TRIGONOMETRIC SERIES APPLIED TO BENDING OF THIN RECTANGULAR PLATES

by

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In Part I, the sine transform is used to obtain the solution of the problem of a thin rectangular plate on an elastic foundation. A table of inverse sine and cosine transforms is given, with emphasis on inverse transforms of simple rational functions.

In Part II, a double sine series is used to solve problems of a rectangular plate which has every edge either clamped, supported, or free. All 81 possible combinations of edge conditions are considered. Considerable attention is given to the role of corner deflections and concentrated loads at corners where two free edges meet.

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INTRODUCTION

An attempt is made to solve new problems involving the bending of plates. In Part I single Fourier sine series are used and in Part II double Fourier sine series are used.

Part I

In the study of boundary value problems a solution is often obtained in terms of a Fourier sine or cosine series. In certain cases these series can be summed and considerable information can be obtained from the summed form. Roettinger[29] and others have indicated the desirability of having an extended table of known inverse transforms. Some known transforms are listed by Churchill[5] and Roettinger.[29]

In this paper the existing tables are extended. Explicit expressions have been obtained for inverse transforms of rational functions of n whose denominators are of 5th degree or less, and the methods are clear to obtain results for any desired rational function.

The problem of an infinite plate on an elastic foundation has been treated for several loads and boundary conditions.[16A] Murphy has solved the problem of a finite plate simply supported on all edges, and has given numerical results.[26] Timoshenko gives the solution when two edges are clamped and two edges are simply supported.[34]

This paper treats the rectangular plate with two opposite edges having prescribed moments and deflections, and the other two edges having any of the six most important boundary conditions. The method of solution is the same as that used by Deverall and Thorne.[10] Numerical results are given for the two edges clamped, free and simply supported. Deflections, moments and shears are given for constant load and strip load.

Part II

The problem of a plate which is free, simply supported or clamped at every edge has been treated by several authors. The case of all edges clamped has been discussed by Timoshenko[34] and others. The case of two opposite edges simply supported has been handled by Deverall and Thorne.[10] The case of two adjacent edges clamped and the other two both simply supported or both free has been treated by Huang and Conway.[17] The method of superposition was used by Timoshenko and Huang and Conway to obtain an infinite set of equations with an infinite number of unknowns. Other cases have been solved by using the Ritz Method.

In this paper we shall show how to find a Fourier series solution for those cases in which every edge is either free, simply supported, or clamped. The set of equations for the clamped plate is identical with those obtained by Timoshenko. The present method with two opposite edges simply supported gives the solutions of Deverall and Thorne, after summation on one variable.

For the problem of Huang and Conway a different set of equations is obtained. The convergence of an example is discussed. The extension to given deflections, slopes, etc. at the edges offers no formal difficulties.

I. THE SINE TRANSFORM

A. General Results

1. Definition of finite sine and cosine transforms[5*]

Let f(x) denote a function that is sectionally continuous over the interval (0, a). The finite sine transformation of f(x) on that interval is the operation

$$S \cdot f(x) \equiv f_s(\alpha) \equiv \int_0^a f(x) \sin \alpha x \, dx \qquad (1)$$

where $\alpha = \frac{n\pi}{a}$ and $n = 1, 2, 3, \ldots\ldots$

Closely allied with the finite sine transformation is the finite cosine transformation. This operator acts on a sectionally continuous function f(x) defined over the interval (0,a) to give

$$C \cdot f(x) \equiv f_c(\alpha) \equiv \int_0^a f(x) \cos \alpha x \, dx \ . \qquad (2)$$

For brevity we will refer to functions obtained by these operators as the sine transform and the cosine transform respectively.

2. Transforms of derivatives

Let f(x) have sectionally continuous derivatives of all orders, but a jump of amount μ_i in its i^{th} derivative at the point $x = x_i$. We can

* References are to the bibliography.

express the sine and cosine transforms of the i^{th} derivative of f(x) in terms of the transforms of f(x), by using integration by parts.[2]

For example:

$$S \cdot \frac{d^2f(x)}{dx^2} \equiv S \cdot f''(x) = -a^2 f_s(a) - a[(-1)^n f(a) - f(0)]$$

$$- \mu_1 \sin ax_1 + a \mu_0 \cos ax_0 \qquad (3)$$

$$S \cdot \frac{d^4f(x)}{dx^4} \equiv S \cdot f''''(x) = a^4 f_s(a) - a[(-1)^n f''(a) - f''(0)]$$

$$+ a^3[(-1)^n f(a) - f(0)] - \mu_3 \sin ax_3 + a\mu_2 \cos ax_2 + a^2 \mu_1 \sin ax_1$$

$$- a^3 \mu_0 \cos ax_0 \qquad (4)$$

where $f''(a)$ is defined as the second derivative of f(x) evaluated at $x = a$, and $f''(0)$ is the second derivative evaluated at $x = 0$.

3. Advantages over non transform methods

Many authors have used the sine transform to solve special differential equations.[4, 18,22,29,31] In solving such equations, we have found that the sine transform method has the following advantages. First, the solution is often less difficult to find. For example, to solve an equation of the type

$$f^{(10)}(x) + 2 f^{(8)}(x) + f(x) = 1 \qquad (5)$$

subject to the boundary conditions

$$f^{(8)} = f^{(6)} = f^{(4)} = f'' = f = 0 \text{ at } x = 0 \text{ and } x = a .$$

The common method assumes a solution of the homogeneous equation of the form e^{mx}. This leads to the auxilliary equation

$$m^{10} + 2\,m^{8} + 1 - 0$$

which has to be solved by numerical methods.

The sine transform method is to write the sine transform of Equation (5) and solve for $f_s(\alpha)$. There results

$$f_s(\alpha) = \frac{1 - (-1)^n}{-\alpha^{11} + 2\,\alpha^9 + \alpha} \quad .$$

By comparing Equation (1) with the Fourier sine series for f(x), we note that

$$f(x) = \frac{2}{a} \sum_{n=1}^{\infty} f_s(\alpha) \sin \alpha x \qquad (6)$$

so that the solution of the problem is

$$f(x) = \frac{2}{a} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{-\alpha^{11} + \alpha^9 + \alpha} \sin \alpha x \quad .$$

The approximate value of f(x) can be calculated by adding the first few terms of this series for some value of x. This involves less labor than the finding of the roots of an algebraic equation which has a degree greater than four.

A second advantage of using the sine transform is that for some boundary conditions, the solving for undetermined constants is unnecessary. For example, in proceeding with the common method to solve equation (5), it is necessary to use the ten boundary conditions to solve for the ten constants of the complimentary solution which involves the solving of ten linear equations.

Since the even derivatives in the above problem were given at the boundaries, all constants of the transforms are determined. However, if in some problems the even derivatives of f(x) were not given at the boundaries, we would use the given boundary conditions to solve for these unknowns. The labor involved in solving for these unknowns might overbalance the advantage in using the sine transform.

A third advantage can be illustrated by examining the following equation:

$$f''(x) + f(x) = g(x)$$

where $g(x) = x$ when $0 \leq x \leq a/2$

$= a - x$ when $a/2 \leq x \leq a$

and $f(0) = f(a) = 0$.

The common method involves the solving of two equations and the matching of boundary conditions at $x = a/2$. The sine transform method gives

$$f(x) = \frac{2}{a} \sum_{n=1}^{\infty} f_s(\alpha) \sin \alpha x$$

for the whole interval (0, a); hence, no solving for constants at $x = a/2$ is involved.

A fourth advantage of using the sine transform is that partial differential equations may be solved which are not readily solvable by other methods. This is illustrated in Section D.

4. Advantage over other transforms

Suppose we wished to solve a differential equation with constant coefficients involving only even derivatives of f(x) and subject to the boundary conditions that

$$f^{(i)}(a) = f^{(i)}(0) = 0$$

where

$$i = 0, 2, 4, \ldots, k - 2$$

where k is the degree of the equation. It is very convenient to have the transform of even derivatives of f(x) equal to a linear function in the transform of f(x). Using a transform of the type

$$\int_a^b f(x) \sin \alpha x \, dx \qquad \text{or} \qquad \int_0^a f(x) \cos \alpha x \, dx$$

would involve the determination of odd derivatives of f(x) at the boundaries. These boundary conditions do not occur frequently in physical problems.

Transforms of the type

$$\int_0^a H_n (x/a) f(x) \, dx \qquad \text{or} \qquad \int_0^a P_n (x/a) f(x) \, dx$$

$$\int_0^a N (\alpha x) f(x) \, dx \qquad \text{or} \qquad \int_0^a x J_\mu(x \xi_n) f(x) \, dx$$

($H_n(x)$ are Hermite Polynomials, $P_n(x)$ are Legendre Polynomials, $N(\alpha x)$ are Normal Functions,[8] and $J_\mu(x)$ are Bessell's Functions[31]) do not have the property that the transform of $f''(x)$ is a linear function in the transform of f(x), whereas the sine transform does have this property.

B. Sine Transform Tables

1. Need for a table

In using the sine transform to solve differential equations, a solution is obtained in the form of a sine series. Often such a series is equal to an elementary function in the interval (0, a). We call this the summed form of the solution. A table of inverse sine transforms lists the summed form of various series for various coefficients $f_s(\alpha)$. It enables us to compare many series solutions with solutions found by other methods.

The summed form often lends itself more easily to numerical calculation. Furthermore, certain properties can be more readily recognized and proved from the summed form; for example, the value of x which will make the third derivative of f(x) a maximum.

Several tables of inverse sine transforms have been published,[5,29] but there is a need to have these tables extended. There is a particular need to include in an extended table inverse transforms which occur in plate problems. Such a table follows.

2. Scope of Tables I and II

The summed form has been tabulated for many rational functions $f_s(n)$. (Note that $f_s(n) = f_s(\alpha)$ if $a = \pi$.) Methods used in the preparation of Tables I and II are presented in the next section. If $f_s(n)$ is an odd rational function of n, we observe from Table IA that the summed form is a rational combination of powers of x, sines, cosines, hyperbolic sines, and hyperbolic cosines. If $f_s(n)$ is an even rational function of n, we observe from Table IB that the summed form is in the form of an integral. Because of this fact, the even transforms were tabulated separately.

For a given value of $f_s(n)$, Table I lists the value of f(x) where

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} f_s(n) \sin nx \ .$$

The function

$$g(x) = \frac{2}{a}\sum_{n=1}^{\infty} f_s(n) \sin \frac{n\pi x}{a}$$

(over an interval (0, a)), can be calculated from the equation

$$g(x) = \frac{\pi}{a} f(\pi x/a) \ .$$

For example, to calculate

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{1}{\alpha} \sin \alpha x$$

we see from Table IA that

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} \sin nx = \frac{\pi - x}{\pi} \equiv f(x)$$

so that

$$\frac{2}{a}\sum_{n=1}^{\infty}\frac{1}{\alpha} \sin \alpha x = \frac{a}{\pi}\left[\frac{2}{a}\sum_{n=1}^{\infty}\frac{1}{n} \sin \alpha x\right] = \frac{a}{\pi} g(x) = \frac{a}{\pi}\left[\frac{\pi}{a}\left(1 - \frac{x}{a}\right)\right] = 1 - \frac{x}{a}$$

3. Methods used in obtaining inverse sine transforms

There are many methods for summing sine and cosine series.[2,5,18] Most of the methods are formal, but if the formal summed form, $g(x)$, is continuous and satisfies the equation

$$\int_o^a g(x) \sin nx \, dx = f_s(n)$$

then it is the inverse transform of $f_s(n)$. (i. e. $f(x) = g(x)$)

The following methods for finding the inverse transform of $f_s(n)$ were used in constructing Tables I and II.

(Method a) Often $f_s(n)$ can be expressed as a sum of coefficients of known series by the method of partial fractions. For example, if the inverse transform of $\frac{1}{n^2(n^2+a^2)}$ is unknown but the inverse transforms of $\frac{1}{n^2}$ and $\frac{1}{n^2+a^2}$ are tabulated, then since the inverse sine operator S^{-1} is linear, the above unknown can be calculated from the equation

$$S^{-1} \frac{1}{n^2(n^2+a^2)} = S^{-1} \left[\frac{1}{a^2 n^2} - \frac{1}{a^2 (n^2+a^2)} \right]$$

$$\frac{1}{a^2} S^{-1} \frac{1}{n^2} - \frac{1}{a^2} S^{-1} \frac{1}{n^2+a^2} .$$

(Method b) By changing parameters, we can obtain inverse transforms of a different type. For example, suppose it is known that

$$S^{-1} \frac{n}{n^2+a^2} \quad \frac{\sinh a(\pi - x)}{\sinh a\pi} .$$

To find the summed form of $S^{-1} \dfrac{n}{n^2 - b^2}$ we let $a = ib$ and obtain

$$S^{-1} \frac{n}{n^2 - b^2} = \frac{\sin b(\pi - x)}{\sin b\pi} .$$

(Method c) We can often obtain the sum of a sine series from the known sum of a power series. Suppose the power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \quad (a_n \text{ real}) \tag{7}$$

can be expressed as an elementary function. We can express trigonometric series in the form

$$R[f(e^{ix})] = \sum_{n=1}^{\infty} a_n \cos nx \tag{8}$$

$$I[f(e^{ix})] = \sum_{n=1}^{\infty} a_n \sin nx . \tag{9}$$

For example, it is known that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z .$$

From Equation (9) we obtain

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = e^{\cos x} \sin(\sin x) .$$

(Method d) Some series can be obtained from known series by a change of variable. Knopp[19] gives, for example,

$$P_{2\lambda+1}(x) = (-1)^{\lambda-1} \sum_{n=1}^{\infty} \frac{2 \sin 2n\pi x}{(2n\pi)^{2\lambda-1}}$$

where $P_{2\lambda+1}(x)$ are Bernoulli Polynomials. By changing the variable x to x' where $x' = 2\pi x$, we can calculate the inverse transform of $\dfrac{1}{n^{2\lambda+1}}$.

Another example: deHaan[7] gives

$$S^{-1}\frac{1}{n+q} = 2q\int_0^{\pi} \frac{\sin^{\frac{x-\pi}{\pi}} t - \sin^{\frac{\pi-x}{\pi}} t}{q^2 + (\ln \sin t)^2} \frac{dt}{\cos t} .$$

By changing variables $u = \ln\sin t$, we get

$$S^{-1}\frac{1}{n+q} = 2q\int_0^{\infty} \frac{\sinh u\,(1-\frac{x}{\pi})\,du}{(\pi^2 q^2 + u^2)\sinh u} .$$

(Method e) If $f_s(n)$ is a rational function, it may be expressed as

$$f_s(n) = \frac{a_o + a_1 n + \ldots + a_s n^s}{b_o + b_1 n + \ldots + b_t n^t} . \tag{10}$$

For convergence, the Fourier coefficients must approach zero as $n \to \infty$ and therefore $s < t$. We can factor the denominator into linear factors and break $f_s(n)$ into partial fractions of the type $\frac{n^p}{(n-a)^q}$ $(p < q)$, and $\frac{n^p}{[n^2 - bn - c]^{p+q+1}}$ where a, b, c are real numbers and p and q are non negative integers. In general, a, b, c must be found by numerical methods.

Multiplying the series

$$\frac{1}{1-z} = \sum_{n=1}^{\infty} z^{n-1} \quad \text{by} \quad z^{-a}$$

and integrating, we obtain

$$\sum_{n\,1} \frac{z^{n-a}}{n-a} = \int_o^z \frac{z_1^{-a}}{1-z_1} dz_1 .$$

Using the integral operator $\int_o^z \frac{1}{z_q}(\ )\,dz_q$ q times and then multiplying by z^a we obtain

$$z^a \int_0^z \frac{dz_q}{z_q} \cdots \int_0^{z_3} \frac{dz_2}{z_2} \int_0^{z_2} \frac{z_1^{-a}}{1 - z_1} dz_1 = \sum_{n=1}^{\infty} \frac{z^n}{(n-a)^q} .$$

Using the differential operator $z \frac{d}{dz}$ ( ) p times, we obtain

$$z \frac{d}{dz} \overset{p}{\cdots} z \frac{d}{dz} \left[z^a \int_0^z \frac{dz_q}{z_q} \cdots \int_0^{z_2} \frac{z_1^{-a}}{1 - z_1} dz_1 \right] = \sum_{n=1}^{\infty} \frac{n^p z^n}{(n-a)^q} . \tag{11}$$

Placing $z = e^{ix}$ and taking the imaginary part of this expression we obtain the inverse sine transform of $\frac{n^p}{(n-a)^q}$ in terms of a repeated integral.

Let $I(b, c, x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 - nb - c}$. $I(b, c, x)$ can be summed in terms of integrals by other methods. Differentiating with respect to b and c we obtain

$$\frac{1}{(p+q)} \frac{\partial^{p+q} I(b, c, x)}{\partial b^p \partial c^q} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{n^p \sin nx}{(n^2 - bn - c)^{p+q+1}} = S^{-1} \frac{n^p}{(n^2 - bn - c)^{p+q+1}} .$$

Any rational function $f_s(n)$ can be expressed as a sum of terms of this type, and, therefore, its summed form can be readily calculated.

<u>(Method f)</u> The use of trigonometric identities simplifies transforms of the type $f_s(n) \sin na$. For example:

$$\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin na}{n^2} \sin nx = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left[\cos n(a-x) - \cos n(a+x) \right]$$

$$\frac{1}{2} \left[g(a-x) - g(a+x) \right]$$

where $g(x)$ is the even extension of $C^{-1} \frac{1}{n^2} = \frac{x^2}{2\pi} - x + \frac{\pi}{3}$.

(Method g) Some useful relationships exist between sine and cosine transforms. If f(x) and its derivatives are continuous for $0 \leq x \leq a$, we obtain by integration by parts the following formulae:

$$C^{-1}\, n\, f_s(n) = f'(x) \tag{12}$$

$$C^{-1}\, \frac{f_s(n)}{n} = -\int_0^x f(t)\, dt \tag{13}$$

$$S^{-1}\, n\, f_c(n) = g(x) \tag{14}$$

$$S^{-1}\, \frac{f_c(n)}{n} = \int_0^x g(t)\, dt \tag{15}$$

where $S^{-1} f_s(n) = f(x)$ and $C^{-1} f_c(n) = g(x)$. For example:

$$S^{-1}\, \frac{1}{n\,(n^2 + a^2)} = \frac{1}{a^2}\left[\frac{\pi - x}{\pi} - \frac{\sinh a\,(\pi - x)}{\sinh a\pi}\right] = f(x)$$

$$C^{-1}\, n\, f_s(n) = C^{-1}\, \frac{1}{n^2 + a^2} = f'(x) = -\frac{1}{\pi a^2} + \frac{\cosh a\,(\pi - x)}{a \sinh a\pi}$$

These formulae enable us to determine inverse cosine transforms from the table of sine transforms and conversely.

By rearranging Equation (3) and substituting $F(x) = \int_0^x \int_0^{x_1} f(t)\, dt\, dx_1$ for f(x) we find

$$S^{-1}\, \frac{f_s(n)}{n^2} = -\int_0^x \int_0^{x_1} f(t)\, dt\, dx_1 + \frac{x}{\pi} \int_0^{\pi} \int_0^{x_1} f(t)\, dt\, dx_1 \tag{16}$$

provided $\mu_i = 0$. For example, suppose we wished to find $S^{-1}\, \frac{1}{n^3}$ by using the known relation $S^{-1}\, \frac{1}{n} = \frac{\pi - x}{\pi}$. Using equation (16) we find

$$S^{-1}\, \frac{1}{n^3} = \int_0^x \int_0^{x_1} \frac{\pi - t}{\pi}\, dt\, dx_1 + \frac{x}{\pi} \int_0^{\pi} \int_0^{x_1} \frac{\pi - x}{\pi}\, dt\, dx_1 = \frac{x}{6}\,(\pi - x)\,(2\pi - x)\,.$$

(Method h) The inverse transform of a product can be evaluated by use of a convolution integral.[5] Thus

$$S^{-1} \; f_s(n) \, g_c(n) \quad \frac{1}{2} \, F_1^{*} \, G_2$$

where

$$F^{*} G = \int_{-\pi}^{\pi} F(x - \xi) \; G(\xi) \; d\xi$$

and where F_1 is the odd extension of $S^{-1} f_s(n)$ and where G_2 is the even extension of $C^{-1} f_c(n)$.

(Method i) By using contour integration, inverse transforms can often be obtained in terms of definite integrals. One useful integral[24] is

$$\frac{1}{2\pi i} \oint \frac{f(z) \, \sin xz \, dz}{\sin \pi z} \; . \qquad (17)$$

By the theory of residues this becomes $\sum_{n=1}^{k} f(n) \sin nx$ plus the residues of the integrand of Equation (17) evaluated at those poles of f(z) which are included in the contour. If f(z) has only a finite number of poles inside the contour, we obtain an expression for a sine series in terms of a definite integral and a finite sum of terms.

As an example, consider one of the sums found in plate problems, $\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{\cosh n\pi}$. If we use a contour integral $\frac{1}{2\pi i} \oint_{C_n} \frac{\sin xz}{\sin \pi z} \frac{dz}{\cosh \pi z}$

where the contour C_k is indicated in Figure 1

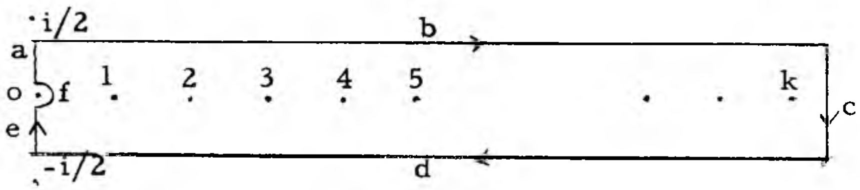


Fig. 1

we obtain

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{\cosh n\pi} = S^{-1}\operatorname{sech} n\pi = \frac{2}{\pi}\int_0^{1/4}\frac{\sinh x\lambda\, d\lambda}{\sinh \pi\lambda \cos\pi\lambda}$$

$$+ \frac{4\sqrt{2}}{\pi}\int_0^{\infty}\frac{d\lambda}{\cosh 2\pi\lambda(\cosh\frac{\pi}{2} - \cos 2\pi\lambda)}\Big\{\cosh\frac{x}{4}\sin x\lambda[-\sinh\pi\lambda\cosh\pi/4\,\sin\pi\lambda$$

$$-\cosh\pi\lambda\sinh\frac{\pi}{4}\cos\pi\lambda] + \sinh\frac{x}{4}\cos x\lambda[-\sinh\pi\lambda\sinh\pi/4\cos\pi\lambda$$

$$+\cosh\pi\lambda\cosh\pi/4\,\sin\pi\lambda] \ .$$

It is seen that the integral form in this case is more difficult to evaluate than the series form.

(Method j) Some series may be summed directly. For example:

$$\Sigma\ e^{-nt}\sin nx = \text{imaginary}\ (\Sigma\ e^{-nt+inx})$$

which is a geometric series of ration e^{-t+ix} . Thus

$$\sum_{n=1}^{\infty} e^{-nt}\sin nx = \frac{e^t \sin x}{e^{2t} - 2e^t\cos x + 1} \ . \tag{18}$$

(Method k) New inverse transforms can be obtained by integrating and differentiating kn own transforms with respect to a parameter. For example, if we multiply Equation (18) by t^p and integrate, using the fact that

$$\Gamma(p+1) = \int_0^{\infty} t^p e^{-nt}\,dt$$

we obtain

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin nx}{n^{p+1}} = \frac{2\sin x}{\pi\Gamma(p+1)}\int_0^{\infty}\frac{t^p e^t\,dt}{e^{2t} - 2e^t\cos x + 1} \ .$$

Another example: Differentiation of the series

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{p^n}{n}\sin nx = \frac{2}{\pi}\arctan\frac{p\sin x}{1 - p\cos x}$$

gives the formula

$$\frac{2}{\pi} \sum_{n=1}^{\infty} p^n \sin nx = \frac{1}{1 - 2p \cos x + p^2}.$$

(Method 1) Sometimes differentiating a series to obtain a differential equation is useful. For example, if it is known that

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

and we wish to sum the function

$$p(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2 (n^2 + a^2)}$$

we can differentiate p (x) and obtain

$$p''(x) - a^2 p(x) = - f(x)$$

with boundary conditions that $p(0) = p(\pi) = 0$. The solution of this equation is

$$p(x) = -\frac{1}{a} \int_0^x \sinh a(x-t) f(t)\, dt + \frac{\sinh ax}{a \sinh a\pi} \int_0^\pi \sinh a(\pi - t) f(t)\, dt .$$

4. Series with missing terms

Note the following functions:

$$f_2(n) = \frac{1}{2} [1 + (-1)^n] = 0, 1, 0, 1, 0, 1, \ldots \quad \text{as } n = 1, 2, 3, \ldots \tag{19}$$

$$f_3(n) = \frac{1}{3} [1 + 2 \cos 2 \frac{n\pi}{3}] = 0, 0, 1, 0, 0, 1, \ldots \text{ as } n = 1, 2, 3, \ldots \tag{20}$$

$$f_4(n) = \frac{1}{4} [1 + (-1)^n + 2 \cos \frac{n\pi}{2}] = 0, 0, 0, 1, 0, 0, 0, 1, \ldots \text{ as } n = 1, 2, 3, \ldots \tag{21}$$

$$f_5(n) = \frac{1}{2\sin^2\frac{\pi}{5}\left(1+\cos\frac{\pi}{5}\right)}\left[\frac{1}{2} - \cos\frac{\pi}{5}\cos\frac{2\pi}{5}\right.$$

$$\left. + \left(\cos\frac{\pi}{5} - \cos\frac{2\pi}{5}\right)\cos\frac{2n\pi}{5} + \frac{1}{2}\cos\frac{4n\pi}{5}\right] = 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, \ldots$$

as $n = 1, 2, 3, \ldots$ (22)

$$f_k(n) = (-1)^{k-1}\prod_{i=1}^{k-1}\frac{\sin(n+i)\pi/k}{\sin i\pi/k} = 0, 0, 0, \ldots, 1, 0, 0, 0, \ldots, 1, \ldots$$

as $n = 1, 2, 3, \ldots$ (23)

This last function could be written

$$f_k(n) = \begin{cases} 1 & \text{if } n = k,\ 2k,\ 3k, \ldots \\ 0 & \text{if } n \neq k,\ 2k,\ 3k, \ldots \end{cases}$$

By expansion of a product of sines in terms of a sum of cosines, we find that Equations (19) → (22) are special cases of Equation (23).

It might be desired to express a sequence 1, 0, 1, 0, 1, 0, ... or a sequence 0,1, 0, 0,1, 0, 0,1, ... as a sum of cosines, and sines. These can be represented as $f_2(n-1)$ and $f_3(n+1)$ or in general $f_k(n+s)$ (where s is the number of slots which the sequence is shifted to the left). For example, from Equation (20) we find

$$f_3(n+1) = \frac{1}{3}\left[1 + 2\cos\frac{2\pi}{3}(n+1)\right]$$

$$= \frac{1}{3}\left[1 - \cos\frac{2\pi n}{3} - \sqrt{3}\sin\frac{2\pi n}{3}\right]$$

$$= 0, 1, 0, 0, 1, 0, 0, 1, \ldots \quad \text{as } n = 1, 2, 3, \ldots$$

We could express a sequence which is 0 on every k^{th} term and 1 elsewhere; i.e., 1, 1, 1, ..., 0, 1, 1, 1, ..., 0, 1, 1, 1, ... as $1 - f_k(n)$.

Thus, $1, 1, 0, 1, 1, 0, \ldots = 1 - f_3(n) = \frac{2}{3}\left(1 - \cos\frac{2n\pi}{3}\right)$.

We could express a sequence 1, 1, 1, 0, 0, 1, 1, 1, 0, 0, ... as $f_5(n + 4) + f_5(n + 3) + f_5(n + 2)$. This procedure will allow us to express any periodic sequence of zeros and ones in terms of a finite sum of sines and cosines.

To sum a Fourier series which has terms missing (in a periodic fashion) we multiply the coefficient $f_s(n)$ by the appropriate formula indicated above. We will obtain terms of the type $f_s(n) \cos \alpha p \sin \alpha x$ which can be summed by #42 - 45 in Table I. For example, suppose we wished to sum the series

$$\frac{2}{\pi} \sum_{\substack{\text{mult}\\ \text{of}\\ 3}} \frac{\sin nx}{n} = \frac{2}{\pi}\left[\frac{\sin 3x}{3} + \frac{\sin 6x}{6} + \ldots\right].$$

Let $F(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n}$.

Then

$$f(x) = \begin{cases} \frac{1}{3}\left[F(x) + F\left(\frac{2}{3}\pi + x\right) - F\left(\frac{2}{3}\pi - x\right)\right] & 0 < x < \frac{\pi}{3} \\ \frac{1}{3}\left[F(x) - F\left(\frac{4}{3}\pi - x\right) - F\left(\frac{2}{3}\pi - x\right)\right] & \pi/3 < x < \frac{2}{3}\pi \\ \frac{1}{3}\left[F(x) - F\left(\frac{4}{3}\pi - x\right) + F\left(x - \frac{2}{3}\pi\right)\right] & \frac{2\pi}{3} < x < \pi \end{cases} \qquad (24)$$

From the tables $F(x) = \frac{\pi - x}{\pi}$ so that

$$f(x) = \begin{cases} \frac{1}{3} - \frac{x}{\pi} & 0 < x < \frac{2\pi}{3} \\ 1 - \frac{x}{\pi} & \frac{2\pi}{3} < x < \pi \end{cases}$$

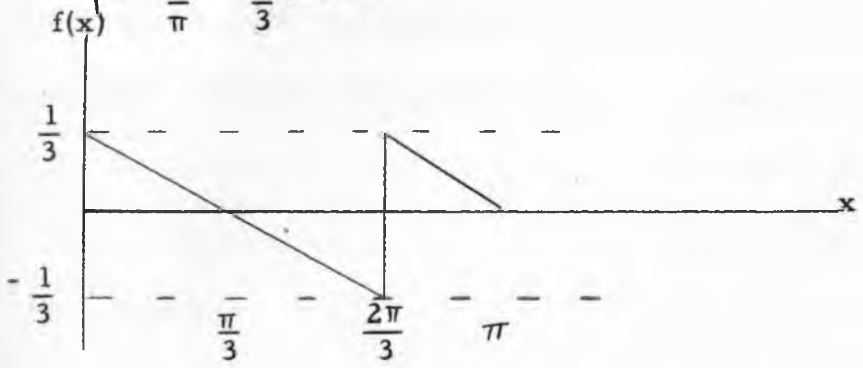


Fig. 2

Equation (24) is the general expression for $S^{-1} f_s(n) f_3(n)$. Two other formulae are useful:

$$S^{-1} f_s(n) f_2(n) = \frac{1}{2}[F(x) - F(\pi - x)] \tag{25}$$

$$S^{-1} f_s(n) f_4(n) = \begin{cases} \frac{1}{4}[F(x) - F(\pi - x) + 2F(\frac{\pi}{2} + x) - 2F(\frac{\pi}{2} - x)] & \text{if } 0 < x < \frac{\pi}{2} \\ \frac{1}{4}[F(x) - F(\pi - x) - 2F(\frac{3\pi}{2} - x) + 2F(x - \frac{\pi}{2})] & \text{if } \frac{\pi}{2} < x < \pi \end{cases} \tag{26}$$

Suppose that we wished to sum the series $1 + \frac{1}{4^2} + \frac{1}{7^2} + \frac{1}{10^2} + \frac{1}{13^2} + \ldots$.

By the above formulae this would become

$$\cos x + \frac{\cos 4x}{4^2} + \frac{\cos 7x}{7^2} + \ldots \Big|_{\text{at } x = 0}$$

$$\frac{\pi}{2} C^{-1} f_c(n) f_3(n-1) \Big|_{\text{at } x = 0}$$

$$\frac{\pi^2}{18} + \frac{1}{\sqrt{3}} \int_{\frac{2}{3}\pi}^{\pi} \log \left(2 \sin \frac{t}{2}\right) dt \quad .$$

5. Abbreviations and notations

Let the capital letter S and C stand for hyperbolic sine and hyperbolic cosine. Let the small letters s and c stand for sine and cosine. If no subscript is used, the argument is understood to be $a(\pi - x)$. Bars over the letters indicate the function is to be evaluated at $x = 0$. Thus

$S \equiv S_a \equiv \sinh a(\pi - x)$ $\qquad$ $C = \cosh a(\pi - x)$

$S_b \equiv \sinh b(\pi - x)$ $\qquad$ $c_b = \cos b(\pi - x)$

$\bar{S} = \sinh a\pi$ $\qquad$ $\bar{c}_b = \cos b\pi$

For all formulae in the cosine tables, $f_c(0) = 0$.

