geq d, \forall x \in \partial C^n, \text{ and } \frac{K}{8d} \geq 1\}. \end{aligned}$$

Our problem is to find the topological degree, $\deg(f, C^n, \theta)$ of $f$ relative to $C^n$ at $\theta$, see Ortega and Rheinboldt (1970), for every $f$ in $F$. To solve this problem we use *information* $N_m$ and an algorithm $\varphi$ using $N_m$. These are defined as in Traub and Woźniakowski (1980): Let $f \in F$ and

$$(2.2)\qquad N_m(f) = [f(x_1),\ldots,f(x_m)]$$

where $x_1 \in \partial C^n$ is given a priori, $x_j = \tilde{x}_j(f(x_1),\ldots,f(x_{j-1}))$ and $\tilde{x}_j$ is a transformation $\tilde{x}_j : R^{n\cdot(j-1)} \to \partial C^n, j = 2,\ldots,m$. If $\tilde{x}_j$ are constant, i.e. all $x_j$ are given a priori, then the information is called *parallel (nonadaptive)*, otherwise it is called *sequential (adaptive)*.

By *minimal cardinality number* $m_{\min}$ we mean the minimal $m$ for which there exists information $N_m$ which uniquely determines the degree of any $f$ in $F$, i.e.

$$N_m(f) = N_m(f') \Rightarrow \deg(f', C^n, \theta) = \deg(f, C^n, \theta), \forall f, f' \in F.$$

Knowing $N_m$ we approximate $\deg(f, C^n, \theta)$ by an algorithm $\varphi$, which is an arbitrary mapping

$$(2.3)\qquad \varphi : N_m(F) \to I.$$

We exhibit an algorithm $\varphi^*$, using information $N^*_m$( mentioned in the Introduction), which was developed by Kearfott (1979) and is based on his parity theorem.

## 3. INFORMATION $N_A^*$

In this section we prove that the computation of function values on a uniform grid with diameter less than $2\frac{d}{K}$ uniquely determines the degree.

Namely let $M = \lfloor \frac{K}{2d} + 1 \rfloor$ and $R = 1/M$. Subdivide each $(n-1)$ face of $C^n$ into $M^{n-1}$ equal cubes of diameter $R$, by subdividing each edge into $M$ equal intervals of length $R$. In this way we obtain a subdivision of $\partial C^n$ into $2nM^{n-1}$ cubes $C_i$ of diameter $R$:

$$\partial C^n = \bigcup_{i=1}^{2nM^{n-1}} C_i. \tag{3.1}$$

Let $X = \{x_1, \ldots, x_A\}$ be the set of all vertices of cubes $C_i$. Observe that

$$A = (M+1)^n - (M-1)^n.$$

Then define the information operator

$$N_A^* = [f(x_1), \ldots, f(x_A)], \quad \forall f \in F.$$

We show

LEMMA 3.1. *The information $N_A^*$ uniquely determines the degree for every $f$ in $F$, i.e.*

$$N_A^*(f) = N_A^*(g) \text{ implies } \deg(f, C^n, \theta) = \deg(g, C^n, \theta), \quad \forall f, g \in F.$$

∎

PROOF: To prove Lemma 3.1 we use the Poincaré-Bohl Theorem, see Ortega and Rheinboldt (1970). Namely let $h(t,z) = tf(z) + (1-t)g(z), \forall t \in [0,1]$ and $\forall z \in \partial C^n$. To conclude that $\deg(f, C^n, \theta) = \deg(g, C^n, \theta), \forall f, g \in F$ such that $N_A^*(f) = N_A^*(g)$, it is enough to show that the homotopy $h(t,z)$ is non zero for every $t \in [0,1]$ and every $z \in \partial C^n$. To show this take an arbitrary $z \in \partial C^n$. Then there exists an $x_j$ such that

$||x_j - z|| \le R/2 < \frac{d}{K}$. Since $x_j \in \partial C^n$ and $f \in F$ we get $||f(x_j)|| = |f_i(x_j)| \ge d$ for some $i, 1 \le i \le n$. Then we have $|f_i(z) - f_i(x_j)| \le ||f(z) - f(x_j)|| \le K||z - x_j|| < d$. This implies that $f_i(z) \ne 0$ and $\text{sign} f_i(z) = \text{sign} f_i(x_j)$. Since $f(x_j) = g(x_j)$ and $g \in F$, then $g_i(z) \ne 0$ and sign $g_i(z) =$ sign $f_i(z)$. Therefore for every $t \in [0,1]$ we have

$$\begin{aligned} ||h(t,z)|| &\ge |tf_i(z) + (1-t)g_i(z)| \\ &= t|f_i(z)| + (1-t)|g_i(z)| \\ &\ge \min(|f_i(z)|, |g_i(z)|) > 0 \end{aligned}$$

which completes our proof. ∎

## 4. Algorithm Using Information $N_A^*$

We exhibit here an algorithm $\varphi^*$, due to Kearfott (1979), using the information $N_A^*$ to compute the degree. The algorithm $\varphi^*$ and information $N_A^*$ are implemented in a Fortran subroutine in Boult and Sikorski (1985b), where a number of numerical tests are also reported. Fortran Code can be found in the appendices of Boult(1986).

First we show that the evaluation points $x_i, i = 1, \ldots, A$, yield an *impartial refinement* of $\partial C$ relative to the sign of $f$, for every $f$ in $F$.

Impartial refinement, see Stynes (1979a), is defined as follows:

**Definition 4.1:** If $n = 1$ then $\partial[0,1] = \{0\} \cup \{1\}$ is impartially refined relative to sign of $f$ *iff* $f(0) \cdot f(1) < 0$.

