DISCREPANCIES OF NORMAL VARIETIES

by

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A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

May 2012

The University of Utah Graduate School

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ABSTRACT

We give an example of a non $\mathbb{Q}$-Gorenstein variety whose canonical divisor has an irrational valuation and an example of a non $\mathbb{Q}$-Gorenstein variety which is canonical but not klt. We also give an example of an irrational jumping number and we prove that there are no accumulation points for the jumping numbers of normal non-$\mathbb{Q}$-Gorenstein varieties with isolated singularities. We prove that the canonical ring of a canonical variety in the sense of [dFH09] is finitely generated. We prove that canonical varieties are klt if and only if $\mathscr{R}(-K_X)$ is finitely generated. We introduce a notion of nefness for non-$\mathbb{Q}$-Gorenstein varieties and study some of its properties. We then focus on the properties of non-$\mathbb{Q}$-Gorenstein toric varieties, with particular attention to minimal log discrepancies.

To Marylinda,

the love of my life.

Because she always believed in us,

and followed me in this crazy adventure making it special...

CONTENTS

ACKNOWLEDGMENTS

I would like to thank my supervisor C. D. Hacon for his constant support and the several helpful discussions and suggestions. I would also like to thank E. Macrì and T. de Fernex for many useful comments.

Thanks to all the members of my committee and to all the professors, the students and the staff of the Mathematics department for creating a nice enviroment throughout these years.

A special thanks to my family and friends in Italy for being constantly present and supportive no matter the distance.

CHAPTER 1

INTRODUCTION

The aim of this work is to investigate some surprising features of singularities of normal varieties in the non-$\mathbb{Q}$-Gorenstein case as defined by T. de Fernex and C. D. Hacon (cf. [dFH09]). In this paper the authors focus on the difficulties of extending some invariants of singularities in the case that the canonical divisor is not $\mathbb{Q}$-Cartier. Instead of the classical approach where we modify the canonical divisor by adding a boundary, an effective $\mathbb{Q}$-divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, de Fernex and Hacon introduce a notion of *pullback* of (Weil) $\mathbb{Q}$-divisors which agrees with the usual one for $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. In this way, for any birational morphism of normal varieties $f : Y \to X$, they are able to define relative canonical divisors $K_{Y/X} = K_Y + f^*(-K_X)$ and $K^-_{Y/X} = K_Y - f^*(K_X)$. The two definitions coincide when K_X is $\mathbb{Q}$-Cartier and using $K_{Y/X}$ and $K^-_{Y/X}$ de Fernex and Hacon extended the definitions of canonical singularities, klt singularities and multiplier ideal sheaves to this more general context. In this dissertation we investigate in detail the properties of these singularities in the non-$\mathbb{Q}$-Gorenstein setting.

In the first chapter we review some basic notions in algebraic geometry that we will heavily use. In particular we recall the notion of divisor and some results in intersection theory.

In the second chapter we will give the basic definitions and recall the main properties of ample and nef divisors.

In the third chapter we will discuss the classical theory of singularities of $\mathbb{Q}$-Gorenstein algebraic varieties. This will be the building block for the rest of the work. In fact the main purpose of this dissertation is to understand how to extend these definitions to varieties with non-$\mathbb{Q}$-Gorenstein singularities.

In the fourth chapter we finally introduce the definition of singularities for non-$\mathbb{Q}$-Gorenstein varieties of [dFH09]. In this setting some of the properties characterizing the usual notions of singularity (see [KM98, Section 2.3]) seem to fail due to the asymptotic nature of the definition of the canonical divisors.

We focus on three properties that for $\mathbb{Q}$-Gorenstein varieties are straightforward:

- The relative canonical divisor always has rational valuations (cf. [Kol08, Theorem 92]).
- A canonical variety is always Kawamata log terminal (cf. [KM98, Definition 2.34]).
- The jumping numbers are a set of rational numbers that have no accumulation points (cf. [Laz04b, Lemma 9.3.21]).

In the second section, we show that if X is klt in the sense of [dFH09], then the relative canonical divisor has rational valuations. We give an example of a (non klt) variety X with an irrational valuation and we use it to find an irrational jumping number (Theorem 5.20).

In the third section we give an example of a variety with canonical but not klt singularities (Theorem 5.21) and we prove that the finite generation of the canonical ring implies that the relative canonical model has canonical singularities (Proposition 5.24).

In the last section, using one of the main results in [dFH09], namely that every effective pair (X, Z) admits m-compatible boundaries for $m \geq 2$ (see Theorem 5.10 below), we show that for a normal variety whose singularities are either klt or isolated, it is never possible to have accumulation points for the jumping numbers (Theorem 5.26).

The last chapter will be focused on properties related to the finite generation of the canonical ring.

In the first section we show that if X is canonical in the sense of [dFH09], then the relative canonical ring $\mathscr{R}_X(K_X)$ is a finitely generated $\mathcal{O}_X$-algebra (Theorem 6.2). Thus, if X is canonical, there exists a small proper birational morphism $\pi : X' \to X$ such that $K_{X'}$ is $\mathbb{Q}$-Cartier and π-ample. As a corollary we obtain that the canonical ring of any normal variety with canonical singularities (in the sense of [dFH09]) is finitely generated.

We next turn our attention to log-terminal singularities. In the fourth chapter, we gave an example of canonical singularities that are not log-terminal. In this section we show that, if X is canonical, then finite generation of the relative anti-canonical ring $\mathscr{R}_X(-K_X)$ is equivalent to X being log-terminal (Proposition 6.5).

In the second section we introduce a notion of nefness for Weil divisors (on non-$\mathbb{Q}$-factorial varieties). We call such divisors quasi-nef (q-nef) and we study their basic properties. We prove that if X is a normal variety with canonical singularities such that K_X is q-nef, then $X' = \mathrm{Proj}_X(\mathscr{R}_X(K_X))$ is a minimal model.

In the last section, we focus our attention on toric varieties. We give a new natural definition of minimal log discrepancies (MLD) in the new setting and we prove that even in the toric case they do not satisfy the ACC conjecture.

CHAPTER 2

DIVISORS AND INTERSECTION THEORY

In this first chapter we review the main definitions and properties that we will use throughout the dissertation.

2.1 Notation and Conventions

- We work over the complex numbers $\mathbb{C}$.
- A *scheme* is a separated complete algebraic scheme of finite type over $\mathbb{C}$.
- A *variety* is a reduced and irreducible scheme. A point will always be a closed point.
- Throughout this work, X will denote a complex proper quasi-projective variety.

2.2 Integral Divisors

We will denote $\mathcal{M}_X = \mathbb{C}(X)$ the constant sheaf of rational functions on X. It contains the structure sheaf $\mathcal{O}_X$ as a subsheaf, and so there is an inclusion $\mathcal{O}_X^* \subseteq \mathcal{M}_X^*$ of sheaves of multiplicative abelian groups.

Definition 2.1 (Cartier divisors)**.** *A* Cartier divisor *on X is a global section of the quotient sheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. We denote by $Div(X)$ the group of all such, so that*

$$Div(X) = \Gamma(X, \mathcal{M}_X^*/\mathcal{O}_X^*).$$

A Cartier divisor $D \in \mathrm{Div}(X)$ can be described by giving an open cover $\{U_i\}$ of X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{M}_X^*)$, such that for each i, j,

$$f_i = g_{ij} f_j \text{ for some } g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*).$$

The function f_i is called a *local equation* for D at any point $x \in U_i$. Two such collections determine the same Cartier divisor if there is a common refinement

$\{V_k\}$ of the open coverings on which they are defined so that they are given by data $\{(V_k, f_k)\}$ and $\{(V_k, f'_k)\}$ with

$$f_k = h_k f'_k \text{ on } V_k \text{ for some } h_k \in \Gamma(V_k, \mathcal{O}_X^*).$$

If $D, D' \in \mathrm{Div}(X)$ are represented respectively by data $\{(U_i, f_i\}$ and $\{U_i, f'_i\}$, then $D + D'$ is given by $\{U_i, f_i f'_i\}$. The *support* of a divisor $D = \{U_i, f_i\}$ is the set of points $x \in X$ at which a local equation of D at x is not a unit in $\mathcal{O}_{X,x}$. D is effective if $f_i \in \Gamma(U_i, \mathcal{O}_X)$ is regular on U_i.

Definition 2.2 (Weil divisors). *Let X be a noetherian integral separated scheme such that every local ring $\mathcal{O}_{X,x}$ of X of dimension 1 is regular ([Har77]). A* prime divisor *on X is a closed integral subscheme Y of codimension one. A* Weil divisor *is an element of the free abelian group $WDiv(X)$ generated by the prime divisors. We write a divisor as a finite sum $D = \sum n_i Y_i$ where the Y_i are prime divisors, the n_i are integers. If all the $n_i \geq 0$, we say that D is effective.*

If D is a prime divisor on X, let $\eta \in D$ be its generic point. Then the local ring $\mathcal{O}_{X,\eta}$ is a discrete valuation ring with quotient field K, the function field of X. We call the corresponding valuation v_D the *valuation of* D. Now let $f \in K^*$ be a non-zero rational function on X. Then $v_D(f)$ is an integer.

Definition 2.3. *We define the* divisor *of f, denoted $div(f)$, by*

$$div(f) = \sum v_D(f) \cdot D,$$

where the sum is taken over all prime divisors on X. Any divisor which is equal to the divisor of a function is called a principal *divisor.*

Definition 2.4. *Two Weil divisors D and D' are said to be* linearly equivalent, *written $D \equiv_{lin} D'$, if $D - D'$ is a principal divisor.*

Proposition 2.5 (Weil & Cartier). *([Har77] II.6.11) Let X be an integral, separated, noetherian scheme, all of whose local rings are unique factorization domains. The group $WDiv(X)$ of Weil divisors on X is isomorphic to the group of Cartier divisors $Div(X)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

Another important aspect of divisor theory is the relation with the concept of line bundles.

A Cartier divisor $D \in \mathrm{Div}(X)$ determines a line bundle $\mathcal{O}_X(D)$ on X leading to a canonical homomorphism

$$\mathrm{Div}(X) \to \mathrm{Pic}(X) \quad , \quad D \mapsto \mathcal{O}_X(D)$$

of abelian groups, where $\mathrm{Pic}(X)$ denotes the Picard group of isomorphism classes of line bundles on X.

If D is given by the data $\{U_i, f_i\}$, then one can build $\mathcal{O}_X(D)$ by using the g_{ij} of Definition 2.1 as transition functions.

One can also view $\mathcal{O}_X(D)$ as the image of D under the connecting homomorphism

$$\mathrm{Div}(X) = \Gamma(\mathcal{M}_X^*/\mathcal{O}_X^*) \to H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$$

determined by the exact sequence $0 \to \mathcal{O}_X^* \to \mathcal{M}_X^* \to \mathcal{M}_X^*/\mathcal{O}_X^* \to 0$ of sheaves on X, where

$$\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \Leftrightarrow D_1 \equiv_{lin} D_2.$$

If D is effective then $\mathcal{O}_X(D)$ carries a non-zero global section $s = s_D \in \Gamma(X, \mathcal{O}_X(D))$ with $\mathrm{div}(s) = D$. In general $\mathcal{O}_X(D)$ has a rational section with the analogous property.

Note 2.6. *There are natural hypotheses to guarantee that every line bundle arises from a divisor:*

- *If X is reduced and irreducible, then the homomorphism $Div(X) \to Pic(X)$ is surjective.*
- *If X is projective then the same statement holds even if it is nonreduced.*

Let $\mathscr{L}$ be a line bundle on X, and $V \subseteq H^0(X, \mathscr{L})$ a non-zero subspace of finite dimension. We denote by $|V| = \mathbb{P}_{sub}(V)$ the projective space of one-dimensional subspaces of V. When X is a complete variety there is a correspondence between this set and the complete linear system of D (where $\mathscr{L} = \mathcal{O}_X(D)$ and

$V = H^0(X, \mathscr{L})$), that is the set of all effective divisors linearly equivalent to the divisor D and is denoted $|D|$.

Evaluation of sections in V gives rise to a morphism:

$$\mathrm{eval}_V : V \otimes_{\mathbb{C}} \mathscr{L}^* \to \mathcal{O}_X$$

of vector bundles on X.

Definition 2.7. *The* base ideal *of $|V|$, written*

$$\mathbf{b}(|V|) = \mathbf{b}(X, |V|) \subseteq \mathcal{O}_X,$$

is the image of the map $V \otimes_{\mathbb{C}} \mathscr{L}^* \to \mathcal{O}_X$ determined by eval_V. The base locus

$$\mathbf{Bs}(|V|) \subseteq X$$

of $|V|$ is the closed subset of X cut out by the base ideal $\mathbf{b}(|V|)$ (set of points at which all the sections in V vanish). When $V = H^0(X, \mathscr{L})$ or $V = H^0(X, \mathcal{O}_X(D))$ are finite-dimensional, we write respectively $\mathbf{b}(|\mathscr{L}|)$ and $\mathbf{b}(|D|)$ for the base ideals of the indicated complete linear series.

Definition 2.8 (Free linear series)**.** *We say that $|V|$ is* free*, or* base-point free*, if its base locus is empty (that is $\mathbf{b}(|V|) = \mathcal{O}_X$). A divisor D or line bundle $\mathscr{L}$ is* free *if the corresponding complete linear series is so. In the case of line bundles we say that $\mathscr{L}$ is* generated by its global sections *or* globally generated *(for each point $x \in X$ we can find a section $s = s_x \in V$ such that $s(x) \neq 0$).*

Assume now that $\dim V \geq 2$, and set $\mathbf{B} = \mathbf{Bs}(|V|)$. Then $|V|$ determines a morphism

$$\varphi : \varphi_{|V|} : X - B \to \mathbb{P}(V)$$

from the complement of the base locus in X to the projective space of one-dimensional quotients of V. Given $x \in X - B$, $\varphi(x)$ is the hyperplane in V consisting of those sections vanishing at x. If we choose a basis $s_0, \ldots, s_r \in V$, this amounts to saying that φ is given in homogeneous coordinates by the expression

$$\varphi(x) = [s_0(x), \ldots, s_r(x)] \in \mathbb{P}^r.$$

When X is a variety it is useful to view $\varphi_{|V|}$ as a rational mapping $\varphi : X \dashrightarrow \mathbb{P}(V)$. If $|V|$ is free then the morphism is globally defined.

