NUMERICAL ANALYSIS IN L-SPACES

by

Vira Babenko

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics
The University of Utah

May 2016

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Vira Babenko**

has been approved by the following supervisory committee members:

		Date Approved
Peter Alfeld	, Chair	**03/04/2016**
Elena Cherkaev	, Member	**03/04/2016**
Alexander Balk	, Member	**03/04/2016**
Jingyi Zhu	, Member	**03/04/2016**
Robert Kirby	, Member	**03/04/2016**

and by **Peter Trapa** , Chair/Dean of

the Department/College/School of **Mathematics**

and by David B. Kieda, Dean of The Graduate School.

ABSTRACT

This dissertation consists of two parts that focus on two interrelated areas of Applied Mathematics. The first part explores fundamental properties and applications of functions with values in L-spaces. The second part is connected to Approximation Theory and dives deeper into the analysis of functions with values in specific classes of L-spaces (in particular, L-spaces of sets).

In the first project devoted to the theory and numerical methods for the solution of integral equations, we explore linear Volterra and Fredholm integral equations for functions with values in L-spaces (which are generalizations of set-valued and fuzzy-valued functions). In this study, we prove the existence and uniqueness of the solution for such equations, suggest algorithms for finding approximate solutions, and study their convergence. The exploration of these equations is of great importance given the wide variety of their applications in biology (population modeling), physics (heat conduction), and engineering (feedback systems), among others. We extend the aforementioned results of existence and uniqueness to nonlinear equations. In addition, we study the dependence of solutions of such equations on variations in the data. In order to be able to better analyze the convergence of the suggested algorithms for the solutions of integral equations, we develop new results on the approximation of functions with values in L-spaces by adapted linear positive operators (Bernstein, Schoenberg, modified Schoenberg operators, and piecewise linear interpolation).

The second project is devoted to problems of interpolation by generalized polynomials and splines for functions whose values lie in a specific L-space, namely a space of sets. Because the structure of such a space is richer than the structure of a general L-space, we have additional tools available (e.g., the support function of a set) which allow us to obtain deeper results for the approximation and interpolation of set-valued functions. We are working on defining various methods of approximation based on the support function of a set. Questions related to error estimates of the approximation of set-valued functions by those novel methods are also investigated.

To the memory of my grandfather P. Movchan

CONTENTS

LIST OF FIGURES

ACKNOWLEDGEMENTS

I am very grateful to all members of my PhD committee. I would first like to thank my advisor and mentor, Professor Peter Alfeld, for his support and encouragement, patience, and understanding during my work on this project. From my first year in the graduate program, he pushed me beyond my comfort zone in a lot of different ways. For example, he helped me expand my teaching experience to huge classes during my first year as a TA, which I would definitely not have done without him. Now I see that it helped me to succeed and I am very thankful for it.

I would also like to thank Professor Elena Cherkaev, for uncountable hours she dedicated to meeting and talking with me, for always encouraging and endorsing me during my work, for numerous helpful suggestions she gave me, and also for her positive attitude about everything.

I would like to thank Professors Alexander Balk and Jingyi Zhu for teaching great courses that helped me develop as a mathematician and broaden my understanding of applications. I also want to thank Professor Robert Kirby for introducing me to uncertainty quantification problems that I now find fascinating. I would like to thank all my committee members for their encouraging discussions, valuable suggestions, and careful reading of my dissertation.

I am grateful to all of the faculty and staff at the Department of Mathematics at the University of Utah with whom I have had the pleasure and honor to work during these five years. I am also happy that I was able to meet a lot of great friends while in the graduate programm. I am especially grateful to Veronika, Jie, Katrina, Rebecca, and Yuan for their constant support and friendship.

But most importantly, I would like to thank my family. Their continuous help, faith in me, and encouragement are invaluable. I am indebted to my parents for all that they are giving to me. My mother taught me to be persistent and forceful in research and work, but thoughtful and compliant in everyday life. She taught me to set goals and achieve them. She is an amazing example of a researcher and wife who found a balance between career and family. I admire her curiosity and constant need to learn something new, and hope I will have it too. My father is a role model for me. From my very young age, he treated me with mathematical puzzles. Each time we were going for a walk in the park I was very excited to get a problem to solve, and while it felt really good to resolve the mystery, it felt even better to just discuss the problems and listen to his explanation. He showed me the beauty of discovery. His attitude and passion about research are what made me make my career choice. He is a great mathematician, excellent teacher, caring advisor, and I just hope to become some day a little bit like him. His patience and support are invaluable.

My husband Oleksii is always there for me. He supports me day to day and shares all ups and downs. He is my best friend, guardian, my love, and I can not imagine going throughout graduate school without his permanent help and invigoration.

My sister Yuliya was always an example for me. She inspired and helped me in every way to come to study in the United States. Nowadays, she continues to support me every day together with all her family: Florin is giving me career advice and my nephew Peter is sending me the cutest smiles that brighten my day.

CHAPTER 1

INTRODUCTION

This dissertation is devoted to the development of some topics of numerical analysis in L-spaces. This gives a unified approach to the numerical solution of various problems connected with set-valued and fuzzy-valued functions. In particular, we consider problems of approximation of functions with values in L-spaces and obtain error estimates for quadrature formulas. We prove also theorems of existence and uniqueness of solution of linear and nonlinear integral equations and we study in detail questions of the numerical solution of linear integral equations for such functions.

In the second chapter of this dissertation, we describe some applications of set-valued and fuzzy-valued functions and give the main definitions related to spaces of sets and fuzzy sets.

The third chapter is devoted to the development of a calculus of functions with values in L-spaces. We present here definitions and examples of L-spaces. Further we define the Hukuhara type difference and derivative and investigate their properties. Finally, we describe the Riemannian integral and its main properties for functions with values in L-space.

Chapter 4 is devoted to methods of an approximate solution of linear Fredholm and Volterra integral equations for functions with values in L-spaces. In that chapter, we obtain estimations of the error in a piecewise linear approximation and the error of quadrature formulas. We also describe collocation and quadrature formula methods for the solution of such equations and perform the convergence analysis for the proposed methods. We also obtain estimates of the rate of convergence of these algorithms. To do this, we essentially use the results on the approximation theory and the theory of quadrature formulas developed in the third chapter. In addition we illustrate our results with some numerical examples.

In Chapter 5, we study nonlinear Fredholm and Volterra integral equations. We prove theorems of existence and uniqueness of their solutions and investigate the data dependence of their solutions.

We perform the adaptation of classical approximation operators to the case of functions with values in L-spaces in Chapter 6. In particular, we consider the adaptation of Bernstein, Schoenberg, and modified Schoenberg operators. These classic operators can be used to construct methods for solving integral equations and for the analysis of their convergence.

Finally, in Chapter 7, we consider some problems of approximation of set-valued functions. In the spaces of sets, we have additional tools available (e.g., the support function of a set) which allow us to obtain deeper results for the approximation and interpolation of set-valued functions. The results presented here can be applied to the numerical analysis of the various problems connected to set-valued functions. We illustrate this with some examples.

CHAPTER 2

SET-VALUED AND FUZZY-VALUED FUNCTIONS

In this chapter, we present some applications of set-valued and fuzzy-valued functions. We also list here definitions of basic concepts related to set-valued and fuzzy-valued functions.

2.1 Applications of set-valued and fuzzy-valued functions

In many applied fields, we are not able to require maps to be single-valued. A wide variety of questions which range from social and economic sciences to physical and biological sciences lead to functions with values that are sets in finite or infinite dimensional spaces, or that are fuzzy sets. The need for set-valued maps was recognized and they were first investigated during the first three decades of the 20^{th} century (see [55], [20], [44]), but for a long time, they were mainly considered a generalization for its own sake of the ordinary function case, because of a lack of motivating applications. The rapid development of the field first occurred during the 1960s (see [39], [51], [53]) due to the many new questions that arose in other fields of science and that required mathematical analysis. This need of the concept of set-valued mappings in applied fields helped overcome pure treatment of this new branch of mathematics and maintains a constant development of it nowadays due to more and more new questions that involve set-valued analysis (for example decision-making problems see [23] and [33]).

As the name suggests, a set-valued function is a function whose values are sets instead of numbers. One can form an arithmetic of sets and an associated calculus of set-valued functions that parallels in part ordinary arithmetic and calculus, but that also exhibits some significant differences. There is now a large body of literature devoted to set-valued maps (see for example [37], [38], [8] and references therein). If one wants to take into account uncertainties, modeling errors, disturbances, etc., one is led naturally to set-valued maps and inclusions. Also, set-valued maps arise when we wish to treat a problem qualitatively, by looking for solutions common to a set of data that share the same qualitative properties with applications in qualitative physics (branch of Artificial Intelligence). Another area where set-valued functions are very useful is control theory with the first contributions made by Wazewski and Filippov in the early sixties and a wide variety of modern applications nowadays. Motivation to use set-valued functions comes also from problems with constraints and mathematical programming as well as from optimization problems for which the solution is not unique (see [40]).

More examples of modern applications that involve set-valued maps include 3-D printing and 3-D scanning, image processing, game theory, biomathematics, and economic analysis (problems of competitive equilibrium, mean demand, coalition production economies). In econometric applications, respondents may report a salary bracket instead of the exact salary or the profit of a firm may be intentionally converted to an interval to ensure anonymity. These are only a few of a wide range of problems where set-valued maps arise.

The idea of a fuzzy set was first proposed by Lotfi Zadeh (see [68]) in the 1960s as a way of handling uncertainty that is due to inaccuracy or indeterminacy rather then to randomness (see [27] and references therein). The basic idea is that each element $x \in X$ is assigned a membership grade $u(x)$ taking values in $[0, 1]$, with $u(x) = 1$ corresponding to full membership, $u(x) = 0$ to non-membership, and $0 < u(x) < 1$ to partial membership. The function u itself is often used synonymously for the fuzzy set. Fuzzy-valued functions and fuzzy logic are widely used nowadays in a great range of problems. We name a few here.

We start with applications in biology. One-sided fuzzy numbers are used in mathematical models for the spread of infectious diseases with a rate of contact that varies seasonally (see [19]). One-sided fuzzy numbers form a particular case of fuzzy numbers (for example right-sided fuzzy numbers are those numbers that are at least x and no more than y). In [19], right-sided numbers represent a real description of the number of infected individuals with infectious disease such as grippe. The reason to use fuzzy numbers rather than regular numbers here is that the real number of infected individuals is greater than the number of registered infectious individuals, but does not exceed a reasonable proportion in the whole population. This approach can be used regarding any other medical statistic when the number of actual cases is greater than those registered by the authorities. In this model, a Volterra integral equation (delay integral equation) is used.

There are many other biological application of fuzzy sets. For instance, [49] presents a method for incorporating fuzzy-set-based spatial relations in registering temporal mammogram pairs. Breast cancer is the most common cancer among women and its early detection is the key to reducing mortality. The most effective method of detection is intensive monitoring and in many countries, women are advised to take a mammogram every two years after a certain age. Those images are kept for future reference and radiologists compare a current mammogram with a previous one to detect any medically relevant changes. Extracting such information by computer programs has not been very successful so far due to the problem of identifying objects in one image with the matching objects in the other image. Differences in equipment, positioning of the breast, natural changes of the breast over time, and finally the fact that a mammograph is a 2D projection of a 3D object all cause difficulties in reading and comparing those images. In the above-mentioned article the authors propose using fuzzy-set-based spatial relation representation for which the breast boundary can provide a global reference.

Fuzzy logic is also used in models that involve uncertainty. For example, in [52], an interval fuzzy logic controller is proposed to consider daily meals as distribution which is imposed to diabetic system. The method proposed by the authors' is shown to control the glucose level in those patients effectively.

Another example of the application of fuzzy sets is the problem of decision making. The authors of [23] claim that many organizational tasks such as budget plans, policies, etc., frequently involve group discussions and meetings. Moreover, research in social psychology shows that decisions made in groups are more effective than those made by individuals. By using fuzzy preference relations to represent the opinions of the decision makers, the authors of [23] develop a concept of a specific (granular) fuzzy relation that offers the required flexibility in models to increase the level of agreement within the group. Moreover, fuzzy logic is also used in models that, for example, compute the power of political groups (for example in the European Parliament, see [33]).

Fuzzy logic also provides an effective framework for Recommendation Systems whose number has been growing rapidly since the development of the internet and e-commerce. Recommender Systems provide a rating or a preference for each user, but what if you are undecided about what you want to buy? The work in [64] suggests a model for discovering user preferences from item characteristics.

Another practical example of the use of fuzzy-logic models is proposed in the paper [24]. Because the amount of fuel consumption for transport jet aircrafts constitutes a large part of operational costs, the airlines constantly look for a practical method to reduce fuel usage within the constrains of accomplishing the mission. The above-mentioned work discusses the problem of enhancing fuel efficiency for commercial transports. Finally, various applications of fuzzy sets and fuzzy logic to economics are discussed, for example in [35], [42], and [69].

Many applied problems can be formulated in the form of integral equations. Those equations can also be obtained by reformulating partial or ordinary differential equations. Thus, the study of those equations and methods of finding their solutions is very useful. In this work, we choose to explore Fredholm and Volterra integral equations because of their wide applicability. Fredholm integral equations are used in image and signal processing, astronomy, geophysics, mathematical economics, etc.; Volterra integral equations are of great importance in population dynamics and demography, infection propagation models, heat transfer problems, potential theory, and actuarial mathematics. We consider set and fuzzy integral equations from a common point of view, as integral equations for functions with values in L-spaces. Some results are new also for specific cases of sets and fuzzy sets.

2.2 The spaces of sets

We present here necessary definitions and facts related to the space of nonempty compact subsets of the space $\mathbb{R}^m$ and set-valued functions.

Definition 1 *A set-valued function F is a mapping with values that are sets.*

Denote by $\mathcal{K}(\mathbb{R}^n)$ the set of all compact, nonempty subsets of the space $\mathbb{R}^n$. Also denote by $\mathcal{K}^c(\mathbb{R}^n)$ the set of all compact, convex, nonempty subsets of the space $\mathbb{R}^n$.

Let $A, B \in \mathcal{K}(\mathbb{R}^n)$, $\alpha \in \mathbb{R}$. Then

$$A + B := \{x + y \ : \ x \in A, \ y \in B\}, \quad \alpha A := \{\alpha x \ : \ x \in A\}.$$

$A + B$ is called the Minkowski sum of sets A and B.

Example. See an example of a Minkowski sum of two sets in Figure 2.1.

If A and B are sets of only one number, their Minkowski sum is the usual sum of two numbers. It they are vectors, it is a regular vector sum. In general, as opposite to the case of numbers, no additive inverse exist. Furthermore, $A + B = A + C$ does not imply that $B = C$.

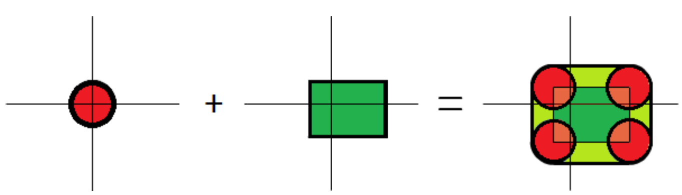


Figure 2.1: Minkowski sum of two sets.

While,

$$\lambda(A+B) = \lambda A + \lambda B$$

for any sets $A, B \in \mathcal{K}(\mathbb{R}^n)$ and any $\lambda \in \mathbb{R}$, the second distributive law

$$(\lambda + \mu)A = \lambda A + \mu A$$

holds only for $A \in \mathcal{K}^c(\mathbb{R}^n)$ and $\lambda, \mu \geq 0$. A generalization of the last property can be proved by induction; thus, if A is convex and $\lambda_1, \lambda_2, ..., \lambda_n \geq 0$

$$(\lambda_1 + \lambda_2 + ... + \lambda_n)A = \lambda_1 A + \lambda_2 A + ... + \lambda_n A.$$

Example. If $A = \{1, 2\}$, we have $A + A = \{2, 3, 4\}$ and therefore, $A + A \neq 2A$.

Definition 2 *The convex hull, denoted by* $\mathrm{co}A$, *of a set* $A \subset \mathcal{K}(\mathbb{R}^n)$ *is the set of all elements of the form* $\sum_{i=1}^{r} \lambda_i a_i$, *where* $r \geq 2$, $a_i \in A$, $\lambda_i \in \mathbb{R}$, $\lambda_i \geq 0$ *for* $i = 1, ..., r$, *and* $\sum_{i=1}^{r} \lambda_i = 1$.

The convex hull has the following properties

$$\mathrm{co}\,(\mu A) = \mu\,\mathrm{co}A, \qquad \forall \mu \in \mathbb{R},\ A \subset \mathcal{K}(\mathbb{R}^n),$$

$$\mathrm{co}\,(A + B) = \mathrm{co}A + \mathrm{co}B, \qquad \forall A, B \subset \mathcal{K}(\mathbb{R}^n).$$

Definition 3 *The Hausdorff distance* $\delta^H(A, B)$ *between* $A, B \in \mathcal{K}(\mathbb{R}^n)$ *is defined by the following relation*

$$\delta^H(A, B) = \max\left\{\sup_{x \in A} \inf_{y \in B} |x - y|,\ \sup_{x \in B} \inf_{y \in A} |x - y|\right\},$$

where $|\cdot|$ *is the Euclidian norm in* $\mathbb{R}^n$.

See Figure 2.2 for an example.

It is well known that all metric conditions hold for δ^H. $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{K}^c(\mathbb{R}^n)$ are complete metric spaces with this metric.

Besides the usual properties of the metric, $\delta^H\,(A, B)$ has the following properties

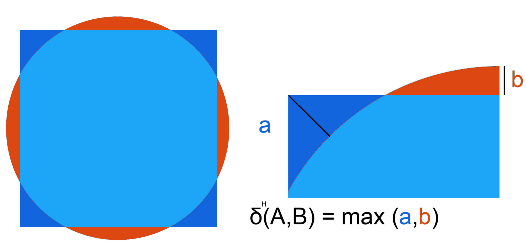


Figure 2.2: Hausdorff distance.

1. $\delta^H(\lambda A, \lambda B) = \lambda \delta^H(A, B),\ \forall \lambda > 0,\ \forall A, B \in \mathcal{K}(\mathbb{R}^n)$;

2. $\delta^H(A + B, C + D) \leq \delta^H(A, C) + \delta^H(B, D),\ \forall A,\ B,\ C,\ D \in \mathcal{K}(\mathbb{R}^n)$;

3. $\delta^H(\text{co } A, \text{co } B) \leq \delta^H(A, B),\ \forall A, B \in \mathcal{K}(\mathbb{R}^n)$;

4. $\delta^H(\alpha A, \beta A) \leq |\alpha - \beta| \|A\|,\ \forall \alpha, \beta \in \mathbb{R}, \forall A \in \mathcal{K}^c(\mathbb{R}^n)$,

   where $\|A\| := \delta^H(A, \{\theta\})$ (here $\theta = (0, \ldots, 0)$ – null element of the space $\mathbb{R}^n$).

One can find proofs of all properties presented above in [60] and [61].

2.2.1 Difference of two sets

Since the spaces $\mathcal{K}(\mathbb{R}^n)$ and $\mathcal{K}^c(\mathbb{R}^n)$ are not linear and therefore, in general, there is no inverse element, we cannot define a difference in these spaces as

$$A - B := A + (-1)B.$$

However, the ability to take a difference is often very important and there exist various other ways to define the concept of a difference of two sets. We give here the definition of the Hukuhara difference since we will use only this notion throughout this work (for more definitions of the difference see, for example, [58]). The notion of Hukuhara differences between sets was introduced by Hukuhara in [39].

Definition 4 *We say that an element $C \in X$ is the Hukuhara difference of the sets $A, B \in X$, if*

$$A = B + C.$$

If such C exists, then this C is unique. We denote this difference by

$$C = A \overset{h}{-} B.$$

One can find properties of Hukuhara difference, for example, in [58].

2.2.2 Derivative of a set-valued function

Because there exist various ways to define difference for set-valued functions, there also exist various ways to define derivatives of such functions. All of them have their advantages and disadvantages. For this work, we consider the approach that was proposed by Hukuhara in [39].

Definition 5 *Let $f(t): [a,b] \to \mathcal{K}(\mathbb{R}^n)$. If $t \in (a,b)$, and for all small enough $h > 0$ there exist differences*

$$f(t+h) \overset{h}{-} f(t) \ \text{ and } \ f(t) \overset{h}{-} f(t-h),$$

and both limits exist and are equal to each other

$$\lim_{h \to 0^+} \frac{f(t+h) \overset{h}{-} f(t)}{h} = \lim_{h \to 0^+} \frac{f(t) \overset{h}{-} f(t-h)}{h}$$

then the function f has the Hukuhara derivative $D_H f(t)$ at point t (if $t = a$ or $t = b$, then there exists only one limit) and

$$D_H f(t) = \lim_{h \to 0^+} \frac{f(t+h) \overset{h}{-} f(t)}{h}.$$

One can find properties of Hukuhara derivative for example in [58] or [39] (also see references therein).

2.2.3 Integral of a set-valued function

There exist various definitions of the integral of a set-valued function (see for example [61], [51], [9], [59]). In particular, there exists an analog of the Riemann integral.

The Riemann-Minkowski sum of f is defined in the following way. Let $P = \{x_0, x_1, ..., x_n\}$, $0 = x_0 < x_1 < ... < x_n = 1$, be some partition of the interval $[0,1]$. We set $\Delta x_i = x_i - x_{i-1}$, $\lambda(P) = \max\{|\Delta x_i| : \ i = 1, ..., n\}$, and $\xi = \{\xi_1, ..., \xi_n\}$, $\xi_i = [x_{i-1}, x_i]$, $i = 1, ..., n$. The Riemann-Minkowski sum of f relative to the pair (P, ξ) is defined as

$$\sigma(f; (P, \xi)) := \sum_{i=1}^{n} \Delta x_i f(\xi_i).$$

We define the standard base $\lambda(P) \to 0$ in the set of all pairs (P, ξ) as follows:

$$\lambda(P) \to 0 := \{B_\epsilon\}_{\epsilon > 0}, \ \ B_\epsilon := \{(P, \xi) : \lambda(P) < \epsilon\}.$$

A function f is integrable in the Riemann-Minkowski sense if (see [51], [59]) there exists an element $I(f) \in \mathcal{K}(\mathbb{R}^n)$ such that

$$\delta(\sigma(f; (P, \xi)), I(f)) \to 0 \ \text{ as } \ \lambda(P) \to 0.$$

However, one of the most flexible and convenient definitions is the definition of the Aumann's integral.

The Aumann's integral of set-valued function $F : [a, b] \to \mathcal{K}(\mathbb{R}^n)$ is defined as the set of all integrals of integrable selections (see Figure 2.3) of function F:

$$I(F) = \int_a^b F(x)dx := \left\{ \int_a^b \phi(x)dx \; : \; \phi(x) \in F(x) \text{ a. e., } \phi \text{ is integrable} \right\}$$

If the function $F : [a, b] \to \mathcal{K}(\mathbb{R}^n)$ is measurable (see definition in [6] or [8]) and the function $\delta^H(F(\cdot), \theta)$ is integrable, then the Aumann's integral exists and has the following properties:

1. $\int_a^b F(x)dx \in \mathcal{K}^c(\mathbb{R}^n)$,
2. $\displaystyle\int_a^b \text{co}(F(x))dx = \int_a^b F(x)dx$,
3. $\displaystyle\int_a^b (\lambda F(x) + \mu G(x))dx = \lambda \int_a^b F(x)dx + \mu \int_a^b G(x)dx \quad \forall \lambda, \mu \in \mathbb{R}$,
4. $\displaystyle\delta^H \left(\int_a^b F(x)dx, \int_a^b G(x)dx \right) \leq \int_a^b \delta^H(F(x), G(x))dx$,

It is proved in [59] that the Riemann-Minkowski integral for any continuous and bounded set-valued function exists and coincides with Aumann integral.

2.3 Spaces of fuzzy sets

Fuzzy sets are a generalization of ordinary (or crisp) sets that have only non-membership and full membership possibilities (see [27] for details). Any crisp subset A of a fuzzy set X can be identified with a fuzzy set on X by the following characteristic function $u(x) : X \to [0, 1]$

$$u(x) = \begin{cases} 0 & \text{if} \quad x \notin A \\ 1 & \text{if} \quad x \in A. \end{cases}$$

Example. Figure 2.4 shows the characteristic function of the interval $[1, 2]$.

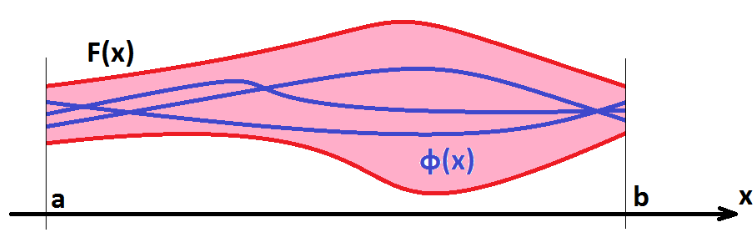


Figure 2.3: Integrable selections

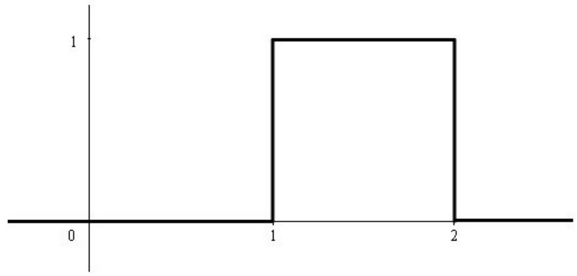


Figure 2.4: The fuzzy set $[1, 2]$

Example. A simple example for a non-crisp set is the function $u : \mathbb{R} \to [0, 1]$ with

$$u(x) = \begin{cases} 0 & \text{if} \quad x \leq 1 \\ \frac{1}{99}(x-1) & \text{if} \quad 1 < x \leq 100 \\ 1 & \text{if} \quad 100 < x \end{cases}$$

that provides one of the choices of a membership function of a fuzzy set of real numbers $x \gg 1$ (see Figure 2.5).

Let us make this notion more precise. Consider (see, e.g., [27]) the class of fuzzy sets $\mathcal{E}^n$ consisting of functions $u : \mathbb{R}^n \to [0, 1]$ such that

1. u is normal, i.e., there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$;

2. u is fuzzy convex, i.e., for any $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$,

$$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\};$$

3. u is upper semicontinuous;

4. the closure of $\{x \in \mathbb{R}^n : \ u(x) > 0\}$, denoted by $[u]^0$, is compact.

We will need the following notion of an α-level set.

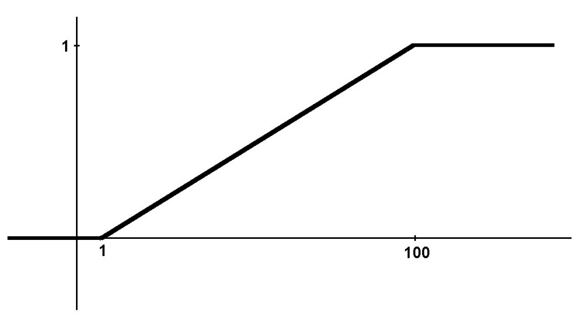


Figure 2.5: The fuzzy set of real numbers $x \gg 1$.

Definition 6 *For each* $0 < \alpha \leq 1$*, the* α*-level set* $[u]^\alpha$ *of a fuzzy set* u *is defined as* $[u]^\alpha = \{x \in \mathbb{R}^n : \ u(x) \geq \alpha\}$.

Example. Figure 2.6 shows the 0.7 level set of a given fuzzy set.

The addition $u+v$ and scalar multiplication cu, $c \in \mathbb{R} \setminus \{0\}$, on $\mathcal{E}^n$ are defined, in terms of α-level sets, by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [cu]^\alpha = c[u]^\alpha \ \text{ for each } 0 < \alpha \leq 1$$

(see Figure 2.7).

Define also $0 \cdot u$ by the equality $[0 \cdot u]^\alpha = \{\theta\}$ (here $\theta = (0, ..., 0) \in \mathbb{R}^n$).

There are many different ways to define a metric for fuzzy sets (see for example [27]). One of the possible metrics in $\mathcal{E}^n$ is defined in the following way

$$d_p(u,v) = \left(\int_0^1 \delta([u]^\alpha, [v]^\alpha)^p d\alpha \right)^{1/p}, \quad 1 \leq p < \infty.$$

Then the space $(\mathcal{E}^n, d_p)$ is (see [27, Theorem 3]) a complete separable metric space.

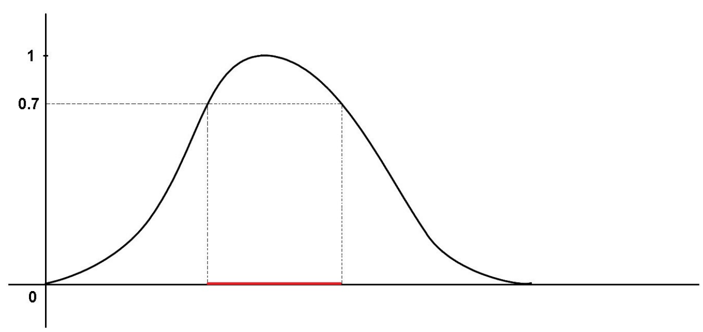


Figure 2.6: The 0.7 level set of a fuzzy set.