If $n > 1$ then $\partial C^n$ is impartially refined relative to the sign of $f$ *iff* $\partial C^n$ may be written as a union of a finite number of $(n-1)$ regions $\beta_1, \ldots, \beta_q$ (by an $(n-1)$ region we mean a union of a finite number of $(n-1)$ dimensional simplices) in such a way that:

(4.1) the $(n-1)$*dimensional* interiors of the regions ($\beta_i' s$) are pairwise disjoint;

(4.2) $\forall i \in [1, \ldots, q], \exists r_i \in [1, \ldots, n] : f_{r_i}$ is of constant sign on $\beta_i$;

(4.3) $\quad\quad$ if $\beta_i \cap \beta_j \neq \emptyset$ for $i \neq j$ then $r_i \neq r_j$,

(4.4) $\quad$ if $S_i$ is an $(n-1)$ simplex in $\beta_i$ such that $S_i$ has an $(n-2)$face in $\partial \beta_i$ then this face is also an $(n-2)$ face of some $(n-1)$ simplex $S_j$ in $\beta_j, i \neq j$.

∎

Now consider the subdivision (3.1) of $\partial C^n$ into $2nM^{n-1}$ $(n-1)$ dimensional cubes $C_i$, and subdivide each $C_i$ into $(n-1)!$ $(n-1)$ dimensional simplices (hereafter we shall use the term (n-1) simplices) as described in Jeppson (1972). This forms a simplicial subdivision of $\partial C^n$, see Allgower et. al. (1971) and Jeppson (1972), into $2nM^{n-1}(n-1)!$ $(n-1)$-simplices:

$$\partial C^n = \sum_{j=i}^{L} t_j S_j, \quad t_j = \pm 1, \quad L = 2n\, M^{n-1}(n-1)!, \tag{4.5}$$

where $S_j$ are oriented $(n-1)$-simplices, see Kearfott (1979), and Stynes (1979a,1979b,1981). Note that the vertices of $S_j$ are uniquely determined by this subdivision and the evaluation points $x_i$. The explicit formulas for the vertices of $S_j$'s are given by Allgower et. al. (1971) Jeppson (1972) and Fortran code generating them can be found in Boult and Sikorski (1985b) and Boult (1986).

We are now ready to prove:

LEMMA 4.1. *The subdivision (4.5) yields an impartial refinement of $\partial C^n$ relative to the sign of $f$, for every $f$ in $F$.* ∎

PROOF: We construct the regions $\beta_i$ from Definition 4.1. For an arbitrary $f$ in $F$ and for each cube $C_i$ in the subdivision (3.1) choose a component $f_{j_i}$ of $f$ which is of constant sign on $C_i$. Such a component exists since for some $j_i$, $|f_{j_i}(z_i)| \geq d$ where $z_i$ is the center of $C_i$. Thus $f_{j_i}$ is of constant sign on $C_i$ since the radius of $C_i$ is less than $\frac{d}{K}$ and $f$ is in $F$, i.e. $|f_{j_i}(z) - f_{j_i}(z_i)| \leq ||f(z) - f(z_i)|| \leq K||z - z_i|| < d$ for $||z - z_i|| < \frac{d}{K}$, which yields $\text{sgn}(f_{j_i}(z)) = \text{sgn}(f_{j_i}(z_i))$. Then group the cubes $C_i$ to form connected regions $\beta_{j,1}, \ldots, \beta_{j,k_j}$ such that $f_j$ is of constant sign on each $\beta_{j,l}, l = 1, \ldots, k_j$, and $\beta_{j,l_1} \cap \beta_{j,l_2} = \emptyset, \quad l_1 \neq l_2$. In this way we obtain a decomposition of $\partial C^n$

$$\partial C^n = \bigcup_{j=1}^{n} \bigcup_{l=1}^{k_j} \beta_{j,l}, \tag{4.6}$$

which satisfies (4.1)-(4.3) of Def. 4.4. Since each cube in every $\beta_{j,l}$ is subdivided into $(n-1)$ simplices forming a simplicial subdivision of $\partial C$ then (4.4) of defintion (4.1) is also met. This completes the proof. ∎

**Remark 4.1:** Since the impartial refinement of $\partial C^n$ is also a sufficient refinement (see Kearfott (1979) and Stynes (1979a, 1979b, 1981) for the definition of sufficient refinement, and Stynes (1979a, Th. 3.3) for the above result) then we can use Kearfott's Parity theorem, see Kearfott (1979), to compute the degree. ∎

Let $S = [S_1, \ldots, S_n]$ be an $(n-1)$ simplex in $R^n$ with vertices $S_i, i = 1, \ldots, n$. The range matrix $R(S, f)$ associated with $S$ and $f \in F$ is an $n$ x $n$ matrix:

$$R(S, f) = [r_{i,j}]_{i,j=1}^{n}, \quad r_{i,j} = \text{sgn}(f_j(S_i)),$$

where

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The range matrix $R(S, f)$ is called feasible if and only if

$$\begin{aligned} & r_{ij} = 1, \quad \forall i \geq j \quad \text{and} \\ & r_{i,i+1} = 0, \quad i = 1, \ldots, n-1. \end{aligned} \tag{4.7}$$

Define the parity of the range matrix $R(S,f)$ by

$$\mathrm{Par}(R(S,f)) = \begin{cases} 1 & \text{if } R(S,f) \text{ is feasible after an even permutation of rows;} \\ -1 & \text{if } R(S,f) \text{ is feasible after an odd permutation of rows;} \\ 0 & \text{otherwise.} \end{cases}$$

We remark that the parity can be computed with roughly $n^2/2$ comparisons.

