THE SIMPLE TRIOD

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A thesis submitted to the faculty of the University
of Utah in partial fulfillment of the requirements
for the degree of

Master of Science

Department of Mathematics

University of Utah
August, 1961

This Thesis for the Masters Degree

by

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August, 1961

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ACKNOWLEDGEMENT

I would like to express my sincere appreciation for the time and guidance Dr. C. E. Burgess has given me during the writing of this thesis. I also wish to thank Dr. Nelson and Dr. Davis for their help as members of my committee.

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SECTION I

INTRODUCTION

In this thesis we wish to show that in a metric space a continuum M is a simple triod if M contains three points a, b, and c such that every point of M - (a+b+c) separates M and a, b, and c do not separate M, and we wish to give some properties of this continuum.

It will first be helpful to know something of the simple arc, usually called an arc. A continuum T is a simple arc if there exist two points a and b of T such that every point of T - (a+b) separates T but a and b do not separate T. By a method of proof similar to that used in Theorem 2.1 of this thesis, it can be shown that if p is a point of T - (a+b) and T - p is the sum of two mutually separated sets H and K, then H and K contain a and b respectively. Here we show that if M is a continuum and p is a point of M - (a+b+c), then all three of a, b, and c cannot be in the same one of the two mutually separated sets. This fact is used to prove that M is irreducible among a, b, and c, which is similar to the fact that an arc is irreducible between its end points. After we show that M is a triod, we use some properties of the arc to show that M has some of the properties of an arc, such as, an arc does not contain a simple closed curve, and for each pair of points of an arc there is a point of the arc which separates the two points.

The following definitions will be used without specific reference to them.

Definition 1. A continuum is a compact connected set in a metric space.

Definition 2. A continuum Q is said to be irreducible between the points a and b if Q contains a + b and no proper subcontinuum of Q contains a + b.

Definition 3. If H, K, and T are proper subsets of the connected point set Q and Q - T is the sum of two mutually separated sets containing H and K respectively, then T is said to separate H from K in Q.

Definition 4. A continuum T is a simple arc if there exist two points a and b of T such that every point of T - (a+b) separates T but a and b do not separate T.

Definition 5. A simple triod is the sum of three arcs such that there exists a point x which is the intersection of each two of the three arcs and which is an end point of each arc.

Definition 6. A continuum which is the sum of two simple arcs having just their end points in common is called a simple closed curve.

The following basic theorems will be assumed without proof, and in some places they will be used without specific reference to them.

Theorem 1.1. If Q is a connected subset of two mutually separated sets, then Q is a subset of one of these two sets.

<u>Theorem 1.2</u>. If Q is a continuum intersecting two disjoint closed sets H and K, then Q contains a continuum Q' which is irreducible between H and K.

<u>Theorem 1.3.</u> If Q is an irreducible continuum between two disjoint closed sets H and K, then Q - H and Q - K are connected.

<u>Theorem 1.4</u>. If Q is an irreducible continuum between two disjoint closed sets H and K, then every point of H·Q is a limit point of Q - H and every point of Q·K is a limit point of Q - K.

<u>Theorem 1.5</u>. If T is a connected subset of the connected point set Q and Q - T is the sum of two mutually separated point sets H and K, then H + T and K + T are connected.

<u>Theorem 1.6</u>. If the point c separates the point <u>a</u> from the point b in the connected point set Q, then b does not separate <u>a</u> from c in Q.

<u>Theorem 1.7</u>. If a point p separates a continuum Q into two mutually separated sets H and K, then H + p and K + p are continua.

Theorem 1.8. A simple arc does not contain a simple closed curve.

The main theorem developed here (Theorem 2.3) can be obtained as a consequence of Theorem 1.1 in Chapter V of Whyburn's *Analytic Topology* [3]. However, Whyburn's theorem follows a more general development than we have undertaken here. Also, in Whyburn's book can be found the theorems we have stated above about arcs and irreducible continua, and original sources can be found in some of the references in Whyburn's book.

An expository development of the properties of an arc which we use here can be found in a thesis by Kenneth Hillam [1]. Also a similar development of irreducible continua can be found in a thesis by Ernest Milton [2].

Throughout this thesis, we will let M be a continuum which contains three points a, b, and c such that every point of M - (a+b+c) separates M and no one of the points a, b, and c separates M.

