POLYNOMIAL REPRESENTATIONS AND ASSOCIATED CYCLES FOR INDEFINITE UNITARY GROUPS

by

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ABSTRACT

The associated variety is a geometric invariant attached to each Harish-Chandra module of a real reductive Lie group. The associated cycle is a finer invariant that gives additional algebraic data for each component of the associated variety. The main result of this thesis is a set of formulas for associated cycles of a large class of Harish-Chandra modules for the real Lie group $U(p,q)$. These formulas give the associated cycle polynomials for the coherent family containing a module X when elements of the dense orbit in the associated variety of X have a single nontrivial Jordan block or exactly two Jordan blocks.

CONTENTS

CHAPTER 1

INTRODUCTION

In this dissertation, we will consider four closely related questions in the representation theory of the Lie group $U(p,q)$ and its Weyl group S_n.

Harish-Chandra modules are fundamental objects in the representation theory of real reductive Lie groups [40]. These modules convert infinite-dimensional representations into algebraically tractable objects. By appropriate constructions, we can attach geometric invariants to these modules. For example, Vogan developed a way to construct an *associated variety* and *associated cycle* for each Harish-Chandra module [41].

These invariants have a number of interesting properties. They relate to the Beilinson–Bernstein geometry of Harish-Chandra modules in striking ways [10, 7] and are connected to analytic invariants such as wave front cycles [33]. The multiplicities in associated cycles extend to polynomials when we view Harish-Chandra modules in the context of coherent families.

The real group $U(p,q)$ is an attractive settings for computing these polynomials as the relevant geometry is relatively simple. Barchini and Zierau developed methods to compute associated cycles for discrete series representations of $U(p,q)$ in [3]. In the appendix to [3], Trapa demonstrates techniques for computing the associated cycle of any Harish-Chandra module of $U(p,q)$; these latter techniques rely on the computationally intensive Kazhdan–Lusztig algorithm and must be carried out on a case by case basis.

Question 1 *Can we write down closed formulas for associated cycle polynomials in the setting of $U(p,q)$?*

Fiber polynomials are closely related to associated cycles [10, 37]. Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{h}$ its Cartan subalgebra. Take the Borel–Weil line bundle derived from a parameter λ in the dual subalgebra $\mathfrak{h}^*$. We can restrict this line bundle to a Springer fiber component and compute the Euler characteristic of the resulting variety. By allowing λ to vary, we obtain a polynomial function on $\mathfrak{h}^*$. These polynomials were originally introduced by Joseph in terms of a different construction. Taking the span of all polynomials for a fixed Springer fiber yields a representation of the Weyl group that is isomorphic to the representation attached to the fiber by the Springer correspondence. These polynomials are building blocks for associated cycle polynomials via results of Chang [10].

Question 2 *Can we write down closed formulas for fiber polynomials in the context of* $\mathfrak{gl}(n, \mathbb{C})$*?*

The conjugacy classes in S_n are determined by cycle structure and so the irreducible representations of S_n stand in correspondence to partitions of n; these in turn correspond to Young diagrams with n boxes. Young made this correspondence explicit by constructing representations for each Young diagram [44, 16].

More recently, Kazhdan and Lusztig constructed irreducible symmetric group representations via a graph theoretic construction that consolidates a number of representation theoretic ideas in a remarkable unified theory [21]. These so called Kazhdan–Lusztig representations are isomorphic to Young's earlier constructions; they come equipped with canonical bases which are ubiquitous in representations theory. For instance, these bases control the structure of Harish-Chandra cells for $U(p, q)$ [29].

We can easily construct Young's irreducible representations in terms of polynomials. This provides a setting for practical computation. We can then attempt to construct the isomorphism between a polynomial representation and the corresponding Kazhdan–Lusztig representation.

Question 3 *Can we compute the image of the Kazhdan–Lusztig basis inside an irreducible polynomial representation of the symmetric group?*

As it turns out, the answers to these three questions are the same. In the setting of $U(p, q)$, experts have known for some time that associated cycle polynomials are equal to certain fiber polynomials. (This is discussed in the appendix to [3].) Fiber polynomials in this setting are computed by taking the image of Kazhdan–Lusztig basis elements under an isomorphism from a Kazhdan–Lusztig left cell representation to a polynomial representation. The principal difficulty in computing these polynomials is the complexity of the relevant Kazhdan–Lusztig combinatorics. In certain cases, these combinatorics are well understood.

By work of Beilinson and Bernstein [5], the Harish-Chandra modules of $U(p,q)$ at fixed regular integral infinitesimal character are parameterized by K-orbits on the flag variety. The main results of this paper give closed formulas for associated cycle polynomials in three broad cases: Theorem 29 gives a closed formula for the associated cycle polynomial for any module X whose annihilator is an induced primitive ideal. Theorem 65 gives a closed form when elements of the dense orbit in the associated variety of X have a single nontrivial Jordan block. Theorem 70 gives a formula when elements of the dense orbit have exactly two Jordan blocks. Equivalently, these formulas yield fiber polynomials for all components of Springer fibers parameterized by hook or two row Young diagrams and for any component that is isomorphic to a flag variety.

This extends associated cycle polynomial calculations to another large family of Harish-Chandra modules in addition to the discrete series covered by the work of Barchini and Zierau [3]. In Section 9.3, we give an example where our methods overlap with theirs; the same shape pair for the Harish-Chandra module considered is hook shaped and corresponds to a closed K-orbit. As expected, the results from the two methods agree.

Question 4 *Can we construct combinatorial descriptions of Kazhdan–Lusztig representations?*

This last question is a little vague; the Kazhdan–Lusztig algorithm is combinatorial, though highly recursive. In effect, the questions asks whether we can describe the Kazhdan–Lusztig representations while bypassing the computational demands of the Kazhdan–Lusztig algorithm. This has been an important question in combinatorics and representation theory for roughly the last 30 years, principally because such combinatorial descriptions would have numerous applications [6, 16, 23, 28]. As mentioned above, the answer is yes thus far only in certain special cases, such as for tableaux consisting of only two rows. Once we have the Kazhdan–Lusztig graph for a two row Young diagram, we can construct the associated Kazhdan–Lusztig representation by working with the presentation of S_n in terms of simple transpositions.

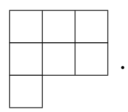
In the last chapter of this paper, we present a new way of constructing Kazhdan–Lusztig representations for two row Young diagrams in terms of skein theory. This construction is originally due to Russell and Tymoczko in a different context [31, 32]; we provide here a combinatorial proof that the resulting based representation is in fact the Kazhdan–Lusztig representation. The construction relates in an intriguingly elegant way to the fiber polynomials for two row cases. In fact, our proofs rely on polynomials. We can speculate that there is a broader link between Kazhdan–Lusztig and skein theory.

CHAPTER 2

POLYNOMIAL REPRESENTATIONS

The irreducible representations of the symmetric group can be constructed in terms of polynomials. (Refer to [13, Problem 4.47].) To understand the construction, we will need the following definition:

Definition 1 *A* Young diagram *is a collection of finitely many boxes organized in left justified rows such that a row is not longer than any of the rows above it, for example*



Given a Young diagram Y with n boxes, a standard Young tableau *with shape Y is a labeling of the boxes in Y with the integers $\{1, 2, \ldots, n\}$ in such a way that labels strictly increase down each column and from left to right in each row. For example,*

1	2	4
3	5	
6		

is a standard tableau. The column superstandard tableau *of shape Y is the unique standard tableau of shape Y where labels are consecutive as one moves down each column, e.g.,*

1	4	6
2	5	
3		

is a column superstandard tableau.

To construct a polynomial representation isomorphic to the irreducible representation corresponding to the Young shape Y, begin with the column superstandard tableau of shape Y. (Refer to this tableau by the notation Γ.) Construct a polynomial

$$p_\Gamma = \prod (x_i - x_j)$$

where the product is over all pairs (i, j) with i and j in the same column of Γ and $i < j$. Letting S_n permute variables, define

$$P_Y = \text{span}_{\mathbb{C}}\{\sigma \cdot p_\Gamma \mid \sigma \in S_n\}. \tag{2.1}$$

This space is isomorphic as an S_n representation to other standard constructions of the irreducible representation for the shape Y, such as the one in terms of Specht modules.

Given a complex finite-dimensional vector space P consisting of polynomials, define $\deg(P)$ to be the maximal degree of polynomials in P.

Theorem 2 *Let Y be a Young diagram and P'_Y a complex vector space of polynomials in the variables $x_1, x_2, \ldots, x_n$ such that P'_Y is an irreducible representation of S_n under the permutation action on variables and is isomorphic as a representation to P_Y. Furthermore, for any other polynomial representation Q_Y of S_n satisfying these conditions, assume that $\deg(P'_Y) \leq \deg(Q_Y)$. Then, $P'_Y = P_Y$ as defined above.*

Proof. There is an isomorphism of representations

$$\phi : P_Y \to P'_Y.$$

Let $p'_\Gamma = \phi(p_\Gamma)$, where p_Γ is the generating polynomial for P_Y defined above. For any pair i, j in the same column of Γ, $(i,j)\cdot p'_\Gamma = -p'_\Gamma$, so no monomial $p'_{\Gamma,k}$ in p'_Γ is fixed by exchanging x_i and x_j. It follows that x_i and x_j must have different exponents in $p'_{\Gamma,k}$. Let ℓ_i be the length of the ith column of Y. The minimum possible degree of p'_Γ is

$$\sum_i \frac{(\ell_i)(\ell_i - 1)}{2}.$$

(This is obtained by letting the variables corresponding to the labels in a column have exponents $0, 1, 2, \ldots$) This minimum degree is the degree of p_Γ. Exchanging x_i and x_j sends p'_Γ to $-p'_\Gamma$, so p'_Γ is 0 when $x_i = x_j$. It follows that $(x_i - x_j)$ divides p'_Γ. Considering all factors $(x_i - x_j)$ for i and j in the same column of Γ, it is clear that p'_Γ equals p_Γ up to scale. It follows that $P'_Y = P_Y$. ■

The work of Kazhdan and Lusztig offers another way to form a basis for the representation corresponding to Y [21]. The basis vectors for the Kazhdan–Lusztig representation are parameterized by standard tableaux of shape Y. Unlike some other bases, such as the Specht module basis, the Kazhdan–Lusztig basis is exceptionally hard to compute.

The Kazhdan–Lusztig basis elements do satisfy certain conditions that can be directly read off from their corresponding tableaux. Denote by KL_Y the Kazhdan–Lusztig representation corresponding to the Young diagram Y and by w_T the Kazhdan–Lusztig basis element in KL_Y corresponding to the tableau T. Denote by s_i the simple transposition $(i, i+1) \in S_n$.

Definition 3 *Given a basis element $w_T \in \mathrm{KL}_Y$, the* τ-invariant *of w_T, denoted by $\tau(w_T)$, is the set of simple transpositions s_i such that*

$$s_i \cdot w_T = -w_T.$$

The τ-invariant of a standard tableau T is the set $\tau(T)$ of simple transpositions s_i such that $i+1$ appears in a row below the row of i in T. A fundamental property of Kazhdan–Lusztig bases specifies that $\tau(w_T) = \tau(T)$.

Once again, let Γ be the column superstandard tableau. There are often two or more tableaux of the same shape that have a common τ-invariant. However, one can show that Γ is the unique tableau with τ-invariant $\tau(\Gamma)$. In fact, if T is a tableau not equal to Γ, then

$$\tau(\Gamma) \nsubseteq \tau(T).$$

Additionally, for any Kazhdan–Lusztig basis element w_T with $s_i \notin \tau(w_T)$, w_T appears with coefficient 1 in $s_i \cdot w_T$ when we express it in terms of the Kazhdan–Lusztig basis. (See Chapter 8.) Then, we have the following:

Lemma 4 *For any vector v in an S_n representation, define $\tau(v)$ to be the set of simple transpositions s_i such that $s_i \cdot v = -v$. If v is a vector in KL_Y satisfying $\tau(v) = \tau(w_\Gamma)$, then v equals w_Γ up to scale.*

Let

$$\phi : \mathrm{KL}_Y \to P_Y$$

be an isomorphism. By Schur's lemma, this isomorphism is unique up to scale. We can immediately say something else about ϕ: Note that $\tau(p_\Gamma) = \tau(\Gamma)$. Thus, by Lemma 4

$$\phi(w_\Gamma) = C p_\Gamma$$

where C is a nonzero constant. More generally, we can ask the following question:

Question 5 *Given a tableau T, what is $\phi(w_T)$?*

As we shall see, answering this question is essential if we wish to compute fiber polynomials and associated cycles in the setting of $U(p,q)$. In certain cases, we will be able to write down closed forms of the Kazhdan–Lusztig basis inside the polynomial representation P_Y.

CHAPTER 3

HARISH-CHANDRA MODULES

Let $G_{\mathbb{R}}$ be a real semisimple Lie group. We frequently study Lie groups via their *representations.*

Definition 5 *A* finite-dimensional representation *of a Lie group $G_{\mathbb{R}}$ is a map*

$$\pi : G_{\mathbb{R}} \to GL(V)$$

where V is a finite-dimensional complex vector space, $GL(V)$ is the group of all invertible linear transformations of V and π is a continuous group homomorphism. We say that π is irreducible *if V has no proper nontrivial subspaces invariant under the action of $G_{\mathbb{R}}$.*

A central problem in understanding representations of $G_{\mathbb{R}}$ is to classify all irreducible representations up to equivalence. For reductive Lie groups, the solution to this problem is known. (See [26] for an exposition.) To motivate a broader class of representations, we consider an example.

Let μ be a left $G_{\mathbb{R}}$-invariant measure on $G_{\mathbb{R}}$. Consider the set $L^2(G_{\mathbb{R}})$ of all square integrable complex functions on $G_{\mathbb{R}}$; this is a complex vector space, generally not finite-dimensional. We can define a group homomorphism

$$\Phi : G_{\mathbb{R}} \to GL(L^2(G))$$

by

$$\Phi(g)f(x) = f(g^{-1}x).$$

This is the *left regular representation of* $G_{\mathbb{R}}$ and is an example of a *unitary representation* [25].

Definition 6 *A* unitary representation *of $G_{\mathbb{R}}$ is a homomorphism Φ from $G_{\mathbb{R}}$ to the group of bounded invertible linear operators on a Hilbert space V such that $\Phi(g)$ is a unitary operator on V for each $g \in G_{\mathbb{R}}$ and the map $g \times v \mapsto \Phi(g) \cdot v$ from $G_{\mathbb{R}} \times V$ to V is continuous.*

By a well known theorem, finite-dimensional representations are smooth, so we can differentiate them to obtain representations of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $G_{\mathbb{R}}$. Complexifying $\mathfrak{g}_{\mathbb{R}}$ to $\mathfrak{g}$ yields a complex Lie algebra representation. Lie algebra representations are an essential tool in the classification of $G_{\mathbb{R}}$-representations.

In general, we cannot differentiate a representation that is not finite-dimensional. However, we can convert unitary representations into *Harish-Chandra modules.* These modules were introduced by Harish-Chandra to study the *unitary dual* of a real semisimple Lie group $G_{\mathbb{R}}$, i.e., the set of all irreducible unitary representations of $G_{\mathbb{R}}$ up to equivalence. We give an abridged description of the construction of Harish-Chandra modules here. See [4] and [40] for more details.

Let $K_{\mathbb{R}}$ be a maximal compact subgroup of $G_{\mathbb{R}}$. All choices for $K_{\mathbb{R}}$ are equivalent in the sense that all maximal compact subgroups are conjugate. Since $K_{\mathbb{R}}$ is compact, it is linear and can be complexified to a complex group K. To construct a Harish-Chandra module, we take a subspace of the representation space V consisting of K-*finite vectors*, i.e., the set of all vectors $v \in V$ such that $\text{span}_{\mathbb{C}}\{K \cdot v\}$ is finite-dimensional. Denote this subspace by X.

Theorem 7 (Harish-Chandra) *The K-finite vectors are smooth in that for a fixed $v \in X$, the map from G to X given by*

$$g \mapsto \Phi(g)v$$

is smooth. Differentiating this map turns X into a $\mathfrak{g}_{\mathbb{R}}$-representation and hence by complexification a $\mathfrak{g}$-representation. In this way, X is a $(\mathfrak{g}, K)$-module.

See [25, Chapter 8] and [42, Chapter 3] for expositions. A $(\mathfrak{g}, K)$-module is defined as follows:

Definition 8 *A* $(\mathfrak{g}, K)$-module *is a complex vector space X equipped with representations of K and $\mathfrak{g}$ (both denoted by π) satisfying the following properties:*

1. *Every vector in X is K-finite.*

2. *Differentiating the K-representation π yields a representation of the Lie algebra $\mathfrak{k}$ of K that is the same as the representation obtained by restricting the $\mathfrak{g}$-representation π to $\mathfrak{k}$.*

3. If $g \in \mathfrak{g}$ and $k \in K$, then

$$\pi(\mathrm{Ad}_k(g)) = \pi(k)\pi(g)\pi(k)^{-1}.$$

4. As a K-representation, π is algebraic.

Every finite-dimensional $G_{\mathbb{R}}$-representation automatically gives rise to a $(\mathfrak{g}, K)$-module by complexification of $\mathfrak{g}_{\mathbb{R}}$ and $K_{\mathbb{R}}$. In [40], it is shown that if X is constructed from an irreducible unitary representation, then X is an irreducible $(\mathfrak{g}, K)$-module. Irreducible modules also satisfy the following broader condition:

Definition 9 *A $(\mathfrak{g}, K)$-module X is said to have* finite-length *if it admits a chain of submodules*

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_N = X$$

such that X_i/X_{i-1} is irreducible for $i = 1, \ldots, N$. Write $\mathcal{F}(\mathfrak{g}, K)$ for the category of $(\mathfrak{g}, K)$-modules of finite-length. By definition, every irreducible module has finite-length. A finite-length $(\mathfrak{g}, K)$-module is referred to as a Harish-Chandra module.

