VARIETIES FIBERED BY GOOD MINIMAL MODELS AND BOUNDING VOLUMES OF SINGULAR FANO THREEFOLDS

by

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ABSTRACT

We study the geometry of higher dimensional algebraic varieties according to the dichotomy of Kodaira dimensions, negative or nonnegative, and the corresponding pictures in the Minimal Model Conjecture.

On the one hand, according to the Minimal Model Conjecture, a variety with nonnegative Kodaira dimension is birational to a minimal model, which has semiample canonical class. This has been done if dimension is less than or equal to three and for varieties of general type in any dimension. In general, the Minimal Model Conjecture is still open. As the first result, we show that the Minimal Model Conjecture for varieties with nonnegative Kodaira dimensions follows from the Minimal Model Conjecture for varieties with Kodaira dimension zero. In particular, the Minimal Model Conjecture is reduced to the Minimal Model Conjecture for varieties of Kodaira dimension zero and the Nonvanishing Conjecture.

On the other hand, according to the Minimal Model Conjecture, Fano varieties of Picard number one are the building blocks for varieties with negative Kodaira dimension. The set of mildly singular Fano varieties of given dimension is expected to be bounded. As a second result, we exhibit an effective upper bound of the anticanonical volume for the set of ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefolds with Picard number one. This result is related to a conjecture open in dimension three and higher, the Borisov-Alexeev-Borisov Conjecture, which asserts boundedness of the set of ϵ-klt log $\mathbb{Q}$-Fano varieties.

In the end of this dissertation, we include some partial results of the Nonvanishing Conjecture in the minimal model program. The minimal model program is developed to attack the Minimal Model Conjecture. The Nonvanishing Conjecture is one of the most important missing ingredient for completing the minimal model program.

To my parents.

CONTENTS

CHAPTER 1

INTRODUCTION

For the purpose of studying the classification problem of complex projective varieties, the minimal model program aims to construct a good representative in the birational class of a given variety X. The Minimal Model Conjecture predicts that this construction works in all cases and is also used to describe the geometry of these good representatives. According to the Minimal Model Conjecture, a good minimal model for a variety X with $\kappa(X) \geq 0$ has a semiample canonical class while a variety X with $\kappa(X) < 0$ is birational to a variety Y with a Mori fiber space structure, i.e., there is a morphism $Y \to B$ with a general fiber Y_b a Fano variety of Picard number one. In this dissertation, we have two main results related to two different aspects of the Minimal Model Conjecture.

The minimal model program is trivial for curves and has been done for surfaces by the Italian school. In dimension three, it is established by S. Mori et al. In higher dimensions, C. Birkar, P. Cascini, C. Hacon, and J. M^{c}Kernan have established the existence of good minimal models for varieties of general type. For a variety X of intermediate Kodaira dimension, i.e., $0 < \kappa(X) < \dim X$, one uses the pluricanonical system to construct the Iitaka fibration of X whose general fibers are varieties with Kodaira dimension zero. By utilizing the structure of Iitaka fibrations and techniques developed in the minimal model program, in this thesis we show that the existence of good minimal models for varieties with Kodaira dimension zero implies the existence of good minimal models for varieties with intermediate Kodaira dimension. In particular, the Minimal Model Conjecture is reduced to the Nonvanishing Conjecture and the Minimal Model Conjecture for varieties with Kodaira dimension zero. This answers a question of Mori concerning Iitaka fibrations with fibers possessing good minimal models. As a corollary, we include an application to the Iitaka's Conjecture C on the subadditivity of Kodaira dimensions for algebraic fiber spaces.

According to the Minimal Model Conjecture, Fano varieties are the building blocks for varieties with negative Kodaira dimension. It is expected that the set of mildly singular

Fano varieties satisfies certain boundedness properties. The precise statement is known as the Borisov-Alexeev-Borisov Conjecture, which asserts boundedness of the set of ϵ-klt log $\mathbb{Q}$-Fano varieties of a given dimension. The B-A-B Conjecture relates to the conjectural termination of flips of the minimal model program. The B-A-B Conjecture is established in dimension two by Alexeev and for toric varieties by A. Borisov and L. Borisov. However, the B-A-B Conjecture is still open in dimension three and higher. As a partial result of the B-A-B Conjecture, we show that there is an effective upper bound of the anticanonical volumes for the set of ϵ-klt log $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefolds of Picard number one, which depends only on ϵ. The existence of an upper bound of anticanonical volumes is a necessary condition for the B-A-B Conjecture to hold.

A big question in the minimal model program is the Nonvanishing Conjecture, which asserts that K_X being pseudo-effective, a numerical condition, would imply that $\kappa(X) \geq 0$. The Nonvanishing Conjecture is known in dimensions less than or equal to three but still open in higher dimensions. In the last part of this dissertation, we include two results related to the Nonvanishing Conjecture. The first one attempts to find a conceptual proof of the two-dimensional Nonvanishing Conjecture. Note that the Nonvanishing Conjecture is established in dimension two by classification and a conceptual proof is demanded to provide new insight for higher dimensional geometry. The second result is a Nonvanishing theorem for irregular varieties.

This dissertation is organized as follow. In Chapter 2, we describe the minimal model program and the Minimal Model Conjecture. In Chapter 3, we include the first main result, a reduction theorem of the Minimal Model Conjecture. In Chapter 4, we establish an effective upper bound for the anticanonical volumes for ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefolds of Picard number one. In Chapter 5, we present some partial results related to the Nonvanishing Conjecture.

CHAPTER 2

MINIMAL MODEL PROGRAM

We would like to study the geometry of a complex projective variety X. By a theorem of Nagata, we can always compactify X to be a complete variety. By Chow's Lemma, a complete variety is birational to a projective variety. Since Hironaka's theorem on resolution of singularities applies over a field of complex numbers, we can assume that X is a smooth projective variety. Hence we focus on smooth projective varieties.

We start from reviewing the classical theory of curves and surfaces where we describe the complete minimal model program and the established Minimal Model Conjecture. The generalized minimal model program and the Minimal Model Conjecture for higher dimensional varieties are described in the sequel. In the second part of this chapter, we review the results on the minimal model program from [8] and [30], which contain most of the techniques we will use in the later chapters.

We follow the notations in [8] and [18].

2.1 Curves and surfaces

Let X be a smooth projective variety. There is a canonically associated line bundle $\omega_X = \wedge^{\dim X}\Omega^1_X$ on X where $\Omega^1_X = T_X^\vee$ is the holomorphic cotangent bundle. In particular, there is a canonical divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$. For a given line bundle L on X, we can study the map $X \dashrightarrow \mathbb{P}(|L|)$ defined by sections of L. The first question we ask is whether there is a section or not for a given line bundle.

Definition 1 *Let X be a smooth projective variety. The Kodaira dimension $\kappa(X)$ is defined to be -1 if $H^0(X, mK_X) = 0$ for all $m \geq 1$. Otherwise, there is an integer $m > 0$ such that $H^0(X, mK_X) \neq 0$ and we say that $\kappa(X) \geq 0$.*

In case $\kappa(X) \geq 0$, we define the canonical ring $R(X)$ of X to be the graded $\mathbb{C}$-algebra

$$R(X) = \bigoplus_{m \geq 0} H^0(X, mK_X).$$

We will see that the geometry of projective varieties is classified according to the sign of Kodaira dimension. We start with a review of the geometry of curves and surfaces.

Let $X = C$ to be a smooth projective curves, i.e., a Riemann surface. From the Riemann-Roch formula for curves, we have $\deg(K_C) = 2g - 2$ where $g = g(C)$ is the topological genus of a compact oriented real two-dimensional manifold C.

- If $\kappa(X) = -1$, then $\deg(K_C)$ is negative. Hence $g = 0$ and $C \cong \mathbb{P}^1$;
- If $\kappa(X) \geq 0$, then $\deg(K_C)$ is non-negative and $g \geq 1$. In this case, $|mK_C|$ is base point free for some $m > 0$ and there is a canonical morphism $\Phi : C \to \mathbf{Proj}(R(C))$. In fact, if $g = 1$, then C is elliptic, $\mathcal{O}_C(K_C) \cong \mathcal{O}_C$, and Φ is a constant map. If $g \geq 2$, then $3K_C$ is very ample and Φ is an isomorphism.

Assume that $\dim X = 2$, i.e., $X = S$ is a smooth projective surface. The geometry of surface is more interesting due to the existence of blow-ups.

(i) Blowing up at a smooth point of S gives a morphism of smooth projective surfaces $\mu : \mathrm{Bl}_p S \to S$ which is birational but not isomorphic. The exceptional set $\mathrm{Exc}(\mu)$ is a rational curve of first kind, i.e., a curve $C \cong \mathbb{P}^1$ with $K_X.C < 0$ and $C^2 < 0$. The numerical condition is equivalent to $K_S.C = C^2 = -1$ and we call such a curve a (-1)-curve. We take S as a good substitution for studying the geometry of S' since blow-ups are well understood for surfaces. A natural question to ask is under which conditions we can "simplify" a surface by blow-downs.

(ii) *Castelnuovo's Contraction Theorem* asserts that if there exists a (-1)-curve $C \subseteq S'$, then there exists a morphism $\pi : S' \to S$ to a smooth projective surface S such that $\pi(C) = p$ is a smooth point of S and $S' \cong \mathrm{Bl}_p S$ with $C = \pi^{-1}(p)$ the exceptional curve. This gives a positive answer to the question in (i).

(iii) The Kodaira dimension $\kappa(S)$ is invariant under a contraction of (-1)-curve. This is because sections $H^0(S, mK_S)$ are holomorphic differentials and we can apply Riemann's Extension Theorem. In particular, we can study the birational geometry of simplified algebraic surfaces according to the Kodaira dimensions.

(iv) A contraction of (-1)-curve drops the second Betti number $b_2(S)$ by one. Since $b_2(S)$ is finite, after at most $b_2(S)$ steps we end up with a smooth projective surface that contains no (-1)-curves. We call a surface with no (-1)-curves a *minimal surface*.

Minimal surfaces are well studied in [6] and in summary we have the following results:

- If $\kappa(S) = -1$, then after blowing down all the (-1)-curves, one shows that a minimal surface $S_{\min}$ is either $\mathbb{P}^2$ or a $\mathbb{P}^1$-bundle over a smooth curve B. In particular, $S_{\min}$, as well as S, is covered by rational curves.

- If $\kappa(S) \geq 0$, then after blowing down all the (-1)-curves, one can show that by classification $|mK_{S_{\min}}|$ is base point free for some $m > 0$ and we get a canonical morphism $S_{\min} \to \mathbf{Proj}(R(S))$.

Since we have similar behavior for curves and surfaces, it is natural to ask if we can generalize this dichotomy of geometry according to the Kodaira dimensions to higher dimensional varieties. This generalizes to the Minimal Model Program and the Minimal Model Conjecture in the next section.

We note here that in case $\kappa(S) \geq 0$, $K_{S_{\min}}$ being semiample is a highly nontrivial result. However, it is easy to show that $K_{S_{\min}}$ is nef: Since $\kappa(S) \geq 0$, we can write $K_{S_{\min}} \sim_{\mathbb{Q}} \sum a_i C_i$ for some irreducible curves C_i and some rational numbers $a_i > 0$. If $K_{S_{\min}}$ is not nef, then there exists an irreducible curve C such that $\sum_i a_i(C_i.C) = K_{S_{\min}}.C < 0$. In particular, $C_i = C$ for some i and $C^2 < 0$ as $C.C_i \geq 0$ if $C \neq C_i$. By adjunction formula, this implies that C is a (-1)-curve on $S_{\min}$, a contradiction. It is S. Mori who observes that nefness is the right condition for developing a higher dimensional minimal model program. The minimal model program for higher dimensional varieties is hence also called Mori's program.

Another remark is that in case $\kappa(S) = -1$, blowing down (-1)-curves in different order can lead to different minimal surfaces. However, each surface with $\kappa(S) \geq 0$ does have a unique minimal surface. This is due to the strong factorization property of birational maps of smooth projective surfaces. We will see later that in higher dimensions a minimal model $X_{\min}$ of X, if exists, is not necessarily unique even when $\kappa(X) \geq 0$.

2.2 Higher dimensional varieties

The Minimal Model Program (MMP) or Mori's Program and the Minimal Model Conjecture (MMC) aim to generalize the dichotomy of geometry for curves and surfaces that we have seen in the last section to higher dimensions. Let X be a smooth projective variety with Kodaira dimension $\kappa(X)$. The Minimal Model Conjecture states the following:

- If $\kappa(X) = -1$, then there exists a birational map $\phi : X \dashrightarrow Y$ and a morphism $f : Y \to B$ with $\dim Y > \dim B$ such that a general fiber Y_b of f is a Fano variety of

Picard number one. The morphism $f : Y \to B$ is called a Mori fiber space (MFS). The variety Y, and hence X, is covered by rational curves.

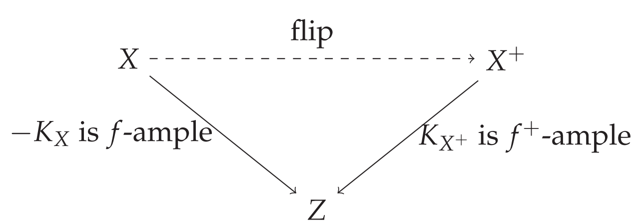
- If $\kappa(X) \geq 0$, then there exists a birational map $\psi : X \dashrightarrow X_{\min}$ such that $|mK_{X_{\min}}|$ is base point free for $m > 0$ sufficiently divisible and this defines a canonical morphism $\Phi : X_{\min} \to X_{\text{can}} := \mathbf{Proj}(R(X))$.

Here are important features of the Minimal Model Conjecture:

(i) The birational map ϕ and ψ are a composition of *K_X-negative maps*. The process of producing these maps is known as the **Minimal Model Program** or **Mori's Program**. This procedure is not unique, e.g., for surfaces we can blow down (-1)-curves in a diferent order. We say that a minimal model program is done if it terminates with a Mori fiber space or a minimal model $X_{\min}$.

(ii) In fact, Castelnuovo's Contraction Theorem is numerical in nature and is generalized to the existence of K_X-negative extremal contractions in higher dimensions via the Cone and Contraction Theorem. A K_X-negative extremal contraction is analogous to "blow-down of a (-1)-curve." It is *divisorial* if the contracting locus is of codimension one, otherwise it is *small*. If $f : X \to Z$ is a small contraction, then K_Z is not $\mathbb{Q}$-Cartier and we can not proceed since the numerical condition $K_Z.C$ is not well-defined. In this case, we have to construct *flips*. A K_X-negative map is either a divisorial contraction or a flip.

(iii) A **flip** occurs only in dimension three or higher. It is a geometric surgery of codimension two or higher (hence a birational map) and is illustrated in the following diagram:



where f and f^+ are small birational morphisms and X^+ is a normal projective $\mathbb{Q}$-factorial variety. Replacing by flips is the key making the higher dimension minimal model program possible. Existence of flips is proved by C. Hacon and J. M^{c}Kernan in [17].

(iv) In a minimal model program, there can be only finitely many divisorial contractions since each time the Picard number drops by one. However, if flips occur, then it is hard to show that there can be only finitely many flips in a minimal model program since we also extract subvarieties. This problem is known as the **Termination of flips.** Termination of flips is established in dimension three.

(v) A Mori fiber space is also given by a K_X-negative extremal contraction, e.g., a $\mathbb{P}^1$-bundle with fiber F is given by contracting the curve class $[F]$ where $K_X.F = -2$. Since (mildly singular) Fano varieties are rationally connected, according to the Minimal Model Conjecture, varieties with negative Kodaira dimension are covered by rational curves and vice versa.

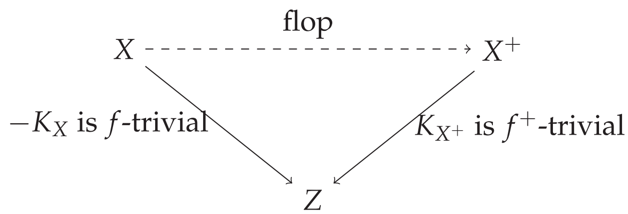
(vi) Recall that in dimension two, nefness of $K_{X_{\min}}$ is easier to establish than semiampleness. This is taken as part of the definition of a minimal model $X_{\min}$, i.e., we ask $K_{X_{\min}}$ to be nef.

(vii) To show that K_X being nef implies that K_X is semiample is known as the **Abundance Conjecture**. This has been established up to dimension three and for varieties of general type in any dimension.

(viii) It is an important theorem of [8] that the canonical ring $R(X)$ of a smooth projective variety is always a finitely generated $\mathbb{C}$-algebra. In case $\kappa(X) \geq 0$, the **canonical model** $X_{\text{can}} = \mathbf{Proj}(R(X))$ is thus well-defined. The canonical ring $R(X)$ is invariant under K_X-negative maps and hence $\mathbf{Proj}(R(X_{\min})) \cong \mathbf{Proj}(R(X))$. In particular, the morphism $X_{\min} \to \mathbf{Proj}(R(X))$ is defined and is called the *Iitaka fibration* of X.

(ix) In case $\kappa(X) \geq 0$, we can understand that a minimal model program aims to eliminate the base locus of $|mK_X|$. Divisorial contractions eliminate the divisorial part of the stable base locus $\mathbf{Bs}(K_X) = \cap_{m \geq 1} \mathrm{Bs}(mK_X)$ while flips take care of codimension two or higher stratum.

(x) Even though X_{can} is unique, there can be more than one minimal model $X_{\min}$. Any two minimal model are isomorphic in codimension one and are connected by a sequence of flops, cf. [22]. A **flop** is a small birational map similar to a flip and is described in the following diagram:



where f and f^+ are small birational morphisms.

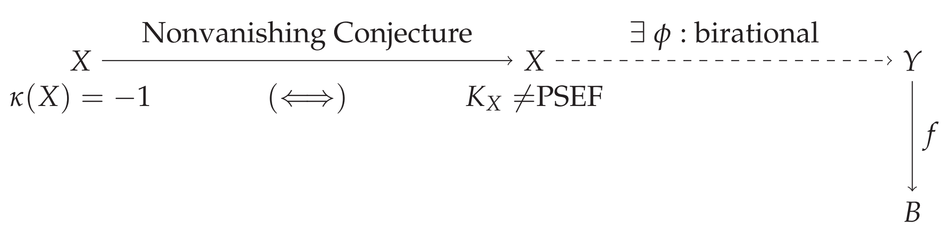
(xi) Singularities arise naturally in the minimal model program. In dimension three or higher, a K_X-negative map creates terminal singularities. It is then necessary to include varieties with terminal singularities to complete the minimal model program. The theory of singularities can be generalized to pairs and we also have the generalized minimal model program for pairs. See Section 2.3.

A very important turning point in the higher dimensional minimal model program is that the dichotomy according to the Kodaira dimensions has been replaced by pseudo-effectiveness (PSEF) of K_X. A divisor is pseudo-effective if numerically it is a limit of effective divisors, a much weaker condition than being effective. It is conjectured that K_X being pseudo-effective is the same as having nonnegative Kodaira dimension. This is known as the **Nonvanishing Conjecture**.

Conjecture 2 *(**Nonvanishing Conjecture**) Let X be a smooth projective variety. If K_X is PSEF, then $\kappa(X) \geq 0$.*

The full Minimal model Conjecture is summarized in the following diagrams:

- If $\kappa(X) = -1$, then we have



where ϕ is a composition of K_X-negative maps and the morphism $f : Y \to B$ is a Mori fiber space.

- If $\kappa(X) \geq 0$, then we have

$$\begin{array}{ccccc} X & \xrightarrow{\text{Nonvanishing Conjecture}} & X & \overset{\exists\, \psi\, :\, \text{birational}}{\dashrightarrow} & X_{\text{min}} \\ K_X = \text{PSEF} & (\Longleftrightarrow) & \kappa(X) \geq 0 & & \Big\downarrow \Phi \\ & & & & X_{\text{can}} = \mathbf{Proj}(R(X)) \end{array}$$

  where ψ is a composition of K_X-negative maps and $K_{X_{\text{min}}}$ is semiample with Φ the induced morphism.

There remain three main problems for completing the minimal model program and hence the Minimal Model conjecture:

(N) **Nonvanishing Conjecture**: K_X =PSEF$\Longrightarrow \kappa(X) \geq 0$;

(T) **Termination of flips**: There exists no infinite sequence of flips;

(A) **Abundance Conjecture**: Let X be a normal projective variety with at worst terminal singularities. Then K_X being nef implies that K_X is semiample.

Remark 3 *In the Abundance Conjecture (A), terminal singularities will be defined in Section 2.3. Since the outcome of each K_X-negative map may possess terminal singularities, it is natural to impose the singularity condition in this conjecture. It is also known that this condition is necessary.*

The Minimal Model Conjecture consists of two parts: The geometric picture of varieties and the completion of full Minimal Model Program. If one only cares about the geometry of varieties, we do not necessarily need to establish the full Minimal Model Program. We have indicated in (i) that the construction of a sequence of K_X-negative maps in a minimal model program is not unique and hence it is difficult to determine whether this process terminates. A technique introduced by V.V. Shokurov, called **the minimal model program with scaling**, enables us to specify a sequence of K_X-negative maps in a particular way that has better chance to terminate. Hence one can establish the existence of minimal models in certain cases without assuming the full minimal model program, or equivalently the conjectural termination of flips. We will talk about the minimal model program with scaling in Section 3.3.

We say that there exists a **good minimal model** for a given variety X if the Minimal Model Conjecture is true for X. In particular, a variety X has a good minimal model if certain minimal model program ends up with a Mori fiber space or terminates with a vareity where the Abundance Conjecture is true. Here are some known results about the existence of good minimal models:

- For threefolds, the Minimal Model Conjecture is established by V.V. Shokurov, Y. Kawamata, S. Mori, et al.
- By [8], the Minimal Model Conjecture is true if K_X is not PSEF. In particular, the Minimal Model Conjecture for varieties with negative Kodaira dimension is reduced to the Nonvanishing Conjecture.
- There exist good minimal models for varieties of general type by [8].
- The existence of good minimal models for varieties of $\kappa(X) = 0$ implies the existence of good minimal models for varieties of $\kappa(X) \geq 0$ by [32].

Another observation is that Fano varieties show up as a significant part of the Minimal Model Program:

(a) Fano varieties of Picard number one are the building blocks for varieties of negative Kodaira dimension.

(b) The exceptional locus of a K_X-negative extremal contraction is covered by rational curves. According to the Minimal Model Conjecture, these subvarieties are also built from Fano varieties (of Picard number one).