SECTION II

Theorem 2.1. If p is a point of M - (a+b+c) and M - p is the sum of two mutually separated sets A and B, then neither A nor B contains all three of the points a, b, and c.

Proof. Suppose that all three of the points a, b, and c are in the same one of the sets A and B, say in A.

Let N be a countable dense set in M - (a+b+c), and let p_1, p_2, p_3, ... be the points of M. As B contains some point of N, let n_1, be the least integer so that p_{n_1} is in B. Now $M - p_{n_1} = A_1 + B_1$, where A_1 and B_1 are mutually separated and the point a is in A_1. Since a is in A_1 and A + p is a continuum in $A_1 + B_1$, it follows that A + p is a subset of A_1. Hence a, b, and c are in A_1, and B_1 is a subset of B.

Let n_2 be the least integer so that p_{n_2} is in B_1. Now $M - p_{n_2} = A_2 + B_2$, where A_2 and B_2 are mutually separated and the point a is in A_2. It can be shown, as above, that a, b, and c are in A_2 and that B_2 is a subset of B_1. By continuing this process, we obtain a sequence of points p_{n_1}, p_{n_2}, ... , a sequence of sets A_1, A_2, ... , and a sequence of sets B_1, B_2, ... such that for each i $(i > 1)$:

(1) n_i is the least integer such that p_{n_i} is in B_{i-1};

(2) $M - p_{n_i}$ is the sum of the two mutually separated sets A_i and B_i where A_i contains a, b, and c;

(3) B_i is a subset of B_{i-1};

(4) $A_i + p_{n_i}$ is a subset of A_{i+1}.

It follows that, for each i, $B_i + p_{n_i}$ is a closed subset of B_{i-1}. Hence the sets $B_1 + p_{n_1}$, $B_2 + p_{n_2}$, ... have a point x in common. Since x is in each B_i, x is different from a, b, and c. Therefore $M - x = A_x + B_x$, where A_x and B_x are mutually separated and a is in A_x. As above, it follows that $A + p$ is a subset of A_x so that a, b, and c are in A_x. Since no p_{n_i} is in all sets of the form $B_j + p_{n_j}$ and x is common to all, x is not in the sequence p_{n_1}, p_{n_2}, p_{n_3}, ... Suppose some point p_{n_k} of p_{n_1}, p_{n_2}, ... is in B_x, then x separates a from p_{n_k}. But $M - p_{n_k} = A_k + B_k$, where A_k and B_k are mutually separated and A_k contains a while B_k contains x. So p_{n_k} separates a from x. As this is contrary to Theorem 1.6, it follows that no point of p_{n_1}, p_{n_2}, p_{n_3}, ... can be in B_x.

There is some point p_t of N in B_x and there exists an integer s such that $n_s < t < n_{s+1}$. Then $M - p_{n_s} = A_s + B_s$, where A_s and B_s are mutually separated and a, b, and c are in A_s while $p_{n_{s+1}}$ and p_t are in B_s. The next point in the sequence p_{n_1}, p_{n_2}, p_{n_3}, ... is $p_{n_{s+1}}$ and n_{s+1} is the least integer such that $p_{n_{s+1}}$ is in B_s. But $t < n_{s+1}$, which means n_{s+1} was not a minimum. This is a contradiction. Therefore neither of the sets A and B contains all three of the points a, b, and c.

<u>Theorem 2.2</u>. The continuum M is irreducible among the points a, b, and c; that is, no proper subcontinuum of M contains a, b, and c.

Proof. Suppose some proper subcontinuum J of M contains a, b, and c; then let p be a point of M - J. Now p is different from a, b, and c since a, b, and c are in J. Hence M - p = A + B, where A and B are mutually separated. By Theorem 2.1, neither A nor B contains all three of a, b, and c; so we will consider the case where a is in A and b and c are in B. Since J contains a and is a connected subset of M - p, it follows that J is a subset of A. But b and c are in J and this involves the contradiction that b and c are in A. Therefore M is irreducible among a, b, and c.

The following eight lemmas are used in proving that M is a simple triod.

Lemma 1. If M contains an arc from a to b, then M is a simple triod.

Proof. Suppose $\widehat{ab}$ contains the point c, then it follows from Theorem 2.2 that $\widehat{ab} = M$. But c does not separate M, whereas every point of $\widehat{ab}$ - (a+b) does separate $\widehat{ab}$. Therefore the point c is not in $\widehat{ab}$. Now since M is a continuum, M contains an irreducible subcontinuum Q from the point c to $\widehat{ab}$.

