VARIATIONS ON A THEME OF SYMMETRIC TROPICAL MATRICES

by

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STATEMENT OF DISSERTATION APPROVAL

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ABSTRACT

Tropical geometry connects the fields of algebraic and polyhedral geometry. This connection has been used to discover much simpler proofs of fundamental theorems in algebraic geometry, including the Brill-Noether theorem. Tropical geometry has also found applications outside of pure mathematics, in areas as diverse as phylogenetic models and auction theory.

This dissertation seeks to answer the question of when the minors of a symmetric matrix form a tropical basis.

The first chapter introduces the relevant ideas and concepts from tropical geometry and tropical linear algebra.

The second chapter introduces different notions of rank for symmetric tropical matrices.

The third chapter is devoted to proving all the cases, outside symmetric tropical rank three, where the minors of a symmetric matrix form a tropical basis.

The fourth chapter deals with symmetric tropical rank three. We prove that the 4×4 minors of an $n \times n$ symmetric matrix form a tropical basis if $n \leq 5$, but not if $n \geq 13$. The question for $5 < n < 13$ remains open.

The fifth chapter is devoted to when the minors of a symmetric matrix do not form a tropical basis. We prove the $r \times r$ minors of an $n \times n$ symmetric matrix do not form a tropical basis when $4 < r < n$. We also prove that, when the minors of a matrix (general or symmetric) define a tropical variety and tropical prevariety that are different, then, with one exception, the two sets differ in dimension. The exception is the 4×4 minors of a symmetric matrix, where the question is still unresolved.

The sixth chapter explores tropical conics. A correspondence between a property of the symmetric matrix of a quadric and the dual complex of that quadric is demonstrated for conics, and proposed for all quadrics.

The seventh chapter reviews the results and proposes possible questions for further study.

The first appendix is devoted to correcting a proof in a paper cited by this dissertation.

The second appendix is a transcript of the Maple worksheets used to perform the computer calculations from the fifth chapter.

To all my friends and family who generously supported me throughout. In particular to my father, Patrick Donovan Zwick, my mother, Rebecca Terry Heal, my stepfather, Gilbert “Scott” Heal, and my adviser, Aaron Bertram.

"City," he cried, and his voice rolled over the metropolis like thunder, "I am going to tropicalize you."

—*The Satanic Verses*, Salman Rushdie

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This dissertation is the culmination of a lengthy, tortuous, and difficult journey. Along the way I've relied upon the help and encouragement of many comrades, and the list that follows is far from complete.

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I mentioned at the beginning that a Ph.D. in math, or in any field, is a difficult journey. I must also acknowledge how extremely fortunate I've been in having the opportunity to make the journey at all. If you, dear reader, are yourself on that journey let me just say that it is worth it. It might not always feel that way, or it might seem the journey will never end, but the experience of discovering something, even something small, at the frontier of man's knowledge is one of the greatest pleasures life has to offer. Mathematics is the pinnacle intellectual construction of mankind, and one could do worse than to contribute a stone, or even a pebble, to its elegant and beautiful structure.

CHAPTER 1

BASICS OF TROPICAL GEOMETRY AND TROPICAL LINEAR ALGEBRA

Tropical geometry is a relatively new area of mathematics that incorporates ideas and methods from both algebraic geometry and polyhedral geometry. As such, it is both interesting in its own right, and a source of possible tools and insights for approaching problems in related areas.

This chapter introduces the fundamental definitions and concepts from tropical geometry and tropical linear algebra that will motivate the rest of the dissertation.

A good general reference for the basics of tropical geometry is the book by Maclagan and Sturmfels [13].

1.1 Ranks of Tropical Matrices

In classical linear algebra there are many equivalent definitions of the rank of a matrix. In particular, the following three are equivalent:

- The rank of a matrix A is the smallest integer r for which A can be written as the sum of r rank one matrices. A matrix has rank one if it is the outer product of a column vector and a row vector.
- The rank of A is the smallest dimension of any linear space containing the columns of A.
- The rank of A is the largest integer r such that A has a nonsingular $r \times r$ minor.

Develin, Santos, and Sturmfels [8] define analogs of these three definitions for tropical matrices, and call them, respectively, the Barvinok rank, the Kapranov rank, and the tropical rank. These three definitions are *not* equivalent, and satisfy the inequalities

$$\text{tropical rank(A)} \leq \text{Kapranov rank(A)} \leq \text{Barvinok rank(A)}.$$

Both inequalities may be strict ([8] Theorem 1.4).

In this dissertation we define and investigate analogs of these notions of rank for symmetric matrices, with particular attention to Kapranov and tropical rank. We work over the *tropical semiring* $(\mathbb{R}, \oplus, \odot)$, whose arithmetic operations are

$$a \oplus b := min(a, b) \quad \text{and} \quad a \odot b := a + b.$$

Note that here we are working over the real numbers $\mathbb{R}$, and not the extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and so for all the definitions of rank the minimum possible rank is one.

In general, letters and variables will be upper case when working in the tropical semiring, and lower case when not. When denoting the element in row i and column j of a matrix these indices will be separated by a comma, so $A_{i,j}$ is element (i, j) of the matrix A. The notation A_{ij} will mean the submatrix formed by eliminating row i and column j from the matrix A.

1.2 Initial Definitions

We first define the basic objects of tropical algebra and tropical geometry.

Definition 1.1. A *tropical monomial* $X_1^{a_1} \cdots X_m^{a_m}$ is a symbol, and represents a function equivalent to the linear form $\sum_i a_i X_i$ (standard addition and multiplication).

Definition 1.2. A *tropical polynomial* is a tropical sum of tropical monomials

$$F(X_1, \ldots, X_m) = \bigoplus_{a \in \mathcal{A}} C_a X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m}, \quad \text{with } \mathcal{A} \subset \mathbb{N}^m,\ C_a \in \mathbb{R}$$

(tropical addition and multiplication), and represents a piecewise linear convex function $F : \mathbb{R}^m \to \mathbb{R}$.

Note, unlike with standard polynomials, it is possible for two distinct tropical polynomials to represent the same linear convex function. For example, the distinct tropical polynomials

$$F_1 = X^2 \oplus Y^2, \quad \text{and} \quad X^2 \oplus 2XY \oplus Y^2$$

represent the same linear convex function.

Definition 1.3. The *tropical hypersurface* $\mathbf{V}(F)$ defined by a tropical polynomial F is the set of all points $P \in \mathbb{R}^m$ such that at least two monomials in F are minimal at P. This is also called the *double-min locus* of F.

So, for example, the tropical hypersurface defined by the tropical polynomial

$$1X^2 \oplus XY \oplus X \oplus 1Y^2 \oplus Y \oplus 1 = min\{2x+1, x+y, x, 2y+1, y, 1\}$$

would include the point $(0,0)$, but not the point $(0,1)$. This hypersurface is graphed in Figure 1.1.

Just as in standard algebraic geometry, there is a tropical notion of projective space.

Definition 1.4. The *tropical projective space* $\mathbb{TP}^{n-1}$ is the quotient of $\mathbb{R}^n$ by the equivalence relation $(a_1, \ldots, a_n) \sim (c \odot a_1, \ldots, c \odot a_n)$, where $c \in \mathbb{R}$.

As in standard algebraic geometry, a homogeneous tropical polynomial defines a projective tropical hypersurface in tropical projective space. We will occasionally be working in tropical projective space, particularly in Chapter 6. When we are it will be made clear.