If one wished an inverse transform of a function $f_s(n)$ not even or odd, one could express it as $f_s(n) = \frac{1}{2}[f_s(n) + f_s(-n)] + \frac{1}{2}[f_s(n) - f_s(-n)]$. The first bracket is even and the second is odd. For example:

$$\frac{1}{n^2 + an + b} = \frac{1}{2}\left[\frac{1}{n^2 + an + b} + \frac{1}{n^2 - an + b}\right] + \frac{1}{2}\left[\frac{1}{n^2 + an + b} - \frac{1}{n^2 - an + b}\right]$$

$$= \frac{n^2 + b}{n^4 + (2b - a^2)n^2 + b^2} - \frac{an}{n^4 + (2b - a^2)n^2 + b^2}.$$

The first term of this expression has an inverse sine function of the type found in Table IC, whereas the second term is of the type found in Table IA.

C. Table of Inverse Sine and Cosine Transforms

Table IA

Inverse Sine Transforms Odd $f_s(n)$

Rational functions whose denominators are polynomials of degree k.

k	#	$f_s(n)$	$f(x) = S^{-1} f_s(n) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx$	Method
1	1	$1/n$	$\frac{\pi - x}{\pi}$	Churchill Table[5]
2	2	$n/(n^2 + a^2)$	$\frac{S}{\bar{S}}$	
	3	$n/(n^2 - a^2)$	$\frac{s}{\bar{s}}$	
3	4	$1/n^3$	$\frac{x}{6\pi} (\pi - x)(2\pi - x)$	
	5	$\frac{1}{n(n^2 + a^2)}$	$\frac{1}{a^2}\left[\frac{\pi - x}{\pi} - \frac{S}{\bar{S}}\right]$	Method a # 1, 2
	6	$\frac{1}{n(n^2 - a^2)}$	$-\frac{1}{a^2}\left[\frac{\pi - x}{\pi} - \frac{s}{\bar{s}}\right]$	Method a # 1, 3
4	7	$\frac{n}{(n^2 - a^2)^2}$	$\frac{\pi \sin ax}{2a\,\bar{s}^2} - \frac{x}{2a}\,\frac{c}{\bar{s}}$	Churchill Table
	8	$\frac{n}{(n^2 + a^2)^2}$	$-\frac{\pi \sinh ax}{2a\,\bar{S}^2} + \frac{x\,C}{2a\,\bar{S}}$	Method b # 7
	9	$\frac{n^3}{(n^2 - a^2)^2}$	$\frac{s}{\bar{s}} + \frac{a\pi \sin ax}{2\,\bar{s}^2} - \frac{axc}{2\bar{s}}$	Method a # 3, 7
	10	$\frac{n^3}{(n^2 + a^2)^2}$	$\frac{S}{\bar{S}} + \frac{a\pi \sinh ax}{2\bar{S}^2} - \frac{axC}{2\bar{S}}$	Method b # 9
4	11	$\frac{n}{n^4 - a^4}$	$\frac{1}{2a^2}\left[\frac{s}{\bar{s}} - \frac{S}{\bar{S}}\right]$	Method a # 2, 3

k	#	$f_s(n)$	$f(x) = S^{-1} f_s(n)$	Method
	12	$\frac{n}{n^4 + 4a^4}$	$\frac{1}{2a^2} \quad \frac{\bar{C}\bar{s}\,Sc - \bar{S}\bar{c}\,Cs}{\bar{C}^2 - \bar{c}^2}$	Method b #11
4	13	$\frac{n^3}{n^4 - a^4}$	$\frac{1}{2}\left[\frac{s}{\bar{s}} + \frac{S}{\bar{S}}\right]$	Method a # 2, 3
	14	$\frac{n^3}{n^4 + 4a^4}$	$\frac{\bar{S}\bar{c}\,Sc + \bar{C}\bar{s}\,Cs}{\bar{C}^2 - \bar{c}^2}$	Method b # 13
	15	$\frac{n}{(n^2 + a^2)(n^2 - b^2)}$	$\frac{1}{a^2 + b^2}\left[\frac{-S_a}{\bar{S}_a} + \frac{s_b}{\bar{s}_b}\right]$	Method a # 2, 3
	16	$\frac{n}{(n^2 - a^2)(n^2 - b^2)}$	$\frac{1}{b^2 - a^2}\left[\frac{-s_a}{\bar{s}_a} + \frac{s_b}{\bar{s}_b}\right]$	Method b # 15
	17	$\frac{n}{(n^2 + a^2)(n^2 + b^2)}$	$\frac{1}{b^2 - a^2}\left[\frac{S_a}{\bar{S}_a} - \frac{S_b}{\bar{S}_b}\right]$	Method b # 15
	18	$\frac{n^3}{(n^2 + a^2)(n^2 - b^2)}$	$\frac{1}{a^2 + b^2}\left[a^2\frac{S_a}{\bar{S}_a} + b^2\frac{s_b}{\bar{s}_b}\right]$	Method a # 2, 3
	19	$\frac{n^3}{(n^2 - a^2)(n^2 - b^2)}$	$\frac{1}{b^2 - a^2}\left[-a^2\frac{s_a}{\bar{s}_a} + b^2\frac{s_b}{\bar{s}_b}\right]$	Method b # 18
	20	$\frac{n^3}{(n^2 + a^2)(n^2 + b^2)}$	$\frac{1}{a^2 - b^2}\left[a^2\frac{S_a}{\bar{S}_a} - b^2\frac{S_b}{\bar{S}_b}\right]$	Method b # 19
	21	$\frac{n}{n^4 + rn^2 + t}$	$\frac{1}{\sqrt{r^2 - 4t}}\left[\frac{s_a}{\bar{s}_a} - \frac{s_b}{\bar{s}_b}\right]$	Method b # 15
	a)	$r < 0$,	$a^2 = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$	
		$r^2 > 4t > 0$	$b^2 = \frac{-r}{2} - \frac{1}{2}\sqrt{r^2 - 4t}$	
			(take positive square roots)	

k	#	$f_s(n)$	$f(x)=S^{-1}f_s(n)$	Method
	b)	$r < 0 \quad r^2 > 4t$ $t < 0$	$\frac{1}{\sqrt{r^2 - 4t}} \left[\frac{s_a}{\bar{s}_a} - \frac{S_b}{\bar{S}_b}\right]$ $a^2 = -\frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$ $b^2 = -\frac{r}{2} - \frac{1}{2}\sqrt{r^2 - 4t}$	
4	c)	$r > 0 \quad r^2 > 4t > 0$	$\frac{1}{\sqrt{r^2 - 4t}} \left[\frac{S_a}{\bar{S}_a} - \frac{S_b}{\bar{S}_b}\right]$ $a^2 = \frac{r}{2} - \frac{1}{2}\sqrt{r^2 - 4t}$ $b^2 = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$	Method b # 15
	d)	$t > 0 \quad r^2 < 4t$	$\frac{a}{\sqrt{4t - r^2}} \frac{}{(\bar{C}^2 - \bar{c}^2)} \left[\bar{C}_a \bar{s}_b S_a c_b - \bar{S}_a \bar{c}_b C_a s_b\right]$ $a^2 = \frac{1}{4}(r + 2\sqrt{t})$ $b^2 = \frac{1}{4}(-r + 2\sqrt{t})$	
	22	$\frac{n^3}{n^4 + rn^2 + t}$	$\frac{1}{\sqrt{r^2 - 4t}} \left[\frac{a^2 s_a}{\bar{s}_a} - \frac{b^2 s_b}{\bar{s}_b}\right]$	Method b # 19
	a)	$r < 0$ $r^2 > 4t > 0$	$a^2 = -\frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$ $b^2 = -\frac{r}{2} - \frac{1}{2}\sqrt{r^2 - 4t}$	

k		$f_s(n)$	$f(x)\ S^{-1}\ f_s(n)$	Method
	b)	$r<0 \quad r^2>4t$ $t<0$	$\frac{1}{\sqrt{r^2-4t}}\left[\frac{a^2 s_a}{\bar{s}_a}+\frac{b^2 S_b}{\bar{S}_b}\right]$ $a^2=\frac{-r}{2}+\frac{1}{2}\sqrt{r^2-4t}$ $b^2=\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$	
	c)	$r>0$ $r^2>4t>0$	$\frac{1}{\sqrt{r^2-4t}}\left[-\frac{a^2 S_a}{\bar{S}_a}+\frac{b^2 S_b}{\bar{S}_b}\right]$ $a^2=\frac{r}{2}-\frac{1}{2}\sqrt{r^2-4t}$ $b^2=\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$	
4	22 d)	$r^2<4t \quad t>0$	$\frac{1}{\sqrt{4t-r^2}}\ \frac{1}{C^2-\bar{c}^2}[r\ (C_a s_b\ \bar{S}_a \bar{c}_b$ $-S_a c_b\ \bar{C}_a \bar{s}_b)+\sqrt{4t-r^2}\ (C_a s_b\ \bar{C}_a \bar{s}_b$ $+S_a c_b\ \bar{S}_a \bar{c}_b)]\quad a^2=\frac{1}{4}(r+2\sqrt{t})$ $b^2=\frac{1}{4}\ (-r+2\sqrt{t})$	Method b # 19
5	23	$\frac{1}{n^5}$	$-\frac{x^5}{120\ \pi}+\frac{x^4}{24}-\frac{\pi\ x^3}{18}+\frac{\pi^3 x}{45}$	Method d Knopp p 522
	24	$\frac{1}{n^3\ (n^2+a^2)}$	$\frac{x}{6\pi\ a^2}\ (\pi-x)\ (2\pi-x)-\frac{1}{\pi\ a^4}\ (\pi-x)$ $+\frac{S}{a^4 \bar{S}}$	Method a # 1, 2, 4
	25	$\frac{1}{n^3\ (n^2-a^2)}$	$-\frac{x}{6\pi\ a^2}\ (\pi-x)\ (2\pi-x)-\frac{1}{\pi\ a^4}\ (\pi-x)$ $+\frac{s}{a^4 \bar{s}}$	Method b # 24

k	#	$f_s(n)$	$f(x)\ S^{-1}\ f_s(n)$	Method
	26	$\frac{1}{n(n^4 - a^4)}$	$-\frac{(\pi - x)}{\pi a^4} + \frac{1}{2a^4}\left[\frac{s}{\bar{s}} + \frac{S}{\bar{S}}\right]$	Method a # 1, 13
	27	$\frac{1}{n(n^4 + 4a^4)}$	$\frac{\pi - x}{4\pi a^4} - \frac{1}{4a^4}\ \frac{1}{\bar{C}^2 - \bar{c}^2}(\bar{S}\bar{c}Sc + \bar{C}\bar{s}Cs)$	Method a # 1, 14
	28	$\frac{1}{n(n^2 - a^2)^2}$	$\frac{\pi - x}{\pi a^4} - \frac{1}{a^4}\frac{s}{\bar{s}} - \frac{1}{2a^3\bar{s}^2}(-\pi \sin ax$ $+ x\, c\, \bar{s})$	Method a # 1, 3, 9
	29	$\frac{1}{n(n^2 + a^2)^2}$	$\frac{\pi - x}{\pi a^4} - \frac{1}{a^4}\frac{S}{\bar{S}} - \frac{1}{2a^3\bar{S}^2}(-\pi \sinh ax$ $+ x\, C\, \bar{S})$	Method b # 28
5	30	$\frac{1}{n(n^2 - a^2)(n^2 - b^2)}$	$\frac{\pi - x}{\pi a^2 b^2} + \frac{s_a}{a^2(a^2 - b^2)\bar{s}_a} - \frac{s_b}{b^2(a^2 - b^2)\bar{s}_b}$	Method b # 1, 3
	31	$\frac{1}{n(n^2 + a^2)(n^2 - b^2)}$	$\frac{-\pi - x}{\pi a^2 b^2} + \frac{S_a}{a^2(a^2 + b^2)\bar{S}_a} + \frac{s_b}{b^2(a^2 + b^2)\bar{s}_b}$	Method c # 30
	32	$\frac{1}{n(n^2 + a^2)(n^2 + b^2)}$	$\frac{\pi - x}{\pi a^2 b^2} + \frac{S_a}{a^2(a^2 - b^2)\bar{S}_a} - \frac{S_b}{b^2(a^2 - b^2)\bar{S}_b}$	Method c # 30
	33	$\frac{1}{n(n^4 + rn^2 + t)}$	$\frac{1}{t}\ S^{-1}\left[\frac{1}{n} - \frac{n^3}{n^4 + rn^2 + t} - \frac{rn}{n^4 + rn^2 + t}\right]$	Method b # 21, 22
k 5	34	$\frac{1}{n^{2\lambda + 1}}$	$(-1)^{\lambda - 1}\ \frac{(2\pi)^{2\lambda + 1}}{\pi}\ P_{2\lambda + 1}\left(\frac{x}{2\pi}\right)$	Method d Knopp p 522

k		$f_s(n)$	$f(x) S^{-1} f_s(n)$	Method
	35	$\frac{1}{n^{2\lambda+1}(n^2-a^2)}$	$\frac{1}{a^{2\lambda+2}}\frac{s}{\bar{s}} + \frac{(-1)^\lambda}{\pi}\sum_{r=0}^{\lambda}\frac{(-1)^r}{a^{2r+2}} (2\pi)^{2\lambda-2r+1} P_{2\lambda-2r+1}\left(\frac{x}{2\pi}\right)$	Method a
	36	$\frac{1}{n^{2\lambda+1}(n^2+a^2)}$	$\frac{(-1)^{\lambda+1}}{a^{2\lambda+2}}\frac{S}{\bar{S}} - \frac{(-1)^\lambda}{\pi}\sum_{r=0}^{\lambda}\frac{(2\pi)^{2\lambda-2r+1}}{a^{2r+2}} P_{2\lambda-2r+1}\left(\frac{x}{2\pi}\right)$	Method b # 35
	37	$\frac{1}{n^{2\lambda+1}(n^4-a^4)}$	$\frac{1}{2a^{2\lambda+4}}\left[\frac{s}{\bar{s}} + (-1)^\lambda \frac{S}{\bar{S}}\right] + \frac{(-1)^\lambda}{2\pi a^2} \sum_{r=0}^{\lambda}\frac{[(-1)^r+1]}{a^{2r+2}} (2\pi)^{2\lambda-2r+1} P_{2\lambda-2r+1}\left(\frac{x}{2\pi}\right)$	Method a
	38	$\frac{1}{n^{4\lambda+3}(n^4+4a^4)}$	$\frac{(-1)^{\lambda+1}}{(2a^2)^{2\lambda+3}} \frac{1}{\bar{C}^2-\bar{c}^2}[\bar{C}\bar{s}Sc - \bar{S}\bar{c}Cs] + \frac{1}{\pi}\sum_{r=0}^{\lambda}\frac{(-1)^r}{(4a^4)^{r+1}} (2\pi)^{4\lambda-4r+3} P_{4\lambda-4r+3}\left(\frac{x}{2\pi}\right)$	Method a
k 5	39	$\frac{1}{n^{4\lambda+1}(n^4+4a^4)}$	$\frac{(-1)^{\lambda+1}}{(2a^2)^{2\lambda+2}} \frac{1}{\bar{C}^2-\bar{c}^2}[\bar{S}\bar{c}Sc + \bar{C}\bar{s}Cs] - \frac{1}{\pi}\sum_{r=0}^{\lambda}\frac{(-1)^r}{(4a^4)^{r+1}} (2\pi)^{4\lambda-4r+1} P_{4\lambda-4r+1}\left(\frac{x}{2\pi}\right)$	Method a
	40	$\frac{1}{n^{2\lambda-1}(n^2-a^2)^2}$	$\frac{\pi a \sin ax - axc\bar{s} - 2\lambda s\bar{s}}{2a^{2\lambda+2}\bar{s}^2} + \frac{(-1)^\lambda}{\pi}\sum_{r=0}^{\lambda-1}\frac{(r+1)(-1)^r}{a^{2r+4}} (2\pi)^{2\lambda-2r-1} P_{2\lambda-2r-1}\left(\frac{x}{2\pi}\right)$	Method a

k	$f_s(n)$	$f(x)=S^{-1} f_s(n)$	Method
41	$\frac{1}{n^{2\lambda-1}(n^2+a^2)^2}$	$(-1)^{\lambda} \frac{[-\pi a \sinh ax + ax C\bar{S} + 2\lambda S\bar{S}]}{2a^{2\lambda+2}\bar{S}^2}$ $+ \frac{(-1)^{\lambda}}{\pi} \sum_{r=0}^{\lambda-1} \frac{(r+1)(2\pi)^{2\lambda-2r-1}}{a^{4+2r}} P_{2\lambda-2r-1}\left(\frac{x}{2\pi}\right)$	Method a

Table IB

Miscellaneous $f_s(n)$

42	$f(n) \sin na$ $0 < a < \pi/2$ $C^{-1}f(n) = F(x)$	$1/2\,[F(a-x) - F(a+x)]$ $0 < x < a$ $1/2\,[F(x-a) - F(a+x)]$ $a < x < \pi-a$ $1/2\,[F(x-a) - F(2\pi-a-x)]$ $\pi-a < x < \pi$	Method f
43	$f(n) \sin na$ $\pi/2 < a < \pi$ $C^{-1}f(n) = F(x)$	$1/2\,[F(a-x) - F(a+x)]$ $0 < x < \pi-a$ $1/2\,[F(a-x) - F(2\pi-a-x)]$ $\pi-a < x < a$ $1/2\,[F(x-a) - F(2\pi-a-x)]$ $a < x < \pi$	Method f
a)	$\frac{1}{n^2} \sin na$ $0 < a < \pi$	$x\left(\frac{\pi-a}{\pi}\right)$ $0 < x < a$ $a\left(\frac{\pi-x}{\pi}\right)$ $a < x < \pi$	Method f
44	$f(n) \cos na$ $0 < a < \pi/2$ $S^{-1}f(n) = F(x)$	$1/2\,[F(a+x) - F(a-x)]$ $0 < x < a$ $1/2\,[F(a+x) + F(x-a)]$ $a < x < \pi-a$ $1/2\,[-F(2\pi-a-x) + F(x-a)]$ $\pi-a < x < \pi$	Method f
45	$f(n) \cos na$ $\pi/2 < a < \pi$ $S^{-1}f(n) = F(x)$	$1/2\,[F(a+x) - F(a-x)]$ $- \; x \; \pi-a$ $1/2\,[-F(2\pi-a-x) - F(a-x)]$ $\pi-a < x < a$ $1/2\,[-F(2\pi-a-x) + F(x-a)]$ $a < x < \pi$	Method f

k #	$f_s(n)$	$f(x) = S^{-1} f_s(n)$	Method
a)	$\frac{1}{n} \cos na$ $0 < a < \pi$	$-\frac{x}{\pi}$ $\quad 0 < x < a$ $1 - \frac{x}{\pi}$ $\quad a < x < \pi$	Method f
46	$f(n)\,(-1)^{n+1}$	$f(\pi - x)$ where $S^{-1} f_s(n) = f(x)$	Churchill
47	P^n $(P^2 < 1)$	$\frac{2}{\pi} \frac{P \sin x}{1 - 2P \cos x + P^2}$	Method d de Haan 65/3
48	$\frac{P^n}{n}$ $(P^2 < 1)$	$\frac{2}{\pi} \arctan \frac{P \sin x}{1 - P \cos x}$	Method k # 47
49	$\frac{P^n}{n^2}$ $(P^2 < 1)$	$\frac{-1}{\pi} \int_0^x \log (1 - 2P \cos x_1 + P^2)\, dx_1$ $= \frac{2}{\pi} \int_0^P \frac{1}{P_1} \arctan \frac{P_1 \sin x}{1 - P_1 \cos x} dP_1$	Method k # 48
50	$\frac{P^n}{n^3}$ $(P^2 < 1)$	$\frac{2}{\pi} \int_0^P \frac{dz_2}{z_2} \int_0^{z_2} \frac{1}{z_1} \arctan \frac{z_1 \sin x}{1 - z_1 \cos x} dz_1$	Method k # 49
51	$\frac{P^n}{n^m}$ $(P^2 < 1)$	$\frac{2}{\pi} \int_0^P \frac{dz_{m-1}}{z_{m-1}} \int_0^{z_{m-1}} \frac{dz_{m-2}}{z_{m-2}} \int_0^{z_{m-3}}$ $\ldots \int_0^{z_2} \frac{1}{z_1} \arctan \frac{z_1 \sin x\, dz_1}{1 - z_1 \cos x}$	Method k # 50
52	nP^n $(P^2 < 1)$	$P \frac{d}{dP} \frac{2}{\pi} \frac{P \sin x}{1 - 2P \cos x + P^2}$	Method k # 47
53	$n^m P^n$ $(P^2 < 1)$	$\left(P \frac{d}{dP}\right)^m \frac{2}{\pi} \frac{P \sin x}{1 - 2P \cos x + P^2}$	Method k # 47
54	$\frac{P^n}{n!}$ $P^2 \leq 1$	$e^{P \cos x} \sin (p \sin x)$	Method c
55	$\binom{a}{n}$	$\frac{2^{a+1}}{\pi} \cos^a \frac{x}{2} \sin \frac{ax}{2}$	Method d de Haan 42/9

Table IC

Table of Even Inverse Sine Transforms

k	#	$f_s(n)$	$f(x)=S^{-1} f_s(n)$	Method
2	56	$\frac{1}{n^2}$	$\frac{-2}{\pi}\int_0^x \log \left\|2\sin \frac{t}{2}\right\| dt = f(x)$ *	Method e
			$\frac{2}{\pi} \sin x \int_0^1 \frac{(\log \frac{1}{u})\, du}{1-2u\cos x + u^2}$	
	57	$\frac{1}{n^2+a^2}$	$a\int_0^x \sinh a(x-t)f(t)dt$	Method 1
			$-a\frac{\sinh ax}{\sinh a\pi}\int_0^\pi \sinh a(\pi-t)f(t)\,dt+f(x)$	
			$=\frac{-2}{\pi}\int_0^x \cosh a(x-t)\log(2\sin\frac{t}{2})dt$	Integration by parts
			$+\frac{2}{\pi}\frac{\sinh ax}{\sinh a\pi}\int_0^\pi \cosh a(\pi-t)\log(2\sin\frac{t}{2})dt$ *	
	58	$\frac{1}{n^2-a^2}$	$-a\int_0^x \sin a(x-t)f(t)dt$	Method b 57
			$+a\frac{\sin ax}{\sin a\pi}\int_0^\pi \sin a(\pi-t)f(t)dt+f(x)$	
			$=\frac{-2}{\pi}\int_0^x \cos a(x-t)\log(2\sin\frac{t}{2})\,dt$	Integration by parts
			$+\frac{2}{\pi}\frac{\sin ax}{\sin a\pi}\int_0^\pi \cos a(\pi-t)\log(2\sin\frac{t}{2})dt$ *	
	59	$\frac{1}{n^4}$	$-\int_0^x (x-t)f(t)dt+\frac{x}{\pi}\int_0^\pi(\pi-t)f(t)dt$ *	Limit of #60 as a→0
	60	$\frac{1}{n^2(n^2+a^2)}$	$\frac{-1}{a}\int_0^x \sinh a(x-t)\,f(t)\,dt$	Method 1
			$+\frac{\sinh ax}{a\sinh a\pi}\int_0^\pi \sinh a(\pi-t)\,f(t)\,dt$ *	

* The f(t) defined in #56 is used in #56 to #70

k	#	$f_s(n)$	$f(x) = S^{-1} f_s(n)$	Method
4	61	$\frac{1}{n^2(n^2 - a^2)}$	$\frac{-1}{a} \int_0^x \sin a(x-t) f(t) dt$	Method c # 60
			$+ \frac{\sin ax}{a \sin a\pi} \int_0^\pi \sin a(\pi-t) f(t) dt$	
	62	$\frac{1}{n^4 - a^4}$	$\frac{-1}{2a} \int_0^x [\sinh a(x-t) + \sin a(x-t)] f(t) dt$	Method a
			$- \frac{\sinh ax}{\sinh a\pi} \int_0^\pi \sinh a(\pi-t) f(t) dt$	
			$- \frac{\sin ax}{\sin a\pi} \int_0^\pi \sin a(\pi-t) f(t) dt$	
	63	$\frac{1}{(n^2 - a^2)(n^2 - b^2)}$	$\frac{-a}{a^2 - b^2} [\int_0^x \sin a (x-t) f(t) dt$	Method a
			$- \frac{\sin ax}{\sin a\pi} \int_0^\pi \sin a (\pi - t) f(t) dt]$	
			$+ \frac{b}{a^2 - b^2} [\int_0^x \sin b(x-t) f(t) dt$	
			$- \frac{\sin bx}{\sin b\pi} \int_0^\pi \sin b(\pi-t) f(t) dt]$	
	64	$\frac{1}{(n^2+a^2)(n^2 + b^2)}$	$\frac{a}{a^2 - b^2} [\int_0^x \sinh a (x-t) f(t) dt$	Method a
			$- \frac{\sinh ax}{\sinh a\pi} \int_0^\pi \sinh a(\pi-t) f(t) dt]$	
			$- \frac{b}{a^2 - b^2} [\int_0^x \sinh b(x-t) f(t) dt$	
			$- \frac{\sinh bx}{\sinh b\pi} \int_0^\pi \sinh b(\pi-t) f(t) dt]$	
	65	$\frac{1}{(n^2 - a^2)(n^2 + b^2)}$	$\frac{-a}{a^2+b^2} [\int_0^x \sin a(x-t) f(t) dt$	Method a

k	#	$f_s(n)$	$f(x) = S^{-1} f_s(n)$	Method
4	65		continued $- \frac{\sin ax}{\sin a\pi} \int_0^x \sin a(\pi-t) f(t) dt]$ $- \frac{b}{a^2+b^2} [\int_0^x \sinh b(x-t) f(t)dt$ $- \frac{\sinh bx}{\sinh b\pi} \int_0^\pi \sinh b(\pi-t) f(t)dt]$	
	66	$\frac{n^2}{(n^2+a^2)(n^2+b^2)}$	$\frac{a^3}{a^2-b^2} [\int_0^x \sinh a(x-t) f(t) dt$ $- \frac{\sinh ax}{\sinh a\pi} \int_0^\pi \sinh a(\pi-t) f(t) dt]$ $- \frac{b^3}{a^2-b^2} [\int_0^x \sinh b(x-t) f(t) dt$ $- \frac{\sinh bx}{\sinh b\pi} \int_0^\pi \sinh b(\pi-t) f(t)dt] + f(x)$	Method a
	67	$\frac{n^2}{(n^2-a^2)(n^2-b^2)}$	$\frac{-a^3}{a^2-b^2} [\int_0^x \sin a(x-t) f(t) dt$ $- \frac{\sin ax}{\sin a\pi} \int_0^\pi \sin a(\pi-t) f(t) dt]$ $+ \frac{b^3}{a^2-b^2} [\int_0^x \sin b(x-t) f(t) dt$ $- \frac{\sin bx}{\sin b\pi} \int_0^\pi \sin b(\pi-t) f(t) dt] + f(x)$	Method a
	68	$\frac{n^2}{(n^2+a^2)(n^2-b^2)}$	$\frac{a^3}{a^2+b^2} [\int_0^x \sinh a(x-t) f(t) dt$ $- \frac{\sinh ax}{\sinh a\pi} \int_0^\pi \sinh a(\pi-t) f(t) dt]$ $- \frac{b^3}{a^2+b^2} [\int_0^x \sin b(x-t) f(t) dt$	Method a

k		$f_s(n)$	$f(x)\ S^{-1}\ f_s(n)$	Method
4	68		continued $- \frac{\sin bx}{\sin b\pi} \int_0^{\pi} \sin b(\pi-t)\ f(t)\ dt] + f(x)$	
	69	$\frac{n^2}{n^4 - a^4}$	$\frac{a}{2} [\int_0^x \sinh a(x-t)\ f(t)\ dt$ $- \frac{\sinh ax}{\sinh a\pi} \int_0^{\pi} \sinh a(\pi-t) f(t)\ dt$ $- \int_0^x \sin a(x-t)\ f(t)\ dt$ $+ \frac{\sin ax}{\sin a\pi} \int_0^{\pi} \sin a(\pi-t)\ f(t)dt] + f(x)$	Method a

Table ID

Miscellaneous Transforms in Terms of Definite Integrals
Sine Transforms from "Tables of Definite Integrales"[7]

#	$f_s(n)$	$f(x) = S^{-1} f_s(n)$	Method
70	$\frac{1}{n+q}$	$8q \int_0^1 \frac{\sinh[\log t \frac{x-\pi}{2\pi}]}{(1-t)\, t^{1/2}} \frac{dt}{4\pi^2q^2+(\log t)^2}$	132/13 Examen
71	$\frac{1}{n+qr}$	$-2q \int_0^1 \frac{\sinh\left(\frac{x-\pi}{\pi} r\log t\right)}{\sinh(r\log t)} \frac{1}{q^2+(\log t)^2} \frac{dt}{t}$	131/12
72	$\frac{1}{n^q}$	$\frac{2\sin x}{\pi\Gamma(q)} \int_0^1 \frac{(\log t)^{q-1}dt}{1-2t\cos x+t^2}$	113/9
73	$\frac{1}{\sqrt{n}}$	$\frac{1}{\pi^{3/2}} \sin x \int_0^\infty \frac{dt}{(\cosh t-\cos x)t^{1/2}}$	98/28
74	$\frac{1}{n+a}$	$2a \int_0^\infty \frac{\sinh \frac{\pi-x}{\pi} t}{\sinh t} \frac{dt}{a^2\pi^2 + t^2}$	97/18
75	$\frac{1}{n^q}$ (q odd)	$\frac{\sin x}{\pi\Gamma(q+1)} \int_0^\infty \frac{(\sinh t)\, t^q dt}{(\cosh t-\cos x)^2}$	88/14
76	$\frac{1}{n^q}$	$\frac{\sin x}{\pi\Gamma(q)} \int_0^\infty \frac{t^{q-1}dt}{\cosh t-\cos x}$	88/5
77	$\frac{1}{n(n+1)}$	$\frac{2}{\pi} \sin x \int_0^{\pi/4} \frac{1-\tan t\, dt}{1-\cos x \sin 2t}$	36/7
78	$\frac{1}{n(n+1)}$	$\frac{2}{\pi} \sin x \int_0^1 \frac{(1-t)\, dt}{1-2t\cos x+t^2}$ $= \frac{2}{\pi} \sin x \left[\frac{\pi-x}{2} \tan \frac{x}{2} - \log(2 \sin \frac{x}{2})\right]$	6/5
79	$\frac{1}{n+r}$	$\frac{2}{\pi} \sin x \int_0^1 \frac{t^r dt}{1-2t\cos x + t^2}$	6/10

	$f_s(n)$	$f(x)=S^{-1}f_s(n)$	Method
80	$n!\,\frac{q^n}{\Gamma(n+P+1)}$	$\frac{2q}{\pi\Gamma(P+1)}\int_0^1 \frac{(\sin x)(1-t)^P dt}{1-2qt\cos x+q^2t^2}$	6/13
81	$\frac{1}{2n-1}$	$\frac{\sin x}{\pi}\int_0^1 \frac{dt}{(1-2t\cos x+t^2)\,t^{1/2}}$	10/17
82	$\frac{1}{2n+1}$	$\frac{\sin x}{\pi}\int_0^1 \frac{t^{1/2}dt}{1-2t\cos x+t^2}$	14/1
83	$\frac{e^{-nP}}{n^2-r^2}$	$\frac{e^{-Pr}\sin xr}{r\sin\pi r}+2\int_0^1 \frac{\cos(\frac{P}{\pi}\log t)}{\pi^2r^2+(\log t)^2}\,\frac{t^{\frac{\pi-x}{\pi}}-t^{\frac{x-\pi}{\pi}}}{1-t^2}\,dt$	407/11
84	$\frac{ne^{-nP}}{n^2-r^2}$	$\frac{e^{-Pr}\sin xr}{\sin\pi r}+\frac{2}{\pi}\int_0^1 \frac{\sin\frac{P}{\pi}\log t)}{\pi^2r^2+(\log t)^2}\,\frac{t^{\frac{\pi-x}{\pi}}-t^{\frac{x-\pi}{\pi}}}{1-t^2}\log t\,dt$	407/7
85	$\frac{e^{-nP}}{n^2-r^2}$	$\frac{e^{-Pr}\sin xr}{r\sin\pi r}-\frac{2}{\pi}\int_0^\infty \frac{\sinh(\pi-x)t}{\sinh\pi t}\,\frac{\cos Pt}{r^2+t^2}dt$	389/22
86	$\frac{ne^{-nP}}{n^2-r^2}$	$\frac{e^{-Pr}\sin xr}{\sin\pi r}-\frac{2}{\pi}\int_0^\infty \frac{\sinh(\pi-x)t}{\sinh\pi t}\,\frac{t\sin Pt}{r^2+t^2}dt$	389/23
87	$\frac{(-1)^{n+1}}{n^q}$	$-\frac{1}{\pi}\frac{\sin x}{\Gamma(q)}\int_0^\infty \frac{q(\cosh t-\cos x-t\sinh t)}{(\cosh t-\cos x)^2}\,t^{q-1}\log t\,dt$	356/17
88	$\frac{1}{n+q}$	$2q\int_0^{\pi/2} \frac{\cos^{\frac{x-\pi}{\pi}}t-\cos^{\frac{\pi-x}{\pi}}t}{\pi^2q^2+(\log\cos t)^2}\,\frac{dt}{\sin t}$	328/3