Remark 4 *Since it is a general fact that Fano varieties (with mild singularities) are covered by rational curves, this justifies saying that the geometry of varieties is complicated by the existence of rational curves.*

The above observation motivates the study of Fano varieties, especially the boundedness problem. The precise question is the **Borisov-Alexeev-Borisov Conjecture**, which asserts boundedness of ϵ-klt log $\mathbb{Q}$-Fano varieties. Heuristically, boundedness of Fano varieties would imply termination of flips by (b). A result of C. Birkar and V.V. Shokurov says that the B-A-B conjecture together with the ascending chain condition of minimal log discrepancies and the lower dimensional minimal model program implies termination of flips. We will come back to the B-A-B conjecture in Chapter 4 where we study the upper bound of the anticanonical volumes for ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefolds of Picard number one.

2.3 Singularities of pairs

We have noted that from (xi) in Section 2.2, singularities arise naturally in the minimal model program. Also, pairs arise naturally in the study of geometry of varieties. Surprisingly, we can combine these two ingredients, singularities and pairs, to establish a theory of singularities of pairs. The theory of singularities of pairs is very important in the minimal model program, which enables us to induct on dimensions via subadjunction, inversion of subadjunction, and the canonical bundle formula.

The main tool for studying singularities is Hironaka's theorem on the resolution of singularities over algebraic closed field of characteristic zero, see [19]. As an application, we can always assume the existence of log resolutions.

Definition 5 *Let X be a normal quasi-projective variety and $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. A log resolution of $(X, \mathfrak{a})$ is a projective birational morphism $f : Y \to X$ such that*

(i) Y is smooth and $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(D)$ for some divisor D on Y;

(ii) the exceptional set $\mathrm{Exc}(f)$ is of pure dimension one and the $\mathrm{Supp}(D) \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor.

A log resolution $f : Y \to X$ always exists. In fact, we can construct a log resolution by a composition of blow-ups along smooth centers of codimension greater than or equal to two.

We start with singularities of normal varieties.

Definition 6 *Let X be a normal projective variety and assume that K_X is a $\mathbb{Q}$-Cartier divisor. Let $\pi : Y \to X$ be a log resolution of X and write $K_{Y/X} = K_Y - \pi^*(K_X) = E_Y = \sum_i a_i E_i$ as $\mathbb{Q}$-divisors. Then we say that*

$$X \text{ has } \left\{ \begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{log canonical} \end{array} \right. \text{ singularities if } \left\{ \begin{array}{l} E_Y > 0 \text{ and } \mathrm{Supp}(E_Y) = \mathrm{Exc}(\pi); \\ E_Y \geq 0; \\ E_Y > -1; \\ E_Y \geq -1. \end{array} \right.$$

Note that smaller a_i's correspond to worse singularities.

Surfaces with at worst terminal singularities are smooth. Indeed, let X be a surface with at worst terminal singularities. Let $\pi : Y \to X$ be a resolution of X and write $K_Y = \pi^* K_X + \sum_i a_i E_i$, where E_i's are irreducible π-exceptional curves and a_i's are positive rational numbers. If $E = \sum_i a_i E_i$, then $\sum_i a_i (K_Y . E_i) = K_Y . E = E^2 < 0$ implies that $K_Y . E_l < 0$

for some l. Since $E_l^2 < 0$, E_l is a (-1)-curve and by Castelnuovo's contraction theorem $\pi : Y \to X$ factors through the blow down $\mu : Y \to Y'$ of E_l. Note that Y' is smooth and hence $\pi' : Y' \to X$ is again a resolution where $\pi = \pi' \circ \mu$. Inductively, we see that X is smooth.

Example 7 *For examples of singular surfaces, let $X_{g,d}$ be the projective cone over a curve C of genus g and degree $d \geq 2$. Then $X_{g,d}$ is a normal projective surface of Picard number one with vertex $O \in X_{g,d}$ the unique singularity. Blowing up the vertex $O \in X_{g,d}$ is a log resolution $\pi : Y = \mathrm{Bl}_O X_{g,d} \to X_{g,d}$ of $X_{g,d}$ which has a unique exceptional divisor $E_{g,d} \cong C$ over P with $E^2 = -d$. It is easy to compute by adjunction formula on Y that*

$$K_Y = \pi^* K_{X_{g,d}} + (-1 + \frac{2-2g}{d}) E_{g,d}.$$

Hence

$$X_{g,d} \text{ has } \begin{cases} \textit{canonical} \\ \textit{klt} \\ \textit{log canonical} \end{cases} \text{ singularities if and only if } \begin{cases} g = 0 \text{ and } d = 2 \\ g = 0 \text{ and } d \geq 2 \\ g = 1. \end{cases}$$

If $g \geq 2$, then $X_{g,d}$ is not log canonical. When $g = 0$, the singularities get worse as d increases. This justifies the last comment in the above definition.

From the examples below, we will see that pairs also arise naturally in the study of geometry. The idea is that for a morphism $f : X \to Y$ of varieties, we want to relate the canonical divisors K_X and K_Y. If f is an closed embedding, then we get a (sub)adjunction formula.

Example 8 *(**Adjunction**) Let X be a smooth divisor of a smooth variety Y, then*

$$(K_Y + X)|_X = K_X.$$

Here the pair is (Y, X) and we have a log canonical divisor $K_Y + X$.

In general, if $f : X \to Y$ is an embedding but neither X nor Y is smooth, then we get only subadjunction, i.e., a correction term is necessary for the adjunction formula to hold.

Example 9 *(**Subadjunction**) Let X be the projective cone over a quadratic curve in $\mathbb{P}^2$ and let $O \in X$ be the vertex of cone, the unique singularity of X. Let l be a ruling of X, then l is not Cartier but $2l$ is Cartier. It is easy to see that $l|_l = \frac{1}{2}(2l|_l) = \frac{1}{2}O$. If $f : Y = \mathrm{Bl}_O X \to X$ is the* *cone resolution with unique exceptional divisor E and let l' be the proper transform of l on Y, then we get*

$$(f|_l)^*(K_l + Diff) \overset{(\star)}{=} f^*(K_X + l)|_{l'} = (K_Y + l' + \frac{1}{2}E)|_{l'} = K_{l'} + \frac{1}{2}O$$

where $(\star)$ is true since we are using a log resolution. Since $f|_l : l' \to l$ is the identity, we must define $Diff = \frac{1}{2}O$ for the adjunction formula, i.e.,

$$(K_X + l)|_l = K_l + \frac{1}{2}O.$$

The following example considers the case where $f : X \to Y$ is an algebraic fiber space with a general fiber F and $K_F \sim_{\mathbb{Q}} 0$. In this case, we expect to have a canonical bundle formula.

Example 10 ***(Canonical bundle formula)*** *Let S be a minimal surface with $\kappa(S) = 1$. The Iitaka fibration is a morphism $p : S \to B$ with $\dim B = 1$ where a general fiber S_b is an elliptic curve. In this case, we call S an elliptic surface. It can be shown that the canonical divisor K_S is a fractional combination of fibers and hence we can write*

$$K_S \sim_{\mathbb{Q}} p^*(K_B + \Delta_B)$$

where Δ_B is a $\mathbb{Q}$-divisor on B, cf. [6, Proposition IX.3]. This is the canonical bundle formula for elliptic surfaces that relates K_S to the smaller dimensional pair $K_B + \Delta_B$.

Now we combine these two ingredients and study the singularities of pairs.

Definition 11 *Let X be a normal projective variety and Δ be a $\mathbb{Q}$-divisor with coefficients in $[0, 1]$ so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Y \to X$ be a log resolution of (X, Δ) and let Δ_Y be a $\mathbb{Q}$-divisor on Y so that $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ as $\mathbb{Q}$-divisors. Then we say that*

$$(X, \Delta) \text{ has } \begin{cases} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{log canonical} \end{cases} \text{singularities if } \begin{cases} \text{mult}_E \Delta_Y < 0 \; \forall \, E \subseteq \text{Exc}(\pi); \\ \text{mult}_E \Delta_Y \geq 0 \; \forall \, E \subseteq \text{Exc}(\pi); \\ \Delta_Y < 1; \\ \Delta_Y \leq 1. \end{cases}$$

These conditions generalize Definition 5 and can be verified on a single log resolution.

The divisor Δ in the definition with $\Delta \in [0, 1]$ is called a *boundary*. In general, we can allow Δ to have arbitrary coefficients with $K_X + \Delta$ being $\mathbb{Q}$-Cartier when studying singularities of pairs. Note that being log canonical implies that $\Delta \leq 1$. Also, we will always assume that Δ is a boundary in the study of minimal model program. We can generalize the minimal model program to log pairs with mild singularities. Here are some categories where people study the minimal model program:

- **Min**: $\mathbb{Q}$-factorial normal projective varieties with at worst terminal singularities;

- **Max**: $\mathbb{Q}$-factorial normal projective pairs with at worst log canonical singularities.

Because the minimal model program in dimension three and higher starts to produce varieties with terminal singularities, we must enlarge the category of smooth varieties to carry out the program. The category **Min** is the smallest category where we can carry out the minimal model program when starting with smooth varieties. Most theorems related to the minimal model program in the category **Min** generalize without much difficulty to the category of $\mathbb{Q}$-factorial normal projective pairs with at worst klt singularities. For example, the Kawamata-Viehweg vanishing theorem for klt pairs is the generalization of the classical Kodaira vanishing theorem for smooth projective varieties. Hence the minimal model program naturally generalizes to the category of $\mathbb{Q}$-factorial normal projective pairs with at worst klt singularities. The category **Max** is the largest category where people expect the minimal model program to be true. However, passing theorems from klt singularities to log canonical singularities is typically technical. This is because actually a log canonical singularity is not a limit of klt singularities as it seems to be from the definition.

If we start with a smooth variety, or more generally a normal and $\mathbb{Q}$-factorial variety, then it is known that a variety as an outcomes of K_X-negative map in a minimal model program remains normal and $\mathbb{Q}$-factorial. Hence *normality* and $\mathbb{Q}$-*factoriality* are two natural conditions to impose on varieties when we work with the minimal model program.

2.4 BCHM

The work [8] of C. Birkar, P. Cascini, C. Hacon, and M^{c}Kernan (BCHM) on the minimal model program is a great advance on the study of higher dimensional geometry. We include here without proofs some main results in [8] that are relevant to this dissertation. A main theorem proved in [8] is the following long standing conjecture:

Theorem 12 *Let X be a smooth projective variety, then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$$

is a finitely generated graded $\mathbb{C}$-algebra.

In particular, as we pointed out earlier in Section 2.2, this implies that there exists a canonical model $X_{\text{can}} := \mathbf{Proj}(R(X))$ canonically associated to a given variety X.

It is known in [15] that by the canonical bundle formula the problem on finite generation of canonical rings for any smooth projective varieties can be reduced to the cases where we have klt pairs of general type. Hence Theorem 12 is actually a corollary of the following theorem in [8] and the base point freeness theorem in [30]:

Theorem 13 *There exists a good minimal model for a klt pair with a big boundary. In particular, there exist good minimal models for klt pairs of general type.*

Another important question in the minimal model program is about the termination. A global approach to solve the termination problem is to show that in a minimal model program all the possible outcomes of K_X-negative maps starting from a given variety are finite. Since it is known that varieties appearing in a sequence of K_X-negative maps do not repeat, it follows that the minimal model program must terminate. Thus Theorem 13 can be thought of as a formal consequence of the following result on finiteness of models.

Theorem 14 *The set of weak log canonical models for a given log pair with big boundary is finite.*

We do not define the technical term *weak log canonical model* here, but the key point is that each outcome of a K_X-negative map is a weak log canonical model. Also, in a minimal model program with scaling of an ample divisor, each outcome of a K_X-negative map is a weak log canonical model for a log pair with big boundary. Hence, the minimal model program with scaling of an ample divisor has a better chance to terminate.

A very important fact about the minimal model program and hence the Minimal Model Conjecture is that we can generalize them to log pairs and also to a relative setting. This is known as the relative log minimal model program. All the above theorems are true in the relative log setting. This has significant applications to the moduli problem of higher dimension varieties. Also, if one start with a birational morphism $f : X \to Y$, e.g., a log resolution, then a log pair is always relative big and we can apply results for pairs with big boundary from [8]. This enables us to create a better behaved birational model for singular varieties, e.g., $\mathbb{Q}$-factorial models or dlt models. Many studies of higher dimension varieties rely on the existence of good birational models. For example, we will use dlt models in Chapter 4.

CHAPTER 3

VARIETIES FIBERED BY GOOD MINIMAL MODELS

In this chapter, I present my first result on a reduction theorem of the Minimal Model Conjecture for varieties of intermediate Kodaira dimensions.

3.1 Kodaira dimension and Iitaka fibration

Let X be a projective $\mathbb{Q}$-Gorenstein variety, i.e., K_X is a $\mathbb{Q}$-Cartier divisor. In Section 2.1, we have defined the Kodaira dimension $\kappa(X)$ to be -1 if $H^0(X, mK_X) = 0$ for all the $m \geq 1$, and $\kappa(X) \geq 0$ if there is an integer $m > 0$ such that $H^0(X, mK_X) \neq 0$. When $\kappa(X) \geq 0$, we also define the canonical ring $R(X)$ of X to be the graded $\mathbb{C}$-algebra

$$R(X) = \bigoplus_{m \geq 0} H^0(mK_X).$$

In fact we can refine the definition of Kodaira dimension.

Definition 15 *Assume that $H^0(X, lK_X) \neq 0$ for some $l > 0$. Then the Kodaira dimension is the unique integer $0 \leq \kappa(X) \leq \dim X$ such that there are positive real numbers α, β with*

$$\alpha \cdot m^{\kappa(X)} \leq P_m(X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(mK_X)) \leq \beta \cdot m^{\kappa(X)}$$

for $m > 0$ divisible. An equivalent definition is that

$$\kappa(X) = \begin{cases} -1 & \text{if } H^0(mK_X) = 0 \text{ for any } m \geq 1; \\ \text{tr.deg}_{\mathbb{C}}(\bigoplus_{m \geq 0} H^0(mK_X)) & \text{if } H^0(mK_X) \neq 0 \text{ for some } m \geq 1. \end{cases}.$$

Geometrically, we associate for each integer $m > 0$ the map defined by the m-th pluricanonical system $|mK_X| \neq \emptyset$:

$$\Phi_m : X \dashrightarrow \Phi_m(X) \subseteq \mathbf{P}(|mK_X|).$$

For $m > 0$ sufficiently divisible, the Kodaira dimension $\kappa(X)$ is given by

$$\kappa(X) = \dim \Phi_m(X),$$

where we put $\kappa(X) = -1$ if $|mK_X| = \emptyset$ for all $m > 0$. We say that X is of general type if K_X is big, i.e., the map Φ_m is birational for $m > 0$ sufficiently divisible. In general, we have the Iitaka fibrations.

Theorem 16 *([33, Theorem 2.1.33]) Let X be a normal projective variety, $\mathbb{Q}$-Gorenstein with $\kappa(X) \geq 0$. Consider the semigroup $\mathbf{N}(X, K_X) = \{m \in \mathbb{N} | H^0(X, mK_X) \neq 0\}$. For all sufficiently large $m \in \mathbf{N}(X, K_X)$, the rational maps $\Phi_m : X \dashrightarrow Y_m = \phi_m(X) \subseteq \mathbb{P}(|mK_X|)$ are birationally equivalent to a fixed algebraic fiber space $\Phi_\infty : X_\infty \to Y_\infty$ of normal varieties. Moreover, a very general fiber of ϕ_∞ has Kodaira dimension zero and $\dim Y_\infty = \kappa(X)$.*

We have seen that a minimal model program consists of K_X-negative maps, i.e., divisorial contractions and flips. Since a flip is an isomorphism in codimension one, the pluri-canonical system $|mK_X|$ is invariant under flips by the normality condition. Let $\phi : X \to X'$ be a divisorial contraction. By the Negativity Lemma (Lemma 17), we can write $K_X \sim_{\mathbb{Q}} K_{X'} + E$ for some effective ϕ-exceptional divisor E on X. In particular, by the projection formula and Fujita's lemma, we see that the pluri-canonical system is also invariant under divisorial contractions. As a consequence, the Kodaira dimension is invariant under the minimal model program, and we will study the outcomes of the minimal model program accroding to different Kodaira dimensions.

Lemma 17 *(**Negativity of contraction**) Let $\pi : Y \to X$ be a proper birational morphism of normal quasi-projective varieties. Let L be an $\mathbb{R}$-Cartier divisor on X such that $\pi^* L \equiv M + G + E$ where M is a π-nef $\mathbb{R}$-Cartier divisor on Y, $G \geq 0$, E is π-exceptional, and G and E have no common components. Then $E \geq 0$. Furthermore, if E_i is a component of E such that there is a component $E_j \neq E_i$ of E with the same center on X as E_i and with the restriction of M to E_j not numerically π-trivial, then the coefficient of E_i is strictly positive.*

3.2 Good minimal models

A pair (X, Δ) over U consists of a $\mathbb{Q}$-factorial normal projective variety X with an effective $\mathbb{R}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and a projective morphism $X \to U$ to a quasi-projective variety U. We recall the definition of a minimal model.

Definition 18 *For a log canonical pair (X, Δ) over U, a **minimal model** of (X, Δ) over U is proper birational map $\phi : (X, \Delta) \dashrightarrow (X', \Delta' = \phi_* \Delta)$ over U with the following properties:*

(1) X' is normal and $\mathbb{Q}$-factorial,

(2) *ϕ extracts no divisors,*

(3) *$K_{X'} + \Delta'$ is nef over U, and*

(4) *$a(F, X, \Delta) < a(F, X', \Delta')$ for each ϕ-exceptional divisor F.*

Moreover, we say that abundance *holds on (X', Δ') if $K_{X'} + \Delta'$ is semiample over U, i.e. $K_{X'} + \Delta'$ is an $\mathbb{R}$-linear sum of $\mathbb{Q}$-Cartier divisors which are semiample over U. A* good minimal model *of a pair (X, Δ) over U is a minimal model such that abundance holds.*

Remark 19 *A minimal model in this paper is a log terminal model as defined in [8].*

Definition 20 *Let $X \to U$ and $Y \to U$ be two projective morphisms of normal quasi-projective varieties. Let $\phi : X \dashrightarrow Y$ be a proper birational contraction (so that ϕ^{-1} contracts no divisors) over U. Let D and D' be $\mathbb{R}$-Cartier divisors such that $D' = \phi_* D$. Then we say that ϕ is* discrepancy-negative *with respect to D if and only if for any common resolution $p : W \to X$ and $q : W \to Y$, we may write*

$$p^* D = q^* D' + E,$$

where $E \geq 0$ and the support of $p_* E$ contains all the ϕ-exceptional divisors (cf. [8, Lemma 3.6.3]).

Remark 21 *Note that if D' in the above definition is nef over U and $p_* E$ is effective, then E is effective by the negativity lemma (cf. Lemma 3.5.2 [8]). Hence in this case ϕ is discrepancy-negative with respect to D if and only if $p_* E \geq 0$ and its support contains all the ϕ-exceptional divisors. Condition (4) of Definition 18 is then equivalent to ϕ being discrepancy-negative with respect to $K_X + \Delta$.*

We start with some preliminary results on good minimal models.

Lemma 22 *Let (X_i, Δ_i), $i = 1, 2$, be two klt pairs over U and $\alpha : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)$ be a birational map over U with $\alpha_* \Delta_1 = \Delta_2$. Suppose that α satisfies the condition (4) of Definition 18 with respect to (X_1, Δ_1) and extracts no divisors. Then (X_1, Δ_1) has a good minimal model over U if (X_2, Δ_2) does.*

Proof. This is [8, Lemma 3.6.9]. ■

Lemma 23 *Let (X, Δ) be a terminal pair over U. For any resolution $\mu : (X', \Delta') \to (X, \Delta)$ with $\Delta' := \mu_*^{-1} \Delta$, a good minimal model of (X', Δ') is also a good minimal model of (X, Δ).*

Proof. Note that if we write $K_{X'} + \Delta' = \mu^*(K_X + \Delta) + E$, then E is effective and its support equals to the set of all μ-exceptional divisors. Hence the same argument as in [8, Lemma 3.6.10] applies (without adding extra any μ-exceptional divisors). ■

Theorem 24 *Let $\phi_i : (X, \Delta) \dashrightarrow (X_i, \Delta_i)$, i=1,2, be two minimal models of a klt pair (X, Δ) over U with $\Delta_i = (\phi_i)_*\Delta$. The natural birational map $\psi : (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2)$ over U can be decomposed into a sequence of $(K_{X_1} + \Delta_1)$-flops over U.*

Proof. By [30, Theorem 3.52], (X_i, Δ_i) are isomorphic in codimension one, and hence the argument in [22] applies. ■

Proposition 25 *Let (X, Δ) be a klt pair over U. If (X, Δ) has a good minimal model over U, then any other minimal model of (X, Δ) over U is also good.*

Proof. Suppose that (X_g, Δ_g) is a good minimal model of (X, Δ) over U and $(\tilde{X}, \tilde{\Delta})$ is another minimal model of (X, Δ) over U. Let W be a common resolution of (X_g, Δ_g) and $(\tilde{X}, \tilde{\Delta})$ over U with maps $p : W \to X_g$ and $q : W \to \tilde{X}$. Following from Lemma 17 (or [30, Lemma 3.39]), we have $p^*(K_{X_g} + \Delta_g) = q^*(K_{\tilde{X}} + \tilde{\Delta})$ where $K_{X_g} + \Delta_g$ is semiample over U as (X_g, Δ_g) is a good minimal model of (X, Δ) over U. By the projection formula $K_{\tilde{X}} + \tilde{\Delta}$ is then semiample over U, and hence $(\tilde{X}, \tilde{\Delta})$ is also a good minimal model of (X, Δ) over U. ■

3.3 Minimal model program with scaling

A pair (X, Δ) over U consists of a $\mathbb{Q}$-factorial normal projective variety X with an effective $\mathbb{R}$-Weil divisor Δ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier and a projective morphism $X \to U$ to a quasi-projective variety U

Start with a $\mathbb{Q}$-factorial klt pair (X, Δ) over U and H an ample $\mathbb{R}$-divisor over U. Assume that $K_X + \Delta + H$ is nef over U and let

$$\lambda = \inf\{t \geq 0 | K_X + \Delta + tH \text{ is nef over } U\}.$$

If $\lambda = 0$, then $K_X + \Delta$ is nef over U and (X, Δ) is a minimal model over U. If $\lambda > 0$, then for fixed $0 < \lambda' < \lambda$ there are only finitely many $(K_X + \Delta + \lambda' H)$-negative extremal rays over U. Let R be one of these extremal rays such that $(K_X + \Delta + \lambda H).R = 0$. We consider the corresponding contraction $\mathrm{cont}_R : X \to Z$ over U. If $\dim Z < \dim X$, then we have a Mori fiber space and we are done. Otherwise, we replace (X, Δ) by the corresponding flip or divisorial contraction $\phi : X \dashrightarrow X'$. Let $H' = \phi_* H$ and $\Delta' = \phi_*\Delta$. Since $K_X + \Delta + \lambda H$ is nef over U and $(K_X + \Delta + \lambda H).R = 0$, it follows from the Cone and Contraction Theorem that $K_{X'} + \Delta' + \lambda H'$ remains nef and we can repeat the process. This is called a minimal model program with scaling of an ample divisor.