Let $Q' = Q - \widehat{ab}$ and let x be a point of $Q' - c$. Suppose x fails to separate Q, then Q - x is connected. Since $x \neq$ a, b, c, it follows that $M - x = A_1 + B_1$, where A_1 and B_1 are mutually separated and A_1 contains a. Then since $\widehat{ab}$ is a subset of M - x, $\widehat{ab}$ is a subset of A_1 so that A_1 contains both a and b. It follows from Theorem 2.1 that B_1 must contain the point c, and since Q - x contains c and is connected, Q - x is a subset of B_1. Hence $\widehat{ab}$ and Q - x are mutually separated. But since M is irreducible among a, b, and c, it follows that $M = \widehat{ab} + Q$. Hence x is common to $\widehat{ab}$ and Q, contrary to the choice of x in Q'. This means that any point that fails to separate Q other than c is in $\widehat{ab}$.

Suppose two points z and y of Q - c fail to separate Q.

Then z and y are in $\widehat{ab}$ and they are limit points of Q'.

There is an arc $\widehat{az}$ from $\underline{a}$ to z and an arc $\widehat{by}$ from b to y such that $\widehat{az} \cdot \widehat{by} = \emptyset$ (if not reverse z and y). Since $\widehat{az} \cdot \widehat{by} = \emptyset$, $\widehat{ab} \neq \widehat{az} + \widehat{by}$, and so there is some point e in $\widehat{ab}$ which is not in $\widehat{az} + \widehat{by}$. Then $Q' + \widehat{az} + \widehat{by}$ is connected since Q' is connected, and $Q' + \widehat{az} + \widehat{by}$ contains $\underline{a}$, b, and c. The point e is neither $\underline{a}$, b, nor c, so $M - e = A_2 + B_2$, where A_2 and B_2 are mutually separated and A_2 contains $\underline{a}$. Since A_2 contains $\underline{a}$ and $Q' + \widehat{az} + \widehat{by}$ is connected, A_2 contains $Q' + \widehat{az} + \widehat{by}$. Hence $\underline{a}$, b, and c are in A_2, contrary to Theorem 2.1. Therefore only one point d of Q - c fails to separate Q. Since Q contains at least two non-separating points, d and c are these two, and hence Q is an arc from c to d. The point d is in $\widehat{ab}$, so $\widehat{ab}$ contains an arc, $\widehat{ad}$ from $\underline{a}$ to d and an arc $\widehat{bd}$ from b to d such that $\widehat{ad} \cdot \widehat{bd} = d$. This gives M as the sum of three arcs $\widehat{ad}$, $\widehat{bd}$, and $\widehat{cd}$ such that d is the intersection of each two of them. Hence M is a simple triod.

Lemma 2. If M contains no arc from a to b and K is a proper subcontinuum of M irreducible from a to b, then there is an arc in M which is irreducible from c to K .*

Proof. Since M is irreducible among a, b, and c, K cannot contain c. Since K is not an arc, there is a point d of K - (a+b) which fails to separate K. Now M - d = A + B, where A and B are mutually separated and a is in A. Then since K - d is connected, it is a subset of A, and so b is also in A. By Theorem 2.1, the point c must be in B. It follows that B + d is a continuum which contains both d and c.

Suppose B + d is not an arc from c to d. Then some point x of B - c fails to separate B + d. Since x $\neq$ d, x is not a point of K. Hence B + d - x + K is connected. But M - x is not connected, and M - x = B + d - x + K.

This contradiction shows that B + d must be an arc from c to d. Furthermore, B + d is irreducible from c to K since K·(B+d) = d.

* Our complete development will show that the hypothesis of Lemma 2 is false. However, we find that this lemma is useful in developing a proof of our main theorem.

Lemma 3. If K is a subcontinuum of M irreducible from a to b, d is a point of K, and $\widehat{cd}$ is an arc in M irreducible from c to K, then every point of K - (a+b+d) separates K.

Proof. Suppose some point x in K - (a+b+d) fails to separate K; then K - x is connected. By Theorem 2.2, M is irreducible among a, b, and c, so $M = K + \widehat{cd}$. Since the point d is in K - x and $K \cdot \widehat{cd} = d$, $\widehat{cd}$ being irreducible from c to K, it follows that $K - x + \widehat{cd}$ is connected. Since $x \neq a,b,c$, then by definition M - x is not connected. But this involves a contradiction as $M - x = K - x + \widehat{cd}$. Therefore, every point of K - (a+b+d) separates K.