1.3 Tropical Varieties and Prevarieties

Let k be an algebraically closed field. Let $f \in k[x_1, \ldots, x_m]$ be a polynomial. The set of points $p \in k^m$ such that $f(p) = 0$ is a *hypersurface*, and is denoted $\mathbf{V}(f)$. Let I be a prime ideal of $k[x_1, \ldots, x_m]$. The ideal I defines a *algebraic variety*, (or *variety*, for short) $\mathbf{V}(I)$, in k^m, which is the set of points $p \in k^m$ such that $f(p) = 0$ for all $f \in I$. If $I = (f_1, \ldots, f_n)$ then the set $\{f_1, \ldots, f_n\}$ is a *basis* for I, and $\mathbf{V}(I)$ is equal to the set of points $p \in k^m$ such that $f_i(p) = 0$ for all f_i in the basis. Put succinctly

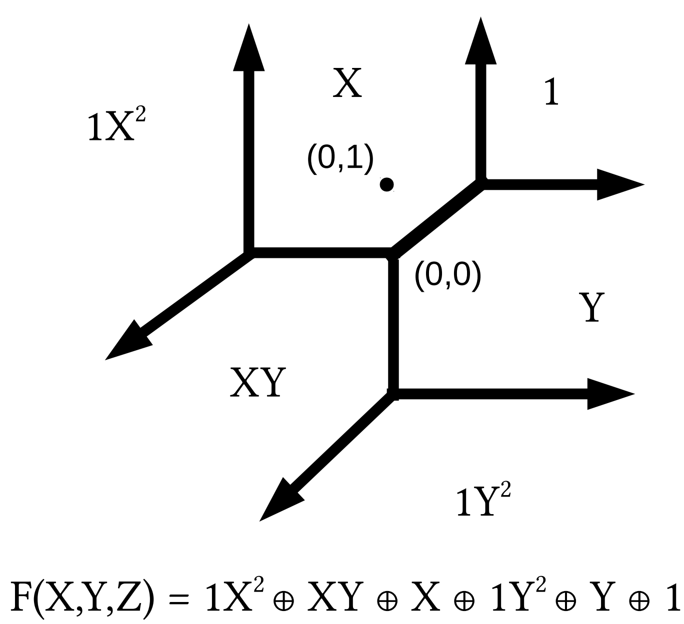


Figure 1.1. This tropical hypersurface contains the point $(0,0)$, but not the point $(0,1)$.

$$\mathbf{V}(I) = \bigcap \mathbf{V}(f_i).$$

So, a variety is an intersection of hypersurfaces. By the Hilbert basis theorem every ideal of $k[x_1, \ldots, x_m]$ is finitely generated, so any variety is a finite intersection of hypersurfaces.

In the tropical setting there is an analog of a hypersurface, and we would like an analog of a variety. It might seem natural to define a tropical variety as the intersection of a finite set of tropical hypersurfaces, but these sets do not always have the properties we need in order for them to be useful analogs of algebraic varieties, and we instead call these sets tropical prevarieties.

Definition 1.5. A *tropical prevariety* $\mathbf{V}(F_1, \ldots, F_n)$ is a finite intersection of tropical hypersurfaces:

$$\mathbf{V}(F_1, \ldots, F_n) = \bigcap_{i=1}^{n} \mathbf{V}(F_i).$$

A tropical variety is defined differently. First, one must define the field of Puiseux series. The use of this field goes all the way back to Isaac Newton [15], although the field is named after Puiseux, because he was the first to prove it is algebraicaly closed [17]. Let $K = \mathbb{C}\{\{t\}\}$ be the set of formal power series $a = c_1 t^{a_1} + c_2 t^{a_2} + \cdots$, where $a_1 < a_2 < a_3 < \cdots$ are rational numbers that have a common denominator. This set is an algebraically closed field of characteristic zero ([21], Theorem 2.4.3), and for any nonzero element a in this set we define the degree of a to be the value of the leading exponent a_1. This gives us a degree map $deg : K^* \to \mathbb{Q}$. For any two elements $a, b \in K^*$ we have

$$deg(ab) = deg(a) + deg(b) = deg(a) \odot deg(b).$$

Generally, we also have

$$deg(a + b) = min(deg(a), deg(b)) = deg(a) \oplus deg(b).$$

The only case when this addition relation is not true is when a and b have the same degree, and the coefficients of the leading terms cancel.

We would like to do tropical arithmetic over $\mathbb{R}$, and not just over $\mathbb{Q}$, so we enlarge the field of Puisieux series to allow this. Define the field $\tilde{K}$ by

$$\tilde{K} = \left\{ \sum_{\alpha \in A} c_\alpha t^\alpha | A \subset \mathbb{R} \text{ well-ordered}, c_\alpha \in \mathbb{C} \right\}.$$

This field contains the field of Puisieux series, and is also an algebraically closed field of characteristic zero. We will define a tropical variety in terms of a variety over $\tilde{K}$.

Definition 1.6. The degree map on $(\tilde{K}^*)^m$ is the map $\mathcal{T}$ taking points $(p_1,\ldots,p_m) \in (\tilde{K}^*)^m$ to points $(deg(p_1), deg(p_2),\ldots,deg(p_m)) \in \mathbb{R}^m$. A tropical variety is the image of a variety in $(\tilde{K}^*)^m$ under the degree map. We call taking this image the *tropicalization* of a set of points in $(\tilde{K}^*)^m$. The tropicalization of a polynomial $f \in \tilde{K}[x_1,\ldots,x_m]$ is the tropical polynomial $\mathcal{T}(f)$ formed by tropicalizing the coefficients of f, and converting addition and multiplication into their tropical counterparts.

For example, the tropicalization of the polynomial

$$f = 3t^2xy - 7tx^3$$

is the tropical polynomial

$$\mathcal{T}(f) = 2XY \oplus 1X^3.$$

Sturmfels [20] proved that the tropicalization of a d-dimensional variety in $(\tilde{K}^*)^m$ is a pure d-dimensional polyhedral fan. That the dimension of the tropicalization is the dimension of the variety originates with Bieri and Groves [2].

In an unpublished manuscript from the early 1990s, Mikhail Kapranov proved the following useful and fundamental result.

Theorem 1.7 (Kapranov's Theorem)**.** *For $f \in \tilde{K}[x_1,\ldots,x_m]$ the tropical variety $\mathcal{T}(\boldsymbol{V}(f))$ is equal to the tropical hypersurface $\boldsymbol{V}(\mathcal{T}(f))$ determined by the tropical polynomial $\mathcal{T}(f)$.*

Given Kapranov's theorem if $I = (f_1,\ldots,f_n)$, then obviously the tropical prevariety determined by the set of tropical polynomials $\{\mathcal{T}(f_1),\ldots,\mathcal{T}(f_n)\}$ contains the tropical variety determined by I:

$$\mathcal{T}(\mathbf{V}(I)) \subseteq \bigcap_{i=1}^{n} \mathbf{V}(\mathcal{T}(f_i)).$$

While Kapranov's theorem gives us the two sets are equal if $N = 1$, in general the containment may be strict. For example, the lines in $(\tilde{K}^*)^2$ defined by the linear equations

$$f = 2x + y + 1, \quad \text{and} \quad g = tx + ty + 1,$$

intersect at the point $(t^{-1} - 1, -2t^{-1} + 1)$. The tropicalization of this point is $(-1,-1)$, and so if $I = (f,g)$ then

$$\mathcal{T}(\mathbf{V}(I)) = (-1,-1).$$

However, is we tropicalize the linear equations we get:

$$\mathcal{T}(f) = X \oplus Y \oplus 0, \quad \text{and} \quad \mathcal{T}(g) = 1X \oplus 1Y \oplus 0.$$

Each of $\mathbf{V}(\mathcal{T}(f))$ and $\mathbf{V}(\mathcal{T}(g))$ is a tropical line, and their intersection is the tropical prevariety consisting of all points (a, a) with $a \leq -1$. This intersection is graphed in Figure 1.2.

This tropical prevariety properly contains the tropical variety $(-1, -1)$, but the prevariety is clearly much larger than the variety. That the intersection of two distinct tropical lines is not necessarily a point is an example of why we do not want to define a tropical variety to be a finite intersection of tropical hypersurfaces.

1.4 Tropical Bases

There are sets of polynomials $f_1, \ldots, f_n \in \tilde{K}[x_1, \ldots, x_m]$ generating a prime ideal for which the tropical variety

$$\mathcal{T}\left(\bigcap_{i=1}^{n} \mathbf{V}(f_i)\right)$$

and tropical prevariety

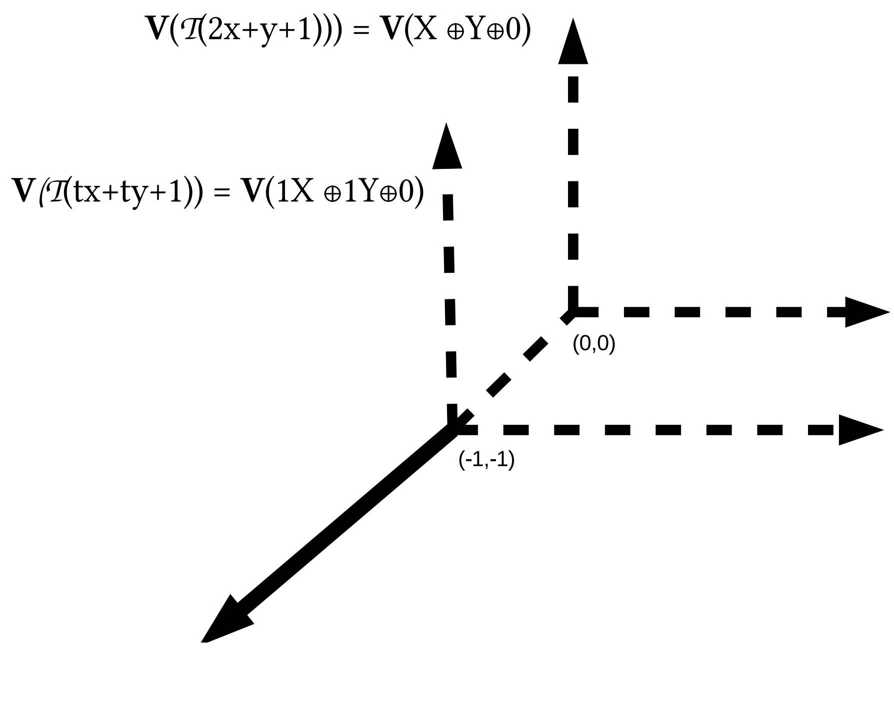


Figure 1.2. Two tropical lines intersecting at a ray.