This process terminates with a minimal model or a Mori fiber space unless we get an infinite sequence of flips $X_i \dashrightarrow X_{i+1}$. Let Δ_i and H_i be the strict transforms of Δ and H on X_i. Then there is a sequence of real numbers $\lambda = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots 0$ such that $K_{X_n} + \Delta_n + \lambda H_n$ is nef over U. In particular, $X \dashrightarrow X_n$ is a minimal model for $(X, \Delta + \lambda_n H)$ over U.

Note that by the finiteness of models for klt pairs with big boundary in [8], eventually we get a strictly decreasing sequence of real numbers $\lambda = \lambda_1 > \lambda_2 > \lambda_3 > \cdots 0$ with $\lim_n \lambda_n = 0$ for any minimal model program with scaling of an ample divisor.

Proposition 26 *If a klt pair (X, Δ) over U has a good minimal model over U, then any $(K_X + \Delta)$ minimal model program scaling of an ample divisor A over U terminates.*

Proof. Let $\phi : (X, \Delta) \dashrightarrow (X_g, \Delta_g)$ with $\Delta_g = \phi_*\Delta$ be a good minimal model of (X, Δ) over U and $f : X_g \to Z = \mathbf{Proj}_U(K_{X_g} + \Delta_g)$ the corresponding morphism over U. Note that ϕ contracts exactly the divisorial part of $\mathbf{B}(K_X + \Delta/U)$ (cf. [8, Lemma 3.6.3]).

Pick $t_0 > 0$ such that $(X_g, \Delta_g + t_0 A_g)$ with $A_g = \phi_* A$ is klt and an ample divisor H on X_g. By [8], the outcome of running a $(K_{X_g} + \Delta_g + t_0 A_g)$-minimal model program with scaling of H over Z exists and is a minimal model $\psi : X_g \dashrightarrow X'$ of $(X_g, \Delta_g + t_0 A_g)$ over Z. As $K_{X_g} + \Delta_g \equiv_Z 0$, we have $K_{X'} + \Delta' \equiv_Z 0$ where $\Delta' = \psi_*\Delta_g$. Hence those curves contracted in each step of this minimal model program over Z have trivial intersection with $K_{X_g} + \Delta_g$ and negative intersection with A_g. In particular, this shows that X' is a minimal model of $(X_g, \Delta_g + tA_g)$ over Z for all $t \in (0, t_0]$. Since $\Delta' + t_0 A'$ with $A' = \psi_* A_g$ is big over U, there exists only finitely many $(K_{X'} + \Delta' + t_0 A')$-negative extremal rays in $\overline{\mathrm{NE}}(X'/U)$ by [8, Corollary 3.8.2]. Hence by considering smaller $t_0 > 0$, we can assume that X' is a minimal model of $(X_g, \Delta_g + tA_g)$ over U for all $t \in (0, t_0]$. Since being discrepancy-negative is an open condition (cf. Definition 20), we may choose $t_0 > 0$ sufficiently small such that $\psi \circ \phi$ is discrepancy-negative with respect to $(X, \Delta + tA)$ for all $t \in (0, t_0]$, and hence X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in (0, t_0]$. This implies that $\psi \circ \phi$ contracts exactly the divisorial part of $\mathbf{B}(K_X + \Delta + t_0 A/U)$ which is contained in $\mathbf{B}(K_X + \Delta/U)$ and is contracted by ϕ. Hence ψ contracts no divisors, and in particular $\psi \circ \phi$

is discrepancy-negative with respect to $(X, \Delta + tA)$ for all $t \in [0, t_0]$. This implies that X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. Note that then $\mathbf{B}(K_X + \Delta + tA/U)$ has the same divisorial components for all $t \in [0, t_0]$.

Now choose $0 < t_1 < t_0$ such that $(X, \Delta + t_1 A)$ is klt and run a minimal model program of $(X, \Delta + t_1 A)$ with scaling of A over U. By [8], the outcome $\phi : X \dashrightarrow \tilde{X}$ exists and is a minimal model of $(X, \Delta + t_1 A)$ over U. Since ϕ being discrepancy-negative with respect to $(X, \Delta + tA)$ is an open condition and $K_{\tilde{X}} + \tilde{\Delta} + t\tilde{A} := \phi_*(K_X + \Delta + tA)$ is nef over U for $t \in [t_1, t_0]$, by picking $t_0 > 0$ smaller if necessary we can assume that $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [t_1, t_0]$. Since $\mathbf{B}(K_X + \Delta + tA/U)$ has the same divisorial components for all $t \in [0, t_0]$, X' and $\tilde{X}$ are isomorphic in codimension one. For each $t \in [t_1, t_0]$, by Theorem 24 we may decompose the birational map $X' \dashrightarrow \tilde{X}$ over U into possibly different sequences S_t of $(K_{X'} + \Delta' + tA')$-flops over U as X' and $\tilde{X}$ are both minimal models of $(X, \Delta + tA)$ over U. Since $\Delta' + tA'$ is big over U for any $t \in [t_1, t_0]$ and each outcome of a $(K_{X'} + \Delta' + tA')$-flop over U is also a minimal model of $(X, \Delta + tA)$ over U, by finiteness of models in [8] we can only have finitely many $(K_{X'} + \Delta' + tA')$-flop over U as t ranges in $[t_1, t_0]$. In particular, there is an uncountable subset $T_1 \subseteq [t_1, t_0]$ such that for all $t \in T_1$, the first $(K_{X'} + \Delta' + tA')$-flops over U of the corresponding sequences S_t's are all the same. Note that those curves contracted by this flop then have trivial intersection with A' and hence this flop is a $(K_{X'} + \Delta')$-flop over U. As each sequence S_t is finite, inductively we can find a $t^* \in [t_1, t_0]$ such that all the steps of the sequence S_{t^*} connecting X' and $\tilde{X}$ are $(K_{X'} + \Delta')$-flops over U. Since X' is a minimal model of (X, Δ) over U, we then also have that $\tilde{X}$ is a minimal model of (X, Δ) over U. In particular, this shows that the minimal model program of (X, Δ) with scaling of A over U terminates. ■

Corollary 27 *Let (X, Δ) be a klt pair over U. Suppose that (X, Δ) has a good minimal model over U, then there exists a $t_0 > 0$ such that: if $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [\alpha, \beta]$ for some $0 \leq \alpha < \beta \leq t_0$, then $\tilde{X}$ is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. In particular, the set of all such minimal models $\tilde{X}$ is finite.*

Proof. By Proposition 26, there exists a $t_0 > 0$ and a birational map $X \dashrightarrow X'$ over U such that X' is a minimal model of $(X, \Delta + tA)$ over U for all $t \in [0, t_0]$. By the proof of Proposition 26, there is a finite sequence of $(K_{X'} + \Delta')$-flops over U connecting $X' \dashrightarrow \tilde{X}$ which are also A'-trivial and hence $(K_{X'} + \Delta' + tA')$-flops over U for all $t \in [0, t_0]$, where Δ' and A' are the proper transforms of Δ and A on X'. Therefore the corollary follows.

Note that X' and the varieties given by $(K_{X'} + \Delta')$-flops over U appearing in the proof are all minimal models of the big pair $(X, \Delta + t_0 A)$ over U and hence by [8] there can be only finitely many of these. ■

A proper morphism $f : X \to Y$ of normal varieties is an *algebraic fiber space* if it is surjective with connected fibers.

Proposition 28 *Let $f : X \to Y$ be an algebraic fiber space of normal quasi-projective varieties such that X is $\mathbb{Q}$-factorial with klt singularities and projective over Y. Suppose that the general fiber F of f has a good minimal model, then X is birational to some X' over Y such that the general fiber of $f' : X' \to Y$ is a good minimal model.*

Proof. Pick an ample divisor H on X and run a minimal model program of X with scaling of H over Y. Suppose that $\mathrm{cont}_R : X \to W$ is the contraction morphism corresponding to an extremal ray $R \in \overline{\mathrm{NE}}(X/Y)$. If R does not give an extremal contraction of F, then we have $\mathrm{cont}_R|_F = \mathrm{id}_F$. Otherwise it is easy to see that cont_R and $\mathrm{cont}_R|_F$ must be of the same type (divisorial or small). However, note that $\mathrm{cont}_R|_F$ may correspond to the contraction of a face of $\overline{\mathrm{NE}}(F)$ (instead of an extremal ray). Suppose that we have a sequence of infinitely many flips which are nontrivial on the general fiber F with $t_i > t_{i+1} > 0$ such that $K_{F_i} + tH_i|_{F_i}$ is nef for all $t \in [t_{i+1}, t_i]$. Since F has a good minimal model, by Corollary 27 the set of such F_i's is finite (modulo isomorphisms) and each F_i is a good minimal model of F. We get a contradiction by the same argument as in the last step of the proof of [8, Lemma 4.2]. Hence after finitely many steps, we may assume that all flips are trivial on the general fiber, and so we get an algebraic fiber space $f' : X' \to Y$ such that the general fiber is a good minimal model. ■

3.4 Main theorem

We start with a lemma concerning the negativity property of a "degenerate" divisor. The following definition is taken from [41].

Definition 29 *Let $f : X \to Y$ be a proper surjective morphism of normal varieties and let D be an effective Weil $\mathbb{R}$-divisor. Then*

- *D is f-exceptional if* $\mathrm{codim}(\mathrm{Supp}(f(D))) \geq 2$.
- *D is of insufficient fiber type if* $\mathrm{codim}(\mathrm{Supp}(f(D))) = 1$ *and there exists a prime divisor* $\Gamma \not\subseteq \mathrm{Supp}(D)$ *such that* $f(\Gamma) \subseteq \mathrm{Supp}(f(D))$ *has codimension one in Y.*

In either of the above cases, we say that D is degenerate. *In particular, a degenerate divisor is always assumed to be effective.*

Lemma 30 *Let $f : X \to Y$ be an algebraic fiber space of normal projective varieties such that X is $\mathbb{Q}$-factorial. For a degenerate Weil divisor D on X, we can always find a component $\tilde{D} \subseteq \operatorname{Supp}(D)$ which is covered by curves contracted by f and intersecting D negatively. In particular, we have $\tilde{D} \subseteq \mathbf{B}_{-}(D/Y)$, the diminished base locus of D over Y.*

Proof. Write $D = \Sigma r_i D_i$ with $r_i > 0$ and $D_i \in \operatorname{Div}(X)$ prime.

Case 1: Suppose D is f-exceptional, and hence $\dim Y \geq 2$. Cutting by $\dim f(D)$ general hyperplanes on Y and by $\dim X - \dim f(D) - 2$ general hyperplanes on X, we reduce to a birational morphism of surfaces with $E = \Sigma r_j \tilde{E}_j$, where $\tilde{E}_j = D_j \cap H_1 \cap ... \cap H_n$ may be nonreduced and reducible and $E = D \cap H_1 \cap ... \cap H_n$. Note that we may assume $P := f(E)$ is a point, i.e. E is exceptional. By the Hodge index theorem (cf. [5, Corollary 2.7]), the intersection matrix of irreducible components of $f^{-1}(P)$ is negative-definite. So $E^2 < 0$, and hence $(\tilde{E}_j.D) = (\tilde{E}_j.E) < 0$ for some j. In particular, $(\tilde{E}_j.D_j) < 0$ and D_j is covered by curves intersecting D negatively.

Case 2: Suppose D is of insufficient fiber type. Cutting by $\dim Y - 1$ general hyperplanes on Y and then by $\dim X - 2$ general hyperplanes on X, we reduce to a morphism from a surface to a curve with $E = \Sigma r_j \tilde{E}_j$ supported on fibers, where $\tilde{E}_j = D_j \cap H_1 \cap ... \cap H_n$ may be non-reduced and reducible and $E = D \cap H_1 \cap ... \cap H_n$. By [5, Corollary 2.6], we have $(E)^2 \leq 0$. But $\operatorname{Supp}(E)$ cannot be the whole fiber. Hence we can find Γ an effective divisor having no common components with E such that $\operatorname{Supp}(E + \Gamma) = f^{-1}(f(E))$. For $P := f^*(f_*(E))$, we can find a and b two positive real numbers such that $aP \leq E + \Gamma \leq bP$. If $E^2 = 0$, then E is nef and hence $E.P = 0$ implies $E.(E + \Gamma) = 0$. But we have $E.\Gamma > 0$ which implies $E^2 < 0$, a contradiction. Thus $E^2 < 0$ and the same argument as in case 1 applies.

To prove that $D_j \subseteq \mathbf{B}_{-}(D/Y)$, we pick an ample divisor A on X and $\epsilon > 0$ a small rational number such that $\tilde{E}_j.(D + \epsilon A) < 0$. Note that we also have $\tilde{E}_j.(D + \epsilon A + f^*R) < 0$ for any $\mathbb{R}$-Cartier divisor R on Y. In particular, this shows that $\tilde{E}_j \subseteq \mathbf{B}(D + \epsilon A/Y)$. As $\tilde{E}_j$ passes through a general point of D_j, we have $D_j \subseteq \mathbf{B}(D + \epsilon A/Y) \subseteq \mathbf{B}_{-}(D/Y)$. ∎

Let $f : X \to Y$ be an algebraic fiber space. For an effective divisor Γ on X, we write $\Gamma = \Gamma_{\text{hor}} + \Gamma_{\text{ver}}$ where Γ_{hor} and Γ_{ver} are effective without common components such that Γ_{hor} dominates Y and $f_*\Gamma_{\text{ver}} = 0$ on Y, respectively.

Theorem 31 *Let X be a $\mathbb{Q}$-factorial normal projective variety with nonnegative Kodaira dimension and at most terminal singularities. Suppose the general fiber F of the Iitaka fibration has a good minimal model. Then X has a good minimal model.*

Proof. The theorem is certainly true for the case $\kappa(X) = 0$. For varieties of general type, the theorem follows from [8] and the base point free theorem in [30]. Hence we may assume $0 < \kappa(X) < \dim(X)$.

By [8], the canonical ring $R(X)$ is a finitely generated $\mathbb{C}$-algebra and hence there is an integer d such that the truncated ring $R^{[d]}(X)$ is generated in degree 1. Take a resolution $\mu : X' \to X$ of X and $|dK_X|$, then

- $\mu^*|mdK_X| = |mM| + mG$ with $|mM|$ base point free and $mG \geq 0$ the fixed divisor for all $m > 0$,
- $f := \phi_{|M|} : X' \to Y$ is birationally equivalent to the Iitaka fibration,
- $K_{X'} = \mu^* K_X + E$ with E effective and μ-exceptional,
- $dK_{X'} \sim M + G + dE$ with $G + dE$ effective and $G + dE \subseteq \mathbf{B}(K_{X'})$.

Write $\Gamma := G + dE = \Gamma_{\text{hor}} + \Gamma_{\text{ver}}$ with respect to f. By Proposition 28, after running a minimal model program of X' with scaling of an ample divisor over Y, we may assume that the general fiber of f is a good minimal model. Moreover, we may assume that $\mathbf{B}_-(K_{X'}/Y)$ contains no divisorial components. As the general fiber F of f has Kodaira dimension zero, we have $\Gamma_{\text{hor}}|_F = (M + G + dE)|_F \sim (dK_{X'})|_F \sim dK_F \sim_{\mathbb{Q}} 0$ and hence $\Gamma_{\text{hor}} = 0$. In particular, we may assume $G + dE$ consists of only vertical divisors. Note that the condition $G + dE \subseteq \mathbf{B}(K_{X'})$ still holds after steps of a minimal model program.

Consider T an effective divisor with $\text{Supp}(T) \subseteq \text{Supp}(G + dE)$ and the exact sequences on Y

$$0 \to f_*\mathcal{O}_{X'}((j-1)T) \to f_*\mathcal{O}_{X'}(jT) \to Q_j \to 0,$$

with $j \geq 1$ and Q_j the cokernel. After tensoring with $\mathcal{O}_Y(k)$ for k sufficiently large, we have $Q_j(k)$ is generated by global sections and $H^1(Y, f_*\mathcal{O}_{X'}((j-1)T) \otimes \mathcal{O}_Y(k)) = 0$. As $T \subseteq \mathbf{B}(K_{X'})$ and $\mathcal{O}_{X'}(M) = f^*\mathcal{O}_Y(1)$, we have for any $j \geq 0$

$$\begin{aligned} H^0(Y, f_*\mathcal{O}_{X'}(jT) \otimes \mathcal{O}_Y(k)) =& H^0(X', \mathcal{O}_{X'}(kM + jT)) \\ =& H^0(X', \mathcal{O}_{X'}(kM)) = H^0(Y, \mathcal{O}_Y(k)). \end{aligned}$$

Hence the exact sequence of cohomological groups shows that $H^0(Y, Q_j(k)) = 0$ and then $Q_j = 0$. In particular, $f_*\mathcal{O}_{X'}(jT) = \mathcal{O}_Y$ for any $j \geq 0$. Suppose that $P := f_*(T)_{\mathrm{red}}$ is a codimension one point on Y such that $\mathrm{Supp}(T)$ contains all divisors in X' dominating P. Note that we can find a big open subset $U \subseteq Y$ such that the image of the exceptional divisors contained in $f^*(P)$ is disjoint from U as it has codimension greater or equal to two. Hence, there is a positive integer j such that $f_*\mathcal{O}_{X'}(jT)|_U \supseteq \mathcal{O}_Y(P)|_U$. Since both sheaves $f_*\mathcal{O}_{X'}(jT) = \mathcal{O}_Y$ and $\mathcal{O}_Y(P)$ are reflexive, we have an inclusion $\mathcal{O}_Y(P) \subseteq \mathcal{O}_Y$, which is impossible. In particular, this shows that $G + dE$ is of insufficient fiber type over Y.

By Lemma 30, we can find a component of $G + dE$ which is contained in $\mathbf{B}_-(K_{X'}/Y)$. But this is impossible as $\mathbf{B}_-(K_{X'}/Y)$ contains no divisorial components. Then $dK_{X'} \sim M$ with $\mathcal{O}_{X'}(M) = f^*\mathcal{O}_Y(1)$ is base point free and hence X' is a good minimal model of X by Lemma 23 (as μ is a resolution of a terminal variety). ∎

3.5 Iitaka's Conjecture C

The original motivation of this work is **Iitaka's Conjecture C** ([43, §11]).

Conjecture 32 *If $f : X^n \to Y^m$ is an algebraic fiber space of smooth projective varieties with general fiber F, then we have*

- $C_{n,m} : \kappa(X) \geq \kappa(F) + \kappa(Y)$, *and*
- $C^+_{n,m} : \kappa(X) \geq \kappa(F) + \mathrm{Max}\{\mathrm{Var}(f), \kappa(Y)\}$ *if* $\kappa(Y) \geq 0$, *where* $\mathrm{Var}(f)$ *is the variation of* f *(cf. [37, §6 and §7]).*

Iitaka's Conjecture C has been established in many cases. For example,

- $C^+_{n,m}$ holds if the general fiber F of f has a good minimal model by [20], and
- $C_{n,m}$ holds if the general fiber F of f is of maximal Albanese dimension by [14].

A related conjecture, **Viehweg's Question Q(f)** (cf. [37, §7]) asks whether $f_*(\omega^k_{X/Y})$ big for some positive integer k, where $f : X \to Y$ is an algebraic fiber space of maximal variation, i.e., $\mathrm{Var}(f) = \dim(Y)$? It is known that a positive answer to $Q(f)$ implies $C^+_{n,m}$. Kawamata proved in [20] that $Q(f)$ holds when the general fiber F has a good minimal model. A question of Mori in [37, Remark 7.7] then asks if $Q(f)$ holds by assuming that the general fiber of the Iitaka fibration of F has a good minimal model. Hence as a corollary of the Theorem 31, we obtain a positive answer to Mori's question:

Corollary 33 *Let $f : X \to Y$ be an algebraic fiber space of normal projective varieties with general fiber F. Suppose that the general fiber of the Iitaka fibration of F has a good minimal model. Then Iitaka's Conjecture C holds on f.*

CHAPTER 4

BOUNDING VOLUMES OF SINGULAR FANO THREEFOLDS

Throughout this chapter, we work over field of complex numbers $\mathbb{C}$. We recall the definition of singularities of pairs and log $\mathbb{Q}$-Fano pairs.

Definition 34 *A pair (X, Δ) consists of a normal projective variety X and a boundary Δ, i.e., a $\mathbb{Q}$-divisor Δ with coefficients in $[0,1]$, such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\pi : Y \to X$ be a log resolution of (X, Δ), the discrepancy $a(E, X, \Delta)$ of a divisor E on Y with respect to the pair (X, Δ) is defined by $a(E, X, \Delta) = \mathrm{mult}_E(K_Y - \pi^*(K_X + \Delta))$. We say that (X, Δ) has only terminal (resp. canonical) singularities if $a(E, X, \Delta) > 0$ (resp. ≥ 0) for any π-exceptional divisor E on Y. We say that (X, Δ) is klt (resp. ϵ-klt for some $0 < \epsilon < 1$) if $a(E, X, \Delta) > -1$ (resp. $> -1 + \epsilon$) for any divisor E on Y. Note that smaller ϵ corresponds to worse singularities.*

A pair (X, Δ) is (weak) log $\mathbb{Q}$-Fano if the $\mathbb{Q}$-Cartier divisor $-(K_X + \Delta)$ is ample (resp. nef and big).

For a klt pair (X, Δ) with $\kappa(K_X + \Delta) = -\infty$, according to the log minimal model program, there exists a birational map $\phi : X \dashrightarrow Y$ and a morphism $Y \to Z$ such that for $\Delta' = \phi_*\Delta$, the pair (Y_z, Δ'_z) is log $\mathbb{Q}$-Fano with $\rho(Y_z) = 1$ for general $z \in Z$. In particular, log $\mathbb{Q}$-Fano pairs are the building blocks for pairs with negative Kodaira dimension. It is also expected that the set of mildly singular $\mathbb{Q}$-Fano varieties is bounded.