Define the algorithm $\varphi^*$ using $N_A^*$ by

$$\varphi^*(N_A^*(f)) = \sum_{j=1}^{L} \mathrm{Par}(R(t_j S_j, f)), \tag{4.8}$$

where $L$ and $t_j S_j$ are as in (4.5). Then Remark 4.1 and the Parity Theorem, see Kearfott (1979), imply that

$$\deg(f, C^n, \theta) = \varphi^*(N_A^*(f)), \qquad \forall f \in F.$$

Observe that implementation of $\varphi^*$ requires computing the parities of $L = 2nM^{n-1}(n-1)!$ $(n-1)$simplices. Thus the complexity of $\varphi^*$ is at most

$$\mathrm{comp}(\varphi^*) \leq Ac + 2nM^{n-1} \cdot \frac{n^2}{2}(n-1)! \leq A\left(c + \frac{n^2}{2}(n-1)!\right)$$

where $c$ is the cost of one function evaluation and arithmetic operations and comparisons cost unity.

## 5. A Lower Bound

In this section we find a lower bound on the number of function evaluations needed to compute the topological degree of functions from the class $F$.

THEOREM 5.1. *For any information* $N_m$, *with* $m \leq 2n\lfloor \frac{K}{8d} \rfloor^{n-1} - 1$, *there exist two functions* $f^*$, $f^{**}$ *in* $F$ *such that* $N_m(f^{**}) = N_n(f^*)$, $|deg(f^*, C^n, \theta)| = 1$ *and* $deg(f^{**}, C^n, \theta) = 0$. ∎

Note that Theorem 5.1 implies (1.3), i.e. to compute the degree for any $f \in F$ using arbitrary information $N_m$ we must use at least $m = 2n(\lfloor \frac{K}{8d} \rfloor)^{n-1}$ function evaluations. This lower bound is exponential in the dimension $n$, thus for large $n$ and/or large $\frac{K}{8d}$ the problem is intractable.

In order to prove Theorem 5.1 we need the following lemma.

LEMMA 5.1. *Let* $H^n$ *be an* $n$*-cube in* $C^n$ *with diameter* $8\frac{d}{K} \leq 1$ *such that:*

(5.1)
$$B^n = H^n \cap \partial C^n \text{ is an } (n-1) \text{ face of } H^n, \text{ and corresponding } (n-1) \text{ faces of } H^n \text{ and } C^n \text{ are parallel.}$$

*Then there exist a function* $f^n \in F, f^n = (f_1^n, \ldots, f_n^n)$, *such that:*

(5.2)
$$\text{there exists exactly one zero } \alpha^n \text{ of } f^n,\ ||\alpha^n - b^n|| = d/K, \text{ where } b^n \text{ is the center of } B^n, \text{ and } dist(\alpha^n, B^n) = d/K;$$

(5.3)
$$f_j^n(z) = d \text{ for } z \in C^n - H^n, \quad \forall_j;$$

(5.4)
$$||f^n(z)|| = d \text{ for } z \in \partial C^n;$$

(5.5)
$$\left.\frac{\partial f_j^n}{\partial z_i}\right|_{\alpha^n} = \pm K\delta_{ij} \text{ where } \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j, \end{cases}$$

*which implies that* $\alpha$ *is a simple zero;*

(5.6)
$$-d \leq f_j^n(z) \leq d, \quad \forall z \in C^n, \quad \forall j;$$

(5.7)
$$\forall z \in C^n : ||z - b^n|| \geq 2\frac{d}{K}, \quad \exists j : f_j^n(z) = d.$$ ∎

PROOF: The proof is by induction on $n$. Let $n = 2$ and let $H^2$ be a square satisfying (5.1). Without loss of generality assume that $B^2 \subset [0,1]$, $B^2 = [b_1, b_2]$, so $b^2 = ((b_1 + b_2)/2, 0)$. Let $c_1 = b^2 + (\frac{d}{K}, \frac{d}{K})$ and $c_2 = b^2 - (\frac{d}{K}, \frac{d}{K})$. Define the function $f^2 : C^2 \to R^2$ by:

$$f^2(z) = (f_1^2(z), f_2^2(z))$$

$$f_1^2(z) = \min(d, \max(-d, -2d + K\|z - c_1\|)),$$

$$f_2^2(z) = \min(d, \max(-d, -2d + K\|z - c_2\|)),$$

see Fig. 5.1.

Observe that $f^2$ satisfies a Lipschitz condition with constant $K$ and that $\alpha^2 = b^2 + (-\frac{d}{K}, \frac{d}{K})$ is the unique zero of $f^2$. Thus $\mathrm{dist}(\alpha^2, B^2) = \frac{d}{K}$ and $\|\alpha^2 - b^2\| = \frac{d}{K}$, which implies (5.2). The definition of $f^2$ directly yields (5.3), (5.4), (5.6), and (5.7). For (5.5) observe that

$$\left.\frac{\partial f_1^2}{\partial z_i}\right|_{\alpha^2} = K\delta_{i,1} \text{ and } \left.\frac{\partial f_2^2}{\partial z_i}\right|_{\alpha^2} = K\delta_{i,2}, i = 1, 2.$$

Thus the lemma holds for $n = 2$.

INDUCTION STEP.

Now assume that Lemma 5.1 holds for $n - 1$. Let $H^n \subset C^n$, $\mathrm{diam}(H_n) = 8\frac{d}{K}$, be an $n$-cube such that (5.1) holds. Without loss of generality assume that all points in $B^n$ have the $l$-th ($l \neq n$) component equal to 1. (If $l = n$ then the same construction follows with the $n$-th dimension replaced by the first dimension).