$$\bigcap_{i=1}^{n} \mathbf{V}(\mathcal{T}(f_i))$$

are equal, and these sets are given special distinction.

Definition 1.8. A basis for a prime ideal $I = (f_1, \ldots, f_n) \subseteq \tilde{K}[x_1, \ldots, x_m]$ is a *tropical basis* if

$$\mathcal{T}(\mathbf{V}(I)) = \bigcap_{i=1}^{n} \mathbf{V}(\mathcal{T}(f_i)).$$

In [3] it is proven that every prime ideal I in $\tilde{K}[x_1, \ldots, x_m]$ has a tropical basis, but not every basis is a tropical basis.

A question of central importance to this dissertation, first posed by Chan, Jensen, and Rubei [7], is when the $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis. In Chapters 2 through 5 we will answer this question entirely, apart from the 4×4 minors of an $n \times n$ symmetric matrix with $5 < n < 13$.

We saw earlier an example of a tropical prevariety that is not a tropical variety. Namely, two tropical lines intersecting at a ray. In this case the tropical prevariety corresponding to the basis is not just larger than the tropical variety, but is in fact of greater dimension. Generally, if a basis for a prime ideal is not a tropical basis, a natural question to ask is whether the corresponding tropical prevariety has a larger dimension than the corresponding tropical variety. That is to say, if $I = (f_1, \ldots, f_n)$ is a prime ideal, and the containment

$$\mathcal{T}(\mathbf{V}(I)) \subset \bigcap_{i=1}^{n} \mathbf{V}(\mathcal{T}(f_i))$$

is proper, is it the case that

$$dim\left(\mathcal{T}(\mathbf{V}(I))\right) < dim\left(\bigcap_{i=1}^{n} \mathbf{V}(\mathcal{T}(f_i))\right)?$$

In general, the answer is no [16], as can be seen with the ideal

$$I = ((x-1)(x-t), (x-1)(x-2t)) \subset \tilde{K}[x].$$

The tropical variety $\mathcal{T}(\mathbf{V}(I))$ is the point $\{0\}$, while the tropical prevariety

$$\mathbf{V}(X^2 \oplus X \oplus 1) \cap \mathbf{V}(X^2 \oplus X \oplus 1)$$

is the set of points $\{0, 1\}$. This variety and prevariety are graphed in Figure 1.3.

In this last example the tropical prevariety is disconnected, but this is not always the case. We can modify this example slightly to get an example of a connected tropical

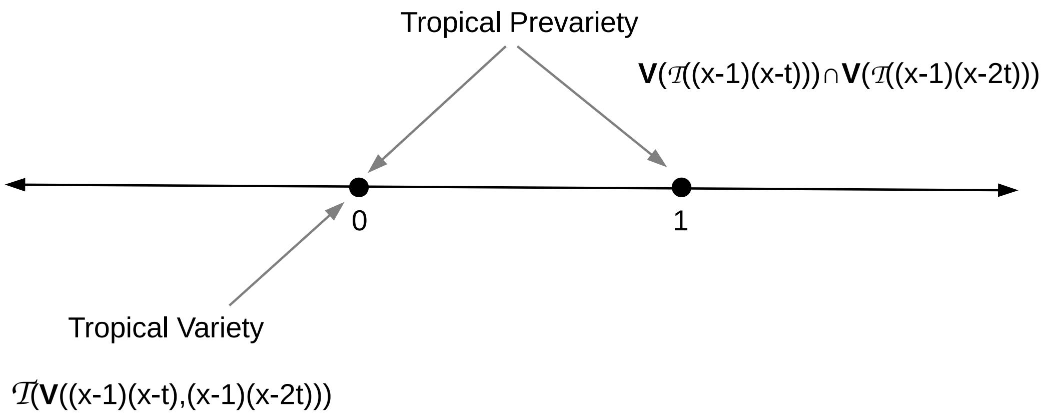


Figure 1.3. An example of a basis that is not a tropical basis, but in which both the tropical variety and tropical prevariety have the same dimension.

prevariety that is larger than its corresponding tropical variety, but does not have greater dimension. Specifically, the ideal

$$J = ((x-y)(x-t), (x-y)(x-2t)) \subset \tilde{K}[x,y]$$

Here the tropical variety $\mathcal{T}(\mathbf{V}(J))$ is the line $X = Y$, while the tropical prevariety

$$\mathbf{V}((X^2 \oplus XY \oplus 1X \oplus 1Y)) \cap \mathbf{V}((X^2 \oplus XY \oplus 1X \oplus 1Y))$$

is the union of the two lines $X = Y$ and $X = 1$. This variety is graphed in Figure 1.4.

For determinantal varieties of standard matrices, however, it is true that every time the $r \times r$ minors of an $m \times n$ matrix of variables are not a tropical basis the tropical prevariety they define has greater dimension than the tropical variety they define.

For determinantal varieties of symmetric matrices the same is true when $r > 4$. When $r = 4$ it is unknown whether it is true or not. As will be proven in Chapter 3, when $r \leq 3$ the minors are always a tropical basis.

These dimension inequalities will all be proven in Chapter 5.

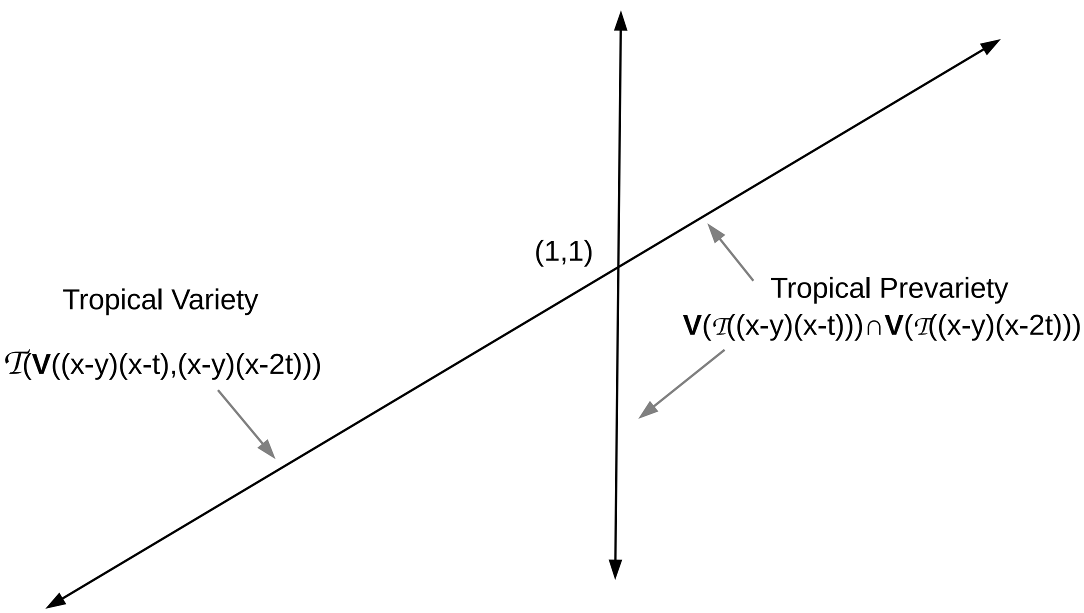


Figure 1.4. A connected example of a basis that is not a tropical basis, but in which both the tropical variety and tropical prevariety have the same dimension.