	$f_s(n)$	$f(x)\ \ S^{-1} f_s(n)$	Method
89	$\frac{1}{n+q}$	$2q \int_0^{\pi/2} \frac{\sin^{\frac{x-\pi}{\pi}} t - \sin^{\frac{\pi-x}{\pi}} t}{\pi^2 q^2 + (\log \sin t)^2} \frac{dt}{\cos t}$	327/2
90	$\frac{(-1)^{n+1}}{n^q}$	$\frac{2\sin x}{\pi\Gamma(q)} \int_0^{\pi/4} \frac{(\log \cot t)^{q-1}}{1-\cos x \sin 2t}\, dt$	300/15
91	$\frac{P^{2n}}{(n!)^2}$	$\frac{2}{\pi} \int_0^{\pi} e^{2P\cos t \cos x/2} \sin\left(2P\cos t \sin \frac{x}{2}\right) dt$	277/10
92	$\frac{1}{n^q}$	$\frac{2\sin x}{\pi\Gamma(q)} \int_0^{\pi/2} \frac{\tan^q t}{\cosh(\tan t)-\cos x} \frac{dt}{\sin 2t}$	275/20
93	$\frac{1}{n+q}$	$\frac{2}{\pi} \int_0^{\pi/2} \frac{\sinh[q(\pi-x)\tan t]}{\sinh(q\pi \tan t)}\, dt$	274/13
94	$\frac{1}{\sqrt{n}}$	$\frac{2\sin x}{\pi^{3/2}} \int_0^1 \frac{dt}{(1-2t\cos(x)+t^2)\sqrt{\log 1/t}}$	133/5
95	$\frac{r^n}{(n+P)^q}$	$\frac{2r}{\pi} \frac{\sin x}{\Gamma(q)} \int_0^{\infty} \left(\log \frac{1}{t}\right)^{q-1} \frac{t^P dt}{1-2rt\cos(x)+r^2t^2}$	140/14

Table IIA

Inverse Cosine Transforms (even $f_s n$)

In each case $f_c(0)=0$

k	#	$f_c(n)$	$f(x) = C^{-1} f_c(n) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_c(n) \cos nx$	Method
1		none		
2	1	$\frac{1}{n^2}$	$\frac{x^2}{2\pi} - x + \frac{\pi}{3}$	Churchill Tables[5]
	2	$\frac{1}{n^2+a^2}$	$\frac{C}{a\bar{S}} - \frac{1}{\pi a^2}$	
	3	$\frac{1}{n^2-a^2}$	$-\frac{c}{a\bar{s}} + \frac{1}{\pi a^2}$	
3		none		
4	4	$\frac{1}{n^4}$	$-\frac{x^4}{24\pi} + \frac{x^3}{6} - \frac{\pi x^2}{6} + \frac{\pi^3}{45}$	Method g #23 in sine tables
	5	$\frac{1}{n^4-a^4}$	$\frac{1}{\pi a^4} - \frac{1}{2a^2}\left[\frac{c}{a\bar{s}} + \frac{C}{a\bar{S}}\right]$	Method a
	6	$\frac{1}{n^4+4a^4}$	$-\frac{1}{4\pi a^4} + \frac{1}{8a^3}\left[\frac{\bar{S}\bar{c}Cc+\bar{C}\bar{s}Ss-\bar{S}\bar{c}Ss+\bar{C}\bar{s}Cc}{\bar{C}^2-\bar{c}^2}\right]$	Method b # 5
	7	$\frac{1}{(n^2-a^2)^2}$	$-\frac{1}{\pi a^4} + \frac{1}{2a^3\bar{s}^2}[\bar{s}c+\pi a\bar{c}c+(\pi-x)a\bar{s}s]$	Method b # 9
	8	$\frac{1}{(n^2+a^2)^2}$	$-\frac{1}{\pi a^4} + \frac{1}{2a^3\bar{S}^2}[\bar{S}C+\pi a\bar{C}C-(\pi-x)a\bar{S}S]$	Method b # 7
	9	$\frac{1}{(n^2-a^2)(n^2-b^2)}$	$-\frac{1}{\pi a^2 b^2} + \frac{1}{a^2-b^2}\left[\frac{-c_a}{a\bar{s}_a} + \frac{c_b}{b\bar{s}_b}\right]$	Method a
	10	$\frac{1}{(n^2-a^2)(n^2+b^2)}$	$\frac{1}{\pi a^2 b^2} - \frac{1}{a^2+b^2}\left[\frac{c_a}{a\bar{s}_a} + \frac{C_b}{b\bar{S}_b}\right]$	Method b # 9

k	#	$f_c(n)$	$f(x)=C^{-1}f_c(n)$	Method
	11	$\frac{1}{(n^2+a^2)(n^2+b^2)}$	$-\frac{1}{\pi a^2 b^2}+\frac{1}{a^2-b^2}\left[\frac{-C_a}{a\bar{S}_a}+\frac{C_b}{b\bar{S}_b}\right]$	Method b #9
4	12	$\frac{1}{n^2(n^2-a^2)}$	$\frac{1}{\pi a^4}+\frac{1}{a^2}\left[\frac{-x^2}{2\pi}+x-\frac{\pi}{3}-\frac{c}{a\bar{s}}\right.$	Method a
	13	$\frac{1}{n^2(n^2+a^2)}$	$\frac{1}{\pi a^4}+\frac{1}{a^2}\left[\frac{x^2}{2\pi}-x+\frac{\pi}{3}-\frac{C}{a\bar{S}}\right]$	Method b #12
	14	$\frac{1}{n^4+rn^2+t}$	$\frac{1}{\sqrt{r^2-4t}}\left[\frac{-c_a}{a\bar{s}}+\frac{c_b}{b\bar{s}_b}\right]-\frac{1}{\pi t}$	Method a
	a)	$r^2-4t>0$ $r<0 \quad t>0$	$a^2=-\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$ $b^2=-\frac{r}{2}-\frac{1}{2}\sqrt{r^2-4t}$	
	b)	$r^2-4t>0$ $r<0 \quad t<0$	$\frac{-1}{\sqrt{r^2-4t}}\left[\frac{c_a}{a\bar{s}}+\frac{C_b}{b\bar{S}}\right]-\frac{1}{\pi t}$ $a^2=-\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$ $b^2=\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$	Method b # 14a
	c)	$r^2-4t>0$ $r>0 \quad t>0$	$\frac{1}{\sqrt{r^2-4t}}\left[\frac{C_a}{a\bar{S}_a}-\frac{C_b}{b\bar{S}_b}\right]-\frac{1}{\pi t}$ $a^2=\frac{r}{2}-\frac{1}{2}\sqrt{r^2-4t}$ $b^2=\frac{r}{2}+\frac{1}{2}\sqrt{r^2-4t}$	Method b #14a
	d)	$r^2-4t>0$ $r>0 \quad t<0$	$\frac{-1}{\sqrt{r^2-4t}}\left[\frac{c_a}{a\bar{s}_a}+\frac{C_b}{b\bar{S}_b}\right]-\frac{1}{\pi t}$	Method b #14a

k	#	$f_c(n)$	$f(x) = C^{-1} f_c(n)$	Method
	d)		continued $a^2 = -\frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$ $b^2 = \frac{r}{2} + \frac{1}{2}\sqrt{r^2 - 4t}$	
4	e)	$r^2 - 4t < 0$ $a < 0$ or $a > 0$ $b > 0$	$\frac{2}{t^{1/2}\sqrt{4t-r^2}} \frac{1}{\bar{C}_a^2 - \bar{c}_b^2}[b(S_a s_b \bar{C}_a \bar{s}_b$ $+ C_a c_b \bar{S}_a \bar{c}_b) + a(C_a c_b \bar{C}_a \bar{s}_b - S_a s_b \bar{S}_a \bar{c}_b)]$ $a^2 = 1/4\ (r+2t^{1/2})$ $b^2 = 1/4\ (-r+2t^{1/2})$	
	15	$\frac{n^2}{(n^2-a^2)(n^2-b^2)}$	$\frac{1}{a^2-b^2}\left[\frac{-ac_a}{\bar{s}_a} + \frac{bc_b}{\bar{s}_b}\right]$	Method a
	16	$\frac{n^2}{(n^2+a^2)(n^2+b^2)}$	$\frac{1}{a^2-b^2}\left[\frac{aC_a}{\bar{S}_a} - \frac{bC_b}{\bar{S}_b}\right]$	Method b # 15
	17	$\frac{n^2}{(n^2+a^2)(n^2-b^2)}$	$\frac{1}{a^2+b^2}\left[\frac{aC_a}{\bar{S}_a} - \frac{bc_b}{\bar{s}_b}\right]$	Method b # 15
	18	$\frac{n^2}{n^4-a^4}$	$\frac{1}{2a}\left[\frac{C}{\bar{S}} - \frac{c}{\bar{s}}\right]$	Method a
	19	$\frac{n^2}{n^4+4a^4}$	$\frac{1}{2a}\frac{1}{\bar{C}^2-\bar{c}^2}[-Cc\bar{S}\bar{s}+Ss\bar{S}\bar{c}+Cc\bar{S}\bar{c}+Ss\bar{C}\bar{s}]$	Method b # 18
	20	$\frac{n^2}{(n^2-a^2)^2}$	$\frac{-1}{2a\bar{s}^2}[c\bar{s} - a(\pi-x)s\bar{s} - a\pi c\bar{c}]$	Method b # 15
	21	$\frac{n^2}{(n^2+a^2)^2}$	$\frac{1}{2a\bar{S}^2}[C\bar{S} + a(\pi-x)S\bar{S} - a\pi C\bar{C}]$	Method b # 20
	22	$\frac{n^2}{n^4+an^2+b}$		

k	#	$f_c(n)$	$f(x) = C^{-1} f_c(n)$	Method
	a)	$a^2 > 4b$	(a) see # 15, 16, 17	
	b)	$a^2 < 4b$	$\frac{2}{\sqrt{4t-a^2}(\bar{C}^2-\bar{c}^2)}[b(S_a s_b \bar{C}_a \bar{s}_b + C_a c_b \bar{C}_a \bar{s}_b$ $+ a(-C_a c_b \bar{C}_a \bar{s}_b + S_a s_b \bar{S}_a \bar{c}_b)]$ $a^2 = 1/4\ (r + 2t^{1/2})$ $b^2 = 1/4\ (-r + 2t^{1/2})$	Method b # 22a
5		none		
$k>5$	23	$\frac{1}{n^{2\lambda}}$	$(-1)^{\lambda-1}\frac{(2\pi)^{2\lambda}}{\pi}P_{2\lambda}\left(\frac{x}{2\pi}\right)$ where $P_\lambda(x)$ is Bernoulli Polynomial	Method d Knopp p 522
	24	$\frac{1}{n^{2\lambda}(n^2-a^2)}$	$\frac{1}{a^{2\lambda+1}}\left(\frac{-c}{\bar{s}} + \frac{1}{\pi a}\right) + \frac{(-1)^\lambda}{\pi}\sum_{S=0}^{\lambda-1}\frac{(-1)^S}{a^{2S+2}}$ $(2\pi)^{2\lambda-2S}P_{2\lambda-2S}\left(\frac{x}{2\pi}\right)$	Method a # 23
	25	$\frac{1}{n^{2\lambda}(n^2+a^2)}$	$\frac{(-1)^\lambda}{a^{2\lambda+1}}\left(\frac{C}{\bar{S}} - \frac{1}{\pi a}\right) - \frac{(-1)^\lambda}{\pi}\sum_{S=0}^{\lambda-1}\frac{1}{a^{2S+2}}$ $(2\pi)^{2\lambda-2S}P_{2\lambda-2S}\left(\frac{x}{2\pi}\right)$	Method a # 23
	26	$\frac{1}{n^{2\lambda}(n^4-a^4)}$ λ odd	$\frac{1}{2a^{2\lambda+3}}\left(\frac{C}{\bar{S}} - \frac{c}{\bar{s}}\right) - \frac{1}{\pi}\sum_{S=0}^{(\lambda-1)/2}\frac{1}{a^{4S+4}}$ $(2\pi)^{2\lambda-4S}P_{2\lambda-4S}\left(\frac{x}{2\pi}\right)$	Method a # 23
	27	$\frac{1}{n^{2\lambda}(n^4-a^4)}$ λ even	$\frac{1}{\pi a^{2\lambda+4}} - \frac{1}{2a^{2\lambda+3}}\left(\frac{c}{\bar{s}} + \frac{C}{\bar{S}}\right)$ $+\frac{1}{\pi}\sum_{S=0}^{\lambda/2-1}\frac{(2\pi)^{2\lambda-4S}}{a^{4s+4}}P_{2\lambda-4S}\left(\frac{x}{2\pi}\right)$	Method a # 23

k	#	$f_c(n)$	$f(x)\quad C^{-1} f_c(n)$	Method
	28	$\frac{1}{n^{2\lambda}(n^4+4a^4)}$ λ odd	$\frac{(-1)^{(\lambda+1)/2}}{2^{\lambda+2}a^{2\lambda+3}} \frac{1}{\bar{C}^2-\bar{c}^2}[-Cc\bar{S}\bar{s}+Ss\bar{S}\bar{c}+Cc\bar{S}\bar{c}$ $+Ss\bar{C}\bar{s}] + \frac{1}{\pi} \sum_{S=0}^{(\lambda-1)/2} (-1)^S \frac{(2\pi)^{2\lambda-4S}}{(4a^4)^{S+1}}$ $P_{2\lambda-4S}\left(\frac{x}{2\pi}\right)$	Method a # 23
k>5	29	$\frac{1}{n^{2\lambda}(n^4+4a^4)}$ λ even	$-\frac{(-1)^{\lambda/2}}{\pi(2a^2)^{\lambda+2}} + \frac{(-1)^{\lambda/2}}{2^{\lambda+3}a^{2\lambda+3}}$ $\frac{\bar{S}\bar{c}Cc+\bar{C}\bar{s}Ss-\bar{S}\bar{c}Ss+\bar{C}\bar{s}Cc}{\bar{C}^2-\bar{c}^2} - \frac{1}{\pi}\sum_{S=0}^{\lambda/2-1} (-1)^S$ $\frac{(2\pi)^{2\lambda-4S}}{(4a^4)^{S+1}} P_{2\lambda-4S}\left(\frac{x}{2\pi}\right)$	Method a # 23
	30	$\frac{1}{n^{2\lambda}(n^2+a^2)^2}$	$-\frac{(-1)^\lambda}{\pi}\sum_{S=0}^{\lambda-1} \frac{(S+1)(2\pi)^{2\lambda-2S}}{a^{4+2S}} P_{2\lambda-2S}\frac{x}{2\pi}$ $-\frac{(-1)^\lambda(\lambda+1)}{\pi a^{2\lambda+4}} + \frac{(-1)^\lambda}{2a^{2\lambda+3}\bar{S}^2}$ $[\bar{S}C(2\lambda+1) + \pi aC\bar{C} - (\pi-x)aS\bar{S}]$	Method a # 23
	31	$\frac{1}{n^{2\lambda}(n^2-a^2)^2}$	$-\frac{(-1)^\lambda}{\pi}\sum_{S=0}^{\lambda-1} \frac{(S+1)(-1)^S(2\pi)^{2\lambda-2S}}{a^{4+2S}}$ $P_{2\lambda-2S}\left(\frac{x}{2\pi}\right) - \frac{\lambda+1}{\pi a^{2\lambda+4}}$ $+\frac{(2\lambda+1)\bar{s}c+\pi ac\bar{c}+(\pi-x)as\bar{s}}{2a^{2\lambda+3}\bar{s}^2}$	Method a # 23
	32	$\frac{1}{(n^2+a^2)^K}$	$\frac{1}{2}F_2(x,1)^{*}F_2(x,K-1)$ $C^{-1}\frac{1}{(n^2+a^2)^K} = F(x,K)$	Method h

Table IIB

Miscellaneous $f_c(n)$

k	#	$f_c(n)$	$f(x)=C^{-1} f_c(n)$	Method
	33	f(n) sin na	$\frac{1}{2}[F(a+x) + F(a-x)]$ $0<x<a$	Method f
		$0<a<\pi/2$ where	$\frac{1}{2}[F(a+x) - F(x-a)]$ $a<x<\pi-a$	
		$S^{-1} f(n) = F(x)$	$\frac{1}{2}[-F(2\pi-a-x) - F(x-a)]$ $\pi-a<x<\pi$	
	34	f(n) sin na, $\pi/2<a<\pi$	$\frac{1}{2}[F(a+x) + F(a-x)]$ $0<x<\pi-a$	Method f
		$S^{-1} f(n) = F(x)$	$\frac{1}{2}[-F(2\pi-a-x) + F(a-x)]$ $\pi-a<x<a$	
			$\frac{1}{2}[-F(2\pi-a-x) - F(x-a)$ $a<x<\pi$	
	a)	$\frac{\sin na}{n}$	$1 - \frac{a}{\pi}$ $0<x<a$	Method f
		$0<a<\pi$	$-\frac{a}{\pi}$ $a<x<\pi$	
	35	f(n) cos na	$\frac{1}{2}[F(a+x) + F(a-x)]$ $0<x<a$	Method f
		$0<a<\pi/2$, $C^{-1} f(n) = F(x)$	$\frac{1}{2}[F(a+x) + F(x-a)]$ $a<x<\pi-a$	
			$\frac{1}{2}[F(2\pi-a-x) + F(x-a)]$ $\pi-a<x<\pi$	
	36	f(n) cos na, $\pi/2<a<\pi$	$\frac{1}{2}[F(a+x) + F(a-x)]$ $0<x<\pi-a$	Method f
		$C^{-1} f(n) = F(x)$	$\frac{1}{2}[F(2\pi-a-x) + F(a-x)]$ $\pi-a<x<a$	
			$\frac{1}{2}[F(2\pi-a-x) + F(x-a)]$ $a<x<\pi$	

k	#	$f_c(n)$	$f(x) \quad C^{-1} f_c(n)$	Method
	a)	$\frac{\cos na}{n^2}$ $0<a<\pi$	$\frac{(\pi-a)^2 + x^2}{2\pi} - \frac{\pi}{6} \quad 0<x<a$ $\frac{(\pi-x)^2 + a^2}{2\pi} - \frac{\pi}{6} \quad a<x<\pi$	Method f
	37	$f_c(n) \quad (-1)^n$	$f(\pi-x)$ where $C^{-1} f_c(n) = f(x)$	Churchill Tables #36
	38	P^n $(P^2<1)$	$\frac{2}{\pi} \frac{P(\cos x-P)}{1-2P\cos x+P^2}$	Method g #48 Sine Tables
	39	$\frac{P^n}{n}$ $(P^2<1)$	$-\frac{1}{\pi} \log (1-2P\cos x+P^2)$	Method K # 38
	40	$\frac{P^n}{n^2}$ $(P^2<1)$	$-\frac{1}{\pi}\int_0^P \frac{1}{z} \log (1-2z\cos (x) + z^2)dz$	Method K # 39
	41	$\frac{P^n}{n^3}$ $(P^2<1)$	$-\frac{1}{\pi}\int_0^P \frac{dz_2}{z_2} \int_0^{z_2} \frac{1}{z_1} \log (1-2z_1\cos x$ $+ z_1^2)\, dz_1$	Method k # 40
	42	$\frac{P^n}{n^m}$ $(P^2<1)$	$-\frac{1}{\pi}\int_0^P \frac{dz_{m-1}}{z_{m-1}} \int_0^{z_{m-1}} \frac{dz_{m-2}}{z_{m-2}} \cdots$ $\int_0^{z_2} \frac{1}{z_1} \log(1-2z_1\cos (x) + z_1^2)\, dz_1$	Method k 41
	43	nP^m $(P^2<1)$	$\frac{2}{\pi} \frac{\cos x\,(P^2+1) - 2P}{(1-2P\cos x+P^2)^2}$	Method g #47 Sine Tables
	44	$n^m P^n$ $(P^2<1)$	(m times) $P\frac{d}{dP}\, P\frac{d}{dP} \cdots\cdots \quad P\frac{d}{dP}\, \frac{2}{\pi}$ $\frac{P(\cos(x)-P)}{1-2P\cos x+P^2}$	Method g #47 Sine Tables

k	#	$f_c(n)$	$f(x) \quad C^{-1} f_c(n)$	Method
	45	$\frac{P^n}{n!}$ $P^2 \leq 1$	$\frac{2}{\pi}[e^{P\cos x}\cos(P\sin x) - 1]$	Method c
	46	$\binom{a}{n}$	$\frac{2}{\pi}(2^a\cos^a\frac{x}{2}\cos\frac{ax}{2} - 1)$	Method d de Haas 42/20

Table IIC

Table of Odd Inverse Cosine Transforms

k	#	$f_c(n)$	$f(x) \quad C^{-1} f_c(n)$	Method
1	47	$\frac{1}{n}$	$-\frac{2}{\pi}\log(2\sin\frac{x}{2}) = \frac{2}{\pi}\int_0^1 \frac{\cos(x)-t}{1-2t\cos(x)+t^2}dt$	Method e
2	48	$\frac{n}{n^2+a^2}$	$a^2\int_0^x \cosh a(x-t) f(t)\,dt$	Method g
			$-a^2\frac{\cosh ax}{\sinh a\pi}\int_0^\pi \sinh a(\pi-t) f(t)\,dt$	
			$-\frac{2}{\pi}\log(2\sin\frac{x}{2})$	
			$-\frac{2a}{\pi}\int_0^x \sinh a(x-t)\log(2\sin\frac{t}{2})dt$	Integration by parts
			$+\frac{2a}{\pi}\int_0^\pi \cosh a(\pi-t)\log(2\sin\frac{t}{2})dt$	
			$-\frac{2}{\pi}\log(2\sin\frac{x}{2})$	
	49	$\frac{n}{n^2-a^2}$	$-a^2\int_0^x \cos a(x-t) f(t)\,dt$	Method g
			$+a^2\frac{\cos ax}{\sin a\pi}\int_0^\pi \sin a(\pi-t) f(t)\,dt$	
			$-\frac{2}{\pi}\log(2\sin\frac{x}{2})$	

k	$f_c(n)$	$f(x) \quad C^{-1} f_c(n)$	Method
50	$\frac{1}{n^3}$	$-\int_0^x f(t)\,dt + \frac{1}{\pi}\int_0^\pi (\pi-t)\, f(t)\,dt$	limit # 52 $a \to 0$
51	$\frac{1}{n(n^2+a^2)}$	$-\int_0^x \cosh a(x-t)\, f(t)\,dt + \frac{\cosh ax}{\sinh a\pi}\int_0^\pi \sinh a(\pi-t)\, f(t)\,dt$	Method g
52	$\frac{1}{n(n^2-a^2)}$	$-\int_0^x \cos a(x-t)\, f(t)\,dt + \frac{\cos ax}{\sin a\pi}\int_0^\pi \sin a(\pi-t)\, f(t)\,dt$	Method g
53	$\frac{n}{n^4-a^4}$	$\frac{-1}{2}\int_0^x \cos a(x-t)\, f(t)\,dt - \frac{\cos ax}{\cos a\pi}\int_0^\pi \sin a(\pi-t)\, f(t)\,dt + \int_0^x \cosh a(x-t)\, f(t)\,dt - \frac{\cosh ax}{\cosh a\pi}\int_0^\pi \sinh a(x-t)\, f(t)\,dt$	Method g
54	$\frac{n}{(n^2-a^2)(n^2-b^2)}$	$\frac{-a^2}{a^2-b^2}\int_0^x \cos a(x-t)\, f(t)\,dt - \frac{\cos ax}{\cos a\pi}\int_0^\pi \sin a(\pi-t)\, f(t)\,dt + \frac{b^2}{a^2-b^2}\int_0^x \cos b(x-t)\, f(t)\,dt - \frac{\cos bx}{\cos b\pi}\int_0^\pi \sin b(\pi-t)\, f(t)\,dt$	Method g
55	$\frac{n}{(n^2+a^2)(n^2+b^2)}$	$\frac{a^2}{a^2-b^2}\int_0^x \cosh a(x-t)\, f(t)\,dt - \frac{\cosh ax}{\cosh a\pi}\int_0^\pi \sinh a(\pi-t)\, f(t)\,dt$	Method g

k		$f_c(n)$	$f(x)\ C^{-1}\ f_c(n)$	Method
	55	continued	$-\frac{b^2}{a^2-b^2}\int_o^x \cosh b(x-t)\, f(t)\, dt$	
			$-\frac{\cosh bx}{\cosh b\pi}\int_o^\pi \sinh b(\pi-t)\, f(t)\, dt$	
4	56	$\frac{n}{(n^2-a^2)(n^2+b^2)}$	$\frac{-a^2}{a^2+b^2}\ [\int_o^x \cos a(x-t)\, f(t)\, dt$	Method g
			$-\frac{\cos ax}{\cos a\pi}\int_o^\pi \sin a(\pi-t)\, f(t)\, dt]$	
			$-\frac{b^2}{a^2+b^2}\ [\int_o^x \cosh b(x-t)\, f(t)\, dt$	
			$-\frac{\cosh bx}{\cosh b\pi}\int_o^\pi \sinh b(\pi-t)\, f(t)\, dt\ ]$	
	57	$\frac{n^3}{n^4-a^4}$	$\frac{a^2}{2}[\int_o^x \cosh a(x-t)\, f(t)\, dt$	Method g
			$-\frac{\cosh ax}{\cosh a\pi}\int_o^\pi \sinh a(\pi-t)\, f(t)\, dt$	
			$-\int_o^x \cos a(x-t)\, f(t)\, dt$	
			$+\frac{\cos ax}{\sin a\pi}\int_o^\pi \sin a(x-t)\, f(t)\, dt]$	
			$-\frac{2}{\pi}\log(2\sin\frac{x}{2})$	
	58	$\frac{n^3}{(n^2-a^2)(n^2-b^2)}$	$\frac{-a^4}{a^2-b^2}\ [\int_o^x \cos a(x-t)\, f(t)\, dt$	Method g
			$-\frac{\cos ax}{\cos a\pi}\int_o^\pi \sin a(\pi-t)\, f(t)\, dt\ ]$	
			$+\frac{b^4}{a^2-b^2}\ [\int_o^x \cos b(x-t)\, f(t)\, dt$	

k	$f_c(n)$	$f(x) \quad C^{-1} f_c(n)$	Method
58	continued	$- \frac{\cos bx}{\cos b\pi} \int_o^{\pi} \sin b(x-t)\, f(t)\, dt$	
		$- \frac{2}{\pi} \log\left(2 \sin \frac{x}{2}\right)$	
59	$\frac{n^3}{(n^2+a^2)(n^2+b^2)}$	$\frac{a^4}{a^2-b^2} \left[\int_o^x \cosh a(x-t)\, f(t)\, dt\right.$	Method g
		$\left. - \frac{\cosh ax}{\cosh a\pi} \int_o^{\pi} \sinh a(\pi-t)\, f(t)\, dt\right]$	
		$- \frac{b^4}{a^2-b^2} \left[\int_o^x \cosh b(x-t)\, f(t)\, dt\right.$	
		$\left. - \frac{\cosh bx}{\cosh b\pi} \int_o^{\pi} \sinh b(x-t)\, f(t)\, dt\right]$	
		$- \frac{2}{\pi} \log\left(2 \sin \frac{x}{2}\right)$	
60	$\frac{n^3}{(n^2+a^2)(n^2-b^2)}$	$\frac{a^4}{a^2+b^2} \left[\int_o^x \cosh a(x-t)\, f(t)\, dt\right.$	Method g
		$\left. - \frac{\cosh ax}{\sinh a\pi} \int_o^{\pi} \sinh a(\pi-t)\, f(t)\, dt\right]$	
		$- \frac{b^4}{a^2+b^2} \left[\int_o^x \cos b(x-t)\, f(t)\, dt\right.$	
		$\left. - \frac{\cos bx}{\cos b\pi} \int_o^{\pi} \sin b(x-t)\, f(t)\, dt\right]$	
		$- \frac{2}{\pi} \log\left(2 \sin \frac{x}{2}\right)$	

Table IID

Miscellaneous Transforms in Terms of Definite Integrals
Cosine Transforms from "Tables of Definite Integrales"[7]

k	#	$f_c(n)$	$C^{-1} f_c(n)$	Method d
	61	$\frac{1}{n+q}$	$-\frac{1}{\pi q} - \frac{4}{\pi}\int_0^1 \frac{\cosh \frac{t(x-\pi)}{2\pi}}{(1-t)t^{1/2}} \frac{\log t\, dt}{4\pi^2 q^2+(\log t)^2}$	132/14 Examen
	62	$\frac{1}{n+qr}$	$-\frac{1}{qr\pi} + \frac{2}{\pi}\int_0^1 \frac{\cosh[(x-\pi)\, r\log t]}{\sinh(4\pi\log t)} \frac{\log t}{q^2+(\log t)^2}\frac{dt}{t}$	131/11
	63	$\frac{P^n}{(q+n)^r}$	$\frac{2P}{\pi\Gamma(r)}\int_0^1 \frac{(\log\frac{1}{t})^{r-1}(\cos x - Pt)}{1-2Pt\cos x + P^2t^2} t^q dt$	113/11 Examen
	64	$\frac{1}{n^q}$	$\frac{2}{\pi\Gamma(q)}\int_0^1 \frac{(\log\frac{1}{t})^{q-1}(\cos x - t)}{1-2t\cos x + t^2} dt$	130/1 113/10 Examen
	65	$\frac{1}{\sqrt{n}}$	$\frac{1}{\pi^{3/2}}\int_0^\infty \frac{\cos x - e^{-t}}{\cosh t - \cos x}\frac{dt}{t^{1/2}}$	98/27
	66	$\frac{1}{n+qr}$	$-\frac{1}{\pi qr} + \frac{2}{\pi}\int_0^\infty \frac{\cosh(\pi-x)rt}{\sinh \pi rt}\frac{t\, dt}{t^2+q^2}$	97/19
	67	$\frac{P^n}{(q+n)^r}$	$\frac{2P}{\pi\Gamma(r)}\int_0^\infty \frac{(\cos(x) - Pe^{-t})}{e^t+P^2e^{-t}-2P\cos x} e^{-qt}t^{r-1}dt$	88/10
	68	$\frac{1}{n^q}$	$\frac{1}{\pi\Gamma(q)}\int_0^\infty \frac{\cos(x) - e^{-t}}{\cosh t - 2\cos x} t^{q-1} dt$	88/6
	69	$\frac{1}{n+r}$	$-\frac{2}{\pi r} + \int_0^1 \frac{1-t\cos x}{1-2t\cos x+t^2} t^{r-1} dt$	6/9
	70	$n!\,\frac{q^n}{\Gamma(n+P+1)}$	$\frac{2q}{\pi\Gamma(P+1)}\int_0^1 \frac{(\cos(x) - qt)(1-t)^P dt}{1-2qt\cos x + q^2t^2}$	6/14

k #	$f_c(n)$	$C^{-1} f_c(n)$	Method d
71	$\frac{ne^{-nP}}{n^2-r^2}$	$-\frac{2}{\pi}\int_0^1 \frac{\cos(\frac{P}{\pi}\log t)}{\pi^2 r^2+(\log t)^2}\,\frac{t^{\frac{\pi-x}{\pi}}+t^{\frac{x-\pi}{\pi}}}{1-t^2}\log t\,dt$ $-\frac{e^{-Pr}\cos(\pi-x)r}{\sin \pi r}$	407/12
72	$\frac{e^{-nP}}{n^2-r^2}$	$\frac{1}{\pi r^2}-\frac{e^{-Pr}\cos(\pi-x)r}{r\sin \pi r}$ $+2\int_0^1 \frac{\sin(\frac{P}{\pi}\log t)}{\pi^2 r^2+(\log t)^2}\,\frac{t^{\frac{\pi-x}{\pi}}+t^{\frac{x-\pi}{\pi}}}{1-t^2}\,dt$	407/6
73	$\frac{ne^{-nP}}{n^2-r^2}$	$\frac{2}{\pi}\int_0^\infty \frac{\cosh(\pi-x)t}{\sinh \pi t}\,\frac{t\cos Pt}{r^2+t^2}\,dt$ $-\frac{e^{-Pr}\cos(\pi-x)r}{\sin \pi r}$	389/24
74	$\frac{e^{-nP}}{n^2-r^2}$	$-\frac{2}{\pi}\int_0^\infty \frac{\cosh(\pi-x)t}{\sinh \pi t}\,\frac{\sin Pt}{r^2+t^2}\,dt$ $+\frac{1}{\pi r^2}-\frac{1}{r}e^{-Pr}\,\frac{\cos(\pi-x)r}{\sin \pi r}$	389/21
75	$\frac{1}{n+q}$	$-\frac{1}{\pi q}-\frac{2}{\pi}\int_0^{\pi/2}\frac{\cos^{\frac{x-\pi}{\pi}} t+\cos^{\frac{\pi-x}{\pi}} t}{\pi^2q^2+(\log\cos t)^2}$ $\frac{\log\cos t}{\sin t}\,dt$	328/11
76	$\frac{1}{n+q}$	$-\frac{1}{\pi q}-\frac{2}{\pi}\int_0^{\pi/2}\frac{\sin^{\frac{x-\pi}{\pi}} t+\sin^{\frac{\pi-x}{\pi}} t}{\pi^2q^2+(\log\sin t)^2}$ $\frac{\log\sin t}{\cos t}\,dt$	327/10
77	$\frac{1}{(n+q)^P}$	$\frac{2}{\pi\Gamma(P)}\int_0^{\pi/4}(\log\cot t)^{P-1}$ $\frac{\cos x-\tan t}{1-\cos x\sin 2t}\tan^q t\,dt$	292/29 Examen

k	#	$f_c(n)$	$C^{-1} f_c(n)$	Method
	78	$\frac{P^{2n}}{(n!)^2}$	$-\frac{2}{\pi}+\frac{2}{\pi}\int_0^{\pi} e^{2P\cos t\cos x/2}\cos(2P\cos t\sin x/2)dt$	277/11
	79	$\frac{1}{n+q}$	$-\frac{1}{\pi q}+\frac{2}{\pi}\int_0^{\pi/2}\frac{\cosh[q(\pi-x)\tan t]}{\sinh(q\pi\tan t)}\tan t\,dt$	274/14
	80	$\frac{1}{\sqrt{n}}$	$\frac{2}{\pi^{3/2}}\int_0^1\frac{(\cos x-t)dt}{(1-2t\cos x+t^2)\sqrt{\log 1/t}}$	133/6
	81	$\frac{r^n}{(P+n)^q}$	$-\frac{2}{\pi P^q}+\frac{2}{\pi\Gamma(q)}\int_0^{\infty}(\log\frac{1}{x})^{q-1}\frac{(1-r t\cos x)t^{P-1}}{1-2r t\cos x+r^2t^2}dt$	140/15

D. Application of the Sine Transforms to a Plate Problem

1. The differential equation of a plate on an elastic foundation

If a thin, homogeneous, isotropic, rectangular plate rests on an elastic foundation where the restoring force is proportional to the deflection and supports a transverse distributed load of $q(x, y)$ per unit area, the defection $w(x, y)$ at any point of the plate must satisfy the Equation[34]

$$\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \frac{q(x, y)}{D} - \frac{k' w}{D} \quad . \tag{27}$$

Here k' = foundation modulus, D = modulus of rigidity of the plate. x subscripts indicate partial derivatives with respect to x, and y subscripts indicate partial derivatives with respect to y. We have assumed that the foundation produces a reaction on every point of the plate which is proportional to the deflection at that point.