Note that the K-finite construction can be used to build Harish-Chandra modules from a much larger class of (not necessarily unitary) representations [40]. However, the aims of this paper are principally algebraic, so we will from now on work directly with Harish-Chandra modules without reference to the representations from which they were derived.

CHAPTER 4

ASSOCIATED VARIETIES AND CYCLES

In a seminal 1991 paper, David Vogan introduced the notion of associated varieties and associated cycles for Harish-Chandra modules [41].

4.1 Associated Varieties

Let X_0 be a finite-dimensional K-invariant generating subspace of X; Such a subspace always exists. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and filter X by degree, i.e., let $\mathcal{U}_n(\mathfrak{g})$ be the subspace of $\mathcal{U}(\mathfrak{g})$ obtained by taking the span of all products of at most n elements in $\mathfrak{g}$. Define a filtration on X by

$$X_n = U_n(\mathfrak{g}) \cdot X_0.$$

Using the degree filtration on $\mathcal{U}(\mathfrak{g})$, we obtain an associated graded module $\operatorname{gr}\mathcal{U}(\mathfrak{g})$. An important corollary to the Poincare–Birkoff–Witt theorem states that $\operatorname{gr}\mathcal{U}(\mathfrak{g})$ is naturally isomorphic to $S(\mathfrak{g})$, the symmetric algebra constructed by treating $\mathfrak{g}$ as a vector space; the symmetric algebra is in turn isomorphic to the ring of polynomial functions on the dual vector space $\mathfrak{g}^*$. This construction hints at the imminent prospect of algebro-geometric machinery. In fact, the filtration on X is compatible with the degree filtration on $\mathcal{U}(\mathfrak{g})$ in that

$$\mathcal{U}_p(\mathfrak{g}) \cdot X_q \subset X_{p+q}.$$

Thus, we can construct an associated graded module $\operatorname{gr} X$ over $\operatorname{gr}\mathcal{U}(\mathfrak{g}) \cong S(\mathfrak{g})$.

We are now ready to define the associated variety of X. Let $\operatorname{Ann}(\operatorname{gr} X)$ be the annihilator of $\operatorname{gr} X$. The annihilator is an ideal in $\operatorname{gr}\mathcal{U}(\mathfrak{g})$ and the associated variety of X is given by

$$\operatorname{AV}(X) = \{y \in \mathfrak{g}^* \mid p(y) = 0 \text{ for all } p \in \operatorname{Ann}(\operatorname{gr} X)\}$$

where $\operatorname{Ann}(\operatorname{gr} X)$ is identified with polynomial functions on $\mathfrak{g}^*$. Equivalently, we can treat $\operatorname{AV}(X)$ as the set of all prime ideals containing $\operatorname{Ann}(\operatorname{gr} X)$.

We gain more insight into the filtration of X, and hence the module $\operatorname{gr} X$, by applying the compatibility properties in Definition 8. In particular, the fact that $k \cdot e \cdot k^{-1} \cdot v = \operatorname{Ad}_k(e) \cdot v$

for $k \in K$, $e \in \mathfrak{g}$, $v \in X$ implies that each X_n is K-stable. Thus, $\operatorname{gr} X$ has a K-action compatible with its $S(\mathfrak{g})$-module structure. By differentiation, we also see that X_n is $\mathfrak{k}$ stable so that $\mathfrak{k}$ annihilates $\operatorname{gr} X$. It follows that $\operatorname{gr} X$ is an $S(\mathfrak{g}/\mathfrak{k})$-module and $\mathrm{AV}(X)$ is a K-invariant subvariety of $(\mathfrak{g}/\mathfrak{k})^*$.

If $\lambda \in \mathrm{AV}(X)$, then λ is nilpotent, i.e., $0 \in \overline{G \cdot \lambda}$. (The bar indicates Zariski closure.) We will use the notation $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$ to indicate the cone of nilpotent elements in $(\mathfrak{g}/\mathfrak{k})^*$. A corollary to a result of Kostant and Rallis [27] tells us that K acts on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$ with finitely many orbits. Since $\mathrm{AV}(X)$ is closed, we may write

$$\mathrm{AV}(X) = \overline{\mathcal{O}_1^K} \cup \cdots \cup \overline{\mathcal{O}_j^K}$$

where each $\mathcal{O}_i^K$ is a K-orbit on $\mathcal{N}(\mathfrak{g}/\mathfrak{k})^*$ and the $\overline{\mathcal{O}_i^K}$ are the components of $\mathrm{AV}(X)$.

We will see in the sequel that all modules in a coherent family or cell share a common associated variety. We need a finer invariant to separate modules in these classifications.

4.2 Associated Cycles

After outlining the construction of associated varieties, [41] introduces an enhancement known in current parlance as the associated cycle [3]. Each component $\overline{\mathcal{O}_i^K}$ of the associated variety corresponds to some minimal prime ideal P_i in the set of primes containing $\operatorname{Ann}(\operatorname{gr} X)$. The associated cycle is written

$$\sum_i m_i \overline{\mathcal{O}_i^K}$$

where roughly speaking m_i is equal to the number of copies of $S(\mathfrak{g})/P_i$ that appear in $\operatorname{gr} X$. To be precise, we give a version of Definition 2.4 in [41]:

Definition 10 *Let* $\overline{\mathcal{O}_1^K}, \ldots, \overline{\mathcal{O}_j^K}$ *be the components of* $\mathrm{AV}(X)$ *and* $P_1, \ldots, P_j$ *the corresponding prime ideals in* $S(\mathfrak{g})$ *containing* $\operatorname{Ann}(\operatorname{gr} X)$. *The* associated cycle *of* X *is the formal sum*

$$\mathrm{AC}(X) = \sum_i m_i \overline{\mathcal{O}_i^K}$$

where the multiplicities m_i *are positive integers determined as follows: choose a finite filtration of* $\operatorname{gr} X$ *so that each subquotient* $(\operatorname{gr} X)_k/(\operatorname{gr} X)_{k-1}$ *is of the form* $S(\mathfrak{g})/Q_k$ *for some prime ideal* Q_k *in* $\operatorname{gr} X$. *Then,* m_i *is the number of times that* $Q_k = P_i$.

As required, the associated cycle is well defined regardless of the choices we make in its computation.

CHAPTER 5

COHERENT FAMILIES AND CELLS

Recall that $\mathcal{F}(\mathfrak{g}, K)$ denotes the category of finite-length $(\mathfrak{g}, K)$-modules (or Harish-Chandra modules) for $(\mathfrak{g}, K)$.

5.1 Virtual Characters

Definition 11 *The* Grothendieck group *of* $\mathcal{F}(\mathfrak{g}, K)$ *is the Abelian group generated by finite-length* $(\mathfrak{g}, K)$*-modules modulo the equivalences*

$$X \sim Y + Z$$

whenever there is a short exact sequence

$$0 \to Y \to X \to Z \to 0.$$

We denote the Grothendieck group by $\mathcal{V}(\mathfrak{g}, K)$; it is also referred to as the group of *virtual* $(\mathfrak{g}, K)$*-modules* or *virtual characters* of $G_{\mathbb{R}}$. Denote by $\widehat{G}_{\mathbb{R}}$ the set of all irreducible $(\mathfrak{g}, K)$-modules. Then, $\mathcal{V}(\mathfrak{g}, K)$ is a free $\mathbb{Z}$-module with a basis given by $\widehat{G}_{\mathbb{R}}$ [40]. Note that $\mathcal{V}(\mathfrak{g}, K)$ contains objects that do not correspond to any legitimate Harish-Chandra module. For example, if X is an irreducible Harish-Chandra module, $\mathcal{V}(\mathfrak{g}, K)$ contains $-3X$.

5.2 Infinitesimal Character

Let $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$.

Definition 12 *A module* X *is called* quasisimple *if* $\mathcal{Z}(\mathfrak{g})$ *acts by scalars in* X*. The corresponding homomorphism*

$$\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

given by

$$z \cdot x \mapsto \chi(z)x$$

for $z \in \mathcal{Z}(\mathfrak{g})$ *and* $x \in X$ *is known as the* infinitesimal character *of* X*.*

By a theorem of Dixmier [12], every irreducible $(\mathfrak{g}, K)$-module is quasisimple.

The following is a corollary to a theorem of Harish-Chandra [40, Theorem 0.2.8].

Theorem 13 *Infinitesimal characters*

$$\xi_\lambda : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$$

are parameterized by Weyl group orbits of elements $\lambda \in \mathfrak{h}^*$.

From now on, we will treat infinitesimal characters as equivalence classes of elements in $\mathfrak{h}^*$. Note that if X is a finite-dimensional irreducible $\mathcal{U}(\mathfrak{g})$-module with highest weight λ, then X has infinitesimal character $\lambda + \rho$ [26].

Definition 14 *Let* $(\cdot\, , \cdot\,)$ *denote the inner product on* $\mathfrak{h}^*$ *derived from the trace form. Then, given two elements* $\alpha, \beta \in \mathfrak{h}^*$, *define*

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)}.$$

Definition 15 *Let* $\Delta(\mathfrak{g}, \mathfrak{h})$ *denote the set of roots of* $\mathfrak{g}$ *relative to the Cartan subalgebra* $\mathfrak{h}$. *If a* $(\mathfrak{g}, K)$*-module* X *has infinitesimal character* λ *such that*

$$\langle \lambda, \alpha \rangle \neq 0 \text{ for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}),$$

then we say that X *has* regular *infinitesimal character. Otherwise,* X *has* singular *infinitesimal character.*

Definition 16 *Call* $\lambda \in \mathfrak{h}^*$ integral *if*

$$\langle \lambda, \alpha \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Delta(\mathfrak{g}, \mathfrak{h}).$$

We say that a $(\mathfrak{g}, K)$*-module* X *has* integral *infinitesimal character if its infinitesimal character corresponds to some integral* $\lambda \in \mathfrak{h}^*$. *The set of all integral* λ *is called the* integral weight lattice.

In what follows, we will principally be interested in modules with regular integral infinitesimal character.

5.3 Coherent Families

As is well known, modules in the set of irreducible finite-dimensional $(\mathfrak{g}, K)$-modules correspond to a lattice of infinitesimal characters in $\mathfrak{h}^*$. *Coherent families* give an analogue of this picture for infinite-dimensional modules [1, 40].

Fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ that contains a Cartan subalgebra of $\mathfrak{k}_{\mathbb{R}}$, the Lie algebra of $K_{\mathbb{R}}$. (Such a Cartan subalgebra is called *fundamental*. This technical property will be important in subsequent machinery.) Assume also that $K_{\mathbb{R}}$ is connected. Complexify $\mathfrak{h}_{\mathbb{R}}$ to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and choose a Borel subalgebra $\mathfrak{b}$ determined by a system of positive roots relative to $\mathfrak{h}$. Let $H_{\mathbb{R}}$ be the centralizer in $G_{\mathbb{R}}$ of $\mathfrak{h}_{\mathbb{R}}$. Since $\mathfrak{h}_{\mathbb{R}}$ is fundamental, $H_{\mathbb{R}}$ is a connected Cartan subgroup of $G_{\mathbb{R}}$. Let $\Lambda \subset \widehat{H}_{\mathbb{R}}$ be the lattice of weights of finite-dimensional representations for $G_{\mathbb{R}}$. Because $H_{\mathbb{R}}$ is connected, we can identify Λ with a sublattice of the integral weight lattice in $\mathfrak{h}^*$ via differentiation.

Definition 17 *Given a finite-dimensional representation F of $G_{\mathbb{R}}$, denote by $\Delta(F)$ the multiset of weights of F in $\mathfrak{h}^*$ counted with multiplicity. (Regard each weight in $\widehat{H}_{\mathbb{R}}$ as an element of $\mathfrak{h}^*$ by differentiation.)*

We need an appropriate means of taking the tensor product of a finite-dimensional representation and a virtual character.

Definition 18 *Each virtual character Θ can be uniquely expressed as a finite integral linear combination of irreducible characters:*

$$\Theta = \sum_{X \in \widehat{G}_{\mathbb{R}}} m_X X.$$

where $\widehat{G}_{\mathbb{R}}$ is the set of irreducible Harish-Chandra modules of $G_{\mathbb{R}}$. Given a finite-dimensional representation F of $G_{\mathbb{R}}$, let

$$\Theta \otimes F = \sum_{X \in \widehat{G}_{\mathbb{R}}} m_X X \otimes F.$$

Definition 19 *Let λ be any integral weight in $\mathfrak{h}^*$. Define the* translate *of Λ by λ to be the sublattice of the integral weight lattice given by*

$$\Lambda + \lambda = \{\mu + \lambda \mid \mu \in \Lambda\}.$$

Definition 20 *[40, Definition 7.2.5] A* coherent family *of virtual characters is a map*

$$\Phi : (\Lambda + \lambda) \to \mathcal{V}(\mathfrak{g}, K)$$

satisfying the following properties:

1. *$\Phi(\xi)$ has infinitesimal character ξ.*

2. *If ξ is dominant, then $\Phi(\xi)$ is either 0 or the character of an irreducible $(\mathfrak{g}, K)$-module; in particular, $\Phi(\xi)$ corresponds to an irreducible $(\mathfrak{g}, K)$-module whenever ξ is regular and dominant.*

3. *For any finite-dimensional representation F*

$$\Phi(\xi) \otimes F = \sum_{\mu \in \Delta(F)} \Phi(\xi + \mu)$$

where the sum counts multiplicity.

Note that this definition depends on a choice of positive roots. Each irreducible $(\mathfrak{g}, K)$-module at regular integral infinitesimal character lies in the dominant chamber of a unique coherent family [40, Theorem 7.2.7].

5.4 The Coherent Continuation Representation

The coherent family structure allows us to define a representation of the Weyl group. Denote by $\mathcal{V}(\mathfrak{g}, K)_\lambda$ the formal $\mathbb{Z}$-span of all irreducible Harish-Chandra modules at infinitesimal character λ, where λ is regular and integral; $\mathcal{V}(\mathfrak{g}, K)_\lambda$ is naturally viewed as a free $\mathbb{Z}$-submodule of $\mathcal{V}(\mathfrak{g}, K)$. Let X be any irreducible Harish-Chandra module at infinitesimal character λ, Φ the coherent family containing X and w an element of the Weyl group W. Then, we define a representation on $\mathcal{V}(\mathfrak{g}, K)_\lambda$ by

$$w \cdot X = w \cdot \Phi(\lambda) = \Phi(w^{-1} \cdot \lambda).$$

This is called the coherent continuation representation. Any choice of dominant, regular, integral λ yields an equivalent construction.

5.5 Associated Varieties and Cycles on the Level of Coherent Families

As we will now see, the structural relationships between modules in a coherent family are reflected in associated varieties and cycles. The following is a version of Lemma 4.1 in [8].

Lemma 21 *Suppose that X is a finite-length $(\mathfrak{g}, K)$-module and F a finite-dimensional representation. Then,* $\mathrm{AV}(X) = \mathrm{AV}(X \otimes F)$.

Proof. Given an appropriate filtration X_i of X, we can filter $X \otimes F$ by $X_i \otimes F$. As an $S(\mathfrak{g})$-module, $\mathrm{gr}\,(X \otimes F)$ is a sum of copies of $\mathrm{gr}\, X$. ■

Lemma 22 *If X and Y are irreducible $(\mathfrak{g}, K)$-modules in the same coherent family, then* $\mathrm{AV}(X) = \mathrm{AV}(Y)$.

Proof. Suppose that the $(\mathfrak{g}, K)$-module Z is a subquotient of a module X. If X_i is a good filtration of X as per the definitions given in Chapter 4, then Z inherits a good filtration Z_i. We have the containment

$$\mathrm{Ann}(\mathrm{gr}\, X) \subseteq \mathrm{Ann}(\mathrm{gr}\, Z).$$

Hence,

$$\mathrm{AV}(X) \supseteq \mathrm{AV}(Z).$$

If X and Y are irreducible modules in the same coherent family, then there exists a finite-dimensional representation F such that X is a subquotient of $Y \otimes F$ and Y is a subquotient of $X \otimes F$. Applying the previous lemma, $\mathrm{AV}(X) \subseteq \mathrm{AV}(Y)$ and $\mathrm{AV}(Y) \subseteq \mathrm{AV}(X)$. ■

We would like to extend the associated variety and associated cycle construction to virtual characters that do not correspond to bona fide $(\mathfrak{g}, K)$-modules. Observe that if

$$0 \to Y \to X \to Z \to 0$$

then

$$\mathrm{Ann}(\mathrm{gr}\, X) = \mathrm{Ann}(\mathrm{gr}\, Y) \cap \mathrm{Ann}(\mathrm{gr}\, Z)$$

and

$$\mathrm{AV}(X) = \mathrm{AV}(Y) \cup \mathrm{AV}(Z).$$

We extend this additivity to all of $\mathcal{V}(\mathfrak{g}, K)$ by specifying that

$$\mathrm{AV}(-X) = \mathrm{AV}(X)$$

and

$$\mathrm{AV}(X) = \mathrm{AV}(Y) \cup \mathrm{AV}(Z)$$

for any virtual characters X, Y, Z such that

$$X = Y + Z$$

in the Grothendieck group.