CHAPTER 2

RANKS OF SYMMETRIC TROPICAL MATRICES

In this chapter we define three notions of rank for symmetric tropical matrices: the symmetric Kapranov rank, the symmetric tropical rank, and the symmetric Barvinok rank. These are the symmetric analogs of the corresponding three notions of rank for tropical matrices from [8].

Like their standard matrix analogs, these ranks are not equivalent, and satisfy the inequalities

$$\text{symmetric tropical rank} \leq \text{symmetric Kapranov rank} \leq \text{symmetric Barvinok rank}.$$

Both these inequalities may be strict. In this chapter we will prove all these inequalities, and prove the right inequality may be strict. We will prove in Chapter 5 that the left inequality may also be strict.

We will focus mostly on symmetric tropical rank and symmetric Kapranov rank, describing how they differ from their general matrix counterparts, and investigating some of their basic properties.

2.1 Symmetric Kapranov Rank

Like the Kapranov rank, the symmetric Kapranov rank relates the rank of a symmetric real matrix to the smallest rank of an appropriate "lift" of that matrix.

2.1.1 Definition

A *lift* of a real matrix $A = (A_{i,j}) \in \mathbb{R}^{d \times n}$ is a matrix $\tilde{A} = (\tilde{a}_{i,j}) \in (\tilde{K}^*)^{d \times n}$ such that $deg(\tilde{a}_{i,j}) = A_{i,j}$ for all i, j. The *Kapranov rank* of a matrix, as defined in [8], is the smallest rank of any lift of the matrix.

For the symmetric Kapranov rank of a real symmetric matrix $B \in \mathbb{R}^{n \times n}$ we require this lift $\tilde{B}$ to be symmetric.

Definition 2.1. The *symmetric Kapranov rank* of a real symmetric matrix is the smallest rank of any symmetric lift.

Denote the set of $m \times n$ real matrices with Kapranov rank less than r by $\tilde{T}_{m,n,r}$. Denote the set of $n \times n$ real symmetric matrices with symmetric Kapranov rank less than r by $\tilde{S}_{n,r}$.

2.1.2 Elementary Properties

Obviously, for any symmetric matrix A,

$$\text{Kapranov rank(A)} \leq \text{symmetric Kapranov rank(A)},$$

and, as demonstrated in the example below, this inequality may be strict. Viewing both $\tilde{S}_{n,r}$ and $\tilde{T}_{n,n,r}$ as subsets of $\mathbb{R}^{n\times n}$ we can write this relation as

$$\tilde{S}_{n,r} \subset \tilde{T}_{n,n,r}.$$

An example of a matrix with different Kapranov and symmetric Kapranov ranks is

$$C_3 := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The reason for the terminology C_3 will be explained later in this chapter in the section on Barvinok rank.

C_3 lifts to the rank two matrix

$$\begin{pmatrix} t & 1 & 1+t \\ 1 & t & 1+t \\ 1+t & -1 & t \end{pmatrix},$$

and so has Kapranov rank two. However, C_3 does not lift to any symmetric rank two matrix.

To prove this, for the sake of contradiction suppose there is a lift to a symmetric rank two matrix

$$\tilde{C}_3 := \begin{pmatrix} c_{1,1}t + \cdots & c_{1,2} + \cdots & c_{1,3} + \cdots \\ c_{1,2} + \cdots & c_{2,2}t + \cdots & c_{2,3} + \cdots \\ c_{1,3} + \cdots & c_{2,3} + \cdots & c_{3,3}t + \cdots \end{pmatrix},$$

where $c_{i,j} \in \mathbb{C}$.

If the first column of $\tilde{C}_3$ were a $\tilde{K}$-multiple of the second,

$$\tilde{\mathbf{c}}_1 = \tilde{k}\tilde{\mathbf{c}}_2,$$

then the relation from the first row

$$c_{1,1}t + \cdots = \tilde{k}(c_{1,2} + \cdots)$$

would require $deg(\tilde{k}) = 1$, while the relation from the second row

$$c_{1,2} + \cdots = \tilde{k}(c_{2,2}t + \cdots)$$

would require $deg(\tilde{k}) = -1$. This is a contradiction, and so the second column of $\tilde{C}_3$ is linearly independent of the first.

As the first two columns are linearly independent, if $\tilde{C}_3$ has rank two there must be a linear combination of the first two columns equal to the third

$$\tilde{k}_1\tilde{\mathbf{c}}_1 + \tilde{k}_2\tilde{\mathbf{c}}_2 = \tilde{\mathbf{c}}_3.$$

Explicitly, this equality is the three equalities:

$$\begin{aligned}
\tilde{k}_1(c_{1,1}t + \cdots) + \tilde{k}_2(c_{1,2} + \cdots) &= c_{1,3} + \cdots;\\
\tilde{k}_1(c_{1,2} + \cdots) + \tilde{k}_2(c_{2,2}t + \cdots) &= c_{2,3} + \cdots;\\
\tilde{k}_1(c_{1,3} + \cdots) + \tilde{k}_2(c_{2,3} + \cdots) &= c_{3,3}t + \cdots.
\end{aligned}$$

If $deg(\tilde{k}_2) < deg(\tilde{k}_1)$ then from the first equality we must have $deg(\tilde{k}_2) = 0$, but this would make the third equality impossible. If $deg(\tilde{k}_1) < deg(\tilde{k}_2)$ then from the second equality we must have $deg(\tilde{k}_1) = 0$, but this would also make the third equality impossible. If $deg(\tilde{k}_1) = deg(\tilde{k}_2)$ then from the first equality (or the second) we must have $deg(\tilde{k}_1) = deg(\tilde{k}_2) = 0$.

Suppose $deg(\tilde{k}_1) = deg(\tilde{k}_2) = 0$, and denote the leading terms of $\tilde{k}_1$ and $\tilde{k}_2$ by, respectively, k_1 and k_2. Then the first, second, and third equalities above, respectively, require:

$$\begin{aligned}
k_2c_{1,2} &= c_{1,3};\\
k_1c_{1,2} &= c_{2,3};\\
k_1c_{1,3} &= -k_2c_{2,3}.
\end{aligned}$$

Substituting the first of these equalities into the left side of the third, and the second into the right side of the third, we derive the equality

$$k_1k_2c_{1,2} = -k_1k_2c_{1,2}.$$

This cannot be as neither k_1, k_2, nor $c_{1,2}$ is 0. So, C_3 has no rank two symmetric lift, and its symmetric Kapranov rank is three.

2.1.3 Tropical Determinantal Varieties

An equivalent definition of the Kapranov and symmetric Kapranov rank of a matrix can be given in terms of tropical varieties. A basic result in algebraic geometry is that the $r \times r$ minors of an $m \times n$ matrix of variables are a basis for a prime ideal, and the variety of this ideal corresponds with the set of $m \times n$ matrices of rank less than r. Similarly, the $r \times r$ minors of an $n \times n$ symmetric matrix of variables are a basis for a prime ideal, and the variety of this ideal corresponds with the set of $n \times n$ symmetric matrices of rank less than r. See Harris [11], for example, as a general reference for these results.

Proposition 2.2. *The Kapranov rank of an $m \times n$ real matrix is the smallest $r \leq min(m,n)$ such that the matrix is not in the set $\mathcal{T}(\boldsymbol{V}(I_r))$, where I_r is the ideal formed by the $r \times r$ minors of an $m \times n$ matrix of variables.*

Similarly, the symmetric Kapranov rank of an $n \times n$ real symmetric matrix is the smallest $r \leq n$ such that the matrix is not in the set $\mathcal{T}(\boldsymbol{V}(J_r))$, where J_r is the ideal formed by the $r \times r$ minors of an $n \times n$ symmetric matrix of variables.

Proof. If A is an $m \times n$ real matrix and r is the smallest $r \leq min(m,n)$ such that A is not in the set $\mathcal{T}(\mathbf{V}(I_r))$, then if $r < min(m,n)$ by definition there does not exist a lift of A to an $m \times n$ matrix over $\tilde{K}$ with rank less than r, while there exists a lift of A to an $m \times n$ matrix over $\tilde{K}$ with rank less than $r+1$, and therefore r is the smallest rank of a lift of A. A generic lift of A has rank $min(m,n)$, and if $r = min(m,n)$ this is the smallest rank lift of A.

If A is an $n \times n$ symmetric real matrix the corresponding proof is identical. □

2.2 Symmetric Tropical Rank

Like the tropical rank, the symmetric tropical rank is an intrinsically tropical property of a symmetric real matrix, and does not depend upon any lift to a matrix over a valued field.