When $\mathbf{B} = \emptyset$ a morphism to projective space gives rise to a linear series. Suppose given a morphism

$$\varphi : X \to \mathbb{P} = \mathbb{P}(V),$$

then the pullback of sections via φ realizes $V = H^0(\mathbb{P}, \mathcal{O}_P(1)))$ as a subspace of $H^0(X, \varphi^*(\mathcal{O}_{\mathbb{P}}(1))$, and $|V|$ is a free linear series on X. Moreover, φ is identified with the corresponding morphism $\varphi_{|V|}$.

2.3 Intersection Theory

Given Cartier divisors $D_1, \dots, D_k \in \mathrm{Div}(X)$ together with an irreducible subvariety $V \subseteq X$ of dimension k, we want to define the *intersection number*

$$(D_1 \cdot \ldots \cdot D_k \cdot V) \overset{\text{not}}{=} \int_V D_1 \cdot \ldots \cdot D_k.$$

We know that each of the line bundles $\mathcal{O}_X(D_i)$ has a Chern class $c_1(\mathcal{O}_X(D_i)) \in H^2(X, \mathbb{Z})$, the cohomology group being ordinary singular cohomology of X from the classical topology. The cup product of these classes is then an element

$$c_1(\mathcal{O}_X(D_1)) \wedge \ldots \wedge c_1(\mathcal{O}_X(D_k)) \in H^{2k}(X, \mathbb{Z}).$$

Denoting by $[V] \in H_{2k}(X, \mathbb{Z})$ the fundamental class of V, cap product leads finally to an integer

$$(D_1 \cdot \ldots \cdot D_k \cdot V) \overset{\text{def}}{=} (c_1(\mathcal{O}_X(D_1)) \wedge \ldots \wedge c_1(\mathcal{O}_X(D_k))) \cap [V] \in \mathbb{Z}.$$

that is the intersection number.

Note 2.9. *Let $n = \dim X$, then*

$$(D_1 \cdot \ldots \cdot D_n) = \int_X D_1 \cdot \ldots \cdot D_n,$$

$$(D^n) = \int_X \underbrace{D \cdot \ldots \cdot D}_{n-times}.$$

The most important features of this product are:

- the integer $(D_1 \cdot \ldots \cdot D_n)$ is symmetric and multilinear as a function of its arguments;
- $(D_1 \cdot \ldots \cdot D_n)$ depends only on the linear equivalence classes of the D_i;
- if $D_1, \ldots, D_n$ are effective divisors that meet transversely at smooth points of X, then $(D_1 \cdot \ldots \cdot D_n) = \#\{D_1 \cap \ldots \cap D_n\}$.

Note 2.10. *Given an irreducible subvariety $V \subseteq X$ of dimension k, $(D_1 \cdot \ldots \cdot D_k \cdot V)$ is then defined by replacing each divisor D_i with a linearly equivalent divisor D_i' whose support does not contain V (assuming these exist), and intersecting the restrictions of the D_i' on V.*

Furthermore, the intersection product satisfies the projection formula*: if $f : Y \to X$ is a generically finite surjective proper map, then*

$$\int_Y f^*D_1 \cdot \ldots \cdot f^*D_n = (degf) \cdot \int_X D_1 \cdot \ldots \cdot D_n.$$

Definition 2.11. *Two Cartier divisors D_1, D_2 are* numerically equivalent*, $D_1 \equiv_{num} D_2$, if $(D_1.C) = (D_2.C)$ for every irreducible curve $C \subseteq X$.*
Equivalently if $(D_1.\gamma) = (D_2.\gamma)$ for all one-cycles γ in X.

Definition 2.12. *A divisor or line bundle is* numerically trivial *if it is numerically equivalent to zero, and $Num(X) \subseteq Div(X)$ is the subgroup consisting of all numerically trivial divisors.*

The Néron-Severi group *of X is the free abelian group*

$$N^1(X) = DivX/NumX$$

of numerical equivalence classes of divisors on X.

Proposition 2.13. *The Néron-Severi group $N^1(X)$ is a free abelian group of finite rank.*

The rank of $N^1(X)$ is called the Picard number *of X and denoted $\rho(X)$.*

Lemma 2.14. *Let X be a variety, and let $D_1, \ldots, D_k, D_1', \ldots, D_k' \in DivX$ be Cartier divisors on X. If $D_i \equiv_{num} D_i'$ for each i, then*

$$(D_1 \cdot \ldots \cdot D_k \cdot [V]) = (D_1' \cdot ... \cdot D_k' \cdot [V])$$

for every subscheme $V \subseteq X$ of pure dimension k.

The Lemma allows the following:

Definition 2.15. *Given classes $\delta_1, \ldots, \delta_k \in N^1(X)$, we denote by $(\delta_1 \cdot \ldots \cdot \delta_k \cdot [V])$ the intersection number of any representatives of the classes in question.*

Definition 2.16. *Let X be a variety of dimension n, and let $\mathscr{F}$ be a coherent sheaf on X. Then the rank $rank(\mathscr{F})$ of $\mathscr{F}$ is the length of the stalk of $\mathscr{F}$ at the generic point of X.*

Theorem 2.17 (Asymptotic Riemann-Roch, I). *Let X be a projective variety of dimension n and let D be a divisor on X. Then the Euler characteristic $\chi(X, \mathcal{O}_X(mD))$ is a polynomial of degree $\leq n$ in m, with*

$$\chi(X, \mathcal{O}_X(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

More generally, for any coherent sheaf $\mathscr{F}$ on X,

$$\chi(X, \mathscr{F} \otimes \mathcal{O}_X(mD)) = rank(\mathscr{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Proof. [Kol96] Let Y/S be a Noetherian scheme. Let $\mathscr{G}$ be a coherent sheaf on Y whose support is proper over a 0-dimensional subscheme of S. We define the *Grothendieck group* of Y, $K(Y)$, as the abelian group generated by the symbols $\overline{\mathscr{G}}$ where for every short exact sequence

$$0 \to \mathscr{G}_1 \to \mathscr{G}_2 \to \mathscr{G}_3 \to 0$$

we have

$$\overline{\mathscr{G}}_2 = \overline{\mathscr{G}}_1 + \overline{\mathscr{G}}_3.$$

We denote by $K_r(Y) \subseteq K(Y)$ the subgroup generated by those $\mathscr{G}$ whose support has dimension at most r.

Let $\mathscr{L}$ be an invertible sheaf on X. We define an endomorphism of $K(X)$

$$c_1(\mathscr{L}).\overline{\mathscr{F}} = \overline{\mathscr{F}} - \overline{\mathscr{L}^{-1} \otimes \mathscr{F}}.$$

Let us assume that $m \geq r = \dim \operatorname{Supp}(\mathscr{F})$. The intersection number of $\mathcal{O}_X(D_1), \ldots, \mathcal{O}_X(D_m)$ with $\mathscr{F}$ is defined by

$$(\mathcal{O}_X(D_1) \cdot \ldots \cdot \mathcal{O}_X(D_m) \cdot \mathscr{F}) = \chi(X, c_1(\mathcal{O}_X(D_1)) \cdot \ldots \cdot c_1(\mathcal{O}_X(D_m)) \cdot \mathscr{F}).$$

Claim 2.18. *In $K_r(Y)$ we have the following equivalence:*

$$\mathscr{F}(mD) = \sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{O}_X(D))^i.\mathscr{F}.$$

Proof of the Claim: Setting $n = -m$ we want to calculate $\mathscr{F}(mD)$ considering $\mathscr{F} \in K_r(X)$.

We have the formal identity

$$(1+x)^n = \sum_{i \geq 0} \binom{n}{i} x^i.$$

If we substitute $x = y^{-1} - 1$ and use that $\binom{-m}{i} = (-1)^i \binom{m+i-1}{i}$ we obtain that

$$y^m = \sum_{i \geq 0} \binom{m+i-1}{i} (1 - y^{-1})^i.$$

If we consider y as the operator $\mathscr{F} \mapsto \mathscr{F}(D)$ we obtain that $1 - y^{-1} = c_1(\mathcal{O}_X(D))$. Also $c_1(\mathcal{O}_X(D))^i.\mathscr{F} = 0$ for $i > r$ by the properties of intersection theory, we have

$$\mathscr{F}(mD) = \sum_{i=0}^{r} \binom{m+i-1}{i} c_1(\mathcal{O}_X(D))^i.\mathscr{F},$$

and the Claim is proved. $\square$

Let us now consider the Euler characteristic, then, if $n = \dim X$,

$$\chi(\mathscr{F}(mD)) = \sum_{i=0}^{n} \binom{m+i-1}{i} \chi(c_1(\mathcal{O}_X(D))^i.\mathscr{F}),$$

where the right hand side is a polynomial in m of degree at most n and the degree n term is

$$\binom{m+n-1}{n} \chi(\mathscr{F}.c_1(\mathcal{O}_X(D))^n) = \frac{(D^n.\mathscr{F})}{n!} m^n + O(m^{n-1}).$$

$\square$

Corollary 2.19. *In the setting of the theorem, if $H^i(X, \mathscr{F} \otimes \mathscr{L}(mD)) = 0$ for $i > 0$ and $m \gg 0$, or more generally, if for $i > 0$, $h^i(X, \mathscr{F} \otimes \mathscr{L}(mD)) = O(m^{n-1})$, then*

$$h^0(X, \mathscr{F} \otimes \mathscr{L}(mD)) = rank(\mathscr{F}) \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

2.4 $\mathbb{Q}$ and $\mathbb{R}$-divisors

Definition 2.20. *Let X be an algebraic variety. A Cartier* $\mathbb{Q}$-divisor *on X is an element of the $\mathbb{Q}$-vector space*

$$Div_{\mathbb{Q}}(X) \stackrel{def}{=} Div(X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Equivalently $D \in Div_{\mathbb{Q}}(X) \Leftrightarrow D = \sum c_i D_i | \ c_i \in \mathbb{Q},\ D_i \in Div(X)$.

Definition 2.21 (Equivalence and operations on $\mathbb{Q}$-divisors)**.** *Assume henceforth that X is complete.*

- *Given a subscheme $V \subseteq X$ of pure dimension k, a $\mathbb{Q}$-valued intersection product*
$$Div_{\mathbb{Q}}(X) \times \ldots \times Div_{\mathbb{Q}}(X) \to \mathbb{Q},$$
$$(D_1, \ldots, D_k) \mapsto \int_{[V]} D_1 \cdot \ldots \cdot D_k = (D_1 \cdot \ldots \cdot D_k \cdot [V])$$
is defined via extension of scalars from the analogous product on $Div(X)$.
- *Two $\mathbb{Q}$-divisors $D_1, D_2 \in Div_{\mathbb{Q}}(X)$ are* numerically equivalent*, written $D_1 \equiv_{num} D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{Q}}$ the resulting finite-dimensional $\mathbb{Q}$-vector space of numerical equivalence classes of $\mathbb{Q}$-divisors.*
We will denote $[D]_{num}$ the numerical equivalence class of D.
- *Two $\mathbb{Q}$-divisors $D_1, D_2 \in Div_{\mathbb{Q}}(X)$ are* linearly equivalent*, written $D_1 \equiv_{lin} D_2$ if there is an integer r such that rD_1 and rD_2 are integral and linearly equivalent in the usual sense.*

Definition 2.22. *Let X be an algebraic variety. A Cartier* $\mathbb{R}$-divisor *on X is an element of the $\mathbb{R}$-vector space*

$$Div_{\mathbb{R}}(X) \stackrel{def}{=} Div(X) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Equivalently $D \in Div_{\mathbb{R}}(X) \Leftrightarrow D = \sum c_i D_i | \ c_i \in \mathbb{R},\ D_i \in Div(X)$.

Definition 2.23. *Let $D \in Div_{\mathbb{R}}(X)$, we say that D is* effective *if*

$$D = \sum c_i A_i$$

with $c_i \in \mathbb{R},\ c_i \geq 0$ and A_i is an effective integral divisor.

Definition 2.24 (Numerical euivalence for $\mathbb{R}$-divisors)**.** *Two $\mathbb{R}$-divisors $D_1, D_2 \in Div_{\mathbb{R}}(X)$ are* numerically equivalent*, written $D_1 \equiv_{num} D_2$ if $(D_1 \cdot C) = (D_2 \cdot C)$ for every curve $C \subseteq X$. We denote by $N^1(X)_{\mathbb{R}}$ the resulting finite-dimensional $\mathbb{R}$-vector space of numerical equivalence classes of $\mathbb{R}$-divisors.*
We will denote $[D]_{num}$ the numerical equivalence class of D.

CHAPTER 3

AMPLE AND NEF DIVISORS

In this chapter we review the definitions and the classical propertier of the divisors that play a central role in the study of intersection theory.