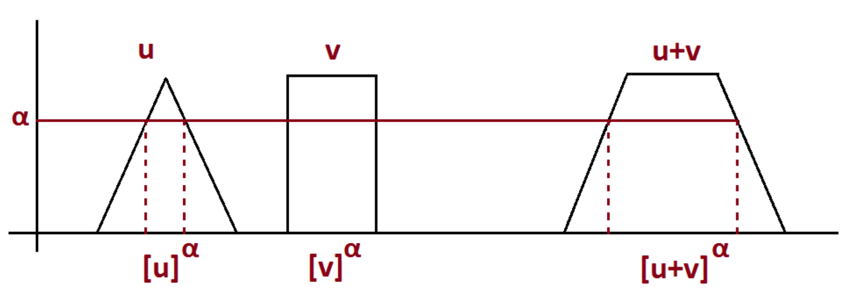


Figure 2.7: Sum of two fuzzy sets.

CHAPTER 3

CALCULUS OF FUNCTIONS WITH VALUES IN L-SPACES

In this chapter, we introduce the definition of L-spaces that encompass the spaces of sets and fuzzy sets described above. We also develop a calculus for functions with values in L-spaces.

3.1 L-spaces: definition and examples

The following definition was introduced by Vahrameev in [65]:

Definition 7 *A complete separable metric space X with metric δ is said to be an $L-$* **space** *if in X, operations of addition of elements and their multiplication with real numbers are defined, and the following axioms are satisfied:*

Axiom 1. $\forall x, y \in X \quad x + y = y + x;$

Axiom 2. $\forall x, y, z \in X \quad x + (y + z) = (x + y) + z;$

Axiom 3. $\exists \theta \in X \ \forall x \in X \ \ x + \theta = x$ *(where θ is called a zero element in X);*

Axiom 4. $\forall x, y \in X \quad \lambda \in \mathbb{R} \quad \lambda(x + y) = \lambda x + \lambda y;$

Axiom 5. $\forall x \in X \quad \lambda, \mu \in \mathbb{R} \quad \lambda(\mu x) = (\lambda\mu)\, x;$

Axiom 6. $\forall x \in X \quad 1 \cdot x = x, \quad 0 \cdot x = \theta;$

Axiom 7. $\forall x, y \in X \quad \lambda \in \mathbb{R} \quad \delta(\lambda x, \lambda y) = |\lambda| \delta(x, y);$

Axiom 8. $\forall x, y, u, v \in X \quad \delta(x + y, u + v) \leq \delta(x, u) + \delta(y, v).$

Examples of L-spaces:

1. The spaces $(\mathcal{K}(\mathbb{R}^n), \delta^H)$ and $(\mathcal{K}^c(\mathbb{R}^n), \delta^H)$ are complete, separable metric spaces (see [30]) and since the Axioms 1-8 hold, these spaces are L-spaces.

2. The space $(\mathcal{E}^n, d_p)$ is a complete separable metric space and, since Axioms 1-8 hold, an L-space.

3. Any real Banach space $(Y, \|\cdot\|_Y)$ endowed with the metric $\delta(x, y) = \|x - y\|_Y$ is an L-space.

4. The set of all closed bounded subsets of a given Banach space, endowed with the Hausdorff metric, is an L-space.

   Let X be a Banach space. Denote by $\Omega(X)$ the space of nonempty closed bounded subsets of X. The operation of addition of elements of $\Omega(X)$ we define as

   $$A + B = \overline{\{a + b : a \in A, b \in B\}}$$

   ($\overline{A}$ denotes the closure of the set A), and the multiplication of a real number α and the set A we define as

   $$\alpha \cdot A = \{\alpha \cdot a : a \in A\}.$$

   In the space $\Omega(X)$, we define the Hausdorf metric as usual:

   $$\forall A, B \in \Omega(X) \quad h_\Omega(A, B) = \inf\{r \geq 0 : \ A \subset B + S_r(\theta), \ B \subset A + S_r(\theta)\},$$

   where $S_r(\theta)$ is the closed ball of radius r and centered at the point $\theta \in X$ in the space X. It is clear that the space $(\Omega(X), h_\Omega)$ as well as its subspace $(\Omega_{\text{conv}}(X), h_\Omega)$ that consist of nonempty, closed, bounded subsets of the space X are L-spaces.

   If $X = \mathbb{R}^m$, then the elements of the spaces $(\Omega(X), h_\Omega)$ and $(\Omega_{\text{conv}}(X), h_\Omega)$ are compact sets.

5. The set X is called a quasilinear space (see [6]), if we can define in it a relation of partial order $\leq$, and operations of algebraic addition and multiplication by real numbers such that for any elements $x, y, z, v \in X$ and for any real numbers α, β the following conditions hold:

   (a) $x \leq x$;

   (b) $x \leq y$ and $y \leq z$ imply that $x \leq z$;

   (c) if $x \leq y$ and $y \leq x$, then $x = y$;

   (d) $x + y = y + x$;

   (e) $x + (y + z) = (x + y) + z$;

   (f) $\exists\, \theta \in X$, such that $x + \theta = x$;

   (g) $\alpha(\beta x) = (\alpha\beta)x$;

   (h) $\alpha(x + y) = \alpha x + \alpha y$;

   (i) $1 \cdot x = x$;

(j) $0 \cdot x = \theta$;

(k) $(\alpha + \beta)x \leq \alpha x + \beta x$;

(l) $x \leq y$ and $z \leq v$ imply that $x + z \leq y + v$;

(m) if $x \leq y$, then $\alpha x \leq \alpha y$.

Let X be a quasilinear space. A real-valued function $\|\cdot\|_X : X \to \mathbb{R}$ is called a norm, if the following conditions hold

(a) $\|x\|_X > 0$, if $x \neq \theta$;

(b) $\|x + y\|_X \leq \|x\|_X + \|y\|_X$;

(c) $\|\alpha x\|_X = |\alpha| \cdot \|x\|_X$;

(d) if $x \leq y$, then $\|x\|_X \leq \|y\|_X$;

(e) if for any $\varepsilon > 0$ exists such an element $x_\varepsilon \in X$, that $x \leq y + x_\varepsilon$ and $\|x_\varepsilon\| \leq \varepsilon$, then $x \leq y$.

A quasilinear space X endowed with a norm is called a **quasilinear normed space**.

Next let X be quasilinear normed space. Define on X the Hausdorff metric by the following equality

$$h_X(x, y) = \inf\{r \leq 0 : \ x \leq y + a_1^r, \ y \leq x + a_2^r, \ \|a_i^r\|_X \leq r\}.$$

Any quasilinear normed space X with the above-defined Hausdorff metric that is a complete separable metric space is an L-space. For example, the spaces $(\Omega(X), h_\Omega)$ and $(\Omega_{\text{conv}}(X), h_\Omega)$ defined above in example 3 are quasilinear normed spaces and L-spaces.

6. The structure of L-spaces arises naturally in some spaces of mappings with values in L-spaces. For example, let a compact topological space T and L-space X be given. By $C(T, X)$ we denote the space of all continuous mappings $f : T \to X$. Operations of addition and multiplication by real numbers are defined in $C(T, X)$ in the standard way. The metric in $C(T, X)$ is defined by the relation

$$h_{C(T,X)}(f_1, f_2) := \max_{t \in T} h_X(f_1(t), f_2(t))$$

It is clear that $(C(T, X), h_{C(T,X)})$ is an L-space.

3.2 Hukuhara type difference and its properties

As we already mentioned, the notion of the Hukuhara difference of two sets was introduced by Hukuhara in [39]. Here we generalize it as the Hukuhara type difference for elements in L-spaces.

Definition 8 *We say that an element $z \in X$ is the* **Hukuhara type difference** *of elements $x, y \in X$, if $x = y + z$.*

We denote this difference by $z = x \overset{h}{-} y$.

The following properties of Hukuhara difference of sets are known (see for ex. [58]). We will show similar properties for a general case of Hukuhara type diference in L-spaces.

Suppose X consists only of convex elements. Let $x, y, u, v \in X$. The Hukuhara type difference has the following properties

1. If $x \overset{h}{-} u$ and $y \overset{h}{-} v$ exist, then $(x+y) \overset{h}{-} (u+v)$ exists and the following equality holds

$$(x+y) \overset{h}{-} (u+v) = \left(x \overset{h}{-} u\right) + \left(y \overset{h}{-} v\right). \tag{3.1}$$

   In particular,

$$(x+y) \overset{h}{-} x = y. \tag{3.2}$$

   Proof. Let $x \overset{h}{-} u = a$. That implies that $x = u + a$. Let also $y \overset{h}{-} v = b$ and then $y = v + b$. Therefore,

$$x + y = u + a + v + b \ \text{ or } \ x + y = u + v + a + b.$$

   Thus,

$$(x+y) \overset{h}{-} (u+v) = a + b$$

   or

$$(x+y) \overset{h}{-} (u+v) = \left(x \overset{h}{-} u\right) + \left(y \overset{h}{-} v\right). \quad \square$$

2. If the Hukuhara type difference $x \overset{h}{-} y$ exists it is unique.

   Proof. Let $x \overset{h}{-} y = z_1$ and let also $x \overset{h}{-} y = z_2$. Then $x = y + z_1$ and using Property (3.2) we have

$$(y + z_1) \overset{h}{-} y = z_2$$

$$z_1 = z_2. \quad \square$$

3. If $x \overset{h}{-} y$, $v \overset{h}{-} u$, and $y \overset{h}{-} v$ exist, then

$$\left(x \overset{h}{-} u\right) \overset{h}{-} \left(y \overset{h}{-} v\right) = \left(x \overset{h}{-} y\right) + \left(v \overset{h}{-} u\right). \tag{3.3}$$

Proof. Let $x \overset{h}{-} y = a$. This implies that $x = y + a$. Let also $v \overset{h}{-} u = b$ which implies that $v = u + b$. Moreover, let $y \overset{h}{-} v = c$, then $y = v + c$. Therefore, $x = v + c + a$. According to property (3.1)

$$x \overset{h}{-} u = \left((v + c + a) \overset{h}{-} u\right) = \left(v \overset{h}{-} u\right) + c + a.$$

Since $y \overset{h}{-} v = c$ and using property (3.1) again we have

$$\left(x \overset{h}{-} u\right) \overset{h}{-} \left(y \overset{h}{-} v\right) = \left(\left(v \overset{h}{-} u\right) + c + a\right) \overset{h}{-} c$$

$$= \left(v \overset{h}{-} u\right) + a = \left(v \overset{h}{-} u\right) + \left(x \overset{h}{-} y\right). \quad \square$$

4. If $(x + u) \overset{h}{-} v$ exist, then

$$x + \left(u \overset{h}{-} v\right) = (x + u) \overset{h}{-} v \tag{3.4}$$

Proof. Let $u \overset{h}{-} v = c$, then $u = v + c$. And therefore, due to the property (3.2)

$$(x + u) \overset{h}{-} v = (x + v + c) \overset{h}{-} v$$

$$= (v + x + c) \overset{h}{-} v = x + c = x + \left(u \overset{h}{-} v\right). \quad \square$$

5. If $y \overset{h}{-} v$ and $\left(x \overset{h}{-} u\right) \overset{h}{-} \left(y \overset{h}{-} v\right)$ exist, then

$$\left(x \overset{h}{-} u\right) \overset{h}{-} \left(y \overset{h}{-} v\right) = \left(x \overset{h}{-} y\right) \overset{h}{-} \left(u \overset{h}{-} v\right). \tag{3.5}$$

Proof. Let $\left(x \overset{h}{-} y\right) \overset{h}{-} \left(u \overset{h}{-} v\right) = a$. Let also $y \overset{h}{-} v = b$, and thus $y = v + b$. Then

$$x \overset{h}{-} y = a + \left(u \overset{h}{-} v\right),$$

$$x \overset{h}{-} (b + v) = a + \left(u \overset{h}{-} v\right),$$

$$x = b + v + a + \left(u \overset{h}{-} v\right).$$

Thus, due to the properties (3.2) and (3.4), we have

$$\left(x \overset{h}{-} u\right) \overset{h}{-} \left(y \overset{h}{-} v\right) = \left(\left(b + v + a + \left(u \overset{h}{-} v\right)\right) \overset{h}{-} u\right) \overset{h}{-} b$$

$$= \left(v + a + \left(u \overset{h}{-} v\right)\right) \overset{h}{-} u = \left((v + a + u) \overset{h}{-} v\right) \overset{h}{-} u$$

$$= (a + u) \overset{h}{-} u = a. \quad \square$$

6. If $x \overset{h}{-} u$, $x \overset{h}{-} v$ and $v \overset{h}{-} u$ exist, then

$$x \overset{h}{-} u = \left(x \overset{h}{-} v\right) + \left(v \overset{h}{-} u\right) \tag{3.6}$$

Proof. Using property (3.3), we have

$$\left(x \overset{h}{-} v\right) + \left(v \overset{h}{-} u\right) = \left(x \overset{h}{-} u\right) \overset{h}{-} \left(v \overset{h}{-} v\right) = x \overset{h}{-} u. \ \square$$

7. If $x \overset{h}{-} y$ exist, then for any number α

$$\alpha x \overset{h}{-} \alpha y = \alpha \left(x \overset{h}{-} y\right). \tag{3.7}$$

Proof. Let $x \overset{h}{-} y = z$, then $x = z + y$ and $\alpha x = \alpha z + \alpha y$. $\square$

8. If $x \overset{h}{-} y$ exists, then

$$\delta\,(x, y) \leq \delta \left(x \overset{h}{-} y, \theta\right). \tag{3.8}$$

Moreover, if space X is such that for any $x, y, z \in X$

$$\delta\,(x + z, y + z) = \delta\,(x, y)\,, \tag{3.9}$$

then

$$\delta\,(x, y) = \delta \left(x \overset{h}{-} y, \theta\right). \tag{3.10}$$

Proof. Let $x \overset{h}{-} y = a$, and therefore $x = y + a$. Then using Axiom 8, we have

$$\delta\,(x, y) = \delta(x + a, x) \leq \delta\,(a, \theta)\,.$$

Thus, (3.8) is proved. In order to prove (3.10), we need to show that

$$\delta \left(x \overset{h}{-} y, \theta\right) \leq \delta\,(x, y)\,.$$

We have using (3.9) and obvious equality $\left(x \overset{h}{-} y\right) + y = x$ that

$$\delta \left(x \overset{h}{-} y, \theta\right) = \delta \left(\left(x \overset{h}{-} y\right) + y, y\right) = \delta\,(x, y)\,. \quad \square$$

9. If $x \overset{h}{-} u$, $y \overset{h}{-} v$ exist and property (3.9) holds, then

$$\delta\left(x \overset{h}{-} u, y \overset{h}{-} v\right) \leq \delta\left(x, y\right) + \delta\left(u, v\right). \tag{3.11}$$

Proof. We have due to the properties (3.9) and (3.4) and Axiom 8

$$\delta\left(x \overset{h}{-} u, y \overset{h}{-} v\right) = \delta\left(\left(x \overset{h}{-} u\right) + u, \left(y \overset{h}{-} v\right) + u\right)$$

$$= \delta\left(x, \left(y \overset{h}{-} v\right) + u\right) = \delta\left(x, (y+u) \overset{h}{-} v\right)$$

$$= \delta\left(x + v, \left((y+u) \overset{h}{-} v\right) + v\right)$$

$$= \delta(x+v, y+u) \leq \delta(x, y) + \delta(v, u). \quad \square$$

3.3 Limits of functions in L-spaces

1. Suppose the functions $f(t)$ and $g(t)$ have limits when $t \to t_0$. Moreover $\lim\limits_{t\to t_0} f(t) = x$, $\lim\limits_{t\to t_0} g(t) = y$. Then

$$\lim_{t\to t_0} (f(t) + g(t)) \to x + y.$$

Proof. We have

$$\delta(f(u) + g(u), x + y) \leq \delta(f(u), x) + \delta(g(u), y).$$

By assumption, both terms go to zero when $t \to t_0$ and this proves what we need. $\square$

2. Let two functions be given $f : [a, b] \to X$ and $g : [a, b] \to \mathbb{R}$. If each of these functions has a limit when $t \to t_0$, and moreover $\lim\limits_{t\to t_0} f(t) = x$, $\lim\limits_{t\to t_0} g(t) = c$. Then

$$\lim_{t\to t_0} f(t)g(t) = cx.$$

Proof. We have

$$\delta(f(t)g(t), xc) \leq \delta(f(t)g(t), xg(t)) + \delta(xg(t), xc)$$

$$\leq |g(t)|\delta(f(t), x) + |g(t) - c|\delta(x, \theta).$$

Both terms in the right-hand side of inequality go to zero, when $t \to t_0$. $\square$

3. If for any t close enough to t_0 there exists $f(t) \overset{h}{-} g(t)$ and

$$\lim_{t\to t_0} f(t) = a, \quad \lim_{t\to t_0} g(t) = b,$$

then $\lim_{t\to t_0}\left(f(t) \overset{h}{-} g(t)\right)$ exists and

$$\lim_{t\to t_0}\left(f(t) \overset{h}{-} g(t)\right) = a \overset{h}{-} b. \tag{3.12}$$

Proof.

$$\delta\left(f(t') \overset{h}{-} g(t'), f(t'') \overset{h}{-} g(t'')\right) \le \delta\left(f(t'), f(t'')\right) + \delta\left(g(t'), g(t'')\right).$$

Therefore, due to existence of limits of $f(t)$ and $g(t)$ for $t \to t_0$, for t' and t'' close enough to t_0, the right side is less than ε (ε - arbitrary, fixed, positive number). Here we use the Cauchy criterion. Then for $f(t) \overset{h}{-} g(t)$, the conditions of the Cauchy criterion for the existence of a limit as $t \to t_0$ hold. Now we show that $\lim_{t\to t_0}\left(f(t) \overset{h}{-} g(t)\right) = a \overset{h}{-} b$. Let $\Delta(t) = f(t) \overset{h}{-} g(t)$. Then

$$f(t) = g(t) + \Delta(t) \tag{3.13}$$

For $t \to t_0$, we have $f(t) \to a$, $g(t) \to b$ and let c be a limit of $\Delta(t)$ when $t \to t_0$ (its existence we just proved). Taking the limit when $t \to t_0$ in (3.13), we have $a = b + c$, and therefore $c = a \overset{h}{-} b$. □

3.4 Hukuhara type derivative and its properties

In this section, we define the Hukuhara type derivative of a function with values in an L-space $f : [a, b] \to X$, and prove necessary properties connected with this notion.

Definition 9 *If $t \in (a, b)$, and for all small enough $h > 0$ there exist differences*

$$f(t+h) \overset{h}{-} f(t) \;\; \textit{and} \;\; f(t) \overset{h}{-} f(t-h),$$

and both limits exist and are equal to each other

$$\lim_{h\to 0^+} \frac{f(t+h) \overset{h}{-} f(t)}{h} = \lim_{h\to 0^+} \frac{f(t) \overset{h}{-} f(t-h)}{h}$$

then the function has a Hukuhara type derivative $D_H f(t)$ at the point t (if $t = a$ or $t = b$, then there exists only one limit) and

$$D_H f(t) = \lim_{h\to 0^+} \frac{f(t+h) \overset{h}{-} f(t)}{h}.$$

3.4.1 Derivative of a sum

Lemma 1 *If we have functions $f(t)$ and $g(t)$ with values in L-spaces that have Hukuhara type derivatives, then the derivative of a sum exists and*

$$D_H\left(f(t)+g(t)\right)=D_Hf(t)+D_Hg(t). \tag{3.14}$$

Proof. By definition, the following two limits have to exist and be equal to each other and the derivative

$$D_H\left(f(t)+g(t)\right)=\lim_{h\to 0+}\frac{\left(f(t+h)+g(t+h)\right)\overset{h}{-}\left(f(t)+g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)+g(t)\right)\overset{h}{-}\left(f(t-h)+g(t-h)\right)}{h}.$$

According to the property (3.1), we can write

$$\lim_{h\to 0+}\frac{\left(f(t+h)+g(t+h)\right)\overset{h}{-}\left(f(t)+g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}f(t)\right)+\left(g(t+h)\overset{h}{-}g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}f(t)\right)}{h}+\lim_{h\to 0+}\frac{\left(g(t+h)\overset{h}{-}g(t)\right)}{h}.$$

Similarly, we have

$$\lim_{h\to 0+}\frac{\left(f(t)+g(t)\right)\overset{h}{-}\left(f(t-h)+g(t-h)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}f(t-h)\right)+\left(g(t)\overset{h}{-}g(t-h)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}f(t-h)\right)}{h}+\lim_{h\to 0+}\frac{\left(g(t)\overset{h}{-}g(t-h)\right)}{h}.$$

Therefore, the property (3.14) holds. □

3.4.2 Derivative of Hukuhara type difference

Lemma 2 *If we have functions $f(t)$ and $g(t)$ with values in L-spaces that have Hukuhara type derivatives, and their difference $f(t)\overset{h}{-}g(t)$ has a Hukuhara type derivative, and difference of the derivatives exists, then*

$$D_H\left(f(t)\overset{h}{-}g(t)\right)=D_Hf(t)\overset{h}{-}D_Hg(t). \tag{3.15}$$

Proof. By definition, the following two limits have to exist and be equal to each other and the derivative

$$D_H\left(f(t)\overset{h}{-}g(t)\right)=\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}g(t+h)\right)\overset{h}{-}\left(f(t)\overset{h}{-}g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}g(t)\right)\overset{h}{-}\left(f(t-h)\overset{h}{-}g(t-h)\right)}{h}.$$

According to the properties (3.5) and (3.12), we can write

$$\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}g(t+h)\right)\overset{h}{-}\left(f(t)\overset{h}{-}g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}f(t)\right)\overset{h}{-}\left(g(t+h)\overset{h}{-}g(t)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t+h)\overset{h}{-}f(t)\right)}{h}\overset{h}{-}\lim_{h\to 0+}\frac{\left(g(t+h)\overset{h}{-}g(t)\right)}{h}.$$

Similarly, we have

$$\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}g(t)\right)\overset{h}{-}\left(f(t-h)\overset{h}{-}g(t-h)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}f(t-h)\right)\overset{h}{-}\left(g(t)\overset{h}{-}g(t-h)\right)}{h}$$

$$=\lim_{h\to 0+}\frac{\left(f(t)\overset{h}{-}f(t-h)\right)}{h}\overset{h}{-}\lim_{h\to 0+}\frac{\left(g(t)\overset{h}{-}g(t-h)\right)}{h}.$$

Therefore, the property (3.15) holds. $\square$

3.4.3 The chain rule

Lemma 3 *Let $f:[a,b]\to X$ have a Hukuhara type derivative $D_H f$ on the interval $[a,b]$. Let also $\varphi:[c,d]\to[a,b]$ be a strictly increasing real-valued function, differentiable at any point on $[c,d]$. Then $f\circ\varphi$ has a Hukuhara type derivative D_H at every point on $[c,d]$ and*

$$D_H(f\circ\varphi)(t)=D_H f(\varphi(t))\varphi'(t). \tag{3.16}$$

Proof. Consider

$$\lim_{h\to 0+}\frac{f(\varphi(t+h))-f(\varphi(t))}{h}=\lim_{h\to 0+}\left(\frac{f(\varphi(t+h))\overset{h}{-}f(\varphi(t))}{\varphi(t+h)-\varphi(t)}\cdot\frac{\varphi(t+h)-\varphi(t)}{h}\right).$$

Next,

$$\lim_{h\to 0+}\frac{f(\varphi(t+h))\overset{h}{-}f(\varphi(t))}{\varphi(t+h)-\varphi(t)}\cdot\frac{\varphi(t+h)-\varphi(t)}{h}$$
$$=\lim_{h\to 0+}\left(\frac{f(\varphi(t+h))\overset{h}{-}f(\varphi(t))}{\varphi(t+h)-\varphi(t)}\right)\cdot\lim_{h\to 0+}\left(\frac{\varphi(t+h)-\varphi(t)}{h}\right)=D_Hf(\varphi(t))\varphi'(t).$$

Similarly, we can show that

$$\lim_{h\to 0+}\frac{f(\varphi(t))-f(\varphi(t-h))}{h}$$
$$=\lim_{h\to 0+}\left(\frac{f(\varphi(t))\overset{h}{-}f(\varphi(t-h))}{\varphi(t)-\varphi(t-h)}\right)\cdot\lim_{h\to 0+}\left(\frac{\varphi(t)-\varphi(t-h)}{h}\right)$$
$$=D_Hf(\varphi(t))\varphi'(t)$$

And therefore,

$$D_H(f\circ\varphi)(t)=D_Hf(\varphi(t))\varphi'(t).\ \square$$

3.4.4 Derivative of a product of a real-valued function and function with values in an L-space

Lemma 4 *Let a real-valued function $f(t)$ (differentiable, nonnegative, and nondecreasing) and a differentiable in Hukuhara sense function $F(t)$ with convex values in L-space be given. Then the following formula for the derivative of such product holds*

$$D_H(fF)(t)=f'(t)F(t)+f(t)D_HF(t).\tag{3.17}$$

Proof. Using property (3.6) of the Hukuhara type difference, we can write

$$f(t+h)F(t+h)\overset{h}{-}f(t)F(t)$$
$$=\left(f(t+h)F(t+h)\overset{h}{-}f(t)F(t+h)\right)+\left(f(t)F(t+h)\overset{h}{-}f(t)F(t)\right)$$

Since we assumed function f to be nondecreasing

$$f(t+h)F(t+h)\overset{h}{-}f(t)F(t+h)=(f(t+h)-f(t))F(t+h).$$

Moreover,

$$\lim_{h\to 0+}\frac{f(t+h)F(t+h)\overset{h}{-}f(t)F(t+h)}{h}=\lim_{h\to 0+}\frac{f(t+h)-f(t)}{h}F(t+h)=f'(t)F(t).$$

Since $F(t+h) \overset{h}{-} F(t)$ is defined, and due to the property (3.7) of the Hukuhara type difference, we have

$$f(t)F(t+h) \overset{h}{-} f(t)F(t) = f(t)\left(F(t+h) \overset{h}{-} F(t)\right).$$

Therefore,

$$\lim_{h\to 0+} \frac{f(t)F(t+h) \overset{h}{-} f(t)F(t)}{h} = \lim_{h\to 0+} f(t)\frac{F(t+h) \overset{h}{-} F(t)}{h} = f(t)D_H F(t).$$

Similarly, we can show that

$$f(t)F(t) \overset{h}{-} f(t-h)F(t-h)$$

$$= \left(f(t)F(t) \overset{h}{-} f(t-h)F(t)\right) + \left(f(t-h)F(t) \overset{h}{-} f(t-h)F(t-h)\right)$$

and

$$\lim_{h\to 0+} \frac{f(t)F(t) \overset{h}{-} f(t-h)F(t)}{h} = \lim_{h\to 0+} \frac{f(t)-f(t-h)}{h}F(t) = f'(t)F(t).$$

Finally,

$$\lim_{h\to 0+} \frac{f(t-h)F(t) \overset{h}{-} f(t-h)F(t-h)}{h}$$

$$= \lim_{h\to 0+} f(t-h)\frac{F(t) \overset{h}{-} F(t-h)}{h} = f(t)D_H F(t).$$

Thus, we have proved the desired formula for the derivative of a product. □

3.5 Integral of functions with values in L-spaces

We start with the following notion of convex elements in L-spaces:

Definition 10 *An element $x \in X$ is* **convex** *if*

$$\lambda x + \mu x = (\lambda + \mu)x \ \ \forall\ \lambda, \mu \geq 0.$$

Remark 1 *Note that if the element x is convex, then it follows from Axioms 7 and 8 that $\forall \lambda, \mu \in \mathbb{R}$*

$$\delta(\lambda x, \mu x) \leq |\lambda - \mu|\delta(x, \theta). \tag{3.18}$$

We denote as X^c the set of all convex elements of a given L-space X.

Remark 2 *Note that X^c is a closed subset of X.*

We need the definition of a convexifying operator (see [65]) which we give in a somewhat modified form.

Definition 11 *Let X be an L-space. The operator $P: X \to X^c$ is called a* **convexifying operator** *if*

1. $\forall x, y \in X \ \delta(P(x), P(y)) \leq \delta(x, y)$;

2. $P \circ P = P$;

3. $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y), \ \forall x, y \in X, \ \alpha, \beta \in \mathbb{R}$.

Examples of convexifying operators:

1. The identity operator in the space $\mathcal{K}^c(\mathbb{R}^n)$ is a convexifying operator.

2. The operator, that on the space $\mathcal{K}(\mathbb{R}^n)$ is defined by the formula $P(A) = co\ (A)$, is a convexifying operator. Here by the $co\ (A)$, we denote the convex hull of a set A.