Let $H^{n-1}$ be the orthogonal projection of $H^n$ onto $C^{n-1}$. From the induction assumption there exists $f^{n-1}$ for $H^{n-1}$ such that (5.1)-(5.6) hold. Define, (see Fig. 5.2),

$$\alpha^n = (\alpha_1^n, \ldots, \alpha_n^n), \tag{5.8}$$

where $\alpha_j^n = \alpha_j^{n-1}$, $j = 1, \ldots, n-1$, $\alpha_n^n = b_n^n - \frac{d}{K}$ and $b^n = (b_1^n, \ldots, b_n^n)$ is the center of $B^n$.

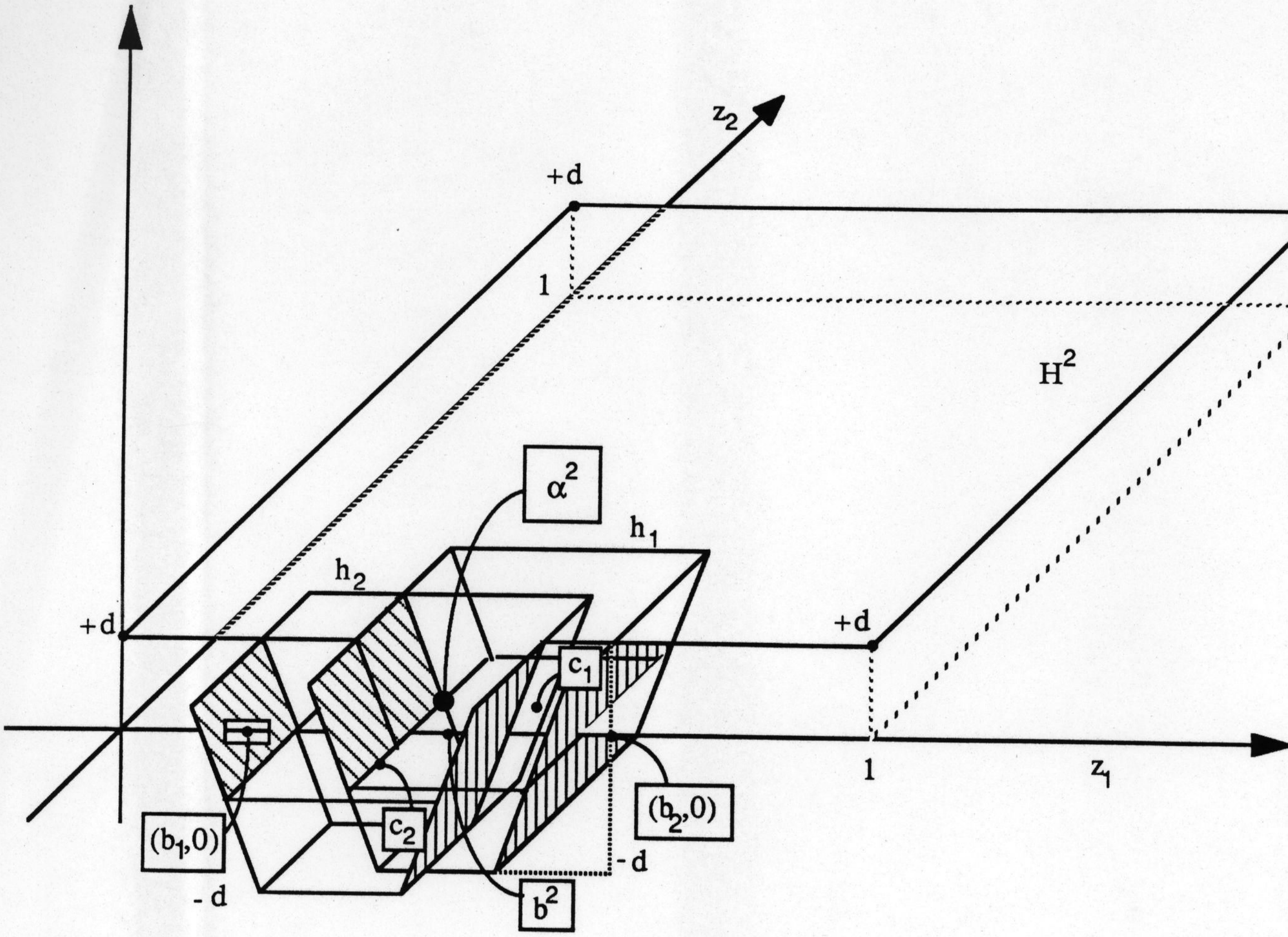


**Figure 5.1**

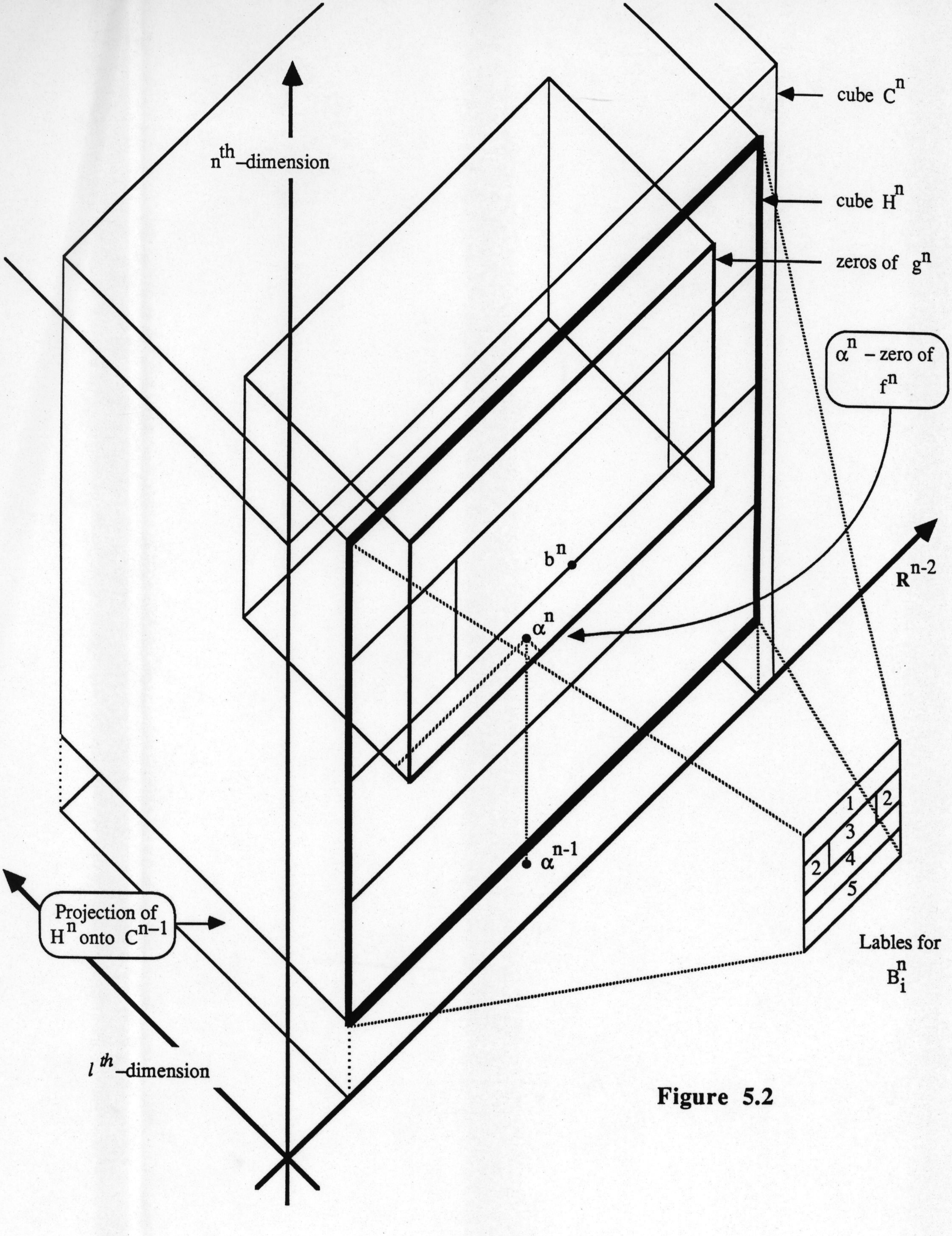
cube $C^n$
$n^{th}$–dimension
cube $H^n$
zeros of $g^n$
$\alpha^n$ – zero of $f^n$
$b^n$
$R^{n-2}$
$\alpha^n$
1
2
3
2
4
5
$\alpha^{n-1}$
Projection of $H^n$ onto $C^{n-1}$
Lables for $B_i^n$
$l^{th}$–dimension

**Figure 5.2**

Let

$$y(i_1,\ldots,i_{n-2}) = \Big(b_1^n + i_1 \cdot \frac{d}{K}, \ldots, b_{l-1}^n + i_{l-1} \cdot \frac{d}{K},$$

$$1,\, b_{l+1}^n + i_{l+1} \cdot \frac{d}{K}, \ldots, b_n^n + \frac{d}{K}\Big) \tag{5.9}$$

where $i_j \in \{+1,-1\}, j = 1,\ldots,n-2$, i.e. these are $2^{n-2}$ points in $B^n$.

Define the function $g^n, g^n : C^n \to R$, by

$$g^n(z) = \min(d, \tilde{y}_1(z), \ldots, \tilde{y}_{n-2}(z)), \tag{5.10}$$