Definition 35 *We say that a collection of varieties $\{X_\lambda\}_{\lambda \in \Lambda}$ is bounded if there exists $h : \mathcal{X} \to S$ a morphism of finite type of Noetherian schemes such that for each X_λ, $X_\lambda \cong \mathcal{X}_s$ for some $s \in S$.*

For example, the set of all the n-dimensional smooth Fano manifolds is bounded by [28]. Boundedness is also known for terminal $\mathbb{Q}$-Fano $\mathbb{Q}$-factorial threefolds of Picard number one by [21] and for canonical $\mathbb{Q}$-Fano threefolds by [29]. However, if one considers the set of all klt $\mathbb{Q}$-Fano varieties with Picard number one of a given dimension, [35] and [39] have shown that birational boundedness fails. The problem is that the category of klt singularities is too big to be bounded since, for example, it contains finite quotients of arbitrarily large order. To get boundedness, one restricts to a smaller class of singularities, known as ϵ-klt singularities. Precisely we have the following conjecture due to A. Borisov, L. Borisov, and V. Alexeev, which is still open in dimension three and higher.

Conjecture 36 (***Borisov-Alexeev-Borisov Conjecture***) *Fix $0 < \epsilon < 1$, an integer $n > 0$, and consider the set of all n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pairs (X, Δ). The set of underlying varieties $\{X\}$ is bounded.*

A. Borisov and L. Borisov establish the B-A-B Conjecture for toric varieties in [10]. V. Alexeev establishes the two-dimensional B-A-B Conjecture in [1] with a simplified argument given in [2]. Our original motivation for studying the B-A-B Conjecture is that it is related to the conjectural termination of flips in the minimal model program. According to [9], the log minimal model program, the a.c.c.[1] for minimal log discrepancies, and the B-A-B Conjecture in dimension $\leq d$ implies termination of log flips in dimension $\leq d+1$ for effective pairs.

The following questions concerning log $\mathbb{Q}$-Fano pairs (X, Δ) are relevant to the B-A-B Conjecture:

(i) The Cartier index of $K_X + \Delta$ of an n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pair (X, Δ) is bounded from above by a fixed integer $r(n, \epsilon)$ depending only on $n = \dim X$ and ϵ;

(ii) The volume $\mathrm{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n$ of an n-dimensional ϵ-klt log $\mathbb{Q}$-Fano pair (X, Δ) is bounded from above by a fixed integer $M(n, \epsilon)$ depending only on $n = \dim X$ and ϵ;

(iii) (**Batyrev Conjecture**) For given positive integers n and r, consider the set of all n-dimensional klt log $\mathbb{Q}$-Fano pairs (X, Δ) with $r(K_X + \Delta)$ a Cartier divisor. The set of underlying varieties $\{X\}$ is bounded.

It is clear that the B-A-B Conjecture follows from (i) and (iii). Note that recently C. Hacon, J. M^{c}Kernan, and C. Xu have announced a proof of the Batyrev Conjecture (iii). In general it is very hard to establish (i). Ambro in [4] has proved (i) for toric singularities when the boundaries have standard coefficients $\{1 - \frac{1}{\ell} | \ell \in \mathbb{Z}_{\geq 1}\} \cup \{1\}$. A necessary condition for

[1] An a.c.c. (respectively d.c.c.) set is a set of real numbers satisfying the ascending (descending) chain condition, i.e., it contains no infinite strictly increasing (decreasing) sequences.

(i) to hold is that we need to restrict the coefficients of boundaries to be in a fixed d.c.c. set. A counterexample for the general statement is given by the set of pairs $(\mathbb{P}^1, \frac{1}{N}\{\mathrm{pt}\})$ for $N \geq 1$.

For the convenience of the reader, we include a well-known argument (to the experts) establishing the B-A-B Conjecture via condition (i) and (ii) in the cases $\Delta = 0$ or $\rho(X) = 1$.

Proposition 37 *Suppose that $\Delta = 0$ or $\rho(X) = 1$, then the B-A-B Conjecture holds if both (i) and (ii) above are true.*

Proof. Suppose that $\Delta = 0$ and let X be any ϵ-klt $\mathbb{Q}$-Fano variety of dimension n. The following statements together imply the B-A-B conjecture in this case:

1. The divisor $N(-K_X)$ is a very ample line bundle for a fixed N depending only on n and ϵ;
2. The set of Hilbert polynomials $\mathfrak{F} = \{P(t) = \chi(\mathcal{O}_X(-NK_X)^{\otimes t})\}$ associated to all n-dimensional ϵ-klt $\mathbb{Q}$-Fano varieties is finite.

Indeed, statements (1) and (2) imply that the set of n-dimensional ϵ-klt $\mathbb{Q}$-Fano varieties is contained in a finite union of Hilbert schemes $\coprod_{P(t)\in\mathfrak{F}} \mathcal{H}_{P(t)}$, where each $\mathcal{H}_{P(t)}$ is Noetherian.

From (i), there is an upper bound $r(n,\epsilon)$ of the Cartier index of K_X depending only on n and ϵ. It follows that rK_X is a line bundle for $r = r(n,\epsilon)$. By [23], $|-mrK_X|$ is base point free for any $m > 0$ divisible by a constant $N_1(n) > 0$ depending only on $n = \dim X$. Since $|-mrK_X|$ is ample and base point free for $m > 0$ sufficiently divisible, it defines a finite morphism. By [25, Theorem 5.9], the map induced by $|-lrK_X|$ is birational for any $l > 0$ divisible by a constant $N_2(n) > 0$ depending only on $n = \dim X$. Since a finite birational morphism of normal varieties is an isomorphism, it follows that there exists an effective embedding by $|M(-rK_X)|$ for some fixed $M > 0$ depending only on $n = \dim X$. Take $N = Mr$, we have (1).

By [27], the coefficients of the Hilbert polynomial $P(t) = h^0(\mathcal{O}_X(tH))$ of a polarized variety (X, H) with H an ample line bundle can be bounded by the intersection numbers $|H^n|$ and $|H^{n-1}.K_X|$. Since by (i) there exists an integer $r = r(n,\epsilon) > 0$ depending only on $n = \dim X$ and ϵ such that $-rK_X$ is an ample line bundle, set $H = -rK_X$ and apply (ii). It follows that there are only finitely many Hilbert polynomials for the set of anti-canonically polarized ϵ-klt Fano varieties $\{(X, -rK_X)\}$.

If $\rho(X) = 1$, then $-(K_X + \Delta)$ being ample implies that $-K_X$ is also ample. It is clear that X is also ϵ-klt and hence boundedness follows from the same proof as above. ■

An effective upper bound in (ii) is obtained for smooth Fano n-folds in [28] and for canonical $\mathbb{Q}$-Fano threefolds in [29]. In this work, we obtain an effective answer to question (ii) in dimension two, i.e., for log del Pezzo surfaces.

Theorem 38 *(Theorem 55) Let (X, Δ) be an ϵ-klt weak log del Pezzo surface. The anticanonical volume* $\mathrm{Vol}(-(K_X + \Delta)) = (K_X + \Delta)^2$ *satisfies*

$$(K_X + \Delta)^2 \leq \max\{64, \frac{8}{\epsilon} + 4\}.$$

Moreover, this upper bound is in a sharp form: There exists a sequence of ϵ-klt del Pezzo surfaces whose volume grows linearly with respect to $1/\epsilon$.

Let (X, Δ) be an ϵ-klt weak log del Pezzo surface and $X_{\min}$ be the minimal resolution of (X, Δ). Alexeev and Mori have shown in [2, Theorem 1.8] that $\rho(X_{\min}) \leq 128/\epsilon^5$. Also from [2, Lemma 1.2] (or see the proof of Theorem 55), an exceptional curve E on $X_{\min}$ over X has degree $1 \leq -E^2 \leq 2/\epsilon$. When $\Delta = 0$, since the Cartier index of K_X is bounded from above by the determinant of the intersection matrix $(E_i.E_j)$ of the exceptional curves E_i's on $X_{\min}$ over X, it follows that the Cartier index bound $r(2, \epsilon)$ in the statement (i) satisfies

$$r(2, \epsilon) \leq 2(2/\epsilon)^{128/\epsilon^5}. \qquad (\Diamond)$$

An upper bound of $(K_X + \Delta)^2$ is implicitly mentioned in [1] but not clearly written down. It is also not clear if the upper bound $(\Diamond)$ is optimal. In view of Theorem A, this seems unlikely.

As a second result, we also obtain an upper bound of the volumes for ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefolds of Picard number one. Recall that a variety X is $\mathbb{Q}$-factorial if each Weil divisor is $\mathbb{Q}$-Cartier.

Theorem 39 *(Theorem 72) Let (X, Δ) be an ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefold of $\rho(X) = 1$. The degree $-K_X^3$ satisfies*

$$-K_X^3 \leq \left(\frac{24M(2, \epsilon)R(2, \epsilon)}{\epsilon} + 12\right)^3,$$

where $R(2, \epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S) = 1$ and $M(2, \epsilon)$ is an upper bound of the volume $\mathrm{Vol}(-K_S) = K_S^2$ *for S any $\epsilon/2$-klt log* *del Pezzo surface of* $\rho(S) = 1$*. Note that* $M(2,\epsilon) \leq \max\{64, 16/\epsilon + 4\}$ *from Theorem A and* $R(2,\epsilon) \leq 2(4/\epsilon)^{128\cdot 2^5/\epsilon^5}$ *from* ($\diamondsuit$).

For a $\mathbb{Q}$-factorial ϵ-klt log $\mathbb{Q}$-Fano pair (X,Δ) of $\rho(X) = 1$, since $-(K_X + \Delta)^3 \leq -K_X^3$ and X is also ϵ-klt, by Theorem B we get an upper bound of the anticanonical volume $\mathrm{Vol}(-(K_X + \Delta)) = -(K_X + \Delta)^3$. However, it is not expected that the bound in Theorem B is sharp or in a sharp form.

Note that $\mathbb{Q}$-factoriality is a technical assumption. However, this condition is natural in the sense that starting from a smooth variety, each variety constructed by a step of the minimal model program remains $\mathbb{Q}$-factorial. In dimension two, normal surfaces with rational singularities, e.g., klt singularities, are always $\mathbb{Q}$-factorial.

Instead of using deformation theory of rational curves as in [29], the Riemann-Roch formula as in [21], or the sandwich argument of [1], we aim to create isolated non-klt centers by the method developed in [36]. The point is that deformation theory for rational curves on klt varieties is much harder and so far no effective Riemann-Roch formula is known for klt threefolds.

The rest of this chapter is organized as follows: In Section 4.1, we study non-klt centers. In Section 4.2, we illustrate the general method in [36] for obtaining an upper bound of anticanonical volumes in Theorem A and B. In Section 4.3, we review the theory of families of non-klt centers in [36]. In Section 4.4, we study weak log del Pezzo surfaces and prove Theorem A. In Section 4.5, we prove Theorem B.

4.1 Non-klt centers

For the theory of the singularities in the minimal model program, we refer to [30].

Definition 40 *Let* (X,Δ) *be a pair. A subvariety* $V \subseteq X$ *is called a* ***non-klt center*** *if it is the image of a divisor of discrepancy at most* -1*. A* ***non-klt place*** *is a valuation corresponding to a divisor of discrepancy at most* -1*. The* ***non-klt locus*** $\mathrm{Nklt}(X,\Delta) \subseteq X$ *is the union of the non-klt centers. If there is a unique non-klt place lying over the generic point of a non-klt center* V*, then we say that* V *is* ***exceptional****. If* (X,Δ) *is not klt along the generic point of a non-klt center* V*, then we say that* V *is* ***pure****.*

The non-klt places/centers here are the log canonical (lc) places/centers in [36].

A standard way of creating a non-klt center on an n-dimensional variety X is to find a very singular divisor: Fix $p \in X$ a smooth point, if Δ is a $\mathbb{Q}$-Cartier divisor on X with

$\mathrm{mult}_p\Delta \geq n$, then $p \in \mathrm{Nklt}(X, \Delta)$. Indeed, consider the blow up $\pi : Y = \mathrm{Bl}_p X \to X$ and let E be the unique exceptional divisor with $\pi(E) = p$, then the discrepancy

$$a(E, X, \Delta) = \mathrm{mult}_E(K_Y - \pi^*(K_X + \Delta)) = (n-1) - \mathrm{mult}_E(\pi^*(\Delta)) \leq -1,$$

as $n - 1 = \mathrm{mult}_E(K_Y - \pi^* K_X)$ and $\mathrm{mult}_E(\pi^*\Delta) = \mathrm{mult}_p\Delta \geq n$.

We can find singular divisors by the following lemma.

Lemma 41 *Let X be an n-dimensional complete complex variety and D be a divisor with the property that $h^i(X, \mathcal{O}(mD)) = O(m^{n-1})$ for all $i > 0$, e.g., D is big and nef. Fix a positive rational number α with $0 < \alpha^n < D^n$. For $m \gg 0$ and any $x \in X_{\mathrm{sm}}$, there exists a divisor $E_x \in |mD|$ with* $\mathrm{mult}_x(E_x) \geq m \cdot \alpha$.

Proof. This is [33, Proposition 1.1.31]. ∎

We will apply Lemma 41 to the case where (X, Δ) is an n-dimensional log $\mathbb{Q}$-Fano pair: Write $(-(K_X + \Delta))^n > (\omega n)^n$ for some rational number $\omega > 0$, then as the cohomology $h^i(X, \mathcal{O}(-m(K_X + \Delta))) = 0$ for $m > 0$ sufficiently divisible by the Kawamata-Viehweg vanishing theorem, we can find for each $p \in X_{\mathrm{sm}}$ an effective $\mathbb{Q}$-divisor Δ_p such that $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ and $\mathrm{mult}_p(\Delta_p) \geq n$. In particular, $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$.

The non-klt centers satisfy the following Connectedness Lemma of Kollár and Shokurov, which is simply a formal consequence of the Kawamata-Viehweg vanishing theorem and is the most important ingredient in this work.

Lemma 42 *Let (X, Δ) be a log pair. Let $f : X \to Z$ be a projective morphism with connected fibers such that the image of every component of Δ with negative coefficient is of codimension at least two in Z. If $-(K_X + \Delta)$ is big and nef over Z, then the intersection of* $\mathrm{Nklt}(X, \Delta)$ *with each fiber $X_z = f^{-1}(z)$ is connected.*

Proof. For simplicity, we assume that $Z = \mathrm{Spec}(\mathbb{C})$ is a point and (X, Δ) is log smooth, i.e., X is smooth and Δ has simple normal crossing support. Then the identity map of X is a log resolution of (X, Δ) and $\mathrm{Nklt}(X, \Delta) = \llcorner \Delta \lrcorner$. Consider the exact sequence

$$\cdots \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_{\llcorner \Delta \lrcorner}) \to H^1(X, \mathcal{O}_X(-\llcorner \Delta \lrcorner)) \to \cdots.$$

Since $-\llcorner \Delta \lrcorner = K_X + \{\Delta\} - (K_X + \Delta)$ and $(X, \{\Delta\})$ is klt, we have $H^1(X, \mathcal{O}_X(-\llcorner \Delta \lrcorner)) = 0$ by the Kawamata-Viehweg vanishing theorem as $-(K_X + \Delta)$ is nef and big. Since we know $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$, we see that $\mathrm{Nklt}(X, \Delta) = \llcorner \Delta \lrcorner$ is connected.

For the general case, see [13, Theorem 17.4]. ■

Here is an example showing that $-(K_X+\Delta)$ being nef and big is necessary in the Connectedness Lemma 42.

Example 43 *Let X be $\mathbb{P}^1\times\mathbb{P}^1$ and denote by F (resp. G) the fiber of the first (resp. second) projection to $\mathbb{P}^1$. Consider $\Delta_1=F_1+F_2$ the sum of two distinct fibers of the first projection to $\mathbb{P}^1$ and $\Delta_2=F+G$ the sum of two fibers with respect to the two different projections to $\mathbb{P}^1$. Then $\mathrm{Nklt}(X,\Delta_1)=F_1+F_2$ is not connected while $\mathrm{Nklt}(X,\Delta_2)=F+G$ is connected. Note that $-(K_X+\Delta_1)$ is nef but not big while $-(K_X+\Delta_2)$ is nef and big.*

Later on, we will produce not only non-klt centers but *isolated* non-klt centers. The following theorem is the main technique that allows us to cut down the dimension of non-klt centers.

Theorem 44 *([25, Theorem 6.8.1]) Let (X,Δ) be klt, projective and $x\in X$ a closed point. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $(X,\Delta+D)$ is log canonical in a neighborhood of x. Assume that $\mathrm{Nklt}(X,\Delta+D)=Z\cup Z'$ where Z is irreducible, $x\in Z$, and $x\notin Z'$. Set $k=\dim Z$. If H is an ample $\mathbb{Q}$-divisor on X such that $(H^k.Z)>k^k$, then there is an effective $\mathbb{Q}$-divisor $B\equiv H$ and rational numbers $1\gg\delta>0$ and $0<c<1$ such that*

(1) $(X,\Delta+(1-\delta)D+cB)$ is non-klt in a neighborhood of x, and

(2) $\mathrm{Nklt}(X,\Delta+(1-\delta)D+cB)=W\cup W'$ where W is irreducible, $x\in W$, $x\notin W'$ and $\dim W<\dim Z$.

4.2 A guiding example

The idea in [36] for obtaining an upper bound for the anticanonical volumes is to create isolated non-klt centers and then use the Connectedness Lemma 42: For simplicity, we assume that $\Delta=0$. Write $(-K_X)^n>(\omega n)^n$ for a positive rational number ω. For each $p\in X_{\mathrm{sm}}$, we can find an effective $\mathbb{Q}$-divisor $\Delta_p\sim_{\mathbb{Q}}-K_X/\omega$ such that $\mathrm{mult}_p\Delta_p\geq n$ and hence $p\in\mathrm{Nklt}(X,\Delta_p)$. The observation is that if $\omega\gg 0$, then for general $p\in X$, $p\in\mathrm{Nklt}(X,\Delta_p)$ can not be an isolated point. Indeed, if this is not true, then for two general points $p,q\in X$, the set $\mathrm{Nklt}(X,\Delta_p+\Delta_q)$ would contain $\{p,q\}$ as isolated non-klt centers. But the divisor $K_X+\Delta_p+\Delta_q\sim_{\mathbb{Q}}(1-\frac{2}{\omega})(-K_X)$ is nef and big for $\omega>2$. By the Connectedness Lemma 42, $\mathrm{Nklt}(X,\Delta_p+\Delta_q)$ must be connected; a contradiction.

Therefore, for general $p \in X$ the minimal non-klt center $V_p \subseteq \mathrm{Nklt}(X, \Delta_p)$ passing through p is typically positive dimensional. We would like to show that the restricted volume $\mathrm{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p is large when $\omega \gg 0$. Hence, we can cut down the dimension of non-klt centers by Theorem 44. After doing this finitely many times, we get isolated non-klt centers and we are done.

In general, it is hard to find a lower bound of the restricted volume $\mathrm{Vol}(-K_X|_{V_p})$ on the minimal non-klt center V_p. We illustrate M^cKernan's method by studying families of non-klt centers to obtain a lower bound of the restricted volumes on the non-klt center of an ϵ-klt log $\mathbb{Q}$-Fano variety via the following guiding example, cf. [36].

Example 45 *Let X be the projective cone over a rational normal curve of degree $d \geq 2$ with the unique singular point $O \in X$. The blow up $\pi : Y = \mathrm{Bl}_O X \to X$ is a resolution of X where Y is a $\mathbb{P}^1$-bundle $f : Y \to \mathbb{P}^1$ over $\mathbb{P}^1$:*

$$\begin{array}{ccccc} X & \xleftarrow{\ \pi\ } & Y & \supseteq & F_t \cong \mathbb{P}^1 \\ & & {\scriptstyle f}\downarrow & & \downarrow \\ & & \mathbb{P}^1 & \ni & t. \end{array}$$

It is easy to show that

(a) $K_Y = \pi^* K_X + (-1 + 2/d)E$, where E is the unique exceptional divisor and hence X is ϵ-klt for $\epsilon = 1/d$;

(b) X is $\mathbb{Q}$-factorial of Picard number one and $-K_X \sim_{\mathbb{Q}} (d+2)l$ is an ample $\mathbb{Q}$-Cartier divisor, where l is the class of a ruling of X. Hence X is an ϵ-klt del Pezzo surface;

(c) $\mathrm{Vol}(-K_X) = d + 4 + 4/d$ is a linear function of $d = 1/\epsilon$ and provides the required example in Theorem A.

Let $p \in X$ be a general point. Then p is not the vertex O and the unique ruling l_p passing through p is the non-klt center of the log pair (X, l_p), i.e., $l_p = \mathrm{Nklt}(X, l_p)$. Moreover, the proper transform F_p of l_p on Y is a fiber of the $\mathbb{P}^1$-bundle $f : Y \to \mathbb{P}^1$. In this case, the $\mathbb{P}^1$-bundle structure of Y is a covering family of non-klt centers of X since the map $\pi : Y \to X$ is dominant.

For $p, q \in X$ two general points, let l_p and l_q be the rulings passing through p and q respectively. Consider the pair $K_Y + (1-2/d)E = \pi^* K_X$. By the Connectedness Lemma 42, the non-klt locus $\mathrm{Nklt}(K_Y + (1-2/d)E + \pi^*(l_p + l_q))$ *containing $F_p \cup F_q$ is connected as*

$$-(K_Y + (1-2/d)E + \pi^*(l_p + l_q)) = -\pi^*(K_X + l_p + l_q) \equiv d\pi^* l$$

is nef and big. In fact, the fibers F_p and F_q are connected in $\mathrm{Nklt}(K_Y + (1-2/d)E + \pi^*(l_p + l_q))$ *by E as*

$$F_p \cup F_q \subseteq \mathrm{Nklt}(K_Y + (1-2/d)E + \pi^*(l_p + l_q)) \subseteq \pi^{-1}(\mathrm{Nklt}(K_X + l_p + l_q)) = F_p \cup F_q \cup E,$$

where the second inclusion follows from the definition of non-klt centers. In particular,

$$\mathrm{mult}_E(\pi^*(l_p + l_q)) \geq \frac{2}{d} = 2\epsilon.$$

By symmetry, $\pi^* l_p$ must contribute multiplicity at least $1/d = \epsilon$ to the component E (and in fact is exactly $1/d$ in this case), i.e.,

$$\pi^* l_p \geq \epsilon E. \tag{4.1}$$

Note that

$$l_p \sim_{\mathbb{Q}} \frac{-K_X}{\sqrt{d \cdot \mathrm{Vol}(-K_X)}}. \tag{4.2}$$

By intersecting both sides of (4.1) *with a general fiber F of $f : Y \to \mathbb{P}^1$, we get for the ruling $l = \pi_*(F)$,*

$$\frac{1}{\sqrt{d \cdot \mathrm{Vol}(-K_X)}} \deg_l(-K_X) = \pi^* l_p . F \geq \epsilon E . F. \tag{4.3}$$

Since F is a general fiber meeting the horizontal divisor E at a smooth point, $E.F \geq 1$. (In this case $E.F = 1$.) Combining all of these, we obtain a lower bound of the restricted volume $\deg_l(-K_X)$*,*

$$\deg_l(-K_X) \geq \epsilon \sqrt{d \cdot \mathrm{Vol}(-K_X)}.$$

Note that since in this case $\deg_l(-K_X) = -K_X.l = -K_Y.\pi^* l \leq 2$*, it follows that the anticaonical volume* $\mathrm{Vol}(-K_X) = K_X^2 \leq 4d = 4/\epsilon$*.*

In summary, the method of getting an upper bound of the anticanonical volumes is to obtain a lower bound of the restricted volume $\mathrm{Vol}(-(K_X + \Delta)|_{V_p})$ on the non-klt centers V_p, which can be outlined in the following steps:

- Suppose that $\mathrm{Vol}(-(K_X + \Delta)) = (-(K_X + \Delta))^n > (\omega n)^n$ for a positive rational number ω. We will show that $\omega > 0$ can not be arbitrarily large.

