Tiling the Sphere with Rational Bezier Patches

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July 11, 1988

UUCS-88-009

1 Introduction

One of the fundamental problems in Computer Aided Geometric Design (CAGD) is the representation of shapes. Two representation schemes that have proved useful for modeling free-form shapes are parametric Bezier and B-spline surfaces [2,8]. In fact the Bezier patch is a special case of the B-spline surface. Therefore remarks below about B-spline surfaces apply as well to Bezier patches. For some modeling systems the B-spline or Bezier representation is the base upon which other shape descriptions rest. For example, the Unisurf system [2] uses Bezier patches, and the Alpha_1 system relies on B-splines. For such a modeling system it is necessary to provide adequate representation of simple shapes (e.g. spheres, ellipsoids, and cones) in terms of the more general scheme. One would like the underlying representation to be exact, with accuracy limited only by the numeric representation within the computer, not by the choice of representation. Furthermore, this representation should avoid degeneracies that would impair the robustness of the modeling system.

One primitive shape that has been somewhat of a problem in this regard is the sphere. It is impossible to exactly represent a portion of a sphere using a polynomial B-spline. To have an exact representation it is necessary to use rational patches. The easiest B-spline description of the sphere is a surface of revolution of a semicircle. This forms a bi-quadratic rational B-spline surface. If the semicircle and the circle of rotation are both made up of rational Bezier quarter circles joined together, the corresponding Bezier patch covers an octant of the sphere. Unfortunately, this patch is degenerate—an entire edge of the parameter space collapses to a single point (the "north pole"). This degeneracy makes it difficult to compute the surface normal and other surface properties at the degenerate point. Normal computations are necessary for shade calculations when rendering a surface, for providing information about tool placement when generating a program for a numerically controlled milling machine, and for local classification of space for boolean operations [10].

In this paper we develop alternative patches for tiling the sphere, which avoid such degeneracies. The technique of function composition of Bezier patches with a polynomial map is employed to construct these alternatives.

*This work was supported in part by DARPA (DAAK1184K0017) and the National Science Foundation (MCS-8121750). All opinions, findings, conclusions or recommendations expressed in this document are those of the author and do not necessarily reflect the views of the sponsoring agencies.

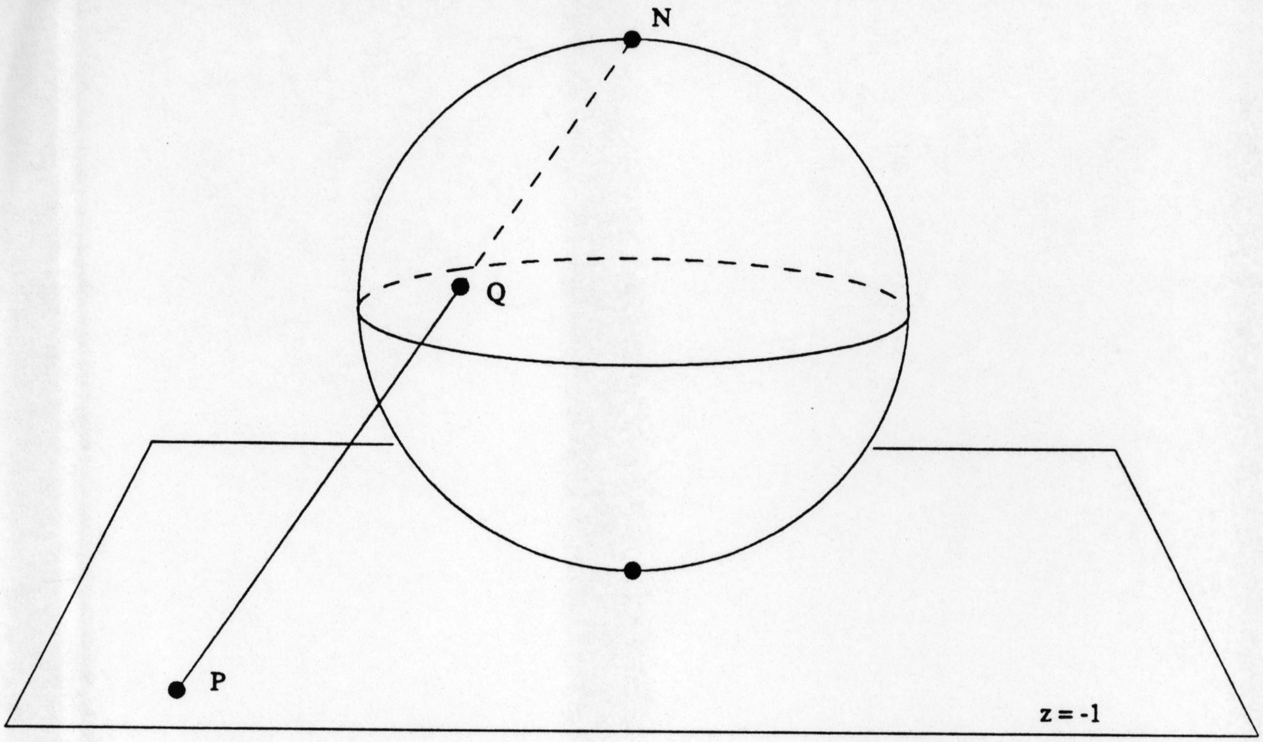


Figure 1: Stereographic mapping.