where $\tilde{y}_i(z) = \max(-d, -2d, +K||z - y_i||)$, and $y_i, i = 1,\ldots,2^{n-2}$, are all of the points $y(i_1,\ldots,i_{n-2})$. Observe that $g^n$ satisfies a Lipschitz condition with constant $K$ since it is obtained by taking the minimum of Lipschitz functions with constant K. Also note that the zero set of $g^n, Z_0 = \{z \in C^n : g^n(z) = 0\}$, (see Figure 5.2) is given by

$$Z_0 = \left\{z \in C^n : \exists i : 2\frac{d}{K} = ||z - y_i|| \le ||z - y_j||,\ \forall j = 1,\ldots,2^{n-2}\right\}.$$

Finally for $z \in C^n, z = (z_1,\ldots,z_n)$, let $\tilde{z} = (z_1,\ldots,z_{n-1})$ be the orthogonal projection of $z$ onto $C^{n-1}$.

Define

$$f^n(z) = (f_1^n(z),\ldots,f_n^n(z)),$$

where

$$f_n^n(z) = \begin{cases} g^n(z), & \forall z \in H^n \\ d, & \forall z \in C^n - H^n, \end{cases}$$

and

$$f_i^n(z) = \begin{cases} \min(d, \max(f_i^{n-1}(\tilde{z}), -d + K|z_n - (b_n^n - \frac{2d}{K})|)), & \text{for } z \in H^n : \\ & b_n^n - 4\frac{d}{K} \le z_n \le b_n^n - 2\frac{d}{K}, \\ f_i^{n-1}(\tilde{z}), & \text{for } z \in H^n : \\ & b_n^n - 2\frac{d}{K} < z_n < b_n^n; \\ \min(d, \max(f_i^{n-1}(\tilde{z}), -d + K|z_n - b_n^n|)), & \text{for } z \in H^n : \\ & b_n^n \le z_n \le b_n^n + \frac{4d}{K}; \\ d & \text{for } z \in C^n - H^n, \end{cases}$$

for $i = 1, 2, \ldots, n-1$.