2.2.1 Tropical Determinantal Prevarieties

The definition of when a square matrix is tropically singular is the analog of the definition from classical linear algebra.

Definition 2.3 ([8] Definition 1.3)**.** A square matrix $A = (A_{i,j}) \in \mathbb{R}^{d \times d}$ is *tropically singular* if the minimum

$$tropdet(A) := \bigoplus_{\sigma \in S_d} A_{1,\sigma(1)} \odot A_{2,\sigma(2)} \odot \cdots \odot A_{d,\sigma(d)}$$

is attained at least twice. Here S_d denotes the symmetric group on $\{1, 2, \ldots, d\}$. We call the number $tropdet(A)$ defined above the *tropical determinant* of A, and any permutation σ such that

$$tropdet(A) = A_{1,\sigma(1)} \odot A_{2,\sigma(2)} \odot \cdots \odot A_{d,\sigma(d)}$$

realizes the tropical determinant. So, equivalently, a square matrix A is tropically singular if more than one permutation realizes the tropical determinant. The *tropical rank* of a matrix $A \in \mathbb{R}^{m \times n}$ is the largest integer r such that A has a tropically nonsingular $r \times r$ submatrix.

An equivalent definition of the tropical rank of a matrix can be given in terms of tropical prevarieties. In particular, the tropical prevariety defined by the minors of an $m \times n$ matrix of variables.

Proposition 2.4. *The tropical rank of an $m \times n$ real matrix is the largest $r \leq min(m, n)$ such that the matrix is not in the tropical prevariety*

$$\bigcap_i \boldsymbol{V}(\mathcal{T}(m_i)),$$

where $\{m_i\}$ is the set of $r \times r$ minors of an $m \times n$ matrix of variables.

Proof. If X' is an $r \times r$ submatrix of the $m \times n$ matrix of variables

$$X := \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix},$$

then the determinant of X' is the polynomial

$$f := \sum_{\sigma} (-1)^{sgn(\sigma)} \prod_i x_{i,\sigma(\rho(i))},$$

where i runs over the row indices of X', σ runs over all permutations of the column indices, and ρ is the order-preserving bijection from the row indices to the column indices. The tropicalization of this polynomial will be the tropical polynomial

$$F := \mathcal{T}(f) = \bigoplus_{\sigma} \bigodot_i X_{i,\sigma(i)},$$

where here addition and multiplication are their tropical counterparts. If A is an $m \times n$ real matrix then A', the submatrix of A with the same row and column indices as X', is by definition tropically singular if and only if A' is on the tropical hypersurface $\mathbf{V}(F)$. As the tropical rank of A is the largest r such that A contains a nonsingular $r \times r$ submatrix the proposition is immediate. □

Suppose A is an $m \times n$ real matrix, and $\{i_1, \imath_2, \ldots, i_r\}$ and $\{j_1, j_2, \ldots, j_r\}$ are subsets of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. These subsets define an $r \times r$ submatrix A' of A, with column indices $\{i_1, \ldots, i_r\}$ and row indices $\{j_1, \ldots, j_r\}$. A tropical monomial of the form

$$\bigodot_{k=1}^{r} X_{i_k, \rho(i_k)},$$

where ρ is a bijection from the column indices to the row indices, is a *minimizing monomial* for the submatrix A' if, over all monomials defined by bijections from $\{i_1, i_2, \ldots, i_r\}$ to $\{j_1, j_2, \ldots, j_r\}$, this monomial is minimal under the valuation $X_{i,j} \mapsto A_{i,j}$. In particular, an $r \times r$ submatrix of A is tropically nonsingular if and only if it has a unique minimizing monomial.

For example, the upper-left principal 3×3 submatrix of

$$\begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & 0 & 5 & 0 \\ 2 & 1 & 0 & 7 \\ 1 & 2 & 4 & 0 \end{pmatrix}$$

has the unique minimizing monomial $X_{1,2}X_{2,1}X_{3,3}$, while the upper-right 3×3 submatrix has two minimizing monomials, $X_{1,2}X_{2,4}X_{3,3}$ and $X_{1,4}X_{2,2}X_{3,3}$.

2.2.2 Definition

Along the lines of Proposition 2.4, we will define the symmetric tropical rank of a real symmetric matrix in terms of tropical prevarieties. However, before we do so, let us examine the symmetric matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

Viewed as a standard tropical matrix, the matrix has two minimizing monomials;

$$X_{1,2}X_{2,3}X_{3,1}X_{4,5}X_{5,4}, \text{ and } X_{1,3}X_{2,1}X_{3,2}X_{4,5}X_{5,4}.$$

The upper-left 4×4 principal submatrix has three minimizing monomials;

$$X_{1,2}X_{2,3}X_{3,1}X_{4,4},\ X_{1,3}X_{2,1}X_{3,2}X_{4,4},\ \text{and}\ X_{1,2}X_{2,1}X_{3,4}X_{4,3}.$$

The submatrix with columns $\{1,2,3,5\}$ and rows $\{1,2,3,4\}$ has two minimizing monomials;

$$X_{1,2}X_{2,3}X_{3,1}X_{4,5},\ \text{and}\ X_{1,3}X_{2,1}X_{3,2}X_{4,5}.$$

So, viewed as a standard tropical matrix, both the matrix and these two submatrices would be tropically singular. However, if we view these monomials as coming from determinants of submatrices of a *symmetric* matrix of variables, then we have the equivalence $X_{i,j} = X_{j,i}$, and the matrix has only one minimizing monomial, namely

$$X_{1,2}X_{2,3}X_{1,3}X_{4,5}^2.$$

The upper-left 4×4 principal submatrix has two, and not three, minimizing monomials;

$$X_{1,2}X_{2,3}X_{3,1}X_{4,4},\ \text{and}\ X_{1,2}^2X_{3,4}^2.$$

The submatrix with columns $\{1,2,3,5\}$ and rows $\{1,2,3,4\}$ has one minimizing monomial,

$$X_{1,2}X_{2,3}X_{1,3}X_{4,5}.$$

So, when viewed specifically as a symmetric matrix, in this example we would like for both the matrix and the given nonprincipal submatrix to be nonsingular, while the given principal submatrix would still be singular. Our definition of symmetric tropical rank is formulated with this in mind.

Definition 2.5. The *symmetric tropical rank* of a symmetric $n \times n$ real matrix is the largest $r \leq n$ such that the matrix is not in the tropical prevariety

$$\bigcap_i \mathbf{V}\left(\mathcal{T}(m_i)\right),$$

where $\{m_i\}$ is the set of $r \times r$ minors of a symmetric $n \times n$ matrix of variables. An $n \times n$ symmetric real matrix is *symmetrically tropically singular* if it is in the tropical hypersurface defined by the tropicalization of the determinant of an $n \times n$ symmetric matrix of variables. An $r \times r$ submatrix of an $n \times n$ symmetric real matrix is symmetrically tropically singular if it is on the tropical hypersurface defined by the tropicalization of the determinant of the corresponding $r \times r$ submatrix of a symmetric $n \times n$ matrix of variables. So, the symmetric tropical rank of a symmetric real matrix A is the largest r such that A contains an $r \times r$ submatrix that is not symmetrically tropically singular.

You can view this as saying that a submatrix is symmetrically tropically singular if there are two permutations that realize its tropical determinant, and these permutations are not required to be equal given the symmetry of the matrix. In terms of minimizing monomials, a submatrix of a symmetric matrix is symmetrically tropically singular if it has two minimizing monomials that are distinct given the equivalence $X_{i,j} = X_{j,i}$.

Denote the set of $m \times n$ real matrices with tropical rank less than r by $T_{m,n,r}$. Denote the set of $n \times n$ real symmetric matrices with symmetric tropical rank less than r by $S_{n,r}$.

$\tilde{S}_{n,r}$ is the tropical variety defined by the $r \times r$ minors of a symmetric $n \times n$ matrix of variables, $S_{n,r}$ is the tropical prevariety defined by these same minors, and as a tropical variety defined by a basis is always contained in the corresponding tropical prevariety defined by that basis, we must have

$$\text{symmetric tropical rank(A)} \leq \text{symmetric Kapranov rank(A)}.$$

Equivalently,

$$\tilde{S}_{n,r} \subseteq S_{n,r}.$$

This is just a specific case of the tropical variety and tropical prevariety containment relation from Chapter 1.