2 The Stereographic Map from the Plane to the Sphere

Our foundation for building sphere patches will be the stereographic mapping of a plane to the sphere [7, pp. 248-251]. Let S be the unit sphere in $\mathbf{E}^3$ (euclidean 3-space)

$$S = \left\{(x, y, z) \in \mathbf{E}^3 \mid x^2 + y^2 + z^2 = 1\right\}.$$

Let N be the north pole $(0,0,1)$. And let $\mathcal{P}$ be the plane

$$\mathcal{P} = \left\{(x, y, z) \in \mathbf{E}^3 \mid z = -1\right\}.$$

The stereographic mapping $s : \mathbf{E}^2 \rightarrow S$ is defined as follows. For a point $(u, v) \in \mathbf{E}^2$, there is a corresponding point $P = (u, v, -1) \in \mathcal{P}$. The line $\ell = \overline{PN}$ intersects S in two points, one of which is N. Call the other point of intersection Q (Figure 1). The stereographic map, s, maps $(u, v) \in \mathbf{E}^2$ to Q.[1]

With some elementary algebra we can compute this map explicitly. For $(u, v) \in \mathbf{E}^2, P = (u, v, -1)$. The line ℓ is described parametrically as

$$\begin{aligned} N + t(P - N) &= (0,0,1) + t(u, v, -2) \\ &= (tu, tv, 1 - 2t). \end{aligned}$$

For what values of t does ℓ intersect S? To find this, we substitute the parametric equation for the line ℓ into the implicit equation for S. That is,

$$\begin{aligned} 1 &= x^2 + y^2 + z^2 \\ &= (tu)^2 + (tv)^2 + (1 - 2t)^2 \\ &= t^2u^2 + t^2v^2 + 1 - 4t + 4t^2. \end{aligned}$$

[1] The map described here is actually the *inverse* of the stereographic map as presented in [7].

This last equation can be written

$$t\left[(u^2+v^2+4)t-4\right] \quad = \quad 0.$$

This has two solutions: $t = 0$, which corresponds to N, and $t = 4/(u^2+v^2+4)$, which corresponds to Q (since Q is, by definition, the other point of intersection). Therefore, $Q = (tu, tv, 1-2t)$, for $t = 4/(u^2+v^2+4)$. Putting these together, we conclude

$$s(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4}\right). \tag{1}$$

It will be useful to have an explicit formula for s^{-1}. Using the same techniques as above, it can be shown that

$$s^{-1}(x,y,z) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right). \tag{2}$$

It is interesting to observe what freedom we have in our choice of the plane $\mathcal{P}$. We can choose any plane that does not contain our center of projection, N. Let us explore this a little further in the case of planes of the form $\mathcal{P}_{z_0} = \{(x,y,z) \in \mathbf{E}^3 \mid z = z_0\}$, where $z_0 \neq 1$. The above construction gives the result

$$s_{z_0}(u,v) = \left(\frac{2(1-z_0)u}{u^2+v^2+(1-z_0)^2}, \frac{2(1-z_0)v}{u^2+v^2+(1-z_0)^2}, \frac{u^2+v^2-(1-z_0)^2}{u^2+v^2+(1-z_0)^2}\right)$$

in this general case. And the inverse is given by

$$s_{z_0}^{-1}(x,y,z) = \left(\frac{1-z_0}{1-z}x, \frac{1-z_0}{1-z}y\right).$$

The s_{z_0} maps differ from each other only by a scale factor in the plane. In fact,

$$s_{z_0}(u,v) = s_{z_1}\left(\frac{1-z_1}{1-z_0}u, \frac{1-z_1}{1-z_0}v\right).$$

The two most common choices are $z_0 = -1$ [7] and $z_0 = 0$ [1].
When $z_0 = 0$,

$$s_0(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right),$$

and

$$s_0^{-1}(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

For the remainder of this paper we will use the stereographic map corresponding to $z_0 = -1$.

3 The Stereographic Map to Projective 3-space

We have demonstrated that s is a rational map. (We have also shown the same for s^{-1}). In CAGD it is common to represent a rational map into euclidean space as a polynomial map into the corresponding projective space [5, p. 138 ff.]. One of the reasons for treating a map this way is that one is allowed to work with polynomial maps, which are intrinsically simpler.

Following the development in [9], a point in $\mathbf{P}^3$ (real projective 3-space) is given by four coordinates $[x_0, x_1, x_2, x_3]$ such that $x_i \in \Re$ and not all x_i are 0. Two points $[x_0, x_1, x_2, x_3]$ and $[y_0, y_1, y_2, y_3]$ are taken to be identical if and only if there exists $\lambda \in \Re$, $\lambda \neq 0$ such that $y_i = \lambda x_i$, $i = 0, \ldots, 3$. The coordinates $[x_0, x_1, x_2, x_3]$ are called *homogeneous coordinates* for the point they represent. We will use square brackets ('[' and ']') for homogeneous coordinates to distinguish them from normal euclidean coordinates.