Now we show that $f^n$ is in $F$ and satisfies (5.2)-(5.7).

First we check that $f^n$ is continuous. Since for every $z \in C^n -$ Int$(H^n), ||z - b^n|| \geq 4\frac{d}{K}$, then $\tilde{y}_j(z) \geq -2d + K||z - y_i|| \geq -2d + K \cdot 3\frac{d}{K} = d \ \forall j = 1, \ldots, 2^{n-2}$, and therefore $g^n(z) = d$. This and continuity of $g^n$ implies that $f_n^n$ is continuous. Thus we must only check the continuity of $f_i^n$, $i = 1, \ldots, n-1$, at all $z \in C^n - H^n$ and $z \in H^n$ such that $z_n = b_n^n$ or $z_n = b_n^n - 2\frac{d}{K}$. First let $z \in C^n - H^n$. If $z_n = b_n^n - 4\frac{d}{K}$ then $f_i^n(z) = \min(d, \max(f_i^{n-1}(\tilde{z}), d)) = d$. If $z_n = b_n^n + 4\frac{d}{K}$ then $f_i^n(z) = \min(d, \max(f_i^{n-1}(\tilde{z}), 3d)) = d$. If $b_n^n - 4\frac{d}{K} < z_n < b_n^n + 4\frac{d}{K}$ then $\tilde{z} \in \overline{(C^{n-1} - H^{n-1})}$ and from the induction assumption $f_i^{n-1}(\tilde{z}) = d$ which implies $f_i^n(z) = d$. For $z \in H^n$ such that $z_n = b_n^n$ we have $f_i^n(z) = \min(d, \max(f_i^{n-1}(\tilde{z}), -d)) = f_i^{n-1}(\tilde{z})$, i.e. $f_i^n$ is continuous. For $z \in H^n$ such that $z_n = b_n^n - 2\frac{d}{K}$ we have $f_n^n(z) = \min(d, \max(f_i^{n-1}(\tilde{z}), -d)) = f_i^{n-1}(\tilde{z})$, i.e. $f_i^n$ is continuous.

Thus all of $f_i^n$ are continuous which implies continuity of $f^n$. The function $f^n$ satisfies a Lipschitz condition with constant $K$ since it is defined by taking minima and maxima of Lipschitz functions with constant $K$. Now we show that $\alpha^n$ is the only zero of $f^n$. Obviously $f^n$ can have zeros only inside $H^n$. Let $z \in H^n$ be such that:

$$z_n \leq b_n^n - \frac{2d}{K}. \tag{5.11}$$

Then $|z_n - (b_n^n + \frac{d}{K})| \geq 3\frac{d}{K}$, so $||z - y_i|| \geq 3\frac{d}{K}$, $\forall y_i, i = 1, \ldots, 2^{n-2}$. This yields that $g^n(z) = \min(d, d, \ldots, d) = d$, thus $f^n$ has no zeros in this domain.

Take

$$z_n \geq b_n^n + 2\frac{d}{K}. \tag{5.12}$$

Then $|z_n - b_n^n| \geq 2\frac{d}{K}$ which combined with the induction assumption $f_i^{n-1}(\tilde{z}) \leq d$ yields $f_i^n(z) = \min(d, \max(f_i^{n-1}(\tilde{z}), d)) = d$. Thus $f_n$ has no zeros in this domain.

Take now

$$b_n^n - 2\frac{d}{K} \leq z_n \leq b_n^n. \tag{5.13}$$

In this domain, by the induction assumption the only zeros of $f_j^n, j = 1, \dots, n-1$ are $(\alpha_1^{n-1}, \dots, \alpha_{n-1}^{n-1}, \dots, \alpha_{n-1}^{n-1}, z_n)$. But $g^n$ is zero only for one of these points, namely with $z_n = b_n^n - \frac{d}{K}$. To see this recall that $|\alpha_j^{n-1} - b_j^n| \le \frac{d}{K}$, $j = 1, \dots, n-1$ and $\alpha_n^n = b_n^n - \frac{d}{K}$, thus by the definition (5.9) $||\alpha^n - y_i|| = 2\frac{d}{K}$ for $i = 1, \dots, 2^{n-2}$ so $g^n(\alpha^n) = 0$. For every $z \in H^n$ such that $b_n^n - \frac{d}{K} < z_n < b_n^n$ there exists a $y_i$ with $i_q = 1$ for $\alpha_q^{n-1} \ge b_q^n$ and $i_q = -1$ for $\alpha_q^{n-1} < b_q^n$ such that $||y_i - z|| < 2\frac{d}{K}$, thus $g^n(z) < 0$. For every $z \in H^n$ such that $b_n^n - 2\frac{d}{K} < z_n \le b_n^n - \frac{d}{K}$ and for every $y_i$ we have $||y_i - z|| > 2\frac{d}{K}$, thus $g_n^n(z) > 0$. Therefore $\alpha^n$ is the only zero of $f_n^n$ in this domain.