Expressions for the deflection $w(x, y)$ have been given by Timoshenko[34] when two opposite edges are supported and the other two edges are clamped. We wish to solve the problem when two opposite edges have specified deflections and moments, and the other two any boundary conditions.[11]

2. Boundary conditions

In all cases we will specify the moments and deflections on the edges $x = 0$ and $x = a$, (Fig. 3). These conditions may be expresses as follows:[34]

$$w(0, y) = w_1(y) \tag{28}$$

$$w(a, y) = w_2(y) \tag{29}$$

$$M_x(0, y) = -D[w_{xx} + \nu w_{yy}]_{at\ x = 0} = M_1(y) \tag{30}$$

$$M_x(a, y) = -D[w_{xx}(a, y) + \nu\, w_{yy}(a, y)] = M_2(y) \tag{31}$$

where $w_1(y)$, $w_2(y)$, $M_1(y)$, $M_2(y)$ are specified and ν is Poisson's Ratio. The six standard cases for edge conditions at the other two edges, $y = \pm b$ are as follows:

Case 1: Two edges fixed:

$$w(x, b) = w_3(x) \tag{32}$$

$$w(x, -b) = w_4(x) \tag{33}$$

$$w_y(x, b) = S_1(x) \tag{34}$$

$$w_y(x, -b) = S_2(x) \tag{35}$$

Case 2: Deflections and moments given:

$$M_y(x, b) = -D\,[w_{yy}(x, b) + \nu w_{xx}(x, b)] \tag{36}$$

$$M_y(x, -b) = -D\,[w_{yy}(x, -b) + \nu w_{xx}(x, -b)] \tag{37}$$

Also equations (28), (29)

Case 3: Moments and reactions given:

$$R_y(x, b) = -D\,[w_{yyy}(x, b) + (2-\nu)\, w_{xyy}(x, b)] \tag{38}$$

$$R_y(x, -b) = -D\,[w_{yyy}(x, -b) + (2-\nu)\, w_{xyy}(x, -b)] \tag{39}$$

Also equations (36), and (37).

Case 4: One edge fixed ($y = b$) and one edge with deflections and moments given ($y = -b$):

Equations (32), (33), (34), (37)

Case 5: One edge with moment and reaction given ($y=b$) and one edge fixed ($y = -b$):

Equations (33), (35), (36), (38)

Case 6: One edge with moment and reaction given ($y=b$) and one edge with deflection and moment given ($y = -b$):

Equations (33), (36), (37), (38)

In all six cases $w_3(x)$, $w_4(x)$, $S_1(x)$, $S_2(x)$, $M_3(x)$, $M_4(x)$, $V_1(x)$, $V_2(x)$ are given functions of x.

We can solve these six problems by use of the sine transform. By taking the sine transform of Equation (27) with respect to x, using Equation (3) and (4), we have

$$\frac{d^4 w_s}{dy^4}(\alpha, y) - 2\alpha^2 \frac{d^2 w_s}{dy^2}(\alpha, y) + (\alpha^4 + k^4)\, w_s(\alpha, y) = Q(\alpha, y) \quad (40)$$

where

$$k^2 = \sqrt{\frac{k'}{D}}$$

where

$$Q(\alpha, y) = q_s(\alpha, y) + \alpha \left[(-1)^n \frac{\partial^2 w}{\partial x^2}(a, y) - \frac{\partial^2 w}{\partial x^2}(0, y)\right]$$

$$-2\alpha \left[(-1)^n \frac{\partial^2 w}{\partial y^2}(a, y) - \frac{\partial^2 w}{\partial y^2}(0, y)\right]$$

$$-\alpha^3 \left[(-1)^n w(a, y) - w(0, y)\right] \quad (41)$$

This may be written

$$Q(\alpha, y) = \frac{q_s(\alpha, y)}{D} + \alpha \left\{ (-1)^{n+1} \frac{[M_2(y)}{D} + \nu w_2''(y)] \right.$$

$$\left. + \frac{M_1(y)}{D} + \nu w_1(y) \right\} + 2\alpha [(-1)^n w_2''(y) - w_1''(y)]$$

$$- \alpha^3 [(-1)^n w_2(y) - w_1(y)] . \qquad (42)$$

The solution to equation (27) is

$$w(x, y) = \frac{2}{a} \Sigma w_s (\alpha, y) \sin \alpha x$$

where $w_s(\alpha, y)$ is a solution to equation (40) and is

$$w_s(\alpha, y) = A \sinh uy \sin vy + B \cosh uy \sin vy$$

$$+ D \sinh uy \cos vy + E \cosh uy \cos vy$$

$$+ G (\alpha, y) \qquad (43)$$

where

$$u = \sqrt{\frac{\alpha^2 + \sqrt{\alpha^4 + k^4}}{2}}$$

$$v = \sqrt{\frac{-\alpha^2 + \sqrt{\alpha^4 + k^4}}{2}} \qquad (44)$$

and $G(\alpha, y)$ is a particular solution of equation (40).

The following abbreviations will be used: $S = \sinh ub$, $C = \cosh ub$, $s = \sin vb$, $S_2 = \sinh 2ub$, $S_4 = \sinh 4ub$, $c_2 = \cos 2vb$, etc.

In all cases when k approaches zero

$$C_1 \sin vy \rightarrow C_2' y \quad C_2 \sin vy \rightarrow C_4' y$$

$$C_3 \rightarrow C_1', \quad C_4 \rightarrow C_3', \quad u \rightarrow \alpha, \quad v \rightarrow \frac{k^2}{2\alpha} \rightarrow 0$$

where primes refer to the case of no elastic foundation.

The transformed edge conditions, Equations (28) to (39), inclusive, are

$$w_s(a, b) = w_{3s}(a) \quad (45)$$

$$w_s'(a, b) = S_{1s}(a) \quad (46)$$

$$w_s(a, -b) = w_{4s}(a) \quad (47)$$

$$w_s'(a, -b) = S_{2s}(a) \quad (48)$$

$$-w_s''(a, b) + \nu a^2 w_s(a, b) + a\nu\,[(-1)^n w(a, b) - w(0, b)] = M_{3s}(a)/D \quad (49)$$

$$-w_s''(a, -b) + \nu a^2 w_s(a, -b) + a\nu[(-1)^n w(a, -b) - w(0, -b)] = M_{4s}(a)/D \quad (50)$$

$$-w_s'''(a, b) + (2-\nu)a^2 w_s'(a, b) + (2-\nu)a[(-1)^n w'(a, b) - w'(0, b)] = R_{1s}(a)/D \quad (51)$$

$$-w_s'''(a, -b) + (2-\nu)a^2 w_s'(a, -b) + (2-\nu)a[(-1)^n w(a, -b) - w(0, -b)] = R_{2s}(a)/D \quad (52)$$

where

$$w_s' = \frac{dw_s(a, y)}{dy}, \text{ etc.}$$

From the edge conditions at the edges $y = \pm b$, we can evaluate the constants A, B, D, and E in Equations (43). We will give first the values of these constants for certain of the edge functions required to be zero (cases la - 6a, inclusive). The values of the constants A, B, D, and E for the edge functions not equal to zero will be written in terms of the corresponding constants for the edge functions equal to zero.

The constants for Cases l(a) - 6(a), inclusive, are as follows:

Case 1(a)

$$A\,(us_2 + vS_2) = [G(a,b) + G(a,-b)]\,(uSc - vCs) - Cc[G'(a,b) - G'(a,-b)]$$

$$B\,(us_2 - vS_2) = [-G(a,b) + G(a,-b)]\,(uCc - vSs) + Sc[G'(a,b) + G'(a,-b)]$$

$$D\,(-us_2 + vS_2) = [-G(a,b) + G(a,-b)]\,(uSs + vCc) + Cs[G'(a,b) + G'(a,-b)]$$

$$E\,(-us_2 - vS_2) \quad [+G(a,b) + G(a,-b)]\,(uCs + vSc) - Ss[+G'(a,b) - G'(a,-b)] \tag{53}$$

Case 2(a)

$$k^2A\,(C_2 + c_2) = -Cc[G''(a,b) + G''(a,-b)] - (-Cca^2 + Ssk^2)[G(a,b) + G(a,-b)]$$

$$k^2B\,(C_2 - c_2) = -Sc[G''(a,b) - G''(a,-b)] - (-Sca^2 + Csk^2)[G(a,b) - G(a,-b)]$$

$$k^2D\,(C_2 - c_2) = +Cs[G''(a,b) - G''(a,-b)] - (+Csa^2 + Sck^2)[G(a,b) - G(a,-b)]$$

$$k^2E\,(C_2 + c_2) = +Ss[G''(a,b) + G''(a,-b)] - (Ssa^2 + Cck^2)[G(a,b) + G(a,-b)] \tag{54}$$

Case 3(a)

$$A\left\{2a^2k^2(1-\nu)(vs_2 - uS_2) + [a^4(1-\nu)^2 - k^4](us_2 + vS_2)\right\} = [a^2(1-\nu)Cc - k^2Ss]$$

$$[G'''(a,b) - G'''(a,-b) - (2-\nu)a^2G'(a,b) + (2-\nu)a^2G'(a,-b)]$$

$$+\left\{Sc[a^2(1-\nu)u + k^2\nu] - Cs[a^2(1-\nu)\nu - k^2u]\right\}[G''(a,b) + G''(a,-b)$$

$$-\nu a^2G(a,b) - \nu a^2G(a,-b)].$$

$$-B\quad\left\{2a^2k^2(1-\nu)(vs_2 + uS_2) + [a^4(1-\nu)^2 - k^4](us_2 - vS_2)\right. \quad [a^2(1-\nu)Sc - k^2Cs]$$

$$[G'''(a,b) + G'''(a,-b) - (2-\nu)a^2G'(a,b) - (2-\nu)a^2G'(a,-b)]$$

$$+\left\{Cc[a^2(1-\nu)u + k^2v] - Ss[a^2(1-\nu)v - k^2u]\right\}[G''(a,b) - G''(a,-b)$$

$$-\nu a^2G(a,b) + \nu a^2G(a,-b)]$$

Case 3(a) continued

$$-D\left\{2a^2k^2(1-\nu)(vs_2+uS_2)+[a^4(1-\nu)^2-k^4](us_2-vS_2)\right\}=[-a^2(1-\nu)Cs-k^2Sc]$$

$$[G'''(a,b)+G'''(a,-b)-(2-\nu)a^2G'(a,b)-(2-\nu)a^2G'(a,-b)]$$

$$-\left\{Ss[a^2(1-\nu)u+k^2v]+Cc[a^2(1-\nu)v-k^2u]\right\}[G''(a,b)-G''(a,-b)-\nu a^2G(a,b)$$

$$+\nu a^2G(a,-b)]$$

$$E\left\{2a^2k^2(1-\nu)(vs_2-uS_2)+[a^4(1-\nu)^2-k^4](us_2+vS_2)\right\}=[-a^2(1-\nu)Ss-k^2Cc]$$

$$[G'''(a,b)-G'''(a,-b)-(2-\nu)a^2G'(a,b)+(2-\nu)a^2G'(a,-b)]$$

$$-\{Cs[a^2(1-\nu)u+k^2v]+Sc[a^2(1-\nu)v-k^2u][G''(a,b)+G''(a,-b)$$

$$-\nu a^2G(a,b)-\nu a^2G(a,-b)] \qquad (55)$$

Case 4(a)

$$k^2(us_4-vS_4)A=-2k^2G(a,b)[uSc(2s^2+C_2)+vCs(2c^2-C_2)]$$

$$+2G(a,-b)[+uk^2Scc_2+vk^2CsC_2+aCc(us_2-vS_2)]$$

$$+4k^2CcG'(a,b)(s^2+S^2)-2CcG''(a,-b)(us_2-vS_2)$$

$$k^2(us_4-vS_4)B=-2k^2G(a,b)[uCc(-2s^2+C_2)+vSs(-2c^2-C_2)]$$

$$+2G(a,-b)[+uk^2Ccc_2-vk^2S_sC_2+a^2Sc(us_2+vS_2)]$$

$$+4k^2ScG'(a,b)(c^2+S^2)-2ScG''(a,-b)(us_2+vS_2)$$

$$k^2(us_4-vS_4)D=-2k^2G(a,b)[uSs(-2c^2-C_2)+vCc(2s^2-C_2)]$$

$$+2G(a,-b)[-uk^2Ssc_2-vk^2CcC_2-a^2Cs(us_2+vS_2)]$$

$$-4k^2CsG(a,b)(c^2+S^2)+2CsG(a,-b)(us_2+\nu S_2)$$

Case 4(a) continued

$$k^2(us_4 - vS_4)E = -2k^2G(a, b)\,[uCs(2c^2 - C_2) + vSc(-2s^2 - C_2)]$$
$$+ 2G(a, -b)\,[-uk^2Csc_2 + vk^2ScC_2 - a^2Ss(us_2 - vS_2)]$$
$$-4k^2SsG'(a, b)\,(s^2 + S^2) + 2SsG''(a, -b)\,(us_2 - vS_2) \tag{56}$$

Case 5(a)

$$\Delta\, A = -k^2G(a, -b)\{2a^2(1-\nu)\,[u^2Cc(-s^2 + C_2) + uvS_s(c^2 - C^2)$$
$$+ v^2Cc(-2s^2 + C^2)] + k^2\,[-u^2Scc_2 + 2uvCc(-c^2 + C^2) - v^2S_sC_2]\}$$
$$+ k^2G'(a, -b)\,\{2a^2(1-\nu)\,[uSc(c^2 - 2C_2) + vCs(-2c^2 + C^2)$$
$$+ k^2\,[uCsc_2 - vScC_2]\} + [G''(a, b) - a^2\nu G(a, b) + a^2(1-\nu)G(a, -b)]$$
$$\{2a^2(1-\nu)\,(uSc - \nu Cs)\,(-usc + vSC) + k^2[u^2Ccc_2 + v^2CcC^2$$
$$+ 2\,uv\,Ss(-c^2 + C^2)] - [G'''(a, b) - (2-\nu)a^2G'(a, b) + a^2(1-\nu)G'(a, -b)]$$
$$-2a^2(1-\nu)Cc(-usc + \nu SC) + k^2uScc_2 + k^2vCsC_2$$

where Δ is given by

$$\Delta = a^4(1-\nu)^2(-u^2{s_2}^2 + v^2{S_2}^2) + a^2(1-\nu)\,k^2uv(c_4 - C_4) - k^4(u^2{c_2}^2 + v^2{C_2}^2)$$

and the other C s are given in terms of C_1

$$2E\,\{k^2[uScc_2 + vCsC_2] + a^2\,(1-\nu)Cc(us_2 - vS_2)\} = G(a, -b)$$
$$[-k^2(uS_2 + vs_2) + a^2(1-\nu)\,(-us_2 + vS_2)] - 2k^2G'(a-b)\,(-c^2 + C^2)$$
$$+ [G''(a, b) - a^2\nu G(a, b) + 2A\,a^2(1-\nu)Ss]\,(-us_2 + vS_2)$$
$$+ 2k^2A\,(-uCsc_2 + vScC_2)$$
$$B(-usc + vSC) = -(uCc - vSs)G(a, -b) + AvS^2 - Euc^2 - ScG'(a, -b)$$
$$DSc = ASs - BCs + ECc + G(a, -b) \tag{57}$$

Case 6(a)

$$\Delta A = -G(a, -b)\ [a^2(1-\nu)\ Cc - k^2Ss] \Big\{ -[a^2(1-\nu)\ u + k^2v]$$

$$[a^2(1-\nu)\ s_2 + k^2S_2] + [k^2u - a^2(1-\nu)v]\ [-a^2(1-\nu)S_2 + k^2s_2]\Big\}$$

$$+ [G''(a, b) - a^2\nu G(a, b)]k^2 \Big\{ -[a^2(1-\nu)\ u + k^2\nu]\ (Sss_2 + ScC_2)$$

$$+ [k^2u - a^2\ (1-\nu)v]\ [Ccs_2 - CsC_2] + [G''(a, b) - \nu a^2G(a, -b)]$$

$$\Big\{-[a^2(1-\nu)\ u + k^2v]\ [2Csc^2a^2(1-\nu) + k^2ScC_2] + [k^2u - a^2(1-\nu)v]$$

$$[-2SC^2ca^2(1-\nu) + k^2Csc_2] + 2k^2[G'''(a, b) - (2-\nu)\ a^2G'(a, b)]$$

$$[a^2(1-\nu)Cc - k^2Ss]\ (c^2 - C^2)$$

$$2\Delta = k^2 \Big\{[ua^2(1-\nu) + k^2\nu]\ [a^2(1-\nu)s_4 + k^2S_4] + [a^2(1-\nu)v - k^2u]$$

$$[-a^2(1-\nu)S_4 + k^2s_4]\Big\}\ 2k^2D\ [-a^2(1-\nu)\ Cc + k^2Ss]\ (c^2 - C^2)$$

$$= -G''(a, -b)\ [k^2c_2S + s_2Ca^2(1-\nu)]\ C - k^2SCG''(a, b) + G(a, -b)$$

$$\Big\{k^2SCa^2\nu - 2\ [-a^2(1-\nu)\ Cc + k^2Ss]\ [a^2Cs + k^2Sc] + a^2\nu k^2SCG(a, b)$$

$$+ 2Ak^2(s^2 - C^2)\ [a^2\ (1-\nu)\ Cs + k^2Sc]$$

$$2E\ [-a^2\ (1-\nu)\ Cc + k^2Ss] = \Big\{G''(a, b) + G''(a, -b) - a^2\nu G(a, b)$$

$$-a^2\nu G(a, -b) - 2A\ [-a^2(1-\nu)\ Ss - k^2Cc]$$

$$CsB = + ASs - DSc + ECc + G(a, b) \tag{58}$$

We shall now find the constants A, B, D, and E for Cases 1 to 6, inclusive:

Case 1 In the constants of Case 1(a), replace

$$G(a, b) \text{ by } G(a, b) - w_{3s}(a) \tag{59}$$

$$G(a, -b) \text{ by } G(a, -b) - w_{4s}(a) \tag{60}$$

$G'(a, b)$ by $G'(a, b) - S_{1s}(a)$ (61)

$G'(a, -b)$ by $G'(a, -b) - S_{2s}(a)$ (62)

Case 2 In the constants of Case 2(a), use Equations (59) and (60) and replace

$G''(a, b)$ by $G''(a, b) - a^2\nu w_{3s}(a) - a\nu\,[(-1)^n w(a, b) - w(0, b)] + M_{3s}(a)/D$ (63)

$G''(a, -b)$ by $G''(a, -b) - a^2\nu w_{4s}(a) - a\nu\,[(-1)^n w(a, -b) - w(0, -b)]$
$+ M_{4s}(a)/D$ (64)

Case 3 In the constants of Case 3(a), replace

$G''(a, b)$ by $G''(a, b) - a\nu[(-1)^n w(a, b) - w(0, b)] + M_{3s}(a)/D$ (65)

$G''(a, -b)$ by $G''(a, -b) - a\nu[(-1)^n w(a, -b) - w(0, -b)] + M_{4s}(a)/D$ (66)

$G'''(a, b)$ by $G'''(a, b) - (2-\nu)\, a[(-1)^n w'(a, b) - w'(0, b)] + R_{1s}(a)/D$ (67)

$G'''(a, -b)$ by $G'''(a, -b) - (2-\nu) a[(-1)^n w'(a, -b) - w'(0, -b)] + R_{2s}(a)/D$ (68)

Case 4 In the constants of Case 4(a) use Equations (59), (60), (61), and (64).

Case 5 In the constants of Case 5(a) use Equations (65), (67), (60), and (62).

Case 6 In the constants of Case 6(a) use Equations (65), (67), (60), and (64).

For example, if a constant moment were applied at $x=0$, $G''(a, b)$ would be replaced by $G''(a, b) + \frac{M_{3s}(a)}{D}$ where

$$M_{3s}(\alpha) = \int_0^a M_3(x) \sin \alpha x \, dx = \frac{M}{\alpha}[1 - (-1)^n] \quad \text{so that}$$

$B = D = G(\alpha, y) = 0$ and

$$k^2 A (C_2 + c_2) = \frac{2M}{\alpha}[1 - (-1)^n] (\alpha^2 C_c - k^2 S_s)$$

$$k^2 E (C_2 + c_2) = \frac{-2M}{\alpha}[1 - (-1)^n] (k^2 C_c + \alpha^2 S_s)$$

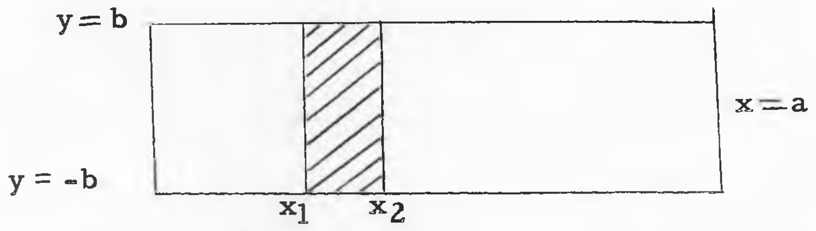
$$w(x, y) = \frac{4M}{ak^2} \Sigma \frac{1 - (-1)^n}{\alpha(C_2 + c_2)} [(\alpha^2 C_c - k^2 S_s) \sinh uy \sin vy$$

$$- (k^2 C_c + \alpha^2 S_s) \cosh uy \cos vy] \sin \alpha x \quad .$$

The factor depending on the load is $G(\alpha, y)$. This factor is evaluated for several specific examples from Equation (23c):

(a) Rectangular strip

$q(x, y) = q$ as indicated is shaded area of Fig. 3.



Constant Load Applied Between x_1 and x_2 Fig. 3

$$G(\alpha, y) = \frac{q_o}{D} \frac{1}{\alpha^4 + k^4} \frac{\cos \alpha x_1 - \cos \alpha x_2}{\alpha} \quad (69)$$

(b) Concentrated line strip

$q(x, y) = \frac{q_o}{dx}$ when $x = \xi$, $q(x, y) = 0$ elsewhere

$$G(\alpha, y) = \frac{q_o}{D} \frac{1}{\alpha^4 + k^4} \sin \alpha \xi \quad . \quad (70)$$

This was obtained by letting x_2 approach x_1 in Equation (69).

(c) Constant load

$$q(x, y) = q_o$$

$$G(\alpha, y) = \frac{q_o}{D} \frac{1}{\alpha^4 + k^4} \frac{1 - (-1)^n}{\alpha} . \tag{71}$$

This was obtained by letting $x_1 \rightarrow 0$ and $x_2 \rightarrow a$ in Equation (69).

(d) Hydrostatic pressure in x-direction

$$q(x, y) = \frac{q_o x}{a}$$

$$G(\alpha, y) = \frac{q_o}{D} \frac{1}{\alpha^4 + k^4} \frac{1 - (-1)^{n-1}}{\alpha} \tag{72}$$

(e) Hydrostatic pressure in y-direction

$$q(x, y) = \frac{q_o y}{b}$$

$$G(\alpha, y) = \frac{q_o}{D} \frac{1}{\alpha^4 + k^4} \frac{1 - (-1)^n}{\alpha} \frac{y}{b} \tag{73}$$

(f) No load but constant moments at $x = 0$, and $x = a$

$$M_1(y) = M_1, \quad M_2(y) = M_2$$

$$G(\alpha, y) \quad \frac{\alpha(-1)^n M_2 + \alpha M_1}{D(\alpha^4 + k^4)} \tag{74}$$

The constant load solution of Case 1 is the same as given by Timoshenko[34], if $w_1 = w_2 = w_4 = M_1 = M_2 = S_1 = S_2 = 0$.

3. Summing of series

The particular solution $G(\alpha, y)$ can often be summed by using Table I. Thus, for a constant load the particular solution is

$$w(x, y) = \frac{2}{a} \sum_{n=1}^{\infty} G(\alpha, y) \sin \alpha x \quad \frac{2q_o}{aD} \sum_{n=1}^{\infty} \frac{1}{\alpha^4 + k^4} \quad \frac{1 - (-1)^n}{\alpha} \sin \alpha x \quad .$$

By using formulae # 27 and #46 in Table IA, we obtain

$$w(x, y) = \frac{q_o}{k^4 D} \left[\frac{a-x}{a} - \frac{(S_c \bar{S}_{\bar{c}} + C_s \bar{C}_{\bar{s}})}{\bar{C}^2 - \bar{c}^2} \right] \tag{75}$$

where

$$S = \sinh \frac{k}{\sqrt{2}} (a-x) \qquad c = \cos \frac{k}{\sqrt{2}} (a-x) \qquad \bar{s} = \sin \frac{ka}{\sqrt{2}} \qquad \text{etc.}$$

and bars indicate evaluation of $x = 0$. $w(x, y)$ as given by Equation (75) is the solution to the problem of a simply supported beam supporting a constant load and resting on an elastic foundation. In examples a→ f we can also sum the series by use of tables and obtain corresponding beam solutions.

4. Numerical solution

The cases where two opposite edges ($x = 0$ and $x = a$) are simply supported, and both the other two edges ($y = \pm b$) are clamped free or simply supported, are treated numerically for constant load and for strip load.

The following constants are used:

$a = 60''$ $\quad x_2 = 35''$ $\quad x_1 = 25''$ $\quad b = 30''$ $\quad h = 6''$

$\nu = 0.15$ $\quad E = 3 \cdot 10^6$ lb/in^2 $\quad k' = 100$ lbs/in^3

$D = \frac{Eh^3}{12(1-\nu^2)}$ (see Fig. 4)

The units of moments are 33.33Q lb-in/in. The units of Q are lb/in^2. The units of deflections are $6.034 \cdot 10^{-4}\ Q_o$ lb/in^2. Deflections and moments are symmetric about the two center lines; hence graphs are plotted only in one quadrant.