We associate $\mathbf{E}^3$ with $\mathbf{P}^3$ in the following way: The standard inclusion map $i : \mathbf{E}^3 \hookrightarrow \mathbf{P}^3$ is defined by $i(x, y, z) = [x, y, z, 1]$. The standard projection map $\pi : \mathbf{P}^3 \rightarrow \mathbf{E}^3$ is defined by

$$\pi([x_0, x_1, x_2, x_3]) = \left(\frac{x_0}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

Notice that π is defined only for points in $\mathbf{P}^3$ with non-zero x_3 coordinate (the points with $x_3 = 0$ form "the plane at infinity"). Note also that $\pi \circ i$ is the identity function (on $\mathbf{E}^3$), and $i \circ \pi$ is the identity function on all points where it is defined. The composition map $i \circ \pi$ will in general change the homogeneous coordinates of a point by a factor of $1/x_3$, but by the definition of $\mathbf{P}^3$ these new coordinates represent the same point.

We can now redefine s as a polynomial map $\mathbf{E}^2 \rightarrow \mathbf{P}^3$. This new map is given by

$$s(u, v) = [\ 4u,\quad 4v,\quad u^2 + v^2 - 4,\quad u^2 + v^2 + 4\ ].$$

Observe that $\pi \circ s$ gives us the original definition of s in Equation (1).

4 Rational Bezier Sphere Patches

The Bernstein polynomial basis has proved itself to be particularly useful for CAGD. Parametric maps built with this basis are called Bezier maps, or Bezier patches. The Bernstein basis of degree n defined over the interval $[a, b]$ is

$$\theta_{i,n}(t) = \frac{\binom{n}{i}(t-a)^i(b-t)^{(n-i)}}{(b-a)^n}, \qquad \text{for } i = 0, \ldots, n.$$

A degree m by n rational parametric surface can be described using a mesh of homogeneous *control points* $P_{i,j} = [x_{ij}, y_{ij}, z_{ij}, w_{ij}]$ for $i = 0, \ldots, m$, and $j = 0, \ldots, n$, by

$$f(u, v) = \frac{\sum_{i=0}^{m} \sum_{j=0}^{n} (x_{ij}, y_{ij}, z_{ij}) \theta_{i,m}(u) \theta_{j,n}(v)}{\sum_{i=0}^{m} \sum_{j=0}^{n} w_{ij} \theta_{i,m}(u) \theta_{j,n}(v)}$$

as a map to $\mathbf{E}^3$, or simply

$$f(u, v) = \sum_{i=0}^{m} \sum_{j=0}^{n} [x_{ij}, y_{ij}, z_{ij}, w_{ij}] \theta_{i,m}(u) \theta_{j,n}(v)$$

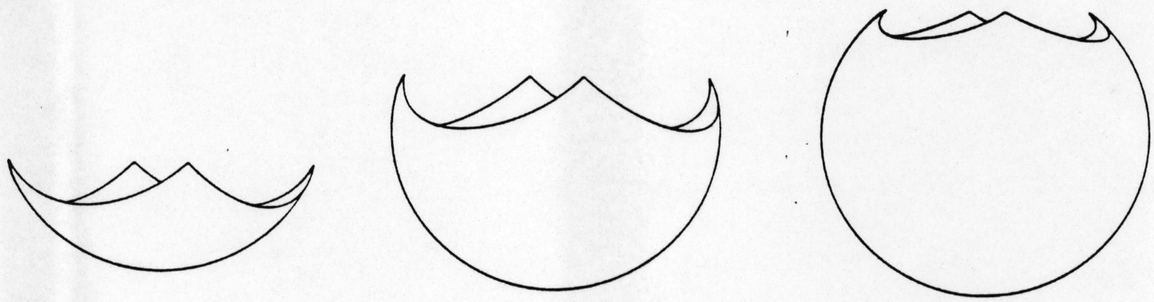
as a map to $\mathbf{P}^3$. Such a function is called a rational tensor product Bezier surface.

We can directly convert the stereographic map s to the Bezier representation to describe a patch that covers a portion of the unit sphere. See section 8 for a discussion on how to compute this. Figure 2 shows the patches for the domains $[-1, 1] \times [-1, 1]$, $[-2, 2] \times [-2, 2]$, and $[-4, 4] \times [-4, 4]$.

However, there is a limitation to this approach. It does not give a way to cover the whole sphere with a single patch, nor does it give us patches which will fit together to tile the entire sphere.

There is a fundamental reason for this difficulty: tensor product Bezier patches are only defined over finite rectangular domains, and there is no such domain that maps onto a tile for the sphere under the stereographic map.