3.1 Ample Divisors

Definition 3.1. *If X is any scheme over Y, a line bundle $\mathscr{L}$ on X is said to be* very ample *relative to Y, if there is an immersion $i : X \to \mathbb{P}^r_Y$ for some r, such that $i^*(\mathcal{O}(1)) \cong \mathscr{L}$. We say that a morphism $i : X \to Z$ is an* immersion *if it gives an isomorphism of X with an open subscheme of a closed subscheme of Z.*

Definition 3.2. *Let X be a finite type scheme over a noetherian ring A, and let $\mathscr{L}$ be a line bundle on X. Then $\mathscr{L}$ is said to be* ample *if $\mathscr{L}^m$ is very ample over $SpecA$ for some $m > 0$.*

Proposition 3.3. *Let $f : Y \to X$ be a finite mapping of complete schemes, and $\mathscr{L}$ an ample line bundle on X. Then $f^*\mathscr{L}$ is an ample line bundle on Y.*

Note 3.4. *In particular, if $Y \subseteq X$ is a subscheme of X, then the restriction $\mathscr{L}|_Y$ is ample.*

Corollary 3.5. *Let $\mathscr{L}$ be a globally generated line bundle on a complete scheme X, and let $\varphi = \varphi_{|\mathscr{L}|} : X \to \mathbb{P} = \mathbb{P}H^0(X, \mathscr{L})$ be the resulting map to projective space defined by the complete linear system $|\mathscr{L}|$. Then $\mathscr{L}$ is ample if and only if φ is a finite mapping, or equivalently if and only if $(C \cdot c_1(\mathscr{L})) > 0$ for every irreducible curve $C \subseteq X$.*

Proposition 3.6 (Asymptotic Riemann-Roch, II)**.** *Let D be an ample Cartier divisor on a projective variety X of dimension n. Then*

$$h^0(X, \mathscr{L}(mD)) = \frac{(D^n)}{n!} m^n + O(m^{n-1}).$$

Example 3.7 (Upper bounds on h^0). *If E is any divisor on a variety X of dimension n, there exists a constant $C > 0$ such that:*

$$h^0(X, \mathcal{O}_X(mE)) \leq Cm^n \ \text{ for all } \ m.$$

Proposition 3.8. *Let $f : Y \to X$ be a finite and surjective mapping of projective schemes, and $\mathscr{L}$ be a line bundle on X. If $f^*\mathscr{L}$ is ample on Y, then $\mathscr{L}$ is ample on X.*

Definition 3.9 (Amplitude for $\mathbb{Q}$ and $\mathbb{R}$-divisors). *A $\mathbb{Q}$-divisor $D \in Div_{\mathbb{Q}}(X)$ (resp. $\mathbb{R}$-divisor $D \in Div_{\mathbb{R}}(X)$) is said to be* ample *if it can be written as a finite sum*

$$D = \sum c_i A_i$$

where $c_i > 0$ is a positive rational (resp. real) number and A_i is an ample Cartier divisor.

Note 3.10 (A useful way to write divisors). *Let D be an $\mathbb{R}$-divisor, and suppose that $D = \sum a_i D_i$ where $a_i \in \mathbb{R}$ and $D_i \in Div(X)$,* not necessarily prime. *For every integer $m \geq 1$ we can write*

$$mD = m \sum a_i D_i = \sum ([ma_i]D_i + \{ma_i\}D_i)$$

so that we obtain:

$$[mD] = \left[\sum ([ma_i]D_i + \{ma_i\}D_i)\right] = \sum [ma_i]D_i + \left[\sum \{ma_i\}\right] D_i.$$

Now $\{[\sum \{ma_i\}]\, D_i\} = \{T_m\}$ is a finite set of integral divisors, $\{T_m\} = \{T_{k_1}, \ldots, T_{k_s}\}$.

Remark 3.11. *If D is an integral divisor, D is ample in the sense of $\mathbb{Z}$-divisors if and only if it is ample in the sense of $\mathbb{R}$-divisors.*

Proof. If D is ample in the sense of $\mathbb{Z}$-divisors, obviously D can be written as $1 \cdot D$ where D is an ample divisor, so that it is an ample real divisor.
If $D = \sum c_i A_i$ in an ample $\mathbb{R}$-divisor, by Note 3.10 we can write $[mD] = \sum [ma_i] A_i + T_k$ for finitely many divisors T_k. As A_1 is ample, there exists an integer $r > 0$ such

that $rA_1 + T_k$ is globally generated for every k and there exists an iteger $s > 0$ such that tA_i is very ample for all i and for all $t \geq s$. Then, if $m \geq \frac{r+s}{a_i}$ $\forall i$, we have

$$[mD] = \sum_{i \geq 2} [ma_i] A_i + ([ma_1] - r) A_1 + (rA_1 + T_k)$$

that is a sum of a very ample and a globally generated integral divisor, that is very ample. But in this case $[mD] = mD$ and we get the statement. $\square$

Proposition 3.12 (Nakai-Moishezon)**.** *D is an ample $\mathbb{R}$-divisor if and only if*

$$(D^{dimV} \cdot V) > 0$$

for every irreducible $V \subseteq X$ of positive dimension.

Remark 3.13. *If $D = \sum c_i A_i$ with $c_i > 0$ and A_i integral and ample, then*

$$(D^{dimV} \cdot V) \geq \left(\sum c_i\right)^{dimV}.$$

Corollary 3.14. *The amplitude of an $\mathbb{R}$-divisor depends only on its numerical equivalence class.*

Proof. We will show that if D and B are $\mathbb{R}$-divisors, with D ample and $B \equiv_{num} 0$, then $D + B$ is ample.
First we want to prove that B is an $\mathbb{R}$-linear combination of numerically trivial integral divisors. Now B is given as a finite sum

$$B = \sum r_i B_i, \; r_i \in \mathbb{R}, \; B_i \in \mathrm{Div}(X).$$

The condition of being numerically trivial is given by finitely many linear equations on the r_i, determined by integrating over a set of generators of the subgroup of $H_2(X, \mathbb{Z})$ spanned by algebraic 1-cycles on X. The assertion then follows from the fact that any real solution of these equations is an $\mathbb{R}$-linear combination of integral ones.

We are now reduced to showing that if A and B are integral divisors, with A ample and $B \equiv_{num} 0$, then $A + rB$ is ample for any $r \in \mathbb{R}$. If r is rational we already know this. In general, we can fix rational numbers $r_1 < r < r_2$, together with a real number $t \in [0,1]$, such that $r = tr_1 + (1-t)r_2$. Then

$$A + rB = t(A + r_1B) + (1-t)(A + r_2B),$$

exhibiting $A + rB$ as a positive $\mathbb{R}$-linear combination of ample $\mathbb{Q}$-divisors.

$\square$

Definition 3.15. *A numerical equivalence class $\delta \in N^1(X)$ is ample if it is the class of an ample divisor.*

Proposition 3.16 (Openness of amplitude for $\mathbb{Q}$ and $\mathbb{R}$-divisors)**.** *Let X be a projective variety and let H be an ample $\mathbb{Q}$-divisor (respectively $\mathbb{R}$-divisor) on X. Given finitely many $\mathbb{Q}$-divisors (resp. $\mathbb{R}$-divisors) $E_1, ..., E_r$, the $\mathbb{Q}$-divisor (resp. $\mathbb{R}$-divisor)*

$$H + \varepsilon_1 E_1 + ... + \varepsilon_r E_r$$

is ample for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof. When H and each E_i are rational, clearing denominators we can assume that H and each E_i are integral. By taking $m \gg 0$ we can arrange for any of the $2r$ divisors $mH \pm E_1, \ldots, mH \pm E_r$ to be ample. Now, provided that $|\varepsilon_i| \ll 1$ we can write any divisor of the form $H + \varepsilon_1 E_1 + ... + \varepsilon_r E_r$ as a positive $\mathbb{Q}$-linear combination of H and some of the $\mathbb{Q}$-divisors $H + \frac{1}{m}E_i$. But a positive linear combination of ample $\mathbb{Q}$-divisors is ample.

Since each E_j is a finite $\mathbb{R}$-linear combination of integral divisors, there is no loss of generality in assuming at the outset that all the E_j are integral. Now write $H = \sum c_i A_i$ with $c_i > 0$ and A_i ample and integral, and fix a rational number $0 < c \leq c_1$. Then

$$H + \sum \varepsilon_j E_j = (cA_1 + \sum \varepsilon_j E_j) + (c_1 - cA_1) + \sum_{i \geq 2} c_i A_i.$$

Here the first term on the right is ample by the above proof and the remaining summands are ample. $\square$

3.2 Nef Divisors

Definition 3.17 (Nef line bundles and divisors)**.** *Let X be a complete variety. A line bundle $\mathscr{L}$ on X is* numerically effective, *or* nef, *if*

$$\int_C c_1(\mathscr{L}) \geq 0$$

for every irreducible curve $C \subseteq X$.
A Cartier $\mathbb{R}$-divisor D on X is nef *if*

$$(D \cdot C) \geq 0$$

for all irreducible curves $C \subset X$.
A Cartier $\mathbb{R}$-divisor D on X is strictly nef *if*

$$(D \cdot C) > 0$$

for all irreducible curves $C \subset X$.

Theorem 3.18 (Kleiman)**.** *Let X be a complete variety. If D is a nef $\mathbb{R}$-divisor on X, then*

$$(D^k \cdot V) \geq 0$$

for every irreducible subvariety $V \subseteq X$ of dimension $k > 0$.

Theorem 3.19 (Higher cohomology of nef divisors)**.** *[Laz04a, 1.4.40]*

Let X be a projective variety of dimension n, and D an integral Cartier divisor on X. If D is nef, then for every coherent sheaf $\mathscr{F}$ on X

$$h^i(X, \mathscr{F}(mD)) = O(m^{n-i}).$$

Corollary 3.20. *Let X be a projective variety, and D a nef $\mathbb{R}$-divisor on X. If H is an ample $\mathbb{R}$-divisor on X, then*

$$D + \varepsilon H$$

is ample for every $\varepsilon > 0$. Conversely, if D and H are any two $\mathbb{R}$-divisors such that $D + \varepsilon H$ is ample for all sufficiently small $\varepsilon > 0$, then D is nef.

Proof. If $D + \varepsilon H$ is ample for $\varepsilon > 0$, then

$$(D.C) + \varepsilon(H.C) = ((D + \varepsilon H).C) > 0$$

for every irreducible curve C. Letting $\varepsilon \to 0$ it follows that $(D.C) \geq 0$, and hence D is nef.

Assume conversely that D is nef and H is ample. Replacing εH by H, it suffices to show that $D + H$ is ample. To this end, by Proposition 3.12, we only need to prove that

$$((D + H)^{\dim V}.V) > 0$$

for every subvariety $V \subseteq X$ of positive dimension.

First suppose that $D + H$ is a rational divisor (then the general case will follow by an approximation argument).

Fix a variety $V \subseteq X$ of dimension $k > 0$. Then

$$\Big((D + H)^k.V\Big) = \sum_{s=0}^{k} \binom{k}{s} (H^s.D^{k-s}.V). \tag{3.1}$$

Since H is a positive $\mathbb{R}$-linear combination of integral ample divisors, the intersection $(H^s.V)$ is represented by an effective $(k-s)$-cycle. Applying Kleiman's theorem to each of the components of this cycle, it follows that $(H^s.D^{k-s}.V) \geq 0$. Thus each of the terms in (3.1) is non-negative for $s \neq k$, and the last intersection number $(H^k.V)$ is strictly positive. Therefore $((D + H)^k.V) > 0$ for every V, and in particular if $D + H$ is rational then it is ample (by Proposition 3.12).

It remains to prove that $D + H$ is ample even when it is irrational. To this end, choose ample divisors $H_1, \dots, H_r$ whose classes span $N_1(X)_{\mathbb{R}}$. By the open nature of amplitude (3.16), the $\mathbb{R}$-divisor $H(\varepsilon_1, \dots, \varepsilon_r) = H - \varepsilon_1 H_1 - \cdots - \varepsilon_r H_r$ remains ample for all $0 < \varepsilon_i \ll 1$. Obviously there exist $0 < \varepsilon_i \ll 1$ such that $D' = D + H(\varepsilon_1, \dots, \varepsilon_r)$ represents a rational class in $N^1(X)_{\mathbb{R}}$. The case of the corollary already treated shows that D' is ample. Consequently so too is

$$D + H = D' + \varepsilon_1 H_1 + \cdots + \varepsilon_r H_r.$$

□

Example 3.21 (Strictly nef but not ample). *[Har70, Appendix 10] Now we will give an example by Mumford of a divisor on a surface, that is strictly nef but not ample.*
Let us consider a nonsingular complete curve C of genus $g \geq 2$ over $\mathbb{C}$; there exists a stable bundle E of rank two and degree zero such that all its symmetric powers $S^m(E)$ are stable. Let $X = \mathbb{P}(E)$ be the ruled surface over C, let $\pi : X \to C$ be the canonical projection and let D be the divisor corresponding to $\mathcal{O}_X(1)$. Then, for every irreducible curve $Y \subseteq X$, we have:

- *If Y is a fiber of π, then $(D.Y) = 1$;*
- *If Y is an irreducible curve of degree m over C, then Y corresponds to a sub-line bundle $M \subseteq S^m(E)$. But $S^m(E)$ is stable of degree zero, so $\deg M < 0$. Therefore $(D.Y) = -\deg M > 0$.*

Thus $(D.Y) > 0$ for every effective curve $Y \subseteq X$, but D is not ample, because

$$(D^2) = 0.$$

We will now give an example of Ramanujan of a divisor strictly nef but not ample on a threefold that is based on the Example 3.21 of Mumford.

Example 3.22 (Strictly nef and big but not ample). *Let X be a nonsingular surface, and D a divisor with $(D.Y) > 0$ for all effective curves, and $(D^2) = 0$ as in the Example 3.21 by Mumford. Let H be an effective ample divisor on X, then we define $\overline{X} = \mathbb{P}(\mathcal{O}_X(D - H) \otimes \mathcal{O}_X)$, and let $\pi : \overline{X} \to X$ be the projection.*
Let X_0 be the zero-section of the associated vector bundle, so that $(X_0^2) = (D - H)_X$. We define $\overline{D} = X_0 + \pi^* H$ which is effective by construction.