- For general $p \in X$, choose
$$\Delta_p \sim_{\mathbb{Q}} \frac{-(K_X + \Delta)}{\omega},$$
so that $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$. Let $V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ be the minimal non-klt center containing p.

- Construct covering families of non-klt centers by "lining up" (part of the) non-klt centers $\{V_p\}$, see Section 4.3. This is the generalization of the $\mathbb{P}^1$-bundle structure in the Example 45 and is called a *covering families of tigers* in [36].

- Use the Connectedness Lemma 42 to obtain a lower bound of the restricted volume
$$\mathrm{Vol}(-(K_X + \Delta)|_{V_p}) = (-(K_X + \Delta)|_{V_p}))^{\dim V_p},$$
on the non-klt center V_p in terms of ω and ϵ. This is the most technical part.

- If $\omega \gg 0$, then we cut down the dimension of non-klt centers by Theorem 44. After finitely many steps, we get isolated non-klt centers and hence a contradiction to the Connectedness Lemma 42.

The difficulty of this argument arises in dimension three in many places. First of all, the non-klt centers can be of dimension one or two and we have to deal with them case by case. When we have one dimensional covering families of tigers, it is subtle to detect the contribution of the ϵ-klt condition from some horizontal subvariety, which is analogous to the exceptional curve E in Example 45. This is done by applying a differentiation argument to construct a better behaved covering family of tigers, see 4.5.3. In case we have two-dimensional non-klt centers, complications arise for computing intersection numbers as the total space Y of a covering family of tigers is in general not $\mathbb{Q}$-factorial. This can be fixed by replacing Y with a suitable birational model. To finish the proof, we also need to run a relative minimal model on the covering family of tigers and study the geometry of all possible outcomes.

4.3 Covering families of tigers

The main reference for this section is [36].

Definition 46 *([36, Definition 3.1]) Let (X, Δ) be a log pair with X projective and D a $\mathbb{Q}$-Cartier divisor. We say that pairs of the form (Δ_t, V_t) form a* ***covering family of tigers*** *of dimension k and weight ω if all of the following hold:*

1. *there is a projective morphism $f : Y \to B$ of normal projective varieties such that the general fiber of f over $t \in B$ is V_t;*

2. *there is a morphism of B to the Hilbert scheme of X such that B is the normalization of its image and f is obtained by taking the normalization of the universal family;*

3. *if $\pi : Y \to X$ is the natural morphism, then $\pi(V_t)$ is a minimal pure non-klt center of $K_X + \Delta + \Delta_t$;*

4. *π is generically finite and dominant;*

5. *$\Delta_t \sim_{\mathbb{Q}} D/\omega$, where Δ_t is effective;*

6. *the dimension of V_t is k.*

Note that by definition $k \leq \dim X - 1$ and $\pi|_{V_t} : V_t \to \pi(V_t)$ is *finite* and *birational*. The covering family of tigers is illustrated in the following diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\ \pi\ } & Y & \supseteq & V_t \\ & & {\scriptstyle f}\downarrow & & \downarrow \\ & & B & \ni & t. \end{array}$$

We will sometimes also refer to V_t as the minimal non-klt center of $(X, \Delta + \Delta_t)$.

For (X, Δ) a log $\mathbb{Q}$-Fano variety, we will always assume that $D = -\lambda(K_X + \Delta)$ for some $\lambda > 0$. In particular, D is assumed to be big and semiample.

The existence of a covering family of tigers is achieved by constructing non-klt centers at general points of X and then fitting a subcollection of them into a fiber space. In order to fit the non-klt centers into a family, we use exceptional non-klt centers so that we patch up the unique non-klt place associated to each of them. The following lemma allows us to create exceptional non-klt centers.

Lemma 47 *Let (X, Δ) be a log pair and let D be a big and semiample $\mathbb{Q}$-Cartier divisor. Write $D^n > (\omega n)^n$ for some positive rational number ω. In order to find an upper bound of ω and hence an upper bound of $\mathrm{Vol}(D) = D^n$, for every $p \in X_{\mathrm{sm}}$ we may assume that there is a divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ such that the unique minimal non-klt center $V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ containing p is exceptional.*

Proof. By Lemma 41, for any $p \in X_{\mathrm{sm}}$ we can find an effective divisor $\Delta'_p \sim_{\mathbb{Q}} \frac{D}{\omega}$ such that $\mathrm{mult}_p\Delta'_p \geq n$ and hence $p \in \mathrm{Nklt}(X, \Delta + \Delta'_p)$.

Fix $p \in X_{\mathrm{sm}}$, pick $0 < \delta_p \leq 1$ the unique rational number such that $(X, \Delta + \delta_p\Delta'_p)$ is log canonical but not klt at p. By [3, Proposition 3.2, Lemma 3.4], we can find an effective divisor $M_p \sim_{\mathbb{Q}} D$ and some rational number $a > 0$ such that for any rational number $0 < \mu < 1$, the pair $(X, (1-\mu)(\Delta + \delta_p\Delta'_p) + \mu\Delta + \mu a M_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. If we write

$$\Delta_p := (1-\mu)\delta_p\Delta'_p + \mu a M_p \sim_{\mathbb{Q}} \frac{1}{\omega'_p} D,$$

then

$$\omega'_p = \frac{\omega}{(1-\mu)\delta_p + \mu a \omega},$$

and $(1-\mu)\delta_p + \mu a\omega < 1 + 1/n$ for any $n \geq 1$ if we pick $0 < \mu \ll 1$ sufficiently small. Hence $\omega'_p > \omega/(1+1/n)$. Since D is semiample, by adding a small multiple of D to Δ_p we have $\Delta_p \sim_{\mathbb{Q}} D/\omega_n$ for $\omega_n = \omega/(1+2/n)$, and $(X, \Delta + \Delta_p)$ has a unique minimal non-klt center V_p passing through p which is exceptional. If there exists an upper bound of ω_n independent of n, then by taking $n \to \infty$, we get the same upper bound of ω. ■

The following proposition is the construction of the covering family of tigers, see [36, Lemma 3.2] or [42, Lemma 3.2].

Proposition 48 *Let (X, Δ) and Δ_p be the same as in Lemma 47. Then there exists a covering family of tigers $\pi : Y \to X$ of weight ω with $V_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ the unique minimal non-klt center passing through p.*

Proof. Choose $m > 0$ an integer such that mD/ω is integral and Cartier and let B be the Zariski closure of points $\{m\Delta_p | p \in X_{\mathrm{sm}}\} \in |mD/\omega|$. Replace B by an irreducible component which contains an uncountable subset Q of B such that the set $\{p \in X | \Delta_p \in Q\}$ is dense in X. This is possible since the Δ_p's cover X. Let $H \subseteq X \times |mD/\omega|$ be the universal family of divisors defined by the incidence relation and $H_B \to B$ the restriction to B. Take a log resolution of $H_B \subseteq X \times B$ over the generic point of B and extend it over an open subset U of B. By assumption the log resolution over the generic point of B has a unique exceptional divisor of discrepancy -1, since this is true over $Q \subseteq B$. Let Y be the image of this unique exceptional divisor in $X \times B$ with the natural projection map $\pi : Y \to X$. By construction $\pi : Y \to X$ dominates X.

Possibly taking a finite cover of B and passing to an open subset of B, we may assume that any fiber V_t of $f : Y \to B$ over $t \in B$ is a non-klt center of $K_X + \Delta + \Delta_t$. Possibly passing to an open subset of B, we may assume that $f : Y \to B$ is flat and B maps into the Hilbert scheme. Replace B by the normalization of the closure of its image in the Hilbert scheme and Y by the normalization of the pullback of the universal family. After possibly cutting by hyperplanes in B, we may assume that π is generically finite and dominant. The resulting family is the required covering family of tigers. ■

In fact, the original construction of covering families of tigers is carried out in a more general setting. For a topological space X, we say that a subset P is *countably dense* if P is not contained in the union of countable many closed subsets of X.

Corollary 49 *Let (X, Δ) be a log pair and let D be a big $\mathbb{Q}$-Cartier divisor. Let ω be a positive rational number. Let P be a countably dense subset of X. If for every point $p \in P$ we may find a pair (Δ_p, V_p) such that V_p is a pure non-klt center of $K_X + \Delta + \Delta_p$, where $\Delta_p \sim_{\mathbb{Q}} D/\omega_p$ for some $\omega_p > \omega$, then we may find a covering family of tigers of weight ω together with a countably dense subset Q of P such that for all $q \in Q$, V_q is a fiber of π.*

Proof. See [36, Lemma 3.2] or [42, Lemma 3.2]. ■

As noted in Example 45, we can assume that the covering families of tigers under our consideration are always positive dimensional.

Lemma 50 *Let (X, Δ) be a projective klt pair and $D = -(K_X + \Delta)$ be a big and nef $\mathbb{Q}$-Cartier divisor. A covering family of tigers (Δ_t, V_t) of weight $\omega > 2$ is positive dimensional, i.e., we have $k = \dim V_t > 0$.*

Proof. This is [36, Lemma 3.4] and we include the proof for the convenience of the reader. Suppose that there exists a zero-dimensional covering family of tigers of weight $\omega > 2$. For p_1 and p_2 general, there are divisors Δ_1 and Δ_2 with $\Delta_i \sim_{\mathbb{Q}} D/\omega$ such that p_i is an isolated non-klt center of $K_X + \Delta + \Delta_i$. As p_1 and p_2 are general, it follows that Δ_2 does not contain p_1 and $\mathrm{Nklt}(X, \Delta + \Delta_1 + \Delta_2)$ contains p_1 and p_2 as disconnected non-klt centers. But $-(K_X + \Delta + \Delta_1 + \Delta_2) \sim (1 - \frac{2}{\omega})D$ is nef and big if $\omega > 2$. This contradicts Lemma 42. ■

Recall that we want to cut down the dimension of non-klt centers via Theorem 44. To do so, we study the associated covering families of tigers and obtain a lower bound of restricted volumes on the non-klt centers. If the new non-klt centers after cutting down the dimension are still positive dimensional, then we have to create new covering families of tigers associated to these new non-klt centers and repeat the process. The following proposition enables us to create covering families of tigers of new non-klt centers after cutting down the dimension.

Proposition 51 *Let (X, Δ) be a log pair and let D be a $\mathbb{Q}$-Cartier divisor of the form $A + E$ where A is ample and E is effective. Let (Δ_t, V_t) be a covering family of tigers of weight ω and dimension k. Let A_t be $A|_{V_t}$. If there is an open subset $U \subseteq B$ such that for all $t \in U$ we may find a covering family of tigers $(\Gamma_{t,s}, W_{t,s})$ on V_t of weight ω' with respect to A_t, then for (X, Δ) we can find a covering family of tigers (Γ_s, W_s) of dimension less than k and weight*

$$\omega'' = \frac{1}{1/\omega + 1/\omega'} = \frac{\omega\omega'}{\omega + \omega'}.$$

Proof. This is [36, Lemma 5.3]. ∎

We will apply Proposition 51 with the ample divisor $D = -(K_X + \Delta)$. In the process of obtaining lower bound of the restricted volume on the non-klt centers, if we have one-dimensional non-klt centers, then we can control the restricted volume of D, cf. [36, Lemma 5.3].

Corollary 52 *Let (X, Δ) be a log pair and let D be an ample divisor. Let (Δ_t, V_t) be a covering family of tigers of weight $\omega > 2$ and dimension one. Then $\deg(D|_{V_t}) \leq 2\omega/(\omega - 2)$.*

Proof. Suppose that $\deg(D|_{V_t}) > 2\omega/(\omega - 2)$. By Lemma 47 and Corollary 49, we may find a covering family $(\Gamma_{t,s}, W_{s,t})$ of tigers of weight $\omega' > 2\omega/(\omega - 2)$ and dimension zero on V_t. By Proposition 51, there exists a covering family of tigers of dimension zero and weight

$$\omega'' = \frac{\omega\omega'}{\omega + \omega'} > 2,$$

for X. This contradicts Lemma 50. ∎

4.4 Log Del Pezzo surfaces

Let (X, Δ) be an ϵ-klt weak log del Pezzo surface. The minimal resolution $\pi : Y \to X$ of (X, Δ) is the unique proper birational morphism such that Y is a smooth projective surface and $K_Y + \Delta_Y = \pi^*(K_X + \Delta)$ for some effective $\mathbb{Q}$-divisor Δ_Y on Y. Note that minimal resolutions always exist for two-dimensional log pairs. It is easy to see that (Y, Δ_Y) is also an ϵ-klt weak log del Pezzo surface with volume

$$\mathrm{Vol}(Y, \Delta_Y) = (K_Y + \Delta_Y)^2 = (K_X + \Delta_X)^2 = \mathrm{Vol}(X, \Delta_X).$$

Replacing (X, Δ) by its minimal resolution, we can assume that X is smooth.

Write $(K_X + \Delta)^2 > (2\omega)^2$. For a general point $p \in X$, let $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ be an effective $\mathbb{Q}$-divisor constructed from Lemma 41 such that $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$. Assume that $\omega > 2$. By Lemma 50, the unique minimal non-klt center F_p of $(X, \Delta + \Delta_p)$ containing p is one dimensional. Note that for general $p \in X$, $F_p \leq \Delta_p$.

Lemma 53 *For a very general point $p \in X$, the numerical class $F := F_p$ on X is well-defined and F is nef.*

Proof. The effective integral one cycles F_p satisfy $F_p \leq \Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ and hence form a bounded set in the Mori cone of curves. As $\mathbb{C}$ is uncountable, for $p \in X$ a very general point the numerical class $F := F_p$ is well-defined. Since $\{F_p\}$ moves, the class F is nef. ∎

The following lemma shows that if we assume the weight ω is large, then the non-klt centers $\{F_p\}$ on X already possess a nearly fiber bundle structure analogous to a covering family of tigers.

Lemma 54 *Assume that $\omega > 3$, then $F^2 = 0$, i.e. $F_p \cap F_q = \emptyset$ for $p, q \in X$ two very general points.*

Proof. Assume that $F_p \cap F_q \neq \emptyset$ for $p, q \in X$ two very general points. We can assume that $p \notin \Delta_q$ as $p \in X$ is very general. Since by Lemma 54 the curve class $F = F_p$ is nef, for $H = -(K_X + \Delta)/\omega$ we have

$$1 \leq F_p.F_q = F_p.F \leq \Delta_p.F = \deg(H|_{F_p}),$$

where the first inequality is true since X is smooth. Since H is big and nef, we can cut down the dimension of the non-klt centers by Theorem 44[2].

To be precise, pick $0 < \delta_1 \leq 1$ such that the pair $(X, \Delta + \delta_1 \Delta_p)$ is log canonical but not klt at p. If $(X, \Delta + \delta_1 \Delta_p) = \{p\}$, then this contradicts the Connected Lemma

[2]By adding a small multiple of $-(K_X + \Delta)$, we may assume that the inequality $\deg(H|_{F_q}) \geq 1$ is strict with a smaller modified ω and hence Theorem 44 applies.

42 as $p \notin \Delta_q$ and the non-klt locus $\mathrm{Nklt}(X, \Delta + \delta_1\Delta_p + \Delta_q)$ containing p and F_q is disconnected, while the divisor $-(K_X + \Delta + \delta_1\Delta_p + \Delta_q)$ is nef and big. Hence we may assume that $\mathrm{Nklt}(X, \Delta + \delta_1\Delta_p)$ is one dimensional in a neighborhood of p. In particular, $F_p \subseteq \mathrm{Nklt}(X, \Delta + \delta_1\Delta_p)$ is the minimal non-klt center containing p. By Theorem 44, there exists rational numbers $0 < \delta \ll 1$, $0 < c < 1$, and an effective $\mathbb{Q}$-divisor $B_p \equiv H$ such that $\mathrm{Nklt}(X, \Delta + (1-\delta)\delta_1\Delta_p + cB_p) = \{p\}$ in a neighborhood of p. It follows that the set of non-klt centers $\mathrm{Nklt}(X, \Delta + (1-\delta)\delta_1\Delta_p + cB_p + \Delta_q)$ containing p and F_q is disconnected but the divisor $-(K_X + \Delta + (1-\delta)\delta_1\Delta_p + cB_p + \Delta_q)$ is nef and big as $\omega > 3$. This again contradicts the Connected Lemma 42. ■

Theorem 55 *Let (X, Δ) be an ϵ-klt weak log del Pezzo surface. Then the anticanonical volume* $\mathrm{Vol}((-K_X + \Delta)) = (K_X + \Delta)^2$ *satisfies*

$$(K_X + \Delta)^2 \leq \max\{64, \frac{8}{\epsilon} + 4\}.$$

Proof. Replacing (X, Δ) by its minimal resolution, we may assume that X is smooth. Write $(K_X + \Delta)^2 > (2\omega)^2$. For each general point $p \in X$, by Lemma 41, there exists an effective $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} -(K_X + \Delta)/\omega$ such that $p \in \mathrm{Nklt}(X, \Delta + \Delta_p)$. From Lemma 50, we may assume that $\omega > 2$ and the unique minimal non-klt center $F_p \subseteq \mathrm{Nklt}(X, \Delta + \Delta_p)$ containing p is one dimensional. Note that $F_p \leq \Delta_p$ for general $p \in X$. By Lemma 53 and 54, we may assume that $\omega > 3$ and for very general $p \in X$ the numerical class F of F_p is well-defined and nef with $F^2 = 0$.

For two very general points $p, q \in X$, $\Delta_p.\Delta_q > 0$ and hence $F_p = \mathrm{Supp}(F_p) \subsetneq \mathrm{Supp}(\Delta_p)$: Otherwise $\Delta_q \equiv \Delta_p \leq NF_p$ for some $N > 0$ and $0 < \Delta_p.\Delta_q \leq N^2F_p^2 = N^2F^2 = 0$, a contradiction. By the Connectedness Lemma 42, $\mathrm{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q$ is connected. Denote $E_p = \mathrm{Supp}(\Delta_p) - F_p \neq 0$. By Lemma 54, $F_p \cap F_q = \emptyset$ and hence E_p must contain a connected curve $E \leq E_p$ such that $F_p.E \neq 0$, $F_q.E \neq 0$, and the set $\mathrm{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q \cup E$. Furthermore, we can assume that E is irreducible since $E.F_q \neq 0$ as $F_q \equiv F_p$ for $q \in X$ a very general point.

Suppose that $E^2 \geq 0$ and hence E is nef. Since $\mathrm{Nklt}(X, \Delta + \Delta_p + \Delta_q) \supseteq F_p \cup F_q \cup E$, we have $\Delta + \Delta_p + \Delta_q \geq E$ and $(\Delta + \Delta_p + \Delta_q - E).E \geq 0$. For $H = -(K_X + \Delta)/\omega$, we see that

$$\begin{aligned} 2 \geq 2 - 2g_a(E) \geq & -(K_X + E).E - (\Delta + \Delta_p + \Delta_q - E).E \\ = & -(K_X + \Delta + \Delta_p + \Delta_q).E \\ = & (\omega - 2)H.E. \end{aligned}$$

Write $\Delta_p = \Delta'_p + \alpha E$ where $\Delta'_p \wedge E = 0$, $\Delta'_p \geq F_p$, and $\alpha > 0$, we have

$$H.E = \Delta_p.E = (\Delta'_p + \alpha E).E \geq F_p.E \geq 1.$$

The last inequality follows from the fact that X is smooth and $F_p.E > 0$. Combine the two inequalities above, we obtain $\omega \leq 4$.

Hence we may assume that $E^2 < 0$, and thus

$$\begin{aligned} -2 \leq 2g_a(E) - 2 &= (K_X + E).E \\ &= (K_X + \Delta).E + (1 - \epsilon - a_E)E^2 - \Delta'.E + \epsilon E^2 \leq \epsilon E^2, \end{aligned}$$

where $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [0, 1-\epsilon)$ by the ϵ-klt condition. This implies that $1 \leq -E^2 \leq 2/\epsilon$, where the first inequality follows from the fact that $E^2 \in \mathbb{Z}$ as X is smooth. Since $F^2 = 0$ for F the numerical class of F_p where $p \in X$ is very general, by Nakai's criterion the divisor $H_s = F + sE$ with $0 < s \leq 1/(-E^2)$ is nef and big. By the Hodge index theorem (see [18, V 1.1.9(a)]), we get the inequality

$$(K_X + \Delta)^2 \leq \frac{(-(K_X + \Delta).H_s)^2}{H_s^2}. \tag{4.4}$$

From $\Delta.F \geq 0$ and $F^2 = 0$, we have that

$$-(K_X + \Delta).F \leq -(K_X + F).F \leq 2. \tag{4.5}$$

Also for $\Delta = \Delta' + a_E E$ with $\Delta' \wedge E = 0$ and $a_E \in [0, 1-\epsilon)$, we have that

$$\begin{aligned} -(K_X + \Delta).E =& - K_X.E - \Delta'.E - a_E E^2 \\ &\leq E^2 + 2 - a_E E^2 = (a_E - 1)(-E^2) + 2 \leq 2 - \epsilon(-E^2). \end{aligned} \tag{4.3}$$

Put $s = 1/(-E^2)$, all together we get

$$\begin{aligned} (K_X + \Delta)^2 &\leq \frac{(-(K_X + \Delta).(F + sE))^2}{H_s^2} \\ &\leq \frac{(2 + s(2 - \epsilon(-E^2)))^2}{2sE.F + s^2E^2} \\ &\leq (-E^2)(2 - \epsilon + \frac{2}{-E^2})^2 \\ &= (-E^2)(2-\epsilon)^2 + 4(2-\epsilon) + \frac{4}{-E^2} \\ &\leq \frac{2}{\epsilon}(2-\epsilon)^2 + 4(2-\epsilon) + 4 \\ &= \frac{8}{\epsilon} + 4 - 2\epsilon \end{aligned}$$

where the first inequality is (4.4), the second inequality follows from (4.5), (4.3), and $F^2 = 0$, the third inequality is given by ignoring the term $sE.F \geq 0$, and the last inequality uses $1 \leq -E^2 \leq 2/\epsilon$. ■

Remark 56 *Note that by applying Corollary 52 one can only obtain an upper bound of order* $1/\epsilon^2$. *Hence Theorem 55 is a nontrivial result.*

4.5 Log Fano threefolds of Picard number one

Let (X, Δ) be an ϵ-klt $\mathbb{Q}$-factorial log $\mathbb{Q}$-Fano threefold of Picard number $\rho(X) = 1$. Note that by hypothesis X is ϵ-klt and $-K_X$ is ample. Moreover, we have the relation that $-K_X^3 \geq \mathrm{Vol}(-(K_X + \Delta)) = -(K_X + \Delta)^3$. Hence it is sufficient to assume that X is an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number $\rho(X) = 1$ and to find an upper bound of $\mathrm{Vol}(-K_X) = -K_X^3$. We will obtain an upper bound of the anticanonical volumes by studying covering families of tigers. The weight of any covering families of tigers in our study will always be the weight with respect to $-K_X$.