However, there are "four-sided" regions in the plane which do map onto a tile for the sphere, and these regions can be described as the image of a Bezier map. We will explore examples of this to get our Bezier sphere tile.



(a) $[-1,1] \times [-1,1]$ (b) $[-2,2] \times [-2,2]$ (c) $[-4,4] \times [-4,4]$

Figure 2: Three stereographic map patches.

	x	y	w
Row 0	2	0	1
	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}/2$
	0	2	1
Row 1	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}/2$
	0	0	1
	$-\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}/2$
Row 2	0	-2	1
	$-\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}/2$
	-2	0	1

Table 1: Control mesh for Bezier disk of radius 2.

5 Tiling the Sphere with Hemispheres

The easiest Bezier tile for the sphere is a hemisphere. Two of these will tile the whole sphere. The disk of radius 2, centered at the origin, goes under the stereographic map onto the southern hemisphere. Table 1 gives control points for such a disk.

This disk map, d, is a rational Bezier patch. Therefore, as a polynomial map it takes values in $\mathbf{P}^2$, $d : \mathbf{E}^2 \rightarrow \mathbf{P}^2$. But our stereographic map s has domain $\mathbf{E}^2$, not $\mathbf{P}^2$. The solution to our dilemma is to use $s \circ \pi$, where $\pi : \mathbf{P}^2 \rightarrow \mathbf{E}^2$ is the standard projection for $\mathbf{P}^2$. We will use the composition:

$$\mathbf{E}^2 \xrightarrow{d} \mathbf{P}^2 \xrightarrow{\pi} \mathbf{E}^2 \xrightarrow{s} \mathbf{P}^3.$$

If $\mathbf{P}^2$ has homogeneous coordinates $[x_0, x_1, x_2]$, and $(u, v) = \pi(x_0, x_1, x_2)$, then $u = \frac{x_0}{x_2}$, $v = \frac{x_1}{x_2}$. Now

$$s(u,v) \quad = \quad [\ 4u, \ \ 4v, \ \ u^2+v^2-4, \ \ u^2+v^2+4\ ]$$

$$s \circ \pi([x_0, x_1, x_2]) = \left[4\left(\frac{x_0}{x_2}\right),\ 4\left(\frac{x_1}{x_2}\right),\ \left(\frac{x_0}{x_2}\right)^2 + \left(\frac{x_1}{x_2}\right)^2 - 4,\ \left(\frac{x_0}{x_2}\right)^2 + \left(\frac{x_1}{x_2}\right)^2 + 4 \right].$$

To simplify, we can multiply all the homogeneous coordinates by $(x_2)^2$. By definition of projective space this will represent the same point. (This is equivalent to multiplying the numerator and denominator by $(x_2)^2$ when s is regarded as a rational map to $\mathbf{E}^3$).

$$s \circ \pi([x_0, x_1, x_2]) = [\ 4x_0x_2,\ \ 4x_1x_2,\ \ (x_0)^2 + (x_1)^2 - 4(x_2)^2,\ \ (x_0)^2 + (x_1)^2 + 4(x_2)^2\ ]$$

We see that the components of $s \circ \pi$ are all homogeneous polynomials of degree 2. Shafarevich [9, p. 35] shows that a necessary and sufficient condition for a map between projective spaces to be regular is that the map be representable with components that are homogeneous polynomials of the same degree. Notice also that $s \circ \pi$ can be obtained from s by replacing u by x_0, v by x_1, and introducing enough x_2 factors to make the polynomial components be homogeneous of the lowest degree possible, all components having the same degree. This technique works in general for converting a map from euclidean space to the corresponding map from projective space of the same dimension.

Now that we have explored this extension to the stereographic map, we will refer to $s \circ \pi$ simply as s. The context will tell us whether we are mapping from euclidean or projective space.

Consider now the composition map $\mathbf{E}^2 \xrightarrow{d} \mathbf{P}^2 \xrightarrow{s} \mathbf{P}^3$. The disk map, d, is a Bezier map, which is simply a polynomial map represented in the Bezier basis. And s is a polynomial map. Therefore the composition, $s \circ d$ is a polynomial map, which can itself be represented as a Bezier map. Techniques for computing this composite Bezier map are discussed later in Section 8. Table 2 shows the control mesh for the hemisphere patch. Notice that the composite map is bi-quartic. This is because it is the composition of two bi-quadratic polynomial maps.

This composite map covers the southern hemisphere of the unit sphere. This is easy to see: the disk map took the $[0,1] \times [0,1]$ square onto the disk of radius 2 centered around the origin, and the stereographic map takes this disk onto the southern hemisphere. Two of these hemisphere patches will tile the sphere, and the patch has no degenerate edges—each edge is a quarter of the equator. However, this map does have a mild degeneracy that makes it unsuitable for some applications in CAGD. At the corners of the parametric space the differential fails to have maximal rank. In other words, the isoparametric tangent vectors are dependent. At a patch corner, $\frac{\partial f}{\partial u} = \pm\frac{\partial f}{\partial v}$. As a result it is difficult to compute a surface normal at the corners. This is a milder degeneracy than the one suffered by the octant patch, since it does not collapse an entire edge to a point. But the difficulty in computing a normal makes it unsuitable for some applications.