For

$$b_n^n < z_n \le b_n^n + 2\frac{d}{K}, \tag{5.14}$$

We shall take any $z$ such that $g^n(z) = 0$ and show that $\exists i \in \{1, \dots, n-1\}$ such that $f_i^n(z) \ne 0$. Observe first that if $|z_j - b_j^n| \ge 2\frac{d}{K}$ for some $j = 1, \dots, n-1$, then $||z - b^{n-1}|| \ge 2\frac{d}{K}$ and from the induction assumption (5.7) there exists an $i$ such that $f_i^{n-1}(\tilde{z}) = d$, which implies $f_i^n(z) = d$ since $|z_n - b_n^n| \le 2\frac{d}{K}$. Thus assume that $|z_j - b_j^n| < 2\frac{d}{K}$, $\forall\, j = 1, \dots, n-1$, and take $z$ such that $g^n(z) = 0$. This means that

$$\forall j, \quad 1 \le j \le 2^{n-2}, \quad ||z - y_j|| \ge 2\frac{d}{K}, \tag{i}$$

and

$$\exists j' : ||z - y_{j'}|| = 2\frac{d}{K}. \tag{ii}$$

Suppose that $y_{j'} = y(i_1, \dots, i_{n-1})$ where

$$\left| z_q - b_q^n + i_q \cdot \frac{d}{K} \right| = 2\frac{d}{K} \text{ for } q \in Q_1,$$

and

$$\left| z_q - b_q^n + i_q \cdot \frac{d}{K} \right| < 2\frac{d}{K}, \text{ for } q \in Q_2,$$

where $Q_1 \neq \emptyset$ and $Q_1 \cup Q_2 = \{1, \ldots, j-1, j+1, \ldots, n-1\}$. Thus for every $q \in Q_1$, we have $|z_q - b_q^n| = 3\frac{d}{K}$ or $|z_q - b_q^n| = \frac{d}{K}$. If $\exists q \in Q_1$ such that $|z_q - b_q^n| = 3\frac{d}{K}$ then $||z - b^n|| \geq 3\frac{d}{K}$ and (5.7) implies that $f_i^n(z) = d$ for some $i$. Otherwise, (i.e. if $|z_q - b_q^n| = \frac{d}{K}$ for all $q \in Q_1$), $z_q = b_q^n \pm \frac{d}{K}$. Then take $y(i_1, \ldots, i_{n-2})$ such that $i_q$ are as above for $q \in Q_2$, and for $q \in Q_1$ take $i_q = +1$ if $z_q = b_q^n + \frac{d}{K}$ and $i_q = -1$ if $z_q = b_q^n - \frac{d}{K}$. This implies that $||y(i_1, \ldots, i_{n-2}) - z|| < 2\frac{d}{K}$ which contradicts (i) and completes the proof of the existence and uniqueness of the zero of $f^n$.

Obviously $||\alpha^n - b^n|| = \frac{d}{K}$ since $\alpha^n = b_n^n - \frac{d}{K}$ and $|\alpha_i^n - b_i^n| \leq \frac{d}{K}$ for $i = 1, \ldots, n-1$. Also note $\operatorname{dist}(\alpha^n, B^n) = \operatorname{dist}(\alpha^{n-1}, B^{n-1}) = \frac{d}{K}$, thus (5.2) holds.

Equations (5.3) and (5.7) follow immediately from the definition of $f^n$ and the continuity.

Now we show that (5.4) holds. Obviously (5.3) implies (5.4) for $z \in \partial C^n - B^n$. Therefore let $z \in B^n$ and subdivide $B^n$ into 5 regions, $B_i^n, i=1, \ldots, 5$ (see Figure 5.2), where

$$B_n = \bigcup_{i=1}^{5} B_i^n$$

and

$$B_1^n = \left\{ z \in B^n : z_n \geq b_n^n + 2\frac{d}{K} \right\},$$

$$B_2^n = \left\{ z \in B^n : b_n^n \leq z_n < b_n^n + 2\frac{d}{K} \text{ and } ||z - b^n|| \geq 2\frac{d}{K} \right\},$$

$$B_3^n = \left\{ z \in B^n : b_n^n \leq z_n < b_n^n + 2\frac{d}{K} \text{ and } ||z - b^n|| < 2\frac{d}{K} \right\},$$

$$B_4^n = \left\{ z \in B^n : b_n^n - 2\frac{d}{K} \leq z_n < b_n^n \right\},$$

$$B_5^n = \left\{ z \in B^n : z_n < b_n^n - 2\frac{d}{K} \right\}.$$

Then recall that (5.6) holds and

(a) $\forall z \in B_1^n$, by an argument similar to that following (5.12) we have $f_i^n(z) = d$, $i = 1, \dots, n-1$, thus $||f^n(z)|| = d$.

(b) $\forall z \in B_2^n$ the same argument as follows (5.14) yields $f_i^n(z) = d$ for some $i \in \{1, \dots, n-1\}$.

(c) $\forall z \in B_3^n$ we have $|z_n - (b_n^n + \frac{d}{K})| \leq \frac{d}{K}, |z_i - b_i^n| < 2\frac{d}{K}$ and obviously $z_l = b_l^n = 1$. Let $Q_1 = \{i : z_i \geq b_i^n\}$ and $Q_2 = \{i : z_i < b_i^n\}$. Then for $i \in Q_1$ we have $|z_i - (b_i^n + \frac{d}{K})| \leq \frac{d}{K}$, and for $i \in Q_2$ we have $|z_i - (b_i^n - \frac{d}{K})| \leq \frac{d}{K}$. Thus for $y(i_1, \dots, i_{n-2})$ such that $i_q = 1$ for $q \in Q_1$ and $i_q = -1$ for $q \in Q_2$ we get $||y(i_1, \dots, i_{n-2}) - z|| \leq \frac{d}{K}$ which yields $\tilde{y}_i(z) = -d$ for some $1 \in \{i, \dots, 2^{n-2}\}$, i.e. $g^n(z) = -d$, thus $||f^n(z)|| = d$.

(d) $\forall z \in B_4^n$ the induction assumption (5.4) and the definition of $f^n$ yield $||f^{n-1}(\tilde{z})|| = d$, therefore $||f^n(z)|| = d$.

(e) $\forall z \in B_5^n$ (5.11) implies that $g^n(z) = d$,therefore $||f^n(z)|| = d$.

Thus $\forall z \in \partial C^n$ we have $||f^n(z)|| = d$, which completes the proof of (5.4).