$\overline{D}.Y \geq 0$ for all effective curves Y:

- *If Y is a fiber of π, then*

$$(\overline{D}.Y) = (X_0.Y) + (\pi^* H.Y) = 1 + 0 = 1.$$

- *If $Y \subset X_0$, then*

$$(\overline{D}.Y) = (\overline{D}|_{X_0}.Y)|_{X_0} = ((D - H + H).Y)_X = (D.Y)_X > 0.$$

- *If $Y \not\subseteq X_0$, and $\pi(Y)$ is a curve Y' in X, then*

$$(\overline{D}.Y) = (X_0.Y) + (\pi^* H.Y)$$

where $(X_0.Y) \geq 0$ and $(\pi^* H.Y) = (H.Y')_X > 0$.

On the other hand, $\overline{D}$ is not ample. In fact

$$(\overline{D}^2) = (\overline{D}|^2_{X_0})_{X_0} = (D^2)_X = 0,$$

and therefore, by Nakai-Moishezon (Proposition 3.12) $\overline{D}$ is not ample.
On the other hand $\overline{D}$ is big, because $(\overline{D}^3) > 0$, in fact:

$$\begin{aligned}(\overline{D}^3) &= (\overline{D}^2.(X_0 + \pi^* H)) = \\ &= ((X_0 + \pi^* H)^2.\pi^* H) = \\ &= (((D-H)_X + 2H_X + \pi^* H^2).\pi^* H) = \\ &= ((D+H).H)_X + (\pi^* H^3) > 0\end{aligned}$$

because $(\pi^* H^3) = 0$ and $(D+H).H > 0$ by Nakai-Moishezon (3.12).

Theorem 3.23 (Fujita's vanishing theorem [Laz04a])**.** *Let X be a variety and let H be an ample integral divisor on X. Given any coherent sheaf $\mathscr{F}$ on X, there exists an integer $m(\mathscr{F}, H)$ such that*

$$H^i(X, \mathscr{F} \otimes \mathcal{O}_X(mH + D)) = 0 \text{ for all } i > 0,\ m \geq m(\mathscr{F}, H),$$

and any nef divisor D on X.

3.3 Ample and Nef Cones

Definition 3.24 (Cones)**.** *Let V be a finite-dimensional real vector space. A* cone *in V is a set $K \subseteq V$ stable under multiplication by positive scalars.*

Definition 3.25 (Ample and nef cones)**.**

- *The* ample cone *$Amp(X) \subset N^1(X)_{\mathbb{R}}$ of X is the convex cone of all ample $\mathbb{R}$-divisor classes on X.*

- *The* nef cone $Nef(X) \subset N^1(X)_{\mathbb{R}}$ *is the convex cone of all nef* $\mathbb{R}$*-divisor classes.*

Theorem 3.26 (Kleiman)**.** *Let X be any projective variety or scheme.*

- *The nef cone is the closure of the ample cone:* $\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)}$
- *The ample cone is the interior of the nef cone:* $\mathrm{Amp}(X) = \mathrm{int}(\mathrm{Nef}(X))$

Definition 3.27 (Numerical equivalence classes of curves)**.** *Let* X *be a variety. We denote by* $Z_1(X)_{\mathbb{R}}$ *the* $\mathbb{R}$*-vector space of* real one cycles *of* X*, consisting of all finite* $\mathbb{R}$*-linear combinations of irreducible curves on* X*. An element* $\gamma \in Z_1(X)_{\mathbb{R}}$ *is thus a formal finite sum*

$$\gamma = \sum a_i \cdot C_i$$

where $a_i \in \mathbb{R}$ *and* $C_i \subset X$ *is an irreducible curve.*
Two one-cycles $\gamma_1, \gamma_2 \in Z_1(X)_{\mathbb{R}}$ *are* numerically equivalent *if* $(D \cdot \gamma_1) = (D \cdot \gamma_2)$ *for every* $D \in Div_{\mathbb{R}}(X)$*.*

The corresponding vector space of numerical equivalence classes *of one-cycles is written* $N_1(X)_{\mathbb{R}}$*. Thus one has a perfect pairing*

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}, \quad (\delta, \gamma) \mapsto (\delta \cdot \gamma) \in \mathbb{R}.$$

Definition 3.28 (Cone of curves)**.** *Let X be a complete variety. The* cone of curves $NE(X) \subseteq N_1(X)_{\mathbb{R}}$ *is the cone spanned by the classes of all effective one-cycles on* X*.*

$$NE(X) = \{\sum a_i[C_i] | C_i \subset X \text{ an irreducible curve}, a_i \in \mathbb{R}, a_i \geq 0\}$$

Proposition 3.29. $\overline{NE}(X)$ *is the closed cone dual to* $Nef(X)$*:*

$$\overline{NE}(X) = \{\gamma \in N_1(X)_{\mathbb{R}} | (\gamma \cdot \delta) \geq 0 \quad \forall \delta \in Nef(X)\}$$

Definition 3.30. *We denote by*

$$D^{\perp} = \{\gamma \in N_1(X)_{\mathbb{R}} | (D \cdot \gamma) = 0\};$$

$$D_{>0} = \{\gamma \in N_1(X)_{\mathbb{R}} | (D \cdot \gamma) > 0\}.$$

CHAPTER 4

SINGULARITIES

In this section we recall the various well known definitions utilized in the study of singularities as well as establish notation for the chapter. Unless otherwise stated all schemes throughout will be excellent, reduced, normal and essentially of finite type are over a field.

4.1 Singularities Defined via Birational Maps

We begin with a review of the theory of singularities of pairs via discrepancies developed in [KM98]. The key ingredient is Hironaka's existence theorem on resolution of singularities [Hir64] over algebraic closed fields of characteristic zero, though much of what is said here works in any characteristic.

Recall for normal schemes X (in any characteristic) a $\mathbb{Q}$-*divisor* D is a $\mathbb{Q}$-linear combination of prime Weil divisors, i.e., irreducible codimension 1 subschemes of X. For a $\mathbb{Q}$-divisor $D = \sum a_i D_i$ we denote by $\lceil D \rceil$ the integral divisor $\sum \lceil a_i \rceil D_i$ and similarly for $\lfloor D \rfloor$. A $\mathbb{Q}$-divisor D is effective provided each $a_i \geq 0$. A $\mathbb{Q}$-divisor D is called $\mathbb{Q}$-*Cartier* provided some multiple mD is an integral Cartier divisor on X. Given a prime divisor $E \subset X$ and $\mathcal{F}$ a sheaf on X we denote by $\mathcal{F}_E$ the stalk of $\mathcal{F}$ at the generic point of E. For a scheme with a dualizing complex $\omega_X^\bullet$ we denote by ω_X the first non-zero cohomology class of $\omega_X^\bullet$. This is a reflexive sheaf and any integral Weil divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$ will be called a *canonical* divisor.

By a pair (X, Δ) we mean a normal variety X and divisor $\mathbb{Q}$-divisor $\Delta = \sum_i a_i \Delta_i$ on X. The pair is called *log* $\mathbb{Q}$-*Gorenstein* provided $K_X + \Delta$ is $\mathbb{Q}$-Cartier where K_X is defined via $\mathcal{O}_X(K_X)|_{X_{\text{reg}}} = \wedge^{\dim X} \Omega_{X_{\text{reg}}} = \omega_{X_{\text{reg}}}$ and X_{reg} is the regular locus of X.

A *log resolution of* (X, Δ) is a projective birational morphism $\rho\colon Y \to X$ such that Y is smooth, $E = \mathrm{Exc}(\rho)$ is of pure dimension one, and $E \cup \rho_*^{-1}(\Delta)$ is a divisor of simple normal crossing support. In characteristic 0, Hironaka's work guarantees that a log resolution can be constructed by composing blowing ups of smooth centers, i.e., irreducible smooth subvarieties.

Assume that (X, Δ) is log $\mathbb{Q}$-Gorenstein and that there is a $\rho : Y \to X$ is a proper birational map. There is a natural isomorphism

$$\mathcal{O}_Y(K_Y + \rho_*^{-1}\Delta)|_{Y-E} \cong \rho^*\mathcal{O}_X(K_X + \Delta)|_{Y-E}.$$

In particular, if $E = \cup_i E_i$ where E_i's are irreducible components, there are rational numbers $a(E_i, X, \Delta)$ such that

$$K_Y + \rho_*^{-1}\Delta = \rho^*(K_X + \Delta) + \sum_{E_i\colon \text{exceptional}} a(E_i, X, \Delta)E_i.$$

Note that one can also write

$$K_Y = \rho^*(K_X + \Delta) + \sum_{D_i\colon \text{arbitrary}} a(D_i, X, \Delta)D_i.$$

For $D \subset Y$ a prime Weil divisor, we call $a(D, X, \Delta)$ the *discrepancy* of D with respect to (X, Δ).

Remark 4.1. *The number $a(D, X, \Delta)$ corresponds to the* algebra valuation *defined by the discrete valuation ring $\mathcal{O}_{Y,D} \subseteq K(Y) \cong K(X)$, where $K(X)$ and $K(Y)$ are the function fields of X and Y, respectively. This shows that discrepancies of a divisor are independent of the choice of resolution. See [KM98, Remark 2.23].*

Definition 4.2. *Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair where X is defined in characteristic 0. The* discrepancy *of (X, Δ) is given by*

$$\mathrm{discrep}(X, \Delta) := \inf_E \{a(E, X, \Delta)\colon\ E \text{ is an exceptional divisor over } X\}.$$

The total discrepancy *of (X, Δ) is defined as*

$$\mathrm{totaldiscrep}(X, \Delta) := \inf_D \{a(D, X, \Delta)\colon\ D \text{ is a divisor over } X\}.$$

Remark 4.3. *Let $\rho\colon Y \to X$ be a particular resolution of a log $\mathbb{Q}$-Gorenstein pair (X, Δ). We also utilize the notation* $\operatorname{discrep}(Y/X, \Delta) = \min_i a(E_i, X, \Delta)$ *where the E_i range over all exceptional prime divisors in Y. So one may take* $\operatorname{discrep}(X, \Delta) = \inf_\rho \operatorname{discrep}(Y/X, \Delta)$*, where the infimum runs over all $\rho\colon Y \to X$ resolutions.*

Here are some properties of discrepancies, see [KM98, Section 2.3] for more detail. These statements hold in any characteristic provided that one can make sense of the infimums of Definition 4.2.

Theorem 4.4. *(c.f., [KM98, 2.31]) Let (X, Δ_X) be a log $\mathbb{Q}$-Gorenstein pair.*

- *Let $\rho\colon Y \to X$ be a proper birational morphism and Δ_Y a $\mathbb{Q}$-divisor on Y defined by*

$$K_Y + \Delta_Y = \rho^*(K_X + \Delta_X) \text{ and } \rho_*\Delta_Y = \Delta_X.$$

  Then for any divisor F over X, $a(F, Y, \Delta_Y) = a(F, X, \Delta_X)$.
- *Either* $\operatorname{discrep}(X, \Delta) = -\infty$ *or* $-1 \leq \operatorname{totaldiscrep}(X, \Delta) \leq \operatorname{discrep}(X, \Delta) \leq 1$*. This can be shown by blowing up smooth centers. It follows that $\Delta \leq 1$ is a necessary condition for* $\operatorname{discrep}(X, \Delta)$ *to be finite. One calls Δ a* sub-boundary *when $\Delta \leq 1$.*
- *If X is smooth, then* $\operatorname{discrep}(X, 0) = 1$.
- *If $a(E, X, \Delta) \geq -1$ for all divisors E on a log resolution Y over X, then* $\operatorname{discrep}(X, \Delta)$ *(and hence also* $\operatorname{totaldiscrep}(X, \Delta)$*) can be computed on Y.*

We assume throughout this work that all pairs (X, Δ) satisfy $\Delta \leq 1$. The singularities relevant to the minimal model program have been defined in [KM98, Section 2.3].

Definition 4.5. *Let (X, Δ) be a log $\mathbb{Q}$-Gorenstein pair. Then we say that*

$$(X, \Delta) \text{ is } \left\{ \begin{array}{c} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{log canonical} \end{array} \right\} \text{ if } \operatorname{discrep}(X, \Delta) \left\{ \begin{array}{l} > 0, \\ \geq 0, \\ > -1 \text{ and } \llcorner \Delta \lrcorner \leq 0, \\ \geq -1. \end{array} \right\}$$

Remark 4.6. *In the study of minimal model program, most of the time we also assume that* $\Delta \geq 0$ *and name such divisors as* boundaries*. The theory of singularities of pairs with* Δ *being a boundary behaves well under perturbation [KM98, Section 2.3]. A crucial fact is that Kodaira vanishing theorem naturally generalizes to klt pairs with* Δ *being a boundary.*

CHAPTER 5

SINGULARITIES OF NORMAL VARIETIES

In this chapter we will introduce singularities for general normal varieties, i.e., we are no longer requiring the canonical divisor to be $\mathbb{Q}$-Cartier.

5.1 Basic Definitions

The following notations and definitions are taken from [dFH09].

Notation 5.1. *Throughout this paper X will be a normal variety over the complex numbers.*

Let us denote by $v = \mathrm{val}_F$ a divisorial valuation on X with respect to the prime divisor F over X. Given a proper closed subscheme $Z \subset X$ we define $v(Z)$ as

$$v(Z) = v(\mathcal{I}_Z) := \min\{v(\phi) | \phi \in \mathcal{I}_Z(U), U \cap c_X(v) \neq \emptyset\}$$

where $\mathcal{I}_Z \subset \mathcal{O}_Z$ is the ideal sheaf of Z. The definition extends to $\mathbb{R}$-linear combinations of proper closed subschemes. The same definition works in a natural way for linear combinations of fractional ideal sheaves.

To any fractional ideal sheaf $\mathcal{I}$ on X, we associate the divisor

$$\mathrm{div}(\mathcal{I}) := \sum_{E \subset X} \mathrm{val}_E(\mathcal{I}) \cdot E$$

where the sum is over all prime divisors E on X and val_E denotes the divisorial valuation with respect to E.