A hemisphere is the portion of a sphere cut off by a plane passing through the center of the sphere. A slight variation on the technique used above will produce a Bezier patch description for any "spherical cap," that is a portion of the sphere cut off by any plane that intersects the sphere in a circle. Given a maximum z value, z_m, $-1 < z_m < 1$, consider the cap $\{(x,y,z) \in S \mid -1 \leq z \leq z_m\}$. Let $x_m = \sqrt{1 - z_m^2}$ (the maximum x value on the sphere when $z = z_m$). Since, by Equation 2,

$$s^{-1}(x_m, 0, z_m) = \left(\frac{2x_m}{1 - z_m}, 0\right)$$

it is clear that the pre-image of this cap under the stereographic map is a disk of radius $2x_m/(1 - z_m)$ centered at the origin. If we use a modified disk map corresponding to this radius and compose with the stereographic map we will produce a bi-quartic Bezier patch describing this cap. Table 3 shows the control mesh for a spherical cap patch. The coefficients in this table have all been multiplied by their least common denominator to make the description more concise.

6 Tiling the Sphere Using the Cube Topology

An approach to the tiling problem that has proved successful is to impose the topology of a cube onto the sphere, and use one of the six faces as a tile. Conceptually, take a cube with edges parallel to

	x	y	z	w
Row 0	8	0	0	8
	$4\sqrt{2}$	$2\sqrt{2}$	0	$4\sqrt{2}$
	4	4	0	$16/3$
	$2\sqrt{2}$	$4\sqrt{2}$	0	$4\sqrt{2}$
	0	8	0	8
Row 1	$4\sqrt{2}$	$-2\sqrt{2}$	0	$4\sqrt{2}$
	4	0	-3	3
	$5\sqrt{2}/3$	$5\sqrt{2}/3$	$-8\sqrt{2}/3$	$4\sqrt{2}/3$
	0	4	-3	3
	$-2\sqrt{2}$	$4\sqrt{2}$	0	$4\sqrt{2}$
Row 2	4	-4	0	$16/3$
	$5\sqrt{2}/3$	$-5\sqrt{2}/3$	$-8\sqrt{2}/3$	$4\sqrt{2}/3$
	0	0	$-16/3$	$8/9$
	$-5\sqrt{2}/3$	$5\sqrt{2}/3$	$-8\sqrt{2}/3$	$4\sqrt{2}/3$
	-4	4	0	$16/3$
Row 3	$2\sqrt{2}$	$-4\sqrt{2}$	0	$4\sqrt{2}$
	0	-4	-3	3
	$-5\sqrt{2}/3$	$-5\sqrt{2}/3$	$-8\sqrt{2}/3$	$4\sqrt{2}/3$
	-4	0	-3	3
	$-4\sqrt{2}$	$2\sqrt{2}$	0	$4\sqrt{2}$
Row 4	0	-8	0	8
	$-2\sqrt{2}$	$-4\sqrt{2}$	0	$4\sqrt{2}$
	-4	-4	0	$16/3$
	$-4\sqrt{2}$	$-2\sqrt{2}$	0	$4\sqrt{2}$
	-8	0	0	8

Table 2: Control mesh for Bezier hemisphere.

	x	y	z	w
Row 0	$72x_m$	0	$72z_m$	72
	$36\sqrt{2}x_m$	$18\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	$36x_m$	$36x_m$	$48z_m$	48
	$18\sqrt{2}x_m$	$36\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	0	$72x_m$	$72z_m$	72
Row 1	$36\sqrt{2}x_m$	$-18\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	$36x_m$	0	$27(z_m-1)$	$-27(z_m-1)$
	$15\sqrt{2}x_m$	$15\sqrt{2}x_m$	$12\sqrt{2}(z_m-2)$	$-12\sqrt{2}(2z_m-1)$
	0	$36x_m$	$27(z_m-1)$	$-27(z_m-1)$
	$-18\sqrt{2}x_m$	$36\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
Row 2	$36x_m$	$-36x_m$	$48z_m$	48
	$15\sqrt{2}x_m$	$-15\sqrt{2}x_m$	$12\sqrt{2}(z_m-2)$	$-12\sqrt{2}(2z_m-1)$
	0	0	$8(z_m-6)$	$-8(6z_m-1)$
	$-15\sqrt{2}x_m$	$15\sqrt{2}x_m$	$12\sqrt{2}(z_m-2)$	$-12\sqrt{2}(2z_m-1)$
	$-36x_m$	$36x_m$	$48z_m$	48
Row 3	$18\sqrt{2}x_m$	$-36\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	0	$-36x_m$	$27(z_m-1)$	$-27(z_m-1)$
	$-15\sqrt{2}x_m$	$-15\sqrt{2}x_m$	$12\sqrt{2}(z_m-2)$	$-12\sqrt{2}(2z_m-1)$
	$-36x_m$	0	$27(z_m-1)$	$-27(z_m-1)$
	$-36\sqrt{2}x_m$	$18\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
Row 4	0	$-72x_m$	$72z_m$	72
	$-18\sqrt{2}x_m$	$-36\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	$-36x_m$	$-36x_m$	$48z_m$	48
	$-36\sqrt{2}x_m$	$-18\sqrt{2}x_m$	$36\sqrt{2}z_m$	$36\sqrt{2}$
	$-72x_m$	0	$72z_m$	72

Table 3: Control mesh for Bezier sphere cap $\left(-1 \leq z \leq z_m,\ x_m = \sqrt{1-z_m^2}\right)$.