2.2.3 Cycle-Similar Permutations

We can construct an equivalent definition for when a symmetric matrix is symmetrically tropically singular by defining an equivalence class on the elements of S_n. We declare two permutations to be in the same class if they have the same disjoint cycle decomposition up to inversion of the cycles. In other words, if τ is a permutation with disjoint cycle decomposition:

$$\tau = \sigma_1 \sigma_2 \cdots \sigma_k,$$

where the σ_i are disjoint cycles, then the other elements in its equivalence class are of the form:

$$\sigma_1^{\pm} \sigma_2^{\pm} \cdots \sigma_k^{\pm}.$$

Note that as the parity of a permutation is determined completely by the sizes of the cycles in its disjoint cycle decomposition, and as a cycle and its inverse have the same size, every element in a given equivalence class has the same parity.

Denote by $\tilde{S}_n$ this equivalence class of permutations in S_n. If two permutations are in the same equivalence class they are *cycle-similar*, and if not they are *cycle-distinct*. Denote the equivalence class containing the permutation τ by $\tilde{\tau}$.

Proposition 2.6. *A symmetric matrix is symmetrically tropically singular if and only if it has two cycle-distinct permutations that realize the determinant.*

Proof. Consider the symmetric matrix of variables:

$$X := \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ x_{1,2} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ x_{1,3} & x_{2,3} & x_{3,3} & \cdots & x_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & x_{2,n} & x_{3,n} & \cdots & x_{n,n} \end{pmatrix}.$$

For any cycle

$$\sigma = (k_1 k_2 \cdots k_m)$$

define the monomial

$$x_\sigma = x_{k_1,k_2} x_{k_2,k_3} \cdots x_{k_m,k_1},$$

and for any permutation $\tau \in S_n$ with disjoint cycle decomposition $\tau = \sigma_1\sigma_2 \cdots \sigma_k$ define the monomial

$$x_\tau = \prod_{i=1}^{n} x_{i,\tau(i)} = \prod_{i=1}^{k} x_{\sigma_i}.$$

We have

$$x_\sigma = x_{k_1,k_2} x_{k_2,k_3} \cdots x_{k_m,k_1}, \quad \text{and} \quad x_{\sigma^{-1}} = x_{k_1,k_m} \cdots k_{k_3,k_2} x_{k_2,k_1}.$$

As $x_{i,j} = x_{j,i}$ we see $x_\sigma = x_{\sigma^{-1}}$, and therefore for any two cycle-similar permutations τ_1 and τ_2 we must have $x_{\tau_1} = x_{\tau_2}$. In other words, the permutations τ_1 and τ_2 produce the same monomial in the determinant of X. Note that as τ_1 and τ_2 have the same parity the monomials x_{τ_1} and x_{τ_2} have the same sign in the determinant, and there is no concern about identical monomials cancelling.

On the other hand, suppose for two distinct permutations τ_1 and τ_2 that, given $x_{i,j} = x_{j,i}$, we have $x_{\tau_1} = x_{\tau_2}$. The permutation τ_1 will have some disjoint cycle decomposition

$$\tau_1 = \sigma_1\sigma_2 \cdots \sigma_t.$$

Suppose

$$\sigma_1 = (k_1 k_2 \cdots k_s).$$

This means the variables

$$x_{k_1,k_2}x_{k_2,k_3}\cdots x_{k_s,k_1}$$

appear in the product of variables defining the monomial x_{τ_1}. If every one of these variables also appear in x_{τ_2}, then the cycle σ_1 also appears in the disjoint cycle decomposition of τ_2. If this is the case for every cycle in the cycle decomposition of τ_1, then $\tau_1 = \tau_2$.

So, assume without loss of generality that σ_1 is not in the disjoint cycle decomposition of τ_2, and the variable x_{k_1,k_2} does not appear in x_{τ_2}. As the only relation between the variables is $x_{i,j} = x_{j,i}$, if x_{k_1,k_2} does not appear in x_{τ_2}, then x_{k_2,k_1} must. This means x_{k_2,k_3} cannot appear in x_{τ_2}, and so x_{k_3,k_2} must. Repeating this argument we see that the product of variables

$$x_{k_2,k_1}x_{k_3,k_2}\cdots x_{k_1,k_s}$$

must appear in x_{τ_2}, which means τ_2 must contain in its disjoint cycle decomposition the cycle

$$(k_s k_{s-1}\cdots k_1) = (k_1 k_2 \cdots k_s)^{-1}.$$

So, for every cycle in the disjoint cycle decomposition of τ_1 either that cycle or its inverse appears in τ_2, and obviously vice-versa. Ergo, τ_1 and τ_2 are cycle-similar.

From this we conclude the distinct monomials occuring in the determinant of X are the cycle-distinct monomials, and therefore a symmetric matrix is symmetrically tropically singular if and only if it has two cycle-distinct permutations that realize the determinant. □

2.2.4 Tropical and Symmetric Tropical Ranks

If an $r\times r$ submatrix of a symmetric $n\times n$ matrix has two distinct minimizing monomials given the equivalence $X_{i,j} = X_{j,i}$ then a fortiori it has two distinct minimizing monomials without that equivalence, and so

$$\text{tropical rank(A)} \leq \text{symmetric tropical rank(A)}.$$

Equivalently, viewing both $S_{n,r}$ and $T_{n,n,r}$ as subsets of $\mathbb{R}^{n\times n}$, we can write the above inequality as

$$S_{n,r} \subset T_{n,n,r}.$$

Just like with Kapranov rank and symmetric Kapranov rank, the tropical rank and symmetric tropical rank of a real symmetric matrix can be different. We can see this for 3×3 symmetric matrices. The determinant of the symmetric matrix of variables

$$\begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{pmatrix}$$

is the polynomial

$$x_{1,1}x_{2,2}x_{3,3} + 2x_{1,2}x_{2,3}x_{1,3} - x_{2,3}^2x_{1,1} - x_{1,2}^2x_{3,3} - x_{1,3}^2x_{2,2}.$$

In particular note that, because the matrix is symmetric, the monomial corresponding with the permutation (123) is *the same* as the monomial corresponding with the permutation (132). The tropicalization of this polynomial is the tropical polynomial

$$X_{1,1}X_{2,2}X_{3,3} \oplus X_{1,2}X_{2,3}X_{1,3} \oplus X_{2,3}^2X_{1,1} \oplus X_{1,2}^2X_{3,3} \oplus X_{1,3}^2X_{2,2}.$$

The symmetric matrix C_3 defined in the last section has tropical rank two, but it is not on the tropical hypersurface defined by the tropical polynomial above, as $X_{1,2}X_{2,3}X_{1,3}$ is the unique minimizing monomial for the entries in C_3. So, C_3 has symmetric tropical rank three.

A more interesting example is provided by the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Using ideas and techniques from matroid theory, Develin, Santos, and Sturmfels proved ([8], Section 7) that this matrix, the cocircuit matrix of the Fano matroid, has tropical rank three but Kapranov rank four. If we permute the rows of this matrix with the permutation (275)(34), and the columns with the permutation (25364), we get the symmetric matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The upper-left 4×4 principal submatrix is tropically singular, but symmetrically tropically *nonsingular*. Consequently, the symmetric 7×7 matrix above has tropical rank three, but *not* symmetric tropical rank three. In particular, it is *not* an example of a symmetric matrix with symmetric tropical rank three and greater symmetric Kapranov rank.

2.2.5 Basic Properties

One situation where tropical rank and symmetric tropical rank are necessarily equal is when both are one.

Proposition 2.7. *A real symmetric matrix A has tropical rank one if and only if it has symmetric tropical rank one.*

Proof. Rank one is the minimum possible rank. As tropical rank cannot be greater than symmetric tropical rank, symmetric tropical rank one implies tropical rank one.

The determinant of a 2×2 submatrix of a symmetric $n \times n$ matrix of variables is the difference of two monomials, the product of the diagonal terms, and the product of the off-diagonal terms, and these monomials cannot be equal. If a matrix has tropical rank one, then for every 2×2 submatrix the sum of the diagonal terms equals the sum of the off-diagonal terms. This means every 2×2 submatrix is symmetrically tropically singular, and the matrix has symmetric tropical rank one. □

Corollary 2.8. *If a real symmetric matrix has symmetric tropical rank two then it has tropical rank two.*

Proof. The tropical rank cannot be greater the symmetric tropical rank, and by the above proposition if the tropical rank were one, the symmetric tropical rank would be one as well. So, the tropical rank must be two. □

We have seen the matrix C_3 has tropical rank two but greater symmetric tropical rank. This is a somewhat special situation.