3. The identity operator in the space $(\mathcal{E}^n, d_p)$ is a convexifying operator.

Below we discuss an L-space X with some fixed convexifying operator P. For any $x \in X$, we will use the notation $\widetilde{x} = Px$. If all elements of an L-space X are convex in the sense of Definition 10, then we choose the identity operator as the convexifying operator.

Next we define the Riemannian integral for a function $f: [a, b] \to X$, where X is an L-space. We again follow Vahrameev [65] for this purpose. Let $\widetilde{f}(t) := \widetilde{f(t)}$.

Definition 12 *The mapping $f: [a, b] \to X$ is called* **weakly bounded**, *if*

$$\delta(\theta, \widetilde{f}(t)) \leq const$$

and **weakly continuous**, *if $\widetilde{f}: [a, b] \to X$ is continuous.*

Remark 3 *Note that if a function $f: [a, b] \to X$ is continuous, then f is weakly continuous. If all elements $x \in X$ are convex (i.e., $P = Id$), then the concepts of continuity and weak continuity coincide.*

Next we introduce the notion of a stepwise mapping from $[a, b]$ to an L-space X.

Definition 13 *The mapping $f: [a, b] \to X$ is called* **stepwise**, *if there exists a set $\{x_k\}_{k=1}^n \subset X$ and a partition $a = t_0 < t_1 < ... < t_n = b$ of the interval $[a, b]$, such that $\widetilde{f}(t) = \widetilde{x}_k$ for $t_{k-1} < t < t_k$.*

Definition 14 *The Riemannian integral of a stepwise mapping $f: [a,b] \to X$ is defined as*

$$\int_a^b f(t)dt = \sum_{k=1}^{n} (t_k - t_{k-1})\widetilde{x}_k.$$

Definition 15 *We say that a weakly bounded mapping $f: [a,b] \to X$ is integrable in the Riemannian sense if there exists a sequence $\{f_k\}$ of stepwise mappings from $[a,b]$ to X, such that*

$$\int^{*} \delta\left(\widetilde{f}(t), \widetilde{f}_k(t)\right) dt \to 0, \quad \text{as} \ \ k \to \infty, \tag{3.19}$$

where $\int^*$ is a regular Riemannian integral for real-valued functions.

It follows from (3.19) that the sequence $\left\{\int_a^b f_k(t)dt\right\}$ is a Cauchy sequence and thus, we can use the following definition.

Definition 16 *Let $f: [a,b] \to X$ be integrable in the Riemannian sense and let $\{f_k\}$ be a sequence of stepwise mappings such that (3.19) holds. Then the* **Riemannian integral** *of f is the limit*

$$\int_a^b f(t)dt = \lim_{k\to\infty} \int_a^b f_k(t)dt.$$

As described in [65], the Riemannian integral for a function $f : [a,b] \to X$ has the following properties:

1. If f and g are integrable, then for any $\alpha, \beta \in \mathbb{R}$, the linear combination $\alpha f + \beta g$ is integrable, and moreover

$$\int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt.$$

2. If f and g are integrable, then the function $t \to \delta\left(\widetilde{f}(t), \widetilde{g}(t)\right)$ is integrable and

$$\delta\left(\int_a^b f(t)dt, \int_a^b g(t)dt\right) \leq \int_a^b \delta\left(\widetilde{f}(t), \widetilde{g}(t)\right) dt.$$

3. If f is integrable, then $\widetilde{f} = Pf$ is also integrable and

$$\int_a^b f(t)dt = P\int_a^b f(t)dt = \int_a^b \widetilde{f}(t)dt.$$

4. If f is integrable on $[a,b]$, and $a \leq c \leq b$, then function f is integrable on $[a,c]$ and $[c,b]$ and

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

The following theorem (see [65], [6]) guarantees that we can consider the integrals which arise below as Riemannian integrals.

Theorem 1 *A weakly bounded mapping $f: [a,b] \to X$ is integrable in the Riemannian sense if and only if it is weakly continuous almost everywhere on $[a,b]$.*

Remark 4 *Note that for the set-valued case, the above-defined integral coincides with the Aumann integral. If a function $f : [a,b] \to X$ is continuous, then f is weakly continuous. If all elements $x \in X$ are convex (i.e., $P = Id$), then the concepts of continuity and weak continuity coincide.*

The following analog of the fundamental theorem of Calculus holds:

Theorem 2 *For any function $F: [a,b] \to X^c$ that has a continuous Hukuhara type derivative on $[a,b]$, the following equality holds*

$$F(t) = F(a) + \int_a^t D_H F(s)ds, \quad t \in [a,b].$$

Proof. For any continuous function $f: [a,b] \to X^c$ and any $m \in X^c$, we can define

$$F(t) = \int_a^t f(s)ds + m, \quad m \in X^c. \tag{3.20}$$

Next we find $D_H F(t)$. It follows from (3.20) and Property 4 of the integral that for $t \in [a,b)$, and $h > 0$ small enough, the difference $F(t+h) \overset{h}{-} F(t)$ is defined and

$$F(t+h) \overset{h}{-} F(t) = \int_t^{t+h} f(s)ds.$$

We consider

$$\delta\left(\frac{1}{h}\int_t^{t+h} f(s)ds, f(t)\right) = \delta\left(\frac{1}{h}\int_t^{t+h} f(s)ds, \frac{1}{h}\int_t^{t+h} f(t)ds\right)$$
$$\leq \frac{1}{h}\int_t^{t+h} \delta(f(s), f(t))ds.$$

Set $\varepsilon > 0$. Since f is continuous at the point t, there exist $\sigma > 0$, such that for $s \in (t, t+\sigma)$, we have $\delta(f(s), f(t)) < \varepsilon$. This implies that for $h < \sigma$, we have

$$\delta\left(\frac{1}{h}\int_t^{t+h} f(s)ds, f(t)\right) \leq \frac{1}{h}\int_t^{t+h} \varepsilon ds = \varepsilon$$

which proves the equality

$$\lim_{h \to 0^+} \frac{1}{h}\int_t^{t+h} f(s)ds = f(t). \tag{3.21}$$

Similarly,

$$\lim_{h\to 0^+} \frac{F(t) \overset{h}{-} F(t-h)}{h} = \lim_{h\to 0^+} \frac{1}{h} \int_{t-h}^{t} f(s)ds = f(t).$$

Thus, $D_H F(t) = f(t)$.

From (3.20), we have that

$$F(t) = m + \int_a^t D_H F(s)ds = F(a) + \int_a^t D_H F(s)ds. \quad \square$$

We continue with properties of integrals in L-spaces, and next we describe the integration by parts formula for such integrals.

3.5.1 Integration by parts formula

Lemma 5 *Let a real-valued function $f(t)$ (differentiable, nonnegative, and nondecreasing) and a differentiable in the Hukuhara sense function $F(t)$ with convex values in an L-space be given. Then the following integration by parts formula holds*

$$\int_a^b f'(t)F(t)dt = \left(f(b)F(b) \overset{h}{-} f(a)F(a) \right) \overset{h}{-} \int_a^b f(t) D_H F(t)dt. \tag{3.22}$$

Proof. In the previous section, we have proved that under our assumptions, the following formula holds

$$D_H(fF)(t) = f'(t)F(t) + f(t)D_H F(t).$$

Therefore,

$$\int_a^b D_H(fF)(t)dt = \int_a^b f'(t)F(t)dt + \int_a^b f(t)D_H F(t)dt$$

and since,

$$\int_a^b D_H(fF)(t)dt = f(b)F(b) \overset{h}{-} f(a)F(a),$$

the integration by parts formula (3.22) holds. $\square$

3.5.2 Change of variable formula

Lemma 6 *Let $f : [a, b] \to X$ have a Hukuhara type derivative $D_H f$ on the interval $[a, b]$. Let also $\varphi : [c, d] \to [a, b]$ be an increasing real-valued function, differentiable at any point on $[c, d]$. Then*

$$\int_c^d f(\varphi(t))\varphi'(t)dt = \int_a^b f(x)dx. \tag{3.23}$$

Proof. Denote $F(x) = \int_a^x f(u)du$. Let also $x = \varphi(t)$ and $\varphi(c) = a,\ \varphi(d) = b$. Then we have

$$F(\varphi(t)) = \int_a^{\varphi(t)} f(u)du.$$

From the cain rule (3.16), we get

$$D_H(F \circ \varphi)(t) = D_H F(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t).$$

Consequently,

$$\int_c^d f(\varphi(t))\varphi'(t)dt = F(\varphi(t))|_c^d = F(b) \overset{h}{-} F(a) = \int_a^b f(x)dx. \ \square$$

3.5.3 Derivative of an integral with a variable upper limit

Lemma 7 *Let $f \in C([a,b], X^c)$. Then*

$$D_H\left(\int_a^x f(t)dt\right) = f(x), \quad x \in [a,b]. \tag{3.24}$$

Proof. We have

$$D_H\left(\int_a^x f(t)dt\right) = \lim_{h\to 0^+} \frac{1}{h}\left(\int_a^{x+h} f(t)dt \overset{h}{-} \int_a^x f(t)dt\right)$$

$$= \lim_{h\to 0^+} \frac{1}{h}\left(\left(\int_a^x f(t)dt + \int_x^{x+h} f(t)dt\right) \overset{h}{-} \int_a^x f(t)dt\right) = \lim_{h\to 0^+} \frac{1}{h}\int_x^{x+h} f(t)dt = f(x).$$

Similarly, we can show that

$$D_H\left(\int_a^x f(t)dt\right) = \lim_{h\to 0^+} \frac{1}{h}\left(\int_a^x f(t)dt \overset{h}{-} \int_a^{x-h} f(t)dt\right)$$

$$= \lim_{h\to 0^+} \frac{1}{h}\left(\left(\int_a^{x-h} f(t)dt + \int_{x-h}^x f(t)dt\right) \overset{h}{-} \int_a^{x-h} f(t)dt\right)$$

$$= \lim_{h\to 0^+} \frac{1}{h}\int_{x-h}^x f(t)dt = f(x). \ \square$$

CHAPTER 4

APPROXIMATE SOLUTION OF LINEAR INTEGRAL EQUATIONS IN L-SPACES

In this chapter, we consider linear Volterra and Fredholm integral equations for functions with values in L-spaces. This includes corresponding problems for set-valued functions, fuzzy-valued functions, and many others. We prove theorems of existence and uniqueness of the solution for such equations and suggest some algorithms for finding approximate solutions. We get initial results in the approximation of functions with values in L-spaces by piecewise linear functions and we also get error estimates of trapezoidal quadrature formulas. For known results on approximation and quadrature formulas for set-valued and fuzzy-valued functions, we refer the reader to [3], [10], [11], [12], [14], [16], [30], [47], [54], [66] and references therein. We use the results on piecewise linear approximation and error estimates of quadrature formulas on convergence analysis.

4.1 Introduction

Fredholm and Volterra integral equations for single valued functions form a classic subject in pure and applied mathematics. Such integral equations have many important applications in biology (e.g., population dynamics, demography, and infection propagation), mathematical economics, actuarial mathematics, physics (e.g., astronomy and geophysics), and engineering (image and signal processing) (see, for example, [25], [48], [36], [34], and the references therein). At the same time, a wide variety of questions lead to integral equations for functions with values that are compact and convex sets in finite or infinite dimensional spaces, or that are fuzzy sets (see [26], [19], [1], [46], [63], [56], [57], [17], [67]).

Despite the large number of papers devoted to numerical methods for the solution of integral equations, we do not know of any work connected with methods of finding approximate solution of integral equations for functions with values in the space of sets or fuzzy sets of dimensions that are greater than one. One of the purposes of this work is to fill this gap. We show that some existing methods for the solution of real-valued integral equations can be adapted to the solution of integral equations in L-spaces. These are methods that have relatively low order of accuracy (such as a collocation method based on "piecewise linear" interpolation and a quadrature formula method based on the trapezoidal rule). Attempts to adapt methods of higher accuracy to solutions of integral equations in L-spaces meet difficulties, because approximation theory and quadrature formulas theory for functions with values in L-spaces are not sufficiently developed yet.

4.2 Piecewise linear approximation and errors of quadrature formulas

As usual, denote by $C[a,b]$ the space of continuous functions $f: [a,b] \to \mathbb{R}$ with the norm $||f||_{C[a,b]} = \max\{|f(t)|: t \in [a,b]\}$. Let X be an L-space. Denote by $C([a,b],X)$ the set of all continuous functions $\varphi : [a,b] \longrightarrow X$. This set endowed with the metric

$$\rho(\varphi,\psi) = \max_{t\in[a,b]} \delta(\varphi(t),\psi(t)) = \|\delta(\varphi(t),\psi(t))\|_{C([a,b])}$$

is a complete metric space (see, e.g., [6]).

For a function $f \in C([a,b],X)$ we define the modulus of continuity by

$$\omega(f,t) = \sup_{\substack{t',t''\in[a,b] \\ |t'-t''|\le t}} \delta(f(t'),f(t'')), \quad t \in [0,b-a].$$

Note that $\omega(f,t) \to 0$, as $t \to 0$.

If

$$\omega(f,t) \le Mt \tag{4.1}$$

then we say that the function f satisfies the Lipschitz condition with constant M.

Denote by $\omega^*(f,t)$ the least concave majorant of the function $\omega(f,t)$. It is well known (see [43]) that the following inequalities hold

$$\omega(f,t) \le \omega^*(f,t) \le 2\omega(f,t).$$

It will be convenient for us to give an estimation of approximation in terms of a function $\omega^*(f,t)$.

4.2.1 Piecewise-linear interpolation

Define an operator P_N that assigns to a function $f \in C([a,b],X)$ the function

$$P_N[f](t) = \sum_{k=0}^{N} f(t_k) l_k(t), \tag{4.2}$$

where $t_k = a + k\frac{b-a}{N}$, $k = 0,1,...,N$, and

$$l_0(t) := \begin{cases} (t_1-t)/(t_1-t_0) & \text{if} \quad t\in[t_0,t_1] \\ 0 & \text{else,} \end{cases}$$

$$l_k(t) = \begin{cases} (t-t_{k-1})/(t_k-t_{k-1}) & \text{if} \quad t \in [t_{k-1}, t_k] \\ (t_{k+1}-t)/(t_{k+1}-t_k) & \text{if} \quad t \in [t_k, t_{k+1}] \\ 0 & \quad\text{else}, \end{cases} \tag{4.3}$$

$$l_N(t) := \begin{cases} (t-t_{N-1})/(t_N-t_{N-1}) & \text{if} \quad t \in [t_{N-1}, t_N] \\ 0 & \text{else}. \end{cases}$$

Theorem 3 *If $f \in C([a,b], X^c)$, then*

$$\rho(f, P_N[f]) \le \omega\left(f, \frac{b-a}{N}\right). \tag{4.4}$$

Moreover, for $t \in [t_{k-1}, t_k]$, $k = 1, 2, ..., N$

$$\delta(f(t), P_N[f](t)) \le \omega^*\left(f, 2\frac{(t-t_{k-1})(t_k-t)}{t_k-t_{k-1}}\right), \tag{4.5}$$

and therefore,

$$\rho(f, P_N[f]) \le \omega^*\left(f, \frac{b-a}{2N}\right). \tag{4.6}$$

If $\omega(f,t) \le Mt$, $t \ge 0$, then

$$\rho(f, P_N[f]) \le M\frac{b-a}{2N}. \tag{4.7}$$

Proof. For $t \in [t_{k-1}, t_k]$,

$$P_N[f](t) = \frac{t_k-t}{t_k-t_{k-1}}f(t_{k-1}) + \frac{t-t_{k-1}}{t_k-t_{k-1}}f(t_k).$$

Consequently, using Axiom 8 and Axiom 7 of an L-space, we have

$$\begin{aligned} &\delta(f(t), P_N[f](t)) \\ &= \delta\left(\frac{t_k-t}{t_k-t_{k-1}}f(t) + \frac{t-t_{k-1}}{t_k-t_{k-1}}f(t), \frac{t_k-t}{t_k-t_{k-1}}f(t_{k-1}) + \frac{t-t_{k-1}}{t_k-t_{k-1}}f(t_k)\right) \\ &\le \frac{t_k-t}{t_k-t_{k-1}}\delta(f(t), f(t_{k-1})) + \frac{t-t_{k-1}}{t_k-t_{k-1}}\delta(f(t), f(t_k)) \\ &\le \frac{t_k-t}{t_k-t_{k-1}}\omega(f, t-t_{k-1}) + \frac{t-t_{k-1}}{t_k-t_{k-1}}\omega(f, t_k-t) \le \omega\left(f, \frac{b-a}{n}\right). \end{aligned} \tag{4.8}$$

Therefore,

$$\delta(f(t), P_N[f](t)) \le \omega\left(f, \frac{b-a}{N}\right)$$

and

$$\rho(f, P_N[f]) \le \omega\left(f, \frac{b-a}{N}\right).$$

The inequality (4.4) is proved. Using (4.8) and applying Jensen's inequality, we have

$$\begin{aligned} \delta(f(t), P_N[f](t)) &\le \frac{t_k-t}{t_k-t_{k-1}}\omega^*(f, t-t_{k-1}) + \frac{t-t_{k-1}}{t_k-t_{k-1}}\omega^*(f, t_k-t) \\ &\le \omega^*\left(f, 2\frac{(t_k-t)(t-t_{k-1})}{t_k-t_{k-1}}\right). \end{aligned}$$

The inequality (4.5) is proved. Inequalities (4.6) and (4.7) are now obvious. □

The following theorem gives an estimate of the error of approximation by piecewise-linear functions for such f that $D_H f(t) \in C([a,b], X)$.

Theorem 4 *Suppose the function $f : [a,b] \to X^c$ has the Hukuhara type derivative $D_H f(t)$ on the interval $[a,b]$. Then if $D_H f \in C([a,b], X^c)$, we have for any $k = 1, 2, ..., N$ and any $t \in [t_{k-1}, t_k]$*

$$\delta(f(t), P_N[f](t)) \leq 2\frac{(t_k - t)(t - t_{k-1})}{(t_k - t_{k-1})^2} \int_0^{\frac{b-a}{2N}} \omega(D_H f, 2u)du, \tag{4.9}$$

and

$$\rho(f, P_N[f]) \leq \frac{1}{2} \int_0^{\frac{b-a}{2N}} \omega(D_H f, 2u)du. \tag{4.10}$$

In particular, if $D_H f$ satisfies the Lipschitz condition with constant M, then

$$\rho\left(f, P_N[f]\right) \leq \frac{M(b-a)^2}{8N^2}. \tag{4.11}$$

Proof. For $t \in [t_{k-1}, t_k]$ using Theorem 2, we have

$$\delta(f(t), P_N[f](t)) = \delta\left(\frac{t_k - t}{t_k - t_{k-1}}\left(f(t_{k-1}) + \int_{t_{k-1}}^{t} D_H f(u)du\right) + \frac{t - t_{k-1}}{t_k - t_{k-1}} f(t),\right.$$

$$\left.\frac{t_k - t}{t_k - t_{k-1}} f(t_{k-1}) + \frac{t - t_{k-1}}{t_k - t_{k-1}}\left(f(t) + \int_t^{t_k} D_H f(u)du\right)\right)$$

$$\leq \delta\left(\frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} D_H f(u)du, \frac{t - t_{k-1}}{t_k - t_{k-1}} \int_t^{t_k} D_H f(v)dv\right)$$

$$= \delta\left(\frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} D_H f(u)du, \frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} D_H f\left(\frac{t - t_k}{t - t_{k-1}} u + \frac{t_k - t_{k-1}}{t - t_{k-1}} t\right) du\right)$$

$$\leq \frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} \delta\left(D_H f(u), D_H f\left(\frac{t - t_k}{t - t_{k-1}} u + \frac{t_k - t_{k-1}}{t - t_{k-1}} t\right)\right) du$$

$$\leq \frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} \omega\left(D_H f, \frac{t - t_k}{t - t_{k-1}} u + \frac{t_k - t_{k-1}}{t - t_{k-1}} t - u\right) du$$

$$= \frac{t_k - t}{t_k - t_{k-1}} \int_{t_{k-1}}^{t} \omega\left(D_H f, \frac{(t_k - t_{k-1})}{t - t_{k-1}}(t - u)\right) du$$

$$= 2\frac{(t_k - t)(t - t_{k-1})}{(t_k - t_{k-1})^2} \int_0^{\frac{b-a}{2N}} \omega(D_H f, 2u)du.$$

Inequality (4.9) is proved. Inequality (4.10) holds since $\frac{(t_k - t)(t - t_{k-1})}{(t_k - t_{k-1})^2} \leq \frac{1}{4}$. Inequality (4.11) now is obvious. $\square$

4.2.2 Estimation of the remainder of a trapezoidal quadrature formula

Next we obtain an estimation of the remainder of the trapezoidal quadrature formula

$$\int_a^b f(t)dt \approx \frac{b-a}{N}\left(\frac{1}{2}\widetilde{f}(t_0) + \sum_{k=1}^{N-1}\widetilde{f}(t_k) + \frac{1}{2}\widetilde{f}(t_N)\right), \;\; t_k = a + k\frac{b-a}{N}, \;\; k = 0, ..., N.$$

Such estimates are well known for real-valued functions (see, for example, [21, ch. 4]).

For $f \in C([a,b], X)$ set

$$R_N(f) = \delta\left(\int_a^b f(t)dt, \frac{b-a}{N}\left(\frac{1}{2}\widetilde{f}(t_0) + \sum_{k=1}^{N-1}\widetilde{f}(t_k) + \frac{1}{2}\widetilde{f}(t_N)\right)\right). \tag{4.12}$$

Note that

$$\int_a^b P_N[f](t)dt = \int_a^b P_N[\widetilde{f}](t)dt = \frac{b-a}{N}\left(\frac{1}{2}\widetilde{f}(t_0) + \sum_{k=1}^{N-1}\widetilde{f}(t_k) + \frac{1}{2}\widetilde{f}(t_N)\right).$$

Therefore,

$$R_N(f) = \delta\left(\int_a^b \widetilde{f}(t)dt, \int_a^b P_N[\widetilde{f}](t)dt\right) \leq \int_a^b \delta(\widetilde{f}(t), P_N[\widetilde{f}](t))dt$$

$$\leq \max_{a\leq t\leq b} \delta(\widetilde{f}(t), P_N[\widetilde{f}](t))(b-a) = (b-a)\rho(\widetilde{f}, P_N[\widetilde{f}]).$$

Note also that due to the Property 1 of a convexifying operator, we have that for any function $f \in C([a,b], X)$:

$$\omega(\widetilde{f}, t) \leq \omega(f, t), \;\; t \in [0, b-a].$$

Therefore, from Theorems 3 and 4, we have

Theorem 5 *Let $f \in C([a,b], X)$. Then*

$$R_N(f) \leq (b-a)\omega\left(f, \frac{b-a}{N}\right).$$

If f satisfies the Lipschitz condition (4.1) with constant M, then

$$R_N(f) \leq \frac{M(b-a)^2}{N}.$$

If function f is such that $\widetilde{f}$ has continuous derivative $D_H\widetilde{f}$ on $[a,b]$, then

$$R_N(f) \leq \frac{b-a}{2}\int_0^{\frac{b-a}{2N}} \omega(D_H\widetilde{f}, 2u)du.$$

In particular, if $D_H\widetilde{f}$ satisfies the condition of (4.1), then

$$R_N(f) \leq \frac{M(b-a)^3}{8N^2}.$$

4.3 Existence and uniqueness of the solution of linear integral equations

In this section, we prove theorems of existence and uniqueness of the solution of linear Fredholm and Volterra integral equations for functions with values in L-spaces. We show that the well-known methods of proving theorems of existence and uniqueness of solutions of integral equations for numerical functions can be adapted to the case of functions with values in L-spaces. There also exist many works on the theory of these integral equations for set-valued and fuzzy-valued functions (see, for example, [63], [56], [57], [26] and references therein).

4.3.1 Fredholm equation

We consider the Fredholm equation of the second kind

$$\varphi(t) = \lambda \int_a^b K(t,s)\varphi(s)ds + f(t), \tag{4.13}$$

where $\varphi : [a,b] \to X$ is the unknown function, $f : [a,b] \to X$ is a known continuous function, the kernel $K(t,s)$ $(t,s \in [a,b])$ is a known real-valued function, λ is a fixed parameter.

Theorem 6 *Let $f \in C([a,b],X)$. Suppose the kernel $K(t,s)$ of the equation (4.13) satisfies the following conditions*

1. *$K(t,s)$ is bounded, i.e., $|K(t,s)| \leq M$ for all $t,s \in [a,b]$;*
2. *$K(t,s)$ is continuous at every point $(t,s) \in [a,b] \times [a,b]$ where $t \neq s$.*

Then for any λ such that $|\lambda| < \frac{1}{M(b-a)}$, the equation (4.13) has a unique solution $\varphi \in C([a,b],X)$.

Proof. Consider the operator $A : C([a,b],X) \to C([a,b],X)$ defined by

$$A\varphi(t) = \lambda \int_a^b K(t,s)\varphi(s)ds + f(t), \quad \varphi \in C([a,b],X).$$

We prove first that under the conditions on the $K(t,s)$, this operator maps $C([a,b],X)$ into $C([a,b],X)$. For this, it is enough to show that the operator

$$B\varphi(t) := \int_a^b K(t,s)\varphi(s)ds$$

maps $C([a,b],X)$ into $C([a,b],X)$. In order to do it, consider $\delta(B\varphi(t'), B\varphi(t''))$, where $a \leq t' < t'' \leq b$. We have

$$\delta(B\varphi(t'), B\varphi(t'')) = \delta\left(\int_a^b K(t',s)\varphi(s)ds, \int_a^b K(t'',s)\varphi(s)ds\right)$$
$$= \delta\left(\int_a^b K(t',s)\widetilde{\varphi}(s)ds, \int_a^b K(t'',s)\widetilde{\varphi}(s)ds\right)$$
$$\leq \int_a^b \delta(K(t',s)\widetilde{\varphi}(s), K(t'',s)\widetilde{\varphi}(s))ds.$$

From the above inequality, using property (3.18), we obtain

$$\delta(B\varphi(t'), B\varphi(t'')) \leq \int_a^b |K(t',s) - K(t'',s)|\delta(\widetilde{\varphi}(s), \theta)ds$$
$$\leq \max_{s\in[a,b]} \delta(\widetilde{\varphi}(s), \theta) \int_a^b |K(t',s) - K(t'',s)|ds$$
$$= \rho(\widetilde{\varphi}(\cdot), \theta) \int_a^b |K(t',s) - K(t'',s)|ds.$$

If $K(t,s)$ satisfies conditions 1 and 2, then for any $\varepsilon > 0$, there exists $\sigma > 0$ such that

$$\forall t', t'' \in [a,b] \quad \left(|t' - t''| < \sigma \ \Rightarrow \ \int_a^b |K(t',s) - K(t'',s)|ds < \varepsilon\right).$$

From this and the above estimate, it follows that the function $B\varphi(t)$ is uniformly continuous, and therefore, $B\varphi(t) \in C([a,b], X)$.

Next we show that the operator A is a contractive operator. We have

$$\delta(A\varphi(t), A\psi(t)) = \delta\left(\lambda\int_a^b K(t,s)\varphi(s)ds + f(t), \lambda\int_a^b K(t,s)\psi(s)ds + f(t)\right)$$
$$\leq |\lambda|\delta\left(\int_a^b K(t,s)\varphi(s)ds, \int_a^b K(t,s)\psi(s)ds\right)$$
$$\leq |\lambda| \int_a^b |K(t,s)|\delta(\varphi(s), \psi(s))ds$$
$$\leq |\lambda| \int_a^b |K(t,s)|ds \max_{s\in[a,b]} \delta(\varphi(s), \psi(s))$$
$$\leq |\lambda|M(b-a)\rho(\varphi(\cdot), \psi(\cdot)).$$

Thus,

$$\rho(A\varphi(\cdot), A\psi(\cdot)) \leq |\lambda|M(b-a)\rho(\varphi(\cdot), \psi(\cdot)).$$

The above inequality means that the mapping A is contractive if $|\lambda| < \frac{1}{M(b-a)}$. This contractive mapping has a unique fixed point which implies that the equation (4.13) has a unique solution in the space $C([a,b], X)$. $\square$

4.3.2 Volterra equation

We consider now the Volterra integral equation

$$\varphi(t) = \int_a^t K(t,s)\varphi(s)ds + f(t), \tag{4.14}$$

where $\varphi:\ [a,b] \to X$ is again an unknown function, and $f:\ [a,b] \to X$ is a known continuous function. The kernel $K(t,s)$ is a known real-valued function defined for $(t,s) \in [a,b]\times[a,b]$ such that $s \le t$. Below we assume that $K(t,s)$ is defined on all square $[a,b]\times[a,b]$, and $K(t,s) = 0$, if $s > t$.