There is also a notion of additivity for associated cycles [10, 36]. Once again, let X, Y and Z satisfy

$$0 \to Y \to X \to Z \to 0.$$

Then, if C is some component of $\mathrm{AV}(X)$, the multiplicity of C in $\mathrm{AC}(X)$ is the sum of multiplicity of C in $\mathrm{AC}(Y)$ and $\mathrm{AC}(Z)$. If C is not a component of $\mathrm{AV}(Y)$ then, for the present purpose, we define the multiplicity of C in $\mathrm{AC}(Y)$ to be 0. In general, $\mathrm{AC}(X)$ loses some information from $\mathrm{AC}(Y)$ and $\mathrm{AC}(Z)$: if C' is a component of $\mathrm{AV}(Y)$ that is properly contained in C, then the multiplicity of C' in $\mathrm{AC}(Y)$ makes no contribution to the multiplicity of C in $\mathrm{AC}(X)$. If $X = -Y$ in $\mathcal{V}(\mathfrak{g}, K)$ and C is a component of $\mathrm{AV}(X) = \mathrm{AV}(Y)$, let the multiplicity of C in X be the opposite of its multiplicity in Y. By applying these additivity rules, we can construct a well defined associated cycle for any virtual character.

Definition 23 *Fix a choice of positive roots, an irreducible module X at regular integral infinitesimal character and some component of* $\mathrm{AV}(X)$. *Let Φ denote the coherent family to which X belongs. Let $p'_X(\lambda)$ denote the multiplicity of the fixed component in* $\mathrm{AC}(\Phi(\lambda))$ *as λ varies over the appropriate sublattice of the integral weight lattice. This function extends to a unique harmonic homogeneous polynomial on $\mathfrak{h}^*$ [38, Lemmas 4.1 and 4.3] which we denote by $p_X(\lambda)$.*

In this notation, nothing indicates which component p_X gives the multiplicity for; this is a practical choice made because $\mathrm{AV}(X)$ is irreducible for any Harish-Chandra module of $U(p, q)$.

5.6 Cells

Cells encode information about the consequences of tensoring an irreducible Harish-Chandra module X and an irreducible finite-dimensional representation with highest weight chosen from the *root lattice* in $\mathfrak{h}^*$ [1]; the root lattice is the sublattice of the integral weight lattice generated by the roots in $\mathfrak{h}^*$. Write $T(\mathfrak{g})$ for the tensor algebra generated by $\mathfrak{g}$. For irreducible $(\mathfrak{g}, K)$-modules X and Y at infinitesimal character λ, write $X > Y$ if Y appears as a subquotient of $X \otimes F$ for some finite-dimensional representation appearing in $T(\mathfrak{g})$. If $X > Y$ and $Y > X$, then write $X \sim Y$. In fact, $\sim$ is an equivalence relation.

Definition 24 *The equivalence classes defined by $\sim$ are called* cells *[1].*

Now we wish to construct an object called a *cell representation.* Let λ be dominant, regular and integral. As discussed in Section 5.4, $\mathcal{V}(\mathfrak{g}, K)_\lambda$ is a W-representation via coherent continuation. Define

$$\operatorname{Cell}(X) = \operatorname{span}_{\mathbb{Z}}\{Y \mid Y < X \text{ and } X < Y\}.$$

We also introduce the related notion of the *cone* over X, consisting of those modules Y such that $Y < X$. We denote the span as

$$\operatorname{Cone}(X) = \operatorname{span}_{\mathbb{Z}}\{Y \mid Y < X\}.$$

We can view $\operatorname{Cell}(X)$ in a slightly different way:

$$\operatorname{Cell}(X) \cong \operatorname{Cone}(X) \Big/ \operatorname{span}_{\mathbb{Z}}\{Y \mid Y < X,\ X \not< Y\}.$$

Both $\operatorname{Cone}(X)$ and $\operatorname{span}_{\mathbb{Z}}\{Y \mid Y < X,\ X \not< Y\}$ are subrepresentations of $\mathcal{V}(\mathfrak{g}, K)_\lambda$, so $\operatorname{Cell}(X)$ is a subquotient of the full coherent continuation representation.

5.7 W-equivariance

As we saw with coherent families, the structural relationships between modules in cells are reflected in associated varieties and cycles. First, observe that the proof of Lemma 22 showing that modules in the same coherent family share an associated variety also shows that modules in a cell have the same associated variety.

Fix a component C of $\operatorname{AV}(X)$. We wish to study the relationship between the associated cycle multiplicity polynomials attached to modules in the cell of X. Define

$$\operatorname{Poly}(X) = \operatorname{span}_{\mathbb{Z}}\{p_Y \mid Y < X \text{ and } X < Y\}$$

where p_Y is the associated cycle multiplicity polynomial for Y and the fixed component C of $\operatorname{AV}(X) = \operatorname{AV}(Y)$.

Theorem 25 *The map from* $\operatorname{Cell}(X)$ *to* $\operatorname{Poly}(X)$ *induced by*

$$Y \mapsto p_Y$$

for irreducible modules Y in the cell of X is W-equivariant.

Proof. We will show that irreducible modules Y such that

$$Y < X,\ X \not< Y$$

satisfy $p_Y = 0$ for the fixed component C. Note first that by the proof of Lemma 22, $\operatorname{AV}(Y) \subset \operatorname{AV}(X)$. But in fact $\dim \operatorname{AV}(Y) < \dim \operatorname{AV}(X)$, therefore C is not a component of

AV(Y) and $p_Y = 0$. (This is a corollary to a result of Borho and Kraft [9, Korollar 4.7]. See also the discussion around Proposition 2.2 in [36].)

Now, induce a map from Cone(X) to associated cycle polynomials by taking

$$Y \mapsto p_Y$$

for the component C. The map induced by extending linearly is W-equivariant since

$$p_{w\cdot Y}(\alpha) = p_Y(w^{-1} \cdot \alpha)$$

where $w \in W$ and $w \cdot Y$ is given by the coherent continuation action. It is now clear that the associated cycle map from Cell(X) to polynomials on $\mathfrak{h}^*$ is W-equivariant; as W-representations,

$$\text{Cell}(X) \cong \text{Cone}(X) \Big/ \text{span}_{\mathbb{Z}}\{Y \mid Y < X, X \not< Y\}$$

but virtual modules in

$$\text{span}_{\mathbb{Z}}\{Y \mid Y < X, X \not< Y\}$$

effectively have associated cycle multiplicity 0 on any component of AV(X). ■

CHAPTER 6

SPRINGER FIBERS

Springer fibers play a central role in the computations we wish to carry out. In fact, we will see that Euler characteristics of certain line bundles over Springer fiber components are building blocks for all associated cycle polynomials.

6.1 Definition

The definition of Springer fibers in the Lie theoretic context relies on the moment map from the cotangent bundle of the flag variety to $\mathfrak{g}^*$. Define the flag variety $\mathfrak{B}$ to be the set of all Borel subalgebras of $\mathfrak{g}$. We can view the cotangent bundle to the flag variety as the set of pairs $\{(\mathfrak{b}, \xi) \mid \mathfrak{b} \in \mathfrak{B},\ \xi \in (\mathfrak{g}/\mathfrak{b})^*\}$. Let $\mathcal{N}^*$ be the cone of nilpotent elements in $\mathfrak{g}^*$. Then, the moment map $\mu : T^*\mathfrak{B} \to \mathcal{N}^*$ carries $(\mathfrak{b}, \xi)$ to $\xi \in \mathfrak{g}^*$.

Definition 26 *Let ξ be a nilpotent element of $\mathfrak{g}^*$. The* Springer fiber *$\mathcal{F}^\xi$ over ξ is the variety $\mu^{-1}(\xi)$.*

In the specific context of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, we can identify ξ with a nilpotent element $N \in \mathfrak{g}$ by using the trace form. The Springer fiber over ξ is then equal to the set of all Borel subalgebras of $\mathfrak{g}$ containing N. Fibers over nilpotent elements in the same nilpotent orbit are isomorphic varieties.

It is computationally useful to view the flag variety in terms of linear algebra. Consider the Borel subalgebra $\mathfrak{b}$ consisting of all upper triangular matrices. If $\{v_1, v_2, \ldots, v_n\}$ is the standard basis of $\mathbb{C}^n$, then $\mathfrak{b}$ stabilizes

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

where each V_k is a k-dimensional subspace of $\mathbb{C}^n$ spanned by the first k standard basis vectors. In general, a complete flag is some sequence of subspaces

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

of $\mathbb{C}^n$ such that $\dim V_k = k$. The variety consisting of all complete flags in $\mathbb{C}^n$ is isomorphic as a variety to $\mathfrak{B}$ as defined in terms of Borel subalgebras above. The isomorphism takes each Borel subalgebra $\mathfrak{b}$ to the unique complete flag stabilized by $\mathfrak{b}$, i.e., some complete flag

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

such that $\mathfrak{b} \cdot V_k \subset V_k$ for for each k.

This leads to a natural linear algebraic construction of the Springer fiber over N. The variety of Borel subalgebras containing N is naturally isomorphic to the variety of complete flags stabilized by N.

In $\mathfrak{gl}(n, \mathbb{C})$, the conjugacy classes of nilpotents, and hence the equivalence classes of Springer fibers, are parameterized by partitions of n or equivalently by Young diagrams with n boxes. In general, Springer fibers are highly reducible. Let N be a nilpotent element of $\mathfrak{gl}(n, \mathbb{C})$. Let f be a flag

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

preserved by N. Define a tableau T on the Young diagram for N by requiring that the diagram obtained from T by restricting to boxes 1 through k has the same shape as the diagram of the Jordan form for the restriction of N to V_k. The standard tableaux of shape given by the Jordan form of N then divide $\mathcal{F}^N$ into subsets. Taking the Zariski closures of these subsets yields the components of $\mathcal{F}^N$ [34].

6.2 Component Geometry in a Special Case

Now we wish to work out the geometry of Springer fiber components attached to column superstandard tableaux. (Recall that these are transposes of reading order tableaux.) This geometry will be the foundation for computation of general fiber polynomials for $\mathfrak{gl}(n, \mathbb{C})$.

Let Y be a Young diagram and T the corresponding column superstandard tableau. Let C_k denote the kth column of T from the left. Define a subspace W_{C_k} of $\mathbb{C}^n$ by

$$W_{C_k} = \bigoplus_{i \in C_k} \langle v_i \rangle$$

where the v_i are the standard basis elements of $\mathbb{C}^n$. Define a nilpotent map $\xi : \mathbb{C}^n \to \mathbb{C}^n$ by these rules:

1. The kernel of N is W_{C_1}.

2. $N \cdot W_{C_k} \subset W_{C_{k-1}}$

There is significant flexibility in this definition, but the Jordan canonical form of N corresponds to the diagram Y regardless of our choices.

Denote by $\mathcal{F}_T^N$ the component of $\mathcal{F}^N$ parameterized by the column superstandard tableau T.

Lemma 27 *Let k_m be the number of boxes in the first m columns of Y. Let f be a flag*

$$V_0 \subset V_1 \subset \cdots \subset V_n$$

in $\mathcal{F}_T^N$ that yields the tableau T via the restriction process described in Section 6.1. Then

$$V_{k_m} = \bigoplus_{i=1}^{m} W_{C_i}$$

for each m from 1 up to the number of columns of Y. Conversely, if a flag f satisfies this condition for any choice of m up to the number of columns in T, then $f \in \mathcal{F}_T^N$

The next theorem essentially completes the geometric description of $\mathcal{F}_T^N$.

Theorem 28 *Let T be the column superstandard tableau. Let f be the standard flag, i.e., the flag whose subspaces V_k are spanned by the first k basis vectors of $\mathbb{C}^n$. Let $|C_i|$ be the number of boxes in the ith column of Y. Then, the Springer fiber component corresponding to Y is given by*

$$\mathcal{F}_Y^N = (SL(|C_1|, \mathbb{C}) \times SL(|C_2|, \mathbb{C}) \times \cdots)\, f$$

where $SL(|C_i|, \mathbb{C})$ acts on the space W_{C_i} defined above.

Proof. The subset defined in the theorem is the same one established in Lemma 27. It remains only to prove that $\mathcal{F}_Y^N$ is Zariski closed as defined. The set is closed because it is a flag variety. ■

Alternatively, define P to be the parabolic subgroup of $GL(n, \mathbb{C})$ consisting of all block upper triangular invertible matrices with blocks of sizes $|C_1|$, $|C_2|, \ldots$ Then, $\mathcal{F}_Y^N = P/B$, where B is the upper triangular Borel subgroup of $GL(n, \mathbb{C})$.

6.3 Fiber Polynomials

Let Z be some component in $\mathrm{Irr}(\mathcal{F}^N)$, the irreducible components of $\mathcal{F}^N$. To compute fiber polynomials, we construct a line bundle over Z by restricting a Borel–Weil line bundle from $\mathfrak{B}$ to Z. Here, we are identifying $\mathfrak{B}$ with G/B; the base Borel is chosen by fixing a Cartan subalgebra and choice of positive roots. In our case, $G = GL(n, \mathbb{C})$.

Let λ be a dominant integral weight. If the base Borel subgroup B has Lie algebra $\mathfrak{b}$, let $\mathbb{C}_\lambda$ be the one-dimensional representation of B induced by λ. Construct the Borel–Weil line bundle $\mathcal{L}_\lambda = G \times_B \mathbb{C}_{-\lambda}$. We define a function $\lambda \mapsto q'_Z(\lambda)$ by taking $q'_Z(\lambda)$ to be the Euler characteristic of $\mathcal{L}_\lambda$ restricted to Z. This function extends to a polynomial on $\mathfrak{h}^*$; we define a homogeneous polynomial q_Z by taking the top degree part of q'_Z. Refer to q_Z as the *Joseph polynomial* or *fiber polynomial* of Z [37, 19].

We mention an alternate method of constructing q_Z for the sake of completeness. Frequently, one sees in the literature the notation

$$q_Z(\lambda) = \int_Z e^\lambda,$$

i.e., the integral over Z of the exponential of the first Chern class of $\mathcal{L}_\lambda$ [3, 10]. For a careful definition of this integral, see [20] or [33].

In our setting, we want to compute the Euler characteristic of $\mathcal{L}_\lambda$ restricted to P/B as defined in the previous section. This will give us the fiber polynomial for the column superstandard tableau. Equivalently, $P/B \cong L/(L \cap B)$ where

$$L = GL(|C_1|, \mathbb{C}) \times GL(|C_2|, \mathbb{C}) \times \cdots$$

is the Levi subgroup of P. The flag variety

$$L/B \cap L = GL(|C_1|, \mathbb{C})/B_1 \times GL(|C_2|, \mathbb{C})/B_2 \times \cdots$$

for appropriately chosen $\{B_1, B_2, \ldots\}$ is identified with P/B by the map γ that takes

$$g_1 B_1 \times g_2 B_2 \times \cdots$$

to

$$(g_1 B_1 \times g_2 B_2 \times \cdots)B.$$

We can construct a Borel–Weil line bundle $\mathcal{L}'_\lambda$ on $L/B \cap L$ by inducing from $-\lambda$ a character of $B \cap L$. The map γ allows us to construct a line bundle on $L/B \cap L$ from $\mathcal{L}_\lambda$; in fact, this line bundle is isomorphic to $\mathcal{L}'_\lambda$. By the Borel–Weil theorem, the Euler characteristic of $\mathcal{L}'_\lambda$ is simply the dimension of the irreducible representation of L induced from λ. The next proposition then follows by applying the Weyl dimension formula and taking the highest degree part of the resulting polynomial.

In the following theorem, take $\mathfrak{h}$ to be the diagonal Cartan in $\mathfrak{gl}(n, \mathbb{C})$; each x_i lies in the dual of $\mathfrak{h}^*$ and hence may be viewed as an element of $\mathfrak{h}$. Let $x_i = E_{i,i}$. If e_i is a basis element in $\mathfrak{h}^*$ that returns the ith diagonal entry of a matrix in $\mathfrak{h}$, each x_i effectively returns the coefficient of e_i in an element of $\mathfrak{h}^*$. Take the positive system determined by the upper triangular Borel.

Theorem 29 *Let Z be a Springer fiber component corresponding to a column superstandard tableau Y. Then,*

$$q_Z = A \prod_{(i,j)\in S_Y} (x_i - x_j).$$

The set S_Y consists of all pairs $i < j$ such that i and j are in the same column of Y and

$$A = \prod A_{|C_k|-1}$$

where

$$A_m = \frac{1}{m!(m-1)!\cdots 1}$$

and the $|C_k|$ are the lengths of the columns of Y. Equivalently, one may choose A so that $q_Z(\rho) = 1$.

Recall that q_Z is, up to scale, the polynomial p_Γ corresponding to the column superstandard tableau in Chapter 2.

Note that these are also associated cycle polynomials: let X be a Harish-Chandra module of $U(p, q)$ at integral regular infinitesimal character supported on $Q \in K\backslash\mathfrak{B}$; let $T(Q)$ be the corresponding standard tableau and suppose that $T(Q)$ is a column superstandard tableau. Then q_Z, the fiber polynomial corresponding to $T(Q)$, is the associated cycle polynomial for X. See Chapter 7.

6.4 Localization

We now present an alternative means of attaching geometry to a Harish-Chandra modules X, the well known localization of Beilinson and Bernstein. (A nice exposition in the setting of Harish-Chandra modules is found in [10].)

Localization was originally developed to prove Kazhdan and Lusztig's conjecture [21] that Kazhdan–Lusztig polynomials describe the change of basis from Verma modules to irreducible modules in the Grothendieck group of category $\mathcal{O}$ [5]. It also has direct relevance to the associated cycle computations we wish to carry out.