5. Related problems

The equation of a thin isotropic homogeneous plate on an elastic foundation subject to an impressed sinusoidal lateral load is

$$\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \frac{q(x, y, t)}{D} - \frac{k'w}{D} - \frac{\rho}{D} w_{tt}$$

where ρ is the density of the plate. If at the time $t = 0$, $q = q_o(x, y)$, and $\frac{dq}{dt} = 0$ we can express the load $q(x, y, t)$ and the steady state deflection $w(x, y, t)$ as

$$q(x, y, t) = q_o(x, y) \cos \omega t$$

$$w(x, y, t) = w_o(x, y) \cos \omega t$$

where w is 2π times the frequency and $w_o(x, y)$ is the amplitude of the deflection at (x, y). w(x, y) will satisfy the equation

$$\nabla^4 w_o = \frac{q_o(x, y)}{D} - \frac{(k' - \rho\omega^2)w}{D}$$

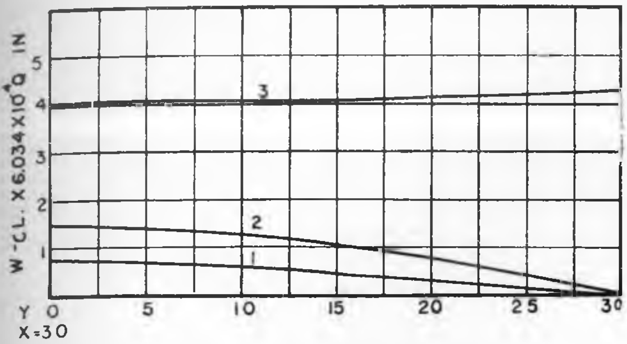


Fig. 4 Deflection at $x = 30$, Constant Load

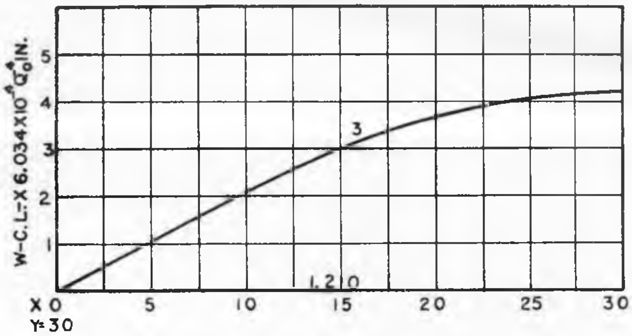


Fig. 8 Deflection at $y = 30$, Constant Load

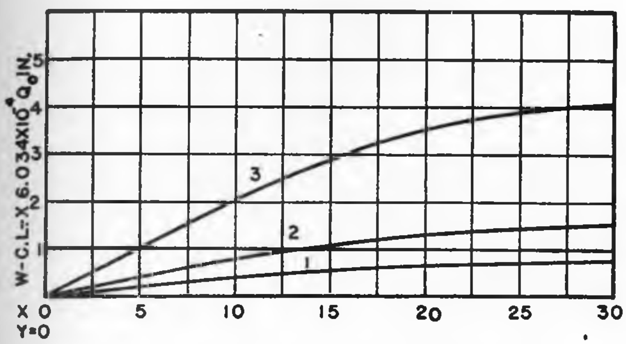


Fig. 5 Deflection at $y = 0$, Constant Load

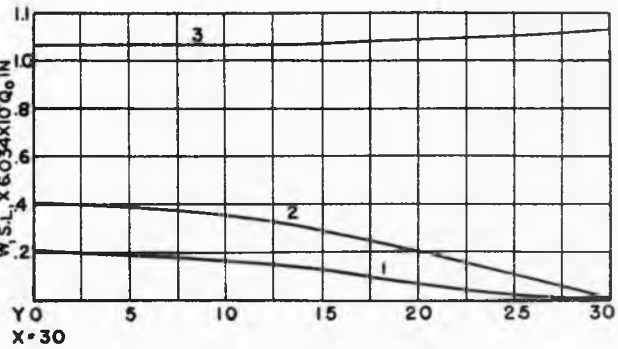


Fig. 9 Deflection at $x = 30$, Strip Load

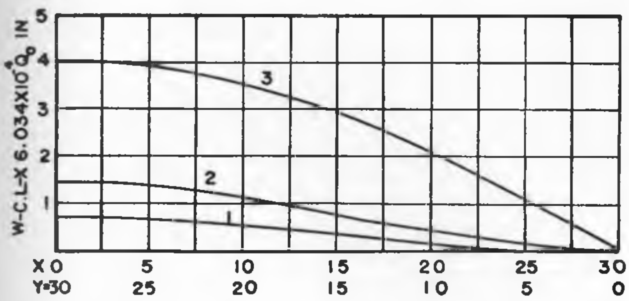


Fig. 6 Deflection on Negative Diagonal Constant Load

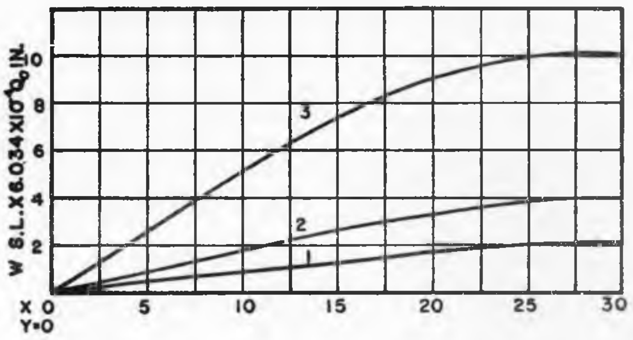


Fig. 10 Deflection at $y = 0$, Strip Load

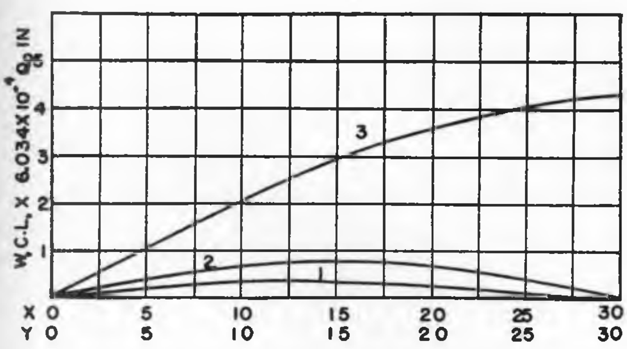


Fig. 7 Deflection on Positive Diagonal, Constant Load

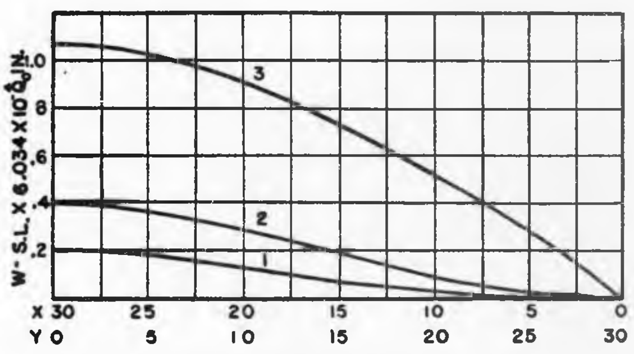


Fig. 11 Deflection on Negative Diagonal, Strip Load

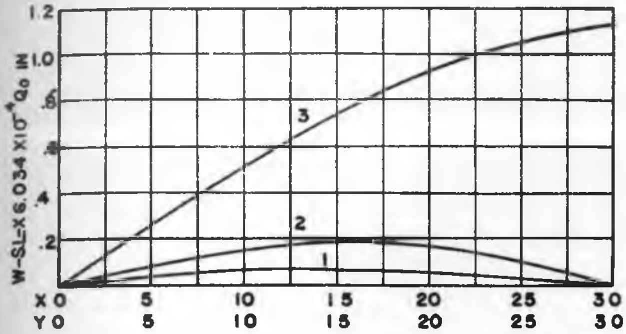


FIG. 12 DEFLECTION ON POSITIVE DIAGONAL, STRIP LOAD

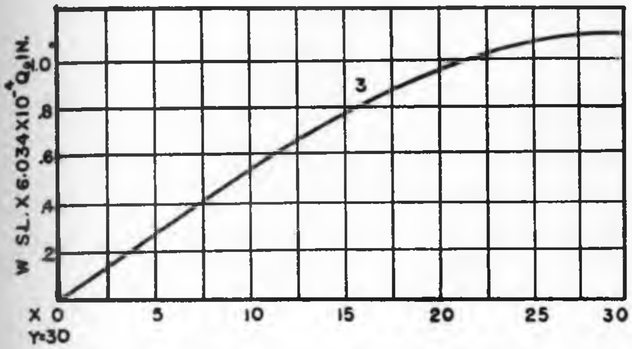


FIG. 13 DEFLECTION AT y = 30, STRIP LOAD

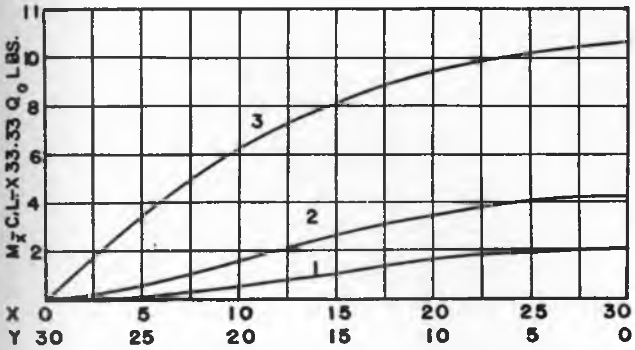


FIG. 14 x-MOMENT ON NEGATIVE DIAGONAL, CONSTANT LOAD

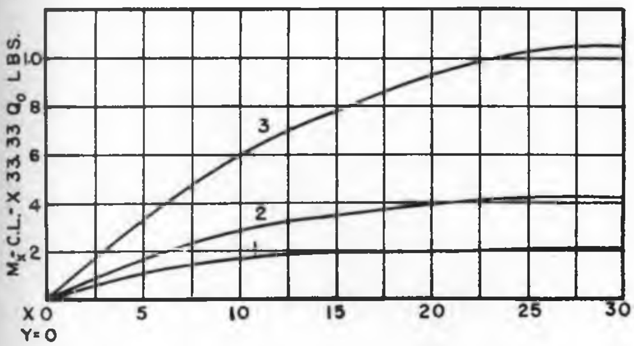


FIG. 15 x-MOMENT AT y = 0, CONSTANT LOAD

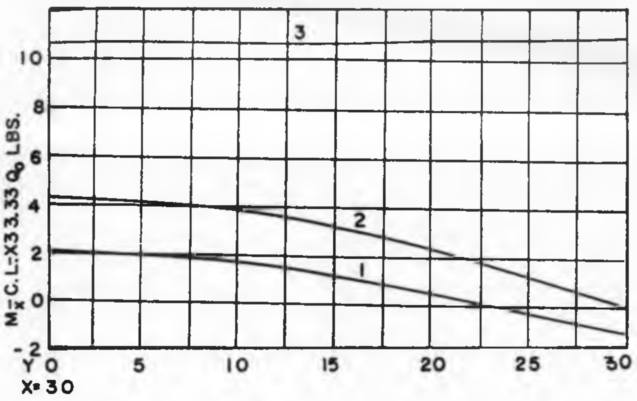


FIG. 16 x-MOMENT AT x = 30, CONSTANT LOAD

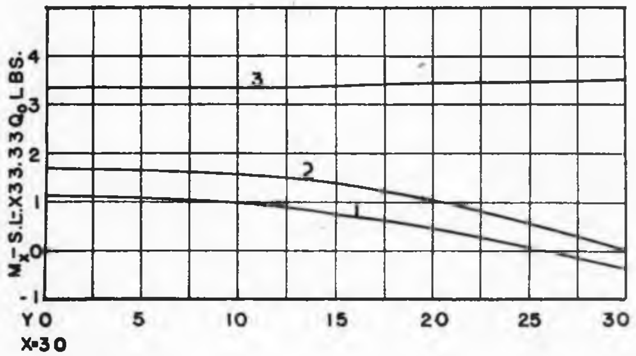


FIG. 17 x-MOMENT AT x = 30, STRIP LOAD

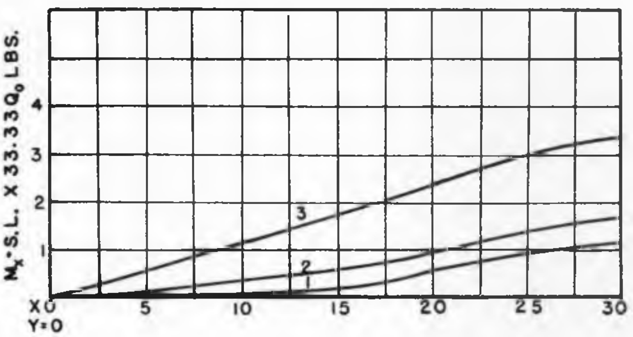


FIG. 18 x-MOMENT AT y = 0, STRIP LOAD

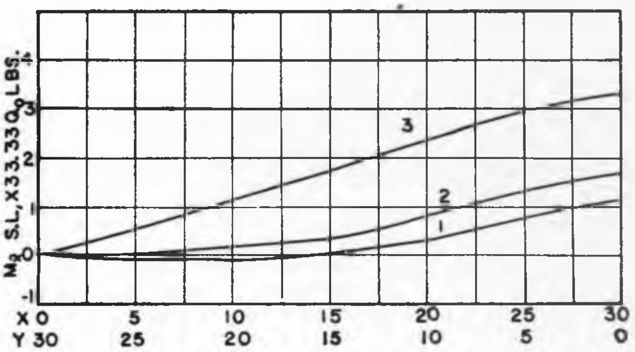


FIG. 19 x-MOMENT ON NEGATIVE DIAGONAL, STRIP LOAD

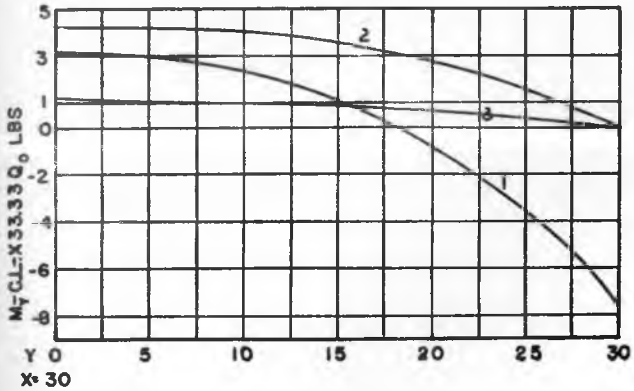


Fig. 20 y-Moment at x = 30, Constant Load

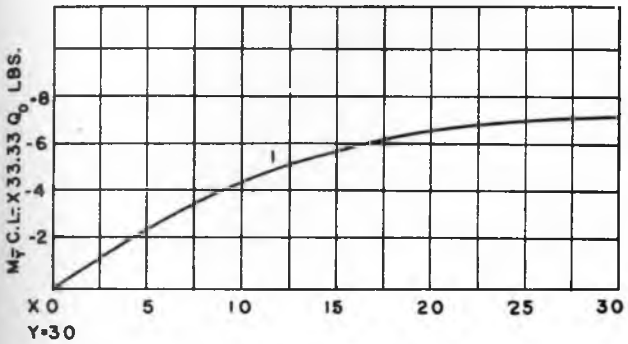


Fig. 21 y-Moment at y = 30, Constant Load

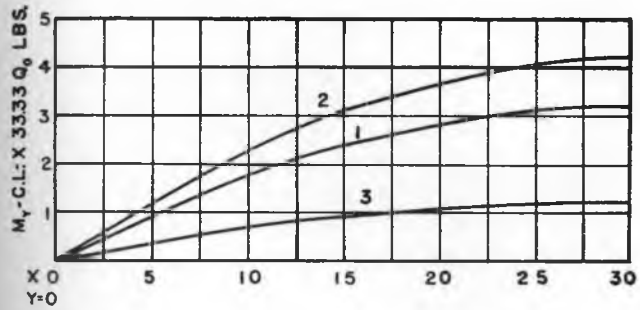


Fig. 22 y-Moment at y = 0, Constant Load

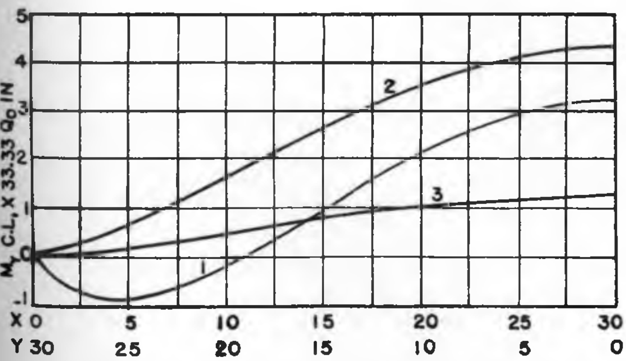


Fig. 23 y-Moment on Negative Diagonal, Constant Load

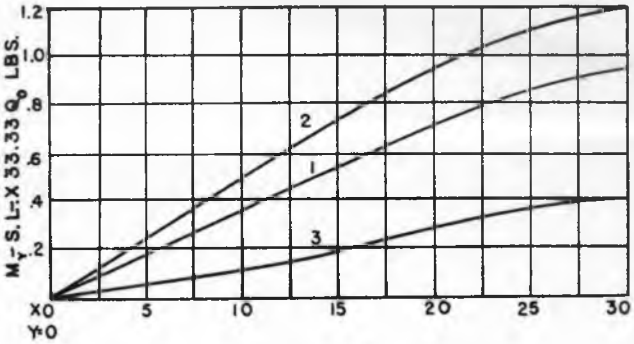


Fig. 24 y-Moment at y = 0, Strip Load

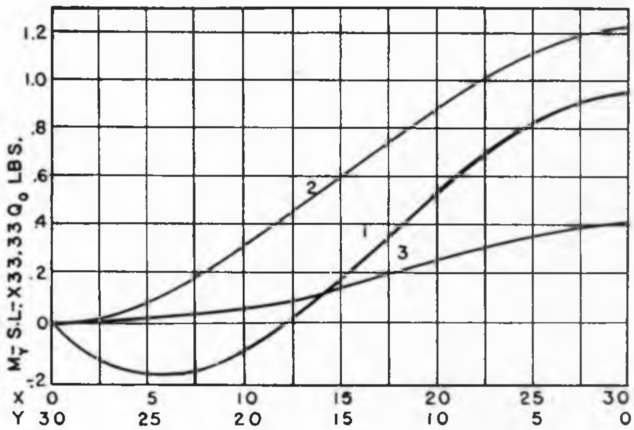


Fig. 25 y-Moment on Negative Diagonal, Strip Load

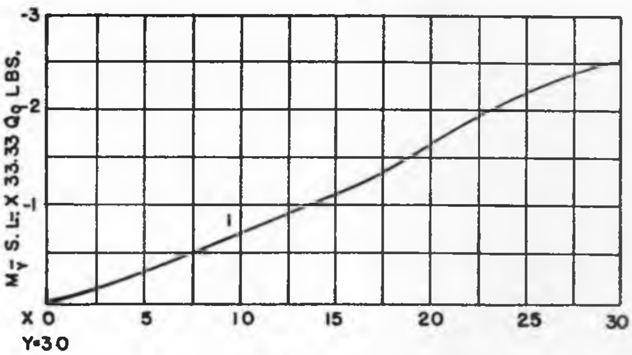


Fig. 26 y-Moment at y = 30, Strip Load

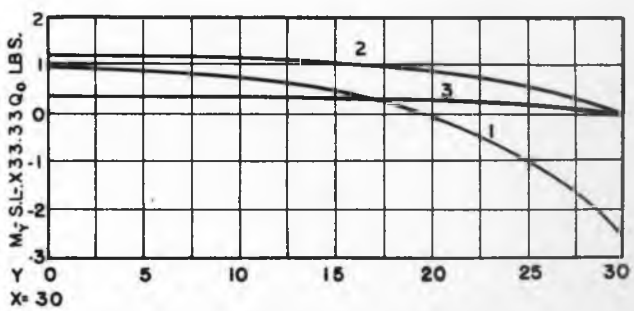


Fig. 27 y-Moment at x = 30, Strip Load

If the impressed frequency is small enough compared to the foundation modulus (i. e. $\omega^2 < \sqrt{k'/\rho}$) this equation is the same as Equation (27) with k' replaced by $k' - \rho\omega^2$.

E. Differentiating Trigonometric Series

To differentiate a sine series of the form

$$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx \qquad 0 < x < \pi$$

(summation from 1 to ∞ unless otherwise specified), we use the cosine transform of $f'(x)$.

$$f'(x) = \frac{1}{\pi} f'_c(0) = \frac{2}{\pi} \sum_{n=1}^{\infty} f'_c(n) \cos nx \quad \text{where} \quad f_c(n) = \int_0^{\pi} f'(\xi) \cos n\xi \, d\xi$$

(See Bromwich.[2]) Integrating by parts we obtain

$$f'(x) = \frac{f(\pi) - f(0) - \mu_0}{\pi} + \frac{2}{\pi} \Sigma \left[nf_s(n) + (-1)^n f(\pi) - f(0) - \mu_0 \cos nx_0 \right] \cos nx \tag{76}$$

where μ_o is the jump of $f(x)$ at $x = x_o$. Similarly, differentiating a cosine series of the form

$$f(x) = \frac{f_c(0)}{\pi} + \frac{2}{\pi} \Sigma f_c(n) \cos nx \tag{77}$$

we obtain

$$f'(x) = \frac{2}{\pi} \Sigma \left[-nf_c(n) - \mu_o \sin nx_o \right] \sin nx \ .$$

Higher derivatives can be obtained by repeated use of Equations (76), (77). It is assumed in the foregoing discussion that f(x) has sectionally continuous derivatives.

II. USE OF DOUBLE SINE SERIES IN SOLVING PLATE PROBLEMS

A. General Equations

In Part I, we solved the problem of a thin rectangular plate, which had specified moments and deflections on two opposite edges, by using a single sine series. In Part II, we will solve more general problems by using a double sine series. The problem is to find a solution of the equation

$$\nabla^4 w = w_{xxxx} + 2w_{xxyy} + w_{yyyy} = \frac{q(x, y)}{D} \qquad (78)$$

subject to rather general boundary conditions on all four edges of the plate. Let us assume that w(x, y) has continuous third derivatives and sectionally continuous fourth derivatives in the region of the plate and is continuous around the boundary. The region of the plate in Part II will be taken as $0 \leq x \leq a$ and $0 \leq y \leq b$. Consider the function

$$w(x, y) = \frac{4ka^2}{b} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn} \sin \alpha x \sin \beta y \qquad (79)$$

where $\alpha = \frac{n\pi}{a}$ $\beta = \frac{m\pi}{b}$ $k = \frac{qa^3}{D}$, D = modulus of rigidity, q = maximum load per unit area, $\{a_{mn}\}$ is a set of coefficients to be determined. A factor a^2 has been introduced to make a_{mn} dimensionless. A dimensionless constant k has been introduced so that a_{mn} will be independent of q/D. We will assume that we can interchange the order of summation in all expressions involving a double sum. A sufficient condition for this assumption is that the series converge absolutely. If the coefficients obtained by using this assumption are calculated, and a test such as given by Bromwich[2] (p 84-85) indicates that the series is absolutely convergent, then our assumption is justified. No study as yet has been made in this regard.

To calculate $w_x(x, y)$ we use the method indicated by Equation (76), and obtain

$$w_x(x, y) = \frac{w(a, y) - w(0, y)}{a} + \frac{2}{a} \sum_{n=1}^{\infty} \Big\{ (-1)^n w(a, y)$$

$$- w(0, y) + \frac{2\alpha k a^3}{b} \sum_{n=1}^{\infty} a_{mn} \sin \beta y \Big\} \cos \alpha x \quad . \tag{80}$$

This involves the boundary functions w(a, y) and w(0, y). It will be convenient to make the following expansions:

$$w(a, y) = \frac{2ka^2}{b} \sum_m b_m \sin \beta y \tag{81}$$

$$w(0, y) = \frac{2ka^2}{b} \sum_m c_m \sin \beta y \tag{82}$$

$$w_{xx}(a, y) = \frac{2k}{b} \sum_m d_m \sin \beta y \tag{83}$$

$$w_{xx}(0, y) = \frac{2k}{b} \sum_m e_m \sin \beta y \tag{84}$$

$$w(x, b) = 2ka \sum_n f_n \sin \alpha x \tag{85}$$

$$w(x, 0) = 2ka \sum_n g_n \sin \alpha x \tag{86}$$

$$w_{yy}(x, b) = \frac{2k}{a} \sum_n h_n \sin \alpha x \tag{87}$$

$$w_{yy}(x, 0) = \frac{2k}{a} \sum_n i_n \sin \alpha x \tag{88}$$

(The symbol $\sum_m$ is an abbreviation for $\sum_{m=1}^{\infty}$.)

To calculate $w_{xy}(x, y)$ we first express $w_x(x, y)$ in the following form

$$w_x(x, y) = \frac{4ka}{b} \sum_m \left\{ \frac{b_m - c_m}{2} + \sum_n \left[(-1)^n b_m - c_m + \alpha a\, a_{mn} \right] \cos \alpha x \right\} \sin \beta y \tag{89}$$

Applying Equation (76), we obtain

$$w_{xy}(x, y) = w_x(x, b) - w_x(x, 0) + \frac{2}{b} \sum_m \Big\{ (-1)^m w_x(x, b) - w_x(x, 0) + 2ka\beta \cdot$$

$$\left[\frac{b_m - c_m}{2} + \sum_n \left((-1)^n b_m - c_m + \alpha a\, a_{mn} \right) \cos \alpha x \right] \Big\} \cos \beta y \; . \tag{90}$$

Applying Equation (76) to Equations (85) and (86), we obtain:

$$w_x(x, b) = \frac{w(a, b) - w(0, b)}{a} + \frac{2}{a} \sum \left\{ (-1)^n w(a, b) - w(0, b) + ka^2 \alpha f_n \right\} \cos \alpha x \tag{91}$$

$$w_x(x, 0) = w(a, 0) - w(0, 0) + \frac{2}{a} \sum \left\{ (-1)^n w(a, 0) - w(0, 0) + ka^2 \alpha g_n \right\} \cos \alpha x \tag{92}$$

so that Equation (90) becomes

$$w_{xy}(x, y) = \frac{w(a, b) - w(0, b) - w(a, 0) + w(0, 0)}{ab}$$

$$+ \frac{2}{ab} \sum_n \Big\{ (-1)^n w(a, b) - (-1)^n w(a, 0) - w(0, b) + w(0, 0)$$

$$+ \alpha ka^2 f_n - \alpha ka^2 g_n \Big\} \cos \alpha x$$

$$+ \frac{2}{ab} \sum_m \Big\{ (-1)^n w(a, b) - (-1)^m w(0, b) - w(a, 0) + w(0, 0)$$

$$+ \beta ka^2 b_m - \beta ka^2 c_m \; \cos \beta y$$

$$+\frac{4}{ab}\sum_{m,n}\Big\{(-1)^{m+n}\, w(a,b) - (-1)^m\, w(0,b) - (-1)^n\, w(a,0)$$

$$+ w(0,0) + (-1)^m\, \alpha k a^2 f_n - \alpha k a^2\, g_n + \beta k a^2 (-1)^n\, b_m$$

$$- k a^2 \beta c_m + \alpha\beta k a^3\, a_{mn}\Big\} \cos \alpha x \cos \beta y \quad . \tag{93}$$

Continuing in this way, we obtain

$$\nabla^4 w(x,y) = \frac{4}{ab}\sum_{m,n}\Big\{2\alpha\beta(-1)^{m+n}\, w(a,b) - 2\alpha\beta(-1)^n\, w(a,0)$$

$$- 2\alpha\beta(-1)^m\, w(0,b) + 2\alpha\beta w(0,0) + k a^2 b_m\, (-1)^n\, (\alpha^2 + 2\beta^2)$$

$$+ k a^2 \alpha\, c_m(\alpha^2 + 2\beta^2) - k a(-1)^n\, d_m + k \alpha e_m + k a^2 \beta\, (-1)^m\, f_n(2\alpha^2 + \beta^2)$$

$$- k a^2 \beta\, g_n(2\alpha^2 + \beta^2) - k\beta(-1)^m h_n + k\beta i_n + (\alpha^2 + \beta^2)^2\, k a^3 a_{mn}\Big\} \sin \alpha x \sin \beta y \tag{94}$$

Suppose q(x, y) can be represented by a Fourier series

$$q(x,y) = \frac{4kD}{a^2 b}\sum_{m,n} q_{mn} \sin \alpha x \sin \beta y \quad . \tag{95}$$

If Equation (78) is valid for all values of x and y, we can equate coefficients and solve for a_{mn}. We obtain

$$a_{mn} = \frac{1}{k a^3(\alpha^2+\beta^2)^2}\Big\{\frac{k}{a}\, q_{mn} - 2\alpha\beta\, (-1)^{m+n}\, w(a,b)$$

$$+ 2\alpha\beta(-1)^n\, w(a,0) + 2\alpha\beta\, (-1)^m\, w\,(0,b) - 2\alpha\beta w\,(0,0)$$

$$- k a^2 b_m(-1)^n\, \alpha(\alpha^2 + 2\beta^2) + k a^2 \alpha c_m\, (\alpha^2 + 2\beta^2) + k a(-1)^n d_m$$

$$- k\alpha e_m - \beta(-1)^m\, k a^2\, f_n\, (2\alpha^2 + \beta^2) + k a^2 \beta\, g_n\, (2\alpha^2 + \beta^2) + k\beta(-1)^m h_n - k\beta i_n\Big\} \tag{96}$$

Let us make the following substitutions:

$$B_m = b_m + \frac{1}{k\beta a^2}\, [(-1)^m\, w(a,b) - w(a,0)] \tag{97}$$

$$C_m = c_m + \frac{1}{k\beta a^2}\, [(-1)^m\, w(0,b) - w(a,0)] \tag{98}$$

$$F_n = f_n + \frac{1}{k\alpha a^2}\left[(-1)^n w(a, b) - w(0, b)\right] \quad (99)$$

$$G_n = g_n + \frac{1}{k\alpha a^2}\left[(-1)^n w(a, 0) - w(0, 0)\right] \quad (100)$$

Using these substitutions and Equation (96), Equation (79) becomes

$$w(x, y) = \frac{4k}{a^2 b} \sum_{m, n} \frac{q_{mn}}{(\alpha^2+\beta^2)^2} \sin \alpha x \sin \beta y$$

$$+ \frac{4k}{ab} \sum_{m, n} \left\{ (-1)^{n+1} \frac{a^2\alpha(\alpha^2+2\beta^2)}{(\alpha^2+\beta^2)^2} B_m + \frac{a^2\alpha(\alpha^2+2\beta^2)}{(\alpha^2+\beta^2)^2} C_m \right.$$

$$+ \frac{(-1)^n \alpha d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha e_m}{(\alpha^2+\beta^2)^2} - \frac{\beta (-1)^m a^2(2\alpha^2+\beta^2)}{(\alpha^2+\beta^2)^2} F_n$$

$$\left. + \frac{\beta a^2(2\alpha^2+\beta^2)}{(\alpha^2+\beta^2)^2} G_n + \frac{\beta(-1)^m h_n}{(\alpha^2+\beta^2)^2} - \frac{\beta i_n}{(\alpha^2+\beta^2)^2} \right\}$$

$$\sin \alpha x \sin \beta y + \frac{4}{ab} \sum_{m, n} \left[\frac{w(a, b)(-1)^{m+n}}{\alpha\beta} - \frac{w(a, 0)(-1)^n}{\alpha\beta} - \frac{w(0, b)(-1)^m}{\alpha\beta} \right.$$

$$\left. + \frac{w(0, 0)}{\alpha\beta} \right] \sin \alpha x \sin \beta y \, . \quad (101)$$

It is convenient to label the three sums in Equation (101) w_1, w_2, w_3 respectively, so that

$$w(x, y) = w_1 + w_2 + w_3$$

$$w_1 = \frac{4k}{a^2 b} \sum_{m, n} \frac{q_{mn}}{(\alpha^2+\beta^2)^2} \sin \alpha x \sin \beta y$$

$$w_2 \quad \text{etc.} \quad (102)$$

w_1 represents the deflection of a simply supported rectangular plate.

w_2 can be summed by Table I to give

$$w_2 = \frac{2ka^2}{b} \sum_m \left\{ \frac{\sinh \beta x}{\sinh \beta a} - \frac{\beta a \sinh \beta(a-x)}{2 \sinh^2 \beta a} + \frac{\beta(a-x) \cosh \beta x}{2 \sinh \beta a} \right\} Bm \sin \beta y$$

$$+ \frac{2ka^2}{b} \sum_m \left\{ \frac{\sinh \beta(a-x)}{\sinh \beta a} - \frac{\beta a \sinh \beta x}{2 \sinh^2 \beta a} + \frac{\beta x \cosh \beta(a-x)}{2 \sinh \beta a} \right\} C_m \sin \beta y$$

$$+ \frac{k}{b} \sum_m \left\{ \frac{a \sinh \beta(a-x)}{\beta \sinh^2 \beta a} - \frac{(a-x) \cosh \beta x}{\beta \sinh \beta a} \right\} d_m \sin \beta y + \frac{k}{b} \sum_m \frac{a \sinh \beta x}{\beta \sinh^2 \beta a}$$

$$- \frac{x \cosh \beta(a-x)}{\beta \sinh \beta a} \Big\} e_m \sin \beta y + 2ka \sum_n \left\{ \frac{\sinh \alpha y}{\sinh \alpha b} - \frac{\alpha b \sinh \alpha(b-y)}{2 \sinh^2 \alpha b} \right.$$

$$+ \frac{\alpha(b-y) \cosh \alpha y}{2 \sinh \alpha b} \Big\} F_n \sin \alpha x + 2ka \sum_n \frac{\sinh \alpha(b-y)}{\sinh \alpha b} - \frac{\alpha b \sinh \alpha y}{2 \sinh^2 \alpha b}$$

$$+ \frac{\alpha y \cosh \alpha(b-y)}{2 \sinh \alpha b} \Big\} G_n \sin \alpha x + \frac{k}{a} \sum_n \left\{ \frac{b \sinh \alpha(b-y)}{\alpha \sinh^2 \alpha b} \right.$$

$$- \frac{(b-y) \cosh \alpha y}{\alpha \sinh \alpha b} \Big\} h_n \sin \alpha x + \frac{k}{a} \sum_n \left\{ \frac{b \sinh \alpha y}{\alpha \sinh^2 \alpha b} \right.$$

$$- \frac{y \cosh \alpha(b-y)}{\alpha \sinh \alpha b} \Big\} i_n \sin \alpha x . \qquad (103)$$

w_3 can be summed to give

$$w_3 = \frac{xy}{ab} [\, w(a, b) - w(a, 0) - w(0, b) + w(0, 0)]$$
$$+ \frac{x}{a} [\, w(a, 0) - w(0, 0)] + \frac{y}{b} [\, w(0, b) - w(0, 0)]$$
$$+ w(0, 0) \quad . \qquad (104)$$

For a given load q(x, y) and for given parameters ν, D, b/a and for given values of the zeroth and second derivatives at the boundaries, the deflection w(x, y) can be calculated numerically, using Equations (101) - (104).