Lemma 4. If M contains two intersecting arcs such that one of them contains a and the other contains c, then M is a triod.

Proof. Let $\widehat{ad_2}$ and $\widehat{cd_1}$ be arcs containing a and c respectively such that $\widehat{ad_2}$ and $\widehat{cd_1}$ have a point x in common. Then $\widehat{ad_2} + \widehat{cd_1}$ contains an arc from a to c. Hence it follows from Lemma 1 that M is a simple triod.

<u>Lemma</u> 5. The continuum M contains two proper subcontinua K_1 and K_2 such that K_1 is irreducible between some pair of the points $\underline{a}$, b, and c and K_2 is irreducible between some other pair of these points.

<u>Proof</u>. Let p_1 be a point of M - (a+b+c). Then M - p_1 = $A_1 + B_1$, where A_1 and B_1 are mutually separated. By Theorem 2.1, $\underline{a}$, b, and c cannot all three be in one of the sets A_1 and B_1. Let us consider the case where $\underline{a}$ and b are in A_1 and c is in B_1. Now $A_1 + p_1$ is a proper subcontinuum of M which contains $\underline{a}$ and b. Therefore M contains a proper subcontinuum K_1 which is irreducible between $\underline{a}$ and b.

If K_1 is an arc, then by Lemma 1, M is a triod; and thus a proper subcontinuum K_2, of M, irreducible from b to c can be found. But if K_1 is not an arc, then by Lemma 2, there is an arc $\widehat{cd_1}$ irreducible from c to K_1. By Lemma 3, every point of K_1 - (a+b+d_1) separates K_1. Choose a point p_2 of K_1 - (a+b+d_1).

Now $K_1 - p_2 = A_2 + B_2$, where A_2 and B_2 are mutually separated and A_2 contains a while B_2 contains b, since K_1 is irreducible between a and b. Either A_2 or B_2 contains d_1, say B_2 does. Then $B_2 + p_2$ is a continuum containing b and d_1. Now $B_2 + p_2 + \widehat{cd_1}$ is a proper subcontinuum of M containing b and c. Therefore M contains a proper subcontinuum K_2 irreducible between b and c.

Lemma 6. If M is not a triod and M contains irreducible subcontinua K_1 and K_2 from a to b and from b to c respectively, and arcs $\widehat{cd_1}$ and $\widehat{ad_2}$ irreducible from c to K_1 and from a to K_2 respectively, such that d_1 and d_2 fail to separate K_1 and K_2 respectively, then $K_1 \supset \widehat{ad_2}$ and $K_2 \supset \widehat{cd_1}$. *

Proof. Since, by Theorem 2.2, M is irreducible among a, b, and c, it follows that $M = K_1 + \widehat{cd_1}$ and $M = K_2 + \widehat{ad_2}$. Since M is not a triod, $\widehat{cd_1}$ and $\widehat{ad_2}$ do not intersect. Therefore K_1 must contain $\widehat{ad_2}$ and K_2 must contain $\widehat{cd_1}$.

* Refer to footnote on page 12.

<u>Lemma 7</u>. If in Lemma 6, H_1 is an irreducible continuum in K_1 from d_2 to b, then $H_1 \cdot \widehat{ad_2} = d_2$.

<u>Proof</u>. Since H_1 is an irreducible continuum from d_2 to b, $H_1 - d_2$ is connected and contains b. Now $K_1 = H_1 + \widehat{ad_2}$, since K_1 is irreducible from <u>a</u> to b. By Lemma 4, H_1 must contain d_1 because K_1 contains d_1.

If $H_1 \cdot \widehat{ad_2} \neq d_2$, then some point x of H_1 is in $\widehat{ad_2}$ such that $x \neq a, d_2$. Since $\widehat{ad_2} - d_2$ is connected, $H_1 - d_2 + \widehat{ad_2} - d_2 + \widehat{cd_1}$ is connected. But this is $M - d_2$ which in not connected. Therefore $H_1 \cdot \widehat{ad_2} = d_2$.