Some choices must be made at this point; we wish to view the flag variety $\mathfrak{B}$ as G/B and so choose a base Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$, a positive system of roots relative to $\mathfrak{h}$ and a corresponding Borel subalgebra $\mathfrak{b}$. For each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein construct in [5] a twisted sheaf of differential operators $\mathcal{D}_\lambda$. The localization construction yields from each Harish-Chandra module a coherent $(\mathcal{D}_\lambda, K)$-modules. In particular, let X be an irreducible Harish-Chandra module with infinitesimal character corresponding to λ dominant. Then, there exists a unique irreducible $(\mathcal{D}_\lambda, K)$-module $\mathcal{X}$ such that X is equal to the sheaf of global sections $\Gamma(\mathfrak{B}, \mathcal{X})$.

Each $(\mathcal{D}_\lambda, K)$-module $\mathcal{X}$ has a characteristic variety and characteristic cycle. The characteristic variety is of the form

$$\overline{T^*_{Z_1}\mathfrak{B}} \cup \cdots \cup \overline{T^*_{Z_k}\mathfrak{B}}$$

where each Z_i is a K-orbit on $\mathfrak{B}$, the notation $\overline{T^*_{Z_i}\mathfrak{B}}$ indicates the closure of a conormal bundle to an orbit and the union is finite; each $\overline{T^*_{Z_i}\mathfrak{B}}$ is a component of the characteristic variety. The characteristic cycle attaches a multiplicity to each component.

The characteristic variety of $\mathcal{X}$ is related to $\mathrm{AV}(X)$ by a result of Borho and Brylinski [7, Theorem 1.9(c)]: the image of the characteristic variety of $\mathcal{X}$ under the moment map

$$\mu : T^*\mathfrak{B} \to \mathfrak{g}^*$$

is equal to $\mathrm{AV}(X)$.

6.5 Application to Associated Cycles

Via results of Chang [10], fiber polynomials are related to associated cycle polynomials in a fundamental way. To understand the relationship, we need the notion of the *leading term* of a characteristic variety. Given a Harish-Chandra module X, denote the characteristic variety of its Beilinson–Bernstein localization by $\mathrm{CV}(X)$. As discussed above, the moment map μ carries $\mathrm{CV}(X)$ onto $\mathrm{AV}(X)$.

Definition 30 *Let $\overline{\mathcal{O}_K}$ be some component of* $\mathrm{AV}(X)$*. The* leading term *of* $\mathrm{CV}(X)$ *over $\overline{\mathcal{O}_K}$, denoted by* $\mathrm{LT}(X, \overline{\mathcal{O}_K})$*, is the set of all components $\overline{T^*_Q\mathfrak{B}}$ of* $\mathrm{CV}(X)$ *such that $\mathcal{O}_K$ is in $\mu(\overline{T^*_Q\mathfrak{B}})$. The leading term of* $\mathrm{CV}(X)$*, denoted* $\mathrm{LT}(X)$*, is the union of the* $\mathrm{LT}(X, \overline{\mathcal{O}_K})$ *over all components of* $\mathrm{AV}(X)$*.*

Now, we must understand the connection between components of Springer fibers and components of $\mathrm{CV}(X)$. Fix some $\overline{T^*_Q\mathfrak{B}}$ and let ξ be a generic element in $\mu(\overline{T^*_Q\mathfrak{B}})$, i.e., an element of the K-orbit dense in $\mu(\overline{T^*_Q\mathfrak{B}})$. In general, $\mu^{-1}(\xi) \cup \overline{T^*_Q\mathfrak{B}}$ is a union of components of the Springer fiber $\mu^{-1}(\xi)$, but these components are all equivalent in the sense that they lie in a single orbit of the component group $A_K(\xi)$ of the centralizer in K of ξ and their fiber polynomials are equal. Denote by $Z(Q)$ the components in $\mathrm{Irr}(\mu^{-1}(\xi) \cap \overline{T_Q^*\mathfrak{B}})$ and by $q_{Z(Q)}$ the sum of the (equal) fiber polynomials obtained from the components in $Z(Q)$.

Write $\mathrm{CC}(X)$ for the characteristic cycle of the localization of X and by

$$m(\mathrm{CC}(X), \overline{T_Q^*\mathfrak{B}})$$

the multiplicity of $\overline{T_Q^*\mathfrak{B}}$ in $\mathrm{CC}(X)$. To make sense of the following theorem, we should note that $\mathrm{CC}(Y) = \mathrm{CC}(X)$ for Y in the coherent family of X [10]. The theorem depends on the choice of a positive system of roots.

Theorem 31 *(Chang's Theorem [10]) Let $\overline{\mathcal{O}_K}$ be some component of* $\mathrm{AV}(X)$. *Then, the multiplicity polynomial for $\overline{\mathcal{O}_K}$ in* $\mathrm{AC}(X)$ *is equal to*

$$\sum_{\{Q \mid \overline{T_Q^*\mathfrak{B}} \in \mathrm{LT}(X, \overline{\mathcal{O}_K})\}} m(\mathrm{CC}(X), \overline{T_Q^*\mathfrak{B}}) q_{Z(Q)}.$$

Thus, the problem of computing associated cycles is reduced to the (nontrivial) problems of computing fiber polynomials and leading terms of characteristic cycles. We will see in the next chapter that this is tractable in the case of the real group $U(p,q)$.

CHAPTER 7

SPECIALIZATION TO INDEFINITE UNITARY GROUPS

In this chapter, we specialize to the case of the indefinite unitary groups, i.e., the real groups $U(p,q)$. We will follow the discussions in the appendix to [3] and in [35]. Recall the definition of $U(p,q)$:

Definition 32 *The real Lie group $U(p,q)$, $p+q=n$, consists of all linear transformations of $\mathbb{C}^n$ that preserve a Hermitian form defined by a diagonal matrix where 1 appears p times and -1 appears q times.*

Note that the appendix to [3] mentions that in the case of $SU(p,q)$, the subgroup of $U(p,q)$ consisting of determinant 1 transforms, there is one block at each regular integral infinitesimal character if $p \neq q$, but two blocks appear if $p = q$. One of these blocks, the one containing a finite-dimensional representation, has what are for our purposes good characteristics; the associated cycles for these representations are relatively easy to compute. By the work of Beilinson and Bernstein applied to Harish-Chandra modules, irreducible Harish-Chandra modules at a fixed regular integral infinitesimal character correspond to pairs $(\mathcal{O}, \phi)$, where $\mathcal{O}$ is a K-orbit on the flag variety $\mathfrak{B}$ and ϕ is a K-equivariant local system on $\mathcal{O}$. If $\mathcal{O} = K \cdot v$, then define $A_K(v) = Z_K(v)/Z_K^\circ(v)$; in other words, $A_K(v)$ is the component group of the centralizer in K of v. We may view ϕ as the choice of an irreducible representation of $A_K(v)$. The "bad" block of modules for $SU(p,p)$ corresponds to nontrivial ϕ; for $U(p,q)$, $A_K(v)$ is always trivial, so this bad block does not appear for $U(p,p)$.

The formula in Theorem 31 simplifies dramatically for $U(p,q)$. As before, let $\mathcal{X}$ be the Beilinson–Bernstein localization of an irreducible Harish-Chandra module X. The support of $\mathcal{X}$ is the closure of a single K-orbit in $\mathfrak{B}$. Denote this K-orbit by $\operatorname{supp}(X)$. By [7, Proposition 2.8(a)], $\overline{T^*_{\operatorname{supp}(X)}\mathfrak{B}}$ is a component of $\operatorname{CV}(X)$ and has multiplicity 1 in $\operatorname{CC}(X)$. The explicit calculations in [35] show that

$$\mu(\overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}) = \mathrm{AV}(X),$$

so $\mathrm{AV}(X)$ is irreducible and $\overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}$ is in $\mathrm{LT}(X)$.

Proposition 33 *If X is an irreducible Harish-Chandra module of $U(p,q)$ at regular, integral infinitesimal character, then*

$$\mathrm{LT}(X) = \overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}.$$

Proof. Let λ be the infinitesimal character of X. Consider the set of primitive ideals $\mathrm{Prim}_\lambda(\mathfrak{g}, \mathcal{O})$, as defined in [37]. This is the set of two sided ideals I in $\mathcal{U}(\mathfrak{g})$ that are annihilators of simple $\mathcal{U}(\mathfrak{g})$ modules at infinitesimal character λ and satisfying $\mathrm{AV}(I) = \overline{\mathcal{O}}$. (The associated variety of I is defined by taking the degree filtration and letting $\mathrm{AV}(I)$ be the zero locus of $\mathrm{gr}\, I$ as an ideal in $S(\mathfrak{g})$.) Let I_X be the annihilator of X as a $\mathcal{U}(\mathfrak{g})$ module and let $\overline{\mathcal{O}} = \mathrm{AV}(I_X)$. (We want I_X to be an element of $\mathrm{Prim}_\lambda(\mathfrak{g}, \mathcal{O})$.) Incidentally, $\mathrm{AV}(I_X) = G \cdot \mathrm{AV}(X)$.

To each element of $\mathrm{Prim}_\lambda(\mathfrak{g}, \mathcal{O})$ is attached a harmonic homogeneous polynomial p_I on $\mathfrak{h}^*$, the so-called *Goldie rank polynomial* [18]. Define

$$\mathrm{Sp}(\mathcal{O}) = \mathrm{span}_{\mathbb{C}}\{p_I \mid I \in \mathrm{Prim}_\lambda(\mathfrak{g}, \mathcal{O})\}.$$

This space of polynomials is a representation of S_n isomorphic to the representation attached to $\mathcal{O}$ by the Springer correspondence. These Goldie rank polynomials are linearly independent. Taking the span of the fiber polynomials for the components of the Springer fiber over an element of $\mathcal{O}$ also gives $\mathrm{Sp}(\mathcal{O})$. Fiber polynomials for the components of a Springer fiber are linearly independent in the setting of $\mathfrak{gl}(n, \mathbb{C})$.

In general, relating these bases of $\mathrm{Sp}(\mathcal{O})$ is difficult [37]. However, in the setting of $\mathfrak{gl}(n, \mathbb{C})$, the bases coincide due to a result of Melnikov [30]. It is also known that the associated cycle polynomial p_X for some component of $\mathrm{AV}(X)$ is a multiple of p_{I_X}. Each fiber polynomial is, up to scale, p_I for some $I \in \mathrm{Prim}_\lambda(\mathfrak{g}, \mathcal{O})$. Thus, on the right side of

$$p_X = \sum_{\{Q \mid \overline{T^*_Q\mathfrak{B}} \in \mathrm{LT}(X, \overline{\mathcal{O}_K})\}} m(\mathrm{CC}(X), \overline{T^*_Q\mathfrak{B}}) q_{Z(Q)}$$

(Chang's theorem), the sum consists of only a single term, the one arising from $\overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}$ with multiplicity 1. ∎

Let $\mathrm{AV}(X) = \overline{\mathcal{O}_K}$. Let $\xi \in \mathcal{O}_K$ and

$$C(X) = \mu^{-1}(\xi) \cap \overline{T^*_{\mathrm{supp}(X)}\mathfrak{B}}.$$

(In the setting of $U(p,q)$, $C(X)$ is irreducible and so consists of a single Springer fiber component.) Then, the formula in Theorem 31 reduces to

$$p_X = q_{C(X)}$$

where p_X is the associated cycle polynomial for X and $q_{C(X)}$ is the fiber polynomial for $C(X)$.

Thus, computation of an associated cycle polynomial for a Harish-Chandra module of $U(p,q)$ at regular integral infinitesimal character is now reduced to the computation of $C(X)$ and $q_{C(X)}$. We will parameterize modules for $U(p,q)$ by their supports, i.e., by orbits in $K\backslash\mathfrak{B}$. The parameterization of orbits in $K\backslash\mathfrak{B}$ is well known [35]. Fix some orbit $Q \in K\backslash\mathfrak{B}$. The moment map image $T^*_Q\mathfrak{B}$ is a closed K-invariant subvariety of $(\mathfrak{g}/\mathfrak{k})^*$ and contains a unique dense K-orbit, which we refer to as $\mu_{\text{orb}}(Q)$. The K-orbits on $(\mathfrak{g}/\mathfrak{k})^*$ are parametrized by signed Young diagrams of signature (p,q) [11].

Definition 34 *A* signed Young diagram *is a Young diagram with a sign in every box such that signs alternate across rows, but not necessarily down columns. (Diagrams that can be obtained from one another by exchange of equal length rows are taken to be equivalent.) A signed Young diagram with* signature (p,q) *is one with p plus signs and q minus signs.*

Given a K-orbit Q on $\mathfrak{B}$, fix some element $\xi \in \mu_{\text{orb}}(Q)$. Then,

$$\mu^{-1}(\xi) \cap \overline{T^*_Q\mathfrak{B}}$$

is a component of the Springer fiber $\mu^{-1}(\xi)$. As discussed in Chapter 6, we assign to this component a standard Young tableau with shape corresponding to the Jordan form of N, the nilpotent map dual to ξ. Label this tableau by $T(Q)$.

The following theorem is discussed extensively in [35]:

Theorem 35 *Orbits $Q \in K\backslash\mathfrak{B}$ correspond bijectively with same shape pairs consisting of a signed Young diagram of signature (p,q) and a standard Young tableau; the bijection is obtained by taking each Q to the pair $(\mu_{\text{orb}}(Q), T(Q))$.*

Given a Harish-Chandra module X at regular integral infinitesimal character supported on an orbit $Q \in K\backslash\mathfrak{B}$, the associated cycle polynomial p_X for the corresponding coherent family is simply the fiber polynomial determined by $T(Q)$. We know this polynomial when $T(Q)$ is a superstandard tableau. It remains for us to understand how to calculate fiber polynomials for other tableaux. For this, we return to cell representations.

The Barbasch–Vogan conjecture [38, 2] relates an invariant known as the wave front cycle to the associated cycle of a Harish-Chandra module. The conjecture was proven by Schmid and Vilonen [33]. This result in conjunction with [1] gives us the following:

Theorem 36 *Fix regular integral infinitesimal character λ. For $U(p,q)$, cells at λ stand in correspondence with the K-orbits on $(\mathfrak{g}/\mathfrak{k})^*$ via the map taking a module X to $\mu_{orb}(Q_X)$, where Q_X is* $\mathrm{supp}(X)$.

Once we fix a K-orbits on $(\mathfrak{g}/\mathfrak{k})^*$ and hence a cell of Harish-Chandra modules, the elements of the cell are parameterized by all standard tableaux of the Young shape for the orbit. The following is a significant application of Kazhdan-Lusztig theory, as discussed in [29].

Theorem 37 *As a based S_n-representation, each Harish-Chandra cell for $U(p,q)$ is isomorphic to a Kazhdan–Lusztig left cell representation; the isomorphism takes each module X in a cell to the Kazhdan–Lusztig basis element parameterized by the tableau $T(Q_X)$, where $Q_X = \mathrm{supp}(X)$.*

Of course, the fiber polynomials attached to a fixed Springer fiber also yield a based S_n-representation; via the W-equivariance results in Section 5.7, the fiber polynomial representation is isomorphic to a Kazhdan–Lusztig left cell representation with the isomorphism taking the Kazhdan–Lusztig basis element parameterized by T to a fiber polynomial q_T. Given that we know q_T when T is a column superstandard tableau, this potentially gives us a method to compute all fiber polynomials or, equivalently, associated cycle polynomials, provided that we can carry out the relevant Kazhdan–Lusztig calculations.

Recall the the polynomial representation P_Y from Chapter 2 and the isomorphism

$$\phi : \mathrm{KL}_Y \to P_Y$$

from the Kazhdan–Lusztig representation to P_Y. The fiber polynomial attached to the column superstandard tableau Γ is equal up scale to p_Γ in P_Y by Theorem 29. Scale ϕ so that it maps w_Γ, the Kazhdan–Lusztig basis element parameterized by Γ, to the corresponding fiber polynomial. With the developments in this chapter, the fiber polynomial for the Springer fiber component parameterized by a tableau T is equal to $\phi(w_T)$, where w_T is the Kazhdan–Lusztig basis element parameterized by T. Our goal in what follows will be the development of closed forms of fiber polynomials, and hence associated cycle polynomials, in special cases where this is possible. For this, we will need to understand some of the combinatorics of Kazhdan–Lusztig representations.

CHAPTER 8

KAZHDAN–LUSZTIG COMBINATORICS

In this chapter, our goal is to develop the exchange rule (Theorem 56) which will be essential in all subsequent calculations. In fact, this rule will be sufficient to calculate fiber polynomials for all hook and two row Young diagrams.

We will assume the definition of Kazhdan–Lusztig polynomials and focus our attention exclusively on the combinatorics of of Kazhdan–Lusztig left cell representations for the symmetric group. The full machinery for general Coxeter groups is introduced in Kazhdan and Lusztig's original paper [21], while [16] and [6] provide thorough expositions of the combinatorial details for the symmetric group. When not otherwise specified, the definitions, lemmas and theorems in this section are discussed in detail in one of the latter two references.

The Kazhdan–Lusztig representations of the symmetric group are irreducible representations that come equipped with distinguished bases. These based representations are remarkable for their ubiquity in interesting representation theoretic contexts.

Left cell representations are constructed from Kazhdan–Lusztig graphs, whose nodes are the elements of the symmetric group. The great combinatorial challenge of this theory is to determine Kazhdan–Lusztig graphs, which in turn may be derived from Kazhdan–Lusztig polynomials. These polynomials can be computed by software for small n, but the resource requirements of the computation grow much faster than n; even a modest calculation can overwhelm a powerful computer. Alternative direct methods for the computation of Kazhdan–Lusztig graphs exist, but these are not known to be less computationally intensive than direct computation of Kazhdan–Lusztig polynomials [6]. All known computational methods are highly nonlocal; the problem of determining the graph at an arbitrary vertex is not in general much simpler than determining the entire graph.