B. Boundary Conditions

If the zeroth and second derivatives are not known on the boundaries we must solve for the coefficients b_m, c_m, d_m, e_m, f_n, g_n, h_n, i_n from given boundary conditions. The twelve standard boundary conditions are:

(1) Clamped at x = a

The equations $w(a, y) = 0$ and $w_x(a, y) = 0$ give rise to the equations

$b_m = B_m = w(a, b) = w(a, 0) = 0$ and

$$0 = \frac{4ka}{b} \sum_m \left\{ -\frac{C_m}{2} + \sum_n (-1)^n \left[\frac{\alpha q_{mn}}{a^3(\alpha^2+\beta^2)^2} - \frac{\beta^4 C_m}{(\alpha^2+\beta^2)^2} \right.\right.$$

$$+ \frac{(-1)^n \alpha^2 d_m}{a^2(\alpha^2+\beta^2)^2} - \frac{\alpha^2 e_m}{a^2(\alpha^2+\beta^2)^2} - \frac{\alpha\beta(-1)^n(2\alpha^2+\beta^2)F_n}{(\alpha^2+\beta^2)^2} + \frac{\alpha\beta(2\alpha^2+\beta^2)\ G_n}{(\alpha^2+\beta^2)^2}$$

$$\left.\left. + \frac{\alpha\beta(-1)^m h_n}{a^2(\alpha^2+\beta^2)^2} - \frac{\alpha\beta i_n}{a^2(\alpha^2+\beta^2)^2} \right]\right\} \sin\beta y + \frac{2}{ab}\left[-w(0,b) + w(0,0)\right]$$

$$\sum \frac{(-1)^m}{\beta} \sin\beta y - \frac{2}{ab}\, w(0,0) \sum \frac{1-(-1)^m}{\beta} \sin\beta y \ . \qquad (105)$$

Since this must hold for all values of y, the coefficient of $\sin\beta y$ must be zero. Summing, where possible by using Table I, we obtain

$$0 = \frac{\beta a C_m}{4S^2}(S + \beta a C) + \frac{d_m(SC - \beta a)}{4\beta a S^2} - \frac{e_m(S - \beta a C)}{4\beta a S^2}$$

$$+ \beta(-1)^m \sum \frac{\alpha(-1)^{n+1} F_n(2\alpha^2+\beta^2)}{(\alpha^2+\beta^2)^2} - \beta \sum \frac{\alpha(-1)^{n+1} G_n}{(\alpha^2+\beta^2)^2}(2\alpha^2+\beta^2)$$

$$- \frac{\beta(-1)^m}{a^2} \sum \frac{\alpha(-1)^{n+1} h_n}{(\alpha^2+\beta^2)^2} + \frac{\beta}{a^2} \sum \frac{\alpha i_n(-1)^{n+1}}{(\alpha^2+\beta^2)^2}$$

$$+ \sum \frac{(-1)^n \alpha\, q_{mn}}{a^3(\alpha^2+\beta^2)^2} + \frac{1}{2ka^2\beta}\left[w(0,b)(-1)^m - w(0,0) \right] \qquad (106)$$

where $S = \sinh\beta a \qquad C = \cosh\beta a$.

(2) Clamped at x 0

In this case $c_m = C_m = w(0, b) = w(0, 0) = 0$ and

$$0 = \frac{B_m \beta a}{4S^2}(S + \beta aC) + \frac{d_m(S - \beta aC)}{4\beta aS^2} - \frac{e_m (SC - \beta a)}{4\beta aS^2}$$

$$- \beta(-1)^m \sum_n \frac{aF_n(2a^2 + \beta^2)}{(a^2 + \beta^2)^2} + \beta \sum_n \frac{a\, G_n(2a^2 + \beta^2)}{(a^2 + \beta^2)^2} + \frac{\beta(-1)^m}{a^2} \sum_n \frac{ah_n}{(a^2+\beta^2)^2}$$

$$-\frac{\beta}{a^2} \sum_n \frac{ai_n}{(a^2+\beta^2)^2} + \frac{1}{a^3} \sum_n \frac{aq_{mn}}{(a^2+\beta^2)^2}$$

$$- \frac{1}{2\beta ka^2}\left[(-1)^m w(a, b) - w(a, 0)\right]. \qquad (107)$$

(3) Clamped at y = b

In this case $f_n = F_n = w(a, b) = w(0, b) = 0$ and

$$0 = \frac{-ab(\bar{S}-ab\bar{C})}{4\bar{S}^2} G_n + \frac{b(\bar{S}\bar{C} - ab)h_n}{4aa^2\bar{S}^2} - \frac{b(\bar{S}-ab\bar{C})}{4aa^2\bar{S}^2} i_n$$

$$+ a(-1)^n \sum_m \frac{\beta(-1)^{m+1}}{(a^2+\beta^2)^2}(a^2+2\beta^2) B_m - a \sum_m \frac{\beta(-1)^{m+1}}{(a^2+\beta^2)^2}(a^2 + 2\beta^2) C_m$$

$$- \frac{a(-1)^n}{a^2} \sum_m \frac{\beta d_m(-1)^{m+1}}{(a^2+\beta^2)^2} + \frac{a}{a^2} \sum_m \frac{\beta e_m(-1)^{m+1}}{(a^2+\beta^2)^2}$$

$$+ \sum_m \frac{\beta(-1)^m q_{mn}}{a^3(a^2+\beta^2)^2} + \frac{1}{2aka^2}\left[(-1)^n w(a, 0) - w(0, 0)\right] \qquad (108)$$

where $\bar{S} = \sinh ab \quad C = \cosh ab$

(4) Clamped at y = 0

In this case $g_n = G_n = 0$ and $w(a, 0) = w(0, 0) = 0$ and

$$0 = \frac{ab}{4\bar{S}^2}(\bar{S}-ab\bar{C})F_n + \frac{b(\bar{S}-ab\bar{C})h_n}{4a^2\, a\bar{S}^2} - \frac{b(\bar{S}\bar{C} - ab)\, i_n}{4a^2\, a\, \bar{S}^2}$$

$$- a(-1)^n \sum \frac{\beta(a^2+2\beta^2)B_m}{(a^2+\beta^2)^2} + a \sum \frac{\beta(a^2 + 2\beta^2)}{(a^2 + \beta^2)^2} C_m + \frac{a(-1)^n}{a^2} \sum \frac{\beta d_m}{(a^2+\beta^2)^2}$$

$$-\frac{a}{a^2}\sum\frac{\beta e_m}{(a^2+\beta^2)^2}+\frac{1}{a^3}\sum\frac{\beta q_{mn}}{(a^2+\beta^2)^2}-\frac{1}{2ka a^2}\left[(-1)^n w(a,b)-w(0,b)\right] \quad (109)$$

(5) Supported at x = a

From the equations $M_x(a,y) = -D[w_{xx}(a,y) + \nu w_{yy}(a,y)] = 0$ and $w(a,y) = 0$, we obtain $b_m = B_m = d_m = w(a,b) = w(a,0) = 0$ (110)

(6) Supported at x = 0

We obtain $c_m = C_m = e_m = w(0,b) = w(0,0) = 0$ (111)

(7) Supported at y = b

We obtain $f_n = F_n = h_n = w(a,b) = w(0,b) = 0$ (112)

(8) Supported at y = 0

We obtain $g_n = G_n = i_n = w(a,0) = w(0,0) = 0$ (113)

(9) Free at x = a

If the edge x = a is free, then

$$M_x(a,y) = -D\left[w_{xx}(a,y) + \nu w_{yy}(a,y)\right] = 0 \quad (114)$$

$$V_x(a,y) = -D\left[w_{xx}(a,y) + (2-\nu)\, w_{xyy}(a,y)\right] = 0 \quad (115)$$

Differentiating Equation 81 and using Equation 97, we obtain

$$w_{yy}(a,y) = -\frac{2ka^2}{b}\sum\beta^2 B_m \sin\beta y$$

Equation (114) becomes

$$d_m = \nu a^2\beta^2 B_m \quad (116)$$

Using Equations (101) and (116) in Equation (115), we obtain

$$0 = -\frac{d_m\beta a(1-\nu)}{4\nu S^2}\left[\beta a(1-\nu) + SC(3+\nu)\right] + \frac{\beta^3 a^3 C_m}{4S^2}\left[(3-\nu)S + (1-\nu)\beta a C\right]$$

$$-\frac{\beta a e_m}{4S^2}\left[(1+\nu)S+(1-\nu)\beta a C\right]+\beta a^2(-1)^m\sum_n(-1)^n\frac{a^3(\beta^2+\nu a^2)}{(a^2+\beta^2)^2}F_n$$

$$-a^2\beta\sum_n\frac{a^3(-1)^n(\beta^2+\nu a^2)}{(a^2+\beta^2)^2}G_n-\beta(-1)^m\sum_n\frac{a h_n(-1)^n}{(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]$$

$$+\beta\sum_n\frac{a i_n(-1)^n}{(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]-\frac{1}{a}\sum_n\frac{a(-1)^n}{(a^2+\beta^2)^2}q_{mn}\left[a^2+(2-\nu)\beta^2\right] \tag{117}$$

(10) Free at $x=0$

A similar argument to that given in (9) gives

$$e_m=\nu\beta^2a^2C_m$$

$$0=-\frac{B_m\beta^3a^3}{4S^2}\left[(3-\nu)S+(1-\nu)\beta a C\right] \tag{118}$$

$$+\frac{e_m\beta a(1-\nu)}{4\nu S^2}\left[(3+\nu)SC+(1-\nu)\beta a\right]+\frac{d_m\beta a}{4S^2}\left[(1+\nu)S+(1-\nu)\beta a C\right]$$

$$+\beta(-1)^m a^2\sum_n\frac{a^3F_n(\beta^2+\nu a^2)}{(a^2+\beta^2)^2}-\beta a^2\sum_n\frac{a^3G_n}{(a^2+\beta^2)^2}(\beta^2+\nu a^2)$$

$$-\beta(-1)^m\sum_n\frac{a h_n}{(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]+\beta\sum_n\frac{a i_n}{(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]$$

$$-\frac{1}{a}\sum_n\frac{a q_{mn}}{(a^2+\beta^2)^2}\left[a^2+\beta^2(2-\nu)\right] \tag{119}$$

(11) Free at $y=b$

We obtain

$$h_n=\nu a^2a^2F_n \tag{120}$$

$$0=(-1)^na^2a\sum_m\frac{(-1)^{m+1}\beta^3(a^2+\nu\beta^2)}{(a^2+\beta^2)^2}B_m-a^2a\sum_m\frac{(-1)^{m+1}\beta^3(a^2+\nu\beta^2)}{(a^2+\beta^2)^2}C_m$$

$$-a(-1)^n\sum_m\frac{(-1)^{m+1}\beta\left[\beta^2+(2-\nu)a^2\right]}{(a^2+\beta^2)^2}d_m+a\sum\frac{(-1)^{m+1}\beta\left[\beta^2+(2-\nu)a^2\right]}{(a^2+\beta^2)^2}e_m$$

$$-\frac{a^2ba^3G_n}{4\bar{S}^2}\left[(3-\nu)\bar{S}+(1-\nu)ab\bar{C}\right]+\frac{abi_n}{4\bar{S}^2}\left[(1+\nu)\bar{S}+(1-\nu)ab\bar{C}\right]$$

$$+\frac{ab(1-\nu)h_n}{4\nu\bar{S}^2}\left[(3+\nu)\,SC+(1-\nu)ab\right]-\frac{1}{a}\Sigma\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] \qquad (121)$$

(12) Free at y = 0

We obtain

$$i_n=\nu\alpha^2 a^2 G_n \qquad (122)$$

$$0=(-1)^n a^2\alpha\,\Sigma\frac{\beta^3(\alpha^2+\nu\beta^2)}{(\alpha^2+\beta^2)^2}B_m-a^2\alpha\,\Sigma\frac{\beta^3(\alpha^2+\nu\beta^2)}{(\alpha^2+\beta^2)^2}C_m$$

$$-(-1)^n\alpha\,\Sigma\frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]+\alpha\,\Sigma\frac{\beta e_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$-\frac{\alpha^3 ba^2F_n}{4\bar{S}^2}[(3-\nu)\bar{S}+(1-\nu)ab\bar{C}]+\frac{abh_n}{4\bar{S}^2}[(1+\nu)\bar{S}+(1-\nu)ab\bar{C}]$$

$$+\frac{i_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab]-\frac{1}{a}\Sigma\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] \qquad (123)$$

For each of these twelve boundary conditions, we have two sets of equations. Since there are four edges, there will be four boundary conditions, or eight sets of equations. An attempt will be made to solve for these eight sets of unknown coefficients.

C. Concentrated Load

We first consider a rectangular load (i. e., a load which is constant over a rectangle) over the region $a_1 < x < a_2$, $b_1 < y < b_2$. From Equation (95), we obtain

$$q_{mn} \quad \frac{1}{a^2 \alpha \beta} (\cos \alpha a_1 - \cos \alpha a_2)(\cos \beta b_1 - \cos \beta b_2)$$

This reduces to a constant load if $a_1 = b_1 = 0$ and $a_2 = a$, $b_2 = b$. From Equations (101) and (102), w_1 becomes

$$w_1 = \frac{4p}{a^2 bD} \sum_{m,n} \frac{\sin \alpha x \sin \beta y (\cos \alpha a_1 - \cos \alpha a_2)}{\alpha\beta(\alpha^2+\beta^2)^2 (a_2 - a_1)(b_2 - b_1)} (\cos \beta b_1 - \cos \beta b_2) \tag{124}$$

where p is the total load. If p, a_1, a_2, x, y are constant, we find the expression for the deflection of a simply supported plate under a concentrated load at (a_1, b_1) to be the limit of w_1 in Equation (123) as $a_2 \rightarrow a_1$, $b_2 \rightarrow b_1$

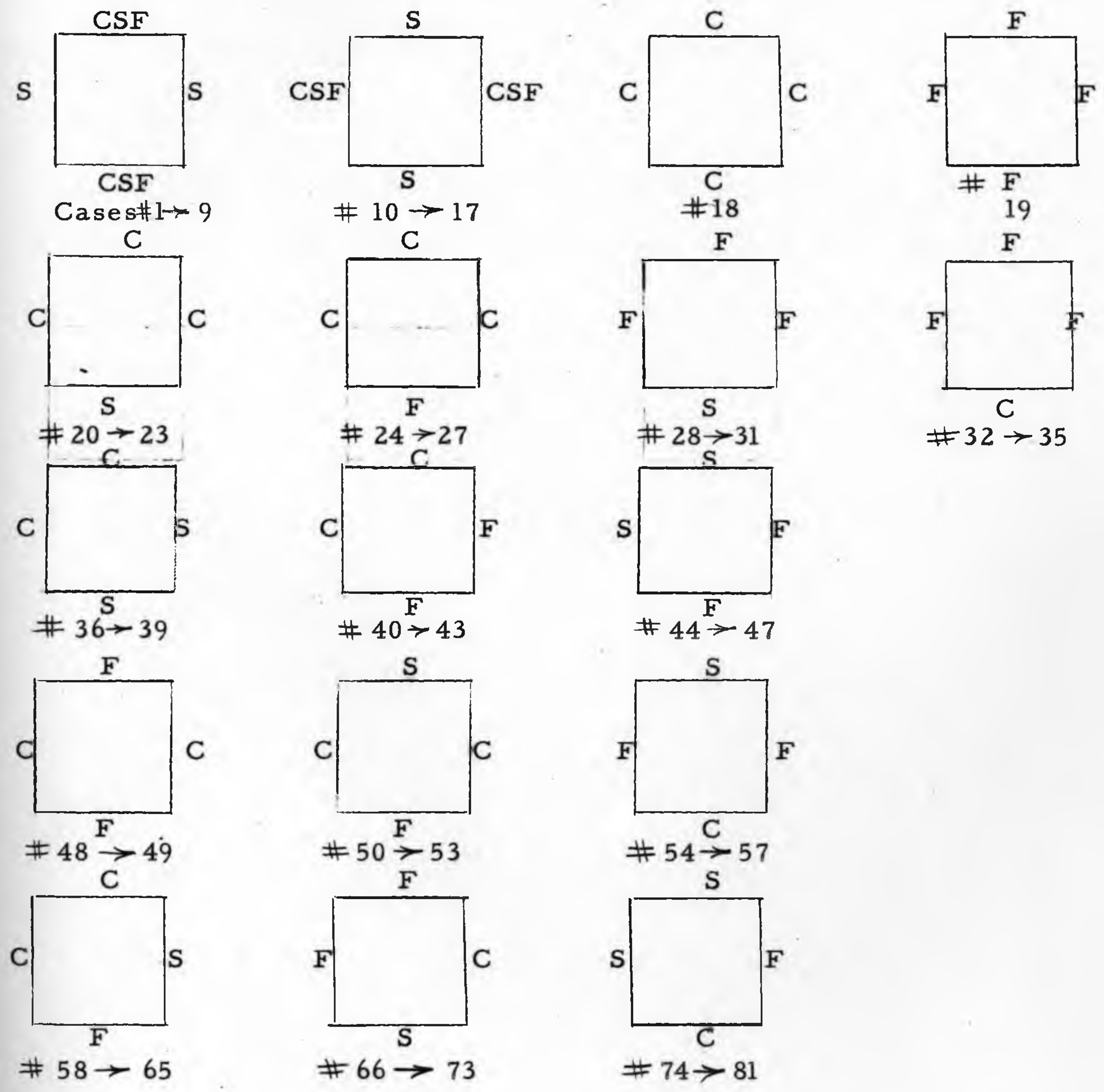
$$w_1 = \frac{4p}{a^2 bD} \sum_{m,n} \frac{\sin \alpha a_1 \sin \beta b_1 \sin \alpha x \sin \beta y}{(\alpha^2 + \beta^2)^2} \tag{125}$$

This can be summed over m or n by use of Table I. The result agrees with the solution given by Timoshenko[34] (p 156) which was obtained by matching boundary conditions along the line $x = a_1$.

D. Solution to Plate Problems Which Have Every Edge Either Clamped, Supported, or Free

There are eighty-one problems which have every edge either clamped, supported, or free. We number these problems as follows:

Fig. 28



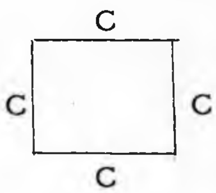
Many of these problems have the same solution except for a transformation of coordinates (e. g., #20, #21, # 22, # 23).

Problems #1 → 17 have been solved by Deverall and Thorne.[10] Problem #18 has been solved by Timoshenko.[34] Problems #36 #43 have been solved by Huang, Conway[17] and Stiles.[32] Approximate methods have been used in many cases (such as the Ritz Method).[15, 34]

Problems # 1 → 17:

If the two x edges are supported, we find that $b_m = c_m = d_m = e_m = 0$. The constants f_n, g_n, h_n, i_n can be found by solving four simultaneous equations. The solutions have been given by Deverall and Thorne.[10]

Problem #18:



Equations (106) → (109) give

$$0 = \frac{d_m(S-\beta aC)}{4\beta aS^2} - \frac{e_m(SC-\beta a)}{4\beta aS^2} + \frac{\beta(-1)^m}{a^2} \Sigma \frac{a h_n}{(a^2+\beta^2)^2} - \frac{\beta}{a^2} \Sigma \frac{a i_n}{(a^2+\beta^2)^2}$$

$$+ \frac{1}{a^3} \Sigma \frac{a q_{mn}}{(a^2+\beta^2)^2}$$

$$0 = \frac{-d_m(SC-\beta a)}{4\beta aS^2} + \frac{e_m(S-\beta aC)}{4\beta aS^2} + \frac{\beta(-1)^m}{a^2} \Sigma \frac{a(-1)^{n+1} h_n}{(a^2+\beta^2)^2} - \frac{\beta}{a^2} \Sigma \frac{a(-1)^{n+1} i_n}{(a^2+\beta^2)^2}$$

$$+ \frac{1}{a^3} \Sigma \frac{a(-1)^{n+1}}{(a^2+\beta^2)^2} q_{mn}$$

$$0 = \frac{h_n(\bar{S}-\alpha bC)b}{4\alpha a^2\bar{S}^2} - \frac{i_n b(\bar{S}\bar{C} - \alpha b)}{4\alpha a^2\bar{S}^2} + \frac{\alpha(-1)^n}{a^2}\Sigma \frac{\beta d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha}{a^2}\Sigma \frac{\beta e_m}{(\alpha^2+\beta^2)^2}$$

$$+\frac{1}{a^3}\Sigma \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{-h_n(\bar{S}\bar{C}-\alpha b)}{4\alpha a^2\bar{S}^2} + \frac{i_n b}{4\alpha a^2\bar{S}^2}(\bar{S}-\alpha bC) + \frac{\alpha(-1)^n}{a^2}\Sigma \frac{\beta(-1)^{m+1} d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha}{a^2}\Sigma \frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}$$

$$+\frac{1}{a^3}\Sigma \frac{\beta(-1)^{m+1}}{(\alpha^2+\beta^2)^2} q_{mn} \qquad (126)$$

Symnetry (a)

If $q(x, y) = q(a-x, y)$, then $e_m = d_m$, $h_n = q_{mn} = i_n = 0$ for even n, and Equation (126) becomes

$$0 = \frac{-d_m}{4\beta aS^2}(S+\beta a)(C-1) + \frac{\beta(-1)^m}{a^2}\sum_{odd} \frac{\alpha h_n}{(\alpha^2+\beta^2)^2} - \frac{\beta}{a^2}\sum_{odd}\frac{\alpha i_n}{(\alpha^2+\beta^2)^2} + \frac{1}{a^3}\Sigma\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4\alpha a^2\bar{S}^2}(\bar{S}-\alpha b\bar{C}) - \frac{i_n b}{4\alpha a^2\bar{S}^2}(\bar{S}\bar{C}-\alpha b) + \frac{\alpha(-1)^n}{a^2}\Sigma\frac{\beta d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha}{a^2}\Sigma\frac{\beta d_m}{(\alpha^2+\beta^2)^2}$$

$$+\frac{1}{a^3}\Sigma\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{-h_n b}{4\alpha a^2\bar{S}^2}(\bar{S}C-\alpha b) + \frac{i_n b}{4\alpha a^2\bar{S}^2}(\bar{S}-\alpha bC) + \frac{\alpha(-1)^n}{a^2}\Sigma\frac{\beta(-1)^{m+1}}{(\alpha^2+\beta^2)^2} d_m$$

$$- \frac{\alpha}{a^2}\Sigma\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} + \frac{1}{a^3}\Sigma\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2} \qquad (127)$$

Symmetry (b)

If $q(x, y) = q(y, x)$ (i.e., a square plate symmetric about a diagonal), then $e_n = i_n$, $d_n = h_n$ and Equation (126) becomes

$$0 = \frac{d_m(S-\beta aC)}{4\beta aS^2} - \frac{e_m(SC-\beta a)}{4\beta aS^2} + \frac{\beta(-1)^n}{a^2} \Sigma \frac{\alpha d_n}{(\alpha^2+\beta^2)^2} - \frac{\beta}{a^2} \Sigma \frac{\alpha e_n}{(\alpha^2+\beta^2)^2}$$

$$+ \frac{1}{a^3} \Sigma \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{-d_m(SC-\beta a)}{4\beta aS^2} + \frac{e_m(S-\beta aC)}{4\beta aS^2} + \frac{\beta(-1)^m}{a^2} \Sigma \frac{\alpha d_n(-1)^{n+1}}{(\alpha^2+\beta^2)^2} - \beta \Sigma \frac{\alpha(-1)^{n+1}}{(\alpha^2+\beta^2)^2} e_n$$

$$+ \frac{1}{a^3} \Sigma \frac{\alpha q_{mn}(-1)^{n+1}}{(\alpha^2+\beta^2)^2} \qquad (128)$$

Symmetry (c)

If $q(x, y) = q(y, x) = q(a-x, y)$, then $e_m = d_m = h_m = i_m$ and Equation (126) becomes

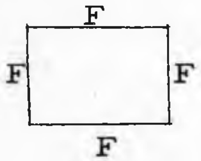
$$0 = \frac{-d_m(S+\beta a)\,(C-1)}{4\beta aS^2} - \frac{2\beta}{a^2} \Sigma_{odd} \frac{\alpha d_n}{(\alpha^2+\beta^2)^2} + \frac{1}{a^3} \Sigma \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2} \qquad (129)$$

For a constant load this is

$$d_m = \frac{2}{m^3\pi^3} \frac{S-m\pi}{S+m\pi} - \frac{8m^2}{\pi} \frac{S^2}{(S+m\pi)(C-1)} \Sigma_{odd} \frac{n d_n}{(m^2+n^2)^2} \qquad m \; 1, 3, 5, \ldots \qquad (130)$$

which checks with the solution given by Timoshenko.[34]

Problem # 19:



Equations (110) → (123) give

$$B_m = \frac{d_m}{\nu\beta^2 a^2} \qquad C_m = \frac{e_m}{\nu\beta^2 a^2} \qquad F_n = \frac{h_n}{\nu\alpha^2 a^2} \qquad G_n = \frac{i_n}{\nu\alpha^2 a^2}$$

$$0 = \frac{-\beta a d_m(1-\nu)}{4\nu S^2}[(3+\nu)\,S + (1-\nu)\beta aC] + \frac{\beta a e_m(1-\nu)}{4\nu S^2}[(3+\nu)SC + (1-\nu)\beta a]$$

$$+ \beta^3(-1)^m \frac{(1-\nu)^2}{\nu} \Sigma \frac{\alpha h_n}{(\alpha^2+\beta^2)^2} - \beta^3\frac{(1-\nu)^2}{\nu} \Sigma \frac{\alpha i_n}{(\alpha^2+\beta^2)^2} - \frac{1}{a} \Sigma \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}[\alpha^2+\beta^2(2-\nu)]$$

$$0 = -\frac{\beta a d_m(1-\nu)}{4\nu S^2}[(3+\nu)SC+(1-\nu)\beta a] + \frac{\beta a e_m}{4\nu S^2}(1-\nu)[(3+\nu)S + (1-\nu)\beta a C]$$

$$+\beta^3(-1)^m\frac{(1-\nu)^2}{\nu}\Sigma \frac{a h_n(-1)^n}{(a^2+\beta^2)^2} - \frac{\beta^3(1-\nu)^2}{\nu}\Sigma \frac{a i_n(-1)^n}{(a^2+\beta^2)^2} - \frac{1}{a}\Sigma \frac{(-1)^n a q_{mn}}{(a^2+\beta^2)^2}$$

$$\cdot [a^2 + \beta^2 (2-\nu)]$$

$$0 = (-1)^n a^3\frac{(1-\nu)^2}{\nu}\Sigma \frac{\beta d_m}{(a^2+\beta^2)^2} - a^3\frac{(1-\nu)^2}{\nu}\Sigma \frac{\beta e_m}{(a^2+\beta^2)^2} - \frac{a b h_n}{4\nu \bar{S}^2}[(3+\nu)\bar{S}+(1-\nu)ab\bar{C}]$$

$$+ \frac{i_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab] - \frac{1}{a}\Sigma \frac{\beta q_{mn}}{(a^2+\beta^2)^2}[\beta^2 + (2-\nu)a^2]$$

$$0 = (-1)^n a^3\frac{(1-\nu)^2}{\nu}\Sigma \frac{\beta d_m(-1)^{m+1}}{(a^2+\beta^2)^2} - a^3\frac{(1-\nu)^2}{\nu}\Sigma \frac{\beta e_m(-1)^{m+1}}{(a^2+\beta^2)^2}$$

$$- \frac{a b i_n}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S} + (1-\nu)ab\bar{C}] + \frac{h_n ab}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S}\bar{C} + (1-\nu)ab]$$

$$- \frac{1}{a}\Sigma \frac{\beta q_{mn}}{(a^2+\beta^2)^2}[\beta^2 + a^2(2-\nu)](-1)^{m+1} \tag{131}$$

Symmetry (a)

If $q(x, y) = q(y, x)$ then

$$e_m = i_m = \nu\beta^2 a^2 C_m = \nu\beta^2 b^2 G_m$$

$$d_m = h_m = \nu\beta^2 a^2 B_m = \nu\beta^2 b^2 F_m$$

$$0 = -\frac{\beta a d_m}{4\nu S^2}(1-\nu)[(3+\nu)S+(1-\nu)\beta a C] + \frac{\beta a e_m}{4\nu S^2}(1-\nu)[(3+\nu)S + (1-\nu)\beta a]$$

$$+ \beta^3(-1)^m\frac{(1-\nu)^2}{\nu}\sum_n \frac{a d_n}{(a^2+\beta^2)^2} - \frac{\beta^3(1-\nu)^2}{\nu}\Sigma \frac{a e_n}{(a^2+\beta^2)^2} - \frac{1}{a}\Sigma \frac{a q_{mn}}{(a^2+\beta^2)^2}$$

$$[a^2 + \beta^2(2-\nu)]$$

$$0 = -\frac{\beta a d_m}{4\nu S^2}(1-\nu)[(3+\nu)SC+(1-\nu)\beta a] + \frac{\beta a e_m}{4\nu S^2}(1-\nu)[(3+\nu)S + (1-\nu)\beta a C]$$

$$+ \beta^3(-1)^m\frac{(1-\nu)^2}{\nu}\Sigma \frac{a d_n(-1)^n}{(a^2+\beta^2)^2} - \frac{\beta^3(1-\nu)^2}{\nu}\Sigma \frac{a e_n(-1)^n}{(a^2+\beta^2)^2} - \frac{1}{a}\Sigma \frac{(-1)^n a\, q_{mn}}{(a^2+\beta^2)^2}$$

$$[a^2+\beta^2(2-\nu)] \tag{132}$$

Symmetry (b)

If $q(x, y) = q(a-x, y)$, then

$$B_m = C_m = \frac{d_m}{\nu\beta^2 a^2} = \frac{e_m}{\nu\beta^2 a^2} \qquad F_n = \frac{h_n}{\nu\alpha^2 a^2} \qquad G_n = \frac{i_n}{\nu\alpha^2 a^2} \qquad i_n = h_n = 0 \quad \text{if}$$

n is even

$$0 = \frac{\beta a d_m}{4\nu S^2}(1-\nu)(C-1)\,[(3+\nu)S-(1-\nu)\beta a]+\beta^3(-1)^m \frac{(1-\nu)^2}{\nu} \Sigma \frac{a h_n}{(\alpha^2+\beta^2)^2}$$

$$-\beta^3 \frac{(1-\nu)^2}{\nu} \sum_{odd} \frac{a i_n}{(\alpha^2+\beta^2)^2} - \frac{1}{a} \sum_{odd} \frac{a q_{mn}}{(\alpha^2+\beta^2)^2} [\alpha^2+\beta^2(2-\nu)]$$

$$0 = \frac{-\beta a d_m}{4\nu S^2}(1-\nu)[(3+\nu)SC+(1-\nu)\beta a] + \frac{\beta a e_m}{4\nu S^2}(1-\nu)[(3+\nu)S+(1-\nu)\beta a C]$$

$$+ \beta^3(-1)^m \frac{(1-\nu)^2}{\nu} \Sigma \frac{a d_n(-1)^n}{(\alpha^2+\beta^2)^2} - \frac{\beta^3(1-\nu)^2}{\nu} \Sigma \frac{a e_n(-1)^n}{(\alpha^2+\beta^2)^2} - \frac{1}{a} \Sigma \frac{(-1)^n a q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$\cdot [\alpha^2 + \beta^2 (2-\nu)] \qquad (133)$$

Symmetry (c)

If $q(x, y) = q(a-x, y) = q(x, b-y)$, then

$$B_m = C_m = \frac{d_m}{\nu\beta^2 a^2} = \frac{e_m}{\nu\beta^2 a^2} \qquad F_n = G_n = \frac{h_n}{\nu\alpha^2 a^2} = \frac{i_n}{\nu\alpha^2 a^2} \qquad i_n = h_n = d_n = e_n = 0$$

if n is even

$$0 = \frac{\beta a d_m}{4\nu S^2}(1-\nu)(C-1)\,[(3+\nu) S-(1-\nu)\beta a] - 2\beta^3\frac{(1-\nu)^2}{\nu} \sum_{odd} \frac{a h_n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{1}{a} \sum_{odd} \frac{a q_{mn}}{(\alpha^2+\beta^2)^2} [\alpha^2+\beta^2(2-\nu)]$$

$$0 = -2\alpha^3\frac{(1-\nu)^2}{\nu} \sum_{odd} \frac{\beta d_m}{(\alpha^2+\beta^2)^2} + \frac{ab(1-\nu)h_n}{4\nu \bar{S}^2} (\bar{C}-1)\,[(3+\nu)\bar{S}-(1-\nu)\alpha b]$$

$$-\frac{1}{a} \sum_{odd} \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2} [\beta^2+\alpha^2(2-\nu)] \qquad (134)$$

Symmetry (d)