Since a Volterra integral equation is a particular case of a Fredholm integral equation, we can use Theorem 6 to guarantee existence and uniqueness of the solution of the equation (4.14) for the kernel $K(t,s)$ such that $M < \frac{1}{b-a}$. However, for Volterra equations, we can prove the following more general theorem without the restriction on M.

Theorem 7 *Let $K(t,s)$ be continuous in the domain $\{(t,s) \in [a,b]\times[a,b]\ :\ s \le t\}$ and let $f \in C([a,b],X)$. Then the equation (4.14) has a unique solution $\varphi \in C([a,b],X)$.*

Proof. To prove this theorem, we adopt the method known for Volterra equations for real-valued functions (see, e.g., [41]).

Consider the operator $A : C([a,b],X) \to C([a,b],X)$ defined by

$$A\varphi(t) = \int_a^t K(t,s)\varphi(s)ds + f(t).$$

Since equation (4.14) is a particular case of equation (4.13), it is clear that this operator maps the space $C([a,b],X)$ into $C([a,b],X)$.

We prove that some integer power of this operator is a contractive operator.

We need to introduce some additional notations. Let

$$K_1(t,s) = K(t,s), \quad K_N(t,s) = \int_a^b K_{N-1}(t,u)K(u,s)du, \quad N \in \mathbb{N},\ N > 1.$$

Note that if $|K(t,s)| \le M$, then (see [41, p. 449])

$$|K_N(t,s)| \le \frac{M^N(b-a)^{N-1}}{(N-1)!}. \tag{4.15}$$

It is easily seen that for any $N \in \mathbb{N}$,

$$A^N\varphi(t) = \int_a^b K_N(t,s)\varphi(s)ds + \int_a^b \sum_{k=1}^{N-1} K_{N-k}(t,s)f(s)ds + f(t). \tag{4.16}$$

Using (4.16) and (4.15) we obtain

$$\delta(A^N\varphi(t), A^N\psi(t)) \le \delta\left(\int_a^t K_N(t,s)\varphi(s)ds, \int_a^t K_N(t,s)\psi(s)ds\right)$$

$$\leq \int_a^t \delta\left(K_N(t,s)\varphi(s), K_N(t,s)\psi(s)\right) ds$$

$$\leq \int_a^t |K_N(t,s)|\delta(\varphi(s),\psi(s))ds \leq \frac{M^N(b-a)^N}{(N-1)!} \max_{a\leq s\leq b} \delta(\varphi(s),\psi(s)).$$

Therefore,

$$\rho(A^N\varphi, A^N\psi) \leq \frac{M^N(b-a)^N}{(N-1)!}\rho(\varphi,\psi).$$

If N is sufficiently large, then $\frac{(b-a)^N}{(N-1)!}M^N < 1$. Fix such an N. We have shown that the operator A^N is contractive. Using the generalized contractive mapping principle (see [41, ch. 9]), we obtain that A has a unique fixed point and that therefore, the equation (4.14) has a unique solution $\varphi \in C([a,b],X)$. $\square$

4.4 Algorithms for approximate solution

In this section, we describe algorithms for the approximate solution of linear Fredholm and Volterra integral equations for functions with values in L-spaces.

We adopt well-known methods for integral equations for real-valued functions, specifically a collocation method (see, for example, [7, ch. 3]) in Section 4.4.1 and 4.4.2, and quadrature formula methods (see [7], [48]) in Section 4.4.3.

4.4.1 Fredholm equation

We start with the Fredholm integral equation (4.13). Let $n \in \mathbb{N}$. Choose a set of knots $a = t_0 < t_1 < ... < t_n = b$ and a set of continuous real-valued functions $\gamma_0, \gamma_1, ..., \gamma_n$ defined in $[a,b]$. Suppose that these functions satisfy the following interpolation conditions

$$\gamma_k(t_j) = \delta_{k,j},\ j,k = 0,1,...,n.$$

Note that we can use cardinal Lagrange interpolation polynomials as well as cardinal interpolation splines as such functions. For example, we can take $\gamma_k(t) = l_k(t)$, where $l_k(t)$ is defined by (4.3), $t_k = a + k\frac{b-a}{n}$.

We look for a solution of (4.13) in the form

$$\varphi^n(t) = \sum_{k=0}^{n} \varphi_k\gamma_k(t), \quad \varphi_k \in X, \quad k = 0,1,...,n. \tag{4.17}$$

Substituting it in the equation (4.13) and setting $t = t_j$ gives

$$\varphi_j = \lambda \sum_{k=0}^{n} \widetilde{\varphi}_k \int_a^b K(t_j,s)\gamma_k(s)ds + f_j, \quad j = 0,1,...,n, \tag{4.18}$$

where $f(t_j) = f_j$.

Set $a_{jk} = \int_a^b K(t_j, s)\gamma_k(s)ds$.

The system can be rewritten in the form

$$\varphi_j = f_j + \lambda \sum_{k=0}^{n} \widetilde{\varphi}_k a_{jk}, \quad j = 0, 1, ..., n. \tag{4.19}$$

Due to the problems related to the existence of differences, the only promising method of solving this system of equations is apparently the method of successive approximations. Under some additional assumptions, we can solve this system by this method. We illustrate this in detail for $\gamma_k(t) = l_k(t),\ k = 0, 1, ..., n$.

Choose an initial approximation $\{x_0, ..., x_n\}$ to the solution of this system (4.19). For $j = 0, 1, ..., n$ set

$$x_j^0 = x_j, \quad x_j^{m+1} = f_j + \lambda \sum_{k=0}^{n} \widetilde{x}_k^m a_{jk}, \quad m = 0, 1, \dots .$$

Let

$$X^{n+1} = \underbrace{X \times \ldots \times X}_{\text{n+1 times}}.$$

Consider the operator $B:\ X^{n+1} \to X^{n+1}$, which is defined according to the rule

$$y_j = f_j + \lambda \sum_{k=0}^{n} x_k a_{jk}, \quad j = 0, 1, ..., n.$$

Let a metric in the space X^{n+1} be defined by

$$\rho\left((x_0, ..., x_n), (y_0, ..., y_n)\right) := \sum_{j=0}^{n} \delta(x_j, y_j).$$

With this metric, the space X^{n+1} is complete.

We have

$$\rho(B(x_0, ..., x_n), B(y_0, ..., y_n)) = \sum_{j=0}^{n} \delta\left(f_j + \lambda \sum_{k=0}^{n} \widetilde{x}_k a_{jk}, f_j + \lambda \sum_{k=0}^{n} \widetilde{y}_k a_{jk}\right)$$

$$\leq \sum_{j=0}^{n} \delta\left(\lambda \sum_{k=0}^{n} \widetilde{x}_k a_{jk}, \lambda \sum_{k=0}^{n} \widetilde{y}_k a_{jk}\right)$$

$$\leq |\lambda| \sum_{j=0}^{n} \sum_{k=0}^{n} \delta(\widetilde{x}_j, \widetilde{y}_j)|a_{jk}| \leq |\lambda| \sum_{j=0}^{n} \delta(x_j, y_j) \sum_{k=0}^{n} |a_{jk}|$$

$$\leq |\lambda| M(b - a)\rho\left((x_0, ..., x_n), (y_0, ..., y_n)\right).$$

The last inequality holds since

$$\sum_{k=0}^{n}|a_{jk}| \leq \sum_{k=0}^{n}\int_a^b |K(t_j,s)|l_k(s)ds \leq M\int_a^b \sum_{k=0}^{n} l_k(s)ds = M(b-a).$$

Therefore, if $|\lambda| < \frac{1}{M(b-a)}$, the operator B is contractive, and consequently, the sequence $\{(x_0^m,...,x_n^m)\}_{m=0}^{\infty}$ converges to the solution $(\varphi_0,...,\varphi_n)$ of the system (4.19) as $m\to\infty$.

The function of the form (4.17) where $\{\varphi_0,...,\varphi_n\}$ is the solution of the system (4.19) is our approximation of the solution of the equation (4.13).

4.4.2 Volterra equation

Because Volterra Equations can be considered as a special case of Fredholm Equations, the method described in the preceding section can be applied for their solution. However, in the case of $X = X^c$ and $\operatorname{supp}\gamma_k(t) \subset [t_{k-1}, t_{k+1}]$, the resulting linear system becomes triangular and can be solved explicitly, and rather simply (here $\operatorname{supp}\gamma(t)$ is the support of a function $\gamma(t)$).

Consider the Volterra Equation (4.14) under the assumption that $K(t,s)$ is nonnegative. As before, we look for a solution of the form (4.17).

As above, let

$$a_{jk} = \int_a^{t_j} K(t_j,s)\gamma_k(s)ds.$$

Assume that $\gamma_k(t) \geq 0,\ k = 0,1,...,n.$ Then $a_{jk} \geq 0$ if $k \leq j$ and $a_{jk} = 0$ if $k > j$. Substituting (4.17) in (4.14) and evaluating at t_j gives the triangular system

$$\varphi_j = \sum_{k=0}^{j} a_{jk}\varphi_k + f_j, \quad j = 0,1,\ldots,n,$$

that define $\varphi_0,...,\varphi_n$. This system can be rewritten as

$$\varphi_j \overset{h}{-} a_{jj}\varphi_j = \sum_{k=0}^{j-1}\varphi_k a_{jk} + f_j.$$

We assume that functions $\gamma_k(t)$ are uniformly bounded (in n) and n is sufficiently large so that

$$0 \leq a_{jj} < 1 \quad \text{for} \quad j = 0,1,\ldots,n.$$

With that assumption and using the fact that φ_j are convex, we obtain

$$\varphi_j \overset{h}{-} a_{jj}\varphi_j = (1-a_{jj})\varphi_j.$$

Thus, we obtain the explicit recursion

$$\varphi_0 = f_0, \quad \varphi_j = \frac{1}{1-a_{jj}}\left(\sum_{k=0}^{j-1}\varphi_k a_{jk} + f_j\right), \quad j = 1,2,\ldots,n. \tag{4.20}$$

4.4.3 Nyström method for Fredholm equations

Consider again the Fredholm equation (4.13). For real-valued functions, the following approach to finding an approximate solution of such equations is well known (see, for example, [7, ch. 4]).

Let a quadrature formula be given:

$$\int_a^b g(s)ds \approx \sum_{j=1}^{n} p_j \widetilde{g}(t_j), \quad g \in C([a,b],X),$$

where $a \le t_1 < t_2 < ... < t_n \le b$, $p_j \in \mathbb{R}$.

Using this quadrature formula, we approximate the integral in (4.13) and obtain a new equation:

$$\varphi(t) = \lambda \sum_{j=1}^{n} p_j K(t,t_j)\widetilde{\varphi}(t_j) + f(t). \tag{4.21}$$

Evaluate (4.21) at $t_k,\ k = 1,2,...,n$:

$$\varphi(t_k) = \lambda \sum_{j=1}^{n} p_j K(t_k,t_j)\widetilde{\varphi}(t_j) + f(t_k).$$

This is the system of equations with unknown $\varphi(t_k) = \varphi_k$, which we rewrite in the form

$$\varphi_k = \lambda \sum_{j=1}^{n} b_{kj}\widetilde{\varphi}_j + f_k^n,\ j = 1,2,...,n, \tag{4.22}$$

where $b_{kj} = p_j K(t_k,t_j)$.

Under some additional assumptions, we can solve (4.22) using the method of consecutive approximations, analogously to the solution of the system (4.19).

The solution $\varphi_1,...,\varphi_n$ of the system (4.22) we can use as approximate values of the solution of the equation (4.13) at points $t_1,...,t_n$.

4.4.4 Nyström method for Volterra equations

Consider now the Volterra equation (4.14). We describe a quadrature formulas method for the approximate solution of the linear Volterra equation (4.14) based on the trapezoidal rule. We assume that kernel $K(t,s)$ is nonnegative. Since we can consider Volterra equations as a particular case of Fredholm equations, the method described in Section 4.4.3 can be applied and for their solutions too. However, if we assume that all elements of the space X are convex, we can obtain an explicit solution of our system.

For simplicity, let $a = 0,\ b = 1,\ t_i = i/n,\ i = 0,1,...,n$.

Once we apply the trapezoidal rule for the approximate calculation of the integral

$$\int_0^{t_k} K(t_k, s)\varphi(s)ds$$

in (4.14) and evaluate the resulting relation at $t = t_k,\ k = 1, ..., n$ we obtain the following

$$\varphi_k = f_k + \frac{1}{n}\left\{\frac{1}{2}K(t_k, t_0)\varphi_0 + \sum_{i=1}^{k-1} K(t_k, t_i)\varphi_i + \frac{1}{2}K(t_k, t_k)\varphi_k\right\}, \quad k = 1, 2, ..., n,$$

(here $\sum\limits_{i=1}^{0} := 0$) with $\varphi_0 = f_0$.

Set $\frac{1}{n}K(t_k, t_i) = c_{ki}$ and note that for n large enough, we have $0 \le c_{ki} < 1$. We obtain the following explicit recursive formula that is analogous to (4.20):

$$\varphi_0 = f_0, \quad \varphi_j = \frac{1}{1 - \frac{1}{2}c_{jj}}\left(\frac{1}{2}c_{j0}\varphi_0 + \sum_{i=1}^{j-1} c_{ji}\varphi_i + f_j\right), \quad j = 1, 2, ..., n.$$

It seems possible to use for the approximate solution of Volterra type equations, as well as for Fredholm type equations (for functions with values in L-spaces), quadrature formulas of higher accuracy (see, for example, [48, ch.7]). An analysis of such methods requires additional tools that have not yet been developed for functions with values in L-spaces, and even for set-valued functions.

4.5 Convergence and error analysis

In Sections 4.5.1 and 4.5.2, we assume that all elements of the space X are convex.

4.5.1 Convergence of collocation algorithm for Fredholm equations

Recall that in Section 4.4.1 the elements $\varphi_0, ..., \varphi_n$ were defined as a solution of the system (4.19). As approximate solution of the Fredholm equation (4.13) we consider the function (4.17) with $\gamma_k = l_k$.

Theorem 8 *Let $K(t, s)$ and $f(t)$ satisfy the condition of Theorem 6, and $|\lambda| < \frac{1}{M(b-a)}$. Let $\varphi(t)$ be the solution of the equation (4.13) and let $\varphi^n(t),\ n \in \mathbb{N}$ be defined by (4.17). Then*

$$\rho(\varphi, \varphi^n) = \max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \to 0, \quad \textit{as } n \to \infty. \tag{4.23}$$

If for any $s \in [a, b]$ the kernel $K(t, s)$ satisfies the Lipschitz condition (for t) with a constant M_1, that does not depend on s, and $f(t)$ satisfies in $[a, b]$ the Lipschitz condition with constant M_2, then there exists a constant C_1 such that for any n

$$\max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \le \frac{C_1}{n}. \tag{4.24}$$

Moreover, if for any $s \in [a,b]$ the kernel $K(t,s)$ is continuously differentiable with respect to t, and $\frac{\partial K(t,s)}{\partial t}$ satisfies the Lipschitz condition with constant M_3 that does not depend on s, $f(t)$ has a Hukuhara type derivative $D_H f(t)$ in $[a,b]$ and $D_H f(t)$ satisfies the Lipschitz condition (4.1) with constant M_4, then there exists a constant C_2 such that for any n

$$\max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \le \frac{C_2}{n^2}. \tag{4.25}$$

Proof. Recall that the operator $P_N[f](t)$ is defined by (4.2). Using (4.17) and (4.18), we have

$$\begin{aligned} P_n[\varphi^n](t) &= \lambda P_n \left[\sum_{k=0}^{n} \varphi_k \int_a^b K(t,s) l_k(s) ds \right] (t) + P_n[f](t) \\ &= \lambda \sum_{j=0}^{n} \left(\sum_{k=0}^{n} \varphi_k \int_a^b K(t_j,s) l_k(s) ds \right) l_j(t) + P_n[f](t) \\ &= \lambda \int_a^b \left[\sum_{j=0}^{n} K(t_j,s) l_j(t) \right] \sum_{k=0}^{n} \varphi_k l_k(s) ds + P_n[f](t) \\ &= \lambda \int_a^b P_{n,t}[K](t,s) \varphi^n(s) ds + P_n[f](t), \end{aligned}$$

where

$$P_{n,t}[K](t,s) = \sum_{j=0}^{n} K(t_j,s) l_j(t).$$

Since $P_n[\varphi^n](t) = \varphi^n(t)$ (the piecewise linear function that interpolates a piecewise linear function at the knots coincides with the interpolated function), we obtain that the function $\varphi^n(t)$ solves the following equation

$$\varphi^n(t) = \lambda \int_a^b P_{n,t}[K](t,s) \varphi^n(s) ds + P_n[f](t). \tag{4.26}$$

Let $\varphi(t)$ be the solution of the equation (4.13) and let $\varphi^n(t)$ solve equation (4.26). We estimate the distance between φ^n and φ. We have

$$\begin{aligned} \delta(\varphi(t), \varphi^n(t)) &\le |\lambda| \delta \left(\int_a^b K(t,s)\varphi(s) ds, \int_a^b P_{n,t}[K](t,s)\varphi^n(s) ds \right) + \delta(f(t), P_n[f](t)) \\ &\le |\lambda| \delta \left(\int_a^b K(t,s)\varphi(s) ds, \int_a^b P_{n,t}[K](t,s)\varphi(s) ds \right) + \\ &+ |\lambda| \delta \left(\int_a^b P_{n,t}[K](t,s)\varphi(s) ds, \int_a^b P_{n,t}[K](t,s)\varphi^n(s) ds \right) + \delta(f(t), P_n[f](t)). \end{aligned}$$

Therefore,

$$\max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t))$$

$$\leq |\lambda| \max_{a\leq t\leq b} \delta\left(\int_a^b P_{n,t}[K](t,s)\varphi(s)ds, \int_a^b P_{n,t}[K](t,s)\varphi^n(s)ds\right) +$$

$$+|\lambda| \max_{a\leq t\leq b} \delta\left(\int_a^b K(t,s)\varphi(s)ds, \int_a^b P_{n,t}[K](t,s)\varphi(s)ds\right) + \max_{a\leq t\leq b} \delta(f(t), P_n[f](t))$$

$$\leq |\lambda| \max_{a\leq t\leq b} \int_a^b |P_{n,t}[K](t,s)|\delta(\varphi(s), \varphi^n(s))ds$$

$$+|\lambda| \max_{a\leq t\leq b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)|\delta(\varphi(s), \theta)ds + \max_{a\leq t\leq b} \delta(f(t), P_n[f](t)).$$

Note that if $|K(t,s)| \leq M$, then $|P_{n,t}[K](t,s)| \leq M$ as well. Therefore,

$$\max_{a\leq t\leq b} \delta(\varphi(t), \varphi^n(t)) \leq |\lambda| M(b-a) \max_{a\leq t\leq b} \delta(\varphi(t), \varphi_n(t))$$

$$+|\lambda| \max_{a\leq t\leq b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)|ds \max_{a\leq t\leq b} \delta(\varphi(t), \theta)$$

$$+ \max_{a\leq t\leq b} \delta(f(t), P_n[f](t)).$$

From the estimation above, we obtain that if $|\lambda| < \frac{1}{M(b-a)}$, then

$$(1 - |\lambda| M(b-a)) \max_{a\leq t\leq b} \delta(\varphi(t), \varphi^n(t)) \tag{4.27}$$

$$\leq |\lambda| \max_{a\leq t\leq b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)|ds \max_{a\leq s\leq b} \delta(\varphi(s), \theta) + \max_{a\leq t\leq b} \delta(f(t), P_n[f](t)).$$

It is easily seen that

$$\max_{a\leq t\leq b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)|ds \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.28}$$

We also have (see Theorem 3),

$$\max_{a\leq t\leq b} \delta(f(t), P_n[f](t)) \to 0, \ \text{ as } n \to \infty.$$

From the last two relations and from (4.27), we see that

$$\rho(\varphi, \varphi^n) = \max_{a\leq t\leq b} \delta(f(t), P_n[f](t)) \to 0, \ \text{ as } n \to \infty.$$

We have proved the relation (4.23). Next we prove the relation (4.24).

Suppose that for any $s \in [a,b]$, the kernel $K(t,s)$ satisfies the Lipschitz condition (for t) with a constant M_1 that does not depend on s. It follows from the real-valued analog of Theorem 3 that

$$\max_{a \le t \le b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)| ds \le \frac{(b-a)^2}{2n} M_1. \tag{4.29}$$

If $f(t)$ satisfies in $[a,b]$ the Lipschitz condition with constant M_2, then it follows from Theorem 3 that

$$\max_{a \le t \le b} \delta(f(t), P_n[f](t)) \le \frac{M_2(b-a)}{2n}. \tag{4.30}$$

Using (4.29) and (4.30), we obtain from (4.27) that there exists a constant C_1 such that for any n

$$\max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \le \frac{C_1}{n}.$$

We have proved the relation (4.24).

Suppose now that for any $s \in [a,b]$, the kernel $K(t,s)$ is continuously differentiable with respect to t, and $\partial K(t,s)/\partial t$ satisfies the Lipschitz condition with constant M_3 that does not depend on s, then (see, e.g., [7, p.60])

$$\max_{a \le t \le b} \int_a^b |K(t,s) - P_{n,t}[K](t,s)| ds \le \frac{M_3(b-a)^3}{8n^2}. \tag{4.31}$$

If the Hukuhara type derivative of $f(t)$ satisfies the Lipschitz condition (4.1) with constant M_4, then due to Theorem 3,

$$\max_{a \le t \le b} \delta(f(t), P_n[f](t)) \le \frac{M_4(b-a)^2}{8n^2}. \tag{4.32}$$

Using (4.31) and (4.32), we obtain from (4.27) that there exists a constant C_2 such that for any n

$$\max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \le \frac{C_2}{n^2}.$$

We have proved the relation (4.25). $\square$

4.5.2 Convergence of collocation algorithm for Volterra equation

Since a Volterra equation is a particular case of a Fredholm equation, we can apply the first statement of Theorem 8 to obtain the convergence of the method presented in Section 4.4.2 under the assumption $M < 1/(b-a)$. However, in the case of a Volterra equation with a nonnegative kernel, we can eliminate the restriction. We obtain

Theorem 9 *Let the kernel $K(t,s)$ of the equation (4.14) be nonnegative and continuous, and suppose $K(t,s) \le M$ in the domain $0 \le s \le t \le b$. Let $\varphi(t)$ be the solution of the equation (4.14) and let $\varphi^n(t)$, $n \in \mathbb{N}$, be the approximate solution (4.17). Then*

$$\rho(\varphi, \varphi^n) = \max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \to 0, \quad as\, n \to \infty.$$

If in addition the kernel $K(t,s)$ satisfies the Lipschitz condition with constant G_1 in t for any s in the domain $a \le s \le t \le b$, and $f(t)$ satisfies the Lipschitz condition with constant G_2 in $[a,b]$, then there exists a constant G_3 such that for any $n \in \mathbb{N}$

$$\frac{1}{2} \max_{a \le t \le b} \delta(\varphi(t), \varphi^n(t)) \le \frac{G_3}{n}. \tag{4.33}$$

Proof. We use notations from the previous section, taking into account that $K(t,s) = 0$ if $s > t$ and $\lambda = 1$. As in Section 4.5.1, we have that $\varphi(t)$ and $\varphi^n(t)$ satisfy (4.13) and (4.26).

We would like to estimate the distance between φ^n and φ. Set

$$P_{n,t;N}[K](t,s) := (P_{n,t}[K])_N(t,s).$$

Since $\max\limits_{a \le t,s \le b} |P_{n,t}[K](t,s)| \le \max\limits_{a \le t,s \le b} |K(t,s)|$ we have

$$|P_{n,t;N}[K](t,s)| \le M^N (b-a)^N. \tag{4.34}$$

Consider the operators

$$\begin{aligned} A\varphi(t) &= \textstyle\int_a^b K(t,s)\varphi(s)ds + f(t), \quad \varphi \in C([a,b],X), \\ A_n\varphi(t) &= \textstyle\int_a^b P_{n,t}[K](t,s)\varphi(s)ds + P_n[f](t), \quad \varphi \in C([a,b],X). \end{aligned}$$

Let $A^N\varphi(t)$ be defined by (4.16) and let

$$A_n^N \varphi(t) = \int_a^b P_{n,t;N}[K](t,s)\varphi(s)ds + \int_a^b \sum_{k=1}^{N-1} P_{n,t;N-k}[K](t,s)P_n[f](s)ds + P_n[f](t).$$

Since φ is a fixed point of the operator A, then φ is a fixed point of operator A^N, for all N. Therefore, φ satisfies the equation

$$\varphi(t) = \int_a^b K_N(t,s)\varphi(s)ds + \int_a^b \sum_{k=1}^{N-1} K_{N-k}(t,s)f(s)ds + f(t). \tag{4.35}$$

Similarly (since φ^n is a fixed point of the operator A_n),

$$\varphi^n(t) = \int_a^b P_{n,t;N}[K](t,s)\varphi^n(s)ds + \int_a^b \sum_{k=1}^{N-1} P_{n,t;N-k}[K](t,s)P_n[f](s)ds + P_n[f](t). \tag{4.36}$$

Next we estimate $\delta(\varphi(t), \varphi^n(t))$. Using (4.35) and (4.36), we have:

$$\max_{a\le t\le b} \delta(\varphi(t), \varphi^n(t)) \le \max_{a\le t\le b} \int_a^b \delta(K_N(t,s)\varphi(s), P_{n,t;N}[K](t,s)\varphi^n(s))ds \tag{4.37}$$

$$+ \sum_{k=1}^{N-1} \max_{a\le t\le b} \int_a^b \delta(K_{N-k}(t,s)f(s), P_{n,t;N-k}[K](t,s)P_n[f](s))ds$$

$$+ \max_{a\le t\le b} \delta(f(t), P_n[f](t))$$

$$=: \Delta_N + \sum_{k=1}^{N-1} \Delta_{N-k} + \Delta_0.$$

For Δ_N, we have

$$\Delta_N = \max_{a\le t\le b} \int_a^b \delta(K_N(t,s)\varphi(s), P_{n,t;N}[K](t,s)\varphi^n(s))ds$$

$$\le \max_{a\le t\le b} \int_a^b \delta(K_N(t,s)\varphi(s), K_N(t,s)\varphi^n(s))ds$$

$$+ \max_{a\le t\le b} \int_a^b \delta(K_N(t,s)\varphi^n(s), P_{n,t;N}[K](t,s)\varphi^n(s))ds$$

$$\le \max_{a\le t\le b} \int_a^b K_N(t,s)\delta(\varphi(s), \varphi^n(s))ds$$

$$+ \max_{a\le t\le b} \int_a^b |K_N(t,s) - P_{n,t;N}[K](t,s)|\delta(\varphi^n(s), \theta)ds.$$

Using (4.15), we obtain

$$\max_{a\le t\le b} \int_a^b K_N(t,s)\delta(\varphi(s), \varphi^n(s))ds \le \frac{M^N(b-a)^{N-1}}{(N-1)!}(b-a) \max_{a\le t\le b} \delta(\varphi(t), \varphi^n(t))$$

$$\le \frac{1}{2} \max_{a\le t\le b} \delta(\varphi(t), \varphi^n(t)),$$

if N is sufficiently large. We fix such an N below. Therefore, we obtain

$$\Delta_N \le \frac{1}{2}\rho(\varphi, \varphi^n) + \max_{a\le t\le b} \int_a^b |K_N(t,s) - P_{n,t;N}[K](t,s)|ds\rho(\varphi^n, \theta) \tag{4.38}$$

Let us estimate Δ_{N-k}. Using (4.15), we have for $k = 1, 2, ..., N-1$

$$\Delta_{N-k} \le \max_{a\le t\le b} \int_a^b \delta(K_{N-k}(t,s)f(s), K_{N-k}(t,s)P_n[f](s))ds$$

$$+ \max_{a\le t\le b} \int_a^b \delta(K_{N-k}(t,s)P_n[f](s), P_{n,t;N-k}[K](t,s)P_n[f](s))ds$$

$$\le \max_{a\le t\le b} \int_a^b K_{N-k}(t,s)\delta(f(s), P_n[f](s))ds$$

$$+\max_{a\le t\le b}\int_a^b |K_{N-k}(t,s)-P_{n,t;N-k}[K](t,s)|\delta(P_n[f](s),\theta)ds$$

$$\le \frac{M^{N-k}(b-a)^{N-k}}{(N-1-k)!}\max_{a\le s\le b}\delta(f(s),P_n[f](s))$$

$$+\max_{a\le t\le b}\int_a^b |K_{N-k}(t,s)-P_{n,t;N-k}[K](t,s)|ds\max_{a\le s\le b}\delta(P_n[f](s),\theta).$$

Thus, for $k=1,2,...,N-1$,

$$\Delta_{N-k}\le \frac{M^{N-k}(b-a)^{N-k}}{(N-1-k)!}\Delta_0 \tag{4.39}$$

$$+\max_{a\le t\le b}\int_a^b |K_{N-k}(t,s)-P_{n,t;N-k}[K](t,s)|ds\rho(P_n[f],\theta).$$

The following statment is easy to prove by induction.