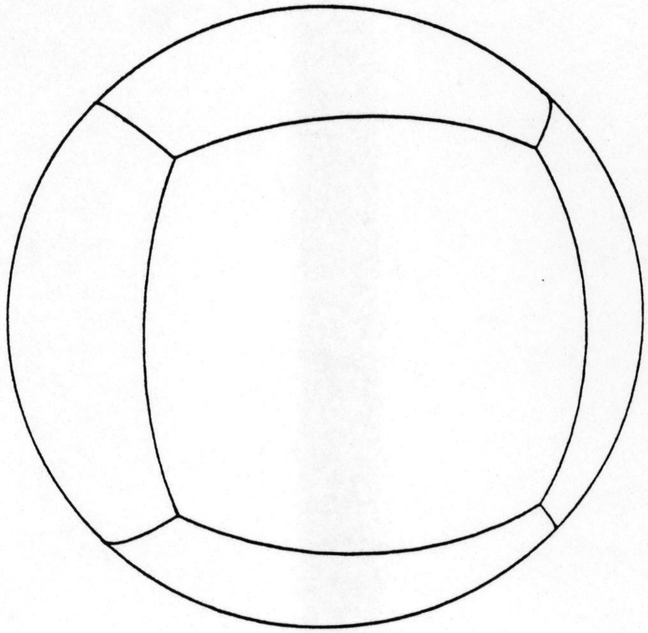


Figure 3: Sphere with cube topology.

the coordinate axes, inscribed inside the unit sphere. This cube has eight vertices: $\left(\pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}, \pm\frac{\sqrt{3}}{3}\right)$. Connect these vertices with great circle arcs on the sphere. (These arcs are the projection of the cube edges if the center of projection is the origin.) See Figure 3.

The arcs divide the sphere into six congruent faces. So we represent a single face as a Bezier surface patch. To do this we determine what region of the plane is mapped onto this face by the stereographic map, describe it as a Bezier patch, and compose the Bezier patch with the stereographic map.

This construction is promising because of a property of the stereographic map: a circle on the sphere has as its pre-image a circle or line in the plane [1, pp. 18-20]. And it is easy to construct a rational Bezier patch in the plane with edges consisting of circular arcs or line segments.

Motivated by symmetry (and the necessity to stay away from the north pole) we will construct the patch containing the south pole $(0,0,-1)$. Two vertices of this patch are $P_1 = \left(\frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$ and $P_2 = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)$.

Using Equation (2) we compute the pre-image, $\hat{P}_1$, of P_1 under the stereographic map.

$$\begin{aligned}
\hat{P}_1 &= s^{-1}(P_1) \\
&= s^{-1}\left(\frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right) \\
&= \left(\frac{2\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}, \frac{-2\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}}\right) \\
&= \left(\sqrt{3}-1, 1-\sqrt{3}\right)
\end{aligned}$$

Similarly,

$$\hat{P}_2 = s^{-1}(P_2)$$

$$= \left(\sqrt{3}-1, \sqrt{3}-1\right).$$

The great circle, C, containing the two points is

$$C = \{(x, y, z) \in S \mid x + z = 0\}.$$

(The equation defining C is the equation for the plane containing P_1, P_2, and the center of the sphere, the origin.)

If we pull back the implicit equation $x + z = 0$ under s we will get an equation for the curve in the plane that is the pre-image of C. Using Equation (1),

$$\begin{aligned} 0 &= x + z \\ &= \frac{4u}{u^2+v^2+4} + \frac{u^2+v^2-4}{u^2+v^2+4} \\ &= 4u + u^2 + v^2 - 4. \end{aligned}$$

This can be written as

$$(u+2)^2 + v^2 = \left(2\sqrt{2}\right)^2$$

which is the equation of a circle with center $(-2, 0)$ and radius $2\sqrt{2}$. Therefore we conclude that one edge of the planar patch will be the arc of that circle from $\hat{P}_1$ to $\hat{P}_2$. This can be represented as a rational quadratic Bezier curve with control points

$$[\ \sqrt{3}-1,\ \ 1-\sqrt{3},\ \ 1\ ],\ [\ \tfrac{1}{2}(3-\sqrt{3})\sqrt{2},\ \ 0,\ \ \tfrac{1}{4}(\sqrt{3}+1)\sqrt{2}\ ],\ \text{and}\ [\ \sqrt{3}-1,\ \ \sqrt{3}-1,\ \ 1\ ].$$

See [3] for a discussion of how to represent circular arcs as Bezier curves.