Definition 5.2. *Let X be as in Notation 5.1. The $\natural$-valuation (or natural valuation) along a valuation v of a divisor F on X is*

$$v^\natural(F) := v(\mathcal{O}_X(-F)).$$

Let D be a $\mathbb{Q}$-divisor on X. The valuation along v of D is

$$v(D) := \lim_{k\to\infty} \frac{v^{\natural}(k!D)}{k!} = \inf_{k\geq 1} \frac{v^{\natural}(kD)}{k} \in \mathbb{R}.$$

Notation 5.3. *Let X be as in Notation 5.1. Let us consider a projective birational morphism $f : Y \to X$ from a normal variety Y.*

We have the following definitions:

Definition 5.4. *Using notation 5.3, for any divisor D on X, the $\natural$-pullback of D to Y is defined to be*

$$f^{\natural}D = div(\mathcal{O}_X(-D) \cdot \mathcal{O}_Y).$$

This is the natural choice to obtain a reflexive sheaf, $\mathcal{O}_Y(-f^{\natural}D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{\vee\vee}$.

We also need a good definition of $pullback$ of D to Y that needs to coincide with the classical one when we restrict to nonsingular varieties. We have:

$$f^*D := \sum val_E(D) \cdot E,$$

where the sum is taken over all the prime divisors E on Y.

We now give the main definitions that characterize multiplier ideal sheaves.

Definition 5.5. *Let $f : Y \to X$ be as in Notation 5.3, for every $m \geq 1$, the m-th limiting relative canonical $\mathbb{Q}$-divisor $K_{m,Y/X}$ of Y over X is*

$$K_{m,Y/X} := K_Y - \frac{1}{m} \cdot f^{\natural}(mK_X).$$

The relative canonical $\mathbb{R}$-divisor $K_{Y/X}$ of Y over X is

$$K_{Y/X} := K_Y + f^*(-K_X)$$

In particular $K_{m,Y/X} \leq K_{mq,Y/X} \leq K_{Y/X}$. Also, taking the limsup of the coefficients of the components of the $\mathbb{Q}$-divisor $K_{m,Y/X}$, one obtains the $\mathbb{R}$-divisor $K^-_{Y/X} := K_Y - f^*K_X$ which satisfies $K^-_{Y/X} \leq K_{Y/X}$ (the two divisors coincide if X is $\mathbb{Q}$-Gorenstein i.e. if K_X is $\mathbb{Q}$-Cartier).

Recall that an effective $\mathbb{Q}$-divisor Δ is a boundary on X if $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. In most cases we require $\Delta = \sum d_i D_i$, D_i prime divisors and $d_i \leq 1$.

Definition 5.6. *Let $f : Y \to X$ as in Notation 5.3, let Δ be a boundary on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let Δ_Y be the proper transform of Δ on Y. The log relative canonical $\mathbb{Q}$-divisor of (Y, Δ_Y) over (X, Δ) is given by:*

$$K^{\Delta}_{Y/X} := K_Y + \Delta_Y - f^*(K_X + \Delta) = K_Y + \Delta_Y + f^*(-K_X - \Delta).$$

In particular, for every boundary Δ on X and every $m \geq 1$ such that $m(K_X + \Delta)$ is Cartier, we have

$$K_{m,Y/X} = K^{\Delta}_{Y/X} - \frac{1}{m} \cdot f^{\natural}(-m\Delta) - \Delta_Y \quad \text{and} \quad K_{Y/X} = K^{\Delta}_{Y/X} + f^*\Delta - \Delta_Y.$$

Note that $K^{\Delta}_{Y/X} \leq K_{m,Y/X} \leq K^{-}_{Y/X}$.

Definition 5.7. *Consider a pair (X, I) where X is a normal quasi-projective variety and $I = \sum a_k \mathcal{I}_k$ is a formal $\mathbb{R}$-linear combination of non-zero fractional ideal sheaves on X. Let us denote by $Z = \sum a_k Z_k$ the associated subscheme, where Z_k is the subscheme generated by $\mathcal{I}_k$.*

We define a log resolution *of this pair as a proper birational morphism $f : Y \to X$, where Y is a smooth variety, such that for every k:*

- *The sheaf $\mathcal{I}_k \dot{\mathcal{O}}_Y$ is an invertible sheaf corresponding to a divisor E_k on Y.*
- *The exceptional locus $Ex(f)$ is a divisor.*
- *The union of the supports of E_k and $Ex(f)$ is simple normal crossing.*

If Δ is a boundary on X, then a log resolution for $((X, \Delta); I)$ is given by a resolution of (X, I) such that $Ex(f)$, E, $Supp f^*(K_X + \Delta)$ are divisors and their union $Ex(f) \cup E \cup Supp f^*(K_X + \Delta)$ has simple normal crossings.

Definition 5.8. *Let (X, Z) be as in Definition 5.7. Let $f : Y \to X$ be a log resolution with Y normal, and let F denote a prime divisor on Y. For any integer $m \geq 1$, we define the m-th limiting log discrepancy of (X, Z) along F to be*

$$a_{m,F}(X; Z) := \mathrm{ord}_F(K_{m,Y/X}) + 1 - val_F(Z).$$

Definition 5.9. *Using the notation above, the pair (X, Z) is said to be* log terminal *if there is an integer m_0 such that $a_{m_0,F}(X, Z) > 0$ for every prime divisor F over X.*

We say that an effective pair is klt if and only if there exists a boundary Δ such that $((X,\Delta);Z)$ is klt (kawamata log terminal) in the usual sense.

In particular the notions of log terminal and klt are equivalent because of the following (cf. [dFH09, Thm 5.4]):

Theorem 5.10. *Every effective pair (X,Z) admits m-compatible boundaries for $m \geq 2$, where, a boundary Δ is said to be m-compatible if:*

i) $m\Delta$ is integral and $\lfloor\Delta\rfloor = 0$.

ii) No component of Δ is contained in the support of Z.

iii) f is a log resolution for the log pair $((X,\Delta);Z+\mathcal{O}_X(-mK_X))$.

iv) $K^{\Delta}_{Y/X} = K_{m,Y/X}$.

Notation 5.11. *Because of the previous Theorem and the genaral notation in the literature, from now on we will abuse our notation saying that a normal variety X is klt whenever it is log terminal according to Definition 5.9. A pair (X,Δ) will be klt in the usual sense.*

Remark 5.12. *Since $K^{\Delta}_{Y/X} \leq K^{-}_{Y/X}$, by Theorem 5.10 and Definition* 5.2

$$val_F(K^{-}_{Y/X}) = \sup\{ord_F(K^{\Delta}_{Y/X})|(X,\Delta) \text{ is a log pair}\}.$$

Note that we are considering a limit and hence we may have irrational valuations.

Definition 5.13. *Let X be an in Notation 5.1. Let $X' \to X$ be a proper birational morphism with X' normal, and let F be a prime divisor on X'. The* log-discrepancy *of a prime divisor F over X with respect to (X,Z) is*

$$a_F(X,Z) := ord_F(K_{X'/X}) + 1 - val_F(Z).$$

Using the notation in Definition 5.7, the pair (X,Z) is said to be canonical *(resp.* terminal*) if $a_F(X,Z) \geq 1$ (resp. > 1) for every exceptional prime divisor F over X.*

Recall that by [dFH09, Proposition 8.2], a normal variety X is canonical if and only if for sufficiently divisible $m \geq 1$, and for every sufficiently high log resolution $f : Y \to X$ of $(X, \mathcal{O}_X(mK_X))$, there is an inclusion

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

We have the following useful lemma:

Lemma 5.14. *Using Notation 5.1, let $f : Y \to X$ be a proper birational morphism such that Y is canonical. If $\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_Y(mK_Y)$ for sufficiently divisible $m \geq 1$, then X is canonical.*

Proof. Let $g : Y' \to X$ be a log resolution of $(X, \mathcal{O}_X(mK_X))$. Without loss of generality we can assume that g factors through f, so that we have $h : Y' \to Y$ with $g = f \circ h$. In particular:

$$\mathcal{O}_{Y'} \cdot \mathcal{O}_X(mK_X) \hookrightarrow \mathcal{O}_{Y'} \cdot \mathcal{O}_Y(mK_Y) \hookrightarrow \mathcal{O}_{Y'}(mK_{Y'}),$$

where the first inclusion is given by assumption and the second by Y being canonical. □

5.2 Irrational Valuations

Given a Weil $\mathbb{R}$-divisor D on a normal variety X, we define the corresponding divisorial ring as

$$\mathscr{R}_X(D) := \bigoplus_{m \geq 0} \mathcal{O}_X(mD).$$

Remark 5.15. *If X is klt as in Notation 5.11, then there exists Δ such that (X, Δ) is klt (in the usual sense) (cf. Theorem 5.10). By [Kol08, Thm 92] $\mathscr{R}_X(D)$ is finitely generated $\mathcal{O}_X$-algebra if and only if D is a $\mathbb{Q}$-divisor.*

Proposition 5.16. *If a normal variety X is klt, then for any prime divisor F over X, the valuation $\mathrm{val}_F(K^-_{Y/X})$ is rational, where $f : Y \to X$ is a projective birational morphism such that F is a divisor on Y.*

Proof. By Remark 5.15 and [Gro61, Lemma 2.1.6] we know that $\mathscr{R}_X(m_0 K_X)$ is generated by $\mathcal{O}_X(m_0 K_X)$ over $\mathcal{O}_X$ for some $m_0 > 0$. It follows that $K^-_{Y/X} =$

$K_{m_0,Y/X}$ and hence $\mathrm{val}_F(K^-_{Y/X}) = \mathrm{val}_F(K_{m_0,Y/X})$ wich is a rational number (Remark 5.12). $\square$

Next we will construct an example of a threefold whose relative canonical divisor $K^-_{Y/X}$ has an irrational valuation. The example is given by the resolution of a cone singularity over an abelian surface.

Let us consider the abelian surface $X = E \times E$ where E is an elliptic curve. For this surface we have that $\overline{NE}(X) = \mathrm{Nef}(X) \subset \mathrm{N}^1(X)$, where $\mathrm{N}^1(X)$ is generated by the classes

$$f_1 = [\{P\} \times E], \quad f_2 = [E \times \{P\}], \quad \delta = [\Delta].$$

The intersection numbers are given by:

$$((f_1)^2) = ((f_2)^2) = (\delta^2) = 0 \quad \text{and} \quad (f_1.f_2) = (f_1.\delta) = (f_2.\delta) = 1.$$

Given a class $\alpha = xf_1 + yf_2 + z\delta$ then α is nef if and only if

$$xy + xz + yz \geq 0, \quad x + y + z \geq 0$$

and we obtain that $\mathrm{Nef}(X)$ is a circular cone (cf. [Kol96, Ch II, Ex 4.16]).

Next, we consider a double covering of this surface ramified over a general very ample divisor $H \in |2\mathscr{L}|$ where $\mathscr{L}$ is an ample line bundle. This cover is given by $W = \mathrm{Spec}_X(\mathcal{O}_X \oplus \mathscr{L}^\vee)$ with projection $p : W \to X$ induced by the inclusion $i : \mathcal{O}_X \hookrightarrow \mathcal{O}_X \oplus \mathscr{L}^\vee$. In particular

$$\omega_W = p^*(\omega_X \otimes \mathscr{L}).$$

There is an induced involution $\sigma : W \to W$. For any Cartier divisor D on W, $D + \sigma^*(D)$ is the pullback of a Cartier divisor on X. Since $H \in |2\mathscr{L}|$ is general, the pullbacks of the generators $p^* f_i$ and $p^*\delta$ are irreducible curves on W. Since the map is finite, the pullback of an ample divisor (resp. nef, effective) on X is ample (resp. nef, effective) on W.

It is easy to see that the map induced at the level of cones $p^* : \mathrm{NE}(X) \to \mathrm{NE}(W)$ is well defined, injective and

$$p^*\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(W) \cap p^*\mathrm{N}^1(X).$$

Let us now consider any ample divisor L on X such that p^*L defines an embedding $W \subset \mathbb{P}^n$. Let $C \subset \mathbb{P}^{n+1}$ be the projective cone over W. We want to investigate the properties of the relative canonical divisor.

Theorem 5.17. *With the above construction, if $H \sim 6(f_1 + f_2)$, $L \sim (3f_1 + 6f_2 + 6\delta)$ and $f : Y \to C$ the blow up of the cone at the vertex, then the relative canonical divisor $K^-_{Y/X}$ has an irrational valuation.*

Proof. p^*L defines an embedding, in fact

$$p_*p^*L \cong L \otimes (\mathcal{O}_X \oplus \mathscr{L}^\vee) = L \oplus (L \otimes \mathscr{L}^\vee) \sim (3f_1 + 6f_2 + 6\delta) \oplus (3f_2 + 6\delta)$$

is a sum of very ample divisors. Since $f : Y \to C$ be the blow-up at the vertex, then Y is isomorphic to the projective space bundle $\mathbb{P}(\mathcal{O}_W \oplus \mathcal{O}_W(p^*L))$, with the natural projection $\pi : Y \to W$. If we denote by W_0 the negative section we have $\mathcal{O}_{W_0}(W_0) \cong \mathcal{O}_W(-p^*L)$. Let us also denote by $W_\infty \sim W_0 + \pi^*p^*L$ the section at infinity. The canonical divisor K_Y is given by $K_Y \sim \pi^*K_W - 2W_0 + \pi^*(-p^*L)$.

Remark 5.17.1. Recall that we have an isomorphism $\mathrm{Cl}W \cong \mathrm{Cl}C$ defined by the map that associates to a divisor $D \subset W$ the cone over D, $C_D \subset C$. A divisor C_D is $\mathbb{R}$-Cartier if and only if $D \sim_{\mathbb{R}} kp^*L$, $k \in \mathbb{R}$.