For any N, there exists a constant $C_N>0$ (independent on n), such that

$$\max_{a\le t\le b}\int_a^b |K_N(t,s)-P_{n,t;N}[K](t,s)|ds\le C_N\max_{a\le u\le b}\int_a^b |K(u,s)-P_{n,t}[K](u,s)|ds. \tag{4.40}$$

Also, the sequences $\{\rho(\varphi^n,\theta)\}$ and $\{\rho(P_n[f],\theta)\}$ are bounded. From this and from the relations (4.37), (4.38), (4.39) and (4.40), it follows that there exist constants C', $C''>0$ such that

$$\frac{1}{2}\rho(\varphi,\varphi^n)\le C'\Delta_0+C''\max_{a\le t\le b}\int_a^b |K_N(t,s)-P_{n,t;N}[K](t,s)|ds,\quad \Delta_0=\rho(P_n[f],f). \tag{4.41}$$

If the kernel $K(t,s)$ of the equation (4.14) is nonnegative, continuous, and $K(t,s)\le M$ in the domain $0\le s\le t\le b$, then

$$\max_{a\le t\le b}\int_a^b |K_N(t,s)-P_{n,t;N}[K](t,s)|ds\to 0,\quad \text{as } n\to\infty. \tag{4.42}$$

If the kernel $K(t,s)$ satisfies the Lipschitz condition with constant G_1 in t for any s in the domain $a\le s\le t\le b$, then there exists a constant G_4 such that for any $n\in\mathbb{N}$

$$\max_{a\le t\le b}\int_a^b |K_N(t,s)-P_{n,t;N}[K](t,s)|ds\le \frac{G_4}{n}. \tag{4.43}$$

Now the statement of the Theorem 14 follows from Theorem 3 and relations (4.41), (4.42), and (4.43). □

4.5.3 Error analysis for quadrature methods

We present here only a theorem that gives an error analysis of the method based on trapezoidal quadrature formula for approximate solution of Fredholm equation.

Let $\varphi(t)$ be an exact solution of the equation (4.13) and let $\varphi_j^n,\ j=0,1,...,n$ be such that

$$\varphi_j^n=\lambda\frac{b-a}{n}\left[\frac{1}{2}K(t_j,t_0)\widetilde{\varphi}_0^n+\sum_{i=1}^{n-1}K(t_j,t_i)\widetilde{\varphi}_i^n+\frac{1}{2}K(t_j,t_n)\widetilde{\varphi}_n^n\right]+f(t_j).$$

Set $\varepsilon_n=\max\limits_{0\le j\le n}\delta(\varphi(t_j),\varphi_j^n)$.

Theorem 10 *1. Let $K(t,s)$ and $f(t)$ be continuous, $|K(t,s)|\le M$ in the domain $[a,b]\times[a,b]$ and $|\lambda|<\frac{1}{M(b-a)}$. Then*

$$\varepsilon_n\to 0,\quad \textit{as}\, n\to\infty. \tag{4.44}$$

2. If for any $s\in[a,b]$ the kernel $K(t,s)$ satisfies the Lipschitz condition (for t) with a constant M_1, that does not depend on s, and $f(t)$ satisfies in $[a,b]$ the Lipschitz condition with constant M_2, then there exists a constant C_1 such that for any n

$$\varepsilon_n\le\frac{C_1}{n}. \tag{4.45}$$

Proof. It is obvious that there exists a constant $C_2>0$ such that

$$\varepsilon_n\le C_2\max_t R_n(K(t,\cdot)\varphi(\cdot))$$

(definition of $R_n(K(t,\cdot)\varphi(\cdot))$ see in (4.12)). If $K(t,s)$ and $f(t)$ satisfy the condition of the statement 1 of the theorem, then $\varphi(t)$ as well as $K(t,\cdot)\varphi(\cdot)$ are continuous.

If $K(t,s)$ and $f(t)$ satisfy the condition of the statement 2 of the theorem, then $K(t,\cdot)\varphi(\cdot)$ satisfies in $[a,b]$ the Lipschitz condition with some constant M_3.

Now the statements of Theorem 10 follow from Theorem 5. $\square$

4.6 Numerical examples

In this section, we discuss some numerical examples for set-valued functions, i.e., functions with values in L-space of convex, compact subsets of $\mathbb{R}^n$: $\mathcal{K}^c(\mathbb{R}^n)$. We used MATLAB to implement the algorithm for the approximate solution of integral equations presented in Section 4.4.

Example 1. We consider first an initial value problem

$$D_HX=\lambda(t)X+A(t),\quad X(a)=X_0,$$

where D_HX is the Hukuhara derivative (see [39]), $\lambda(\cdot):[a,b]\to\mathbb{R}_+$ and $A(\cdot):\ [a,b]\to\mathcal{K}^c(\mathbb{R}^n)$ are continuous functions, $X_0\in\mathcal{K}^c(\mathbb{R}^n)$.

This initial value problem can be rewritten in the form of the Volterra integral equation

$$X(t) = \int_a^t \lambda(s)X(s)ds + F(t), \text{ where } F(t) = X_0 + \int_a^t A(s)ds. \tag{4.46}$$

It is well known (see for example [58]) that the solution of this equation has the form

$$X(t) = e^{\int_a^t \lambda(s)ds}\left(X_0 + \int_a^t A(s)e^{-\int_a^s \lambda(\tau)d\tau}ds\right).$$

We consider the equation (4.46) with $\lambda(s) = s$ and

$$F(t) = [-1, t] \times [0, 1] = [-1, 0] \times [0, 1] + [0, t] \times \{0\},\ t \in [0, 1].$$

The exact solution of the equation (4.46) is

$$X(t) = \left[-e^{t^2/2}, \int_0^t e^{(t^2-s^2)/2}ds\right] \times \left[0, e^{t^2/2}\right].$$

We plot (see Figure 4.1) the Hausdorff distance between the exact solution and the approximate solution. The time-step is 1/50.

Example 2. Here we consider the Volterra integral equation (4.14) with $a = 0,\ b = 1$. Let $K(t, s) = ts$ and

$$f(t) = \left[0, \frac{1}{2+t} + 2t\ln\frac{2+t}{2} - t^2\right] \times \left[0, \frac{1}{3+t} + 3t\ln\frac{3+t}{3} - t^2\right].$$

The exact solution is $\varphi(t) = \left[0, \frac{1}{2+t}\right] \times \left[0, \frac{1}{3+t}\right],\ \ t \in [0, 1]$. We plot the Hausdorff distance

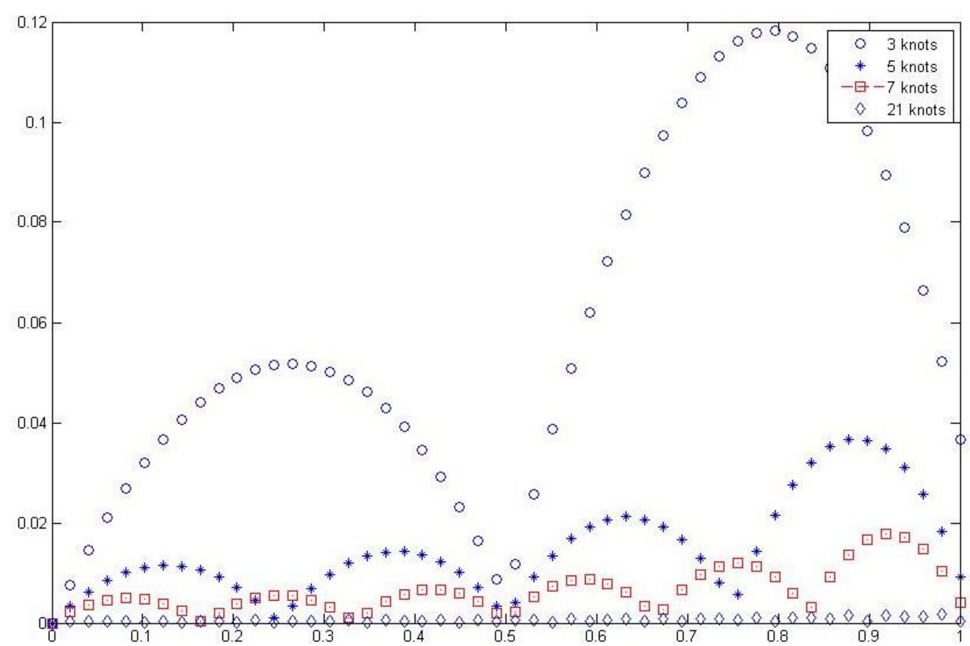


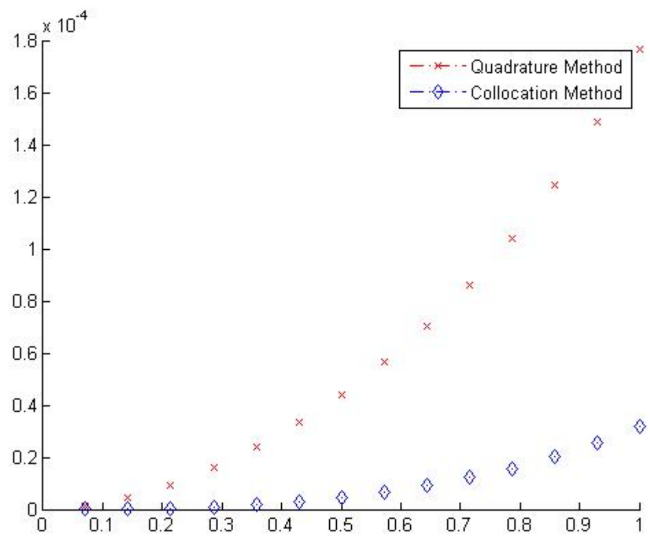
Figure 4.1: The Hausdorff distance between exact solution and approximate solution

(the error) between the exact solution and the solution obtained by both collocation and quadrature algorithms that were described in Sections 4.4.2 and 4.4.4. The time-step is 1/14.

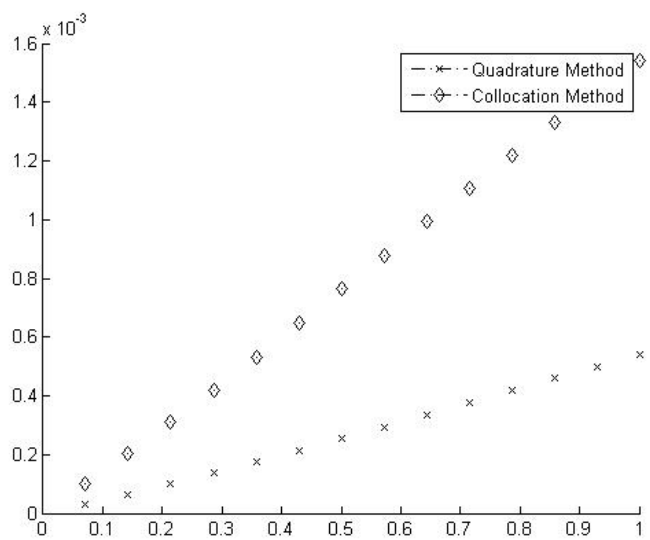
As one can see from the corresponding picture (see Figure 4.2a), the collocation method gives a better approximation then the quadrature method in this case.

Example 3. Let $K(t,s) = e^{-s}$, $t \in [0,1]$ in the equation (4.14) and

$$f(t) = \left[0, e^t(1+\alpha\cos t) - t - \alpha\sin t\right] \times \left[0, e^t(1+\alpha\sin t) - t + \alpha\cos t - \alpha\right].$$



(a) Example 2. Kernel $K(t,s) = ts$



(b) Example 3. Kernel $K(t,s) = e^{-s}$

Figure 4.2: Example of quadrature and collocation algorithms for Volterra Equations

The exact solution has the following form: $\varphi(t) = \left[0, e^t(1+\alpha\cos t)\right] \times \left[0, e^t(1+\alpha\sin t)\right]$. This is the example of the problem for which quadrature method gives better approximation than collocation method (see Figure 4.2b).

Example 4. This is the example of approximate solution of Fredholm equation (4.13) with $a = 0,\ b = 1,\ \lambda = 1$ by the collocation algorithm. We use a kernel $K(t,s) = e^{-(t+s)}$ and

$$f(t) = [0, e^t + 1 - (2 - 1/e)e^{-t}] \times [0, 1 - e^{-t}(1 - 1/e)],\ t \in [0,1].$$

The exact solution of the problem in this case is $\varphi(t) = \left[0, e^t + 1\right] \times [0,1]$ and as it was in all previous examples we plot the Hausdorff distance (the error) between the exact solution and the solution obtained by described in Section 4.4.1 algorithm, though here to plot results we use a logarithmic scale for the y-axis (see Figure 4.3a). We plot error for 8, 16, 32, and 64 knots.

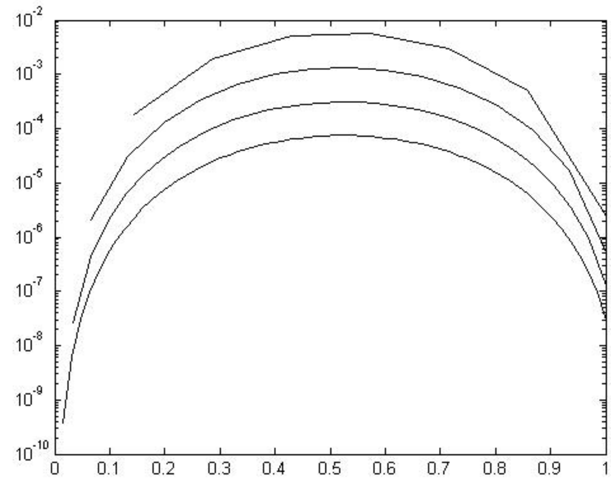
Example 5. This is another example of a collocation algorithm for the Fredholm Equation. This time we use a kernel $K(t,s) = s\sin^4(3t)$ and

$$f(t) = \left[0, 1 - \frac{1}{2}\sin^4(3t)\right] \times \left[0, 1 + t - \frac{5}{6}\sin^4(3t)\right],\ t \in [0,1].$$

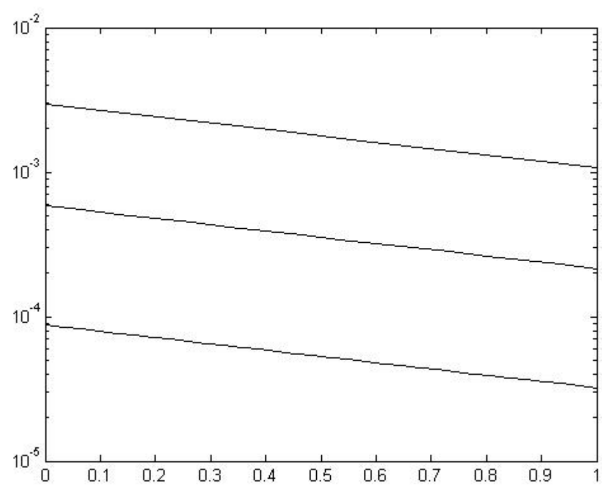
The exact solution has the form $\varphi(t) = [0,1] \times [0, 1+t]$. Results for 8, 16, and 32 knots are shown on Figure 4.3b.

4.7 Discussion

We have shown that many principles and concepts governing single valued integral equations transfer to the more general case of functions with values in L-spaces, particularly for set-valued functions, and functions whose values are fuzzy sets. The algorithms we discussed adopt the collocation method for the approximate solution of integral equations and use "piecewise-linear" functions. These algorithms converge at the rate of $O(1/n)$ if the functions defining the problem have smoothness of order 1, and converge at the rate of $O(1/n^2)$ if the functions have smoothness of order 2. In future work, we hope to obtain methods of the approximate solution of integral equations that will use alternative methods of approximation and will converge faster for functions of greater smoothness. We also plan to investigate the solution of more general, nonlinear integro-differential equations, and integral and differential equation problems involving functions of more than one independent variable, with values in L-spaces.



(a) Example 4. Kernel $K(t,s) = s\sin^4(3t)$. Error for 8, 16, 32, and 64 knots.



(b) Example 5. Kernel $K(t,s) = e^{-(t+s)}$. Error for 8, 16, and 32 knots.

Figure 4.3: Examples of a collocation algorithm for Fredholm Equation. Note we use here logarithmic scale for the y-axis.

CHAPTER 5

NONLINEAR INTEGRAL EQUATIONS IN L-SPACES

In this chapter, we consider nonlinear integral equations of Fredholm and Volterra type with respect to functions having values in L-spaces.

$$x(t) = f(t) + \lambda \int_a^b g(t, s, x(s))ds \quad \text{Fredholm Equation}$$

and

$$x(t) = f(t) + \int_a^t g(t, s, x(s))ds \qquad \text{Volterra Equation.}$$

We prove for these equations theorems of existence and uniqueness of their solutions and investigate data dependence of their solutions. Our results generalize the results of I. Tişe [63] for the case of $X = K^c(\mathbb{R}^n)$ and $f(t) = A,\ A \in K^c(\mathbb{R}^n)$.

5.1 Theorems of existence and uniqueness

5.1.1 Fredholm equation

Consider the set $Y = [a, b] \times [a, b] \times X$. In the space Y, introduce a metric assuming that for points $y = (t, s, x)$ and $y' = (t', s', x')$ from Y

$$d(x, y) = |t - t'| + |s - s'| + \delta(x, x').$$

Consider the Fredholm integral equation

$$x(t) = f(t) + \lambda \int_a^b g(t, s, x(s))ds, \tag{5.1}$$

where $f : [a, b] \to X$ and $g : Y \to X$ are known functions, λ is a fixed real parameter and $x : [a, b] \to X$ is an unknown function.

Theorem 11 *Suppose the function $f(t)$ is continuous on $[a, b]$ and the function $g(t, s, x)$ satisfies the following conditions*

1. *g is weakly continuous on Y, so the function $\widetilde{g} : Y \to X^c$ is continuous;*

2. *There exists a constant K such that for any $(t,s) \in [a,b]\times[a,b]$, the function $\widetilde{g}$ satisfies the Lipschitz condition with constant $K > 0$ on the variable x, so $\forall x', x'' \in X$*

$$\delta\left(\widetilde{g}(t,s,x'), \widetilde{g}(t,s,x'')\right) \le K\delta(x', x''). \tag{5.2}$$

Then if $|\lambda| < \frac{1}{K(b-a)}$, the equation (5.1) has a unique solution $x \in C([a,b], X)$.

Proof. Denote by $C([a,b],X)$ the space of continuous functions $x:\ [a,b] \to X$. Introduce in this space a metric

$$\rho(x,y) = \max_{t\in[a,b]} \delta(x(t), y(t)).$$

It is known (see, for example, [6]) that the obtained space is complete and separable.

We consider an operator A on the space $C([a,b],X)$ defined by

$$Ax(t) := f(t) + \lambda \int_a^b g(t,s,x(s))ds. \tag{5.3}$$

Next we show that $\forall x \in C([a,b],X)$ $Ax \in C([a,b],X)$. For this, it is enough to prove continuity of the operator

$$Bx(t) = \int_a^b g(t,s,x(s))ds.$$

Let the function x be given. Consider a set

$$M = \{(t,s,x(s)) : t,s \in [a,b]\} \subset Y.$$

Due to the continuity of the function x, this set is a compact subset of the space Y. The restriction of the function $\widetilde{g}$ on M is a continuous function on M and thus is uniformly continuous.

Take an arbitrary $\varepsilon > 0$ and choose $\eta > 0$ such that

$$(d((t,s,x(s)),(t',s',x(s'))) < \eta) \Rightarrow \tag{5.4}$$

$$\delta(\widetilde{g}(t,s,x(s)), \widetilde{g}(t',s',x(s'))) < \frac{\varepsilon}{b-a}.$$

Estimate

$$\delta(Bx(t'), Bx(t'')) = \delta\left(\int_a^b \widetilde{g}(t',s,x(s))ds, \int_a^b \widetilde{g}(t'',s,x(s))ds\right)$$

$$\le \int_a^b \delta(\widetilde{g}(t',s,x(s)), \widetilde{g}(t'',s,x(s)))ds.$$

Taking into account (5.4), we have that if $|t'-t''| < \eta$, then

$$\delta(Bx(t'), Bx(t'')) < \varepsilon.$$

Therefore, the function $Bx(t)$ is continuous.

Next we show that with $|\lambda| < \frac{1}{K(b-a)}$ the operator A is contractive. We have

$$\rho(Ax, Ay) = \max_{t\in[a,b]} \delta(Ax(t), Ay(t))$$

$$= \max_{t\in[a,b]} \delta\left(f(t) + \lambda \int_a^b g(t,s,x(s))ds, f(t) + \lambda \int_a^b g(t,s,y(s))ds\right)$$

$$\leq \max_{t\in[a,b]} \delta\left(\lambda \int_a^b \widetilde{g}(t,s,x(s))ds, \lambda \int_a^b \widetilde{g}(t,s,y(s))ds\right)$$

$$\leq |\lambda| \max_{t\in[a,b]} \int_a^b \delta(\widetilde{g}(t,s,x(s)), \widetilde{g}(t,s,y(s)))ds.$$

Due to the second condition of the Theorem 11 $\forall t, s$:

$$\delta(\widetilde{g}(t,s,x(s)), \widetilde{g}(t,s,y(s))) \quad \leq \quad K\delta(x(s), y(s)).$$

Therefore,

$$\rho(Ax, Ay) \leq |\lambda| \int_a^b K\delta(x(s), y(s))ds$$

$$\leq |\lambda| K \max_{s\in[a,b]} \delta(x(s), y(s)) \int_a^b ds$$

$$= |\lambda| K(b-a)\rho(x,y).$$

If $|\lambda| < \frac{1}{K(b-a)}$, then this operator is contractive and thus the equation (5.1) has a unique solution.

Remark 5 *Note that if the known function $f(t)$ in (5.1) is convex-valued ($f : [a,b] \to X^c$), then the solution of equation (5.1) also is convex-valued.*

5.1.2 Volterra equation

Consider the set $Y = [a,b] \times [a,b] \times X$. The nonlinear Volterra integral equation has the following form

$$x(t) = f(t) + \int_a^t g(t,s,x(s))ds, \tag{5.5}$$

where $x : [a,b] \to X$ is an unknown function, $g : Y \to X$ is a known function, and $f : [a,b] \to X$ is a known function.

Theorem 12 *Suppose the function $f(t)$ is continuous on $[a,b]$ and the function $g(t,s,x)$ satisfies the following conditions*

1. *g is weakly continuous on Y, so the function $\widetilde{g} : Y \to X^c$ is continuous.*

2. *There exists a constant K such that for any $(t,s) \in [a,b]\times[a,b]$, the function $\widetilde{g}$ satisfies the Lipschitz condition (5.2) with a constant K on variable x.*

Then the equation (5.5) has a unique solution $x \in C([a,b],X)$.

Proof. Consider an operator $A: \ C([a,b],X) \to C([a,b],X)$:

$$Ax(t) := f(t) + \int_a^t g(t,s,x(s))ds.$$

That fact that $Ax \in C([a,b],X)$ if $x \in C([a,b],X)$ can be derived similarly as in the previous section.

We have

$$\begin{aligned}\delta(Ax(t),Ay(t)) &= \delta\left(f(t) + \int_a^t g(t,s,x(s))ds, f(t) + \int_a^t g(t,s,y(s))ds\right)\\ &\leq \delta\left(\int_a^t g(t,s,x(s))ds, \int_a^t g(t,s,y(s))ds\right)\\ &\leq \int_a^t \delta(\widetilde{g}(t,s,x(s)),\widetilde{g}(t,s,y(s)))ds.\end{aligned}$$

Using the second condition of the Theorem 12, we obtain

$$\begin{aligned}\delta(Ax(t),Ay(t)) &\leq \int_a^t K\delta(x(s),y(s))ds \leq K \max_{s\in[a,b]} \delta(x(s),y(s)) \int_a^t ds \qquad (5.6)\\ &= K(t-a)\rho(x,y).\end{aligned}$$

Therefore,

$$\rho(Ax,Ay) \leq K(b-a)\rho(x,y),$$

and thus if $b-a < \frac{1}{K}$, then this operator is contractive and on any interval $[a,b]$, where $0 < b-a < \frac{1}{K}$ the equation (5.5) has a unique solution.

We now prove by induction, that $\forall n \geq 1$

$$\delta(A^n x(t), A^n y(t)) \leq \frac{(t-a)^n}{n!} K^n \rho(x,y). \qquad (5.7)$$

Inequality (5.6) is the induction base case. Assume that

$$\delta(A^{n-1}x(t), A^{n-1}y(t)) \leq \frac{(t-a)^{n-1}}{(n-1)!} K^{n-1} \rho(x,y).$$

Then,

$$\delta(A^n x(t), A^n y(t)) = \delta\left(f(t) + \int_a^t g(t,s,A^{n-1}x(s))ds, f(t) + \int_a^t g(t,s,A^{n-1}y(s))ds\right)$$

$$\leq \delta\left(\int_a^t \widetilde{g}(t,s,A^{n-1}x(s))ds, \int_a^t \widetilde{g}(t,s,A^{n-1}y(s))ds\right)$$

$$\leq \int_a^t \delta\left(\widetilde{g}(t,s,A^{n-1}x(s)), \widetilde{g}(t,s,A^{n-1}y(s))\right) ds$$

$$\leq \int_a^t K\delta\left(A^{n-1}x(s), A^{n-1}y(s)\right) ds$$

$$\leq \int_a^t K\frac{(s-a)^{n-1}}{(n-1)!}K^{n-1}\rho(x,y)ds$$

$$= \frac{K^n}{(n-1)!}\rho(x,y)\int_a^t (s-a)^{n-1}ds = \frac{(t-a)^n}{n!}K^n\rho(x,y).$$

Therefore, inequality (5.7) is proved. It implies that for any $a < b$ and any n

$$\rho(A^n x, A^n y) = \max_{a\leq t\leq b} \delta(A^n x(t), A^n y(t)) \leq \frac{(b-a)^n}{n!}K^n\rho(x,y).$$

If n is sufficiently large, then $\frac{(b-a)^n}{n!}K^n < 1$, and therefore A^n is a contractive operator. Using the generalized contractive mapping principle (see [41, ch.2 §14]) we see that the operator A has a unique fixed point. Thus, the equation (5.5) has a unique solution in the space $C([a,b],X)$. □

5.2 Initial and boundary value problems for differential equations with Hukuhara type derivatives

Various problems in the theory of ordinary differential equations lead to systems of the following form

$$\dot{x}(t) = A(t)x(t) + f(t,x(t)). \tag{5.8}$$

For the case of real-valued functions in the system (5.8) $x(t)$ is the vector-valued function in $\mathbb{R}^n$, $A(t)$ is a square matrix of order n with real-valued entries, and $f(t,x)$ is a vector-valued function again with values in $\mathbb{R}^n$, defined on some interval $[a,b)$, $b \leq \infty$.

If one is interested in proving the existence of (5.8), satisfying the given initial condition

$$x(a) = x_0, \;\; x_0 \in \mathbb{R}^n, \tag{5.9}$$

then the standard and more convenient approach is to rewrite this problem (5.8),(5.9) as an integral equation with the same unknown function $x(t)$. The obtained integral equation is a Volterra integral equation of the second kind. On details of how to rewrite one form to another, see [25]. For set- and fuzzy-valued cases, differential equations of such type were actively studied in, for example, [46] and [45].

Here we show that the analogous results for L-space also hold, e.g., the solution of the initial value problem is also the solution of the Volterra integral equation.

We consider first the initial value problem in the following linear form

$$D_H x(t) = A(t)x(t) + F(t), \quad x(a) = x_0, \tag{5.10}$$

where $A(t)$ is a real-valued function and $F(t)$ is a function with values in an L-space. In (5.10), we obtain, using the fundamental theorem of calculus

$$x(t) \overset{h}{-} x_0 = \int_a^t (A(s)x(s) + F(s))ds.$$

Further,

$$x(t) = x_0 + \int_a^t A(s)x(s)ds + \int_a^t F(s)ds.$$

Finally, if we set $x_0 + \int_a^t F(s)ds = f(t)$, we have

$$x(t) = f(t) + \int_a^t A(s)x(s)ds,$$

which is the same as Volterra integral equation form (4.14) that we considered earlier.

Now, we consider the initial value problem in the nonlinear form

$$D_H x(t) = A(t)x(t) + F(t, x(t)), \quad x(a) = x_0, \tag{5.11}$$

where $A(t)$ is again a real-valued function, $F(t, s)$ is a function of two variables with values in L-space. We apply the fundamental theorem of calculus and have

$$x(t) \overset{h}{-} x_0 = \int_a^t (A(s)x(s) + F(s, x(s)))ds.$$

Set $A(s)x(s) + F(s, x(s)) = g(s, x(s))$, we have

$$x(t) = x_0 + \int_a^t g(s, x(s))ds,$$

which is the same as nonlinear Volterra integral equation form (5.5).