8.1 Preliminaries

The *Bruhat order* will be essential in what follows.

Definition 38 *The* length *of a permutation x is the length of a minimal expression for x in terms of simple transpositions s_i. We define the* Bruhat order *on S_n by the rule that $x < y$ if some (equivalently any) minimal expression for x can be obtained by deleting letters in some minimal expression for y; $\ell(x, y) = \ell(y, x)$ is given by the number of letters that must be deleted from the expression for y to obtain x.*

We will freely use standard results on the Bruhat order. Thorough expositions are found in [17] and [6]. To construct Kazhdan–Lusztig cell representations we will need, in addition to Kazhdan–Lusztig cell graphs, certain data attached to each element of the symmetric group.

Definition 39 *The* left descent set *$D_L(x)$ of some x in the symmetric group S_n is the subset of $\{1, 2, \ldots, n-1\}$ consisting of all i such that i appears to the right of $i+1$ in the one-line notation for x. The right descent set $D_R(x)$ of x is defined to be the left descent set of x^{-1}.*

The theory we develop here will be specialized to left cells and left cell representations. This theory relies on attaching left descent sets to each permutation in the symmetric group. There is an analogous mirror image theory built by using right descent sets. For the sake of concision, we will assume that a cell is a left cell unless otherwise specified.

Kazhdan–Lusztig theory for the symmetric group stands in a close relationship with the theory of standard Young tableaux. This relationship begins with the Robinson–Schensted correspondence.

Definition 40 *The* Robinson–Schensted correspondence *is a bijection between elements of the symmetric group S_n and all same shape pairs of standard Young tableaux with n boxes in each tableau.*

We now explain the procedure that gives the correspondence. Start with an element of a symmetric group in one line notation. We read the numbers in this expression from left to right. The procedure is defined inductively. Suppose that we have the same shape (nonstandard) tableaux (P_i, Q_i) after reading the first i numbers. We compare the next number k to the first row of P_i; if k is greater than all numbers in the first row, then it is added to the end. If on the other hand $k_j < k < k_{j+1}$ for k_j and k_{j+1} in the first row, then k replaces or "bumps" k_{j+1} from the first row. (If k is less than every number in the first row, then k bumps the first number in the row.) We then take k_{j+1} to the next row and apply the same process applied with k in the first row. This continues until the process terminates, possibly with the creation of a new one box row at the bottom of P_i. Once the shape of P_{i+1} has been determined, we append a box with $i+1$ to Q_i so that Q_{i+1} has the correct shape. The left tableau P is often known as the bumping tableau, while Q is called the recording tableau. This procedure is best understood by example:

Example 41 *We will construct the Robinson–Schensted pair for $x = 315264$ in S_6.*

$$\begin{array}{cc} P_i & Q_i \end{array}$$

$$\begin{array}{|c|}\hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline \end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}$$

$$\begin{array}{cc} 1 & 5 \\ 3 & \end{array} \qquad \begin{array}{cc} 1 & 3 \\ 2 & \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$$

$$\begin{array}{ccc} 1 & 2 & 6 \\ 3 & 5 & \end{array} \qquad \begin{array}{ccc} 1 & 3 & 5 \\ 2 & 4 & \end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline \end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline \end{array}$$

Definition 42 *The* descent set of a tableau *T with n boxes is the set of all i in $\{1, 2, \ldots, n\}$ such that i appears in a row above the row of $i+1$ in T.*

Lemma 43 *The left descent set $D_L(x)$ of $x \in S_n$ is equal to the descent set of the left tableau in the Robinson–Schensted pair of x.*

Given a pair of elements u, v in the symmetric group, Kazhdan and Lusztig defined a polynomial $P_{u,v}(q)$. From these polynomials, we define a multiplicity $\mu(u, v)$ which equals the coefficient of the $q^{\frac{1}{2}(\ell(u,v)-1)}$ term of $P_{u,v}(q)$ if $\ell(u, v)$ is defined and odd; and 0 otherwise. Note that $\mu(u, v)$ is always an integer. When $\mu(u, v)$ is not zero, the multiplicity comes from the highest degree term of $P_{u,v}(q)$. Our goal is to avoid direct reference to Kazhdan–Lusztig polynomials whenever possible; instead, we wish to extract from the literature standard combinatorial results for Kazhdan–Lusztig representations. This will ultimately culminate in the exchange rule.

Definition 44 *The (left)* Kazhdan–Lusztig graph *of the symmetric group S_n (or K-L graph, for short) has as nodes the permutations in S_n. If $\mu(x,y) \neq 0$, then the K-L graph contains an edge $\{x,y\} \in E$. Each vertex is labeled by the left descent set $D_L(x)$. In the right Kazhdan–Lusztig graph, right descent sets $D_R(x)$ replace left descent sets as vertex labels.*

Definition 45 *(Left equivalence)*

1. *Generate a partial order on S_n by specifying that $x \preceq_L y$ if an edge connects x and y and $D_L(x)$ does not contain $D_L(y)$, i.e., in the full partial order, $x \preceq_L y$ if there is a sequence of edges from x to y with the vertices connected by each edge satisfying appropriate conditions on left descent sets.*

2. *Define an equivalence relation $\sim_L$ on S_n by $x \sim_L y$ if $x \preceq_L y$ and $y \preceq_L x$.*

3. *The equivalence classes under $\sim_L$ are called (left)* cells.

4. *The cell graph for a cell $\mathcal{C}$ is the restriction of the full Kazhdan–Lusztig graph on S_n to the nodes in $\mathcal{C}$.*

Proposition 46 *[6, Theorem 6.5.1] Each left cell in S_n is of the form $\{x \in S_n \mid Q(x) = T\}$, where $Q(x)$ is the right tableau in the Robinson–Schensted pair for x and T is a fixed standard tableau.*

Proposition 47 *[6, Theorem 6.5.2] Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two left cells in S_n defined by standard tableaux T_1 and T_2 with the same shape. Then, the map $\phi : \mathcal{C}_1 \to \mathcal{C}_2$ defined by taking a Robinson–Schensted pair (T, T_1) to a pair (T, T_2) is an isomorphism of cell graphs.*

This last proposition tells us specifically that the combinatorics of a cell in S_n are determined completely by the shape of the tableau that defines it.

Definition 48 *The τ-*invariant *of a permutation $x \in S_n$ is the set $\tau(x)$ of simple transpositions s_i such that $i \in D_L(x)$. The τ-invariant of a standard tableau T is the set of s_i such that i is in the descent set of T.*

This terminology is standard in the context of Lie theory. See for instance [39], which is in fact a precursor to Kazhdan and Lusztig's more general developments in [21].

Definition 49 *The* Kazhdan–Lusztig cell representation *of the cell $\mathcal{C}$ has a canonical basis v_x, $x \in \mathcal{C}$. The representation is defined by the following rules:*

1. *If $s_i \in \tau(x)$, then*
$$s_i \cdot v_x = -v_x.$$
2. *If $s_i \notin \tau(x)$, then*
$$s_i \cdot v_x = v_x + \left(\sum_{\{y | s_i \in \tau(y)\}} \mu(x,y) v_y \right).$$
Note in particular that y can only appear in the sum on the right if x and y are connected by an edge.

We then have the following remarkable result:

Theorem 50 *[6, Theorem 6.5.3] Let $\mathcal{C}$ be a Kazhdan–Lusztig left cell whose elements have Robinson–Schensted pairs with shape T. The cell representation of $\mathcal{C}$ is isomorphic to the representation for T constructed by Young.*

Among other things, this means that every cell representation of S_n is irreducible and that every irreducible representation occurs as a cell representation.

8.2 The Exchange Rule

We will now develop what we refer to as the *exchange rule*. This will be essential for all polynomial calculations in this paper and provides a means of bypassing computation of Kazhdan–Lusztig polynomials in certain cases.

As stated earlier, determination of full Kazhdan–Lusztig graphs is a computationally difficult problem. However, we can find certain edges easily. The following lemma follows from the definition of Kazhdan–Lusztig polynomials.

Lemma 51 *[6, Lemma 6.2.2] The elements x and $s_i x$ are always connected by a multiplicity one edge in the Kazhdan–Lusztig graph. Such edges are referred to as* weak Bruhat edges.

By standard rules for the Bruhat order, $s_i x < x$ exactly when $i+1$ occurs to the left of i in the one line expression for x. This leads us to an alternate definition for the τ-invariant of x:

Definition 52 *The τ-invariant of x is the set $\tau(x)$ of all simple transpositions s_i such that $s_i x < x$.*

Suppose that exactly one of s_i, s_{i+1} is in $\tau(x)$. Then, one of the following four patterns occurs in the one line notation for x:

$$\begin{array}{ccccccc} \cdots & i & \cdots & i+2 & \cdots & i+1 & \cdots \\ \cdots & i+1 & \cdots & i+2 & \cdots & i & \cdots \\ \cdots & i+1 & \cdots & i & \cdots & i+2 & \cdots \\ \cdots & i+2 & \cdots & i & \cdots & i+1 & \cdots \end{array}$$

By trading the first and last numbers in the pattern, we exchange s_i and s_{i+1} in $\tau(x)$, e.g., if x satisfies the first pattern above, then $\tau(x)$ contains s_{i+1}, but not s_i, whereas $\tau(s_i x)$ contains s_i but not s_{i+1}.

Definition 53 *If $\tau(x)$ contains exactly one of s_i, s_{i+1}, define $L_i(x)$ to be the permutation obtained by exchanging the first and last elements of the appropriate pattern above in the one line notation for x. There is always a weak Bruhat edge connecting x and $L_i(x)$. We also define L_i for a tableau: suppose that T is a tableau such that $\tau(T)$ contains exactly one element of $\{s_i, s_{i+1}\}$. We obtain the tableau $L_i(T)$ by exchanging either i, $i+1$ or $i+1$, $i+2$ in such a way that $\tau(L_i(T))$ is obtained from $\tau(T)$ by trading s_i with s_{i+1}.*

A quick check shows that $L_i(T)$ is well defined. The L_i operations on permutations and tableaux are directly related:

Proposition 54 *Let x be an element of S_n with corresponding Robinson–Schensted pair*

$$(P(x), Q(x))$$

such that $\tau(x)$ contains exactly one of s_i, s_{i+1}. Then, $L_i(x)$ has Robinson–Schensted pair

$$(L_i(P(x)), Q(x)).$$

Proof. See the discussion around equation (5.9) in [16]. ■

The following lemma will combine with Proposition 54 to yield the exchange rule.

Lemma 55 *Suppose that x is some permutation such that $\tau(x)$ contains the simple transposition s but not the adjacent simple transposition s'. Then, in the left cell of x, there is a unique y connected to x by an edge such that $\tau(y)$ contains s' but not s; the connecting edge has multiplicity 1.*

Proof. Observe that if s and s' are adjacent simple transpositions in S_n, then $ss's = s'ss'$. But, $ss's \cdot v_x = -ss' \cdot v_x$, so $s'ss' \cdot v_x = -ss' \cdot v_x$. This implies that for any Kazhdan–Lusztig basis element v_z appearing in $ss' \cdot v_x$, $s' \in \tau(v_z)$. By the formulas for Kazhdan–Lusztig representations in Definition 49,

$$s' \cdot v_x = v_x + \sum_{i=1}^{k} v_{y_i} + [\text{other terms with } s \text{ and } s' \text{ in } \tau]$$

where $\tau(y_i)$ contains s' but not s and the sum accounts for multiplicity if necessary. Next, we have

$$\begin{aligned} ss' \cdot v_x = -v_x + k'v_x + \sum_{i=1}^{k} v_{y_i} \\ + [\text{other terms with } s \text{ and } s' \text{ in } \tau] + [\text{other terms with } s \text{ but not } s' \text{ in } \tau]. \end{aligned}$$

Of course, each y_i is connected to x by an edge. Because these edges have multiplicity greater than or equal to 1, $k' \geq k$. We will now use the fact that for any Kazhdan–Lusztig basis element v_z appearing in the expansion of $ss' \cdot v_x$, $s' \in \tau(v_z)$; this tells us that

$$[\text{other terms with } s \text{ but not } s' \text{ in } \tau] = 0,$$

$k' = 1$ and $k = 1$. ■

Finally, we have:

Theorem 56 *(The exchange rule.) For a fixed cell $\mathcal{C}$, denote each Kazhdan–Lusztig basis element by its tableau T. Suppose that $\tau(T)$ contains a simple transposition s but not the adjacent simple transposition s'. Then in the Kazhdan–Lusztig representation,*

$$s' \cdot T = T + L_i(T) + \sum_k U_k$$

where $\tau(U_k)$ contains s and s' as well as any $s_j \in \tau(T)$ that is not adjacent to s'. The sum accounts for multiplicity. In this context, we will refer to $L_i(T)$ as the exchange tableau *and the other tableaux in the expansion of $s' \cdot T$ as* nonexchange tableaux.

Proof. This is just a combination of previous lemmas except for the requirement that for each U_k, $\tau(U_k)$ contains any simple transposition in $\tau(T)$ that is not adjacent to s'. If r is a simple transposition in $\tau(T)$ that is not adjacent to s', then s' and r commute, so $r \cdot (s' \cdot T) = -s' \cdot T$. ■

8.3 A Small Example: [3,2,1]

In this section, we will demonstrate application of the exchange rule to computation of fiber polynomials for a small Young diagram. (This approach to computation works only in relatively simple cases, such as the present one and in the setting of hook and two row tableaux which we will discuss later.) Begin with the column superstandard tableau

$$\begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{1}$$

where the superscript 1 indicates that this is T_1 and corresponds to a polynomial p_1. Integer labels on tableaux also indicate the order in which we are first able to calculate their corresponding polynomials. Of course, each calculation depends on previously computed polynomials.

By Proposition 29, we compute

$$p_1 = \frac{1}{2}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_4 - x_5).$$

Notice that $(34) \notin \tau(T_1)$ but (23) and (45) are in $\tau(T_1)$. We thus may apply the exchange rule to the pair $(23), (34)$ and to the pair $(34), (45)$. In this case, the exchange rule will yield the same exchange tableau for both pairs:

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}^{2} .$$

(As we will see, this will not always be the case.) To compute $(34) \cdot T_1$, we must rule out other tableaux that could appear in its expansion. But, the exchange rule tells us that any nonexchange tableau T_k in the expansion must contain the τ-invariant of T_1 and the transposition (34). A quick check shows that no such T_k exist, so

$$(34) \cdot \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{1} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{1} + \begin{array}{|c|c|c|} \hline 1 & 3 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}^{2} .$$

Similar reasoning now gives us the following:

$$(56) \cdot \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{1} = \begin{array}{|c|c|c|} \hline 1 & 4 & 6 \\ \hline 2 & 5 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{1} + \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{3} .$$

$$(34) \cdot \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{3} = \begin{array}{|c|c|c|} \hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}^{3} + \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}^{4} .$$

Now, we wish to compute

$$(23) \cdot \young(136,25,4)^{2} .$$

Again, (23) is not in $\tau(T_2)$ whereas (12) and (34) are. In this case, the pairs $(12),(23)$ and $(23),(34)$ yield different exchange tableaux:

$$\young(126,35,4)^{5}$$

and

$$\young(146,25,3)^{1}$$

respectively. Ruling out nonexchange tableaux in the expansion as before, we have

$$(23) \cdot \young(136,25,4)^{2} = \young(136,25,4)^{2} + \young(126,35,4)^{5} + \young(146,25,3)^{1} .$$

Continuing in like manner, we get

$$(45) \cdot \young(135,26,4)^{4} = \young(135,26,4)^{4} + \young(134,26,5)^{6} .$$

$$(23) \cdot \young(135,26,4)^{4} = \young(135,26,4)^{4} + \young(125,36,4)^{7} + \young(145,26,3)^{3} .$$

$$(45) \cdot \young(126,35,4)^{5} = \young(126,35,4)^{5} + \young(126,34,5)^{8} .$$

$$(45) \cdot \young(125,36,4)^{7} = \young(125,36,4)^{7} + \young(124,36,5)^{9} .$$

$$(34) \cdot \young(124,36,5)^{9} = \young(124,36,5)^{9} + \young(123,46,5)^{10} + \young(125,36,4)^{7} .$$

$$(56) \cdot \young(123,46,5)^{10} = \young(123,46,5)^{10} + \young(123,45,6)^{11} .$$

$$(45) \cdot \young(123,45,6)^{11} = \young(123,45,6)^{11} + \young(124,35,6)^{12} + \young(123,46,5)^{10} .$$

$$(12)\cdot \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{12} = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{12} + \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{13}.$$

$$(34)\cdot \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{13} = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{13} + \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{14}.$$

$$(45)\cdot \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{14} = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{14} + \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5\\\cline{1-2} 6\\\cline{1-1}\end{array}^{13} + \begin{array}{|c|c|c|}\hline 1&3&6\\\hline 2&4\\\cline{1-2} 5\\\cline{1-1}\end{array}^{15}.$$

$$(23)\cdot \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{14} = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{14} + \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}^{16}.$$

We then have the following sequence of equations to compute polynomials:

$$p_2 = (34)\cdot p_1 - p_1.$$

$$p_3 = (56)\cdot p_1 - p_1.$$

$$p_4 = (34)\cdot p_3 - p_3.$$

$$p_5 = (23)\cdot p_2 - p_2 - p_1.$$

$$p_6 = (45)\cdot p_4 - p_4.$$

$$p_7 = (23)\cdot p_4 - p_4 - p_3.$$

$$p_8 = (45)\cdot p_5 - p_5.$$

$$p_9 = (45)\cdot p_7 - p_7.$$

$$p_{10} = (34)\cdot p_9 - p_9 - p_7.$$

$$p_{11} = (56)\cdot p_{10} - p_{10}.$$

$$p_{12} = (45)\cdot p_{11} - p_{11} - p_{10}.$$

$$p_{13} = (12)\cdot p_{12} - p_{12}.$$

$$p_{14} = (34)\cdot p_{13} - p_{13}.$$

$$p_{15} = (45)\cdot p_{14} - p_{14} - p_{13}.$$

$$p_{16} = (23)\cdot p_{14} - p_{14}.$$

This glib recitation of calculations hides the requisite process of trial and error necessary to arriving at these relations. The principal challenge in relating new tableaux to T_1 is that it is difficult to rule out the existence of nonexchange tableaux in each expansion. Only exchange tableaux appear in the expansions above, but this was made possible by careful searching. As Young diagrams grow larger, this becomes an insurmountable problem and it is not possible to compute all the polynomials without recourse to computation of the full Kazhdan–Lusztig graph.