Let X be an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number $\rho(X) = 1$ and write the anticanonical volume $\mathrm{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω. Denote $D = -2K_X$, we have $D^3 > (6\omega)^3$. By Lemma 41, we can fix an affine open subset $U \subseteq X$ such that for each $p \in U$ there exists an effective divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ with $\mathrm{mult}_p \Delta_p \geq 6$. We pick divisors Δ_p's in the following systematic way so that we can control their multiplicities uniformly.

4.5.1 Construction

Let $\Delta_U \subseteq U \times U$ be the diagonal and $\mathcal{I}_{\mathcal{Z}}$ be the ideal sheaf of $\mathcal{Z} = \overline{\Delta_U} \subseteq X \times U$. For each $p \in U$, by the existence of $\mathbb{Q}$-divisor $\Delta_p \sim_{\mathbb{Q}} D/\omega$ with $\mathrm{mult}_p \Delta_p \geq 6$, there exists $m_p > 0$ such that $L_{m_p} = m_p D/\omega$ is Cartier and $H^0(X, L_{m_p} \otimes \mathcal{I}_p^{\otimes 6m_p}) \neq 0$. In particular, we can write $U = \cup U_m$ where $m > 0$ runs through all sufficiently divisible integers such that $L_m = mD/\omega$ is Cartier and $U_m = \{p \in U | H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}) \neq 0\}$. Moreover, each U_m is locally closed in X by [18, III, Theorem 12.8] and $X = \cup \overline{U_m}$. Since the base field $\mathbb{C}$ is uncountable, X can not be a countable union of locally closed subsets. Thus there exists some $m > 0$ such that U_m is dense in X.

Fix an $m > 0$ such that $L_m = mD/\omega$ is Cartier and $U_m = \{p \in U | H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}) \neq 0\}$ is dense in X. Denote $\mathrm{pr}_X : X \times U \to X$ and $\mathrm{pr}_U : X \times U \to U$ the projection maps. Since

$\mathrm{pr}_U : X \times U \to U$ is flat, by [18, III,Theorem 12.11], after restricting to a smaller open affine subset of U, we can assume that the map

$$(\mathrm{pr}_U)_*(\mathrm{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \otimes \mathbb{C}(p) \to H^0(X, L_m \otimes \mathcal{I}_p^{\otimes 6m}),$$

is an isomorphism for each $p \in U$ where $\mathcal{I}_p$ is the ideal sheaf of $p \in U$. Since U_m is dense in U, the sheaf $(\mathrm{pr}_U)_*(\mathrm{pr}_X^* L_m \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ on U and hence $H^0(X \otimes U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ as U is affine. Let $s \in H^0(X \otimes U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with $F = \mathrm{div}(s)$ the corresponding divisor on $X \times U$. For each $p \in U$, denote $F_p = F \cap (X \times \{p\})$ the associated divisor on $X \cong X \times \{p\}$. Since $\mathrm{mult}_{\mathcal{Z}}(F) \geq 6m$, by Lemma 57 below, the $\mathbb{Q}$-divisor $\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega$ on X satisfies $\mathrm{mult}_p \Delta_p \geq 6$ for general $p \in U$.

Lemma 57 *([33, Lemma 5.2.11]) Let $g : M \to T$ be a morphism of smooth varieties, and suppose that $\mathcal{Z} \subseteq M$ is an irreducible subvariety dominating T:*

$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & M \\ & \searrow & \swarrow_{g} \\ & T. & \end{array}$$

Let $F \subseteq M$ be an effective divisor. For a general point $t \in T$ and an irreducible component $\mathcal{Z}'_t \subseteq \mathcal{Z}_t$, $\mathrm{mult}_{\mathcal{Z}'_t}(M_t, F_t) = \mathrm{mult}_{\mathcal{Z}}(M, F)$, *where* $\mathrm{mult}_{\mathcal{Z}}(M, F)$ *is the multiplicity of the divisor F on M along a general point of the irreducible subvariety $\mathcal{Z} \subseteq M$ and similarly for* $\mathrm{mult}_{\mathcal{Z}'_t}(M_t, F_t)$.

For a given collection of $\mathbb{Q}$-divisors $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega | p \in U \text{ general}\}$ associated to a nonzero section in $H^0(X \otimes U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ as above, by Lemma 47, we can modify the Δ_p's so that the unique non-klt centers $V_p \subseteq \mathrm{Nklt}(X, \Delta_p)$ passing through p are exceptional. By Lemma 48 (or in general Corollary 49), we can construct covering families of tigers from these divisors.

In order to obtain an upper bound of ω, which is sufficient for bounding the anticanonical volumes, we will pick up a "well-behaved" nonzero section $s \in H^0(X \otimes U, p^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ and study the corresponding covering families of tigers.

4.5.2 Cases

By Section 4.5.1, there exists an open affine subset $U \subseteq X$ and an integer $m > 0$ such that $H^0(X \otimes U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. Let $s \in H^0(X \times U, \mathrm{pr}_X^* L \times \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a nonzero section with divisor $F = \text{div}(s)$ on $X \times U$ and $\{\Delta_p = F_p/m \sim_{\mathbb{Q}} D/\omega | p \in U\}$ be the associated collection of $\mathbb{Q}$-divisors. We consider two cases:

1. (**Small multiplicity**) For each irreducible component $\mathcal{W}$ of $\text{Supp}(F)$ passing through $\mathcal{Z}$, $\text{mult}_{\mathcal{W}}(F) \leq 3m$, i.e., for general $p \in U$ we have $\text{mult}_W(\Delta_p) \leq 3$ for any irreducible component W of $\text{Supp}(\Delta_p)$ passing through p. After differentiating F, we will construct a "well-behaved" covering family of tigers of dimension one. We will derive an upper bound of ω by studying this covering family of tigers. See Section 4.5.3.

2. (**Big multiplicity**) There exists an irreducible component $\mathcal{W}$ of $\text{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\text{mult}_{\mathcal{W}}(F) > 3m$, i.e., for general $p \in U$ we have $\text{mult}_W(\Delta_p) > 3$ for some irreducible component W of $\text{Supp}(\Delta_p)$ passing through p. We will construct a covering family of tigers of dimension two and derive an upper bound of ω by studying the geometry of this covering family of tigers. See Section 4.5.4.

To pick a "well-behaved" nonzero section in $H^0(X \otimes U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$, we will apply the following proposition.

Proposition 58 *([33, Proposition 5.2.13]) Let X and U be smooth irreducible varieties, with U affine, and suppose that $\mathcal{Z} \subseteq \mathcal{W} \subseteq X \times U$ are irreducible subvarieties such that $\mathcal{W}$ dominates X. Fix a line bundle L on X, and suppose we are given a divisor $F \in |\text{pr}_X^*(L)|$ on $X \times U$. Write $l = \text{mult}_{\mathcal{Z}}(F)$ and $k = \text{mult}_{\mathcal{W}}(F)$. After differentiating in the parameter directions, there exists a divisor $F' \in |\text{pr}_X^*(L)|$ on $X \times U$ with the property that* $\text{mult}_{\mathcal{Z}}(F') \geq l - k$, and $\mathcal{W} \not\subseteq \text{Supp}(F')$.

4.5.3 Small multiplicity

Let X be an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number one and write the anticanonical volume $\text{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω. Denote $D = -2K_X$, we have $D^3 > (6\omega)^3$. By Section 4.5.1, there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \text{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \text{div}(s)$ on $X \times U$.

Proposition 59 *With the set up above. Assume that $\omega > 4$. If we are in the case where all the irreducible components $\mathcal{W}$ of* $\mathrm{Supp}(F)$ *passing through $\mathcal{Z}$ satisfy* $\mathrm{mult}_{\mathcal{W}}(F) \leq 3m$, *then $\omega < 8/\epsilon + 4$. In particular, there is an upper bound for the volume*

$$\mathrm{Vol}(-K_X) = -K_X^3 \leq (\frac{24}{\epsilon} + 12)^3.$$

Proof. Let M be the maximum of $\mathrm{mult}_{\mathcal{W}}(F)$ among all the irreducible components $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$. Then $M \leq 3m$ by the hypothesis. For a fixed irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$, we can apply Proposition 58 to F. We obtain a divisor $F' \in |\mathrm{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ with the property that

$$\mathrm{mult}_{\mathcal{Z}}(F') \geq (6m - M) \geq 3m, \text{ and } \mathcal{W} \nsubseteq \mathrm{Supp}(F').$$

Since there are only finitely many irreducible components of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$, by taking a generic differentiation, it follows that for general $F'' \in |\mathrm{pr}_X^*(L) \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ we have $\mathcal{W} \nsubseteq \mathrm{Supp}(F'')$ for any irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$. In particular, the base locus $\mathrm{Bs}(|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of $\mathcal{Z}$.

Let G be a general divisor in $|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ and $\Delta_p = G_p/m$ for $p \in U$ general the corresponding $\mathbb{Q}$-divisors on X. It follows that $p \in \mathrm{Nklt}(K_X + \Delta_p)$ as $\mathrm{mult}_p \Delta_p \geq 3$. The minimal non-klt center $V_p \subseteq \mathrm{Nklt}(K_X + \Delta_p)$ passing through p must be positive dimensional by Lemma 50 as the weight of Δ_p is $\omega/2 > 2$. Note that we may replace $|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|$ by $|\mathrm{pr}_X^* L^{\otimes k} \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes k(6m-M)}|$ for any $k \geq 1$ and hence we may assume that $m \gg 0$. In particular, we have $0 \leq \mathrm{mult}_W \Delta_p \ll 1$ for W any irreducible component of $\mathrm{Supp}(\Delta_p)$, and V_p can be only one-dimensional.

Let $\pi : Y \to X$ and $f : Y \to B$ be a one dimensional covering family of tigers of weight $\omega' \geq \omega/2$ constructed from the Δ_p's above by Lemma 47 and Lemma 48. By abuse of notation, we still denote Δ_p's the divisors associated to this covering family of tigers.

Choose $p, q \in U \subseteq X$ general. By Lemma 42, $\mathrm{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q)) \supseteq V_p \cup V_q$ on Y is connected and it contains a one-dimensional cycle $C_{p,q}$ connecting V_p and V_q. Since Y is normal, an irreducible component C of $C_{p,q}$ intersecting V_q satisfies $C \cap Y_{\mathrm{sm}} \neq \emptyset$ for $p, q \in X$ general. Since C is in $\mathrm{Nklt}(\pi^*(K_X + \Delta_p + \Delta_q))$, by symmetry, we have that $\mathrm{mult}_C(\pi^*(\Delta_p)) > \epsilon/2$.

Suppose that $\Sigma \subseteq \mathrm{Supp}(\pi^*(\Delta_p))$ is an irreducible component containing C. If the image $f(\Sigma) = f(C)$ is a curve, then $V_p \subseteq \Sigma = f^{-1}(f(C))$ as the general fiber of $f : Y \to B$ is irreducible. Moreover, we can assume that Σ is not π-exceptional as there are only finitely many π-exceptional divisors and we choose $p \in X$, and hence V_p, general. Note that there can only be one such Σ once we fix $p \in X$ and C. In particular, $\Sigma \subseteq \mathrm{Supp}(\pi_*^{-1}(\Delta_p))$ is an irreducible component containing V_p, and we can write $\pi^*(\Delta_p) = \Delta' + \lambda\Sigma$ with $\Delta' \wedge \Sigma = 0$. Moreover, $\lambda \leq 1/m$, where $m \gg 0$ by our choice of Δ_p with $0 \leq \mathrm{mult}_W \Delta_p \ll 1$ for W any irreducible component of $\mathrm{Supp}(\Delta_p)$. Also, $\mathrm{mult}_C \Sigma = 1$ since Σ is smooth along C as $f(C)$ passes through a general point of B and Y is smooth in codimension one.

Choose a general point $b' \in f(C)$, we have that $Y_{b'}$ is a general fiber of $f : Y \to B$ and

$$\frac{2}{\frac{\omega}{2} - 2} \geq \frac{2}{\omega}(-K_X.V_t) = \pi^*(\Delta_p).Y_{b'} = (\Delta' + \lambda\Sigma).Y_b > \frac{\epsilon}{2} - \frac{1}{m},$$

where the first inequality follows from Corollary 52. The second inequality follows from $\Sigma.Y_b \geq 0$ and $\mathrm{mult}_C \Delta' = \mathrm{mult}_C(\pi^*(\Delta_p)) - \lambda \mathrm{mult}_C \Sigma$. Since $m \gg 0$, we get $\omega \leq 8/\epsilon + 4$. ■

Remark 60 *In the proof of Proposition 59, the difficulty arises because in general the one cycle C might be contained in $\mathrm{Supp}(\pi_*^{-1}(\Delta_p))$. In this case, one can not see the contribution of the ϵ-klt condition from the intersection number $\pi^*\Delta_p.Y_b$ for Y_b a general fiber over $f(C) \subseteq B$ as $Y_b \subseteq \mathrm{Supp}(\pi_*^{-1}(\Delta_p))$, cf., Example 45. The differentiation argument eliminates the contribution of irreducible components of $\mathrm{Supp}(\pi_*^{-1}(\Delta_p))$ along Y_b.*

4.5.4 Big multiplicity

Again, let X be an ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefold of Picard number one. Write $\mathrm{Vol}(-K_X) = -K_X^3 > (3\omega)^3$ for some positive rational number ω and denote $D = -2K_X$. As before, by Section 4.5.1, there is an integer $m > 0$ such that $L = mD/\omega$ is Cartier and an open affine subset $U \subseteq X$ such that $H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$. We fix a nonzero section $s \in H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ with $F = \mathrm{div}(s)$ on $X \times U$. We now consider the case where there exists an irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\mathrm{mult}_{\mathcal{W}}(F) > 3m$.

Lemma 61 *If there exists an irreducible component $\mathcal{W}$ of $\mathrm{Supp}(F)$ passing through $\mathcal{Z}$ with multiplicity $\mathrm{mult}_{\mathcal{W}}(F) > 3m$, then there exists a covering family of tigers of dimension two and weight $\omega' \geq \omega/2$.*

Proof. Fix $\mathcal{W}$ to be one of these irreducible components of $\mathrm{Supp}(F)$. We have the inclusions $\mathcal{Z} \subseteq \mathcal{W} \subseteq X \times U$ with the projection map $\mathcal{W} \to U$. Cutting down by hyperplanes on U

and restricting to a smaller open subset of U, we may assume that $\mathcal{W} \to U$ factors through a Hilbert scheme of X and $\mathcal{W} \to X$ is generically finite. Replace U by the normalization of the closure of its image in the Hilbert scheme and $\mathcal{W}$ by the normalization of universal family. We obtain maps $\pi : Y \to X$ and $f : Y \to B$. Note that a general fiber Y_b is two-dimensional. We claim that the pairs $(\Delta_b = \pi_*(Y_b), V_b = Y_b)$ is a two-dimensional covering of tigers of weight $\omega' \geq \omega/2$.

Since X is $\mathbb{Q}$-factorial and $\rho(X) = 1$, the integral divisor $\Delta_b = \pi_*(Y_b)$ for any $p \in B$ on X is $\mathbb{Q}$-linear equivalent to a multiple of $-K_X$. Since $\mathcal{W} \leq F$, we have $\pi_*(Y_b) \leq F_b$ for general $b \in B$. In particular, $\pi_*(Y_b) \sim_{\mathbb{Q}} -K_X/\omega'$ for some $\omega' \geq \omega/2$. Since any two general divisors $\pi_*(Y_{b_i})$, $i = 1,2$, on X are $\mathbb{Q}$-linear equivalent as the base field is uncountable, and it is clear that $V_t = \pi(Y_b)$ is the minimal non-klt center of $\mathrm{Nklt}(X, \Delta_b)$, and the lemma follows. ■

Let $\pi : Y \to X$ with $f : Y \to B$ be a covering family of tigers of dimension two and weight $\omega' \geq \omega/2$ given by Lemma 61. We first deal with case where $\pi : Y \to X$ is not birational.

Proposition 62 *Suppose that the two dimensional covering family of tigers $\pi : Y \to X$ with $f : Y \to B$ of weight $\omega' \geq \omega/2$ is not birational and assume that $\omega > 12$, then $\omega \leq 24/\epsilon + 12$. In particular, there is an upper bound of volume*

$$\mathrm{Vol}(-K_X) = -K_X^3 \leq (\frac{72}{\epsilon} + 36)^3.$$

Proof. Let $d \geq 2$ be the degree of $\pi : Y \to X$. Fix an open subset $U \subseteq X$ such that for a general point $p \in U$ there are d divisors $\Delta_p^{t_i}$, for some $t_1, ..., t_d \in B$, with $\pi(Y_{t_i}) \subseteq \mathrm{Nklt}(X, \Delta_p^{t_i})$ the unique minimal non-klt center passing through p. Consider the collection of $\mathbb{Q}$-divisors $\{\Delta'_p = \frac{6}{d}\sum_{i=1}^{d} \Delta_p^{t_i} | p \in U\}$, then $\mathrm{mult}_p \Delta'_p \geq 6$, $\mathrm{mult}_{W'} \Delta'_p = \frac{6}{d} \leq 3$ for $W' \subseteq \mathrm{Supp}(\Delta'_p)$ any irreducible component, and $\Delta'_p \sim_{\mathbb{Q}} \frac{-K_X}{d\omega'/6}$.

By the same construction as in Section 4.5.1, possibly after shrinking U to a smaller open affine subset, there exists an integer $m > 0$ such that $H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m}) \neq 0$ where $L = 6m(-K_X)/d\omega'$ is Cartier. Let $t \in H^0(X \times U, \mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m})$ be a general nonzero section and $G = \mathrm{div}(t)$ be the associated divisor on $X \times U$. Note that $\mathrm{mult}_{\mathcal{Z}}(G) \geq 6m$ and $\mathrm{mult}_{\mathcal{W}}(G) \leq 6m/d \leq 3m$ for any irreducible component $\mathcal{W}$ of $\mathrm{Supp}(G)$ passing through $\mathcal{Z}$. Indeed, we know that for general $p \in U$ there is the divisor Δ'_p with $\mathrm{mult}_p \Delta'_p \geq 6$ and $\mathrm{mult}_{W'} \Delta'_p = \frac{6}{d} \leq 3$ for any irreducible component $W' \subseteq \mathrm{Supp}(\Delta'_p)$. Since t is a general section, $t_p = t|_{X \times \{p\}}$ is also a general section for general $p \in U$. Using Lemma 57 to compute the multiplicity, we obtain $\mathrm{mult}_{\mathcal{W}}(G) = \mathrm{mult}_{\mathcal{W}_p}(G_p) \leq m \cdot \mathrm{mult}_{W'}\Delta'_p \leq 3m$, where $G_p = \mathrm{div}(t_p)$ and $\mathcal{W}_p$ is any irreducible component of $\mathrm{Supp}(G_p)$.

By a differentiation argument and the same construction as in Proposition 59, there is a covering family of tigers (Δ_t, V_t) of dimension one and weight $\omega'' \geq d\omega'/6 \geq d\omega/12$, which satisfies the property that the base locus $\mathrm{Bs}(|\mathrm{pr}_X^* L \otimes \mathcal{I}_{\mathcal{Z}}^{\otimes 6m-M}|)$ contains no codimension one components in a neighborhood of $\mathcal{Z}$, where M is the maximum of $\mathrm{mult}_{\mathcal{W}}(G)$ amongst all the irreducible components $\mathcal{W}$ of $\mathrm{Supp}(G)$ passing through $\mathcal{Z}$. Hence by Corollary 52, we get

$$\frac{2}{\omega''-2} \geq \frac{1}{\omega''}(-K_X.V_t) = \pi^*\Delta_p.Y_b \geq \frac{\epsilon}{2}.$$

In particular,

$$\frac{4}{\epsilon} + 2 \geq \omega'' \geq \frac{d\omega}{12} \geq \frac{\omega}{6},$$

and $\omega \leq 24/\epsilon + 12$. ■

Assumption: From now on, we assume that $\pi : Y \to X$ with $f : Y \to B$ is a **birational** covering family of tigers of dimension two and weight $\omega' \geq \omega/2$. Write $K_Y + \Gamma - R = \pi^* K_X$ where Γ and R are effective divisors on Y with no common components.

Lemma 63 *There is a π-exceptional divisor E on Y dominating B. In particular, $\pi : Y \to X$ is not small.*

Proof. Suppose that there is no π-exceptional divisors dominating B. Let A_B be a sufficiently ample divisor on B and $A_Y = f^* A_B$ the pull-back. Since $\rho(X) = 1$, the divisor $A_X = \pi_* A_Y$ on X is ample and $\pi^* A_X = A_Y + G$ for some effective π-exceptional divisor G. By assumption $f(G) \subseteq B$ has codimension one and hence $A_Y + G \leq f^* H$ for some divisor H on B. This is a contradiction since then $A_Y + G$ is not big but $\pi^* A_X$ is. ■

The following lemma is crucial for computing the restricted volume. The key point is that it allows us to control the negative part of the subadjunction $-K_X|_{V_t}$. Note that the proof fails in higher dimensions, cf. [36, Lemma 6.2].