We have that $K_C = f_*K_Y = C_{K_W} - C_{(p^*L)}$ and $C_{k(p^*L)}$ is an $\mathbb{R}$-Cartier divisor on C such that $f^*(C_{k(p^*L)}) = \pi^*(k(p^*L)) + kW_0$. Let Γ be a boundary on C, then $\Gamma \equiv C_\Delta$ and since $K_C + \Gamma$ is $\mathbb{Q}$-Cartier we have that $K_C + C_\Delta = C_{K_W} - C_{(p^*L)} + C_\Delta \equiv C_{k(p^*L)}$ for some $k \in \mathbb{Q}$. In particular, given $s = k + 1$, we have $s(p^*L) - K_W \equiv \Delta \geq 0$. So that

$$\Delta \equiv s(p^*L) - K_W \equiv s(p^*L) - \frac{1}{2}p^*H \equiv p^*\left(s(3f_1 + 6f_2 + 6\delta) - \frac{1}{2}(6f_1 + 6f_2)\right).$$

By Remark 5.12, we have that:

$$\mathrm{val}_{W_0}(K^-_{Y/C}) = \sup\{\mathrm{ord}_{W_0}(K^\Gamma_{Y/C}) | (C,\Gamma) \text{ log pair}\} \geq$$

$$\geq \sup\{\mathrm{ord}_{W_0}(K^{C_\Delta}_{Y/C}) | (C, C_\Delta) \text{ log pair}\}.$$

Note that

$$\begin{aligned}
K^{\Gamma}_{Y/C} &\equiv K_Y + f_*^{-1}\Gamma - f^*(K_C + \Gamma) \equiv \\
&\equiv K_Y + f_*^{-1}\Gamma - f^*(C_{K_W} - C_{(p^*L)} + \Gamma) \equiv \\
&\equiv \pi^*K_W - 2W_0 + \pi^*(-p^*L) + f_*^{-1}\Gamma - \pi^*(K_W - p^*L + \Delta) - (s-1)W_0 \equiv \\
&\equiv -(s+1)W_0 + f_*^{-1}\Gamma - \pi^*\Delta.
\end{aligned}$$

In particular $K^{C_\Delta}_{Y/C} = -(s+1)W_0$. Therefore if we let $t = \inf\{s \in \mathbb{R} | \exists\, \Delta \geq 0,\ K_W + \Delta \equiv s(p^*L)\}$, then we have that

$$\mathrm{val}_{W_0}(K^{-}_{Y/C}) \geq -(1+t).$$

Remark 5.17.2. Note that Δ is ample if $s > t$, in particular it is always possible to choose $\Delta = A/m$, with A a smooth very ample Cartier divisor.

Claim 5.17.3. $\mathrm{val}_{W_0}(K^{-}_{Y/C}) = -(1+t)$.

Proof of the claim. Let us consider any effective boundary $\Gamma \geq 0$. It sufficies to show that, in the previous construction, it is always possible to choose a boundary $\Delta \equiv s(p^*L) - K_W \subseteq W$ such that $\mathrm{ord}_{W_0} K^{\Gamma}_{Y/C} = \mathrm{ord}_{W_0} K^{C_\Delta}_{Y/C}$. If $f^*(K_C + \Gamma) = K_Y + f_*^{-1}\Gamma + kW_0$, let $\Delta = f_*^{-1}\Gamma|_{W_0} \geq 0$. Note that

$$\Delta = f_*^{-1}\Gamma|_{W_0} \equiv -(K_Y + kW_0)|_{W_0} \equiv -K_W + (k-1)p^*L.$$

By what we have seen above (with $s = k-1$), $K^{C_\Delta}_{Y/C} = -kW_0$. Hence $\mathrm{ord}_{W_0} K^{\Gamma}_{Y/C} = \mathrm{ord}_{W_0} K^{C_\Delta}_{Y/C}$. □

We now return to the proof of Theorem 5.17.

Since $p^*\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(W) \cap p^*\mathrm{N}^1(X)$ and $\Delta \geq 0$, the sum of the coefficients of $p^*(f_1)$, $p^*(f_2)$ and $p^*\delta$ has to be positive, so that $s \geq \frac{2}{5}$. Again, because of the above isomorphism of cones, we have that Δ is effective if and only if it is nef:

$$(\Delta^2)/4 = 9(8s^2 - 7s + 1) \geq 0 \quad \Leftrightarrow \quad s \geq \frac{7+\sqrt{17}}{16} \left(> \frac{2}{5}\right)$$

and we obtain an irrational valuation of the relative canonical divisor:

$$\mathrm{val}_{W_0}(K^{-}_{Y/C}) = -\frac{23+\sqrt{17}}{16}.$$

□

Using the result of Theorem 5.17, we now give an example of an irrational jumping number. The following are the definitions of multiplier ideal sheaf and jumping numbers in the sense of [dFH09].

Definition 5.18. *As in Definition 5.7, let (X, Z) be an effective pair. The multiplier ideal sheaf of (X, Z), denoted by $\mathcal{J}(X, Z)$, is the unique maximal element of $\{\mathcal{J}_m(X, Z)\}_{m\geq 1}$, where*

$$\mathcal{J}_m(X, Z) := f_{m_*}\mathcal{O}_{Y_m}(\lceil K_{m,Y_m/X} - f_m^{-1}(Z)\rceil),$$

with $f_m : Y_m \to X$ a log resolution of the pair $(X, Z + \mathcal{O}_X(-mK_X))$.

Definition 5.19. *A number $\mu \in \mathbb{R}_{>0}$ is a* jumping number *of an effective pair (X, Z) if $\mathcal{J}(X, \lambda \cdot Z) \neq \mathcal{J}(X, \mu \cdot Z)$ for all $0 \leq \lambda < \mu$.*

A relevant feature of the jumping numbers in the $\mathbb{Q}$-Gorenstein case is that they are always rational.

Theorem 5.20. *With the same construction as in Theorem 5.17, there are irrational jumping numbers for the pair (C, P), where P is the vertex of the projective cone.*

Proof. We are considering $Z = P \subset C$ the vertex of the projective cone. Let us denote by $\mathrm{Bl}_P C := f : Y \to C$ the blow-up of the vertex. Then we have that $f^{-1}(k \cdot Z) = k \cdot W_0$. By Theorem 5.10, for every $m \geq 1$, there exists an m-compatible bounday Γ_m such that $K_{m,Y/X} = K^{\Gamma_m}_{Y/X}$ and in particular $\mathcal{J}_m(X, Z) = \mathcal{J}((X, \Gamma_m); Z)$, hence

$$\mathcal{J}(X, k \cdot Z) = \bigcup_m \mathcal{J}_m(X, k \cdot Z) = \bigcup_{\Gamma_m} \mathcal{J}((X, \Gamma_m); k \cdot Z).$$

Also, because of Remark 5.17.2, we may assume that the blow up is a log resolution of $((X, \Gamma_m); Z)$ for every $m \geq 1$, so that

$$\mathcal{J}((X, \Gamma_m); k \cdot Z) = f_*\mathcal{O}_Y \left(\left\lceil K^{\Gamma_m}_{Y/X} - k \cdot W_0\right\rceil\right)$$

and we conclude that we can compute the jumping numbers just considering the log resolution given by the blow-up $Y \to C$. We have

$$\mathcal{J}(X, k \cdot Z) = \bigcup_{\Gamma_m} f_*\mathcal{O}_Y \left(\left\lceil K^{\Gamma_m}_{Y/X} - k \cdot W_0\right\rceil\right).$$

Since $\mathrm{val}_{W_0}(K^-_{Y/X}) = -\frac{23+\sqrt{17}}{16}$, the jumping numbers are of the form $k = t - \frac{23+\sqrt{17}}{16}$ with t any integer ≥ 1. $\square$

5.3 Canonical Singularities

We begin by giving an example of a canonical singularity which is not klt.

Let us consider a construction similar to the one in the previous section. Let $S = \mathbb{P}^1 \times \mathcal{E}$, where $\mathcal{E}$ is an elliptic curve. The canonical sheaf is

$$\omega_S \sim \mathcal{O}_{\mathbb{P}^1}(-2) \boxtimes \mathcal{O}_{\mathcal{E}}.$$

Let $\mathscr{A}$ be an ample line bundle on $\mathscr{E}$ and let us consider the embedding $S \subseteq \mathbb{P}^n$ given by the very ample divisor $L = \mathcal{O}_{\mathbb{P}^1}(2) \boxtimes \mathscr{A}^{\otimes 2}$. Let $C \subseteq \mathbb{P}^{n+1}$ the projective cone over S.

Theorem 5.21. *With the above construction, the singularity of C at its vertex is canonical but not klt.*

Proof. With the same computation as in Theorem 5.17, let $f : Y \to C$ be the blow up of the cone at the origin P, $\pi : Y \to S$ the natural projection and let us denote by S_0 the negative section. The canonical divisor K_Y is given by $K_Y \sim \pi^*(K_S) - 2S_0 + \pi^*(-L)$. Let us compute s in this case. We have $\Delta \equiv sL - K_S \sim \mathcal{O}_{\mathbb{P}^1}(2s+2) \boxtimes \mathscr{A}^{\otimes 2s}$. In particular Δ is effective if and only if $s > 0$. Hence, we have:

$$\mathrm{val}_{S_0}(K^-_{Y/C}) = -1.$$

In particular C is not klt.

With a similar computation we will show that C has canonical singularities. The relative canonical divisor used to characterize this type of singularities is $K_{Y/X} = K_Y + f^*(-K_X)$ and, by the notion of pullback given in Definition 5.4, it is given by an approximation of the form:

$$K^+_{m,Y/X} = K_Y + \frac{1}{m} f^\natural(-mK_X)$$

where, in this new definition, we have $K^+_{m,Y/X} \geq K^+_{mq,Y/X} \geq K_{Y/X}$. In particular the proof of the existence of an m-compatible boundary given in [dFH09] works also in this case with small modifications (changing the role of K_X with $-K_X$ in the proof of [dFH09, Theorem 5.4]).

We now introduce the following corollary of Lemma 5.14:

Proposition 5.22. *Let $f : Y \to X$ be a proper birational morphism such that Y is canonical. If $\mathrm{val}_F(K_{Y/X}) \geq 0$ for all divisors F on Y, then X is canonical.*

Proof. For all sufficiently divisible $m \geq 1$, $\mathrm{val}_F(K^+_{m,Y/X}) \geq 0$
(i.e., $mK_Y \geq -f^\natural(-mK_X)$), so that:

$$\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X) \hookrightarrow (\mathcal{O}_Y \cdot \mathcal{O}_X(mK_X))^{\vee\vee} = \mathcal{O}_Y(-f^\natural(-mK_X)) \hookrightarrow \mathcal{O}_Y(mK_Y).$$

Lemma 5.14 now implies the claim. □

Let $f_*^{-1}(\Gamma') := \Gamma'_C$. Since $K_Y + f^*(-K_C + \Gamma') - \Gamma'_C = K^+_{m,Y/C} \geq K_{Y/C}$, as in Remark 5.12, if we denote by S_0 the negative section, we obtain that:

$$\mathrm{val}_{S_0}(K_{Y/C}) = \inf\{\mathrm{ord}_{S_0}(K_Y + f^*(-K_C + \Gamma') - \Gamma'_C) | (-K_C + \Gamma') \text{ is } \mathbb{R}\text{-Cartier}, \Gamma' \geq 0\},$$

with $\Gamma' \equiv C_{\Delta'}$, where $\Delta' \equiv rL + K_S$. So, if

$$t = \inf\{r \in \mathbb{R} | \exists \Delta' \geq 0, -K_S + \Delta' \equiv rL\}$$

then

$$\mathrm{val}_{S_0}(K_{Y/C}) = t - 1.$$

As before, we want to control for which values r, Δ' is numerically equivalent to an effective class. In this case $\Delta' \equiv rL + K_S \sim \mathcal{O}_{\mathbb{P}^1}(2r-2) \boxtimes \mathscr{A}^{\otimes 2r}$, hence

$$\Delta' \geq 0 \Leftrightarrow \quad r \geq 1$$

and in particular, $\mathrm{val}_{S_0}(K_{Y/C}) = 0$, and so C is canonical. □

Next we will show that, if X is canonical and $\mathscr{R}_X(K_X)$ is finitely generated, then X has a canonical model with canonical singularities.

Let us introduce an useful Lemma from [KM98, Lemma 6.2]:

Lemma 5.23. *Let Y be a normal algebraic variety and B a Weil divisor on Y. The following are equivalent.*

i) $\mathscr{R}_Y(B)$ *is a finitely generated sheaf of* $\mathcal{O}_Y$*-algebras.*

ii) *There exists a projective birational morphism* $\pi : Y^+ \to Y$ *such that* Y^+ *is normal,* $Ex(\pi)$ *has codimension at least* 2*,* $B' = \pi_*^{-1}B$ *is* $\mathbb{Q}$*-Cartier and* π*-ample over* Y*, where* $Y^+ := Proj_Y \sum_{m \geq 0} \mathcal{O}_Y(mB)$*.*

$\pi : Y^+ \to Y$ *is the unique morphism with the above properties.*

Proposition 5.24. *Let* X *be a normal quasi-projective variety with canonical singularities whose canonical ring* $\mathscr{R}_X(K_X)$ *is a finitely generated* $\mathcal{O}_X$*-algebra. Then the relative canonical model* $X_{can} := Proj_X(\mathscr{R}_X(K_X))$ *exists and it has canonical singularities.*

Proof. Since X is canonical, by [dFH09, Proposition 8.2], we know that for any sufficiently high log resolution $f : Y \to X$, we have $K_Y - \frac{1}{m} f^{\natural}(-mK_X) \geq 0$.
By Lemma 5.23 there exists a small birational morphism $\pi : X^+ \to X$ such that K_{X^+} is a relatively ample $\mathbb{Q}$-Cartier divisor. Also, since the morphism is small, we have that $\pi^{\natural}(-mK_X) = -mK_{X^+}$. Let us now consider $f : Y \to X$ and $g : Y \to X^+$, a common log resolution of both X and X^+.
Let us consider the map $\mathcal{O}_{X^+} \cdot \mathcal{O}_X(mK_X) \to \mathcal{O}_{X^+}(mK_{X^+})$. Since $\pi_*^{-1}(K_X) = K_{X^+}$ is π-ample, $\mathcal{O}_{X^+}(mK_{X^+})$ is globally generated over X for m sufficiently divisible hence, since the map is small, we have an isomorphism of sheaves. Thus

$$K_Y - g^*(K_{X^+}) = K_Y + \frac{1}{m} g^*(-mK_{X^+}) = K_Y + \frac{1}{m} f^{\natural}(-mK_X) \geq 0$$

where the last equality holds by [dFH09, Lemma 2.7]. Therefore the canonical model X^+ has canonical singularities. □

5.4 Accumulation Points for Jumping Numbers

In this last section we use definitions and results from [dFH09].