<u>Lemma 8</u>. If M is not a triod, then M contains a proper subcontinuum H such that H has the same properties as M.*

<u>Proof</u>. Let K_1 and K_2 be irreducible proper subcontinua in M from <u>a</u> to b and from b to c respectively. Since M is not a triod, it follows from Lemma 1 that there exists points d_1 and d_2 which lie in but fail to separate K_1 and K_2 respectively. Then

* Refer to footnote on page 12.

by Lemmas 2 and 3, there exist arcs $\widehat{cd_1}$ and $\widehat{ad_2}$ irreducible from c and a to K_1 and K_2 respectively. And by Lemma 6, $K_1 \supset \widehat{ad_2}$ while $K_2 \supset cd_1$. Then let $H = H_1 + H_2$ where H_1 and H_2 are irreducible continua in K_1 and K_2 from b to d_2 and from b to d_1 respectively. Hence H is the sum of two continua with a common point b, so H is a continuum.

Suppose some point d_4 of H, where $d_4 \neq d_1$, d_2, b, fails to separate H. Then $H - d_4$ is connected, and by Lemma 7, $H - d_4 + \widehat{ad_2} + \widehat{cd_1}$ is connected. Since M is irreducible among a, b, and c, $M = \widehat{ad_2} + \widehat{cd_1} + H$. This means $M - d_4 = H - d_4 + \widehat{ad_2} + \widehat{cd_1}$. But $M - d_4$ is not connected. Therefore d_4 must separate H. Since H is a continuum, at least two of the points d_1, d_2, and b fail to separate H. Suppose one of these does separate H. Then all but two points of H separate H. Hence H is an arc and $\widehat{ad_2} + H$ contains an arc containing a and b. Thus by Lemma 1, M would be a simple triod, but M is not a triod. Therefore d_1, d_2, and b do not separate H, but every point of $H - (b+d_1+d_2)$ separates H.

Theorem 2.3. The continuum M is a triod.

Proof. Suppose M is not a triod. Them M contains two proper subcontinua K_1 and K_2 satisfying the conclusion of Lemma 5. Consider the case where K_1 is irreducible between a and b and K_2 is irreducible between b and c.

If one of K_1 and K_2 were an arc, then by Lemma 1, M would be a triod. Therefore neither K_1 nor K_2 is an arc. Hence there is a point d_1 of K_1 where $d_1 \neq a,b$, such that $K_1 - d_1$ is connected, and there is a point d_2 of K_2, where $d_2 \neq b,c$, such that $K_2 - d_2$ is connected. It follows from Lemmas 2 and 3 that there is an arc $\widehat{cd_1}$ in M irreducible from c to K_1 and there is an arc $\widehat{ad_2}$ in M irreducible from a to K_2. The arcs $\widehat{ad_2}$ and $\widehat{cd_1}$ have no point in common since, if they did, by Lemma 4, M would be a triod.

Since K_1 and K_2 are continua, and by Lemma 6, $K_1 \supset \widehat{ad_2}$ and $K_2 \supset \widehat{cd_1}$, then K_1 contains an irreducible subcontinuum H_1 from d_2 to b and K_2 contains an irreducible subcontinuum H_2 from d_1 to b. Let $H = H_1 + H_2$. Then by Lemma 8, and its proof, H is a continuum such that every point of $H - (d_1 + d_2 + b)$ separates H whereas d_1, d_2, and b do not separate H.

If there is an arc from any one of d_1, d_2, b to one of the other two, then M contains an arc containing two of a, b, and c and so is a triod by Lemma 1. For example, if there is an arc $\widehat{d_1 d_2}$ from d_2 to d_1, then $\widehat{cd_1} + \widehat{d_1 d_2} + \widehat{ad_2}$ is an arc containing a and c. Therefore, there is no arc from one of d_1, d_2, b to one of the remaining two. Now H contains two proper subcontinua N_1 and N_2 satisfying the conclusion of Lemma 5.

We will consider two cases which will take care of all possibilities. First the case where N_1 is irreducible from

d_1 to b and N_2 is irreducible from d_2 to b. Since N_2 is not an arc, there exists a point t_1 of $N_2 - (d_2 + b)$ such that $N_2 - t_1$ is connected. It follows from Lemmas 2 and 3 that there is an arc $\widehat{d_1t_1}$ in H irreducible from d_1 to N_2. The set H is a subset of K_1 since $H \cdot \widehat{cd_1} = d_1$ (Lemma 7) and $M = K_1 + \widehat{cd_1}$. Therefore N_2 is a subset of K_1. Since $\widehat{d_1t_1} + \widehat{cd_1}$ contains $\widehat{cd_1}$ as a proper subset, then $N_2 + \widehat{ad_2}$ is a proper subset of K_1. But $N_2 + \widehat{ad_2}$ is a continuum containing a and b and cannot be a proper subset of K_1 since K_1 is irreducible between a and b. Therefore M is a triod.