If $q(x, y) = q(y, x) = q(a-x, y) = q(x, b-y)$, then

$$B_m = C_m = F_m = G_m = \frac{d_m}{\nu\beta^2a^2} = \frac{e_m}{\nu\beta^2a^2} = \frac{h_m}{\nu\beta^2a^2} = \frac{i_m}{\nu\beta^2a^2}$$

$$0 = \frac{\beta a d_m}{4\nu S^2}(1-\nu)(C-1)[(3+\nu)S-(1-\nu)\beta a] - 2\beta^3\frac{(1-\nu)^2}{\nu}\sum_{odd}\frac{\alpha d_n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{1}{a}\sum_{odd}\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}[\alpha^2+\beta^2(2-\nu)] \quad (135)$$

All coefficients are zero when m is even. For the special case of a constant load we obtain

$$d_m = \frac{8\,m^2(1-\nu)}{\pi}\frac{S^2}{(C-1)[(3+\nu)S-(1-\nu)m\pi]}\sum_{odd}\frac{n d_n}{(m^2+n^2)^2}$$

$$+\frac{2\nu}{1-\nu}\frac{1}{m^3\pi^3}\frac{(3-\nu)S-(1-\nu)m\pi}{(3+\nu)S-(1-\nu)m\pi} \qquad m = 1, 3, 5, \ldots \quad (136)$$

For the special case of a point load, we obtain

$$d_m' = \frac{8\,m^2(1-\nu)S^2}{(C-1)[(3+\nu)S-(1-\nu)m\pi]}\sum_{odd}\frac{n d_n'}{(n^2+m^2)^2}$$

$$+\frac{(-1)^{\frac{m-1}{2}}}{16\cosh^2\frac{m\pi}{2}}\left[4\cosh\frac{m\pi}{2}+(1-\nu)m\pi\sinh m\pi\right] \quad (137)$$

where

$$d_m = d_m'\frac{pa}{kD}$$

The deflection is given by $w(x, y) = w_1+w_2+w_3$, where w_1 and w_3 are given by Equations (102) and (104) and w_2 is given by

$$w_2 = \frac{2ka}{b\nu} \Sigma \left\{ \frac{\sinh \beta x}{\beta^2 a \sinh \beta a} + \frac{(1-\nu) \sinh \beta x \cosh \beta a}{2\beta \sinh^2 \beta a} - \frac{(1-\nu) x \cosh \beta x}{2\beta a \sinh \beta a} \right\} d_m \sin \beta y$$

$$+ \frac{2ka}{b\nu} \Sigma \left\{ \frac{\sinh \beta(a-x)}{\beta^2 a \sinh \beta a} - \frac{(1-\nu) \sinh \beta x}{2\beta \sinh^2 \beta a} + \frac{(1-\nu) x \cosh \beta(a-x)}{2a\beta \sinh ab} \right\} e_m \sin \beta y$$

$$+ \frac{2k}{\nu a} \Sigma \left\{ \frac{\sinh \alpha y}{\alpha^2 \sinh \alpha b} + \frac{(1-\nu) b \sinh \alpha y \cosh \alpha b}{2\alpha \sinh^2 \alpha b} - \frac{(1-\nu) y \cosh \alpha y}{2\alpha \sinh \alpha b} \right\} h_n \sin \alpha x$$

$$+ \frac{2k}{\nu a} \Sigma \left\{ \frac{\sinh \alpha(b-y)}{\alpha^2 \sinh \alpha b} - \frac{b(1-\nu) \sinh \alpha y}{2\alpha \sinh^2 \alpha b} + \frac{(1-\nu) y \cosh \alpha(b-y)}{2\alpha \sinh \alpha b} \right\} i_n \sin \alpha x \qquad (138)$$

If $q(x, y) = q(a-x, y) = q(y, x)$, this reduces to

$$w_2(x, y) = \frac{4k}{b\nu} \sum_m \frac{d_m}{\beta^2} \sin \beta y \left[\frac{\sinh \beta x}{\sinh \beta a} + \frac{\sinh \beta(a-x)}{\sinh \beta a} \right.$$

$$\left. + \frac{(1-\nu)}{2} \left(\frac{-\beta a \sinh \beta(a-x)}{\sinh^2 \beta a} + \frac{\beta (a-x) \cosh \beta x}{\sinh \beta a} - \frac{\beta a \sinh \beta x}{\sinh^2 \beta a} \right) \right] \qquad (139)$$

a. Corner deflections and loads

Equation (104) is

$$w_3(x, y) = \frac{xy}{ab} [w(a, b) - w(a, 0) - w(0, b) + w(0, 0)] + \frac{x}{a} [w(a, 0) - w(0, 0)]$$

$$+ \frac{y}{b} [w(0, b) - w(0, 0)] + w(0, 0)$$

This represents the deflection of a free plate subjected to concentrated corner forces. The moments and reactions are zero throughout the plate. The corner forces are

$$P = \pm M_{xy} = \pm 2D (1-\nu) w_{xy}$$

$$= \pm \frac{2D(1-\nu)}{ab} [w(a, b) - w(a, 0) - w(0, b) + w(0, 0)] \qquad (140)$$

where the plus sign is taken at the corners (a, b) and (0, 0) and the minus sign is taken at the corners (a, 0) and (0, b). If P = 0, the deflection surface is a plane

$$w_3 = \frac{x}{a}[w(a,0) - w(0,0)] + \frac{y}{b}[w(0,b) - w(0,0)] + w(0,0) \;.$$

Let a load P be applied at (a, b) and (0, 0) and a load P be applied in the opposite direction at (0, b) and (a, 0). If δ is the deflection at (a, b), the deflection surface with corner loads becomes

$$\bar{w}_3 = w_3 + \frac{4\delta}{ab}\left(x - \frac{a}{2}\right)\left(y - \frac{b}{2}\right) \;. \qquad (141)$$

This surface is a hyperbolic paraboloid (i. e, a saddle-shaped hyperboloid) with a center at the center of the plate.

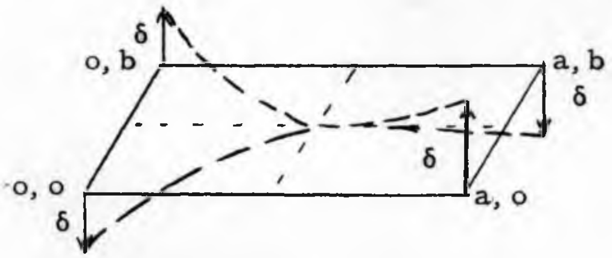


Fig. 29

In the problem of a free plate under a distributed load, we notice from Equations (102) and (138) that w_1 and w_2 do not involve the corner deflections. Four corner deflections may be specified arbitrarily. These deflections will determine the corner loads.

For the forces to be in equilibrium, the corner loads P(a, b), P(a, 0), P(0, b), P(0, 0) must satisfy

$$P(a,b) + P(a,0) + P(0,b) + P(0,0) = \int_0^b \int_0^a q(x,y)\,dx\,dy \;. \qquad (142)$$

For the torques to be in equilibrium, we must have

$$b\,[P(a,b) + P(0,b)] = \int_0^b \int_0^a y\, q(x,y)\, dx\, dy \tag{143}$$

$$a\,[P(a,b) + P(a,0)] = \int_0^b \int_0^a x\, q(x,y)\, dx\, dy\ . \tag{144}$$

If Equations (142), (143), and (144) are independent, only one of the corner loads can be specified arbitrarily. The corner load at (a, b) is

$$P(a,b) = P_1(a,b) + P_2(a,b) + \frac{2D(1-\nu)}{ab}[w(a,b) - w(a,0) - w(0,b) + w(0,0)] \tag{145}$$

where $P_1(a, b)$ is the load due to w_1 and $P_2(a, b)$ is the load due to w_2. Since $P_1(a, b)$ and $P_2(a, b)$ do not depend upon the corner deflection, Equation (145) gives a relation between the corner load and the corner deflections.
If a corner load is specified arbitrarily, then Equation (145) indicates that three rather than four corner deflections may be specified arbitrarily.

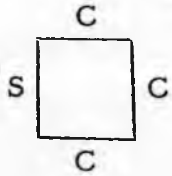
b. <u>Special Example</u>

We could solve the problem of a free, square plate subjected to a uniformly distributed transverse load q by use of Equation (139). We could also find the solution to the problem of a free, square plate subjected to a concentrated force acting upward at the center of the plate of magnitude qa^2. If we specify that the corner deflections of the above two problems are zero, the first case will then have a force at each corner of magnitude $qa^2/4$ acting upwards; the second case will then have a force at each corner of magnitude $qa^2/4$ acting in the opposite direction. If we superimpose these two problems, we will have the problem of a free plate which has no corner loads, which supports a uniform load q, and which is acted upon by a concentrated force at the center equal to $-qa^2$.

The zeroth, first and second derivatives of w(x, y) can be obtained from Equation (19), for $0 < x < a$, $0 < y < b$ except at the center of the plate. At a point in the neighborhood of the concentrated force, the second derivatives will be of the order log r where r is the distance from the force to the point. The third derivatives in the neighborhood of the concentrated force will be of the order $\frac{1}{r}$. Since $\frac{1}{x-a/2}$ does not have a Fourier expansion in the interval $0 < x < a$, we would not expect to easily find an expression for the third derivatives of w(x, y) along either of the two center lines.

We could calculate the deflection at the center of the plate from Equation (79). If we subtract this deflection from the above solution, we obtain the solution to the problem of a plate whose center is not deflected. This is all based on the assumption that the sums converge.

Problem #20:



In this case $b_m = c_m = d_m = f_m = g_m = 0$

$$0 = \frac{e_m a}{4\beta S^2}(S-\beta aC)+\beta(-1)^m \Sigma \frac{\alpha(-1)^{n+1}h_n}{(\alpha^2+\beta^2)^2} - \beta\, \Sigma \frac{\alpha(-1)^{n+1}i_n}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\Sigma \frac{\alpha(-1)^{n+1}q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4\alpha\bar{S}^2}(\bar{S}-\alpha b\bar{C}) - \frac{i_n b}{4\alpha\bar{S}^2}(\bar{S}\bar{C}-\alpha b) - \alpha\, \Sigma \frac{\beta e_m}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\Sigma \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

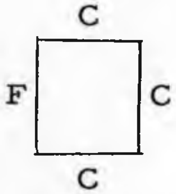
$$0 = \frac{i_n b}{4\alpha\bar{S}^2}(\bar{S}-\alpha b\bar{C}) - \frac{h_n b}{4\alpha\bar{S}^2}(\bar{S}\bar{C}-\alpha b) - \alpha\, \Sigma \frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\Sigma \frac{\beta(-1)^{m+1}q_{mn}}{(\alpha^2+\beta^2)^2} \tag{146}$$

Case (a) If $q(x, y) = q(x, b-y)$ then $e_m = 0$ if m is even and $i_n = h_n$

$$0 = \frac{e_m a}{4\beta S^2}(S-\beta ac) - 2\beta\, \Sigma \frac{\alpha(-1)^{n+1}h_n}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\Sigma \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4\alpha\bar{S}^2}(\bar{S}+\alpha b)(1-\bar{C}) - \alpha \sum_{odd} \frac{\beta e_m}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2} \tag{147}$$

Problem # 24:



$b_m = f_m = g_m = 0 \qquad C_m = e_m/\nu\beta^2 a^2$

$$0 = \frac{e_m a\beta(1-\nu)}{4\nu S^2}[(3+\nu)\, SC+(1-\nu)\beta a] + \frac{d_m a\beta}{4S^2}[(1+\nu)\, S+(1-\nu)\beta aC]$$

$$-\beta(-1)^m\, \Sigma \frac{\alpha h_n}{(\alpha^2+\beta^2)^2}(\alpha^2+(2-\nu)\beta^2) + \beta\, \Sigma \frac{\alpha i_n}{(\alpha^2+\beta^2)^2}(\alpha^2+(2-\nu)\beta^2)$$

$$-\frac{1}{a}\Sigma \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}(\alpha^2+\beta^2(2-\nu))$$

$$0 = \frac{i_m a}{4\nu\beta S^2}\left[(1+\nu)S + (1-\nu)\beta a C\right] - \frac{d_m a}{4\beta S^2}(SC-\beta a) + \beta(-1)^m \Sigma \frac{a(-1)^{n+1}h_n}{(a^2+\beta^2)^2}$$

$$-\beta\,\Sigma \frac{a(-1)^{n+1}i_n}{(a^2+\beta^2)^2} + \frac{1}{a}\,\Sigma \frac{a q_{mn}(-1)^{n+1}}{(a^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4a\bar{S}^2}(\bar{S}-ab\bar{C}) - \frac{i_n b}{4a\bar{S}^2}(\bar{S}\bar{C}-ab) + \frac{a}{\nu}\,\Sigma \frac{i_m\left[a^2+(2-\nu)\beta^2\right]}{\beta(a^2+\beta^2)^2}$$

$$+ a(-1)^n\,\Sigma \frac{\beta d_m}{(a^2+\beta^2)^2} + \frac{1}{a}\,\Sigma \frac{\beta q_{mn}}{(a^2+\beta^2)^2}$$

$$0 = \frac{i_n b(\bar{S}-ab\bar{C})}{4a\bar{S}^2} - \frac{h_n b}{4a\bar{S}^2}(\bar{S}\bar{C}-ab) + \frac{2}{\nu}\,\Sigma \frac{e_m(-1)^{m+1}}{\beta(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]$$

$$+ a(-1)^n\,\Sigma \frac{\beta d_m(-1)^{m+1}}{(a^2+\beta^2)^2} + \frac{1}{a}\,\Sigma \frac{\beta(-1)^{n+1} q_{mn}}{(a^2+\beta^2)^2} \qquad (148)$$

Case (a) $q(x,y) = q(x, b-y)$ $\quad d_m = e_m = C_m = 0,\ h_n = i_n$ if m is even

$$0 = \frac{e_m a\beta(1-\nu)}{4\nu S^2}\left[(3+\nu)SC + (1-\nu)\beta a\right] + \frac{d_m a\beta}{4S^2}\left[(1+\nu)S + (1-\nu)\beta a C\right]$$

$$+ 2\beta\,\Sigma \frac{a h_n}{(a^2+\beta^2)^2}(a^2+(2-\nu)\beta^2) - \frac{1}{a}\,\Sigma \frac{a q_{mn}}{(a^2+\beta^2)^2}\left[a^2+\beta^2(2-\nu)\right]$$

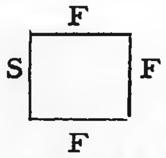
$$0 = \frac{e_m a}{4\nu\beta S^2}\left[(1+\nu)S + (1-\nu)\beta a C\right] - \frac{d_m a}{4\beta S^2}(SC-\beta a) - 2\beta\,\Sigma \frac{a(-1)^{n+1}h_n}{(a^2+\beta^2)^2}$$

$$+ \frac{1}{a}\,\Sigma \frac{a q_{mn}(-1)^{n+1}}{(a^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4aS^2}(\bar{S}+ab)(1-\bar{C}) + \frac{a}{\nu}\sum_{\text{odd}} \frac{e_m}{\beta(a^2+\beta^2)^2}\left[a^2+(2-\nu)\beta^2\right]$$

$$+ a(-1)^n \sum_{\text{odd}} \frac{\beta d_m}{(a^2+\beta^2)^2} + \frac{1}{a}\sum_{\text{odd}} \frac{\beta q_{mn}}{(a^2+\beta^2)^2} \qquad (149)$$

Problem # 28:



$$e_m = C_m = 0, \quad B_m = d_m/\nu\ \beta^2 a^2, \quad G_n = i_n/\nu\alpha^2 a^2, \quad F_n = h_n/\nu\alpha^2 a^2$$

$$0 = \frac{d_m \beta a(1-\nu)}{4\nu S^2}\left[(3+\nu)\, SC + (1-\nu)\,\beta a\right] + \frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\Sigma \frac{\alpha h_n(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{\beta^3}{\nu}(1-\nu)^2\,\Sigma\frac{\alpha i_n(-1)^n}{(\alpha^2+\beta^2)^2} - \frac{1}{a}\,\Sigma\frac{(-1)^n\alpha q_{mn}}{(\alpha^2+\beta^2)^2}\,(\alpha^2+\beta^2(2-\nu))$$

$$0 = \frac{(-1)^n\alpha^3}{\nu}(1-\nu)^2\,\Sigma\frac{\beta d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha b h_n}{4\nu\bar{S}^2}(1-\nu)\left[(3+\nu)\,\bar{S} + (1-\nu)\alpha b\bar{C}\right]$$

$$+\frac{i_n\alpha b(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\,\bar{S}\bar{C} + (1-\nu)\,\alpha b\right] - \frac{1}{a}\,\Sigma\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}\,(\beta^2+(2-\nu)\,\alpha^2)$$

$$0 = \frac{(-1)^n\alpha^3}{\nu}(1-\nu)^2\,\Sigma\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} - \frac{\alpha b i_n}{4\nu\bar{S}^2}(1-\nu)\left[(3+\nu)\,\bar{S} + (1-\nu)\alpha b\bar{C}\right]$$

$$+\frac{h_n\alpha b(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\,\bar{S}\bar{C} + (1-\nu)\,\alpha b\right] - \frac{1}{a}\,\Sigma\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}\,(\beta^2+\alpha^2(2-\nu))(-1)^{m+1} \tag{150}$$

Case (a) If $q(x, y) = q(x, b-y)$ then $h_n = i_n$, $w(a, b) = w(a, 0)$, $d_m = 0$ if m is even

$$0 = -\frac{d_m\beta a(1-\nu)}{4\nu S^2}\left[(3+\nu)\, SC + (1-\nu)\,\beta a\right] - \frac{2\beta^3(1-\nu)^2}{\nu}\,\Sigma\frac{\alpha h_n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{1}{a}\,\Sigma\frac{(-1)^n\alpha q_{mn}}{(\alpha^2+\beta^2)^2}\left[\alpha^2\,(2-\nu)\,\beta^2\right]$$

$$0 = \frac{(-1)^n\alpha^3(1-\nu)^2}{\nu}\sum_{odd}\frac{\beta d_m}{(\alpha^2+\beta^2)^2} - \frac{\alpha b h_n}{4\nu\bar{S}^2}(1-\nu)\,(1-\bar{C})\left[(3+\nu)\,\bar{S} - (1-\nu)\,\alpha b\right]$$

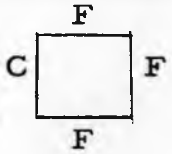
$$-\frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}\left[\beta^2+\alpha^2(2-\nu)\right] \tag{151}$$

Since $w(0, b) = w(0, 0) = 0$, we see that there are only two arbitrary corner deflections. For the moments to be in equilibrium, we must have

$$a\,[P(a, b) + P(0, 0)] = \int_0^b \int_0^a x\, q(x, y)\, dx\, dy \qquad (152)$$

Hence, one rather than two corner loads may be specified arbitrarily. We may, therefore, specify two corner deflections or one corner deflection and one corner load. In the symmetric case, we may specify either the deflection at a corner or the load at a corner.

Problem # 32: The Cantilever Plate



$$c_m = C_m = 0\,,\quad B_m = d_m/\nu\beta^2 a^2\,,\quad G_n = i_n/\nu\alpha^2 a^2\,,\quad F_n = h_n/\nu\alpha^2 a^2$$

$$0 = \frac{d_m a}{4\nu\beta S^2}[(1+\nu)S + (1-\nu)a\beta C] - \frac{e_m a}{4\beta S^2}(SC - \beta a) - \frac{1}{2\beta k}[(-1)^m w(a, b) - w(a, 0)]$$

$$- \frac{\beta(-1)^m}{\nu}\,\Sigma\,\frac{h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2 + (2-\nu)\alpha^2] + \frac{\beta}{\nu}\,\Sigma\,\frac{i_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2 + (2-\nu)\alpha^2]$$

$$+ \frac{1}{a}\,\Sigma\,\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{d_m\beta a(1-\nu)}{4\nu S^2}[(3+\nu)SC + (1-\nu)\beta a] - \frac{e_m a\beta}{4S^2}[(1+\nu)S + (1-\nu)\beta a C]$$

$$+ \frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\,\Sigma\,\frac{\alpha h_n(-1)^n}{(\alpha^2+\beta^2)^2} - \frac{\beta^3(1-\nu)^2}{\nu}\,\Sigma\,\frac{\alpha i_n(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$- \frac{1}{a}\,\Sigma\,\frac{(-1)^n \alpha q_{mn}}{(\alpha^2+\beta^2)^2}[\alpha^2 + \beta^2(2-\nu)]$$

$$0=\frac{(-1)^n a^3(1-\nu)^2}{\nu}\ \Sigma \frac{\beta d_m}{(a^2+\beta^2)^2} + a\Sigma \frac{\beta e_m}{(a^2+\beta^2)^2}(\beta^2+(2-\nu)a^2) - \frac{abh_n(1-\nu)}{4\nu S^2}$$

$$\cdot [(3+\nu)\bar{S}+(1-\nu)ab\bar{C}] + \frac{i_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab] - \frac{1}{a}\Sigma \frac{\beta q_{mn}}{(a^2+\beta^2)^2}(\beta^2+a^2(2-\nu))$$

$$0=\frac{(-1)^n a^3(1-\nu)^2}{\nu}\Sigma \frac{\beta d_m(-1)^{m+1}}{(a^2+\beta^2)^2} + a\Sigma \frac{\beta e_m(-1)^{m+1}}{(a^2+\beta^2)^2}(\beta^2+(2-\nu)a^2)$$

$$-\frac{abi_n(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}+(1-\nu)ab\bar{C}] + \frac{h_n ab(1-\nu)}{4\nu\bar{S}^2}[3+\nu)\bar{S}\bar{C}+(1-\nu)ab]$$

$$-\frac{1}{a}\Sigma \frac{\beta q_{mn}}{(a^2+\beta^2)^2}(\beta^2+(2-\nu)a^2)(-1)^{m+1} \qquad (153)$$

Case (a) Symmetric about $y = b/2$

$q(x, y) = q(x, b-y)$, $h_n = i_n$, $d_m = B_m = 0$, $e_m = 0$ if m is even

$$0=\frac{d_m}{4\nu\beta aS^2}[(1+\nu)S+(1-\nu)a\beta C] - \frac{e_m}{4\beta aS^2}(SC-\beta a) + \frac{1}{2\beta ka^2}[w(a,b)+w(a,0)]$$

$$+\frac{2\beta}{a^2\nu}\Sigma \frac{h_n}{a(a^2+\beta^2)^2}[\beta^2+(2-\nu)a^2] + \frac{1}{a^3}\Sigma \frac{aq_{mn}}{(a^2+\beta^2)^2} \qquad m = 1, 3, 5, \ldots$$

$$0=d_m\frac{\beta a(1-\nu)}{4\nu S^2}[(3+\nu)SC+(1-\nu)\beta a] - \frac{e_m a\beta}{4S^2}[(1+\nu)S+(1-\nu)\beta aC]$$

$$-\frac{2\beta^3(1-\nu)^2}{\nu}\Sigma \frac{ah_n(-1)^n}{(a^2+\beta^2)^2} - \frac{1}{a}\Sigma \frac{(-1)^n aq_{mn}}{(a^2+\beta^2)^2}[a^2+\beta^2(2-\nu)],\ m = 1, 3, 5, \ldots$$

$$0=\frac{(-1)^n a^3(1-\nu)^2}{\nu}\sum_{odd}\frac{\beta d_m}{(a^2+\beta^2)^2} + a\sum_{odd}\frac{\beta e_m}{(a^2+\beta^2)^2}[\beta^2+(2-\nu)a^2]$$

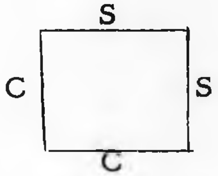
$$-\frac{abh_n}{4\nu\bar{S}^2}(1-\nu)(1-\bar{C})[(3+\nu)\bar{S}-(1-\nu)ab] - \frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(a^2+\beta^2)^2}[\beta^2+a^2(2-\nu)] \qquad (154)$$

As in Problem # 28, the corner deflections at (a, b) and (a, 0) are arbitrary. If we do not specify these deflections, we can arbitrarily specify the corner loads at (a, b) and (a, 0). This problem is different from all previous ones in that two corner loads rather than one can be specified arbitrarily. This is due to the fact that the coefficients depend upon the corner deflections, as can be seen from the equations.

If there are no corner loads and the load is symmetric about $y = b/2$, we must use the equation $w_{xy}(a, b) = 0$ in addition to Equations (154).

We consider w(a, b) as unknown. Thus, if we neglect coefficients for $m, n > 11$, we will have 19 equations with 19 unknowns. The unknowns are w (a, b), d_m, e_m, h_n. (m = 1, 3, 5, 7, 9, 11 and n 1, 2, ... , 11)

Problem # 36:



$b = c = d = f = g = h = 0$

$$0 = \frac{-e_m a(SC-\beta a)}{4\beta S^2} - \beta \sum_n \frac{a i_n}{(a^2+\beta^2)^2} + \frac{1}{a} \sum_n \frac{a\, q_{mn}}{(a^2+\beta^2)^2}$$

$$0 = \frac{-i_n b}{4a\bar{S}^2} (\bar{S}\bar{C}-ab) - a \sum_m \frac{\beta e_m}{(a^2+\beta^2)^2} + \frac{1}{a} \sum_m \frac{\beta q_{mn}}{(a^2+\beta^2)^2} \qquad (155)$$

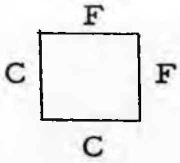
Case (a) $q(x, y) = q(y, x)$ and a square plate $e_m = i_m$

$$0 = \frac{-e_m a(SC-\beta a)}{4\beta S^2} - \beta \sum_n \frac{a e_n}{(a^2+\beta^2)^2} + \frac{1}{a} \sum \frac{a q_{mn}}{(a^2+\beta^2)^2} \qquad (156)$$

Case (b) For constant load

$$e_m = \frac{1 - (-1)^m}{\pi^3 m^3} \frac{(S - m\pi)(C-1)}{SC - m\pi} - \frac{4S^2}{(SC-m\pi)} \sum_{n=1}^{\infty} \frac{n e_n}{(n^2+m^2)^2} \qquad (157)$$

Problem # 40:



$C_m = g_m = 0, \quad B_m = d_m/\nu\beta^2 a^2, \quad F_n = h_n/\nu\alpha^2 a^2$

$$0 = \frac{d_m a}{4\nu\beta S^2}[(1+\nu)S+(1-\nu)\beta a C] - \frac{e_m a}{4\beta S^2}(SC-\beta a) - \frac{(-1)^m w(ab)}{2\beta k}$$

$$-\frac{\beta(-1)^m}{\nu}\Sigma\frac{h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{\beta}{\nu}\Sigma\frac{i_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{1}{a}\Sigma\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{h_n b}{4\nu\alpha\bar{S}^2}[(1+\nu)\bar{S}+(1-\nu)\beta a\bar{C}] - \frac{i_n b}{4\alpha\bar{S}^2}(\bar{S}\bar{C}-\alpha b) - \frac{(-1)^n w(ab)}{2\alpha k}$$

$$-\frac{\alpha(-1)^n}{\nu}\Sigma\frac{d_m[\alpha^2+(2-\nu)\beta^2]}{\beta(\alpha^2+\beta^2)^2} + \frac{\alpha}{\nu}\Sigma\frac{e_m}{\beta(\alpha^2+\beta^2)^2}[\alpha^2+(2-\nu)\beta^2] + \frac{1}{a}\Sigma\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{-d_m\beta a(1-\nu)}{4\nu S^2}[(3+\nu)SC+(1-\nu)\beta a] - \frac{e_m\beta a}{4S^2}[(1+\nu)S+(1-\nu)\beta aC]$$

$$+\frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\Sigma\frac{\alpha h_n(-1)^n}{(\alpha^2+\beta^2)^2} + \beta\Sigma\frac{\alpha i_n(-1)^n}{(\alpha^2+\beta^2)^2}[\alpha^2+(2-\nu)\beta^2]$$

$$-\frac{1}{a}\Sigma\frac{\alpha q_{mn}(\alpha^2+\beta^2(2-\nu))(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{(-1)^n\alpha^3}{\nu}(1-\nu)^2\Sigma\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} + \alpha\Sigma\frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{\alpha b i_n}{4\bar{S}^2}[(1+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}] + \frac{\alpha b h_n(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)\alpha b]$$

$$-\frac{1}{a}\Sigma\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (158)$$

Case (a) $q(x, y) = q(y, x)$, $d_m = h_m$, $e_m = i_m$, square plate

$$0 = \frac{d_m}{4\nu\beta a S^2}[(1+\nu)S + (1-\nu)\beta a C] - \frac{e_m}{4\beta a}\frac{(SC-\beta a)}{S^2} - \frac{(-1)^m w(a,b)}{2\beta a^2 k}$$

$$- \frac{\beta(-1)^m}{\nu a^2}\Sigma\frac{d_n[\beta^2 + (2-\nu)a^2]}{a(a^2+\beta^2)^2} + \frac{\beta}{\nu a^2}\Sigma\frac{e_n[\beta^2+(2-\nu)a^2]}{a(a^2+\beta^2)^2}$$

$$+ \frac{1}{a^3}\Sigma\frac{a q_{mn}}{(a^2+\beta^2)^2}$$

$$0 = -\frac{d_m\beta a(1-\nu)}{4\nu S^2}[(3+\nu)SC + (1-\nu)\beta a] - \frac{e_m\beta a}{4S^2}[(1+\nu)S + (1-\nu)\beta a C]$$

$$+ \frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\Sigma\frac{a d_n(-1)^n}{(a^2+\beta^2)^2} + \beta\Sigma\frac{a e_n(-1)^n}{(a^2+\beta^2)^2}[a^2+(2-\nu)\beta^2]$$

$$- \frac{1}{a}\Sigma\frac{a q_{mn}(-1)^n}{(a^2+\beta^2)^2}[a^2 + \beta^2(2-\nu)] \qquad (159)$$

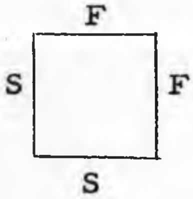
The condition that the corner (a, b) is free (in the symmetric case), implies

$$0 = \frac{4}{ab}\sum_{m,n}\frac{a\beta q_{mn}(-1)^{m+n}}{(a^2+\beta^2)^2} + \frac{2}{\nu}\sum_m(-1)^m\left\{\frac{d_m}{S^2}[SC(1+\nu)+\beta a(1-\nu)]\right.$$

$$\left. - \frac{e_m\nu}{S^2}(S - \beta a C)\right\} + \frac{w(a,b)}{ka} \qquad (160)$$

We can solve this equation for w(a, b) and substitute it into Equation (159) and thus obtain a set of equations not involving w(a, b).