These computations are quite complicated, and it is easy to make mistakes. As such, it is useful to compute polynomials for each tableau as new relations are constructed and then check the τ-invariants for these polynomials. This can easily be carried out using computer algebra software.

CHAPTER 9

FIBER POLYNOMIALS FOR HOOK SHAPES

As in previous sections, we let $\mathfrak{h}$ be the diagonal Cartan and $\mathfrak{b}$ the upper triangular Borel. The variables x_i are functionals on $\mathfrak{h}^*$ given by $x_i = E_{i,i}$ in $\mathfrak{h}$.

Let H be a hook shape Young diagram with $m+1$ rows. If T is the column superstandard tableau of shape H, we know from Theorem 29 that

$$p_T = A_m \left(\prod_{1 \leq i < j \leq m+1} (x_i - x_j) \right) \tag{9.1}$$

where

$$A_m = \frac{1}{m!(m-1)! \cdots 1}.$$

Thus, the polynomial representation for the cell corresponding to H is

$$P_H = \mathrm{span}_{\mathbb{C}} \{ \sigma \cdot p_T \mid \sigma \in S_n \}.$$

9.1 Characteristics of the KL-graph

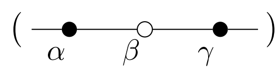
The Kazhdan–Lusztig combinatorics for hook shapes were originally worked out by Kerov in [23]. These shapes essentially give the simplest possible Kazhdan–Lusztig graphs. In particular, the relationship between tableaux of shape H and their corresponding τ-invariants is quite transparent: the τ-invariant of a tableau T is the set of all s_i such that $i+1$ appears as a label below the top row of T. The τ-invariants for tableaux of shape H are the m-element subsets of $\{s_1, s_2, \ldots s_{n-1}\}$ and the map from tableaux to such sets is a bijection. For a polynomial p, define $\tau(p)$ be the set of all s_i such that $s_i \cdot p = -p$. Then, we have the following:

Lemma 57 *The fiber polynomial for a hook shaped tableau T lies in a one-dimensional subspace $P_{\tau(T)}$ of P_H consisting of polynomials p satisfying $\tau(p) = \tau(T)$. (We include 0 in $P_{\tau(T)}$.)*

Proof. One can show that for each m-element set S of simple transpositions, KL_H has a one-dimensional subspace of vectors v satisfying $\tau(v) = S$. (This subspace is the span of a single Kazhdan–Lusztig basis element.) Since P_H is isomorphic to KL_H, the theorem follows. ■

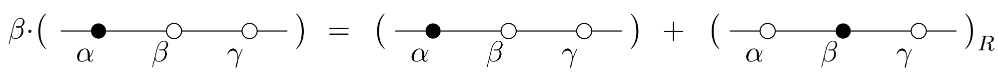
We are now close to understanding the fiber polynomials, but we still need to compute their scale factors. To do this, we need to understand the S_n action on P_H in terms of the Kazhdan-Lusztig basis. As mentioned, the Kazhdan–Lusztig graph that defines the basis is very simple in this case. In particular, given a simple transposition $\alpha \notin \tau(T)$, any nonexchange tableau in $\alpha \cdot T$ would need to contain $\tau(T) \cup \{\alpha\}$. The τ-invariants for all tableaux have the same cardinality so nonexchange tableaux do not exist.

Since tableaux of shape T can be parameterized by their τ-invariants, it will be useful to represent τ-invariants graphically by showing their simple transpositions as shaded nodes on a Dynkin diagram. For instance,

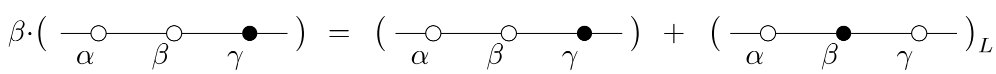


represents a tableau whose τ-invariant contains α and γ, but not β. (The τ-invariant may also contain other simple transpositions not shown.)

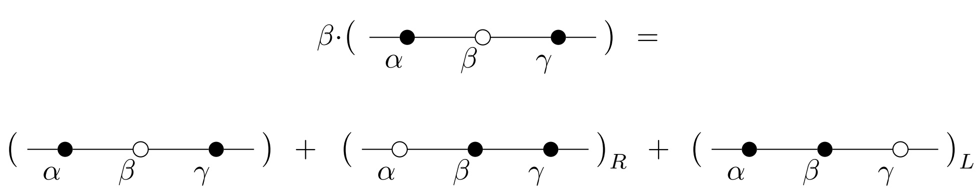
It is a simple exercise to show that for any tableau with τ-invariant containing α but not β and γ, we get



where the R indicates that this τ-invariant is obtained from the original one by “sliding” an element to the right. (Naturally, L will indicate sliding to the left.) Elements of τ-invariants not shown are the same for all diagrams. We also have



and



If $\beta \notin \tau$, we may say in general that

$$\beta \cdot T_\tau = T_\tau + (T_\tau)_R + (T_\tau)_L \tag{9.2}$$

where for instance $(T_\tau)_R$ is 0 if $\alpha \notin \tau$. (We must use the L and R notation carefully since its meaning is clear only in the context of some simple transposition β.) This leads us to the definition of a partial order on τ-invariants for a fixed hook shape H:

Definition 58 *Given m element sets τ and τ' of simple transpositions s_i, $\tau \leq \tau'$ if τ' can be obtained from τ by "sliding" elements of τ to the right; that is, generate a partial order by specifying that $\tau \leq \tau'$ if there is some i such that $s_i \in \tau$, $s_{i+1} \notin \tau$ and τ' can be obtained from τ by replacing s_i with s_{i+1}. We further say that $\tau < \tau'$ if $\tau \neq \tau'$. Note in particular that the column superstandard tableau T satisfies $\tau(T) \leq \tau'$ if τ' is any m element set of simple transpositions.*

Using equation (9.2), we can inductively define polynomials by the equation

$$p_{T_R} = \beta \cdot p_T - p_T - p_{T_L} \tag{9.3}$$

when β is not in the τ-invariant of T. (Note that T and T_L are less than T_R.) We will induct on $<$ in some of our proofs.

Now, we will present a series of lemmas that will allow us to describe the fiber polynomials in detail. Note that any polynomial in the vector space P_H consists of terms of the form

$$x_{i_1}^m x_{i_2}^{m-1} \cdots x_{i_m}^1 \tag{9.4}$$

where $\{i_1, i_2, \dots i_m\}$ is some m element subset of $\{1, 2, \dots n\}$. Given $\sigma \in S_m$, we can act by σ on the exponents of $x_{i_1}^m x_{i_2}^{m-1} \cdots x_{i_m}^1$:

$$\sigma * x_{i_1}^m x_{i_2}^{m-1} \cdots x_{i_m}^1 = x_{i_1}^{\sigma(m)} x_{i_2}^{\sigma(m-1)} \cdots x_{i_m}^{\sigma(1)}.$$

For $p \in P_H$, $\sigma * p$ is defined by applying σ term by term.

Lemma 59 *Given $\sigma \in S_m$ and $p \in P_H$,*

$$\sigma * p = \operatorname{sgn}(\sigma) p$$

Proof. One easily shows that the lemma is true for p_T when T is the column superstandard tableau. It is then also true for $\theta \cdot p_T$, $\theta \in S_n$. (Here, θ acts by the permutation action on variables.) Any $p \in P_H$ is a linear combination of such polynomials and the lemma follows. ■

Definition 60 *Given a set τ of simple transpositions s_i, define S_τ to be the subgroup of S_n generated by the simple transposition in τ. Let p be some polynomial in P_H and let S_τ act by permuting variables; then, for any $\sigma \in S_{\tau(p)}$,*

$$\sigma \cdot p = \operatorname{sgn}(\sigma) p.$$

Definition 61 *Given a set τ of simple transpositions s_i, we define the components of τ to be the maximal sequences*

$$\{s_{k+1}, s_{k+1}, \ldots, s_{k+j}\}$$

of adjacent simple transpositions in τ.

Lemma 62 *Given a polynomial $p \in P_H$ with*

$$\{s_{k+1}, s_{k+2}, \ldots s_{k+j}\}$$

a component of $\tau(p)$, each monomial of p contains exactly j of the variables

$$x_{i+1}, x_{x+2}, \ldots, x_{i+j+1}.$$

Proof. Because any transposition $\sigma \in S_\tau$ must send p_τ to $-p_\tau$, σ cannot fix any monomial term of p_τ. Suppose that a term $x_{k_1}^m x_{k_2}^{m-1} \cdots x_{k_m}^1$ is missing two variables from $x_{i+1}, x_{x+2}, \ldots, x_{i+j+1}$. Then, transposition of these variables fixes $x_{k_1}^m x_{k_2}^{m-1} \cdots x_{k_m}^1$, a contradiction. If on the other hand some term of p_τ contained all of $x_{i+1}, x_{x+2}, \ldots, x_{i+j+1}$, then this term would contain more than m variables. ■

Given $\sigma \in S_m \times S_\tau$ and $p \in P_H$, define $\sigma \cdot p$ by letting S_m act on exponents and S_τ act on variables.

Lemma 63 *Monomials of all forms allowed by Lemma 62 and expression* (9.4) *occur in p_τ. For each $\sigma \in S_m \times S_\tau$, $\operatorname{sgn}(\sigma)\sigma$ permutes the monomials of p_τ; $S_m \times S_\tau$ acts on these monomials with a single orbit.*

9.2 The Main Formula

This leads us to the main theorem for fiber polynomials attached to hook shaped Young tableaux:

Theorem 64 *The fiber polynomial p_T attached to the Young tableau T of hook shape H is the unique polynomial satisfying the following properties:*

1. *p_T consists of monomial terms of the form $x_{k_1}^m x_{k_2}^{m-1} \cdots x_{k_m}^1$.*

2. *With S_m acting on exponents and $S_\tau(T)$ acting on variables as explained above, $\sigma \cdot p_T = \mathrm{sgn}(\sigma) p_T$ for all $\sigma \in S_m \times S_{\tau(T)}$.*

3. *p_T has a monomial term*

$$A_m x_{j_1}^m x_{j_2}^{m-1} \cdots x_{j_m}^1$$

where $s_{j_1}, s_{j_2}, \ldots s_{j_m}$ are the simple transpositions in $\tau(T)$ ordered so that $j_1 < j_2 < \ldots < j_m$.

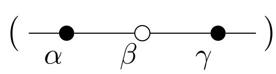
Proof. The first two parts of the theorem are simply restatements of earlier lemmas. It remains only to prove that the coefficient of the term in part 3 is A_m. For this, we apply induction on $<$.

Part 3 is obviously true if T is the column superstandard tableau. Otherwise, there exists some $T' < T$ and a simple transposition β so that

$$p_T = \beta \cdot p_{T'} - p_{T'} - p_{T'_L}$$

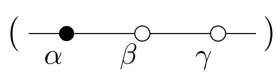
where $p_{T'_L}$ may be 0. (See equation (9.3); p_T equals $p_{T'_R}$ in the context of the simple transposition β.)

We now work in two cases: near β, $\tau(T')$ is either



(Case I)

or



(Case II).

We can understand the proof by working out each case in the $n = 4$ setting.

- Case I:

  In the equation

$$p_T = \beta \cdot p_{T'} - p_{T'} - p_{T'_L}, \tag{9.5}$$

$$p_{T'} = A_m(x_1^2 x_3 - x_1 x_3^2 + x_1 x_4^2 - x_1^2 x_4 + x_2^2 x_4 - x_2 x_4^2 + x_2 x_3^2 - x_2^2 x_3),$$

$$\beta = (23),$$

$$\beta \cdot p_{T'} = A_m(x_1^2 x_2 - x_1 x_2^2 + x_1 x_4^2 - x_1^2 x_4 + x_3^2 x_4 - x_3 x_4^2 + x_3 x_2^2 - x_3^2 x_2),$$

and

$$p_{T'_L} = A_m(x_1^2x_2 - x_1x_2^2 + x_1x_3^2 - x_1^2x_3 + x_2^2x_3 - x_2x_3^2).$$

Adding up all the terms of form $x_2^2x_3$ on the right hand side of equation (9.5) tells us that $A_mx_2^2x_3$ is a term of p_T.

- Case II:

  In this case, equation (9.5) reduces to

$$p_T = \beta \cdot p_{T'} - p_{T'}, \tag{9.6}$$

  and the polynomials on the right-hand side becomes

$$p_{T'} = A_m(x_1 - x_2)$$

  and

$$\beta \cdot p_{T'} = A_m(x_1 - x_3)$$

  so that A_mx_2 is a term of p_T.

■

The next theorem is a constructive version of Theorem 64:

Theorem 65 *Given a tableau T of some hook shape H, the fiber polynomial for T is given by*

$$p_T = \frac{A_m}{B}\left(\sum_{\sigma\in S_m\times S_{\tau(T)}} \operatorname{sgn}(\sigma)\sigma\cdot(x_{j_1}^m x_{j_2}^{m-1}\cdots x_{j_m}^1)\right) \tag{9.7}$$

where $m+1$ is the number of rows in H, $s_{j_1}, s_{j_2}, \ldots s_{j_m}$ are the simple transpositions in $\tau(T)$ ordered so that $j_1 < j_2 < \ldots < j_m$, $A_m = \frac{1}{m!(m-1)!\cdots 1}$ and B is the number of elements in the stabilizer of $x_{j_1}^m x_{j_2}^{m-1}\cdots x_{j_m}^1$ in $S_m \times S_\tau$. Equivalently, B is a positive coefficient chosen so that each monomial term has coefficient plus or minus A_m.

This of course also gives us the associated cycle polynomial for an irreducible module X with support $Q \in K\backslash\mathfrak{B}$ and $T(Q) = H$.

9.3 Comparison with Methods of Barchini and Zierau

In this section, we will reproduce a calculation from [3] to compare our methods with those of Barchini and Zierau. Whereas we have obtained closed forms for hook shapes and will discuss two row shapes in Chapter 10, their techniques compute associated cycles for any Harish-Chandra module X that is constructed from a discrete series representation.

Here, we will reproduce the calculation in Example 6.10 of [3] and carry out the same calculation using the formulas developed in this chapter. Begin with the real form $SU(3,2)$. Then, $\mathfrak{k} = \mathfrak{gl}(3,\mathbb{C}) \times \mathfrak{gl}(2,\mathbb{C})$. Our Cartan subalgebra $\mathfrak{h}$ consists of the diagonal matrices in $\mathfrak{gl}(5,\mathbb{C})$. Our chosen simple system is given by $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5\}$, where e_k is the functional that returns the $E_{k,k}$ entry of a matrix in $\mathfrak{h}$. There is a corresponding simple system $\{e_1 - e_2, e_2 - e_3, e_4 - e_5\}$ for $\mathfrak{k}$. The closed K-orbits on the flag variety stand in correspondence with choices of positive systems in $\Delta(\mathfrak{g},\mathfrak{h})$ that contain the standard positive system in $\Delta(\mathfrak{k},\mathfrak{h})$. Such a system is determined by an ordering of the integers 1 through 5 such that 1 occurs before 2 which is chosen before 3; and 4 is chosen before 5. In this case, we choose the sequence $\{4,1,2,3,5\}$. This corresponds to the simple system $\{e_4 - e_1, e_1 - e_2, e_2 - e_3, e_3 - e_5\}$. Taking the corresponding positive system yields a nilpotent subalgebra $\mathfrak{n}$. Define a Borel subalgebra $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. The corresponding closed K-orbit is given by $Z = K \cdot \mathfrak{b}$. This orbit determines a discrete series representation of $SU(3,2)$.