Next we look for the relation between the boundary value problem and a Fredholm integral equation. For the results in the case of systems of equations for real-valued functions, see [25].

We consider the differential equation in the following form

$$D_H x(t) = f(t, x(t)), \tag{5.12}$$

where $f(t, x(t))$ is continuous mapping on $[a, b] \times D$, $(D \subset X)$ with values in X (X is an L-space). We assume $x(t)$ to be convex for any t, continuously differentiable in Hukuhara sense function with values in L-space, and we look for solutions of the differential equation (5.12) that satisfy

$$\int_a^b x(t)P'(t)dt = \int_a^b F(t,x(t))dt, \tag{5.13}$$

where $P(t)$ is a real-valued function which is differentiable, nonnegative, monotone, and nondecreasing on the interval $[a,b]$.

From (5.12), we obtain

$$x(t) = x(a) + \int_a^t f(s,x(s))ds, \quad t \in [a,b]. \tag{5.14}$$

Due to the conditions imposed above on the function $P(t)$, we can apply the integration by parts formula (3.22) to the left-hand side of (5.13), and it yields the relation

$$\int_a^b x(t)P'(t)dt = \left(P(b)x(b) \overset{h}{-} P(a)x(a)\right) \overset{h}{-} \int_a^b P(t)D_H x(t)dt. \tag{5.15}$$

If $x(t)$ is a solution of (5.12), (5.13), then (5.15) leads to

$$\left(P(b)x(b) \overset{h}{-} P(a)x(a)\right) \overset{h}{-} \int_a^b P(t)D_H x(t)dt = \int_a^b F(t,x(t))dt. \tag{5.16}$$

However,

$$x(b) = x(a) + \int_a^b f(t,x(t))dt$$

and if we substitute it into (5.16), we have

$$\left(P(b)\left(x(a) + \int_a^b f(t,x(t))dt\right) \overset{h}{-} P(a)x(a)\right) \overset{h}{-} \int_a^b P(t)D_H x(t)dt = \int_a^b F(t,x(t))dt.$$

According to the property (3.4) of Hukuhara type difference and since $x(a)$ is convex

$$\left(P(b)\int_a^b f(t,x(t))dt + (P(b)-P(a))x(a)\right) \overset{h}{-} \int_a^b P(t)f(t,x(t))dt = \int_a^b F(t,x(t))dt.$$

According to the property (3.1) of Hukuhara type difference

$$\int_a^b P(b)f(t,x(t))dt \overset{h}{-} \int_a^b P(t)f(t,x(t))dt + (P(b)-P(a))x(a) = \int_a^b F(t,x(t))dt.$$

Furthermore,

$$\int_a^b \left(P(b)f(t,x(t)) \overset{h}{-} P(t)f(t,x(t))\right) dt + (P(b)-P(a))x(a) = \int_a^b F(t,x(t))dt,$$

and

$$(P(b)-P(a))x(a) = \int_a^b F(t,x(t))dt \overset{h}{-} \int_a^b \left(P(b)f(t,x(t)) \overset{h}{-} P(t)f(t,x(t))\right) dt.$$

Next,

$$x(a) = \frac{1}{P(b) - P(a)} \left(\int_a^b F(t, x(t)) \stackrel{h}{-} \left(P(b) f(t, x(t)) \stackrel{h}{-} P(t) f(t, x(t)) \right) dt \right).$$

Once we substitute the last equality in (5.14), we have

$$x(t) = \int_a^b G(t, s, x(s)) ds, \tag{5.17}$$

where

$$G(t, s, x(s)) = \begin{cases} \frac{1}{P(b)-P(a)} \left(F(t, x(t)) \stackrel{h}{-} \left(P(a) f(t, x(t)) \stackrel{h}{-} P(t) f(t, x(t)) \right) \right) & \text{if} \quad a \leq s \leq t \\ \frac{1}{P(b)-P(a)} \left(F(t, x(t)) \stackrel{h}{-} \left(P(b) f(t, x(t)) \stackrel{h}{-} P(t) f(t, x(t)) \right) \right) & \text{if} \quad t \leq s \leq b \end{cases} \tag{5.18}$$

Equation (5.17) is an integral equation of Fredholm type.

Note that with the method of approximate solution of integral equations, we have also the method of approximate solution of the Cauchy problem and boundary value problem for differential equation with Hukuhara type derivative.

5.3 Data dependence

In this section, we consider questions about the dependence of solutions of equations (5.1) and (5.5) on perturbations of the given functions $g(t, s, x)$ and $f(t)$. These questions were considered in [63] for set-valued functions. Upon receipt of the results, we use some of the ideas from the work [63] for our more general setting.

Theorem 13 *Consider the set $Y = [a, b] \times [a, b] \times X$ and let $g_1,\ g_2 :\ Y \ \to\ X$ be weakly continuous. Consider also the following equations:*

$$x(t) = f_1(t) + \int_a^b g_1(t, s, x(s)) ds, \tag{5.19}$$

and

$$y(t) = f_2(t) + \int_a^b g_2(t, s, y(s)) ds. \tag{5.20}$$

Suppose:

1. *For any $(t, s) \in [a, b] \times [a, b]$ the function $\widetilde{g}_1(t, s, x)$ satisfies Lipschitz condition (5.2) on variable x and $K(b - a) < 1$. Denote by $x^*(t)$ the unique solution of the equation (5.19).*

2. *There exist $\eta_1, \eta_2 > 0$ such that $\delta(\widetilde{g}_1(t, s, x), \widetilde{g}_2(t, s, x)) \leq \eta_1$ for all $(t, s, x) \in [a, b] \times [a, b] \times X$ and $\rho(f_1(t), f_2(t)) \leq \eta_2$.*

3. There exists $y^*(t)$ a solution of the equation (5.20).

Then

$$\rho(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - K(b-a)}.$$

Proof. We have

$$\delta(x^*(t), y^*(t)) = \delta\left(f_1(t) + \int_a^b \widetilde{g}_1(t,s,x^*(s))ds, f_2(t) + \int_a^b \widetilde{g}_2(t,s,y^*(s))ds\right)$$

$$\leq \delta\left(\int_a^b \widetilde{g}_1(t,s,x^*(s))ds, \int_a^b \widetilde{g}_2(t,s,y^*(s))ds\right) + \delta(f_1(t), f_2(t))$$

$$\leq \delta\left(\int_a^b \widetilde{g}_1(t,s,x^*(s))ds, \int_a^b \widetilde{g}_1(t,s,y^*(s))ds\right)$$

$$+\delta\left(\int_a^b \widetilde{g}_1(t,s,y^*(s))ds, \int_a^b \widetilde{g}_2(t,s,y^*(s))ds\right) + \eta_2$$

$$\leq \int_a^b \delta(\widetilde{g}_1(t,s,x^*(s)), \widetilde{g}_1(t,s,y^*(s)))ds + \int_a^b \delta(\widetilde{g}_1(t,s,y^*(s)), \widetilde{g}_2(t,s,y^*(s)))ds + \eta_2$$

$$\leq \int_a^b K\delta(x^*(s), y^*(s))ds + \int_a^b \eta_1 ds + \eta_2.$$

By taking the maximum for $t \in [a,b]$, we have:

$$\rho(x^*, y^*) \leq \max_{t\in[a,b]}\left(K\int_a^b \delta(x^*(t), y^*(t))dt + \eta_1(b-a) + \eta_2\right)$$

$$= K \max_{t\in[a,b]} \delta(x^*(t), y^*(t)) \int_a^b dt + \eta_1(b-a) + \eta_2$$

$$= K \max_{t\in[a,b]} \delta(x^*(t), y^*(t))(b-a) + \eta_1(b-a) + \eta_2.$$

Therefore,

$$\rho(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - K(b-a)}. \ \Box$$

Consider now the question of data dependence for Volterra integral equations. We need the following metric on $C([a,b], X)$:

$$\rho_*(x,y) := \max_{t\in[a,b]}[\delta(x(t), y(t))e^{-\tau(t-a)}], \quad \text{with arbitrary } \tau > 0.$$

The pair $(C([a,b], X), \rho_*)$ forms a complete metric space.

It is easily seen that metrics ρ and ρ_* satisfy the following inequalities

$$e^{-\tau(b-a)}\rho(x,y) \leq \rho_*(x,y) \leq \rho(x,y). \tag{5.21}$$

We now prove the following theorem

Theorem 14 *Let $Y = [a,b] \times [a,b] \times X$ and let $g_1,\ g_2 : Y \to X$ be weakly continuous. Consider the following equations:*

$$x(t) = f_1(t) + \int_a^t g_1(t,s,x(s))ds, \tag{5.22}$$

and

$$y(t) = f_2(t) + \int_a^t g_2(t,s,y(s))ds. \tag{5.23}$$

Suppose:

1. *For any $(t,s) \in [a,b] \times [a,b]$ function $\widetilde{g}_1(t,s,x)$ satisfies Lipschitz condition (5.2) on variable x. (Denote by $x^*(t)$ the unique solution of the equation (5.22).)*
2. *There exist $\eta_1, \eta_2 > 0$ such that $\delta(\widetilde{g}_1(t,s,x), \widetilde{g}_2(t,s,x)) \le \eta_1$ for all $(t,s,x) \in [a,b] \times [a,b] \times X$ and $\rho(f_1(t), f_2(t)) \le \eta_2$.*
3. *There exists $y^*(t)$ a solution of the equation (5.23).*

Then

$$\rho_*(x^*, y^*) \le \frac{\eta_2 + \eta_1(b-a)}{1 - \frac{K}{\tau}} e^{-\tau(b-a)} \ (\textit{where } \tau > K) \tag{5.24}$$

and moreover

$$\rho(x^*, y^*) \le \eta_2 + \eta_1(b-a).$$

Proof. We estimate

$$\begin{aligned}
\delta(x^*(t), y^*(t)) &= \delta\left(f_1(t) + \int_a^t \widetilde{g}_1(t,s,x^*(s))ds, f_2(t) + \int_a^t \widetilde{g}_2(t,s,y^*(s))ds\right) \\
&\le \delta\left(\int_a^t \widetilde{g}_1(t,s,x^*(s))ds, \int_a^t \widetilde{g}_2(t,s,y^*(s))ds\right) + \delta(f_1(t), f_2(t)) \\
&\le \delta\left(\int_a^t \widetilde{g}_1(t,s,x^*(s))ds, \int_a^t \widetilde{g}_1(t,s,y^*(s))ds\right) \\
&\quad + \delta\left(\int_a^t \widetilde{g}_1(t,s,y^*(s))ds, \int_a^t \widetilde{g}_2(t,s,y^*(s))ds\right) + \eta_2 \\
&\le \int_a^t \delta(\widetilde{g}_1(t,s,x^*(s)), \widetilde{g}_1(t,s,y^*(s)))ds + \int_a^t \delta(\widetilde{g}_1(t,s,y^*(s)), \widetilde{g}_2(t,s,y^*(s)))ds + \eta_2 \\
&\le \int_a^t K\delta(x^*(s), y^*(s))e^{-\tau(s-a)}e^{\tau(s-a)}ds + \int_a^t \eta_1 ds + \eta_2.
\end{aligned}$$

By taking the maximum for $t \in [a,b]$, we have:

$$\max_{t\in[a,b]}(\delta(x^*(t), y^*(t))e^{-\tau(t-a)}e^{\tau(t-a)})$$

$$\leq \max_{t\in[a,b]} \left(K \int_a^t \delta(x^*(s), y^*(s)) e^{-\tau(s-a)} e^{\tau(s-a)} dt + \int_a^t \eta_1 ds + \eta_2 \right)$$

and, therefore,

$$\begin{aligned} \rho_*(x^*, y^*) e^{\tau(b-a)} &= K \rho_*(x^*, y^*) \int_a^t e^{\tau(s-a)} ds + \eta_1(b-a) + \eta_2 \\ &= \frac{K}{\tau} \rho_*(x^*, y^*)(e^{\tau(t-a)} - 1) + \eta_1(b-a) + \eta_2 \\ &\leq \frac{K}{\tau} \rho_*(x^*, y^*) e^{\tau(b-a)} + \eta_1(b-a) + \eta_2. \end{aligned}$$

From the derived inequality, for $\tau > K$, we obtain

$$\rho_*(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - \frac{K}{\tau}} e^{-\tau(b-a)}.$$

The inequality (5.24) is proved. Using the last inequality and (5.21), we have

$$\rho(x^*, y^*) \leq e^{\tau(b-a)} \rho_*(x^*, y^*) \leq \frac{\eta_2 + \eta_1(b-a)}{1 - \frac{K}{\tau}}.$$

Since this is true for any $\tau > K$, we obtain

$$\rho(x^*, y^*) \leq \eta_2 + \eta_1(b-a). \ \square$$

5.4 Discussion

Along with equations (5.1) and (5.5), equations of the following form are interesting:

$$x(t) + \lambda \int_a^b g(t, s, x(s)) ds = f(t), \quad t \in [a, b], \tag{5.25}$$

and

$$x(t) + \int_a^t g(t, s, x(s)) ds = f(t), \quad t \in [a, b]. \tag{5.26}$$

Equations (5.25) and (5.26) are equivalent to equations (5.1) and (5.5), respectively, in the case of real-valued functions, but not for functions with values in L-spaces.

Equations (5.25) and (5.26) are equivalent to equations

$$x(t) = f(t) \overset{h}{-} \lambda \int_a^b g(t, s, x(s)) ds, \quad t \in [a, b],$$

and

$$x(t) = f(t) \overset{h}{-} \int_a^t g(t, s, x(s)) ds, \quad t \in [a, b].$$

The fact that the Hukuhara type difference is not defined for all elements of the L-spaces brings significant difficulties into the investigation of existence and uniqueness of the solutions of these equations. We know only two references [56] and [57] in which theorems of existence and uniqueness are proved for equations of the form (5.26) in the space $K^c(\mathbb{R}^n)$ for the special case the $f(t) \equiv a$, where a is fixed element of $K^c(\mathbb{R}^n)$. We will study the existence and uniqueness of solutions of (5.25) and (5.26) in future work.

CHAPTER 6

APPROXIMATION IN L-SPACES BY CLASSICAL OPERATORS

In this chapter, we show how classical approximation operators (such as Bernstein, Schoenberg and Modified Schoenberg Operator) can be adapted to functions with values in L-spaces.

6.1 General definitions and estimations

Many classical approximation operators for real-valued functions are defined by the following scheme. Let the set of functions $\lambda = \{\lambda_0(t), ..., \lambda_N(t)\} \subset C([0,1])$ be given, such that $\lambda_i(t) \geq 0$ and

$$\sum_{i=0}^{N} \lambda_i(t) \equiv 1.$$

Let also the set of points $\xi = \{\xi_0, ..., \xi_N\} \subset [0,1]$ be given. For each function $f \in C[0,1]$, we define

$$\Lambda_{\lambda,\xi}[f](t) = \sum_{k=0}^{N} f(\xi_k)\lambda_k(t). \tag{6.1}$$

In this definition, function $f \in C[0,1]$ can be replaced by a function $f \in C([0,1], X)$. We get the operator that is defined on the set of all continuous functions with values in L-spaces. Thus, the definition (6.1) we use for both functions $f \in C[0,1]$ and for $f \in C([0,1], X)$.

The following theorem gives general estimation of approximation of the function $f \in C([0,1], X^c)$ by the operators of the form (6.1).

Theorem 15 *Suppose that for the function $g(t) = t$*

$$\Lambda_{\lambda,\xi}[g](t) \equiv t. \tag{6.2}$$

Then for any function $f \in C([0,1], X^c)$ and for any $t \in [0,1]$, the following inequality holds

$$\delta(f(t), \Lambda_{\lambda,\xi}[f](t)) \leq \omega^* \left(f, \sqrt{\Lambda_{\lambda,\xi}[(\cdot)^2](t) - t^2} \right).$$

Proof. We have

$$\delta(f(t), \Lambda_{\lambda,\xi}[f](t)) = \delta\left(\sum_{k=0}^{N} f(t)\lambda_k(t), \sum_{k=0}^{N} f(\xi_k)\lambda_k(t)\right)$$

$$\leq \sum_{k=0}^{N} \delta(f(t), f(\xi_k))\lambda_k(t) \leq \sum_{k=0}^{N} \omega^*(f, |t-\xi_k|)\lambda_k(t).$$

By using the fact that the function ω^* and the square root are concave and using Jensen's inequality, we have

$$\delta(f(t), \Lambda_{\lambda,\xi}[f](t)) \leq \omega^*\left(f, \sum_{k=0}^{N} \sqrt{(t-\xi_k)^2}\lambda_k(t)\right)$$

$$\leq \omega^*\left(f, \sqrt{\sum_{k=0}^{N}(t-\xi_k)^2\lambda_k(t)}\right)$$

$$= \omega^*\left(f, \sqrt{\sum_{k=0}^{N}\xi_k^2\lambda_k(t) - 2\sum_{k=0}^{N} t\xi_k\lambda_k(t) + t^2}\right).$$

Due to (6.2)

$$\sum_{k=0}^{N} \xi_k\lambda_k(t) = t,$$

and thus, we have

$$\delta(f(t), \Lambda_{\lambda,\xi}[f](t)) \leq \omega^*\left(f, \sqrt{\Lambda_{\lambda,\xi}[(\cdot)^2](t) - t^2}\right). \quad \square$$

6.2 Bernstein operator

Let now

$$\lambda_k(t) = \binom{N}{k} t^k(1-t)^{N-k}, \quad \xi_k = \frac{k}{N}, \quad k = 0, 1, ..., N.$$

We obtain the analog of the Bernstein Operator for functions with values in L-space

$$B_N[f](t) = \sum_{k=0}^{N} f\left(\frac{k}{N}\right)\binom{N}{k} t^k(1-t)^{N-k}.$$

For this operator, the conditions of the Theorem 15 hold.

Moreover,

$$B_N[(\cdot)^2](t) = \frac{t-t^2}{N}.$$

Thus, the following theorem holds.

Theorem 16 *For any $f \in C([0,1], X^c)$ and for any $t \in [0,1]$*

$$\delta(f(t), B_N[f](t)) \leq \omega^* \left(f, \sqrt{\frac{t(1-t)}{N}} \right),$$

and therefore,

$$\rho(f, B_N[f]) \leq \omega^* \left(f, \frac{1}{2\sqrt{N}} \right).$$

6.3 Schoenberg operator

For integers $N, k > 0$, we consider the sequence of knots $\Delta = \{t_j\}_{j=-k}^{N+k}$

$$t_{-k} = ... = t_0 = 0 < t_1 < ... < t_N = ... = t_{N+k} = 1.$$

We denote

$$|\Delta| = \max_{j=-k,...,N+k-1} \{t_{j+1} - t_j\}.$$

Let also,

$$\xi_{j,k} := \frac{t_{j+1} + ... + t_{j+k}}{k}, \quad -k \leq j \leq N-1,$$

and

$$b_{j,k}(t) := (t_{j+k+1} - t_j)[t_j, ..., t_{j+k+1}](\cdot - t)_+^k$$

be the normalized B-splines.

For $f \in C([0,1], X^c)$ let

$$S_{N,k}[f](t) = \sum_{j=-k}^{N-1} f(\xi_{j,k}) b_{j,k}(t), \quad 0 \leq t < 1,$$

and

$$S_{N,k}[f](1) = \lim_{y \nearrow 1} S_{N,k}[f](y).$$

For real-valued functions, the operator $S_{N,k}$ was introduced by Schoenberg in 1965, see [62].

The normalized B-splines form a partition of the unity

$$\sum_{j=-k}^{N-1} b_{j,k}(t) = 1,$$

and the Schoenberg operator reproduces linear functions(see [18], [50]), i.e.,

$$\sum_{j=-k}^{N-1} \xi_{j,k} b_{j,k}(t) = t.$$

For the function $g(t) = t^2$, the error $E(t) = S_{N,k}[g](t) - g(t)$ satisfies ([18], [50])

$$0 \le E(t) = \sum_{j=-k}^{N-1} \xi_{j,k}^2 b_{j,k}(t) - t^2 \le \left(\min \left\{ \frac{1}{\sqrt{2k}}, \sqrt{\frac{k+1}{12}} |\Delta| \right\} \right)^2$$

and besides that for $N \ge 1,\ k \ge 1,\ t \in [0,1]$, the following pointwise estimation holds (see [18])

$$E(t) \le \frac{\min\{2t(1-t), \frac{k}{N}\}}{N+k-1}.$$

Taking into account Theorem 15, we obtain

Theorem 17 *For any* $f \in C([0,1], X^c)$

$$\delta(f(t), S_{N,k}[f](t)) \le \omega^* \left(f, \min \left\{ \frac{1}{\sqrt{2k}}, \sqrt{\frac{k+1}{12}} |\Delta| \right\} \right)$$

and for any $t \in [0,1]$

$$\delta(f(t), S_{N,k}[f](t)) \le \omega^* \left(f, \frac{\min\{2t(1-t), \frac{k}{N}\}}{N+k-1} \right).$$

6.4 Modified Schoenberg operator

Consider now a modification of a Schoenberg operator that we obtain when

$$t_j = \begin{cases} 0 & \text{if} \quad j = -k, ..., 0, \\ \frac{j}{N} & \text{if} \quad j = 1, ..., N, \\ 1 & \text{if} \quad j = N+1, ..., N+k \end{cases}$$

and $\xi_{jk} = t_j$ for $j = -k, ..., N+k$.

The obtained operator has the form

$$\widetilde{S}_{N,k}[f](t) = \sum_{j=-k}^{N-1} f(t_j) b_{j,k}(t)$$

and

$$\widetilde{S}_{N,k}[f](1) = \lim_{t \nearrow 1} \widetilde{S}_{N,k}[f](t).$$

Next we get the estimation of the approximation by such an operator.

Theorem 18 *For any* $f \in C([0,1], X^c)$

$$\rho\left(f, \widetilde{S}_{N,k}[f]\right) \le 2\omega^* \left(f, \frac{k+1}{2N} \right).$$

Proof. For any $t \in [0,1]$, we have

$$\delta\left(f(t), \widetilde{S}_{N,k}[f](t)\right) \leq \delta\left(f(t), S_{N,k}[f](t)\right) + \delta\left(S_{N,k}[f](t), \widetilde{S}_{N,k}[f](t)\right). \tag{6.3}$$

Due to the definition of $\xi_{j,k}$ for any $j = -k, ..., N-1$ we have $|t_j - \xi_{j,k}| \leq \frac{k+1}{N}$. Thus,

$$\begin{aligned}
\delta\left(S_{N,k}[f](t), \widetilde{S}_{N,k}[f](t)\right) &\leq \delta\left(\sum_{j=-k}^{N-1} f(\xi_{j,k}) b_{j,k}(t), \sum_{j=-k}^{N-1} f(t_j) b_{j,k}(t)\right) \\
&\leq \sum_{j=-k}^{N-1} \delta\left(f(\xi_{j,k}), f(t_j)\right) b_{j,k}(t) \qquad (6.4) \\
&\leq \sum_{j=-k}^{N-1} \omega^*(f, |t_j - \xi_{j,k}|) b_{j,k}(t) \leq \omega^*\left(f, \frac{k+1}{2N}\right).
\end{aligned}$$

We use Theorem 17 to estimate first term and inequality (6.4) to estimate second term at (6.3). We have

$$\begin{aligned}
\delta\left(f(t), S_{N,k}[f](t)\right) &\leq \omega^*\left(f, \frac{k+1}{12}\frac{1}{N}\right) + \omega^*\left(f, \frac{k+1}{2N}\right) \\
&\leq 2\omega^*\left(f, \frac{1}{2}\frac{k+1}{12} + \frac{1}{2}\frac{k+1}{2N}\right) \\
&= 2\omega^*\left(f, \frac{k+1}{4N}\left(1 + \frac{1}{\sqrt{k+1}\sqrt{3}}\right)\right) \leq 2\omega^*\left(f, \frac{k+1}{2N}\right). \quad \square
\end{aligned}$$

6.5 Discussion

An adaptation of various classical approximation operators for the set-valued functions was proposed by R.A. Vitale (positive linear operators and in particular Bernstein operator were discussed in detail in [66]), Z. Artstein (piecewise linear approximation see in [5]), and N. Dyn et al. (see [31] for results on approximations of set-valued functions based on the metric average, [28] for functions with compact images, [29] on approximations by metric linear operators). Set-valued approximation was also discussed in [54] and in [30]. M. Mureşan in [54] considered convex and non-convex cases. For a convex case (values of functions are convex), he presented results on the Bernstein approximation, Stone-Weierstrass approximation theorem, and Korovkin-type approximation. In the non-convex case, he showed results for metric Bernstein, Schoenberg, and interpolation operators, as well as metric piecewise linear approximation. N. Dyn and coauthors in [30] considered approximation methods of set-valued functions based on canonical representations and Minkowski convex combinations, as well as methods based on metric average, metric linear combinations, and metric selections. Regarding similar results for the fuzzy positive linear operators acting on fuzzy continuous functions (for example on fuzzy Bernstein operators), see [3] and the references therein. A survey of main results in classical approximation theory for fuzzy functions can be found in [32]; on Korovkin-type approximation, see [4] and [22].

Note also that for the Piecewise-linear interpolation discussed in Section 4.2.1, the interpolant that is written in the form (4.2) is a special case of the operator of the form (6.1). In the case $a = 0$ and $b = 1$, this operator satisfies conditions of the general Theorem 15. Moreover, if $t \in [t_{k-1}, t_k]$, then

$$P_N[(\cdot)^2](t) - t^2 = (t - t_{k-1})(t_k - t) = \frac{(Nt - k + 1)(k - Nt)}{N^2}.$$

Therefore, we obtain the following theorem

Theorem 19 *If $f \in C([0,1], X^c)$, then for $t \in [t_{k-1}, t_k],\ k = 1, ..., N$*

$$\delta(f(t), P_N[f](t)) \le \omega^* \left(f, \sqrt{(t - t_{k-1})(t_k - t)}\right) \tag{6.5}$$

$$= \omega^* \left(f, \frac{\sqrt{(Nt - k + 1)(k - Nt)}}{N}\right),$$

and thus,

$$\rho(f, P_N[f]) \le \omega^* \left(f, \frac{1}{2N}\right).$$

In particular, if f satisfies the Lipschitz condition (4.1) with constant M, then

$$\rho(f, P_N[f]) \le \frac{M}{2N}. \tag{6.6}$$

Remark 6 *Estimation (4.5) in the case $a = 0,\ b = 1$ for $t = (t_{k-1} + t_k)/2$ coincides with estimation (6.5). But for the rest of $t \in (t_{k-1}, t_k)$, the estimation (4.5) is better than (6.5).*

Finally, we expect that ideas that were used for functions of single variable with values in L-spaces in this work can be generalized to the case of functions of multivariables with values in L-spaces.

CHAPTER 7

APPROXIMATION IN THE SPACES OF SETS

This chapter is devoted to problems of approximation (in particular, interpolation) by generalized polynomials and splines for functions whose values lie in a specific L-space, namely a space of sets.

7.1 Introduction

Because the structures of spaces of sets are richer than the structure of general L-spaces, we have additional tools in the former space (e.g., the support function of a set), which allows us to obtain deeper results for the approximation and interpolation of set-valued functions.

We define several methods of approximation based on the concept of a support function of a set. Using these extra tools, it is possible to define, for example, the notion of interpolating polynomials. An early version of the definition of interpolating polynomials was proposed by Lempio in 1995. One of the disadvantages of Lempios definition is the fact that the values of such polynomials can be empty sets, which often makes the approach inapplicable to practical problems. We propose ways to circumvent these disadvantages. One of the approaches is to define interpolation polynomials in a modified way. Another is to define and use generalized interpolating splines instead of polynomials.

These approximation technics can be used, for instance, for the recovery of 3D objects using their cross-sections (with applications in tomography and image processing, among others). In the process, many questions emerged that were related to the error estimations of the approximation of set-valued functions by those novel methods.

7.2 Set-valued analog of linear operators

We define analogs of linear operators on the space $C([a,b])$ for the space $C([a,b],\mathcal{K}^c(\mathbb{R}^n))$. We start with the definition of a support function.

7.2.1 Support function

See the following definition of the support function and its properties for example in [8] and [37]. Below (a, ξ) is the ordinary inner product in $\mathbb{R}^m$ and $||\xi|| = \sqrt{(\xi, \xi)}$.