Problem # 44:



$$C_m = e_m = g_m = i_m = 0, \quad B_m = \frac{d_m}{\nu\beta^2 a^2}, \quad F_n = \frac{h_n}{\nu\alpha^2 a^2}$$

$$0 = \frac{-d_m\beta a(1-\nu)}{4\nu S^2}[(3+\nu)SC+(1-\nu)\beta a] + \frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\,\Sigma\,\frac{\alpha h_n(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{1}{a}\,\Sigma\,\frac{(-1)^n\alpha q_{mn}}{(\alpha^2+\beta^2)^2}\,(\alpha^2+\beta^2(2-\nu))$$

$$0 = \frac{\alpha^3(-1)^n}{\nu}(1-\nu)^2\,\Sigma\,\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} + \frac{h_n\alpha b(1-\nu)}{4\nu\bar{S}^2}[3+\nu)\,\bar{S}\bar{C} + (1-\nu)\alpha b]$$

$$-\frac{1}{a}\,\Sigma\,\frac{\beta q_{mn}[\beta^2 + \alpha^2(2-\nu)]}{(\alpha^2+\beta^2)^2}(-1)^{m+1} \qquad (161)$$

Case (a) $q(x, y) = q(y, x)$, $d_m = h_m$

$$0 = \frac{-d_m\beta a(1-\nu)}{4\nu S^2}[(3+\nu)\,SC + (1-\nu)\,\beta a] + \frac{\beta^3(-1)^m(1-\nu)^2}{\nu}\,\Sigma\,\frac{\alpha d_n(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$-\frac{1}{a}\,\Sigma\,\frac{(-1)^n\alpha q_{mn}}{(\alpha^2+\beta^2)^2}[\alpha^2 + \beta^2\,(2-\nu)] \qquad (162)$$

Case (b) Constant load

$$d_m = \frac{2\nu(C-1)}{(1-\nu)\,m^3\pi^3}\,\frac{(3-\nu)S-(1-\nu)\,m\pi}{(3+\nu)SC+(1-\nu)m\pi} + \frac{4S^2(1-\nu)\,(-1)^m}{m\pi[(3+\nu)SC+(1-\nu)m\pi]}\,\Sigma\,\frac{nd_n(-1)^n}{(n^2+m^2)^2} \qquad (163)$$

For the symmetric case

$$w_2(x, y) = \frac{2k}{b\nu}\,\Sigma\, d_m \sin\beta y \left\{ \frac{2\sinh\beta x}{\beta^2\sinh\beta a} + (1-\nu)\left[-\frac{a}{\beta}\frac{\sinh\beta(a-x)}{\sinh^2\beta a}\right.\right.$$

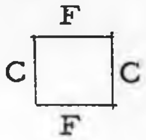
$$\left.\left. + \frac{(a-x)\cosh\beta x}{\beta\sinh\beta a}\right]\right. \qquad (164)$$

$$w_3(x, y) = \frac{xy}{ab}\,w(a, b)$$

For a free corner at (a, b), the deflection can be found from the equation $w_{xy}(a, b) = 0$. This gives

$$w(a,b) = -\frac{4k}{a}\sum_{m,n}\frac{\alpha\beta q_{mn}(-1)^{m+n}}{(\alpha^2+\beta^2)^2} - \frac{2ka}{\nu}\sum_m \frac{d_m(-1)^m}{S^2}\left[SC(1+\nu)+\beta a(1-\nu)\right] \tag{165}$$

Problem # 48:



$$b_m = C_m = 0, \quad G_n = i_n/\nu\alpha^2a^2, \quad F_n = h_n/\nu\alpha^2a^2$$

$$0 = \frac{d_m a}{4\beta S^2}(S-\beta aC) - \frac{e_m a}{4\beta S^2}(SC-\beta a) - \frac{\beta(-1)^m}{\nu}\sum\frac{h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{\beta}{\nu}\sum\frac{i_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{1}{a}\sum\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{e_m a}{4\beta S^2}(S-\beta aC) - \frac{d_m a}{4\beta S^2}(SC-\beta a) - \frac{\beta(-1)^m}{\nu}\sum\frac{(-1)^{n+1}h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{\beta}{\nu}\sum\frac{i_n(-1)^{n+1}}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{1}{a}\frac{\alpha q_{mn}(-1)^{n+1}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n\sum\frac{\beta d_m}{(\alpha^2+\beta^2)^2}(\beta^2+(2-\nu)\alpha^2) + \alpha\sum\frac{\beta e_m}{(\alpha^2+\beta^2)^2}(\beta^2+(2-\nu)\alpha^2) - \frac{ab\,h_n}{4\nu\bar{S}^2}(1-\nu)$$

$$\cdot\left[(3+\nu)\bar{S}+(1-\nu)ab\bar{C}\right] + \frac{i_n ab(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab\right] - \frac{1}{a}\sum\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}(\beta^2+\alpha^2(2-\nu))$$

$$0 = -\alpha(-1)^n\sum\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}(\beta^2+(2-\nu)\alpha^2) + \alpha\sum\frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}(\beta^2+(2-\nu)\alpha^2)$$

$$-\frac{ab i_n(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\bar{S}+(1-\nu)ab\bar{C}\right) + \frac{h_n ab(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab\right]$$

$$-\frac{1}{a}\sum\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}\left[\beta^2+\alpha^2(2-\nu)\right] \tag{166}$$

Case (a) $q(x,y) = q(a-x,y)$, $d_m = e_m$, $h_n = i_n = q_{mn} = 0$, if n = even

$$0 = \frac{d_m a}{4\beta S^2}(S+\beta a)(1-C) - \frac{\beta(-1)^m}{\nu}\sum_{odd}\frac{h_n}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{\beta}{\nu}\sum_{odd}\frac{i_n}{a(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{1}{a}\sum_{odd}\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = 2a\sum\frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{\alpha b h_n}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}]$$

$$+\frac{i_n\alpha b(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)\alpha b] - \frac{1}{a}\sum\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)]$$

$$0 = 2a\sum\frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{\beta b i_n(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}]$$

$$+\frac{h_n\alpha b(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)\alpha b] - \frac{1}{a}\sum\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \quad (167)$$

Case (b) $q(x,y) = q(x,b-y)$, $h_n = i_n$, $d_m = e_m = q_{mn} = 0$, if m = even

$$0 = \frac{d_m a}{4\beta S^2}(S-\beta aC) - \frac{e_m a}{4\beta S^2}(SC-\beta a) + \frac{2\beta}{\nu}\sum\frac{h_n}{a(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{1}{a}\sum\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\frac{d_m a}{4\beta S^2}(SC-\beta a) + \frac{e_m a}{4\beta S^2}(S-\beta aC) + \frac{2\beta}{\nu}\sum\frac{h_n(-1)^{n+1}}{a(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{1}{a}\frac{\alpha q_{mn}(-1)^{n+1}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n\sum_{\substack{odd\\m}}\frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \alpha\sum_{odd}\frac{\beta e_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$-\frac{h_n\alpha b(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}-(1-\nu)\alpha b][1-\bar{C}] - \frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \quad (168)$$

Case (c) $q(x, y) = q(x, b-y)$, $q(x, y) = q(a-x, y)$, $e_m = d_m$, $h_n = i_n$,

$h_n = i_n = d_m = e_m = q_{mn} = 0$, if m or n = 0

$$0 = \frac{d_m a}{4\beta S^2}(S+\beta a)(1-C) + \frac{2\beta}{\nu}\sum_{odd}\frac{h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{1}{a}\sum_{odd}\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = 2\alpha\sum_{odd}\frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{\alpha b h_n(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}-(1-\nu)\alpha b][1-\bar{C}]$$

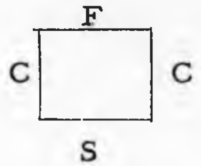
$$+\frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (169)$$

Case (d) For constant load (square plate)

$$d_m = \frac{2}{m^3\pi^3}\,\frac{S-m\pi}{S+m\pi} + \frac{8m^2}{\nu\pi}\,\frac{S^2}{(S+m\pi)(C-1)}\sum_{\substack{odd\\ n}}\frac{h_n[m^2+(2-\nu)n^2]}{n[n^2+m^2]^2}$$

$$h_n = \frac{2\nu}{1-\nu}\,\frac{1}{n^3\pi^3}\,\frac{(3-\nu)\bar{S}-(1-\nu)n\pi}{(3+\nu)\bar{S}-(1-\nu)n\pi} + \frac{8\nu\bar{S}^2\sum m d_m(m^2+n^2)^{-2}[m+(2-\nu)n^2]}{(1-\nu)(1-\bar{C})[(3+\nu)\bar{S}-(1-\nu)m\pi]} \qquad (170)$$

Problem # 50



$b_m = C_m = g_m = i_m = 0$, $F_n = h_n/\nu\alpha^2 a^2$

$$0 = \frac{d_m a}{4\beta S^2}(S-\beta aC) - \frac{e_m a(SC-\beta a)}{4\beta S^2} - \frac{\beta(-1)^m}{\nu}\sum\frac{h_n}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{1}{a}\sum\frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{e_m a}{4\beta S^2}(S-\beta aC) - \frac{d_m a}{4\beta S^2}(SC-\beta a) - \frac{\beta(-1)^m}{\nu}\sum\frac{(-1)^{n+1}h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+\frac{1}{a}\sum\frac{\alpha q_{mn}(-1)^{n+1}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n \Sigma \frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \alpha \Sigma \frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+ \frac{h_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab] - \frac{1}{a}\Sigma \frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (171)$$

Case (a) $d_m = e_m$, $h_n = 0$ if n is even

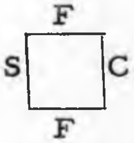
$$0 = \frac{d_m a}{4\beta S^2}(S+\beta a)(1-C) - \frac{\beta(-1)^m}{\nu} \sum_{odd} \frac{h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2]$$

$$+ \frac{1}{a}\sum_{odd} \frac{\alpha q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = 2\alpha \Sigma \frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{h_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab]$$

$$-\frac{1}{a} \Sigma \frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (172)$$

Problem # 54



$b = C = e = 0$, $G_n = i_n/\nu\alpha^2 a^2$, $F_n = h_n/\nu\alpha^2 a^2$

$$0 = -\frac{d_m a(SC-\beta a)}{4\beta S^2} - \frac{\beta(-1)^m}{\nu}\Sigma \frac{h_n(-1)^{n+1}}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{\beta}{\nu}\Sigma \frac{(-1)^{n+1} i_n}{\alpha(\alpha^2+\beta^2)^2}$$

$$\cdot [\beta^2 + (2-\nu)\alpha^2] + \frac{1}{a}\Sigma \frac{\alpha(-1)^{n+1} q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n \Sigma \frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{abh_n}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S}+(1-\nu)ab\bar{C}]$$

$$+ \frac{i_n ab(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)ab] - \frac{1}{a}\Sigma \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)]$$

$$0 = -\alpha(-1)^n \sum \frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{\alpha b i_n}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}]$$

$$+\frac{h_n \alpha b(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\,\bar{S}\bar{C}+(1-\nu)\alpha b] - \frac{1}{a}\sum\frac{\beta q_{mn}(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (173)$$

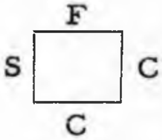
Case (a) $q(x,y) = q(x, b-y)$, $h_n = i_n$, $d_m = 0$ if m is even

$$0 = -\frac{d_m \alpha(SC-\beta\alpha)}{4\beta S^2} + \frac{2\beta}{\nu}\sum\frac{h_n(-1)^{n+1}}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{1}{a}\sum\frac{\alpha(-1)^{n+1}q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n\sum_{odd}\frac{\beta d_m}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \frac{\alpha b h_n}{4\nu\bar{S}^2}(1-\nu)[(3+\nu)\bar{S}-(1-\nu)\alpha b][1-\bar{C}]$$

$$-\frac{1}{a}\sum_{odd}\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)] \qquad (174)$$

Problem #58



$b_m = C_m = d_m = g_m = 0$, $F_n = h_n/\nu\alpha^2 a^2$

$$0 = \frac{e_m \alpha(S-\beta\alpha C)}{4\beta S^2} - \frac{\beta(-1)^m}{\nu}\sum\frac{(-1)^{n+1}h_n}{\alpha(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] - \beta\sum\frac{\alpha(-1)^{n+1}i_n}{(\alpha^2+\beta^2)^2}$$

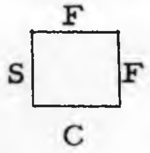
$$+\frac{1}{a}\sum\frac{\alpha(-1)^{n+1}q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \frac{h_n\alpha b}{4\nu\bar{S}^2}[(1+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}] - \frac{i_n b}{4\alpha\bar{S}^2}(\bar{S}\bar{C}-\alpha b) - \alpha\sum\frac{\beta e_m}{(\alpha^2+\beta^2)^2} + \frac{1}{a}\sum\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = \alpha\sum\frac{\beta e_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2}[\beta^2+(2-\nu)\alpha^2] + \frac{\alpha b i_n}{4\bar{S}^2}[(1+\nu)\bar{S}+(1-\nu)\alpha b\bar{C}]$$

$$+\frac{h_n\alpha b(1-\nu)}{4\nu\bar{S}^2}[(3+\nu)\bar{S}\bar{C}+(1-\nu)\alpha b)] - \frac{1}{a}\sum\frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}[\beta^2+\alpha^2(2-\nu)](-1)^{m+1} \qquad (175)$$

Problem # 66:



$$C = e = g = 0, \quad B_m = d_m/\nu\beta^2 a^2, \quad F_n = h_n/\nu\alpha^2 a^2$$

$$0 = \frac{h_n b}{4\nu a \bar{S}^2}\left[(1+\nu)\bar{S} + (1-\nu)ab\bar{C}\right] - \frac{i_n b}{4aS^2}(\bar{S}\bar{C} - ab) - \frac{(-1)^n w(a,b)}{2ak}$$

$$- \frac{\alpha(-1)^n}{\nu} \sum \frac{d_m[\alpha^2 + (2-\nu)\beta^2]}{\beta(\alpha^2+\beta^2)^2} + \frac{1}{a}\sum \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\frac{d_m \beta a(1-\nu)}{4\nu S^2}\left[(3+\nu)SC + (1-\nu)\beta a\right] + \frac{\beta^3(-1)^m(1-\nu)^2}{\nu} \sum \frac{\alpha h_n(-1)^n}{(\alpha^2+\beta^2)^2}$$

$$+ \beta \sum \frac{\alpha i_n(-1)^n}{(\alpha^2+\beta^2)^2}\left[\alpha^2 + (2-\nu)\beta^2\right] - \frac{1}{a}\sum \frac{(-1)^n \alpha q_{mn}}{(\alpha^2+\beta^2)^2}\left[\alpha^2 + \beta^2(2-\nu)\right]$$

$$0 = \frac{(-1)^n \alpha^3(1-\nu)^2}{\nu} \sum \frac{\beta d_m(-1)^{m+1}}{(\alpha^2+\beta^2)^2} + \frac{abi_n}{4\bar{S}^2}\left[(1+\nu)\bar{S} + (1-\nu)ab\bar{C}\right] + \frac{h_n ab(1-\nu)}{4\nu\bar{S}^2}$$

$$\cdot \left[(3+\nu)\bar{S}\bar{C} + (1-\nu)ab\right] - \frac{1}{a}\sum \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}\left[\beta^2 + \alpha^2(2-\nu)\right](-1)^{m+1} \qquad (176)$$

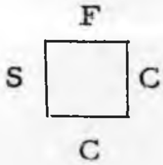
$w(a, b)$ can be eliminated from these equations by using $w_{xy}(a, b) = 0$.

This gives

$$w(a,b) = -\frac{4k}{a}\sum_{m,n} \frac{\alpha\beta q_{mn}(-1)^{m+n}}{(\alpha^2+\beta^2)^2} - \frac{ka}{\nu}\sum_m \frac{d_m(-1)^m}{S^2}\left[SC(1+\nu) + \beta a(1-\nu)\right]$$

$$- \frac{kb}{\nu}\sum_n \left\{ \frac{h_n(-1)^n}{S^2}\left[\bar{S}\bar{C}(1+\nu) + ab(1-\nu)\right] - \frac{i_n \nu}{\bar{S}^2}(\bar{S} - ab\bar{C})(-1)^n \right\} \qquad (177)$$

Problem # 74:



$b = c = e = g = 0, \; F_n = h_n / \nu \alpha^2 a^2$

$$0 = \frac{h_n b}{4\alpha\nu S^2}\left[(1+\nu)\bar{S} + (1-\nu)\alpha b\bar{C}\right] - \frac{i_n b}{4\alpha\bar{S}^2}\left(\bar{S}\bar{C} - \alpha b\right) + \alpha(-1)^n \Sigma \frac{\beta d_m}{(\alpha^2+\beta^2)^2}$$

$$+ \frac{1}{a} \Sigma \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\frac{d_m a}{4\beta S^2}(SC - \beta a) - \frac{\beta(-1)^m}{\nu} \Sigma \frac{(-1)^{n+1} h_n}{\alpha(\alpha^2+\beta^2)^2}\left[\beta^2 + (2-\nu)\alpha^2\right] - \beta\Sigma \frac{\alpha(-1)^{n+1} i_n}{(\alpha^2+\beta^2)}$$

$$+ \frac{1}{a} \Sigma \frac{\alpha(-1)^{n+1} q_{mn}}{(\alpha^2+\beta^2)^2}$$

$$0 = -\alpha(-1)^n \Sigma \frac{\beta d_m (-1)^{m+1}}{(\alpha^2+\beta^2)^2}\left[\beta^2 + (2-\nu)\alpha^2\right] + \frac{\alpha b i_n}{4\bar{S}^2}\left[(1+\nu)\bar{S} + (1-\nu)\alpha b\bar{C}\right]$$

$$+ \frac{h_n \alpha b(1-\nu)}{4\nu\bar{S}^2}\left[(3+\nu)\bar{S}\bar{C} + (1-\nu)\alpha b\right] - \frac{1}{a} \Sigma \frac{\beta q_{mn}}{(\alpha^2+\beta^2)^2}\left[\beta^2 + \alpha^2(2-\nu)\right](-1)^{m+1} \qquad (178)$$

E. Solution of the Equations

To obtain a solution of problems 18 to 81, we must solve an infinite number of equations with an infinite number of unknowns. To obtain an approximate solution, coefficients when $n > N$, $m > M$ are neglected. If M and N are large, the solution is expected to be very accurate, but there will be considerable labor involved in calculating the coefficients. Generally, if the coefficients approach zero rapidly as m and n approach infinity, then the values of M and N need not be chosen very large to obtain a reasonably accurate solution.

The question of convergence of an infinite set of equations has been treated by March,[25] Koch,[20, 21] and Riesz.[28]

We now outline a proof of convergence of Equation (136). It is of the form:

$$d_m = A_m + B_m \sum_{n\ 1} \frac{n d_n}{(n^2+m^2)^2} \qquad m = 1, 2, 3, 4, \ldots \tag{179}$$

If we replace d_m given by this formula back into the series, we obtain formally

$$\begin{aligned} d_m = A_m &+ B_m \sum_{n_1=1}^{\infty} \frac{n_1 A_{n_1}}{(n_1^2+m^2)^2} + B_m \sum_{n_1=1}^{\infty} \frac{n_1 B_{n_1}}{(n_1^2+m^2)^2} \cdot \\ &\sum_{n_2=1}^{\infty} \frac{n_2 A_{n_2}}{(n_2^2+n_1^2)^2} + B_m \sum_{n_1=1}^{\infty} \frac{n_1 B_{n_1}}{(n_1^2+m^2)^2} \sum_{n_2=1}^{\infty} \frac{n_2 B_{n_2}}{(n_2^2+n_1^2)^2} \\ &\sum_{n_3=1}^{\infty} \frac{n_3 A_{n_3}}{(n_3^2+n_2^2)^2} + \ldots + B_m \sum_{n_1=1}^{\infty} \frac{n_1 B_{n_1}}{(n_1^2+m^2)^2} \sum_{n_2=1}^{\infty} \ldots \\ &\sum_{n_{k-1}}^{\infty} \frac{n_{k-1} B_{n_{k-1}}}{(n_{k-1}^2+n_{k-2}^2)^2} \sum_{n_k=1}^{\infty} \frac{n_k A_{n_k}}{(n_k^2+n_{k-1}^2)^2} + \ldots \end{aligned} \tag{180}$$

Koch[20] points out that if Equation (180) converges absolutely, then Equation (179) will have a unique soluution. In Equation (136)

$$A_m < \frac{2\nu}{(1-\nu)} \cdot \frac{1}{\pi^3} \cdot \frac{1}{m^3} = \frac{k_1}{m^3}, \quad \sum_{n=1}^{\infty} \frac{nA_n}{(n^2+m^2)^2} < k_1 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2+m^2)^2}$$

$$< \frac{k_1}{m^4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2 k_1}{6m^4} = \frac{k_2}{m^4} \quad . \tag{181}$$

It can be shown that the sequence $\dfrac{s^2}{(c-1)\left(s-\frac{(1-\nu)}{3+\nu}m\pi\right)}$ (182)

is monotonicly decreasing and hence every term is less than the first. This is less than 1. 4. Hence

$$|B_m| < 1.4\,\frac{8m^2}{3\pi} = k_3 m^2. \tag{183}$$

$$\sum_{n_{k-1}=1}^{\infty} \frac{n_{k-1} B_{n_{k-1}}}{(n_{k-1}^2 + n_{k-2}^2)^2} \sum_{n_k=1}^{\infty} \frac{n_k \quad A_{n_k}}{(n_k^2 + n_{k-1}^2)^2} \tag{184}$$

$$k_2 \sum_{n_k=1}^{\infty} \frac{B_{n_{k-1}}}{(n_{k-1}^2 + n_{k-2}^2)^2\, n_{k-1}^3} < k_2 k_3 \sum_{n_k=1}^{\infty} \frac{1}{n_{k-1}(n_{k-1}^2 + n_{k-2}^2)^2} \tag{185}$$

$$k_2 k_3 \sum_{n_k=1}^{\infty} \frac{1}{n_{k-1}^{.05}(n_{k-1}^2 + n_{k-2}^2)^2} . \tag{186}$$

A sum of the form $\sum_{n\ 1} \frac{1}{n^a} \frac{1}{(n^2+m^2)^2}$ has its terms monotonicly tending to zero, and therefore is less than $\int_0^{\infty} \frac{dx}{x^a(x^2+m^2)^2}$ (187)

where $-3 < a < 1$. By using Dwight Tables[15] (855. 7, 855.1, 850. 2, 850. 3) this becomes $\dfrac{(1+a)}{4\cos\frac{\pi a}{2}} \dfrac{1}{m^{3+a}}$ so that Equation (184) becomes

$$\sum_{n_{k-1}} \sum_{n_k} < k_2 k_3 \frac{1.05\,\pi}{4 \cos 4.5°} \cdot \frac{1}{(n_{k-2}^{3.05})} = \frac{k_2 k_3 k_4}{(n_{k-2}^{3.05})} , \tag{188}$$

where $\sum_{n_{k-1}} \sum_{n_k}$ represents the sum in Equation (184).

$$\sum_{n_{k-2}} \sum_{n_{k-1}} \sum_{n_k} < \frac{k_2 k_3^2 k_4^2}{n_{k-3}^{3.05}}, \tag{189}$$

$$\sum_{n_1} \sum_{n_2} \cdots \sum_{n_k} < \frac{k_2 (k_3 k_4)^{k-1}}{m^{3.05}} , \tag{190}$$

where $\sum_{n_i}$ means $\sum_{n_i=1}^{\infty} \frac{n_i \; B_{n_i}}{(n_i^2 + n_{i-1}^2)^2}$ $i = 1, 2, \ldots k-1$.

Equation (180) becomes

$$|d_m| < \frac{k_1}{m^3} + \frac{k_2 k_3}{m^2} + \frac{k_2 k_3^2 k_4}{m^{1.05}} + \frac{k_2 k_3^3 k_4^2}{m^{1.05}} + \ldots \frac{k_2 k_3 (k_3 k_4)^k}{m^{1.05}} \ldots$$

$$\frac{k_1}{m^3} + \frac{k_2 k_3}{m^{1.05}} (1 + k_3 k_4 + (k_3 k_4)^2 + \ldots + (k_3 k_4)^k + \ldots) . \tag{191}$$

Since $k_3 k_4 = 1.4 \; \frac{8}{3\pi} \; \frac{1.05\,\pi}{4 \cos 4.5^0} = .98 < 1,$ (192)

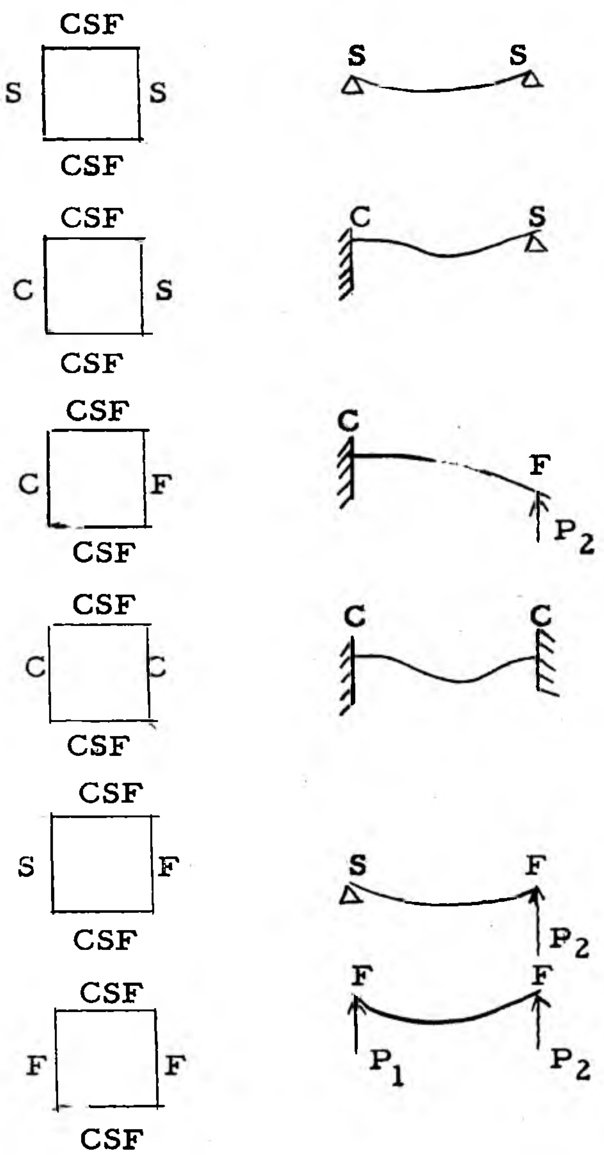
this is a geometric series which converges absolutely for all m. Its sum

is $\frac{k_1}{m^3} + \frac{k_2 k_3}{1 - k_3 k_4} \frac{1}{m^{1.05}}$.

Koch[20] concludes that the d_m, so defined are unique.

F. Comparison With Beam Problems

We indicate in Figure 30 the beam problems which correspond to various plate problems as b



$$w_a = \frac{q}{24D}\left(x^4 - 2ax^3 + a^3x\right)$$

$$w_b = \frac{q}{24D}\left(x^4 - \frac{5ax^3}{2} + \frac{3a^2x^2}{2}\right)$$

$$w_c = \frac{q}{24D}\left(x^4 - 4ax^3 + 6a^2x^2\right) + P_2\frac{ax^2}{2}\left(1 - \frac{x}{3a}\right)$$

$$w_c = \frac{q}{24D}\left(x^4 - \frac{5ax^3}{2} + \frac{3a^2x^2}{2}\right) + \frac{3x^2}{2a^2}\left(1 - \frac{x}{3a}\right)w(a)$$

$$w_d = \frac{q}{24D}\left(x^4 - 2ax^3 + a^2x^2\right)$$

$$w_e = \frac{q}{24D}\left(x^4 - 2ax^3 + a^3x\right) + w(a)\,\frac{x}{a}$$

$$w_f = \frac{q}{24D}\left(x^4 - 2ax^3 + a^3x\right) + \frac{x}{a}\left[w(a) - w(0)\right] + w(0)$$

Fig. 30

where P_2 is the upward force at $x = a$ and P_1 is the upward force at $x = 0$. In Case (c) either P_2 or $w(a)$ is arbitrary. In Case (e) the deflection $w(a)$ is arbitrary but not P_2. In Case (f), $w(0)$ and $w(a)$ are arbitrary but not P_1 or P_2.

We would like to show that most plate problems reduce to one of the above beam problems as $b \to \infty$. The exceptions are those problems with corner loads. If a problem has a corner load, the solution in general will be a function of y (even as $b \to \infty$). We will indicate the reduction in several cases:

In Problems #1 → 9, $B_m = C_m = d_m = e_m = 0$ and the other terms are of order $0(\frac{1}{b})$ so that w(x, y) in Problem# 1 → $w_1(x, y) = \frac{4k}{a^2 b} \sum_{m, n}$.

$\frac{q_{mn}}{(\alpha^2+\beta^2)^2}$ sin αx sin βy. For a constant load, this becomes

$$w_1 = \frac{4q}{abD} \sum_m \frac{1-(-1)^m}{\beta} \sum_n \frac{1-(-1)^n}{\alpha^5} \sin \alpha x$$

which by use of the tables becomes

$$w = \frac{q}{24D} (x^4 - 2ax^3 + a^3 x) \ .$$

This is just the corresponding beam solution, namely w_a.

Next consider the cantilever plates; Problem 32. In this case, if we neglect higher order terms, we can solve for the coefficients explicitly. Thus $C_m = 0$

$$B_m = \frac{-2w(a, b)}{\beta k a^2} + \frac{1}{8a} \frac{1-(-1)^m}{\beta} = \frac{d_m}{\nu \beta^2 a^2}$$

$$e_m = \frac{1}{2a} \quad \frac{1-(-1)^m}{\beta}$$

$$F_n = -\frac{(2-\nu)}{2a^3(1-\nu)(3+\nu)} \quad \frac{1}{\alpha^3} = \frac{h_n}{\nu\alpha^2 a^2} = G_n = \frac{i_n}{\nu\alpha^2 a^2}$$

$$w(x,y) \rightarrow \frac{4ka^2}{b} \quad \Sigma \frac{1}{a^3\alpha^4} \left\{ \frac{q_{mn}}{a} - (-1)^n a^2\alpha^2 B_m + a^2\alpha^2 C_m \right.$$

$$+ (-1)^n \alpha d_m - \alpha e_m - 2\beta(-1)^m \; a^2\alpha^2 F_n + 2\beta a^2\alpha^2 G_n + \beta(-1)^m h_n$$

$$\left. - \beta i_n \right\} + \frac{x}{a} \quad w(a,0) \; .$$

For a constant load, this reduces to

$$w(x,y) \rightarrow w_a + \frac{q}{24D} \quad [-2ax^3 + 6a^2x^2 - a^3x] - 2w(a,b)\frac{x}{a}\;\frac{b-y}{b} + \frac{x}{a}\,w(a,b)$$

$$= \frac{q}{24D} \quad [x^4 - 4ax^3 + 6a^2x^2] - w(a,b)\frac{x}{a}\;\frac{b-2y}{b}$$

$$= \frac{q}{24D} \; [x^4 - 4ax^3 + 6a^2x^2] + \frac{Px(b-2y)}{4D(1-\nu)}$$

when P is the corner load, (it acts upward at (a, b) and down at (0, 0)). If we set the corner loads = 0, then the problem reduces to the corresponding beam problem, namely w_c. If we take values of y away from the corners, for example $y = b/2$, then the problem again reduces to w_c. The corner loads essentially add to the beam solution a twist term of amount

$$\frac{Px(b-2y)}{4D\,(1-\nu)} \; .$$

If we wanted the case of a load on the whole edge $x = a$, we would have to use the boundary condition $V_x(a,y) = P$ in Problem #32 instead of $V_x = 0$. This problem could be solved just as easily as the others.

It should be observed that the beam solution satisfies the plate equation

$$\nabla^4 w = \frac{q}{D}$$

and 6 of the boundary conditions. w_a is not a solution of the corresponding plate problem because

$$M_y(x, 0) = M_y(x, b) = \frac{q\nu x}{2}(a-x)$$

instead of zero. However, at points at large distances from the edges $y = 0$ and $y = b$, by St. Venants principle, the deflection of the plate will not differ appreciably from the corresponding deflection of a beam.

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PROGRAM OF FUTURE RESEARCH

We have solved problems which have one edge clamped, supported, or free. We have not shown that all double sums are absolutely convergent. This is necessary to justify interchanging the order of summation. We have not shown that the sets of infinite equations for the coefficients converge, nor that they allow a unique solution, or if there are any interesting special cases in which the coefficients can be expressed as an elementary function. We have not shown that the formal solution obtained for $w(x, y)$ is a unique solution. We have not investigated the singularities in the derivatives at the corners, or in the neighborhood of concentrated loads. In particular we would like to find expressions for the reactions along the lines $x = a_1$, $y = b_1$ when a concentrated load is applied at (a_1, b_1).

We would like to have numerical solutions, for various values of the parameters $D, \nu, b/a, a, q(x, y), x, y$. We would like to especially investigate deflections, moments, shears and reactions, and find the maximums of all these quantities. The free plate and cantilever plate are the most interesting cases to be studied numerically.

We would like to study problems with non zero boundary conditions, such as constant slope or some realistic boundary condition. We would like to study the effect of different kinds of loads including transient and vibrating loads. The problem is somewhat different if $w(x, y)$ is a function of time, and it would be interesting to know whether the methods presented here are applicable. End thrusts, supporting foundation and body forces alter the above equations, but the problems are very similar. Are there any complications in using the double series method?

We could test the validity of some of the formal solutions of Part II by using a solution

$$w(x, y) = xy \quad \text{or } w(x, y) = \frac{q}{D} \sin^5 \frac{k\pi x}{a} \sin^5 \frac{l\pi a}{b}$$

and using as q(x, y) the values as calculated from the equation

$$q(x, y) = D \nabla^4 w(x, y) .$$

We might find an exact solution of an infinite set of equations in this way.

We would like to see how the above methods might be used in solving skew plates and tapered plates. In cylindrical plates, perhaps Hankel transforms could be used. Does differentiating non uniformly convergent series give any complications in this case?

We would like to investigate those series which have only definite integrals as the summed form. Are there any interesting properties which can be readily obtained from the integrals which are not contained in the series ?