Given an effective pair (X, Z), we want to consider a family of ideal sheaves in the form

$$\mathcal{J}_k = \{\mathcal{J}(X, t_k \cdot Z)\}_k$$

for $k \in \mathbb{N}$, $t_k > 0$.

If t_k is a decreasing sequence, then $\mathcal{I}_k \subset \mathcal{I}_{k+1}$ and by the Noetherian property, the sequence stabilizes.

If we consider an increasing sequence t_k, then $\mathcal{I}_k \supset \mathcal{I}_{k+1}$ and the ascending chain condition does not apply. We will show that (under appropriate hypothesis) even in this case the set of ideals stabilizes. Thus there are no accumulation points for the jumping numbers of the pair (X, Z). We will use the following.

Lemma 5.25. *Let X be a projective variety and $I = \{\mathcal{I}_k\}_k$ the family of ideals defined above. If there exists a line bundle $\mathscr{L}$ on X such that $\mathscr{L} \otimes \mathcal{I}_k$ is globally generated for all k, then it is not possible to have an infinite sequence of ideal sheaves $\mathcal{I}_r \subseteq I$ such that*

$$\mathcal{O}_X \supseteq \cdots \supseteq \mathcal{I}_r \supsetneq \mathcal{I}_{r+1} \supsetneq \mathcal{I}_{r+2} \supsetneq \cdots.$$

Proof. Tensoring by $\mathscr{L}$ and considering cohomology we would have

$$0 \leq \cdots \lneq h^0(\mathscr{L} \otimes \mathcal{I}_{r+1}) \lneq h^0(\mathscr{L} \otimes \mathcal{I}_r) \leq h^0(\mathscr{L}) = n.$$

This is impossible. □

The following is the main result of this section.

Theorem 5.26. *If (X, Z) is an effective pair with X a projective normal variety such that X has either log terminal or isolated singularities. Then the set of jumping numbers has no accumulation points, that is, given any sequence $\{t_i\}_{i \in \mathbb{N}}$ such that $t_i > 0$ and $\lim_{i \to \infty} t_i = t$, then*

$$\bigcap_i \mathcal{J}(X, t_i \cdot Z) = \mathcal{J}(X, t_{i_0} \cdot Z)$$

for some $i_0 > 0$.

We will need the following results.

Theorem 5.27. *[dFH09, Corollary 5.8] Let (X, Z) be an effective pair, where X is a projective normal variety and $Z = \sum a_k \cdot Z_k$. Let $m \geq 2$ be an integer such that $\mathcal{J}(X, Z) = \mathcal{J}_m(X, Z)$, and let Δ be an m-compatible boundary for (X, Z). For each k, let B_k be a Cartier divisor such that $\mathcal{O}_X(B_k) \otimes \mathcal{I}_{Z_k}$ is globally generated, where $\mathcal{I}_{Z_k}$ is the ideal sheaf* *of Z_k, and suppose that L is a Cartier divisor such that $L - (K_X + \Delta + \sum a_k B_k)$ is nef and big. Then*

$$H^i(\mathcal{O}_X(L) \otimes \mathcal{J}(X,Z)) = 0 \quad \text{for } i > 0.$$

Corollary 5.28. *[dFH09, Corollary 5.9] With the same notation and assumptions as in Theorem 5.27, let A be a very ample Cartier divisor on X. Then the sheaf $\mathcal{O}_X(L + kA) \otimes \mathcal{J}(X,Z)$ is globally generated for every integer $k \geq \dim X + 1$.*

Proposition 5.29. *Let X be a projective normal variety that has either log terminal or isolated singularities. Then, for any divisor $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m \geq 1$.*

Proof. If X has log terminal singularities, then by Remark 5.24 $\mathscr{R}_X(D)$ is a finitely generated $\mathcal{O}_X$-algebra. It is then easy to see that the proposition holds.

Let us then assume that X has isolated singularities. We may assume $D \in \mathrm{WDiv}(X)$. Let us fix a log resolution $f : Y \to X$ of (X, D), where $\mathcal{O}_Y \cdot \mathcal{O}_X(D) = \mathcal{O}_Y(\tilde{D} + F)$, with $\tilde{D} = f_*^{-1}D$ and F an exceptional divisor. Let B be a general very ample divisor on X such that $\mathcal{O}_X(D + B)$ and $\mathcal{O}_X(-K_X + B)$ are globally generated, with $\mathcal{O}_Y \cdot \mathcal{O}_X(-K_X + B) = \mathcal{O}_Y(G)$. Then $\tilde{B} = f^*B$ and $\mathcal{O}_Y(\tilde{B} + m\tilde{D} + mF)$ is globally generated, hence nef and big, for every $m > 0$. By the Kawamata-Viehweg vanishing, if $\mathcal{G} = \mathcal{O}_Y(K_Y + m\tilde{B} + m\tilde{D} + mF + G)$, $R^i f_*(\mathcal{G}) = 0$ for all $i > 0$, hence $H^i(Y, \mathcal{G}) \cong H^i(X, f_*\mathcal{G}) = 0$ for all $i > 0$. Then, by Mumford regularity, we may assume that $\mathcal{F} := f_*\mathcal{O}_Y(K_Y + m((n\tilde{B} + \tilde{D} + F) + G))$ is globally generated for all $m > 0$. Since $(f_*\mathcal{F})^{\vee\vee} \cong \mathcal{O}_X(K_X + mD + mnB + B - K_X) \cong \mathcal{O}_X(mD + (mn+1)B)$, we have an induced short exact sequence:

$$0 \to f_*\mathcal{F} \to \mathcal{O}_X(mD + (mn+1)B) \to Q \to 0$$

where the quotient Q is suppoted on points and hence globally generated, therefore $\mathcal{O}_X(mD + (mn+1)B)$ is globally generated for all m. In particular $\mathcal{O}_X(mD + m(n+1)B)$ is globally generated for every m.

□

Remark 5.30. *It seems that it is not known if Proposition 5.29 holds for any divisor $D \in WDiv_{\mathbb{Q}}(X)$ on any projective normal variety (regardless of the singularity). We conjecture that this is the case. Note that by Proposition 5.29 this conjecture holds for surfaces.*

We can now prove Theorem 5.26.

Proof of Theorem 5.26. We follow the proof of [dFH09, Theorem 5.4]. Let us consider an effective divisor D such that $K_X - D$ is Cartier. By Proposition 5.29 we know that there exists an ample line bundle $\mathscr{A}$ such that

$$\mathscr{A}^{\otimes m} \otimes \mathcal{O}_X(-mD)$$

is globally generated for all $m \geq 0$.
For a general element G in the linear system $|\mathscr{A}^{\otimes m} - mD|$, let $G = M + mD$ and we can choose $\Delta_m := \frac{1}{m}M$ as our boundary.
Let B_k be Cartier divisors such that $\mathcal{O}_X(B_k) \otimes \mathcal{I}_{Z_k}$ is globally generated. As in Corollary 5.27, let H be an ample Cartier divisor such that $H - (K_X - D + \sum a_k \cdot B_k)$ is nef and big. Then the Cartier divisor $(\mathscr{A} + H)$ is such that

$$(\mathscr{A} + H) - (K_X + \Delta_m + \sum a_k B_k)$$

is nef and big for all m.
Let B be a very ample Cartier divisor on X. Then for $\mathscr{L} := \mathcal{O}_X(\mathscr{A} + H + sB)$, with $s > \dim X$, we have that

$$\mathscr{L} \otimes \mathcal{I}_k(X, Z)$$

is globally generated for all k.

By Lemma 5.25,

$$\bigcap_i \mathcal{J}(X, t_i \cdot Z) = \mathcal{J}(X, t_{i_0} \cdot Z)$$

for some $i_0 > 0$ and the theorem is proved.

□

CHAPTER 6

FINITE GENERATION

In this last chapter we will consider properties of canonical varieties. In particular, we will prove how even with the definition of canonical singularity given in [dFH09], the finite generation of the canonical ring is preserved.

6.1 Canonical Singularities II

In this section we will show that if X has canonical singularities, then its canonical ring is finitely generated.

de Fernex and Hacon gave the following characterization of canonical singularities:

Proposition 6.1. *[dFH09, Proposition 8.2] Let X be a normal variety. Then X is canonical if and only if for all sufficiently divisible $m \geq 1$, and for every resolution $f : Y \to X$, there is an inclusion*

$$\mathcal{O}_X(mK_X) \cdot \mathcal{O}_Y \subseteq \mathcal{O}_Y(mK_Y)$$

as sub-$\mathcal{O}_Y$-modules of $\mathcal{K}_Y$.

The first result that we will prove is that if X is canonical, then $\mathscr{R}_X(K_X)$ is finitely generated over X. Note that this result is trivial for $\mathbb{Q}$-Gorenstein varieties (c.f. [Kol08]).

Theorem 6.2. *If X is canonical, then $\mathscr{R}_X(K_X)$ is finitely generated over X.*

Proof. We may assume that X is affine. Let $\tilde{X} \to X$ be a resolution. By [BCHM] $\mathscr{R}(K_{\tilde{X}}/X)$ is finitely generated. Running the MMP over X, we obtain $X^c = \mathrm{Proj}_X(\mathscr{R}(K_{\tilde{X}}))$ and let $f : X^c \to X$ be the induced morphism, where X^c is canonical and $\mathbb{Q}$-Gorenstein. Since X is canonical, for any $m > 0$, there is an inclusion $\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X) \to \mathcal{O}_{X^c}(mK_{X^c})$. Pushing this forward we obtain inclusions

$$f_*(\mathcal{O}_{X^c} \cdot \mathcal{O}_X(mK_X)) \subset f_*\mathcal{O}_{X^c}(mK_{X^c}) \subset \mathcal{O}_X(mK_X).$$

Since the left and right hand sides have isomorphic global sections, then $H^0(f_*\mathcal{O}_{X^c}(mK_{X^c})) \cong H^0(\mathcal{O}_X(mK_X))$. Since X is affine, $\mathcal{O}_X(mK_X)$ is globally generated and hence $f_*\mathcal{O}_{X^c}(mK_{X^c}) = \mathcal{O}_X(mK_X)$. But then $\mathscr{R}(K_X/X) \cong \mathscr{R}(K_{X^c}/X)$ is finitely generated. $\square$

Remark 6.3. *Note that we have seen that*

$$\mathscr{R}(K_X/X) \cong \mathscr{R}(K_{X^c}/X) \cong \mathscr{R}(K_{\tilde{X}}/X)$$

hence

$$X^c = Proj_X(\mathscr{R}(K_X/X))$$

and so $X^c \to X$ is a small mophism.

Corollary 6.4. *If X is canonical, then the canonical ring $\mathscr{R}(K_X)$ is finitely generated.*

Proof. Since $f : X^c \to X$ is small, it follows that $\mathscr{R}(K_X) \cong \mathscr{R}(K_{X^c})$. Since X^c is canonical and $\mathbb{Q}$-Gorenstein if follows that $\mathscr{R}(K_{X^c})$ is finitely generated (cf. [BCHM]).

$\square$

The next Proposition strictly relates log terminal singularities with the finite generation of the canonical ring even in the non-$\mathbb{Q}$-Gorenstein case:

Proposition 6.5. *Let X be a normal variety with at most canonical singularities. $\mathscr{R}(-K_X/X)$ is a finitely generated $\mathcal{O}_X$-algebra if and only if X is log terminal.*

Proof. If X is log terminal, then $\mathscr{R}(-K_X/X)$ is a finitely generated $\mathcal{O}_X$-algebra by [Kol08, Theorem 92].

For the reverse implication, since $\mathscr{R}(-K_X/X)$ is finitely generated, by [KM98, Proposition 6.2], there exists a small map $\pi : X^- \to X$, such that $\pi_*^{-1}(-K_X)$ is $\mathbb{Q}$-Cartier and π-ample. For any m sufficiently divisible, consider the natural map $\mathcal{O}_{X^-} \cdot \mathcal{O}_X(-mK_X) \to \mathcal{O}_{X^-}(-mK_{X^-})$ which, since the map is small, is an isomorphism of sheaves. Thus, considering $f : Y \to X$ and $g : Y \to X^-$, a common log resolution of both X and X^-, we have

$$K_Y + \frac{1}{m} g^*(-mK_{X^-}) = K_Y + \frac{1}{m} g^*(\pi^\natural(-mK_X)) = K_Y + \frac{1}{m} f^\natural(-mK_X) \geq 0$$

so that X^- has at most canonical singularities. Since K_{X^-} is $\mathbb{Q}$-Cartier and canonical, X^- is log terminal.

Choosing a general ample $\mathbb{Q}$-divisor $H^- \sim_{\mathbb{Q},X} -K_{X^-}$, let $m \gg 0$ and $G^- \in |mH^-|$ a general irreducible component. Then, picking $\Delta^- := \frac{G^-}{m}$, we have that $K_{X^-} + \Delta^- \sim_{\mathbb{Q},X} 0$ is still log terminal and $\pi^*(K_X + \pi_*\Delta^-) \sim_{\mathbb{Q}} K_{X^-} + \Delta^-$, hence so $(X, \Delta = \pi_*\Delta^-)$ is log terminal. □

6.2 Quasi-nef Divisors

Given a divisor D on a variety X, it is useful to know if the divisor is nef. In particular, varieties such that the canonical divisor K_X is nef are minimal models.