Definition 17 *The support function of the set $A \in \mathcal{K}(\mathbb{R}^m)$ is the function defined on $\mathbb{R}^m$*

$$\delta^*(\xi, A) := \max_{a \in A}(a, \xi), \ \xi \in \mathbb{R}^m.$$

Support functions have the following properties:

1. $\delta^*(\xi, \lambda A) = \delta^*(\lambda\xi, A) = \lambda\delta^*(\xi, A), \quad \lambda \geq 0;$

2. $\delta^*(\xi, A + B) = \delta^*(\xi, A) + \delta^*(\xi, B) \ \ A, B \in \mathcal{K}(\mathbb{R}^m);$

3. $(\forall \xi \in \mathbb{R}^m \ \delta^*(\xi, A) \leq \delta^*(\xi, B)) \ \Rightarrow \ \mathrm{co}A \subset \mathrm{co}B;$

4. $\forall A, B \in \mathcal{K}^c(\mathbb{R}^m)(A = B) \Leftrightarrow (\forall \xi \in \mathbb{R}^m \ \ \delta^*(\xi, A) = \delta^*(\xi, B));$

5. If $A \in \mathcal{K}^c(\mathbb{R}^n)$, then $A = \{z \in \mathbb{R}^n : \ (\xi, z) \leq \delta^*(\xi, A), \ \forall \xi \in \mathbb{R}^n\}$ (see Figure 7.1);

6. $\forall A, B \in \mathcal{K}^c(\mathbb{R}^m) \quad \delta^H(A, B) = \sup\limits_{\xi \in S^{m-1}} |\delta^*(\xi, A) - \delta^*(\xi, B)|$, (here S^{m-1} is the unit sphere in the space $\mathbb{R}^m$);

7. $\forall \xi \in \mathbb{R}^n \ \ \delta^*\left(\xi, \int\limits_a^b F(x)dx\right) = \int\limits_a^b \delta^*(\xi, F(x))dx$ (here the integral is Aumann's integral).

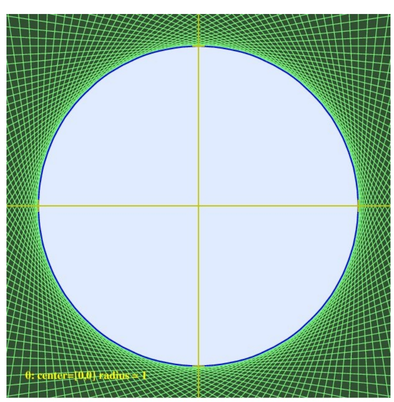


Figure 7.1: Set defined by half spaces

7.2.2 Definition of extended operator

Let $C[a,b]$ be the space of continuous functions $f:[a,b]\to\mathbb{R}$ with the norm

$$||f||_{C[a,b]} = \max_{t\in[a,b]} |f(t)|.$$

Denote by $C([a,b],\mathcal{K}^c(\mathbb{R}^n))$ the space of continuous set-valued functions

$$F:[a,b]\to\mathcal{K}^c(\mathbb{R}^n).$$

Let the linear bounded operator $A:\ C[a,b]\to C[a,b]$ be given. Next we use this operator A to define the operator

$$\widetilde{A}:C([a,b],\mathcal{K}^c(\mathbb{R}^n))\to C([a,b],\mathcal{K}^c(\mathbb{R}^n)).$$

We now investigate the properties of this operator $\widetilde{A}$.

Let function $F\in C([a,b],\mathcal{K}^c(\mathbb{R}^n))$ be given. For any $\xi\in\mathbb{R}^n$, consider the following function of variable $t\in[a,b]$

$$\delta^*(\xi,F(t)) =: F_\xi(t).$$

It is clear that for $\forall\ \xi\ F_\xi\in C[a,b]$. Once we apply the operator A to a function F_ξ, we have

$$A[F_\xi](t) = A[\delta^*(\xi,F(\cdot))](t).$$

For any t, we have a function defined on $\mathbb{R}^n$

$$\xi\to A[F_\xi](t). \tag{7.1}$$

This is a continuous, positively homogeneous, but not necessarily convex function.

Therefore, this function is not a support function of a convex set. Thus, if we try to define operator $\tilde{A}$ by the equality

$$\tilde{A}[F](t) = \{x\in\mathbb{R}^n : (\xi,x)\le A[F_\xi](t)\ \ \forall\xi\in\mathbb{R}^n\}, \tag{7.2}$$

we meet some difficulties. For example, for some t, the set in the right side of (7.2) can be an empty set (see for example [47]). Hence, it is very important to find criteria for when $A[F_\xi](t)$ is convex.

Lemma 8 *If the operator A is positive, then the function (7.1) is convex for $\forall t\in[a,b]$.*

Proof. We have due to convexity of δ^* and positivity of the operator A that

$$A[\delta^*(\alpha\xi_1 + (1-\alpha)\xi_2, F(\cdot))] \le A[\alpha\delta^*(\xi_1, F(t)) + (1-\alpha)\delta^*(\xi_2, F(t))].$$

Since A is linear

$$A[\alpha\delta^*(\xi_1, F(t)) + (1-\alpha)\delta^*(\xi_2, F(t))] = \alpha A[\delta^*(\xi_1, F(t))] + (1-\alpha)A[\delta^*(\xi_2, F(t))]$$

$$= \alpha A[F_{\xi_1}](t) + (1-\alpha)A[F_{\xi_2}](t).$$

Thus,

$$A[F_{\alpha\xi_1+(1-\alpha)\xi_2}](t) \le \alpha A[F_{\xi_1}](t) + (1-\alpha)A[F_{\xi_2}](t).$$

Therefore, for any positive operator, we have convexity on ξ.

Thus, the function (7.1) is positively homogeneous and convex on ξ. This implies that (7.1) is a support function of some nonempty, convex set and we can naturally introduce the next definition.

Definition 18

$$\widetilde{A}[F](t) := \{z \in \mathbb{R}^n : (\xi, z) \le A[F_\xi](t),\ \forall\xi \in \mathbb{R}^n\}.$$

Further, since $\delta^*(\xi, \widetilde{A}[F](t)) = A[\delta^*(\xi, F(\cdot))](t)$, we have for positive operator A:

$$\delta^h(F(t), \widetilde{A}[F](t)) = \sup_{\xi\in S^{n-1}} |\delta^*(\xi, F(t)) - \delta^*(\xi, \widetilde{A}[F](t))|$$

$$= \sup_{\xi\in S^{n-1}} |\delta^*(\xi, F(t)) - A[\delta^*(\xi, F(\cdot))](t)|.$$

Therefore, the problem of finding error estimates for positive operators is reduced to the problem of finding error estimates for real-valued functions.

If the operator A is not positive, we still can use Definition 18 and define

$$\widetilde{A}[F](t) := \{z \in \mathbb{R}^n : (\xi, z) \le A[\delta^*(\xi, F(\cdot))](t),\ \forall\xi \in \mathbb{R}^n\}.$$

However, in this case, operator $\widetilde{A}$ will be defined not for all functions $F \in C([a,b], \mathbb{R}^n)$ since in this case $A[F_\xi](t)$ in general will not be convex on ξ and for some t the set $\widetilde{A}[F](t)$ may be empty.

We denote by $D(\widetilde{A})$ a set of F such that $\widetilde{A}[F](t) \ne \emptyset\ \forall t \in [a,b]$. Set $D(\widetilde{A})$ we call the domain of set-valued linear operator.

7.2.3 Domain of set-valued linear operators

Let $D(\widetilde{A})$ denote the domain of $\widetilde{A}$. In this section, we study some of its properties.

Lemma 9 $F,\ G \in D(A) \Rightarrow\ F + G \in D(A)$

Proof.

$$\delta^*(\xi, F(t) + G(t)) = \delta^*(\xi, F(t)) + \delta^*(\xi, G(t))$$

$$\widetilde{A}[F + G](t) := \{z : (\xi, z) \leq A[\delta^*(\xi, F)] + A[\delta(\xi, G(t))]\ \forall\xi\}$$

$$\forall z_1 \in \widetilde{A}[F] \Longrightarrow \forall z_1 :\ (\xi, z_1) \leq A^*[\delta^*(\xi, F(t))]\ \forall\xi$$

and

$$\forall z_2 \in \widetilde{A}[G] \Longrightarrow \forall z_2 :\ (\xi, z_2) \leq A^*[\delta^*(\xi, G(t))]\ \forall\xi.$$

Therefore,

$$z_1 + z_2 \in \widetilde{A}[F + G](t).$$

This implies

$$\widetilde{A}[F + G](t) \supset \widetilde{A}[F](t) + \widetilde{A}[G](t).$$

The right-hand side is nonempty, and thus the left-hand side is nonempty too. □

Lemma 10 *If* $\alpha \geq 0$ *and* $F \in D(A) \Rightarrow\ \alpha F \in D(A)$.

Proof. We have

$$\tilde{A}[\alpha F](t) = \{z \in \mathbb{R}^n\ : (\xi, z) \leq A[\delta^*(\xi, \alpha F(\cdot))](t)\forall\xi \in \mathbb{R}^n\}$$

$$= \{z \in \mathbb{R}^n\ : (\xi, z) \leq A[\alpha\delta^*(\xi, F(\cdot))](t)\forall\xi \in \mathbb{R}^n\}$$

$$= \alpha\{z \in \mathbb{R}^n\ : (\xi, z) \leq A[\delta^*(\xi, F(\cdot))](t)\forall\xi \in \mathbb{R}^n\} = \alpha\tilde{A}[F](t).\quad \square$$

We now give sufficient conditions of $F \in D(\widetilde{A})$.

If an operator is given, then for any F, define

$$\varepsilon(t) = \varepsilon(F, t) = \sup_{\xi \in S^{n-1}} |\delta^*(\xi, F(t)) - A(\delta^*(\xi, F(t)))|.$$

The following Lemma was proved for interpolation polynomials by Lempio (see [47]).

Lemma 11 *If* $F(t) \supset B(m(t), r(t))$ *and* $\varepsilon(t) < r(t)$, *then* $\tilde{A}[F](t)$ *contains a ball with radius* $r(t) - \varepsilon(t)$ *and therefore* $\tilde{A}[F](t) \neq \emptyset$.

Proof. Since $B(m(t), r(t)) \subset F(t)$, then

$$\delta^*(\xi, B(m(t), r(t))) = (\xi, m(t)) + r(t)||\xi|| \leq \delta^*(\xi, F(t)).$$

Since $\delta^*\left(\frac{\xi}{||\xi||}, F(t)\right) \leq A\left[\delta^*\left(\frac{\xi}{||\xi||}, F(t)\right)\right] + \varepsilon(t)$ we have

$$(\xi, m(t)) + r(t)||\xi|| \leq ||\xi||\delta^*\left(\frac{\xi}{||\xi||}, F(t)\right)$$

$$\leq ||\xi||\left[A\left(\delta^*\left(\frac{\xi}{||\xi||}, F(t)\right) + \varepsilon(t)\right)\right] = A(\delta^*(\xi, F(t))) + \varepsilon(t)||\xi||.$$

Therefore,

$$(\xi, m(t)) + (r(t) - \varepsilon(t))||\xi|| \leq A[\delta^*(\xi, F(t))].$$

And thus,

$$B(m(t), r(t) - \varepsilon(t)) \subset \widetilde{A}[F(t)]. \quad \square$$

7.2.4 Properties of set-valued linear operators

Lemma 12 *If the operator A is positive, then the operator $\widetilde{A}$ is monotone with respect to inclusion, i.e., if for $\forall t$ $F(t) \subset G(t)$, it implies that for $\forall t$ $\widetilde{A}[F](t) \subset \widetilde{A}[G](t)$.*

Proof. We have that for $\forall \xi$ and for $\forall t$

$$\delta^*(\xi, F(t)) \leq \delta^*(\xi, G(t)).$$

Because the operator A is positive, we have

$$A[F_\xi](t) \leq A[G_\xi](t).$$

Thus,

$$\widetilde{A}[F](t) := \{z \in \mathbb{R}^n : (\xi, z) \leq A[F_\xi](t),\ \forall \xi \in \mathbb{R}^n\}$$

$$\subset \{z \in \mathbb{R}^n : (\xi, z) \leq A[G_\xi](t),\ \forall \xi \in \mathbb{R}^n\} = \widetilde{A}[G](t). \quad \square$$

The following property is given for the particular case when A is an interpolation operator.

Lemma 13 *Interpolants are invariant under affine transformations.*

Let points $t_j \in [a, b]$ be given. Consider $F : [a, b] \to \mathcal{K}^c(\mathbb{R}^n)$. We now define "interpolation polynomial".

Take the support function $\delta^*(\xi, F(t))$. We can build for it a numerical interpolation polynomial $p_N(\xi, t)$ that interpolates $\delta^*(\xi, F(t))$ at points t_j.

Define

$$P_N(t) := \{z \in \mathbb{R}^n : (\xi, z) \le p_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^n\}.$$

We also can consider

$$\widetilde{P}_N(t) := \{z \in \mathbb{R}^n : (\xi, z) \le p_N(\xi, t) \ \ \forall \xi \in S^n\},$$

but for our purposes, they are the same since we just need to normalize our function to write it in terms of support function instead of support function.

Let an affine transformation $L(x) = Ax + b$ from $\mathbb{R}^m \to \mathbb{R}^m$ be given, where A is nonsingular matrix and b is a vector from $\mathbb{R}^m$. Now take

$$(L \circ F)(t) = L(F(t)) = A(F(t)) + b = \{Ax + b : \ x \in F(t)\}$$

Find "interpolation polynomial" for $L \circ F$. For this, first, find support function for $(L \circ F)(t)$:

$$\begin{aligned}
\delta^*(\xi, A(F(t)) + b) &= \sup_{z \in A(F(t))+b} (\xi, z) = \sup_{z = Ax+b, x \in F(t)} (\xi, z) \\
&= \sup_{x \in F(t)} (\xi, Ax + b) = \sup_{x \in F(t)} (\xi, Ax) + (\xi, b) \\
&= \sup_{x \in F(t)} (A^*\xi, x) + (\xi, b) = \delta^*(A^*\xi, F(t)) + (\xi, b).
\end{aligned}$$

Let $q_N(\xi, t)$ interpolate $\delta^*(A^*\xi, F(t)) + (\xi, b)$ at points t_j. Note that

$$q_N(\xi, t) = q'_N(\xi, t) + (\xi, b),$$

where $q'_N(\xi, t)$ interpolate $\delta^*(A^*\xi, F(t))$ and therefore $q'_N(\xi, t) = p_N(A^*\xi, t)$.

Consider

$$\begin{aligned}
Q_N(t) &= \{z \in \ \mathbb{R}^m : (\xi, z) \le q_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{z \in \mathbb{R}^m : \ (\xi, z) \le q'_N(\xi, t) + (\xi, b) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{z \in \mathbb{R}^m : \ (\xi, z - b) \le q'_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{z \in \mathbb{R}^m : \ (\xi, AA^{-1}(z - b)) \le q'_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{z \in \mathbb{R}^m : \ (A^*\xi, A^{-1}(z - b)) \le q'_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\}.
\end{aligned}$$

Let $w = A^{-1}(z - b)$, $z = Aw + b$ and

$$\begin{aligned}
Q_N(t) &= \{Aw + b : \ (A^*\xi, w) \le q'_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{Aw + b : \ (A^*\xi, w) \le p_N(A^*\xi, t) \ \ \forall \xi \in \mathbb{R}^m\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
Q_N(t) &= \{Aw + b : \ \ w \in \mathbb{R}^m, \ (A^*\xi, w) \le p_N(A^*\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= \{Aw + b : \ \ w \in \mathbb{R}^m, \ (\xi, w) \le p_N(\xi, t) \ \ \forall \xi \in \mathbb{R}^m\} \\
&= A(P_N(t)) + b = (L \circ P_N)(t).
\end{aligned}$$

Question 1 *Give a precise description of these set valued functions that are reproduced (recovered) exactly.*

Definition 19 *Let X be a linear space and H a subspace of X. A linear operator $\Lambda: X \to H$ is called a projector, if $\Lambda^2 = \Lambda$, so $\forall\, x \in H\ \Lambda(x) = x$.*

Example. The operator that assigns to every continuous real-valued function f the polynomial of degree less or equal then N, that interpolates f at $N+1$ distinct points, is a projector.

Let $H \subset C[0,1]$ be an arbitrary subspace that contains constants and let $\Lambda: C[0,1] \to H$ be a projector. Choose in H an arbitrary finite set of nonnegative functions $\lambda_1(t), ..., \lambda_n(t)$ (we can do this since H contains constants).

Now consider set-valued function

$$F(t) = \sum_{k=1}^{n} \lambda_k(t) A_k, \tag{7.3}$$

where the A_k are arbitrary convex sets.

The operator Λ generates a set-valued operator according to the rule:

Given $G: [0,1] \to \mathcal{K}^c(\mathbb{R}^m)$ for $\forall \xi \in S^{m-1}$ generate a real-valued function $\delta^*(\xi, G(t))$ with variable $t \in [0,1]$.

Apply operator Λ to this function. Get $p(\xi, t) = \Lambda(\delta^*(\xi, G(\cdot)))(t)$.

Define

$$P_\Lambda(G; t) := \{z \in \mathbb{R}^m : (\xi, z) \leq p(\xi, t)\ \ \forall \xi \in S^{m-1}\}.$$

This is a convex set.

If for some t this set is nonempty, we can continue to work with it as Lempio did. In general, to guarantee that $P_\Lambda(G, t) \neq \emptyset$, additional assumptions are needed.

Lemma 14 *For any G of the form (7.3) and any t*

$$P_\Lambda(G, t) = G(t).$$

Proof. For $G(t) = F(t)$ in the form of (7.3), we have

$$\delta^*(\xi, F(t)) = \delta^*\left(\xi, \sum_{k=1}^{n} \lambda_k(t) A_k\right) = \sum_{k=1}^{n} \lambda_k(t) \delta^*(\xi, A_k),$$

since $\lambda_k(t)$ are nonnegative.

This implies

$$p(\xi, t) = \Lambda\left(\sum_{k=1}^{n} \lambda_k(\cdot)\delta^*(\xi, A_k)\right)(t)$$

$$= \sum_{k=1}^{n} \Lambda(\lambda_k(\cdot))(t)\delta^*(\xi, A_k) = \sum_{k=1}^{n} \lambda_k(t)\delta^*(\xi, A_k) = \delta^*(\xi, F(t)).$$

Therefore,

$$P_\Lambda(F, t) = \{z \in \mathbb{R}^m : \ (\xi, z) \le p(\xi, t) \ \ \forall \xi \in S^{m-1}\}$$

$$= \{z \in \mathbb{R}^m : \ (\xi, z) \le \delta^*(\xi, F(t)) \ \ \forall \xi \in S^{m-1}\} = F(t). \quad \Box$$

Corollary 1 *Let $\lambda_1(t), ..., \lambda_{N+1}(t)$ be an arbitrary set of nonnegative algebraic polynomials of degree less or equal then N. Then the interpolation operator $P_N(t)$ reproduces any function of the form*

$$F(t) = \sum_{k=1}^{N+1} \lambda_k(t) A_k,$$

where A_k - arbitrary element of the space $\mathcal{K}^c(\mathbb{R}^m)$. In particular, the interpolation operator reproduces any "algebraical polynomials"

$$F(x) = \sum_{k=0}^{N} x^k A_k$$

on the $[0, 1]$.

7.3 Modified set-valued linear operator

In the event that for some t the set $\widetilde{A}[F](t)$ is empty, we can use the following approach to the construction of the operator that approximates a function $F(t)$.

Instead of

$$\widetilde{A}[F](t) = \{z \in \mathbb{R}^n : (\xi, z) \le A[\delta^*(\xi, F(\cdot))](t) \ \forall \xi \in S^{n-1}\}$$

we will consider and use the following

$$\widetilde{A}_\varepsilon[F](t) = \{z \in \mathbb{R}^n : (\xi, z) \le A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t) \ \forall \xi\},$$

where

$$\varepsilon(t) = \sup_{||\xi||=1} |\delta^*(\xi, F(t)) - A[\delta^*(\xi, F(\cdot))](t)|.$$

We show that for $\forall t$

$$F(t) \subset \widetilde{A}_\varepsilon[F](t),$$

so that $\widetilde{A}_\varepsilon[F](t)$ is necessarily nonempty.

We have

$$F(t) = \{z: \ (\xi, z) \leq \delta^*(\xi, F(t)) \ \ \forall \xi\}.$$

From the definition of $\varepsilon(t)$

$$\delta^*(\xi, F(t)) \leq A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t).$$

Therefore,

$$F(t) \subset \{z: \ \ (\xi, z) \leq A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t) \ \ \forall \xi\} = \widetilde{A}_\varepsilon[F](t).$$

Lemma 15 *For any function $F \in C([a, b], K^c(\mathbb{R}^n))$*

$$\delta^h(F(t), \tilde{A}_\varepsilon[F](t)) \leq 2\varepsilon(t). \tag{7.4}$$

Proof. We have

$$\delta^h(F(t), \widetilde{A}_\varepsilon[F](t)) = \sup_{||\xi||=1} |\delta^*(\xi, \widetilde{A}_\varepsilon[F](t)) - \delta^*(\xi, F(t))|.$$

Since the first term inside the absolute value is greater than the second, we can remove the absolute value. Moreover,

$$\delta^*(\xi, \widetilde{A}_\varepsilon[F](t)) = \sup_{||\xi||=1} \{\delta^*(\xi, z): \ z \in \widetilde{A}_\varepsilon[F](t)\}.$$

But for $z \in \widetilde{A}_\varepsilon[F](t)$

$$(\xi, z) \leq A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t).$$

Thus,

$$\delta^h(F(t), \widetilde{A}_\varepsilon[F](t)) \leq \sup_{||\xi||=1} \{A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t) - \delta^*(\xi, F(t))\}$$

$$\varepsilon(t) + \varepsilon(t) = 2\varepsilon(t).$$

Therefore,

$$\delta^h(F(t), \widetilde{A}_\varepsilon[F](t)) \leq 2\varepsilon(t). \quad \square$$

Advantages of this approach are the simplicity of construction of the operator $\tilde{A}_\varepsilon$ and a good estimate of the approximation error.

7.4 Error estimates

Let the operator A that generates operator $\widetilde{A}$ be the polynomial interpolation operator. In this case, the error estimation $\delta^h(F(t), \widetilde{A}[F](t))$ is given in [15] and [13].

If $\widetilde{A}[F](t))$ contains the ball $B(m(t), r(t))$ and

$$c(t) = \sup_{||\xi||=1} A[\delta^*(\xi, F(\cdot))](t),$$

then

$$\delta^h(F(t), \widetilde{A}[F](t)) \leq \frac{2c(t)}{r(t)}\varepsilon(t).$$

The next theorem gives an estimate of $\delta^h(F(t), \widetilde{A}[F](t))$ in the case of an arbitrary operator A. The formulation and proof we give for functions $F(t)$ with $2D$-values due to applications to recovery of $3D$ bodies from their cross-sections.

Theorem 20 *Suppose $\widetilde{A}[F](t))$ contains the ball $B(m(t), r(t))$. Suppose, also, that $\varepsilon(t) < r(t)$ for any $t \in [a, b]$. Then for any $F \in C([a, b], \mathcal{K}^c(\mathbb{R}^2))$*

$$\delta^h(F(t), \widetilde{A}[F](t)) \leq \left(\frac{2c(t)}{r(t) - \varepsilon(t)} + 2\right)\varepsilon(t).$$

Proof. We have

$$\delta^h(F(t), \widetilde{A}[F](t)) \leq \delta^h(F(t), \widetilde{A}_\varepsilon[F](t)) + \delta^h(\widetilde{A}_\varepsilon[F](t), \widetilde{A}[F](t))$$

$$\leq 2\varepsilon(t) + \delta^h(\widetilde{A}_\varepsilon[F](t), \widetilde{A}[F](t)). \tag{7.5}$$

Thus, it is enough to estimate $\delta^h(\widetilde{A}_\varepsilon[F](t), \widetilde{A}[F](t))$. Because

$$\widetilde{A}[F](t) = \{z \in \mathbb{R}^2 : (\xi, z) \leq A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t)\ \forall\xi\}$$

$$\subset \{z :\ (\xi, z) \leq A[\delta^*(\xi, F(\cdot))](t) + \varepsilon(t)\ \forall\xi\} = \widetilde{A}_\varepsilon[F]$$

it is enough to estimate deviation of $\widetilde{A}_\varepsilon[F](t)$ from $\widetilde{A}[F](t)$.

Since any convex compact set in a plane can be approximated arbitrarily well by a convex polygon, we assume below that

$$\widetilde{A}[F](t) = \{z \in \mathbb{R}^2 :\ (\xi_i, z) \leq a_i\ \forall i \in I\}$$

and

$$\widetilde{A}_\varepsilon[F](t) = \{z \in \mathbb{R}^2 :\ (\xi_i, z) \leq a_i + \varepsilon(t)\ \forall i \in I\}$$

(where I is some finite index set).

Let a_{ij}^1 be the vertex of a polygon $\widetilde{A}_\varepsilon[F](t)$ that we get when the following edges are intersected

$$(\xi_i, z) = a_i \text{ and } (\xi_j, z) = a_j, \tag{7.6}$$

and a_{ij}^2 is the vertex of $\widetilde{A}_\varepsilon[F](t)$ that we get when the following edges are intersected

$$(\xi_i, z) = a_i + \varepsilon \text{ and } (\xi_j, z) = a_j + \varepsilon. \tag{7.7}$$

It is clear that

$$\delta^h(\widetilde{A}[F](t), \widetilde{A}_\varepsilon[F](t)) \leq \max_{ij} |a_{ij}^1 - a_{ij}^2|.$$

For fixed i and j we estimate

$$|a_{ij}^1 - a_{ij}^2|.$$

Let β be an angle between the edges (7.6) and it is the same in (7.7).

We consider a set

$$co(B(m(t), r(t)) \cup \{a_{ij}^2\}).$$

And let α be an angle near the vertex of this set. It is clear that $\alpha \leq \beta < \pi$.

Denote as x a maximum of numbers $|a_{ij}^1 - a_{ij}^2|$. Then we have

$$|m(t) - a_{ij}^2| \leq |m(t) - a_{ij}^1| + |a_{ij}^1 - a_{ij}^2| \leq 2c(t) + x$$

(Ball with radius $c(t)$ and center θ contains set $\widetilde{A}[F](t)$).

We have

$$\sin\frac{\alpha}{2} = \frac{r(t)}{|m(t) - a_{ij}^2|} \geq \frac{r(t)}{2c(t) + x}. \tag{7.8}$$

Next,

$$x = \frac{\varepsilon(t)}{\sin\frac{\beta}{2}}.$$

Since $\alpha \leq \beta < \pi$, then

$$\sin\frac{\beta}{2} \geq \sin\frac{\alpha}{2}.$$

Therefore, due to equation (7.8)

$$x \leq \frac{\varepsilon(t)}{\sin\frac{\alpha}{2}} \leq \frac{\varepsilon(t)}{\left(\frac{r(t)}{2c(t)+x}\right)} = \frac{2c(t) + x}{r(t)}\varepsilon(t).$$

Thus,

$$r(t)x \leq 2c(t)\varepsilon(t) + x\varepsilon(t)$$

and, therefore,

$$(r(t) - \varepsilon(t))x \leq 2c(t)\varepsilon(t).$$

Hence,

$$x \leq \frac{2c(t)}{r(t)-\varepsilon(t)}\varepsilon(t).$$

We have proved that

$$\delta^h\left(\widetilde{A}[F](t), \widetilde{A}_\varepsilon[F](t)\right) \leq \frac{2c(t)}{r(t)-\varepsilon(t)}\varepsilon(t).$$

The last inequality, together with estimation (7.5), gives

$$\delta^h(F(t), \widetilde{A}[F](t)) \leq 2\varepsilon(t) + \frac{2c(t)}{r(t)-\varepsilon(t)}\varepsilon(t). \quad \square$$

We can use finite dimensional operators.

Any operator

$$A_N : C[a,b] \to H_N, \quad \dim H_N = N$$

has the form

$$A_N[f](t) = \sum_{k=1}^{N} \Phi_k(f)\varphi_k(t),$$

where $\varphi_1, ..., \varphi_N$ is the basis in H_N, and $\Phi_1, ..., \Phi_N$ are linear continuous functions on $C[a,b]$.

In particular, as $\Phi_1, ..., \Phi_N$, we can take values of a function at points $t_1, ..., t_N$, i.e.,

$$A_N[f](t) = \sum_{k=1}^{N} f(t_k)\varphi_k(t).$$

In this form, we can write, for example,

- interpolation polynomials;
- interpolation splines;
- Bernstein operator;
- Schoenberg operator.

7.5 Interpolation of functions with 1D-images

As a special case, we describe polynomials and provide more properties and suggest what should be done in practical problems.