By symmetry, the other three edges of the region are simply rotations of the above arc about the origin by multiples of 90°. The control points of these Bezier arcs give us eight of the nine control points for a Bezier patch filling the region. We choose to use the point [0, 0, 1] as the missing (middle) control point. The first two components are zero because we want the patch to have symmetry. The choice of weight 1 is arbitrary. Changing it to any other positive value will affect the *parametrization* of the patch, but not what region in the plane will be covered by the patch. Nor will a change in that weight disturb the symmetry of the patch. See Figure 4.

Now we compose this bi-quadratic Bezier patch with s to obtain a bi-quartic Bezier patch representing our sphere tile. Table 4 has the control points for this patch. Unlike the hemisphere tile developed above, this one has no degeneracies of any kind.

7 Other Sphere Tiles

Other tiles for the sphere can be generated using the same techniques as above. For example, imposing the topology of the icosahedron on the sphere divides it into twenty triangular regions. Joining pairs of regions will make a tiling consisting of ten four-sided patches. One of these patches can be realized as a bi-quartic rational Bezier patch in the same way the cube patch was constructed.

Using the same techniques as above, any four-sided region of the sphere bounded by arcs which maps to a convex region in the plane under the inverse stereographic map can be realized as a bi-quartic rational Bezier patch. A property of the stereographic mapping helps one determine whether such a four-sided region will project to a convex region in the plane. The stereographic map is a conformal (or angle-preserving) mapping [7]. Therefore, if the interior angles on the sphere are less than or equal to 180 degrees, the corresponding region in the plane will also have interior angles less than 180 degrees. If the angles are strictly less than 180 degrees, there will be no normal degeneracies at the corners of

	x	y	z	w
Row 0	$4(1-\sqrt{3})$	$4(1-\sqrt{3})$	$4(1-\sqrt{3})$	$4(3-\sqrt{3})$
	$-\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	0	$4(1-2\sqrt{3})/3$	$4(1-2\sqrt{3})/3$	$4(5-\sqrt{3})/3$
	$\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	$4(\sqrt{3}-1)$	$4(1-\sqrt{3})$	$4(1-\sqrt{3})$	$4(3-\sqrt{3})$
Row 1	$\sqrt{2}(\sqrt{3}-4)$	$-\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	$(2-3\sqrt{3})/2$	$(2-3\sqrt{3})/2$	$-(\sqrt{3}+6)/2$	$(\sqrt{3}+6)/2$
	0	$\sqrt{2}(2\sqrt{3}-7)/3$	$-5\sqrt{6}/3$	$\sqrt{2}(\sqrt{3}+6)/3$
	$(3\sqrt{3}-2)/2$	$(2-3\sqrt{3})/2$	$-(\sqrt{3}+6)/2$	$(\sqrt{3}+6)/2$
	$\sqrt{2}(4-\sqrt{3})$	$-\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
Row 2	$4(1-2\sqrt{3})/3$	0	$4(1-2\sqrt{3})/3$	$4(5-\sqrt{3})/3$
	$\sqrt{2}(2\sqrt{3}-7)/3$	0	$-5\sqrt{6}/3$	$\sqrt{2}(\sqrt{3}+6)/3$
	0	0	$4(\sqrt{3}-5)/3$	$4(5\sqrt{3}-1)/9$
	$\sqrt{2}(7-2\sqrt{3})/3$	0	$-5\sqrt{6}/3$	$\sqrt{2}(\sqrt{3}+6)/3$
	$4(2\sqrt{3}-1)/3$	0	$4(1-2\sqrt{3})/3$	$4(5-\sqrt{3})/3$
Row 3	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	$(2-3\sqrt{3})/2$	$(3\sqrt{3}-2)/2$	$-(\sqrt{3}+6)/2$	$(\sqrt{3}+6)/2$
	0	$\sqrt{2}(7-2\sqrt{3})/3$	$-5\sqrt{6}/3$	$\sqrt{2}(\sqrt{3}+6)/3$
	$(3\sqrt{3}-2)/2$	$(3\sqrt{3}-2)/2$	$-(\sqrt{3}+6)/2$	$(\sqrt{3}+6)/2$
	$\sqrt{2}(4-\sqrt{3})$	$\sqrt{2}$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
Row 4	$4(1-\sqrt{3})$	$4(\sqrt{3}-1)$	$4(1-\sqrt{3})$	$4(3-\sqrt{3})$
	$-\sqrt{2}$	$\sqrt{2}(4-\sqrt{3})$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	0	$4(2\sqrt{3}-1)/3$	$4(1-2\sqrt{3})/3$	$4(5-\sqrt{3})/3$
	$\sqrt{2}$	$\sqrt{2}(4-\sqrt{3})$	$\sqrt{2}(\sqrt{3}-4)$	$\sqrt{2}(3\sqrt{3}-2)$
	$4(\sqrt{3}-1)$	$4(\sqrt{3}-1)$	$4(1-\sqrt{3})$	$4(3-\sqrt{3})$

Table 4: Control Mesh for one face of sphere with cube topology.