Proposition 2.9. *A real symmetric matrix of tropical rank two has greater symmetric tropical rank if and only if a principal* 3×3 *submatrix is not symmetrically tropically singular.*

Proof. If any 3×3 submatrix of a real symmetric matrix is not symmetrically tropically singular, then the matrix has symmetric tropical rank greater than two. So, what must be proven is that for a real symmetric matrix if a 3×3 submatrix is not a principal submatrix then tropically singular implies symmetrically tropically singular.

Take any 3×3 submatrix from an $n \times n$ symmetric matrix of variables

$$\begin{pmatrix} x_{i,p} & x_{i,q} & x_{i,r} \\ x_{j,p} & x_{j,q} & x_{j,r} \\ x_{k,p} & x_{k,q} & x_{k,r} \end{pmatrix},$$

where $i < j < k$, and $p < q < r$. The determinant of this submatrix is the polynomial

$$x_{i,p}x_{j,q}x_{k,r} + x_{i,q}x_{j,r}x_{k,p} + x_{i,r}x_{j,p}x_{k,q} - x_{i,p}x_{j,r}x_{k,q} - x_{i,q}x_{j,p}x_{k,r} - x_{i,r}x_{j,q}x_{k,p}.$$

Suppose, given the symmetry of the $n \times n$ matrix of variables, that two of these monomials are equal. If $i < p$ then i is not the index of any column in our submatrix, and symmetry provides no duplication of variables from row i. This means if a monomial in the 3×3 determinant above is duplicated, the monomials in a 2×2 minor are duplicated. This is impossible. Identical logic applies if $p < i$, and therefore $i = p$. Applying the same argument we get $j = q$ and $k = r$. So, the only situation where tropically singular and symmetrically tropically singular can differ for a 3×3 submatrix is if that submatrix is principal. □

In standard linear algebra if one column (or row) of a square matrix is a multiple of another, then that matrix must be singular. The same is true for symmetric tropical matrices.

Proposition 2.10. *If A is an $r \times r$ submatrix of an $n \times n$ symmetric matrix, and one row of A is a tropical multiple of another, then A is symmetrically tropically singular. The same is true if one column of A is a tropical multiple of another.*

Proof. Suppose A is formed from the row indices $i_1, \ldots, i_r$ and the column indices $j_1, \ldots, j_r$ of the $n \times n$ symmetric matrix. Denote the rows of A by $\mathbf{a}_{i_1}, \mathbf{a}_{i_2}, \ldots, \mathbf{a}_{i_r}$. We may assume without loss of generality that $\mathbf{a}_{i_r} = c \odot \mathbf{a}_{i_{r-1}}$, where $c \in \mathbb{R}$. Suppose the monomial

$$X_1 = X_{i_1,\rho(i_1)} \odot X_{i_2,\rho(i_2)} \odot \cdots \odot X_{i_{r-1},\rho(i_{r-1})} \odot X_{i_r,\rho(i_r)},$$

where ρ is a bijection from the column indices of A to the row indices, is a minimizing monomial for A. Given the equivalence of $\mathbf{a}_{i_r}$ and $c \odot \mathbf{a}_{i_{r-1}}$ the monomial

$$X_2 = X_{i_1,\rho(i_1)} \odot X_{i_2,\rho(i_2)} \odot \cdots \odot X_{i_{r-1},\rho(i_r)} \odot X_{i_r,\rho(i_{r-1})}$$

must have the same valuation as X_1, and therefore also be a minimizing monomial. If $X_1 = X_2$ under the equivalence $X_{i,j} = X_{j,i}$ then this would require one of the four equalities below to be true:

$$X_{i_{r-1},\sigma(i_{r-1})} = X_{i_{r-1},\sigma(i_r)}; \quad X_{i_{r-1},\sigma(i_{r-1})} = X_{\sigma(i_r),i_{r-1}};$$
$$X_{i_{r-1},\sigma(i_{r-1})} = X_{i_r,\sigma(i_{r-1})}; \quad X_{i_{r-1},\sigma(i_{r-1})} = X_{\sigma(i_{r-1}),i_r}.$$

Given $i_{r-1} \neq i_r$ and ρ is a bijection, none of these equalities is possible. So, even under the equivalence $X_{i,j} = X_{j,i}$ the minimizing monomials X_1 and X_2 are distinct, and therefore A is symmetrically tropically singular.

An identical proof applies if one column is a tropical multiple of another. □

The symmetric tropical rank exhibits some interesting properties that are not encountered with the standard tropical rank. For example, suppose A is the real symmetric matrix

$$\begin{pmatrix} 3 & 0 & 4 & 1 \\ 0 & 6 & 1 & 9 \\ 4 & 1 & 7 & 2 \\ 1 & 9 & 2 & 5 \end{pmatrix}.$$

The tropical determinant of A is realized by the permutation (1234), and also by the permutations (12)(34) and (14)(23). These permutations are all cycle-distinct, and therefore A is symmetrically tropically singular. In fact, for any 4×4 real symmetric matrix if the tropical determinant is realized by the permutation (1234), then the matrix is symmetrically tropically singular, and this is an example of a general phenomenon.

Proposition 2.11. *If a permutation has a disjoint cycle decomposition containing an odd-cycle larger than a transposition, then, if this permutation realizes the tropical determinant of a real symmetric matrix, the matrix must be symmetrically tropically singular.*

Proof. Suppose A is an $n \times n$ real symmetric matrix. For a permutation $\sigma \in S_n$ we define the tropical product

$$A_\sigma = \bigodot_{i=1}^{n} A_{i,\sigma(i)}.$$

For a cycle $\sigma' = (k_1 k_2 \cdots k_m)$ we define the tropical product

$$A_{\sigma'} = A_{k_1,k_2} \odot A_{k_2,k_3} \odot \cdots \odot A_{k_m,k_1}.$$

In particular, if σ has the disjoint cycle decomposition

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_p,$$

then

$$A_\sigma = \bigodot_{i=1}^{p} A_{\sigma_i}.$$

Suppose σ has the disjoint cycle decomposition above, and $\sigma_j = (k_1 k_2 \cdots k_m)$ is an odd-cycle larger than a transposition. We define the permutations τ' and τ''

$$\tau' = \sigma_1 \sigma_2 \cdots \sigma_{j-1} (k_1 k_2)(k_3 k_4) \cdots (k_{m-1} k_m) \sigma_{j+1} \cdots \sigma_p,$$
$$\tau'' = \sigma_1 \sigma_2 \cdots \sigma_{j-1} (k_2 k_3)(k_4 k_5) \cdots (k_m k_1) \sigma_{j+1} \cdots \sigma_p.$$

As σ_j is an odd-cycle m must be even, and so τ' and τ'' are well-defined. As σ_j is larger than a transposition, σ, τ', and τ'' are cycle-distinct.

As A is symmetric we have the following sequence of equalities (standard addition)

$$
\begin{aligned}
2A_{\sigma_j} = 2(A_{k_1,k_2}+A_{k_2,k_3}+\cdots+A_{k_m,k_1}) &= A_{k_1,k_2}+A_{k_2,k_1}+A_{k_2,k_3}+A_{k_3,k_2}+\cdots+A_{k_m,k_1}+A_{k_1,k_m} \\
&= (A_{k_1,k_2} + A_{k_2,k_1} + A_{k_3,k_4} + A_{k_4,k_3} + \cdots + A_{k_{m-1},k_m} + A_{k_m,k_{m-1}}) \\
&\quad +(A_{k_2,k_3} + A_{k_3,k_2} + A_{k_4,k_5} + A_{k_5,k_4} + \cdots + A_{k_m,k_1} + A_{k_1,k_m}) \\
&= (A_{(k_1k_2)} + A_{(k_3k_4)} + \cdots A_{(k_{n-1}k_n)}) + (A_{(k_2k_3)} + A_{(k_4k_5)} + \cdots + A_{(k_mk_1)}).
\end{aligned}
$$

From these equalities we get

$$2A_\sigma = A_{\tau'} + A_{\tau''}.$$

If the permutation σ realizes the tropical determinant of A, then we must have

$$A_\sigma \leq A_{\tau'} \text{ and } A_\sigma \leq A_{\tau''}.$$

These inequalities combined with the above equality give us

$$A_\sigma = A_{\tau'} = A_{\tau''}.$$

As σ, τ', and τ'' are all cycle-distinct the matrix A is symmetrically tropically singular. □

So, if the disjoint cycle decomposition of σ contains an odd-cycle larger than a transposition it is impossible for σ to realize the determinant of a symmetric matrix that is not symmetrically tropically singular. A natural question to ask, then, is whether this is the only type of permutation for which this is the case. The answer is yes.