For arbitrary normal varieties, unfortunately, there is no good notion of nefness (this is a numerical property that is well defined if the variety is $\mathbb{Q}$-factorial or at least if D is $\mathbb{Q}$-Cartier). In particular, whenever looking for a minimal model in this case it is always necessary to either pass to a resolution of the singularities or to perturb the canonical divisor adding a boundary (an auxiliary divisor Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier). However both operations are not canonical and in either case different choices lead us to different minimal models. What we would like to do in this section is to define a notion of a minimal model for an arbitrary normal variety.

We will start defining a notion of nefness for a divisor that is not $\mathbb{Q}$-Cartier.

Definition 6.6. *Let X be a normal variety. A divisor $D \subseteq X$ is* quasi-nef *(*q-nef*) if for every ample $\mathbb{Q}$-divisor $A \subseteq X$, $\mathcal{O}_X(m(D + A))$ is generated by global sections for every $m > 0$ sufficiently divisible.*

Remark 6.7. *Let X be a normal $\mathbb{Q}$-factorial variety. A divisor $D \subseteq X$ is nef if and only if it is q-nef.*

Proposition 6.8. *Let D be a divisor on a normal variety X. If $g : Y \to X$ is a small projective birational map such that $\bar{D} := g_*^{-1}D$ is $\mathbb{Q}$-Cartier and g-ample, then D is q-nef if and only if $\bar{D}$ is nef.*

Proof. Let us first assume that D is q-nef. For every ample divisor $A \subseteq X$, by definition there exists a positive integer m such that $\mathcal{O}_X(m(D+A))$ is generated by global sections and $\mathcal{O}_Y(m\bar{D})$ is relatively globally generated. In particular, since g is small,

$$\varphi : \mathcal{O}_Y \cdot \mathcal{O}_X(m(D+A)) \to \mathcal{O}_Y(m(\bar{D}+g^*A))$$

induces an isomorphism at the level of global sections. Now $\bar{D}$ is g-ample, and there exists $k \gg 0$ such that $\mathcal{O}_Y(m(\bar{D}+g^*A)) \otimes \mathcal{O}_Y(kg^*A)$ is also generated by global sections, hence φ must be surjective and hence an isomorphism. Thus $\mathcal{O}_Y(m(\bar{D}+g^*A))$ is generated by global sections. This implies that $\bar{D}+g^*A$ is nef, and since nefness is a closed property, $\bar{D}$ is nef.

Let us now suppose that $\bar{D}$ is a nef divisor on Y. Fix an ample divisor A on X and r an integer such that $rA \sim H$ is very ample. Since $\bar{D}$ is g-ample, $\bar{D}+kg^*(A)$ is an ample divisor for any k big enough. Fix k with this property. In particular, since $\bar{D}$ is nef, by Fujita's vanishing theorem ([Laz04a, Theorem 1.4.35]) we have that

$$\begin{aligned} &H^i(Y, k(m-(n-i))(\bar{D}+g^*(A))) = \\ &\qquad = H^i(Y, (m-(n-i))(\bar{D}+kg^*(A)) + (m-(n-i))(k-1)\bar{D}) = 0 \end{aligned}$$

for $0 < i \leq n = \dim X$, if $m \gg 0$. By [Laz04a, Lemma 4.3.10], this implies that

$$R^j(g_*\mathcal{O}_Y(mk(\bar{D}+g^*(A)) - (n-i)k(\bar{D}+g^*(A))) = 0 \quad \text{for } j > 0 \tag{6.1}$$

and by Castelnuovo-Mumford regularity we conclude that $g_*(\mathcal{O}_Y(mk(\bar{D}+g^*A)))$ is generated by global sections. Let us denote $\mathcal{F} := \mathcal{O}_Y(mk(\bar{D}+g^*A)+nr(\bar{D}+g^*A))$, and $M := (mk+nr)$.

Let us now consider the following exact sequence:

$$0 \to g_*\mathcal{F} \to \mathcal{O}_X(M(D+A)) \to Q \to 0$$

where Q is the cokernel of the first map. We will prove by induction on $d := \dim(\mathrm{Supp}(Q))$, that $\mathcal{O}_X(M(D+A))$ is generated by global sections.
If $\dim(\mathrm{Supp}(Q)) = 0$, then Q is supported on points and hence globally generated. Since $g_*\mathcal{F}$ is globally generated as we observed above, and $H^1(g_*\mathcal{F}) = 0$, it follows that $\mathcal{O}_X(M(D+A))$ is globally generated.
Let us now consider the general case, with $\dim Q = d$. In particular, if $H_1, \dots, H_d \in |rA|$ are general hyperplane sections, $g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d}$ is torsion free and we can construct the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & g_*\mathcal{F} \arrow[r] \arrow[d, "p"] & \mathcal{O}_X(M(D+A)) \arrow[r] \arrow[d, "q"] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d} \arrow[r, "s"] \arrow[d] & \mathcal{O}_X(M(D+A))|_{H_1\cap\cdots\cap H_d} \arrow[r, "v"] \arrow[d] & Q|_{H_1\cap\cdots\cap H_d} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We first need to justify the existence of the map s. It suffices show that $(g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d})^{\vee\vee} \cong \mathcal{O}_X(M(D+A))|_{H_1\cap\cdots\cap H_d}$, where we already know that the two sheaves agree on a big open set. $\mathcal{O}_X(M(D+A))$ is a reflexive sheaf if and only if there exists an associated exact sequence of the form

$$0 \to \mathcal{O}_X(M(D+A)) \to \mathscr{E} \to \mathscr{G} \to 0$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion free [Har80, Proposition 1.1]. We need to show that the restriction to a general hyperplane section H leaves the sequence exact. In particular we need to show that $\mathscr{G}|_H$ is torsion free, and this is true since it is possible to pick H that does not contain any of the associated primes of $\mathscr{G}$. Since the left hand side is reflexive by definition, and the two sheaves agree on a big open set, they have to be the same.
We can now finish the proof via a few simple observations. In fact, we have that $\dim \mathrm{Supp}(Q|_{H_1\cap\cdots\cap H_d}) = 0$, hence the sheaf is generated by global sections. Also, since it g is a small map, we have that s is an isomorphism at the level of global sections and by (6.1) it follows that $H^1(g_*\mathcal{F}|_{H_1\cap\cdots\cap H_d}) = 0$, hence $Q|_{H_1\cap\cdots\cap H_d}$

is trivial, so that Q is trivial and $\mathcal{O}_X(M(D+A)) \cong g_*\mathcal{F}$ is generated by global sections.

We conclude that for every ample divisor A on X, there exists an integer M such that $\mathcal{O}_X(M(D+A))$ is generated by global sections, hence D is q-nef. □

Definition 6.9. *Let X be a normal projective variety, D any divisor on X and A an ample divisor. If there exists a $t \in \mathbb{R}$ such that $D + tA$ is quasi-nef, we define the quasi-nef threshold with respect to A (qnt_A) as:*

$$qnt_A(D) = \inf\{t \in \mathbb{R} \mid \ \mathcal{O}_X(m(D+tA)) \text{ is globally generated for all } m \text{ sufficiently divisible}\}.$$

Remark 6.10. *Let X be a normal projective variety with at most log-terminal singularities. For any divisor D on X and any ample divisor A, then $qnt_A(D)$ exists and it is a rational number. This is a direct consequence of the fact that for any variety with at most klt singularities, every divisorial ring is finitely generated [Kol08, Theorem 92].*

Let us recall the following conjecture from [Urb11]:

Conjecture 6.11. *Let X be a projective normal variety. Then, for any divisor $D \in \mathrm{WDiv}_{\mathbb{Q}}(X)$, there exists a very ample divisor A such that $\mathcal{O}_X(mD) \otimes \mathcal{O}_X(A)^{\otimes m}$ is globally generated for every $m \geq 1$.*

Remark 6.12. *Let X be a normal projective variety for which 6.11 holds. Then for any Weil divisor D and every ample divisor A on X, $qnt_A(D)$ exists.*

The Remark is trivial assuming 6.11. However, we expect the existance of the threshold independently of the Conjecture.

Remark 6.13. *For any Weil divisor D and every ample divisor A on a normal projective variety, if $qnt_A(D)$ exists, then $D + qnt_A(D)A$ is a quasi-nef divisor.*

6.3 MLD's of Toric Threefolds

For the notation and basic properties of toric varieties we refer the reader to [CLS11].

Consider a normal projective toric variety $X = X_\Sigma$ corresponding to a complete fan Σ in $N_\mathbb{R}$ (with no torus factor), with $\dim N_\mathbb{R} = n$. Recall that every T_N-invariant Weil divisor is represented by a sum

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho,$$

where ρ is a one-dimensional subcone (a ray), and D_ρ is the associated T_N-invariant prime divisor. D is Cartier if for every maximal dimension subcone $\sigma \in \Sigma(n)$, $D|_{U_\sigma}$ is locally a divisor of a character ($\operatorname{div}(\chi^m)$ with $m \in N^\vee = M$).

To every divisor we can associate a polytope:

$$P_D = \{m \in M_\mathbb{R} | \langle m, u_\rho \rangle \geq a_\rho \text{ for every } \rho \in \Sigma(1)\}.$$

Even if the divisor is not Cartier, the polytope is still convex and rational but not necessarily integral.

For every divisor D and every cone $\sigma \in \Sigma(n)$, we can describe the local sections as

$$\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[W]$$

where $W = \{\chi^m | \langle m, u_\rho \rangle + a_\rho \geq 0 \text{ for all } \rho \in \sigma(1)\}$.

Let us recall the following Proposition from [Lin03]:

Proposition 6.14. *For a torus invariant Weil divisor $D = \sum a_\rho D_\rho$, the following statements hold.*

i) $\Gamma(X, D) = \bigoplus_{m \in P_D \cap M} \mathbb{C} \cdot \chi^m$.

ii) Given that $\mathcal{O}_X(D)(U_\sigma) = \mathbb{C}[\sigma^\vee \cap M]\langle \chi^{m_{\sigma,1}}, \dots, \chi^{m_{\sigma,r_\sigma}} \rangle$ is a finitely generated $\mathbb{C}[\sigma^\vee \cap M]$ module for every $\sigma \in \Sigma(n)$, $\mathcal{O}_X(D)$ is generated by its global sections if and only if $m_{\sigma,j} \in P_D$ for all σ and j.

We will also need the following result [Eli97].

Theorem 6.15. *Let X be a complete toric variety and let D be a Cartier divisor on X. Then the ring*

$$\mathscr{R}_D := \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$$

is a finitely generated $\mathbb{C}$-algebra.

Corollary 6.16. *Since every toric variety admits a $\mathbb{Q}$-factorialization, a small morphism from a $\mathbb{Q}$-factorial variety ([Fuj01, Corollary 3.6]), the above result holds for Weil divisors as well.*

Remark 6.17. *Conjecture 6.11 holds for $X = X_\Sigma$, a complete toric variety.*

Depending on our choice of the relative canonical divisor, we have two possible definitions for the Minimal Log Discrepancies (MLD's).

Definition 6.18. *Let X be a normal variety over the complex numbers, we will associate two numbers to the variety X:*

$$MLD^{-}(X) = \inf_{E} val_E(K^{-}_{Y/X})$$

and

$$MLD^{+}(X) = \inf_{E} val_E(K^{+}_{Y/X})$$

where $E \subseteq Y$ is a prime divisor and $Y \to X$ is any proper birational morphism of normal varieties.

It is natural to wonder if these MLD's also satisfy the ACC conjecture. If X is assumed to be $\mathbb{Q}$-Gorenstein, then this is conjectured to hold by V. Shokurov. In view of [dFH09, Theorem 5.4], the MLD^{+}'s correspond to MLD's of appropriate pairs (X, Δ). However the coefficients of Δ do not necessarily belong to a DCC set (cf. [Amb06]).

Proposition 6.19. *The set of all MLD^{+}'s for terminal toric threefolds do not satisfy the ACC conjecture.*

Proof. We give an explicit example of a set of terminal toric threefolds whose associeted MLD^{+}'s converge to a number from below. The problem is local, hence we will consider a set of affine toric threefolds given by the following data.

Let X be the affine toric variety associated to the cone $\sigma = \langle u_1, u_2, u_3, u_4 \rangle$, $u_1 = (2, -1, 0)$, $u_2 = (2, 0, 1)$, $u_3 = (1, 1, 1)$, $u_4 = (a, 1, 0)$ with $a \in \mathbb{N}$. The associated toric variety is non-$\mathbb{Q}$-Gorenstein, i.e., the canonical divisor $K_X = \sum -D_i$ is not $\mathbb{Q}$-Cartier.

Let $\Delta = \sum d_i D_i$ be a $\mathbb{Q}$-divisor such that $0 \leq d_i \leq 1$ and $-K_X + \Delta$ is $\mathbb{Q}$-Cartier. This means that there exists $m = (x, y, z)$ such that $-K_X + \Delta = \sum(m, u_i)D_i$. Hence, by an easy computation, $\Delta = (2x - y - 1)D_1 + (2x + z - 1)D_2 + (x + y + z - 1)D_3 + (ax + y - 1)D_4$.

The smallest discrepancy will be obtained blowing-up the lattice point u_E properly contained in the cone that is closer to the origin. By our choice of the cone, this point is given by $u_E = u_1 + u_2 + u_3$. Let us denote $f : Bl_{u_E}X = Y \to X$ the blow-up and E the corresponding exceptional divisor. Since

$$\mathrm{val}_E(f^*(-K_X + \Delta)) = \sum_{i=1}^{3}(m, u_i)$$

with a simple computation we have that

$$\mathrm{val}_E(K^+_{Y/X}) = \inf_{\Delta \text{ boundary}} 5x - 2z.$$

Increasing the value of the parameter a we see that the minimal valuation (solving a problem of minimality with constrains) is given by $\dfrac{4a+5}{a+2}$ that accumulates from below at the value 4. $\square$

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