Following a paper of Yamamoto [43], we can obtain the clan parameterization of Z by assigning a positive sign to $\{1,2,3\}$ and a minus sign to $\{4,5\}$. Then, the sequence $\{4,1,2,3,5\}$ yields the clan $\{-,+,+,+,-\}$. The moment map image of $K \cdot \mathfrak{b}$ contains a dense orbit $\mu_{\mathrm{orb}}(Z)$ obtained by reading the signs in the clan from right to left and adding them to a signed tableau. We start with $-$:

$$\begin{array}{|c|}\hline - \\ \hline \end{array}.$$

The next sign is added to the end of this row:

$$\begin{array}{|c|c|}\hline - & + \\ \hline \end{array}.$$

In a signed Young diagram, two plus signs cannot appear next to each other in a row, so $+$ goes to a new row:

$$\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} \end{array}.$$

This happens again with the next $+$:

$$\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1} \end{array}.$$

The next $-$ goes next to $+$ on the top row:

$$\begin{array}{|c|c|c|}\hline - & + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1}\end{array}\ .$$

This yields one of the tableaux in the same shape pair parameterization of Z. Following work of Garfinkle in [15], we obtain the standard tableau for the Springer fiber component parameterized by $T(Z)$ by numbering boxes in the order in which they appear in the construction of the signed Young diagram for $\mu_{\text{orb}}(Z)$. This gives us the following sequence of same shape pairs:

$$\begin{array}{|c|}\hline - \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline - & + \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|}\hline - & + \\ \hline + \\ \cline{1-1} + \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline - & + & - \\ \hline + \\ \cline{1-1} + \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ .$$

We now use Theorem 64 to compute the associated cycle polynomial. The τ-invariant for $T(Z)$ is $\{s_2, s_3\}$ and the coefficient $A_2 = \frac{1}{2}$. This yields the polynomial

$$p = \frac{1}{2}(x_2^2 x_3 - x_2 x_3^2 + x_2 x_4^2 - x_2^2 x_4 + x_3^2 x_4 - x_3 x_4^2). \tag{9.8}$$

Note that this computation is implicitly carried out with respect to the standard positive system in $\Delta(\mathfrak{g}, \mathfrak{h})$ given by the simple roots $\{e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 - e_5\}$; while we choose a nonstandard positive system to determine the K-orbit Z, p gives the associated cycle multiplicities across a coherent family constructed by taking the standard dominant chamber in $\mathfrak{h}^*$.

Barchini and Zierau in their calculations construct a certain reductive subgroup $L = GL(3) \times GL(1) \times GL(1)$ of $GL(5, \mathbb{C})$. Note however that L is built using the simple roots $\{e_4 - e_1, e_1 - e_2, e_2 - e_3, e_3 - e_5\}$ with the diagonal Cartan subalgebra. Define

$$\lambda = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5.$$

Then, the multiplicity polynomial is computed on the (nonstandard) dominant chamber by

$$p' = \text{gr dim}(L \cdot w_\lambda)$$

where this again is the highest degree part of the polynomial obtained by computing the dimension and w_λ is a highest weight vector in an irreducible representation with highest weight λ. (In the original paper, λ is shifted; we can ignore the shift since we are taking the highest degree part of the dimension polynomial.)

By the Weyl dimension formula, we get

$$\begin{aligned} p'(\lambda) &= \frac{1}{2}(\lambda, e_4 - e_1)(\lambda, e_4 - e_2)(\lambda, e_2 - e_1) \\ &= \frac{1}{2}(x_4 - x_1)(x_4 - x_2)(x_1 - x_2). \end{aligned}$$

This expression does not agree with our earlier calculation. However, this is due to the nonstandard choice of positive system used in the computation of p'. The Weyl group element

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$$

takes the simple system $\{e_4 - e_1, e_1 - e_2, e_2 - e_3, e_3 - e_5\}$ to the standard simple system. Then, we have

$$p'(w^{-1} \cdot \lambda) = \frac{1}{2}(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = p(\lambda).$$

CHAPTER 10

THE TWO ROW CASE

In this chapter, we will use the exchange rule to carry out fiber polynomial calculations for all two row tableaux.

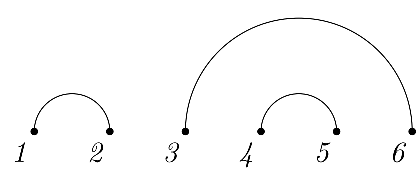
10.1 Two Row Standard Tableaux and Crossingless Matchings

Once again, take $\mathfrak{h}$ to be the diagonal Cartan subalgebra and choose the positive system of roots that gives the upper triangular Borel subalgebra. The variables x_i are functionals on $\mathfrak{h}^*$ specified by letting $x_i = E_{i,i}$ in $\mathfrak{h}$.

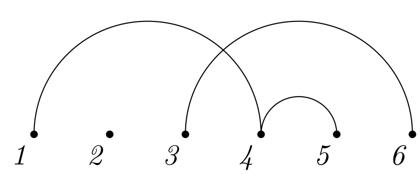
In [14], Fung establishes a bijective correspondence between $[n-p, p]$ standard tableaux and cup diagrams with n nodes and p cups. In current parlance, these diagrams are called crossingless matchings. (See for example [24].)

Definition 66 *An $[n-p, p]$ crossingless matching is a diagram with n nodes arranged along a horizontal line and p arcs above the horizontal line whose ends connect to nodes in such a way that no two arcs connect to the same node, no unconnected node is under an arc and arcs do not cross.*

Example 67 *The diagram*



is a $[3, 3]$ crossingless matching, while



is not a crossingless matching.

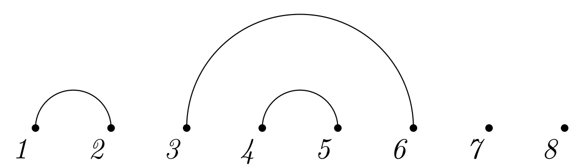
Definition 68 *Given an $[n-p,p]$ standard tableau T, construct an $[n-p,p]$ crossingless matching by starting with n nodes labeled left to right 1 to n. Reading the labels in the lower row of T from left to right, draw an arc from the node specified by the label to the nearest open node to the left.*

Observe that the possible bottom rows of $[n-p,p]$ standard tableaux are exactly the strictly increasing sequences of p integers such that the kth integer is greater than or equal to $2k$ and less than or equal to n. These sequences completely characterize the tableaux. The procedure clearly gives an injective map; every $[n-p,p]$ crossingless matching can be constructed in this way, so the map is bijective.

Example 69 *The tableau*

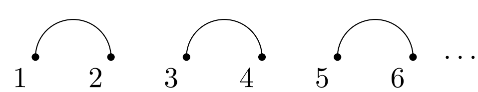
1	3	4	7	8
2	5	6		

corresponds to a $[5,3]$ *crossingless matching*



10.2 Fiber Polynomials

The $[n-p,p]$ column superstandard tableau has corresponding crossingless matching



with p arcs in total.

By Proposition 29, the $[n-p,p]$ column superstandard tableau has fiber polynomial

$$(x_1 - x_2)(x_3 - x_4)(x_5 - x_6)\cdots$$

where the polynomial has p factors. In other words, there is a bijective correspondence between arcs in the crossingless matching and factors in the fiber polynomial.

Given a crossingless matching C, denote by $\mathrm{Arc}(C)$ the set of pairs (i,j) with $i<j$ such that there is an arc in C connecting the ith node to the jth node. The following Theorem gives a complete description of fiber polynomials for two row tableaux.

Theorem 70 *Let T be an $[n-p,p]$ Young tableau and C_T its corresponding crossingless matching. Then, the fiber polynomial p_T is given by*

$$p_T = \prod_{(i,j)\in \mathrm{Arc}(C_T)} (x_i - x_j).$$

This theorem also gives us the associated cycle polynomial of a module X with support $Q \in K\backslash\mathfrak{B}$ and $T(Q) = T$ a two row tableau.

Proof. Define a partial order on $[n-p,p]$ tableaux by specifying that $T \leq T'$ if each label in the bottom row of T is less than or equal to the corresponding label in the bottom row of T'; $T < T'$ if $T \neq T'$. The column superstandard tableau is the unique minimum under this order. We'll proceed by induction on $<$.

Given a nonminimal tableau T, let i be the first label in the bottom row read from the left that differs from the labels in the minimum tableau, i.e., such that i is the kth label and $i > 2k$; both $i-1$ and $i-2$ will be on the top row, so we have something like

$$T = \left(\cdots \begin{array}{|c|c|} \hline i-2 & i-1 \\ \hline i & \\ \hline \end{array} \cdots \right).$$

(Note that $\tau(T)$ contains s_{i-1} but not s_{i-2}.) Define a new tableau

$$T' = \left(\cdots \begin{array}{|c|c|} \hline i-2 & i \\ \hline i-1 & \\ \hline \end{array} \cdots \right)$$

by exchanging i and $i-1$ in T and note that $T' < T$. Observe that $\tau(T')$ contains s_{i-2} but not s_{i-1}. Then,

$$s_{i-1} \cdot T' = s_{i-1} \cdot \left(\cdots \begin{array}{|c|c|} \hline i-2 & i \\ \hline i-1 & \\ \hline \end{array} \cdots \right) =$$

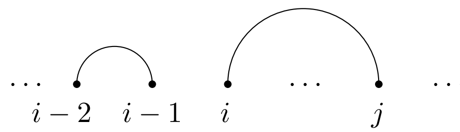
$$\left(\cdots \begin{array}{|c|c|} \hline i-2 & i \\ \hline i-1 & \\ \hline \end{array} \cdots \right) + \left(\cdots \begin{array}{|c|c|} \hline i-2 & i-1 \\ \hline i & \\ \hline \end{array} \cdots \right) = T' + T. \tag{10.1}$$

No nonexchange tableaux occur on the right of equation (10.1) because the τ-invariant for a two row tableau never contains two adjacent simple transpositions. If $s_i \in \tau(T')$, then the exchange tableau for the pair $\{s_{i-2}, s_{i-1}\}$ is also the exchange tableau for $\{s_{i-1}, s_i\}$.

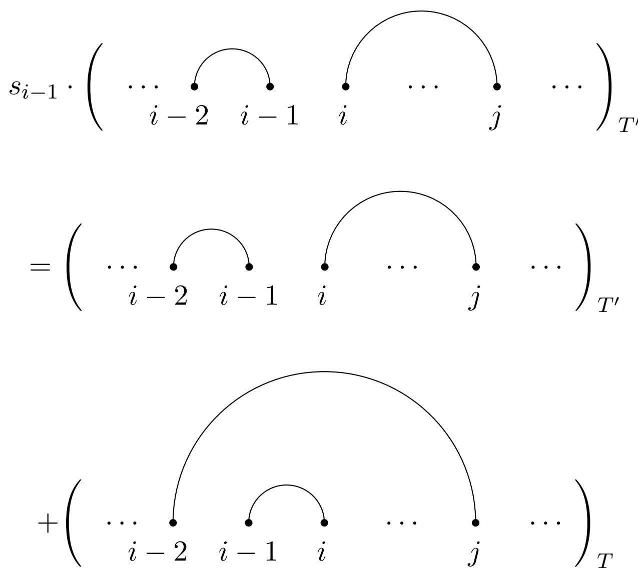
The next step in the proof is to translate equation (10.1) into the language of crossingless matchings and then apply our induction assumption on polynomials. Two cases arise.

- Case I:

In the matching of T', the ith vertex is connected to an arc, i.e., we have



Tracing through the description of the map from tableaux to matchings in Definition 66, we get



where the arcs not shown are the same for both tableaux.

By the induction assumption, the polynomial for T' is given by

$$p_{T'} = q_{T'}(x_{i-2} - x_{i-1})(x_i - x_j)$$

where $q_{T'}$ does not involve the variables x_{i-2}, x_{i-1}, x_i or x_j. Computing the action of s_{i-1}, we get

$$s_{i-1} \cdot p_{T'} = q_{T'}(x_{i-2} - x_i)(x_{i-1} - x_j)$$

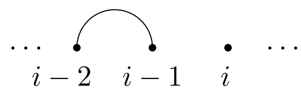
$$= q_{T'}(x_{i-2} - x_{i-1})(x_i - x_j) + q_{T'}(x_{i-2} - x_j)(x_{i-1} - x_i)$$

$$= p_{T'} + p_T$$

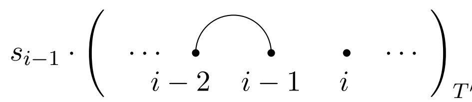
where p_T is the polynomial for T.

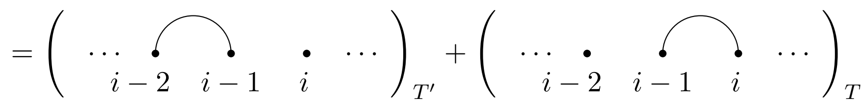
- Case II:

  In the matching corresponding to T', the ith vertex is not connected to an arc:



Using equation (10.1) and the mapping from tableaux to matchings, we get





The polynomial for T' is

$$p_{T'} = q_{T'}(x_{i-2} - x_{i-1})$$

where $q_{T'}$ does not involve the variables x_{i-2}, x_{i-1} or x_i The action of s_{i-1} gives us

$$s_{i-1} \cdot p_{T'} = s_{i-1} \cdot q_{T'}(x_{i-2} - x_{i-1}) = q_{T'}(x_{i-2} - x_i)$$

$$= q_{T'}(x_{i-2} - x_{i-1}) + q_{T'}(x_{i-1} - x_i) = p_{T'} + p_T$$

where p_T is the polynomial corresponding to the matching for T.

■

CHAPTER 11

A SKEIN THEORETIC CONSTRUCTION

The combinatorics of Kazhdan–Lusztig graphs for two row Young tableaux first appear in a paper by Lascoux and Schützenberger [28]. We will now discuss a skein theoretic picture developed by Russell and Tymoczko in a different context [32, 31].

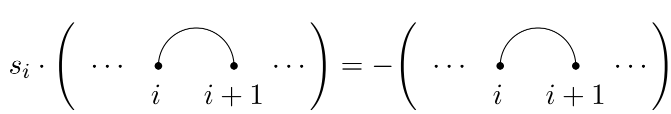
Russell and Tymoczko's work is in the context of Springer representations for Springer fibers over two row nilpotent elements. Kazhdan and Lusztig's original paper defining Kazhdan–Lusztig representations [21] was motivated by earlier work on Springer representations in [22] and it was believed that the Kazhdan–Lusztig basis coincided with the basis of fundamental classes of Springer fiber components in top degree cohomology for the Springer representation in type A. (See for instance [23].) This was later established as a consequence of a result of Melnikov [30].

Independent of this relationship, [32] uses combinatorial means to establish that the crossingless matching basis is equivalent to the Kazhdan–Lusztig basis for all two equal row Young diagrams. We will give a combinatorial proof based on polynomial calculations for the general two row case.

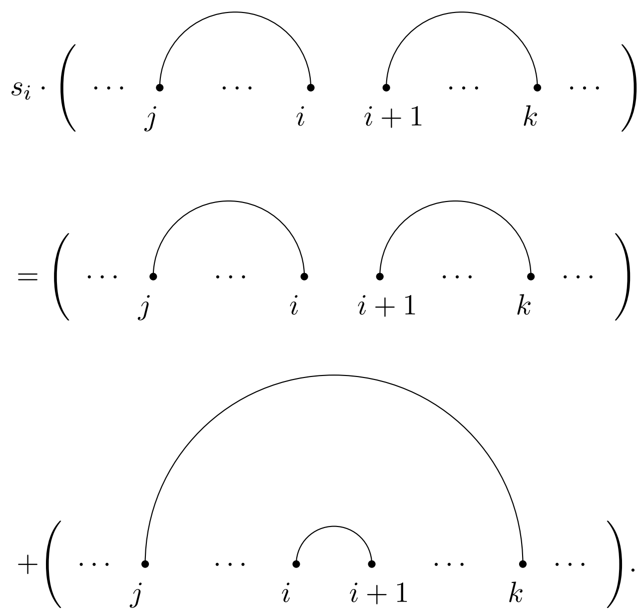
This skein theoretic construction of Kazhdan–Lusztig representations is remarkable because previous approaches to these representations framed computations in terms of the actions of simple transpositions. The skein theoretic picture offers a direct way to compute the image of a Kazhdan–Lusztig basis element under the action of any element of the symmetric group.

To build the full skein theoretic structure, we first use the polynomial structure from the last chapter to make general statements about the actions of simple transpositions on polynomials, crossingless matchings and, equivalently, Kazhdan–Lusztig basis elements. We will state these rules in terms of matchings.

Rule 1 *If nodes i and $i+1$ are ends of the same arc, then*

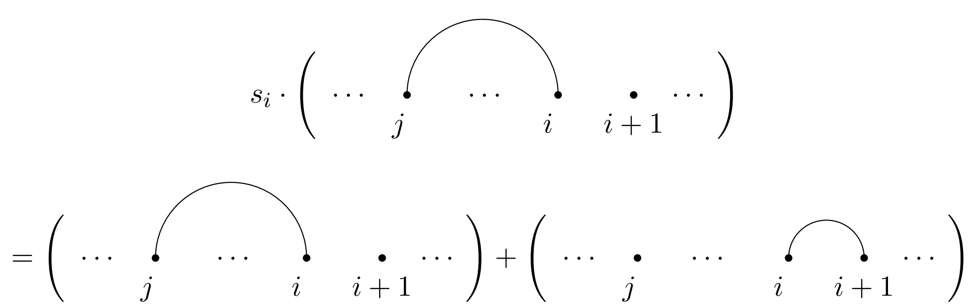


Rule 2 *If the i and $i+1$ nodes are the ends of separate arcs, then*

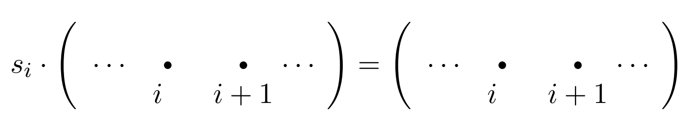


Note that this rule also works when one arc lies over the other; the second matching is obtained by joining the pairs $i, i+1$ and j, k with arcs.