Lemma 64 *Let E be a π-exceptional divisor dominating B. For general points $p, q \in X$ we have that $E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_p + \Delta_q))$. In particular, denote $H = \pi^*(-K_X)$. For any π-exceptional divisor E dominating B we have*

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \epsilon E.$$

Proof. Since the construction of covering families of tigers is done via the Hilbert scheme, π is finite on the general fibers V_t of $f : Y \to B$. Recall that $\pi(V_t) \subseteq X$ is the minimal non-klt center of $(X, \Delta_{p(t)})$ for some $\Delta_{p(t)}$ passing through a general point $p(t) \in X$. We denote $\Delta_{p(t)}$ by Δ_t for simplicity.

Let E be a π-exceptional divisor dominating B. Since $E \cap V_b$ is one-dimensional for general $b \in B$ and $\pi|_{V_b}$ is finite, $\dim \pi(E) > 0$ as $\pi(E) \supseteq \pi(E \cap V_b)$. Since E is irreducible and π-exceptional, $\pi(E)$ is an irreducible curve. Fix $t_1, t_2 \in B$ two general points. Pick a general point $x \in \pi(E)$ and consider its preimage on V_{t_i}. Since π is finite on the general fiber V_t, $\pi^{-1}(x) \cap V_{t_i}$ can be only a discrete finite set. Choose $x_i \in \pi^{-1}(x) \cap V_{t_i}$ over x for $i = 1, 2$. Apply the Connectedness Lemma 42 to the pair $(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ over X. There is a (possibly reducible) curve contained in $\pi^{-1}(x) \cap \mathrm{Nklt}(Y, \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$ connecting x_1 and x_2. The component of this curve containing x_1 cannot lie on V_{t_1} as the map π is finite on V_{t_1}. As $x \in \pi(E)$ is general, this curve deforms into a dimension two subset of E by moving $x \in \pi(E)$. Since E is irreducible, the closure of this two-dimensional subset coincides with E and hence $E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2}))$. In particular, $\mathrm{mult}_E(K_Y + \Gamma - R + \pi^*(\Delta_{t_1} + \Delta_{t_2})) \geq 1$. If $E \nsubseteq \mathrm{Supp}(\Gamma)$, then $\pi^*(\Delta_p + \Delta_q) \geq E$. If $E \subseteq \mathrm{Supp}(\Gamma)$, then $\pi^*(\Delta_p + \Delta_q) \geq \epsilon E$ since $\Gamma \in [0, 1-\epsilon)$ as X is ϵ-klt. ■

To study the geometry of the covering family $f : Y \to B$, we would like to run a relative minimal model program of (Y, Γ) over B. However, Y is normal but possibly not $\mathbb{Q}$-factorial. To get a $\mathbb{Q}$-factorial model of (Y, Γ), we adopt Hacon's dlt models, cf. [26, Theorem 3.1]. In fact, since the volume bound will be obtained by doing a computation on a general fiber Y_b, it suffices to modify Y over an open subset $U \subseteq B$.

Lemma 65 *After restricting to an open subset $U \subseteq B$ and replacing Y by a suitable birational model, we can assume that Y is $\mathbb{Q}$-factorial and (Y, Γ) is $\epsilon/2$-klt. Moreover, we can assume for E any π-exceptional divisor dominating U and $p, q \in X$ general, we have that*

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} E. \tag{4.6}$$

Proof. Fix $p, q \in X$ general and consider the pair

$$K_Y + \Gamma - R_{\mathrm{d}} + \pi^*(\Delta_p + \Delta_q) - R_{\mathrm{e}} \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_p + \Delta_q) \tag{$\sharp$}$$

where $R = R_{\mathrm{d}} + R_{\mathrm{e}}$ with $(-)_{\mathrm{d}}$ the sum of components dominating B and $(-)_{\mathrm{e}}$ the sum of components mapping to points in B. Restricting Y to $Y_U = f^{-1}(U)$ for a suitable nonempty open set $U \subseteq B$, we may assume that $R_{\mathrm{e}} = 0$ and ($\sharp$) becomes

$$K_Y + \Gamma - R_{\mathrm{d}} + \pi^*(\Delta_p + \Delta_q) \sim_{\mathbb{Q}} \pi^*(K_X + \Delta_p + \Delta_q).$$

We abuse the notation: Y is understood to be Y_U if not specified.

Denote $\Gamma_{p,q} = \Gamma - R_{\mathrm{d}} + \pi^*(\Delta_p + \Delta_q)$. Note that $\Gamma_{p,q} \geq 0$ by Lemma 64. Let $\phi : W \to Y$ be a log resolution of $(Y, \Gamma_{p,q})$ and write

$$K_W + \phi_*^{-1}\Gamma_{p,q} + Q \sim_{\mathbb{Q}} \phi^*(K_Y + \Gamma_{p,q}) + P,$$

where $Q, P \geq 0$ are ϕ-exceptional divisors with $Q \wedge P = 0$. We aim to modify W by running a relative minimal model program over Y with scaling of an ample divisor so that it contracts $Q^{<1-\epsilon/2} + P$, where $(\sum_i a_i Q_i)^{<\alpha} := \sum_{a_i<\alpha} a_i Q_i$. Note that we define $(-)^{\alpha\leq\cdot<\beta}$ and $(-)^{\geq\alpha}$ in the same way.

Consider $F = \sum_i F_i$, where the sum runs over all the ϕ-exceptional divisors with log discrepancy in $(\epsilon/2, 1]$ with respect to $(Y, \Gamma_{p,q})$, then

$$(F + P) \wedge Q^{\geq 1-\epsilon/2} = 0, \text{ and } \operatorname{Supp}(F) \supseteq \operatorname{Supp}(Q^{<1-\epsilon/2}).$$

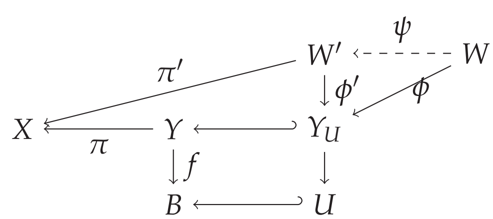
Since $(Y, \Gamma - R)$ is ϵ-klt, the divisor Γ on Y as well as $\phi_*^{-1}\Gamma$ on W has coefficients in $[0, 1-\epsilon)$. For rational numbers $0 < \epsilon < \epsilon' < 1$ and $0 < \delta, \delta' \ll 1$, we have the following $\epsilon/2$-klt pair

$$\begin{aligned} &K_W + \phi_*^{-1}\Gamma + Q^{<1-\epsilon/2} + \delta' Q^{1-\epsilon/2\leq\cdot<1} + (1-\epsilon')(Q^{\geq 1})_{\mathrm{red}} + \delta F \\ \sim_{\mathbb{Q}}\ &\phi^*(K_Y + \Gamma_{p,q}) - (\phi_*^{-1}\Gamma_{p,q} - \phi_*^{-1}\Gamma) - (1-\delta')Q^{1-\epsilon/2\leq\cdot<1} - (Q^{\geq 1} - (1-\epsilon')(Q^{\geq 1})_{\mathrm{red}}) \\ &+ P + \delta F \end{aligned}$$

where $(\sum_j b_j G_j)_{\mathrm{red}} := \sum_{b_j \neq 0} G_j$. We denote the above pair by (W, Ξ) where

$$\Xi = \phi_*^{-1}\Gamma + Q^{<1-\epsilon/2} + \delta' Q^{1-\epsilon/2\leq\cdot<1} + (1-\epsilon')(Q^{\geq 1})_{\mathrm{red}} + \delta F.$$

By [8], a relative minimal model program with scaling of an ample divisor of the pair (W, Ξ) over Y terminates with a birational model $\psi : W \dashrightarrow W'$ over Y with $\phi' : W' \to Y$ the induced map. We obtain the following diagram,



where $\pi' : W' \to X$ is the induced map.

Write $K_{W'} + \Gamma_{W'} - R_{W'} \sim_{\mathbb{Q}} \pi'^* K_X$ where $\pi' = \phi' \circ \pi$. Note that $\Gamma_{W'} \in [0, 1-\epsilon)$ by the ϵ-klt condition and $\Gamma_{W'} - (\phi')_*^{-1}\Gamma \geq 0$ is ϕ'-exceptional. It follows by the construction that $\Gamma_{W'} \leq \psi_* \Xi$. In particular, $(W', \Gamma_{W'})$ is $\epsilon/2$-klt as the pair (W, Ξ) is $\epsilon/2$-klt and the minimal model program does not make singularities worse.

On W', the divisor

$$G = \psi_*(-(\phi_*^{-1}\Gamma_{p,q} - \phi_*^{-1}\Gamma) - (1-\delta')Q^{1-\epsilon/2 \leq \cdot < 1} - (Q^{\geq 1} - (1-\epsilon')(Q^{\geq 1})_{\mathrm{red}}) + P + \delta F)$$

is ϕ'-nef with $\phi'_* G \leq 0$ since $\Gamma_{p,q} \geq \Gamma$. By [30, Negativity Lemma 3.39], we have that $G \leq 0$. Since $F + P$ is ϕ-exceptional and $(F+P) \wedge Q^{\geq 1-\epsilon/2} = 0$, it follows that $\psi_*(P + \delta F) = 0$. In particular, all the ϕ'-exceptional divisors on W' have log discrepancies less than or equal to $\epsilon/2$ with respect to $(Y, \Gamma_{p,q})$.

We now show that for any π'-exceptional divisor E' on W' dominating U, E' satisfies the inequality

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} E',$$

where $H' = \pi'^*(-K_X)$. This easy to see. If $E = \phi'_*(E') \neq 0$ on Y_U, then by Lemma 64, $E \subseteq \mathrm{Nklt}(K_Y + \Gamma - R + \pi^*(\Delta_p + \Delta_q))$ and $E' \subseteq \mathrm{Nklt}(K_{W'} + \Gamma_{W'} - R_{W'} + \pi'^*(\Delta_p + \Delta_q))$. The inequality then follows from the same argument as in Lemma 64. If $\phi'_* E' = 0$, then by construction $\mathrm{mult}_{E'}(K_{W'} + \Gamma_{W'} - R_{W'} + \pi'^*(\Delta_p + \Delta_q)) \geq 1 - \epsilon/2$. Suppose that we have $E' \subseteq \mathrm{Supp}(R_{W'})$, then

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq E' \geq \frac{\epsilon}{2} E'.$$

If $E' \subseteq \mathrm{Supp}(\Gamma_{W'})$, then as $\Gamma_{W'} \in [0, 1-\epsilon)$ we get

$$\frac{2}{\omega'} H' \sim_{\mathbb{Q}} \pi'^*(\Delta_p + \Delta_q) \geq ((1 - \frac{\epsilon}{2}) - (1-\epsilon))E' = \frac{\epsilon}{2} E'.$$

It follows that W' satisfies the required properties. ∎

Remark 66 *Write $\Gamma = \pi_*^{-1}\Delta + \Gamma_{\mathrm{d}} + \Gamma_{\mathrm{e}}$ and $R = R_{\mathrm{d}} + R_{\mathrm{e}}$, where $(-)_{\mathrm{d}}$ is the sum of components dominating B and $(-)_{\mathrm{e}}$ is the sum of components mapping to points in B. From the proof of Lemma 65, we deduce the following two inequalities :*

$$\frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq R_{\mathrm{d}} \text{ and } \frac{2}{\omega'} H \sim_{\mathbb{Q}} \pi^*(\Delta_p + \Delta_q) \geq \frac{\epsilon}{2} \Gamma_{\mathrm{d}}. \tag{5.2}$$

Now let $\pi : Y \to X$ with $f : Y \to U$ be the modified birational covering family of tigers of dimension two and weight $\omega' \geq \omega/2$ given by Lemma 65, where Y is now $\mathbb{Q}$-factorial. Write $K_Y + \Gamma - R \sim_{\mathbb{Q}} \pi^* K_X$, where $\Gamma, R \geq 0$ are π-exceptional and $\Gamma \wedge R = 0$. The pair (Y, Γ) is $\epsilon/2$-klt with $\Gamma \in [0, 1 - \epsilon/2)$ and note that $H = \pi^*(-K_X)$ is semiample and big on Y.

Recall that for a projective morphism $\phi : Z \to U$, a divisor D on Z is pseudo-effective (PSEF) over U if the restriction of D to the generic fiber is pseudo-effective.

Lemma 67 *Assume that $\omega' > 2$ and consider the pseudo-effective threshold of $K_Y + \Gamma$ over U with respect to H*

$$\tau := \inf\{t > 0 | K_Y + \Gamma + tH \text{ is PSEF over } B\}.$$

Then $1 \geq \tau \geq 1 - \frac{2}{\omega'} > 0$.

Proof. Since $K_Y + \Gamma + H \sim_{\mathbb{Q}} R \geq 0$, the first inequality is clear. Restricting to a general fiber Y_u of Y over U, we have

$$\begin{aligned}(K_Y + \Gamma + \tau H)|_{Y_u} =& (R - (1-\tau)H)|_{Y_u} \\ =& (R_d - \frac{2}{\omega'} H)|_{Y_u} - (1 - \tau - \frac{2}{\omega'})H|_{Y_u}\end{aligned}$$

which cannot be PSEF if $\omega' > 2$ and $\tau < 1 - \frac{2}{\omega'}$ since the first term is nonpositive by (5.2) and the second term is negative. ∎

Now we run a relative minimal model program with scaling for the covering family of tigers $f : Y \to U$. Since (Y, Γ) is $\epsilon/2$-klt and H is semiample and big, we may assume that $(Y, \Gamma + \tau' H)$ remains $\epsilon/2$-klt for any rational number $0 < \tau' < \tau$. By [8], a relative minimal model program of $(K_Y + \Gamma + \tau' H)$ with scaling of H over U terminates with a relative Mori fiber space $Y' \to T$ over U with $\dim Y' > \dim T \geq \dim U$. Denote the induced maps by $g : Y \dashrightarrow Y'$, $\psi : Y' \to T$, and $\phi : Y' \to U$. We obtain the following diagram,

$$\begin{array}{ccccc} X & \xleftarrow{\ \pi\ } & Y & \overset{g}{\dashrightarrow} & Y' \\ & & \big\downarrow f & \swarrow \phi & \big\downarrow \psi \\ & & B \supseteq U & \longleftarrow & T. \end{array}$$

For a general fiber Y'_t of $\psi : Y' \to T$, by construction, the Picard number $\rho(Y'_t) = 1$ and the divisor $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ on Y'_t is ample.

Lemma 68 *There exists a divisor E' on Y' which is exceptional over X and dominates T.*

Proof. Recall that there is a natural map $T \to U \to B$. We can extend $\psi : Y' \to T$ to $\overline{\psi} : \overline{Y'} \to \overline{T}$ over B where $\overline{(-)}$ stands for a projective compactification of $(-)$. Take a common resolution $p : W \to X$ and $q : W \to \overline{Y'}$ and let $A_{\overline{T}}$ be a sufficiently ample divisor on $\overline{T}$. Let $A_{\overline{Y'}} = \overline{\psi}^* A_{\overline{T}}$, $A_W = q^* A_{\overline{Y'}}$, and $A_X = p_* A_W$. Then there is an effective divisor E on W which is exceptional over X such that $p^* A_X = A_W + E = q^* A_{\overline{Y'}} + E = q^* \overline{\psi}^* A_{\overline{T}} + E$. Since $\rho(X) = 1$, it follows by the same argument as in Lemma 47 that one of the irreducible components of E maps to a divisor E' on $\overline{Y'}$. By the same argument as in Lemma 47 again, one of the irreducible components of the nonzero divisor $q_*(E)$ dominates $\overline{T}$. ■

Proposition 69 *If* $\dim T = 2$, *then* $\omega' \leq 8/\epsilon + 2$.

Proof. By Lemma 68, there exists a divisor E' on Y' which is exceptional over X and dominates T. Note that Y' is normal and hence $\psi(\mathrm{Sing}(Y'))$ is a proper subset of T. In particular, a general fiber Y'_t of $\psi : Y' \to T$ is a smooth projective curve and hence $E'.Y'_t \geq 1$. Since the divisor $-(K_{Y'} + \Gamma'_d)|_{Y'_t} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_t}$ is ample, a general fiber Y'_t is a smooth rational curve $\mathbb{P}^1$. From (4.6), we know that

$$\frac{2}{\omega'} H' - \frac{\epsilon}{2} E' \sim_{\mathbb{Q}} \text{effective.}$$

Also from (5.2),

$$\begin{aligned} -(K_{Y'} + \Gamma').Y'_t = (H' - R').Y'_t =& (1 - \frac{2}{\omega'}) H'.Y'_t + (\frac{2}{\omega'} H - R').Y'_t \\ \geq& (1 - \frac{2}{\omega'}) H'.Y'_t. \end{aligned}$$

It follows that

$$\begin{aligned} \frac{2}{\omega'} \geq \frac{1}{\omega'}(-(K_{Y'} + \Gamma').Y'_t) &\geq \frac{1}{\omega'}(1 - \frac{2}{\omega'}) H'.Y'_t \\ &\geq (1 - \frac{2}{\omega'}) \frac{\epsilon}{4} E'.Y'_t \\ &\geq (1 - \frac{2}{\omega'}) \frac{\epsilon}{4} \end{aligned}$$

where the first inequality follows by the adjunction formula on $\mathbb{P}^1$. Hence $\omega' \leq \frac{8}{\epsilon} + 2$. ■

Proposition 70 *If* $\dim T = 1$, *then*

$$\omega' \le \frac{4M(2,\epsilon)R(2,\epsilon)}{\epsilon} + 2$$

where $R(2,\epsilon)$ *is an upper bound of the Cartier index of* K_S *for* S *any* $\epsilon/2$*-klt log del Pezzo surface of* $\rho(S) = 1$ *and* $M(2,\epsilon)$ *is an upper bound of the volume* $\mathrm{Vol}(-K_S) = K_S^2$ *for* S *any* $\epsilon/2$*-klt log del Pezzo surface of* $\rho(S) = 1$.

Proof. Since $f : Y \to U$ has connected fibers, $T \cong U$. Since $-(K_{Y'} + \Gamma'_d)|_{Y'_u} \sim_{\mathbb{Q}} (H' - R_d)|_{Y'_u}$ is ample and $\rho(Y'_u) = 1$ for a general point $u \in U$, we see that

$$-K_{Y'_u} \sim_{\mathbb{Q}} (H' + \Gamma'_d - R_d)|_{Y'_u}$$

is ample. By Lemma 68, let E' be a divisor on Y' exceptional over X, which dominates U, then

$$-K_{Y'_u} \equiv (H' + \Gamma'_d - R_d)|_{Y'_u} \ge (1 - \frac{2}{\omega'})H|_{Y'_u} \ge (1 - \frac{2}{\omega'}) \cdot \frac{\omega'\epsilon}{4} E'_u$$

where the second inequality follows by dropping Γ'_d and applying (5.2) while the last one from (4.6). By intersecting with the ample divisor $-K_{Y'_u}$, this implies that

$$(-K_{Y'_u})^2 \ge (\omega' - 2)\frac{\epsilon}{4} E'_u.(-K_{Y'_u}).$$

Now (Y'_u, Γ'_u) is an $\epsilon/2$-klt log del-Pezzo surfaces of Picard number one. Hence Y'_u is an $\epsilon/2$-klt del-Pezzo surface of Picard number $\rho(Y'_u) = 1$. By Theorem 55, $(-K_{Y'_u})^2$ is bounded above by a positive number $M(2,\epsilon)$ satisfying

$$M(2,\epsilon) \le \max\{64, \frac{16}{\epsilon} + 4\}.$$

Also, by $(\Diamond)$ the Cartier index of $K_{Y'_u}$ has an upper bounded

$$R(2,\epsilon) \le r(2, \frac{\epsilon}{2}) \le 2(4/\epsilon)^{128 \cdot 2^5/\epsilon^5}.$$

It follows that

$$M(2,\epsilon) \ge (-K_{Y'_u})^2 \ge \frac{1}{R(2,\epsilon)}(\omega' - 2)\frac{\epsilon}{4} E'_u.(\text{Ample Cartier}) \ge \frac{1}{R(2,\epsilon)}(\omega' - 2)\frac{\epsilon}{4}$$

and hence we get an upper bound

$$\omega' \le \frac{4M(2,\epsilon)R(2,\epsilon)}{\epsilon} + 2.$$

∎

Remark 71 *It has been shown in [7] that a klt log del Pezzo surface has at most four isolated singularities. Also surface klt singularities are classified by Alexeev in [31]. Hence we expect that it is possible to obtain a better upper bound for $R(2,\epsilon)$ and $M(2,\epsilon)$ in Proposition 70.*

Theorem 72 *Let (X,Δ) be an ϵ-klt log $\mathbb{Q}$-Fano threefold of $\rho(X) = 1$. Then the degree $-K_X^3$ satisfies*

$$-K_X^3 \leq \left(\frac{24M(2,\epsilon)R(2,\epsilon)}{\epsilon} + 12\right)^3$$

where $R(2,\epsilon)$ is an upper bound of the Cartier index of K_S for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S) = 1$ and $M(2,\epsilon)$ is an upper bound of the volume $\mathrm{Vol}(S) = K_S^2$ for S any $\epsilon/2$-klt log del Pezzo surface of $\rho(S) = 1$. Note that we have $M(2,\epsilon) \leq \max\{64, 16/\epsilon + 4\}$ from Theorem 55 and $R(2,\epsilon) \leq 2(4/\epsilon)^{128\cdot 2^5/\epsilon^5}$ from $(\diamondsuit)$.

Proof. Recall that $\omega' \geq \omega/2$. The theorem then follows from Propositions 59, 69 and 70. ■

The following example shows that the cone construction analogous to Example 45 only provides ϵ-klt Fano threefolds with volumes of order $1/\epsilon^2$.