Let

$$\varepsilon(t) = \sup_{||\xi||=1} |\delta^*(\xi, F(t)) - p_N(\xi, t)|,$$

so for any $\xi \in S^{n-1}$: $|\delta^*(\xi, F(t)) - p_N(\xi, t)| \leq \varepsilon(t)$. Instead of $P_N(t)$, consider

$$\widetilde{P}_N(t) = \{z \in \mathbb{R}^n : (z, \xi) \leq p_N(\xi, t) + \varepsilon(t),\ \forall \xi \in S^{n-1}\}.$$

Properties that the new polynomial has:

1. For $t = t_i$: $\widetilde{P}_N(t_i) = F(t_i)$

   Since $p_N(\xi, t_i) = \delta^*(\xi, F(t_i))$ then $\varepsilon(t_i) = 0$ and

$$\widetilde{P}_N(t_i) = P_N(t_i) = F(t_i). \ \square$$

2. Constants recovery: $F(t) \equiv A \in \mathcal{K}^c(\mathbb{R}^n)$, then $\widetilde{P}_N(t)) \equiv A$.

   Really, $\delta^*(\xi, F(t)) = \delta^*(\xi, A)$ for any t. It implies that

$$p_N(\xi, t) = \sum_{i=0}^{n} \delta^*(\xi, F(t_i)) L_i(t)$$

$$= \delta^*(\xi, A) \sum_{i=0}^{n} L_i(t) = \delta^*(\xi, A).$$

   So for any t $p_N(\xi, t) = \delta^*(\xi, A)$, $\varepsilon(t) = 0$ and

$$\widetilde{P}_N(t)) = \{z \in \mathbb{R}^n : (z, \xi) \leq p_N(\xi, t) \ \forall \xi \in S^{n-1}\}$$

$$= \{z \in \mathbb{R}^n : (z, \xi) \leq \delta^*(\xi, A) \ \forall \xi \in S^{n-1}\} = A. \ \square$$

3. For $\forall t \quad F(t) \subset \widetilde{P}_N(t)$ and consequently $\widetilde{P}_N(t)$ is nonempty.

   We have

$$\delta^*(\xi, F(t)) \leq p_N(\xi, t) + \varepsilon(t). \tag{7.9}$$

   Therefore,

$$F(t) = \{z \in \mathbb{R}^n : (z, \xi) \leq \delta^*(\xi, F(t)), \ \forall \xi\}$$

$$\subset \{z \in \mathbb{R}^n : (z, \xi) \leq p_N(\xi, t) + \varepsilon(t), \ \forall \xi\} = \widetilde{P}_N(t). \ \square$$

4. $h(F(t), \widetilde{P}_N(t)) \leq 2\varepsilon(t)$.

Due to the property of the Hausdorff metric

$$h(F(t), \widetilde{P}_N(t)) = \sup_{||\xi||=1} |\delta^*(\xi, F(t)) - \delta^*(\xi, \widetilde{P}_N(t))|.$$

Note that

$$\delta^*(\xi, \widetilde{P}_N(t)) \geq \delta^*(\xi, F(t)) \ \ (\text{since } F(t) \subset \widetilde{P}_N(t))$$

$$\delta^*(\xi, \widetilde{P}_N(t)) = \sup_{z \in \widetilde{P}_N(t)} (\xi, z) \leq p_N(\xi, t) + \varepsilon(t).$$

So,

$$h(F(t), \widetilde{P}_N(t)) = \sup_{||\xi||=1} (\delta^*(\xi, \widetilde{P}_N(t)) - \delta^*(\xi, F(t))) \leq$$

$$\leq \sup_{||\xi||=1} (p_N(\xi, t) + \varepsilon(t) - \delta^*(\xi, F(t))) \leq$$

$$\leq \sup_{||\xi||=1} (|p_N(\xi, t) - \delta^*(\xi, F(t))| + \varepsilon(t)) =$$

$$= \varepsilon(t) + \varepsilon(t) = 2\varepsilon(t). \ \square$$

For practical purposes, when we do not know the function $F(t)$ as well as $\varepsilon(t)$, we can use the following approach.

It is well known that the interpolation error can be rewritten in the following Cauchy form

$$\delta^*(\xi, F(t)) - p_N(\xi, t) = \frac{\delta^*(\xi, F(\eta))_t^{(N+1)}}{(N+1)!} \omega_{N+1}(t),$$

where $\omega_{N+1}(t) = \prod_0^N (t - t_k)$.

Suppose $\delta^{*(N+1)}(\xi, F(t))$ exist and also

$$\sup_{|\xi|=1} \sup_{t \in [a,b]} |\delta^{*(N+1)}(\xi, F(t))| \leq M.$$

Then,

$$\varepsilon(t) \leq \frac{M}{(N+1)!} |\omega_{N+1}(t)|$$

and the modified interpolation polynomial can be rewritten for practical purposes in the following way

$$\widetilde{P}_N(t) = \left\{ z : \ \ (\xi, z) \leq p_N(\xi, t) + \frac{M}{(N+1)!} |\omega_{N+1}(t)| \ \ \ \forall \xi \right\}.$$

If for some reasons we cannot find M, we can choose a constant, instead of the factor $\frac{M}{(N+1)!}$, to be a small number (ex. 0.0001) and run a program with $\varepsilon(t) = 0.0001|\omega_{N+1}(t)|$. If the program signals that we still have an empty set, we should increase our constant by a reasonably small number and run it again, and so on till we do obtain a nonempty set.

Next we build a modified interpolation polynomials according to another scheme that is different from the above general scheme. For functions with 1D-images this scheme seems to be more appropriate and convenient.

Let $F: [0,1] \longrightarrow \mathcal{K}^c(\mathbb{R}^1)$. We will work with the interval-valued functions that can be described in the following way:

$$F(t) = [\underline{F}(t), \overline{F}(t)], \quad \underline{F}(t) \leq \overline{F}(t).$$

$\delta^*(\xi, [a,b])$ is the support function and $\xi \in S^0 = \{-1, 1\}$.

Now we find $\delta^*(\xi, F(t))$. It is clear that for the interval $[a,b]$

$$\delta^*(\xi, F(t)) = \begin{cases} \overline{F}(t), & \xi = 1; \\ -\underline{F}(t), & \xi = -1. \end{cases}$$

Choose points on $[0,1]$: $0 \leq t_0 < t_1 < ... < t_N \leq 1$.

For $\forall \xi \in S^0$, we build an interpolation Lagrange polynomial for the function $\delta^*(\xi, F(t))$ as a function of t:

$$p_N(\xi, t) = \sum_{j=0}^{N} \delta^*(\xi, F(t_j)) L_j(t),$$

where

$$L_j(t) = \prod_{\mu=0, \mu\neq j}^{N} \frac{(t - t_\mu)}{(t_j - t_\mu)}.$$

7.5.1 Scheme of construction of interpolant for the function

Consider

$$F(t) = [\underline{F}(t), \overline{F}(t)].$$

We build two Lagrange interpolation polynomials. First $p_N(-1,t)$, that interpolates a function $-\underline{F}(t)$ and second, $p_N(1,t)$, that interpolates a function $\overline{F}(t)$.

Once we have this two polynomials, we can define an interval-valued function:

$$P_N(t) := \{z \in \mathbb{R}^1 : \ \ \xi \cdot z \leq p_N(\xi, t) \ \ \forall \xi = \pm 1\}$$

$$= \{z \in \mathbb{R}^1 : \ \ -p_N(-1,t) \leq z \leq p_N(1,t)\}$$

Next we find out for what t set $P_N(t)$ is empty and for what nonempty.

For the fixed t, let $\alpha = \frac{p_N(-1,t) + p_N(1,t)}{2}$.

It is clear that if $\alpha \geq 0$, then $P_N(t) \neq \emptyset$. If $\alpha < 0$, then $P_N(t) = \emptyset$.

If $P_N(t) \neq \emptyset$, we set

$$\tilde{P}_N(t) = P_N(t).$$

If $P_N(t) = \emptyset$, then we build the following modified polynomial

$$\widetilde{P}_N(t) := \{z \in \mathbb{R}^1 : \ \ \xi \cdot z \leq p_N(\xi, t) - \alpha(t) \ \ \forall \xi = \pm 1\}$$

$$= \{z \in \mathbb{R}^1 : \ \ -p_N(-1, t) + \alpha(t) \leq z \leq p_N(1, t) - \alpha(t)\}$$

Since $\alpha = \frac{p_N(-1,t)+p_N(1,t)}{2}$ the system of inequalities

$$-p_N(-1, t) + \alpha(t) \leq z \leq p_N(1, t) - \alpha(t)$$

can be rewritten in the following form

$$\frac{p_N(1,t) - p_N(-1,t)}{2} = -p_N(-1,t) + \frac{p_N(-1,t) + p_N(1,t)}{2} \leq z$$

$$\leq p_N(1,t) - \frac{p_N(-1,t) + p_N(1,t)}{2} = \frac{p_N(1,t) - p_N(-1,t)}{2}$$

Therefore,

$$\widetilde{P}_N(t) = \begin{cases} P_N(t), & \alpha \geq 0; \\ \left\{\frac{p_N(1,t)-p_N(-1,t)}{2}\right\}, & \alpha < 0. \end{cases}$$

Next we estimate $\delta^h(F(t), \tilde{P}_N(t))$. Note that the polynomial $p_N(1,t)$ interpolates $\overline{F}(t)$, and polynomial $p_N(-1,t)$ interpolates $-\underline{F}(t)$, i.e., $-p_N(-1,t)$ interpolates $\underline{F}(t)$.

Denote as above

$$\varepsilon(t) = \sup_{\xi \in S^0} |\delta^*(\xi, F(t)) - p_N(\xi, t)|,$$

i.e.,

$$\varepsilon(t) = \max\left\{|\overline{F}(t) - p_N(1,t)|, |\underline{F}(t) + p_N(-1,t)|\right\}.$$

Theorem 21 *For any function $F \in C([a,b], \mathcal{K}^c(\mathbb{R}^1))$*

$$\delta^h(F(t), \widetilde{P}_N(t)) \leq \varepsilon(t).$$

Proof. We assume first that $P_N(t) \neq \emptyset$. Then

$$\delta^h(F(t), \widetilde{P}_N(t)) = \delta^h(F(t), P_N(t)) =$$

$$\max\left\{|\overline{F}(t) - p_N(1,t)|, |\underline{F}(t) + p_N(-1,t)|\right\} \leq \varepsilon(t).$$

Now let $P_N(t) = \emptyset$.

This means that the following system of inequalities

$$-p_N(-1,t) \le z \le p_N(1,t)$$

is inconsistent, i.e.,

$$p_N(1,t) < -p_N(-1,t). \tag{7.10}$$

Moreover,

$$\underline{F}(t) \le \overline{F}(t). \tag{7.11}$$

Consider $\varepsilon(t)$ neighborhood of the points $\underline{F}(t)$ and $\overline{F}(t)$. It follows from inequalities (7.10) and (7.11) that intersection of these neighborhoods contains the interval $(p_N(1,t), -p_N(-1,t))$. This means that its midpoint

$$z_0 = \frac{p_N(1,t) + p_N(-1,t)}{2}$$

differs from each of the points $\underline{F}(t)$ and $\overline{F}(t)$ by no more then $\varepsilon(t)$.

Since in this case $\widetilde{P}_N(t) = \{z_0\}$ we have that in the case $P_N(t) = \emptyset$

$$\delta^h(F(t), \widetilde{P}_N(t)) \le \varepsilon(t). \quad \square$$

Figure 7.2 illustrates the result of applying the interpolant $\widetilde{P}_N(t)$ in the case when $P_N(t)$ does not give an empty set.

Figure 7.3 and Figure 7.4 show how the interpolant $\widetilde{P}_N(t)$ works in the case when $P_N(t)$ actually gives an empty set.

7.6 Interpolation of functions with 2D-images

Interpolation of 2D-images has many applications in our day-to-day life. These include, for example, recovery of a 3D body using it cross-sections with applications to tomography, 3D-printing, and 3D-scanning.

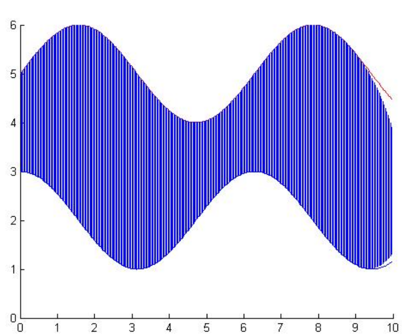


Figure 7.2: Modified interpolant $\widetilde{P}_N(t)$. Example with no emptiness for regular interpolant $P_N(t)$.

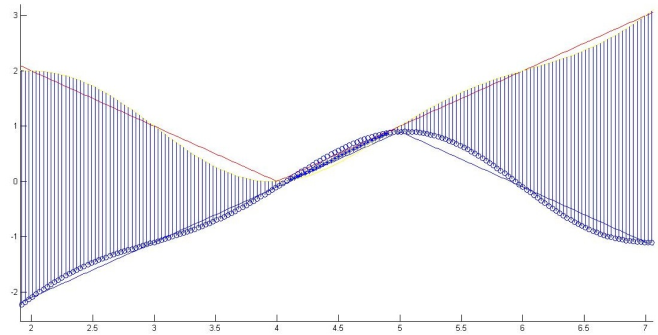


Figure 7.3: Modified interpolant $\widetilde{P}_N(t)$. Example of the problem for which regular approach would give an empty set.

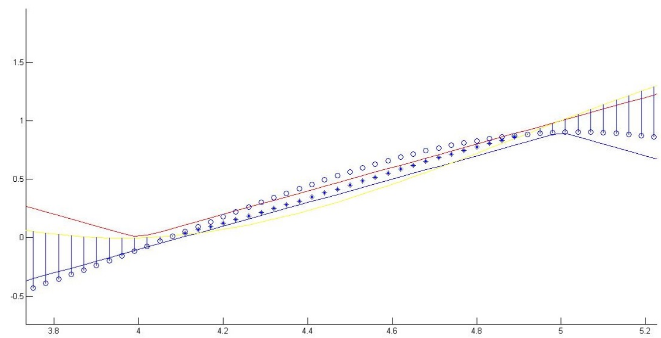


Figure 7.4: Modified interpolant $\widetilde{P}_N(t)$. Figure 7.3 zoomed in.

Using the approach in defining interpolation polynomials introduced by Lempio in [47], it is possible to extend the notion of interpolation polynomials to interpolation splines. P. Alfeld ([2]) has constructed software that applies various interpolation techniques such as polynomial, piecewise linear, natural cubic spline, and cubic Hermite spline to set-valued functions. We illustrate here the importance of having different methods of interpolation.

We considered a problem of interpolation of set-valued function within 12 knots. The given values at knots were:

- a circle with center at (1,1) and radius 0.6;
- a circle with center at (-1,1) and radius 0.3;
- a circle with center at (-1,-1) and radius 0.6;
- a circle with center at (1,-1) and radius 0.3;

and then the same values for 2 more rounds.

One can see from Figure 7.5 that the result given by polynomial interpolation is not satisfying since its value is the empty set in several intervals. This is natural since we computed a polynomial of degree 11 and thus interpolation polynomials $p_{11}(\xi, t)$ oscillate widely.

Figure 7.6 illustrates the piecewise linear interpolation applied to the same data. We have no empty sets in this case, but the interpolation is not smooth.

Finally, Figure 7.7 shows the result of natural cubic spline interpolation applied to the same problem, which in this case obviously gives the best results.

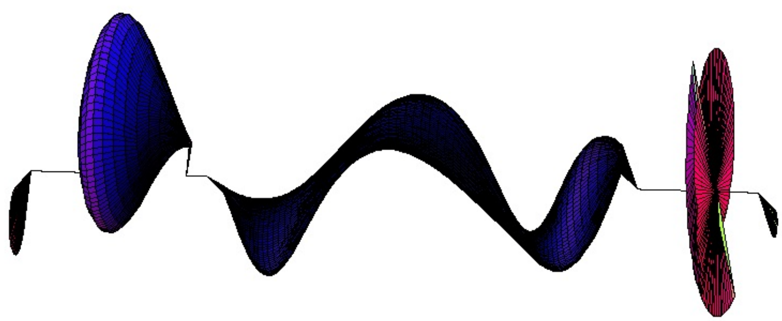


Figure 7.5: Polynomial interpolation

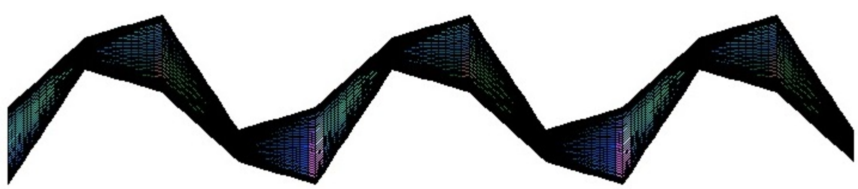


Figure 7.6: Piecewise linear interpolation

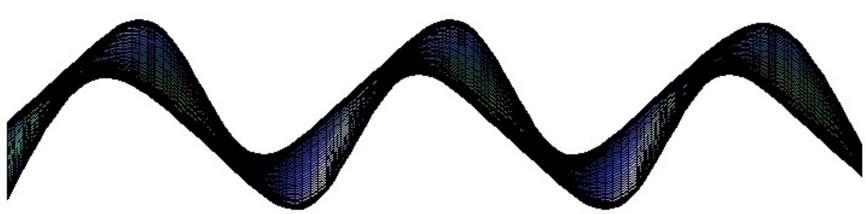


Figure 7.7: Natural cubic spline interpolation

7.7 Discussion

In this chapter, we showed some additional tools that can be used to study questions of approximation of set-valued functions due to the presence of a notion of a support function of a set. In particular, we have shown that any arbitrary linear, bounded operator $A: C[a,b] \to C[a,b]$ can be put into correspondence to an interpolation operator $\widetilde{A} : C([a,b], \mathcal{K}^c(\mathbb{R}^n)) \to C([a,b], \mathcal{K}^c(\mathbb{R}^n))$. We also studied approximative properties of this operator $\widetilde{A}$.

Similar results would be interesting to obtain for fuzzy-valued functions. We plan to investigate this problem in the future.

REFERENCES

[1] *Volterra integrodifferential equations in Banach spaces and applications*, vol. 190 of Pitman Research Notes in Mathematics Series, Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. Papers from the conference held in Trento, February 2–7, 1987, Edited by G. Da Prato and M. Iannelli.

[2] P. Alfeld, *Sets*, Personal communication, (2015).

[3] G. A. Anastassiou, *Fuzzy mathematics: approximation theory*, vol. 251 of Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin, 2010.

[4] G. A. Anastassiou and O. Duman, *Statistical fuzzy approximation by fuzzy positive linear operators*, Comput. Math. Appl., 55 (2008), pp. 573–580.

[5] Z. Artstein, *Piecewise linear approximations of set-valued maps*, J. Approx. Theory, 56 (1989), pp. 41–47.

[6] S. M. Aseev, *Quasilinear operators and their application in the theory of multivalued mappings*, Trudy Mat. Inst. Steklov., 167 (1985), pp. 25–52, 276. Current problems in mathematics. Mathematical analysis, algebra, topology.

[7] K. E. Atkinson, *The numerical solution of integral equations of the second kind*, vol. 4 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1997.

[8] J.-P. Aubin and H. Frankowska, *Set-valued analysis*, vol. 2 of Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1990.

[9] R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl., 12 (1965), pp. 1–12.

[10] V. F. Babenko and V. V. Babenko, *Optimization of approximate integration of set-valued functions monotone with respect to inclusion*, Ukrainian Math. J., 63 (2011), pp. 177–186.

[11] V. F. Babenko, V. V. Babenko, and M. Polischuk, *On optimal recovery of integrals of set-valued functions*, arXiv:1403.0840v1, (2014).

[12] ——, *Approximation of some classes of set-valued periodic functions by generalized trigonometric polynomials*, arXiv:1504.07307v1, (2015).

[13] R. Baier, *Mengenwertige integration und die deskrete approximation erreichbarer mengen*, Dissertation, Universitat Bayreuth, (1994).

[14] R. Baier and F. Lempio, *Computing Aumann's integral*, in Modeling techniques for uncertain systems (Sopron, 1992), vol. 18 of Progr. Systems Control Theory, Birkhäuser Boston, Boston, MA, 1994, pp. 71–92.

[15] R. Baier, F. Lempio, and E. Polovinkin, *Set-valued integration with negative weights*, Preprint, (1995).

[16] E. I. Balaban, *Approximate calculation of the Riemann integral of a multivalued mapping*, Zh. Vychisl. Mat. i Mat. Fiz., 22 (1982), pp. 472–476, 496.

[17] K. Balachandran and P. Prakash, *On fuzzy Volterra integral equations with deviating arguments*, J. Appl. Math. Stoch. Anal., (2004), pp. 169–176.

[18] L. Beutel, H. Gonska, D. Kacsó, and G. Tachev, *On variation-diminishing Schoenberg operators: new quantitative statements*, in Multivariate approximation and interpolation with applications (Almuñécar, 2001), Monogr. Real Acad. Ci. Exact. Fís.-Quím. Nat. Zaragoza, 20, Acad. Cienc. Exact. Fís. Quím. Nat. Zaragoza, Zaragoza, 2002, pp. 9–58.

[19] A. M. Bica, *One-sided fuzzy numbers and applications to integral equations from epidemiology*, Fuzzy Sets and Systems, 219 (2013), pp. 27–48.

[20] G. Bouligand, *Sur la semi-continuite d'inclusions et quelques sujets connexes*, Enseignement Mathematique, 31 (1930), pp. 14–22.

[21] R. L. Burden, J. D. Faires, and A. C. Reynolds, *Numerical analysis*, Prindle, Weber & Schmidt, Boston, Mass., 1978.

[22] M. Burgin and O. Duman, *Approximations by linear operators in spaces of fuzzy continuous functions*, Positivity, 15 (2011), pp. 57–72.

[23] F. J. Cabrerizo, R. Ureña, W. Pedrycz, and E. Herrera-Viedma, *Building consensus in group decision making with an allocation of information granularity*, Fuzzy Sets and Systems, 255 (2014), pp. 115–127.

[24] R. C. Chang, *Examination of excessive fuel consumption for transport jet aircraft based on fuzzy-logic models of flight data*, Fuzzy Sets and Systems, 269 (2015), pp. 115–134.

[25] C. Corduneanu, *Integral equations and applications*, Cambridge University Press, Cambridge, 1991.

[26] P. Diamond, *Theory and applications of fuzzy volterra integral equations*, IEEE Transactions on Fuzzy Systems, 10 (2002), pp. 97–102.

[27] P. Diamond and P. Kloeden, *Metric spaces of fuzzy sets*, Fuzzy Sets and Systems, 35 (1990), pp. 241–249.

[28] N. Dyn and E. Farkhi, *Approximation of set-valued functions with compact images—an overview*, in Approximation and probability, vol. 72 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2006, pp. 71–84.

[29] N. Dyn, E. Farkhi, and A. Mokhov, *Approximations of set-valued functions by metric linear operators*, Constr. Approx., 25 (2007), pp. 193–209.

[30] ——, *Approximation of set-valued functions*, Imperial College Press, London, 2014. Adaptation of classical approximation operators.

[31] N. Dyn and A. Mokhov, *Approximations of set-valued functions based on the metric average*, Rend. Mat. Appl. (7), 26 (2006), pp. 249–266.

[32] S. G. Gal, *Approximation theory in fuzzy setting*, in Handbook of analytic-computational methods in applied mathematics, Chapman & Hall/CRC, Boca Raton, FL, 2000, pp. 617–666.

[33] I. Gallego, J. R. Fernández, A. Jiménez-Losada, and M. Ordóñez, *A Banzhaf value for games with fuzzy communication structure: computing the power of the political groups in the European Parliament*, Fuzzy Sets and Systems, 255 (2014), pp. 128–145.

[34] G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra integral and functional equations*, vol. 34 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990.

[35] M. L. Guerra, C. A. Magni, and L. Stefanini, *Interval and fuzzy average internal rate of return for investment appraisal*, Fuzzy Sets and Systems, 257 (2014), pp. 217–241.

[36] D. Guo, V. Lakshmikantham, and X. Liu, *Nonlinear integral equations in abstract spaces*, vol. 373 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1996.

[37] S. Hu and N. S. Papageorgiou, *Handbook of multivalued analysis. Vol. I*, vol. 419 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 1997. Theory.

[38] ——, *Handbook of multivalued analysis. Vol. II*, vol. 500 of Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2000. Applications.

[39] M. Hukuhara, *Intégration des applications mesurables dont la valeur est un compact convexe*, Funkcial. Ekvac., 10 (1967), pp. 205–223.

[40] A. A. Khan, C. Tammer, and C. Zălinescu, *Set-valued optimization*, Vector Optimization, Springer, Heidelberg, 2015. An introduction with applications.

[41] A. N. Kolmogorov and S. Fomin, *Elements of the Theory of Functions and Functional Analysis*, Dover Books on Mathematics, Dover Publications, 1999.

[42] K. Kolomvatsos, D. Trivizakis, and S. Hadjiefthymiades, *An adaptive fuzzy logic system for automated negotiations*, Fuzzy Sets and Systems, 269 (2015), pp. 135–152.

[43] N. P. Korneĭchuk, *Tochnye konstanty v teorii priblizheniya*, “Nauka”, Moscow, 1987.

[44] K. Kuratowski, *Les fonctions semi-continues dans l’espace des ensembles fermes*, Fund. Math., 18 (1932), pp. 148–159.

[45] V. Lakshmikantham, T. G. Bhaskar, and J. Vasundhara Devi, *Theory of set differential equations in metric spaces*, Cambridge Scientific Publishers, Cambridge, 2006.

[46] V. Lakshmikantham and R. N. Mohapatra, *Theory of fuzzy differential equations and inclusions*, vol. 6 of Series in Mathematical Analysis and Applications, Taylor & Francis, Ltd., London, 2003.

[47] F. Lempio, *Set-valued interpolation, differential inclusions, and sensitivity in optimization*, in Recent developments in well-posed variational problems, vol. 331 of Math. Appl., Kluwer Acad. Publ., Dordrecht, 1995, pp. 137–169.

[48] P. Linz, *Analytical and numerical methods for Volterra equations*, vol. 7 of SIAM Studies in Applied Mathematics, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1985.

[49] F. Ma, L. Yu, M. Bajger, and M. J. Bottema, *Incorporation of fuzzy spatial relation in temporal mammogram registration*, Fuzzy Sets and Systems, 279 (2015), pp. 87–100.

[50] M. J. Marsden, *On uniform spline approximation*, J. Approximation Theory, 6 (1972), pp. 249–253. Collection of articles dedicated to J. L. Walsh on his 75th birthday, VII.

[51] G. Matheron, *Random sets and integral geometry*, John Wiley & Sons, New York-London-Sydney, 1975. With a foreword by Geoffrey S. Watson, Wiley Series in Probability and Mathematical Statistics.

[52] N. Mollaei and R. K. Moghaddam, *A new controlling approach of type 1 diabetics based on interval type-2 fuzzy controller*, J. of Fuzzy Set Valued Analysis, 2014 (2014), pp. 1–14. Collection of articles dedicated to J. L. Walsh on his 75th birthday, VII.

[53] R. E. Moore, *Interval analysis*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1966.

[54] M. Mureşan, *Set-valued approximation of multifunctions*, Stud. Univ. Babeş-Bolyai Math., 55 (2010), pp. 107–148.

[55] P. Painleve, *C.r.a.s.*, Paris, 148 (1909), p. 1156.

[56] A. V. Plotnikov and N. V. Skripnik, *Existence and uniqueness theorem for set Volterra integral equations*, J. Adv. Res. Dyn. Control Syst., 6 (2014), pp. 1–7.

[57] ——, *Existence and uniqueness theorem for set Volterra integral equations*, J. Adv. Res. Dyn. Control Syst., 6 (2014), pp. 1–7.

[58] P. A. V. Plotnikov, V. A. and A. N. Vityuk, *Differential equations with multivalued right-hand side*, Astroprint, 1999. Asymptotic methods.

[59] E. S. Polovinkin, *Riemannian integral of set-valued function*, Optim. Techn., IFIP Techn. Conf. Novosibirsk, Lect. Notes in Comput. Sci., 27 (1974), pp. 405–410.

[60] E. S. Polovinkin and M. Balashov, *Elements of convex and strongly convex analysis [in russian]*, (2004).

[61] G. B. Price, *The theory of integration*, Trans. Amer. Math. Soc., 47 (1940), pp. 1–50.

[62] I. J. Schoenberg, *On spline functions*, in Inequalities (Proc. Sympos. Wright-Patterson Air Force Base, Ohio, 1965), Academic Press, New York, 1967, pp. 255–291.

[63] I. Tişe, *Set integral equations in metric spaces*, Math. Morav., 13 (2009), pp. 95–102.

[64] L. Troiano, L. J. Rodríguez-Muñiz, and I. Díaz, *Discovering user preferences using Dempster-Shafer theory*, Fuzzy Sets and Systems, 278 (2015), pp. 98–117.

[65] S. A. Vahrameev, *Integration in l-spaces*, Book: Applied Mathematics and Mathematical Software of Computers, M.: MSU Publisher, [Russian], (1980), pp. 45–47.

[66] R. A. Vitale, *Approximation of convex set-valued functions*, J. Approx. Theory, 26 (1979), pp. 301–316.

[67] H. Wang and Y. Liu, *Existence results for fuzzy integral equations of fractional order*, Int. J. Math. Anal. (Ruse), 5 (2011), pp. 811–818.

[68] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), pp. 338–353.

[69] P. Zhang and W.-G. Zhang, *Multiperiod mean absolute deviation fuzzy portfolio selection model with risk control and cardinality constraints*, Fuzzy Sets and Systems, 255 (2014), pp. 74–91.