The second case is where N_1 is irreducible from d_1 to d_2 and N_2 is irreducible from d_2 to b. Now since N_2 is not an arc, there exists a point t_2 of $N_2 - (d_2 + b)$ such that $N_2 - t_2$ is connected. It follows from Lemmas 2 and 3 that there is an arc $\widehat{d_1t_2}$ in H irreducible from d_1 to N_2. The set N_2 is a subset of K_1 since H is a subset of K_1. Since $\widehat{cd_1}$ is a proper subset of $\widehat{d_1t_2} + \widehat{cd_1}$, then $N_2 + \widehat{ad_2}$ is a proper subset of K_1. But $N_2 + \widehat{ad_2}$ is a continuum and contains a and b which contradicts the fact that K_1 is irreducible between a and b. Therefore M is a triod.

Theorem 2.4. There exists a point p of M - (a+b+c) such that M - p = A + B + C, where A, B, and C are mutually separated and A contains a, B contains b, and C contains c.

Proof. Since M is a triod, there is a point x such that M is the sum of three arcs $\widehat{ax}$, $\widehat{bx}$, and $\widehat{cx}$, where x is the intersection of each two of them. Then $M - x = (\widehat{ax} - x) + (\widehat{bx} - x) + (\widehat{cx} - x)$. These three sets are mutually separated and each contains one of a, b, and c.

Theorem 2.5. If e and f are two distinct points of M - (a+b+c), then there exists a point p of M such that M - p = A + B, where A and B are mutually separated and A contains e while B contains f.

Proof. Since M is a triod, there is a point x such that M is the sum of three arcs $\widehat{ax}$, $\widehat{bx}$, and $\widehat{cx}$, where x is the intersection of each two of them. If both e and f are in one of the arcs $\widehat{ax}$, $\widehat{bx}$, and $\widehat{cx}$, say in $\widehat{ax}$, then there exists a point p of $\widehat{ax}$ such that $\widehat{ax} - p = A_1 + B_1$, where A_1 and B_1 are mutually separated and A_1 contains e while B_1 contains f. Either A_1 or B_1 contains x, say A_1 does. This means that B_1 contains a.

Then $M - p = A_2 + B_2$, where A_2 and B_2 are mutually separated and B_2 contains x. Since $A_1 + \widehat{cx} + \widehat{bx}$ is connected and contains x but not p, it is a subset of B_2. This set contains b and c, so A_2 must contain a by Theorem 2.1 and therefore, A_2 must contain B_1. Hence e is in B_2 and f is in A_2.

Suppose e and f are in different arcs of $\widehat{ax}$, $\widehat{bx}$, and $\widehat{cx}$, say e is in $\widehat{ax}$ and f is in $\widehat{bx}$ but neither e nor f is x. Then $M - x = (\widehat{ax} - x) + (\widehat{bx} - x) + (\widehat{cx} - x)$ such that $\widehat{ax} - x$, $\widehat{bx} - x$, and $\widehat{cx} - x$ are mutually separated. Since e is in $\widehat{ax} - x$ and f is in $\widehat{bx} - x$, the point x separates e from f in M.

Theorem 2.6. The continuum M does not contain a simple closed curve.

Proof. Suppose M contains a simple closed curve J. Since M is a triod, $M = \widehat{ax} + \widehat{bx} + \widehat{cx}$, where x is the intersection of each two of these three arcs. From the proof of Theorem 2.4, $M - x = (\widehat{ax} - x) + (\widehat{bx} - x) + (\widehat{cx} - x)$ where these three sets are mutually separated. Since J is a simple closed curve, J - x is connected. Hence J - x is a subset of one of the sets $\widehat{ax} - x$, $\widehat{bx} - x$, and $\widehat{cx} - x$, say it is a subset of $\widehat{ax} - x$. Then J is a subset of $\widehat{ax}$. But an arc cannot contain a simple closed curve by Theorem 1.8. Therefore M does not contain a simple closed curve.

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