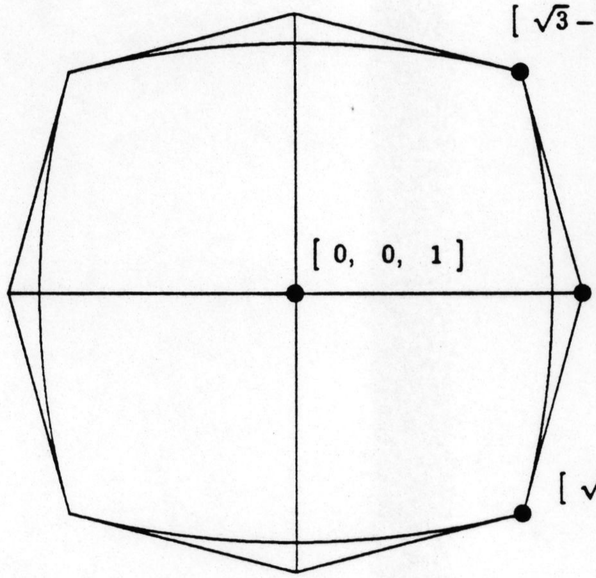


Figure 4: Pre-image of the Cube Topology Patch.

the composition patch. If an interior angle is 180 degrees there will be a corner degeneracy of the same form as in the hemisphere tile.

One might consider using the tetrahedron or octahedron in the same manner as the icosahedron was used above to generate a tiling. Unfortunately, neither of these will give satisfactory solutions. The four-sided region constructed from joining two triangles from the tetrahedral topology has a 240 degree interior angle at two of its vertices. Therefore it does not project to a convex region in the plane. A four-sided region from the octahedral topology has a 180 degree interior angle at two of its vertices. Therefore the resulting patch will have degenerate normals at two of the corner points.

It is possible to subdivide an equilateral triangle into three congruent quadrilaterals by introducing vertices at the midpoints of the sides and the center of the face. Using this technique, any of the triangular-faced platonic solids can be used to tile the sphere.

There is no reason to be limited to topologies imposed by the platonic solids. Regular complex polyhedra [4], or even non-regular polyhedra could be used. A possible example of a regular complex polyhedron that could be used is the rhombicuboctahedron. In this example three different four-sided patches would be used to tile the sphere.

8 Computational Techniques

We have used three different techniques for computing the representation of these composition maps as Bezier patches. We will now discuss the specific problem of representing the hemisphere patch as a rational bi-quartic Bezier patch.

The first technique used was interpolation. Because a rational map to $\mathbf{E}^3$corresponds to a polynomial map to $\mathbf{P}^3$, polynomial techniques can be used to compute the interpolation. It is easy to deduce that the composition of the disk map, d, with the stereographic map, s, will be a bi-quartic surface. To produce the patch, we simply sample the disk map on a five-by-five grid of points, map those points under the stereographic map to $\mathbf{P}^3$, and interpolate the resulting data. Interpolation will produce the unique result which we know exists.

The second technique used involved deriving a closed form solution for the product of scalar-valued Bezier functions. Since any polynomial map can be expressed as products and sums, we are able to use the product algorithm to perform the composition. This approach is good to use in a modeling system as a general tool for function compositions to be computed on demand.

The last technique used was to use Reduce, a symbolic algebra system [6]. We convert the disk map, d, from the Bernstein basis to the standard polynomial basis. Substituting this into s, and converting the composition result to the Bezier basis gives us the Bezier representation of the composition map. This was the technique that produced the tables of control points for this paper. This approach has the advantage of producing exact results rather than floating point approximations. However, it is not as convenient for embedding in a modeling system unless the modeling system already contains a symbolic algebra package.

9 Conclusion

The technique of function composition has been shown to be useful for generating tiles of the sphere using rational Bezier patches. These same techniques can be used in a similar fashion to generate other primitive shapes as well. Furthermore, a wide class of four-sided pieces of a sphere (or other shape) can be realized directly as a patch without the need to resort to trimming the surface.

The same technique of composition can also be used for extracting four-sided sub-pieces of a Bezier patch. Suppose we have a Bezier patch, f, and we desire to represent a four-sided sub-piece, P, as another Bezier patch. We represent the pre-image of P in the parametric domain of f as a Bezier patch, g. It may be impossible (or difficult) to represent this pre-image exactly as a Bezier patch. If it is impossible and an approximation will suffice, approximate it with a Bezier patch, g. Now the composition $f \circ g$ can be expressed as a Bezier patch and will represent the sub-piece.

Function composition can also be useful for representing derived properties associated with a Bezier patch (e.g. curvature, unnormalized normals, and moments of inertia). With some limitations, many of these techniques can be applied to B-spline surfaces as well.

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