Rule 3 *If i and $i+1$ label the end of an arc and an unconnected vertex, then*

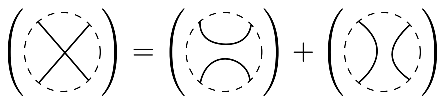


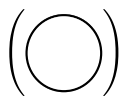
Rule 4 *If i and $i+1$ both label unconnected vertices, then*

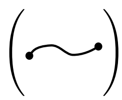


The above rules reduce to the following skein theoretic formulation of the Kazhdan–Lusztig representation in terms of the basis of crossingless matchings:

Theorem 71 *Given a crossingless matching M on n nodes and an element $\sigma \in S_n$, $\sigma \cdot M$ is obtained by gluing a loopless graph corresponding to σ to the bottom of M and reducing to a sum of crossingless matchings using the following relations:*

1. 

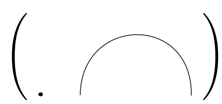
2. $X \sqcup$  $= -2X$

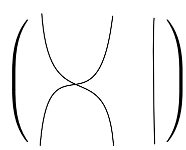
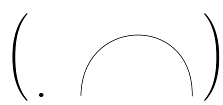
3. $X \sqcup$  $= 0$

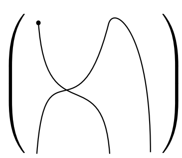
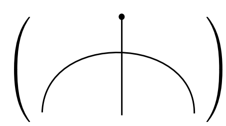
The segment in (3) connects interior vertices*.*

We have not yet defined *interior vertices*. These are most easily understood by looking at an example:

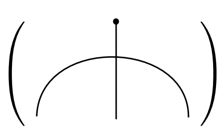
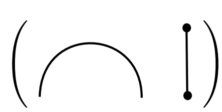
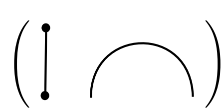
Example 72

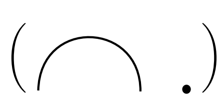
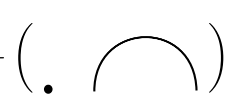
$$\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\cdot$$ 

$=$  $\cdot$ 

$=$  $=$  .

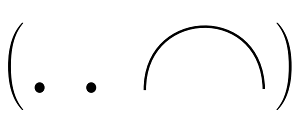
The points to which lines connect on the bottom of the last picture are referred to as base vertices*. Other vertices, such as the lone vertex at the top of the picture, are referred to as* interior vertices*. Of course, the diagram is not yet decomposed into crossingless matchings:*

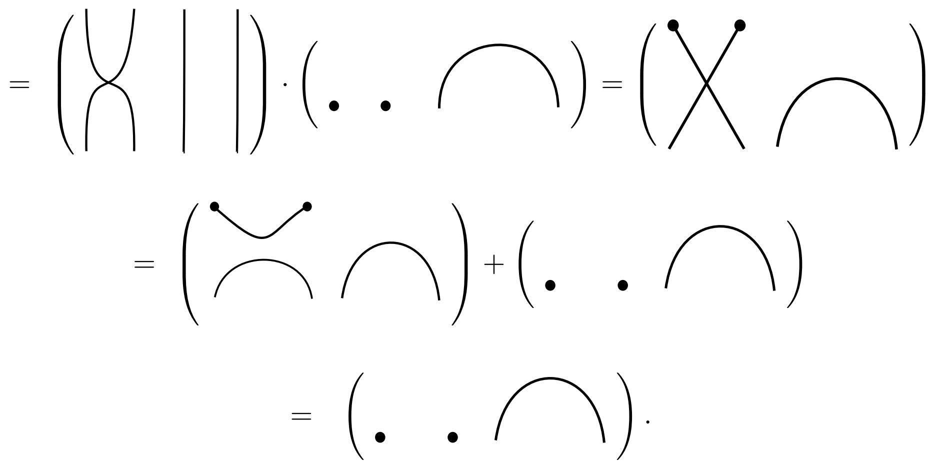
 $=$  $+$ 

$=$  $+$  .

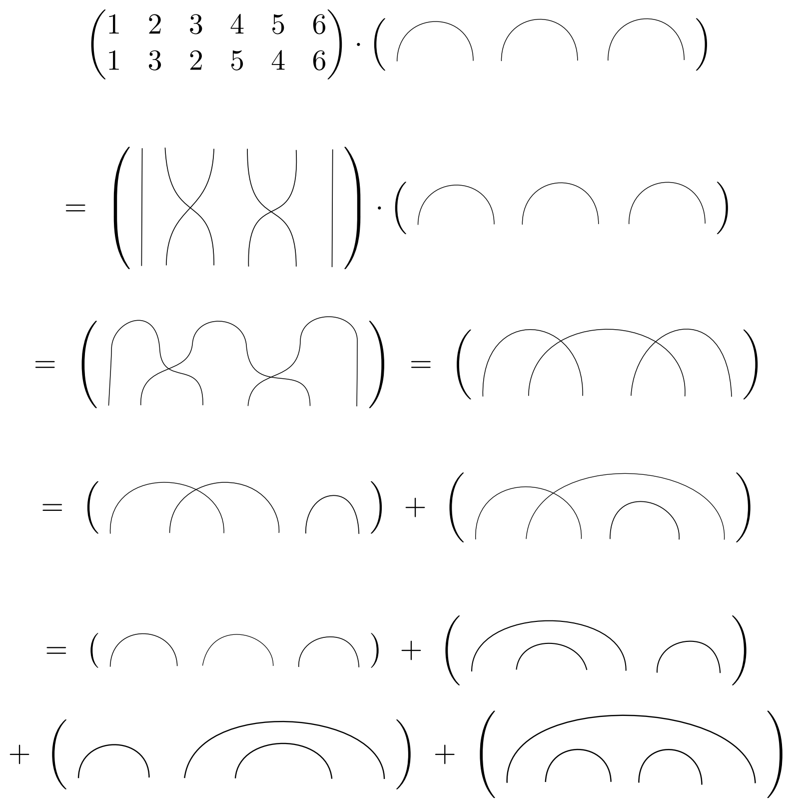
As seen in the final expression, an interior vertex connected to a base vertex by a line that does not cross other lines collapses down to a base vertex.

Example 73

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}\cdot$$ 

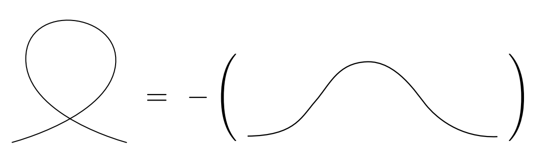


Example 74

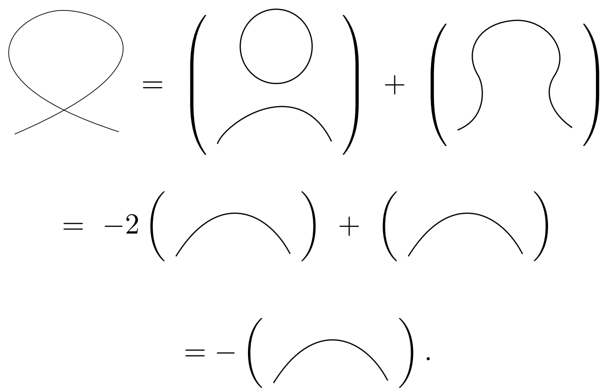


Note that in this context, we do not differentiate between over and under crossings. The next three lemmas will address the applicability of Reidemeister moves in this setting.

Lemma 75 *(The first Reidemeister move.)*



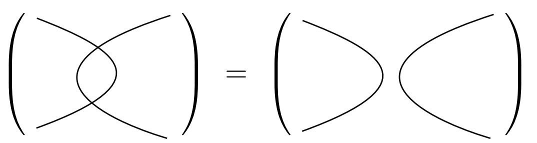
Proof.



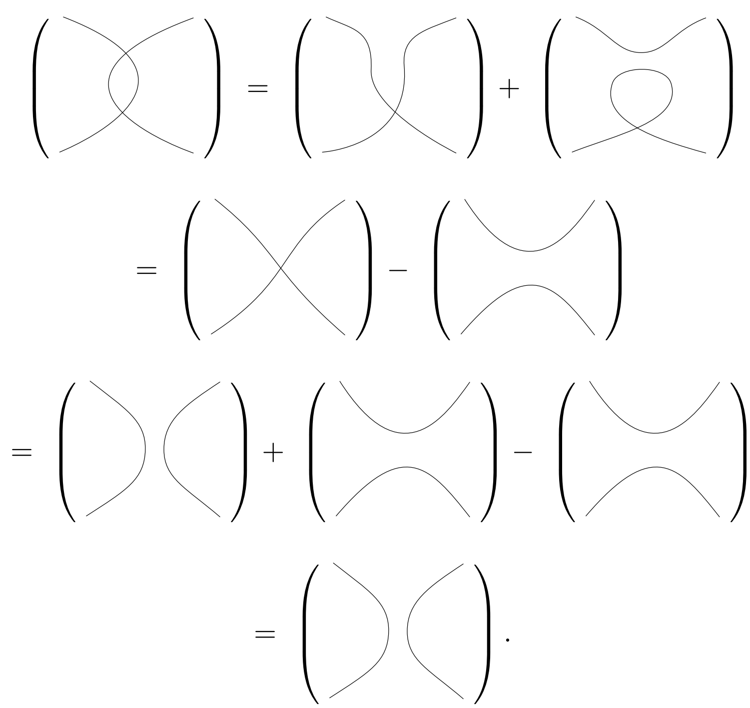
■

The statement of Theorem 71 requires that the graph of a symmetric group element not have loops because, as we have just seen, any loop is equivalent to a -1 in the final calculation.

Lemma 76 *(The second Reidemeister move.)*

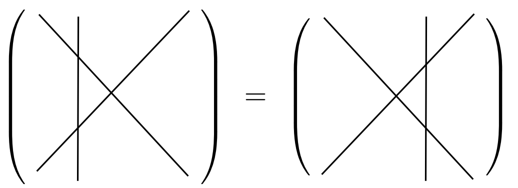


Proof. In this proof, we will make use of the first Reidemeister move as established in Lemma 75



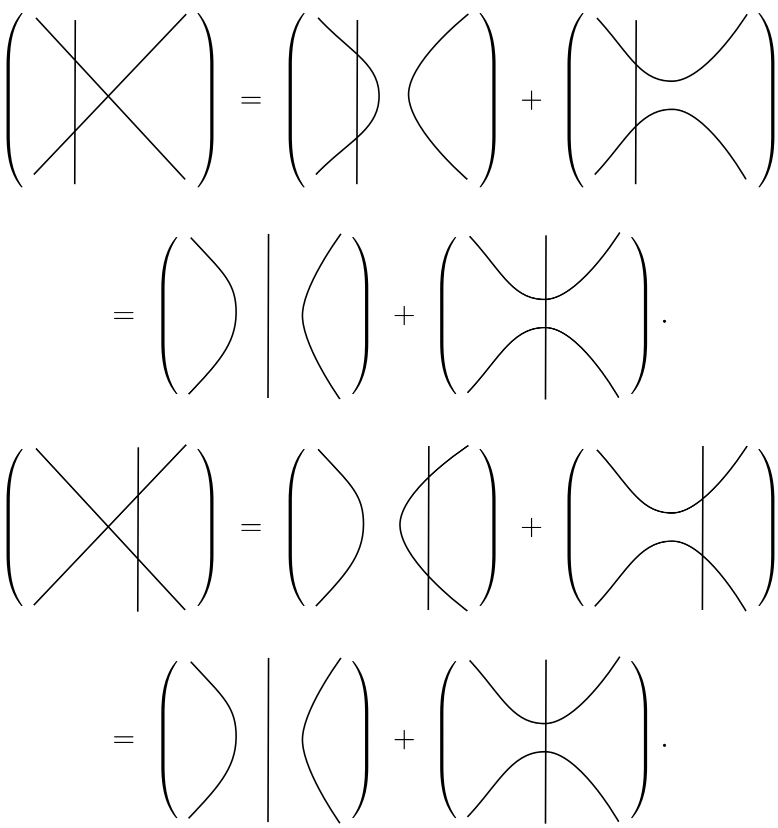
■

Lemma 77 *(The third Reidemeister move.)*



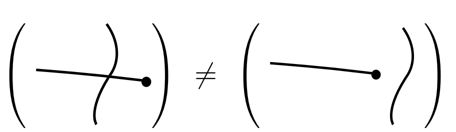
(11.1)

Proof. In this proof, we will show that the left and right hand sides of equation (11.1) can be decomposed to the same expressions. We will make use of the second Reidemeister move.

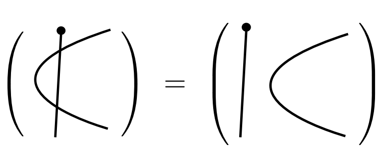


■

We must also consider lines that terminate on interior vertices. In general,

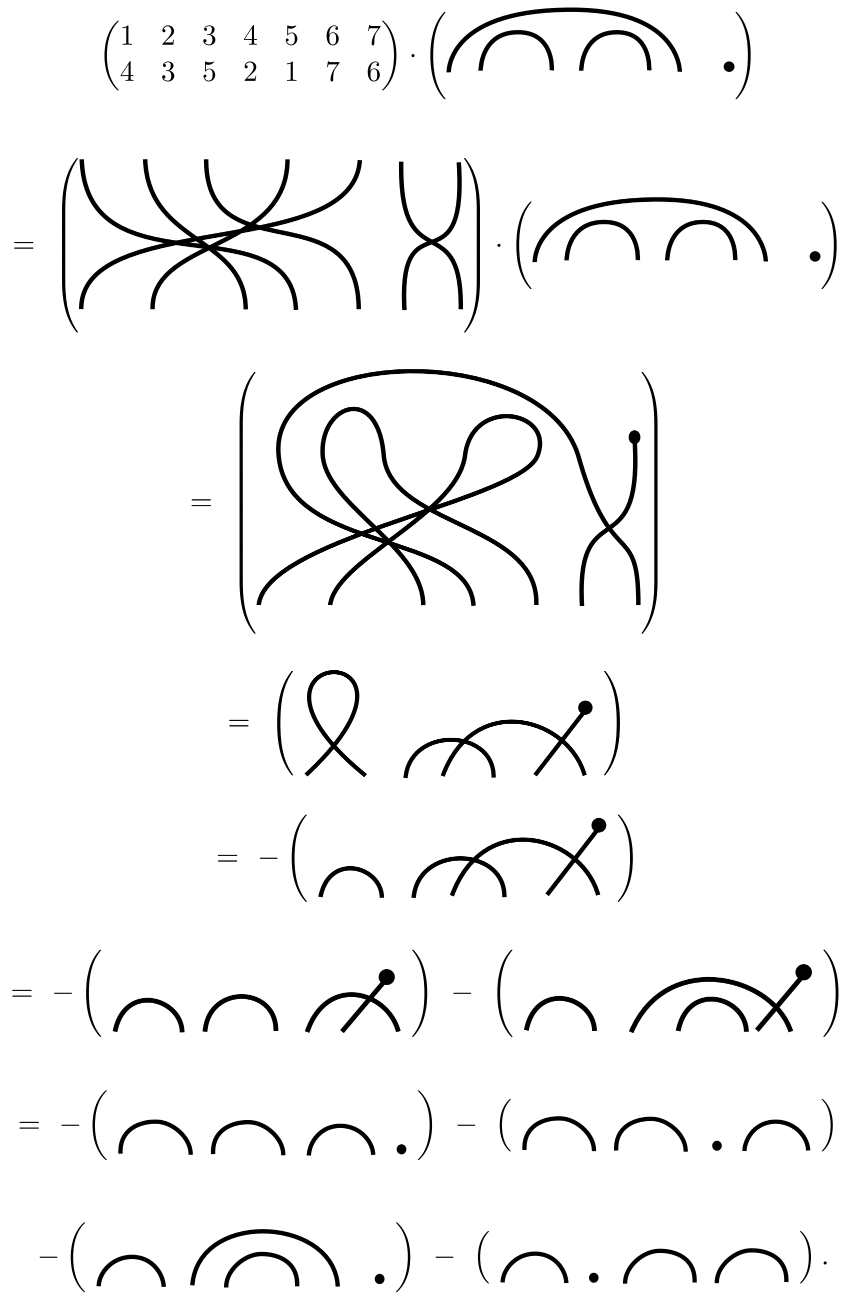


We can however use allowed Reidemeister moves if we are careful. For example,



Example 78 *In this example, we will make extensive use of the Reidemeister moves.*

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 4 & 3 & 5 & 2 & 1 & 7 & 6\end{pmatrix} \cdot$$



One easily checks that

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7\\ 4 & 3 & 5 & 2 & 1 & 7 & 6\end{pmatrix} \cdot \big((x_1 - x_6)(x_2 - x_3)(x_4 - x_5)\big)$$

$$= -(x_1 - x_2)(x_3 - x_4)(x_5 - x_6) - (x_1 - x_2)(x_3 - x_4)(x_6 - x_7)$$

$$-(x_1 - x_2)(x_3 - x_6)(x_4 - x_5) - (x_1 - x_2)(x_4 - x_5)(x_6 - x_7).$$

Proof of theorem 71: First of all, we must prove that $\sigma \cdot C$ is well defined for $\sigma \in S_n$ and a crossingless matching C. We must account for the fact that σ can be represented by infinitely many loopless crossing configurations. However, all the reductions for $\sigma \cdot C$ must be equivalent by application of the second and third Reidemeister moves. Given two elements $\theta, \sigma \in S_n$, applying σ, reducing, applying θ and reducing is equivalent to gluing θ to the bottom of σ, applying the result to the bottom of the matching and reducing. Thus, $(\theta\sigma) \cdot C = \theta \cdot (\sigma \cdot C)$. This tells us that the construction is a group homomorphism from S_n to the general linear group of the complex span of $[n-p, p]$ crossingless matchings. One easily checks that the construction gives the right answer for all simple transpositions. It follows that the construction yields the Kazhdan–Lusztig representation. $\blacksquare$

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