For (5.5) note that for $z$ close to $\alpha^n$ by the induction assumption and definition of $f_i^n$ we have $\frac{\partial f_i^n}{\partial z_n}(z) = 0, \quad \forall i = 1, \dots, n$. Thus we need to show only $\left.\frac{\partial g^n}{\partial z_i}\right|_{\alpha^n} = \pm K \cdot \delta_{in}, i = 1, \dots, n$. Let $y(i_1, \dots, i_{n-2})$ be such that for $\alpha_q^n \geq b_q^n$ we have $i_q = +1$ and for $\alpha_q^n < b_q^n$ we have $i_q = -1$. Then $|\alpha_q^n - (b_q^n + i_q \cdot \frac{d}{K})| \leq \frac{d}{K}$ since $||\alpha^n - b^n|| \leq \frac{d}{K}$, and obviously $|\alpha_j^n - b_j^n| \leq \frac{d}{K}$. For $z$ in a small neighborhood of $\alpha^n$ we have

$$\begin{aligned} g^n(z) &= \min_{i=1,\dots,2^{n-2}} (-2d + K \cdot ||z - y_i||) \\ &= -2d + K \cdot \min_{i=1,\dots,2^{n-2}} ||z - y_i|| \\ &= -2d + K \cdot ||z - y(i_1, \dots, i_{n-2})|| \quad . \end{aligned}$$

Thus $g^n(z) = -2d + K \cdot |z_n - (b_n^n + \frac{d}{K})|$, and therefore

$$\left.\frac{\partial g^n}{\partial z_i}\right|_{\alpha^n} = \left.\frac{\partial}{\partial z_i}[K \cdot (-z_n + (b_n^n + \frac{d}{K}))]\right|_{\alpha^n} = -K \cdot \delta_{\text{in}},$$

which shows (5.5).

Now we show (5.7). Observe that (5.3) implies (5.7) for $z \in C^n - H^n$. For $z \in H^n$ such that $z_n \le b_n^n - 2\frac{d}{K}$ or $z_n \ge b_n^n + 2\frac{d}{K}$ (5.7) follows directly from the proofs following (5.11) and (5.12). For $z \in H^n$ such that $b_n^n - \frac{2d}{K} < z_n \le b_n^n$ and $\|z - b^n\| \ge 2\frac{d}{K}$ we have $\|\tilde{z} - b^{n-1}\| \ge 2\frac{d}{K}$ and then by the induction assumption there exists $j$ such that $f_j^{n-1}(\tilde{z}) = d$, so $f_j^n(z) = d$. For $z \in H_n$ such that $b_n^n < z_n \le b_n^n + 2\frac{d}{K}$ and $\|z - b^n\| \ge 2\frac{d}{K}$ as in (5.14) we have $\|\tilde{z} - b^{n-1}\| \ge 2\frac{d}{K}$ and by the induction assumption there exists $j$ such that $f_j^{n-1}(\tilde{z}) = d$ which combined with the definition of $f^n$ yields $f_j^n(z) = d$, which completes the proof of (5.7).

The function $f^n$ is in $F$ since it satisfies a Lipschitz condition with constant $K$ and its norm is exactly $d$ on the boundary of $C^n$ (see (5.4)).

This finally completes the proof of Lemma 5.1. (We bet you thought it would never end.) ∎

We are now ready to prove Theorem 5.1.

First let $P = \lfloor \frac{K}{8d} \rfloor$ and we show that for every $f$ in $F$ and every sequential (adaptive) information $N_m(f) = [f(x_1), \ldots, f(x_m)]$, with $m \le 2nP^{n-1} - 1$, there exists a cube $H^n \subset C^n$ with $\text{diam}(H^n) = 8\frac{d}{K}$ satisfying (5.1), and such that *no point* $x_i$ belongs to $B^n$. Indeed, subdivide the boundary of $C^n$ into $2nP^{n-1}(n-1)$ cubes of diameter $1/P$ by subdividing uniformly each $(n-1)$ face of $C^n$ into $P^{n-1}(n-1)$-cubes. Then since $m \le 2nP^{n-1} - 1$ there must exist at least one $(n-1)$-cube in this subdivision, say $\tilde{B}^n$, which does not contain any of the $x_i$ points. Since diam $\tilde{B}^n = 1/P \ge 8\frac{d}{K}$, take as $B^n$ any $(n-1)$ cube of diameter $8\frac{d}{K}$, contained in $\tilde{B}^n$, with faces parallel to the corresponding faces of $\tilde{B}^n$.This $B^n$ is obviously an $(n-1)$ face of a cube $H^n$ satisfying (5.1).

for the function $f = f^{**}$

Let $f^{**}(z) = [d, \ldots, d], \forall z \in C^n$, and let $H^n$ be constructed as above. Let $f^* = f^n$ from Lemma 5.1 applied to this cube $H^n$. Observe that

$$N_m(f^{**}) = N_m(f^*), \tag{5.15}$$

since for every $x_i$, $f^{**}(x_i) = f^*(x_i) = [d, \ldots, d]$. Moreover there exists a unique zero $\alpha^n$ of $f^*$. Let $D$ be an open neighborhood of $\alpha^n$ such that $f^*$ is continuously differentiable in $D$. Then since $\alpha^n$ is a simple zero of $f^*$, the degree $\deg(f^*, D, \theta) = \pm 1$. Also since $f^*$ has no zeros in $C^n - D$ then $\deg(f^*, C^n - D, \theta) = 0$. Thus by the additivity of degree we get

$$\deg(f^*, C^n, \theta) = \deg(f^*, D, \theta) + \deg(f^*, C^n - D, \theta) = \pm 1.$$

Obviously $\deg(f^{**}, C^n, \theta) = 0$, which combined with (5.15) completes the proof.

∎

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