Example 73 *(Projective cone of projective spaces) For $n \geq 1$ and $d \geq 2$, let $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the embedding by $|\mathcal{O}(d)|$ and X be the associated projective cone. The projective variety X is normal $\mathbb{Q}$-factorial of Picard number one with unique singularity at the vertex O. Also, X admits a resolution $\pi : Y = Bl_O X \to X$ with the unique exceptional divisor $E \cong \mathbb{P}^n$ of normal bundle $\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$. The variety Y is the projective bundle $\mu : Y \cong \mathbb{P}_{\mathbb{P}^n}(\mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)) \to \mathbb{P}^n$ with tautological bundle $\mathcal{O}_Y(1) \cong \mathcal{O}_Y(E)$. We have:*

- *$\mathcal{O}_E(E) \cong \mathcal{O}_{\mathbb{P}^n}(-d)$ and hence $E^{n+1} = (-d)^n$;*
- *$K_Y = \pi^*K_X + (-1 + \frac{n+1}{d})E$ and hence X is always klt. Also, X is terminal (resp. canonical) if and only if $n+1 > d \geq 2$ (resp. $n+1 \geq d \geq 2$);*
- *$K_Y = \mu^*(K_{\mathbb{P}^n} + \det(\mathcal{E})) \otimes \mathcal{O}_Y(-\mathrm{rk}(\mathcal{E})) \equiv -(n+1+d)F - 2E$ where the vector bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-d)$ and $F = \mu^*\mathcal{O}^n_{\mathbb{P}}(1)$;*
- *$F^{n+1} = 0$ and $F^{n+1-k}.E^k = (-d)^{k-1}$ for $1 \leq k \leq n+1$;*
- *$K_Y^{n+1} = K_X^{n+1} + (-1 + \frac{n+1}{d})^{n+1}E^{n+1}$ and*

$$\begin{aligned} K_Y^{n+1} &= \frac{-1}{d}\sum_{k=1}^{n+1}\begin{pmatrix} n+1-k \\ k \end{pmatrix}(-1+\frac{n+1}{d})^{n+1-k}(2d)^k \\ &= \frac{-1}{d}((d-n-1)^{n+1} - (-(d+n+1)^{n+1})); \end{aligned}$$

- *In summary, $-K_X$ is ample with*

$$(-K_X)^{n+1} = \frac{(d+n+1)^{n+1}}{d}.$$

If $n = 2$, then we have an ϵ-klt Fano threefold of Picard number one with $\epsilon = 1/d$. The volume $\mathrm{Vol}(X) = (-K_X)^3$ is of order $1/\epsilon^2$.

In view of Theorem 72, it is then interesting to see whether ϵ-klt Fano threefolds with big volumes exist.

Question 74 *Can one find ϵ-klt $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano threefolds X of $\rho(X) = 1$ with volume $\mathrm{Vol}(X) = (-K_X)^3 = O(\frac{1}{\epsilon^c})$ for $c \geq 3$?*

CHAPTER 5

NONVANISHING CONJECTURE

Here we provide some partial results toward the following conjecture in the log minimal model program.

Conjecture 75 *(**Nonvanishing Conjecture**) Let (X, Δ) be a $\mathbb{Q}$-factorial projective klt pair. If $K_X + \Delta$ is pseudo-effective, then $K_X + \Delta \sim_{\mathbb{Q}} D$ for some effective divisor D.*

We will focus on the cases where $\Delta = 0$ and X is a $\mathbb{Q}$-factorial normal projective variety with at worst terminal singularities. We would like to solve the following problem introduced in Section 2.2:

Conjecture 76 *Let X be a projective variety with at worst terminal singularities. If K_X is pseudo-effective, then $\kappa(X) \geq 0$.*

Here we include two results related to the Nonvanishing Conjecture. The first one attempts to get a conceptional proof of Nonvanishing Conjecture in dimension two. The second one is a nonvanishing theorem for irregular varieties.

5.1 Surfaces

In dimension two, a surface is terminal if and only if it is smooth. The Nonvanishing Conjecture 76 is solved by classification of surfaces. However, people aim to find a conceptional proof and expect from that we can understand better the same problem in higher dimensions.

Here we discuss the idea coming from the study of Iitaka's conjecture C (Section 3.5): For an algebraic fiber space $f : X \to Y$ of smooth projective varieties with a general fiber F, the Kodaira dimensions are related by

$$H^0(X, \mathcal{O}_X(mK_X)) = H^0(Y, \mathcal{O}_Y(mK_Y) \otimes f_*(\omega_{X/Y}^{\otimes m}))$$

where for a general point $p \in Y$

$$f_*(\omega_{X/Y}^{\otimes m}) \otimes \mathbb{C}(p) \cong H^0(F, \mathcal{O}_F(mK_F)).$$

If $\kappa(Y), \kappa(F) \geq 0$, then one expects to have $\kappa(X) \geq 0$.

In general, for a given projective variety X one can construct a non-trivial algebraic fiber space by taking a (sub)linear system, resolve the indeterminacy of the induced map, and take a Stein factorization. Since pseudo-effectiveness and the Kodaira dimension of K_X are invariant under this construction, we get an extra structure for proving nonvanishing. However, the base of this algebraic fiber space typically is a rational variety and it has negative Kodaira dimension. Hence it is much harder to show that $\kappa(X) \geq 0$. The key point is that we need to establish a stronger positivity property for the sheaves $f_*(\omega_{X/Y}^{\otimes m})$ or $f_*(\omega_X^{\otimes m})$. We include a partial result on the positivity of sheaves $f_*(\omega_X^{\otimes m})$.

Let X be a smooth projective variety over $\mathbb{C}$. Pick a very general pencil from a sufficiently ample linear system, e.g., a Lefschetz pencil. This defines a rational map from X to $\mathbf{P}^1$. We resolve the indeterminacy to get an algebraic fiber space $\pi : \tilde{X} \to \mathbb{P}^1$ whose general fiber $\tilde{X}_p$ is a smooth variety with ample canonical divisor. Assume that K_X is pseudo-effective, then $K_{\tilde{X}}$ is also pseudo-effective. Also $\kappa(X) \geq 0$ if and only if $\kappa(\tilde{X}) \geq 0$.

Since any torsion free sheaf on a smooth curve is locally free and any locally free sheaf on $\mathbb{P}^1$ splits into line bundles, to show the nonvanishing $\kappa(\tilde{X}) \geq 0$ is equivalent to say that the vector bundle $\pi_*(\omega_{\tilde{X}}^{\otimes m})$ contains a line bundle summand of nonnegative degree. In general, it suffices to show that there is a nonnegative degree line bundle summand for some $0 \neq \mathcal{F} \subseteq \pi_*(\omega_{\tilde{X}}^{\otimes m})$. We will use the sheaf

$$\mathcal{F}_m := \pi_* \left(\omega_{\tilde{X}}^{\otimes m} \otimes \mathcal{J}(\|(m-1)K_{\tilde{X}} + \epsilon \pi^* \mathcal{O}_{\mathbb{P}^1}(1)\|) \right),$$

for $m \geq 2$. The point of using this sheaf is that it is related to the Nadel (or Kawamata-Viehweg) vanishing theorem and hence we have the estimation of the degrees of its line bundle decomposition.

From now on, we assume that X is a smooth projective surface, $\mu : \tilde{X} \to X$ is a resolution of a Lefschetz pencil, and $\pi : \tilde{X} \to \mathbb{P}^1$ is the resulted algebraic fiber space.

Lemma 77 *The divisor $(m-1)K_{\tilde{X}} + \epsilon \pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ is big for $\epsilon > 0$ sufficient small. In particular, for $m \geq 2$ the multiplier ideal sheaf*

$$\mathcal{J}_m^\epsilon := \mathcal{J}(\|(m-1)K_{\tilde{X}} + \epsilon \pi^* \mathcal{O}_{\mathbb{P}^1}(1)\|),$$

is defined.

Proof. Since $\pi^*\mathcal{O}_{\mathbb{P}^1}(1) \sim l$ is a general fiber of the algebraic fiber space $\pi : \tilde{X} \to \mathbb{P}^1$, by construction the divisor $\mu_* l$ is ample. As $K_{\tilde{X}} = \mu^* K_X + E$ for some effective μ-exceptional divisor E, the lemma follows from the projection formula. ■

Lemma 78 *Let X be a smooth projective variety. Suppose $\{D_k\}$ is a collection of effective $\mathbb{Q}$-divisors with $k \in \mathbb{N}$ such that the corresponding multiplier ideal sheaves $\mathcal{J}_k := \mathcal{J}(D_k)$ satisfy $\mathcal{J}_k \subseteq \mathcal{J}_{k'}$ whenever $k \geq k'$. If there exists a line bundle L such that $L - D_k$ is nef and big for all $k > 0$, then $\bigcap_{i>0} \mathcal{J}_i = \mathcal{J}_k$ for k sufficiently large.*

Proof. The proof is taken from [16, Proposition 5.1]. We reproduce the proof here for the convenience of the reader. Take a sufficiently ample divisor H on X and consider the line bundle $M = L + (n+1)H$ for $n = \dim(X)$, then

$$M - D_k - (iH) \equiv L - D_k + (n - i + 1)H$$

is nef and big for all $k > 0$ and $1 \leq i \leq n$. Hence $H^i(X, \mathcal{O}_X(K_X + M - iH) \otimes \mathcal{J}_k) = 0$ for all $i > 0$ by Nadel vanishing, and then $\mathcal{O}_X(K_X + M) \otimes \mathcal{J}_k$ is generated by global sections by Mumford regularity. In particular, if $\mathcal{J}_k \neq \mathcal{J}_{k'}$ for $k \leq k'$, then we get a strict inclusion $H^0(X, \mathcal{O}_X(K_X + M) \otimes \mathcal{J}_k) \subsetneq H^0(X, \mathcal{O}_X(K_X + M) \otimes \mathcal{J}_{k'})$ of $\mathbb{C}$ vector spaces. But this can not happen infinitely many times, hence the lemma follows. ■

Corollary 79 *For a fixed $m \geq 2$, the sheaf $\mathcal{J}_m^\epsilon$ stabilizes as ϵ goes to zero. In particular, the sheaf $\mathcal{J}_m := \mathcal{J}_m^\epsilon$ is well-defined by choosing $\epsilon = \epsilon_m$ sufficiently small.*

Proof. Take $L = mK_{\tilde{X}} + \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ and apply Lemma 78.

■

Note that the asymptotic multiplier ideal sheaf $\mathcal{J}_m$ is defined via the multiplier ideal sheaf $\mathcal{J}(\frac{1}{q}|q(m-1)K_{\tilde{X}} + q\epsilon\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|)$ for $q > 0$ sufficiently divisible. For a fixed $m \geq 2$, let $\phi : X' \to \tilde{X}$ be a log resolution of $|q(m-1)K_{\tilde{X}} + q\epsilon\pi^*\mathcal{O}_{\mathbb{P}^1}(1)|$. Then

$$\phi^*|q(m-1)K_{\tilde{X}} + q\epsilon\pi^*\mathcal{O}_{\mathbb{P}^1}(1)| = M_q + F_q,$$

where M_q is big and semi-ample and F_q has simple normal crossing support.

Denote $\tilde{\pi} = \phi \circ \pi$ and consider the sheaf

$$\omega_{\tilde{X}}^{\otimes m} \otimes \mathcal{J} \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) = \phi_*\mathcal{O}_{X'}(m\phi^*K_{\tilde{X}} + K_{X'/\tilde{X}} - \llcorner\frac{1}{q}F_q\lrcorner + \tilde{\pi}^*\mathcal{O}_{\mathbb{P}^1}(1)).$$

The Cartier divisor

$$\begin{aligned}
M =& m\phi^* K_{\tilde{X}} + K_{X'/\tilde{X}} - \llcorner \frac{1}{q} F_q \lrcorner + \tilde{\pi}^* \mathcal{O}_{\mathbb{P}^1}(1) \\
=& K_{X'} + \phi^*(m-1) K_{\tilde{X}} + \tilde{\pi}^* \mathcal{O}_{\mathbb{P}^1}(1) - \llcorner \frac{1}{q} F_q \lrcorner \\
\equiv& K_{X'} + \frac{1}{q} M_q + \tilde{\pi}^* \mathcal{O}_{\mathbf{P}^1}(1-\epsilon) + \{\frac{1}{q} F_q\} \\
=& K_{X'} + (\text{nef and big}) + (\text{fractional SNC})
\end{aligned}$$

and hence by Kawamata-Viehweg vanishings $H^1(X', M) = 0$. Since a nef and big divisor on X' is also $\tilde{\pi}$-nef and $\tilde{\pi}$-big, we have also the relative Kawamata-Viehweg vanishing

$$\mathbf{R}^j \tilde{\pi}_* \mathcal{O}_{X'}(M) = 0, \ \forall\, j > 0.$$

In particular, the spectral sequence for computing $H^1(X', M)$ degenerates and we get

$$H^1(\mathbb{P}^1, \tilde{\pi}_* M) = H^1(\mathbb{P}^1, \pi_*(\omega_{\tilde{X}}^{\otimes m} \otimes \mathcal{J}) \otimes \mathcal{O}_{\mathbb{P}^1}(1)) = 0, \qquad (\heartsuit)$$

where $\tilde{\pi}_* M = \pi_*(\omega_{\tilde{X}}^{\otimes m} \otimes \mathcal{J}) \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. On $\mathbb{P}^1$, torsion free sheaves decompose into line bundles. Hence we can write for any $m \geq 1$, $\pi_*(\omega_{\tilde{X}}^{\otimes m} \otimes \mathcal{J}) = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i^m)$. Then the vanishing cohomology $(\heartsuit)$ implies that $a_i^m \geq -2$ for all i.

We conclude with the following proposition.

Proposition 80 *Let X be a smooth projective surface. Let $\pi : \tilde{X} \to \mathbb{P}^1$ be an algebraic fiber space constructed from a Lefschetz pencil by resolving the indeterminacy. For each $m \geq 1$, write*

$$\pi_*(\omega_{\tilde{X}}^{\otimes m}) = \oplus \mathcal{O}_{\mathbb{P}^1}(a_i^m).$$

If K_X is pseudo-effective, then $c_i^m \geq -2$ for all i.

It is easy to see that for some $m \geq 1$, $c_i^m \geq 0$ for some i is sufficient to conclude the Non-vanishing Conjecture in dimension two. However, from the weak-positivity of $\pi_*(\omega_{\tilde{X}/\mathbb{P}^1}^{\otimes m})$, one can only conclude that for each $m \geq 1$, $c_i^m \geq -2m$ for all i. Thus this is a nontrivial result very close to what we expect.

5.2 Irregular varieties

Here we include a nonvanishing theorem of irregular varieties. The main ingredient is the following theorem on the structure of cohomological loci $V_m(K_X)$.

Theorem 81 *Let X be a smooth projective variety. The cohomological loci*

$$V_m(K_X) := \{P \in \mathrm{Pic}^0(X) |\, h^0(X, \omega_X^{\otimes m} \otimes P) > 0\}$$

for m a positive integer, if non-empty, is a finite union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

Proof. If $m = 1$, then by a result of Simpson [40] the loci $V_1(K_X)$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$. In general, let $\tilde{P} \in V_m(K_X)$. Since $\mathrm{Pic}^0(X)$ is divisible, we can write $\tilde{P} = mP$ for some $P \in \mathrm{Pic}^0(X)$. Let $\mu : X' \to X$ be a log resolution of $|m(K_X + P)|$, and $D \in \mu^*|m(K_X + P)|$ be a divisor with simple normal crossing support. Consider the line bundle $N := \mu^*\mathcal{O}_X((m-1)(K_X + P)) \otimes \mathcal{O}_{X'}(-\lfloor \frac{m-1}{m} D \rfloor)$. It follows from [12, Theorem 8.3] and [40] that the cohomological loci

$$V^0(\omega_{X'} \otimes N) := \{R \in \mathrm{Pic}^0(X') | h^0(\omega_{X'} \otimes R) > 0\},$$

is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X')$. Since X is smooth, $\mathrm{Pic}^0(X') \cong \mathrm{Pic}^0(X)$ and hence we may identify the elements in these two groups (via pulling back by μ). It is easy to see that $P \in V^0(\omega_{X'} \otimes N)$, and hence there exists an abelian subvariety $T \subseteq \mathrm{Pic}^0(X)$ and a torsion element $Q \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$ such that

$$P \in T + Q \subseteq V^0(\omega_{X'} \otimes N).$$

By pushing forward, it is also easy to see that

$$T + Q + (m-1)P \subseteq V_m(K_X).$$

Since $rP \in rT$ for some positive integer r and rT is a group, we have that $r(m-1)P \in rT$ and hence $(m-1)P \in T + Q'$ for some torsion element $Q' \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$. In particular, we have

$$\tilde{P} = mP \in T + Q + (m-1)P = T + Q + Q' \subseteq V_m(K_X),$$

and hence $V_m(K_X)$ is a union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$.

Let V be an irreducible component of $V_m(K_X)$ and denote $\mathrm{Pic}^0(X)$ by A. Note that for any general point of V, there is a torsion translate of an abelian subvariety of A contained in V passing through it. It is well-known that if V is of general type, then there are no nontrivial abelian subvarieties of A contained in V passing through general points of V. In this case, a general point of V must be torsion and hence dim V can only be zero since there are only countably many torsion points in A. It follows that V is a torsion point. If V is not of general type, then by [43, Theorem 10.9] there is an algebraic fiber space $f : V \to B$ with general fiber A_1 induced by $\pi : A \to A/A_1$, where A_1 is an abelian subvariety of A and $B \subseteq A/A_1$ is a subvariety of general type. Since there are also torsion translate of abelian subvarieties of A/A_1 contained in B passing through general points of B, B is a torsion point and so V is a torsion translate of an abelian subvariety of A. Hence we conclude that the algebraic set $V_m(K_X)$, if non-empty, is a *finite* union of torsion translates of abelian subvarieties of $\mathrm{Pic}^0(X)$. ■

Recall that a variety X is irregular if $H^1(X, \mathcal{O}_X) \neq 0$.

Theorem 82 *Let X be a smooth projective irregular variety with $A := Alb(X)$ the Albanese variety. Let $\alpha := alb_X : X \to A := Alb(X)$ be the Albanese morphism and $\alpha' : X \to Y$ with general fiber F be the Stein factorization of $\alpha : X \to \alpha(X) \subseteq A$. Suppose $\kappa(F) \geq 0$, then $\kappa(X) \geq 0$.*

Lemma 83 *With the assumptions as in Theorem 82, K_X is pseudo-effective.*

Proof. We have $\alpha'_* \omega^N_{X/Y} \neq 0$ and is weakly positive by [44]. Hence for any $\epsilon > 0$ and H ample on Y, $\alpha'_* \omega^N_{X/Y} \otimes (\epsilon H)$ is big. As Y is finite over $\alpha(X)$, a subvariety in A, we have $\kappa(Y) \geq 0$ and hence $\alpha'_* \omega^N_X \otimes (\epsilon H)$ is also big. In particular $\kappa(K_X + \frac{\epsilon}{N}(\alpha')^* H) \geq 0$ for any $\epsilon > 0$, and hence K_X is pseudo-effective. ■

For H an ample divisor on A and a nonnegative integer m, it follows from Lemma 78 by taking L to be $mK_X + \alpha^* H$ on X, the multiplier ideal sheaf $\mathcal{J}(\|mK_X + \epsilon\alpha^* H\|)$ is independent of $\epsilon \in \mathbb{Q}$ for any $\epsilon > 0$ sufficiently small. Hence we can define the sheaf

$$\mathcal{F}_m := \alpha_*(\omega^m_X \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|)),$$

on A for $\epsilon > 0$ a sufficiently small rational number.

Lemma 84 *With the above setting, for L any sufficiently ample line bundle on the dual abelian variety $\hat{A}$ with $\hat{L}$ the Fourier-Mukai transform of L on A, we have $H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee) = 0$ for all $i > 0$. From [16, Corollary 3.2], we then have for any nonnegative integer m the inclusions:*

$$V^0(\mathcal{F}_m) \supseteq V^1(\mathcal{F}_m) \supseteq \ldots \supseteq V^n(\mathcal{F}_m).$$

In particular, $V^0(\mathcal{F}_m) = \phi$ implies $\mathcal{F}_m$=0.

Proof. The vanishing of cohomology follows from [16, Theorem 4.1] with a slight modification and hence we reproduce the argument here. Consider the isogeny $\phi_L : \hat{A} \to A$ defined by L, $\hat{\alpha} : \hat{X} \to \hat{A}$, and $f : \hat{X} = X \times_A \hat{A} \to X$. Then as $\phi_L^* \hat{L}^\vee = \oplus_{h^0(L)} L$, we have

$$\begin{aligned} H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee) &\subseteq H^i(A, \mathcal{F}_m \otimes \hat{L}^\vee \otimes \phi_{L*}\mathcal{O}_{\hat{A}}) \\ &= H^i(\hat{A}, \phi_L^*\mathcal{F}_m \otimes \phi_L^*\hat{L}^\vee) \\ &= \oplus H^i(\hat{A}, \hat{\alpha}_* f^*(\omega_X^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|)) \otimes L) \\ &= \oplus H^i(\hat{A}, \hat{\alpha}_*(\omega_{\hat{X}}^m \otimes \mathcal{J}(\|(m-1)K_{\hat{X}} + \epsilon\hat{\alpha}^*\phi_L^* H\|)) \otimes L), \end{aligned}$$

where the last equality is the étale base change of multiplier ideal sheaves in [34, Theorem 11.2.16]. For $i > 0$, the cohomological groups above vanish by Nadel vanishing on $\hat{X}$, or by Kawamata-Viehweg vanishing theorem on a log resolution $\pi : Y \to \hat{X}$. The final statement follows from [38, Theorem 2.2]. ∎

Proof._(of Theorem 82)_ For general point $z \in Y$ and m sufficiently divisible, we have for the sheaves defined by $\mathcal{F}'_m := \alpha'_*(\omega_X^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|))$ on Y:

$$\begin{aligned} (\mathcal{F}'_m)_z &= H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|)|_F) \\ &\supseteq H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_X + \epsilon\alpha^* H\|_F)) \\ &= H^0(F, \omega_F^m \otimes \mathcal{J}(\|(m-1)K_F\|)) \\ &\supseteq H^0(F, \omega_F^m \otimes \mathcal{J}(\|mK_F\|)) \\ &= H^0(F, \omega_F^m) > 0. \end{aligned}$$

The first inclusion follows from the property of the restriction of multiplier ideal sheaves in [34, Theorem 11.2.1]. The second equality follows from the explanation of semipositivity in [24, Proposition 10.2], and the last inequality from $\kappa(F) \geq 0$. Hence $\mathcal{F}'_m$ is nontrivial. In particular, $\mathcal{F}_m$ is also nontrivial for m sufficiently divisible.

For m sufficiently divisible, $\mathcal{F}_m \neq 0$ and hence $V^0(\mathcal{F}_m) \neq \phi$ by Lemma 84. This shows that we can find an element $P \in \mathrm{Pic}^0(X)$ with $H^0(X, \omega_X^m \otimes P) \neq 0$. Following the argument of [11, Theorem 3.2] (cf. Theorem 81), $V_m(K_X)$ is a union of torsion translates of subvarieties in $\mathrm{Pic}^0(X)$ for $m \geq 1$ and in particular we can find an element $P' \in \mathrm{Pic}^0(X)_{\mathrm{tor}}$ with $H^0(X, \omega_X^m \otimes P') \neq 0$. Then $H^0(X, \omega_X^{md}) \neq 0$ for $d = \mathrm{ord}(P')$ in $\mathrm{Pic}^0(X)$ and hence $\kappa(X) \geq 0$. ∎

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