Proposition 2.12. *Suppose that a permutation cannot realize the tropical determinant of any real symmetric matrix that is not symmetrically tropically singular. Then this permutation has a disjoint cycle decomposition containing an odd-cycle larger than a transposition.*

Proof. We first note that for any permutation $\sigma \in S_n$ we can find an $n \times n$ symmetric matrix for which σ realizes the tropical determinant. Using σ we define the matrix A such that $A_{i,\sigma(i)} = A_{\sigma(i),i} = 0$ for $1 \leq i \leq n$, and all other terms in A are 1. Obviously the sum

$$A_\sigma = \bigodot_{i=1}^{n} A_{i,\sigma(i)} = 0$$

is minimal over all permutations in S_n, and so σ realizes the tropical determinant. If σ realizing the tropical determinant requires that A is symmetrically tropically singular, then there is a permutation $\tau \in S_n$ also realizing the tropical determinant, where σ and τ are cycle-distinct.

Suppose σ has the disjoint cycle decomposition

$$\sigma = \sigma_1\sigma_2\cdots\sigma_p.$$

As σ and τ are cycle-distinct there must exist a $\sigma_i = (k_1 k_2 \cdots k_m)$ such that neither σ_i nor σ_i^{-1} is in a cycle-decomposition of τ.

The only 0 terms on row k_1 of A are A_{k_1,k_2}, and A_{k_1,k_m}. So, we must have either $\tau(k_1) = k_2$ or $\tau(k_1) = k_m$.

If $\tau(k_1) = k_2$, then as the only 0 terms on row k_2 of A are A_{k_2,k_3} and A_{k_2,k_1}, either $\tau(k_2) = k_3$ or $\tau(k_2) = k_1$.

Suppose $\tau(k_1) = k_2$ and $\tau(k_2) = k_3$. The only 0 terms on row k_3 of A are A_{k_3,k_4} and A_{k_3,k_2}. As $\tau(k_1) = k_2$ we cannot have $\tau(k_3) = k_2$, and so we must have $\tau(k_3) = k_4$. Repeating this argument we get $\tau(k_{j-1}) = \tau(k_j)$ for all $1 < j \leq m$, and $\tau(k_m) = k_1$. So, τ has a cycle decomposition containing σ_i, which cannot be.

So, if $\tau(k_1) = k_2$, then $\tau(k_2) = k_1$. Using the same reasoning used in the paragraph above we get $\tau(k_3) = k_4$, and either $\tau(k_4) = k_3$ or $\tau(k_4) = k_5$. Again, applying the same reasoning as the paragraph above if $\tau(k_4) = k_5$ we must have $\tau(k_{j-1}) = \tau(k_j)$ for all $3 < j \leq m$, and $\tau(k_m) = k_1$. As $\tau(k_2) = k_1$ this cannot be, and so $\tau(k_4) = k_3$. Repeating this argument we get that τ has a cycle decomposition containing the cycles $(k_1 k_2)(k_3 k_4) \cdots (k_{m-1} k_m)$. This is only possible if m is even, and as τ does not contain σ_i in its cycle decomposition we must have $m > 2$. So, σ_i is an odd-cycle larger than a transposition.

If $\tau(k_1) = k_m$ then an essentially identical argument gives us either τ has a cycle decomposition that contains σ_i^{-1}, which cannot be, or τ has a cycle decomposition containing $(k_2 k_3)(k_4 k_5) \cdots (k_m k_1)$ with $m > 2$, implying σ_i is an odd-cycle larger than a transposition. □

2.3 Symmetric Barvinok Rank

The symmetric Barvinok rank of a symmetric matrix, along with two additional notions of rank for symmetric matrices (tree rank and star tree rank) have been examined in depth by Cartwright and Chan [4], from whom we lift the definition of symmetric Barvinok rank. We will not explore the symmetric Barvinok rank in much detail, except to prove

$$\text{symmetric Kapranov rank(A)} \leq \text{symmetric Barvinok rank(A)}$$

and that this inequality may be strict.

2.3.1 Definition

Definition 2.13. The symmetric Barvinok rank of a tropical symmetric matrix A is the smallest integer r for which A can be written as the sum of r rank one symmetric matrices.

We note that any rank one symmetric matrix can be written as the outer product $X \odot X^T$, where X is an $n \times 1$ column vector.

2.3.2 Example of Inequality

The proof of the above inequality is a straightforward modification of the proof for the corresponding inequality in Develin, Santos, and Sturmfels ([8], Proposition 3.6).

Proposition 2.14. *Every symmetric tropical matrix A satisfies*

$$\text{symmetric Kapranov rank}(A) \leq \text{symmetric Barvinok rank}(A).$$

Proof. If a matrix A has symmetric Barvinok rank one, then $A = X \odot X^T$. If we pick a vector $\tilde{X}$ that tropicalizes to X, then the matrix $\tilde{X} \odot \tilde{X}^T$ will be a rank 1 symmetric matrix that tropicalizes to A. So, A has symmetric Kapranov rank 1 as well.

Suppose the matrix A has symmetric Barvinok rank r. Write

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_r.$$

Each A_i has symmetric Kapranov rank one, so for each i there exists a rank one matrix $\tilde{A}_i$ that tropicalizes to A_i. By multiplying the matrices $\tilde{A}_i$ by random complex numbers, we can choose $\tilde{A}_i$ such that there is no cancallation of leading terms in t when we form the matrix

$$\tilde{A} = \tilde{A}_1 + \cdots + \tilde{A}_r.$$

This matrix $\tilde{A}$ has rank at most r, and tropicalizes to A. □

To prove this inequality can be strict we examine the *classical identity matrix* introduced in [8]:

$$C_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Note this is certainly *not* the identity matrix in tropical linear algebra, but it is symmetric. Develin, Santos, and Sturmfels ([8], Examples 3.5 and 4.4) demonstrate the tropical and Kapranov ranks of C_n are both 2 for any $n \geq 2$. We now prove the symmetric tropical and symmetric Kapranov ranks of C_n are both 3 for any $n \geq 3$.

Consider the matrix

$$\begin{pmatrix} t & 1 & 1 & \frac{16}{5}+t & \cdots & \frac{n^2}{n+1}+t \\ 1 & t & 1 & \frac{1}{5}+4t & \cdots & \frac{1}{n+1}+nt \\ 1 & 1 & t & 5-\frac{4}{5}t & \cdots & (n+1)-\frac{n}{n+1}t \\ \frac{16}{5}+t & \frac{1}{5}+4t & 5-\frac{4}{5}t & \frac{409}{25}t & \cdots & \frac{(4-n)^2}{5(n+1)}+\frac{5+21n+20n^2}{5(n+1)}t \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{n^2}{n+1}+t & \frac{1}{n+1}+nt & (n+1)-\frac{n}{n+1}t & \frac{(n-4)^2}{5(n+1)}+\frac{5+21n+20n^2}{5(n+1)}t & \cdots & \frac{1+2n+n^2+2n^3+n^4}{(n+1)^2}t \end{pmatrix}.$$

This matrix, which we will denote by $\tilde{C}_n$, is defined as follows. The upper-left 3×3 submatrix is given. For the rest of the entries in the first three columns we define ($i > 3, j \leq 3$)

$$c_{i,j} = c_{1,j} + ic_{2,j} - \frac{i}{1+i}c_{3,j},$$

which gives us

$$c_{i,1} = \frac{i^2}{1+i} + t,\ c_{i,2} = \frac{1}{1+i} + it,\ c_{i,3} = (1+i) - \frac{i}{1+i}t.$$

In particular, as $i > 3$, the constant term for all these elements is never 0.

The remaining columns of $\tilde{C}_n$ are defined in terms of the first three columns, in a matter exactly analogous to how we completed the first three columns above. For $j > 3$

$$\begin{aligned} c_{i,j} &= c_{i,1} + jc_{i,2} - \frac{j}{1+j}c_{i,3} \\ &= \frac{(i-j)^2}{(1+i)(1+j)} + \frac{1+i+j+ij+i^2j+ij^2+i^2j^2}{(1+i)(1+j)}t. \end{aligned}$$

The matrix $\tilde{C}_n$ has rank three by construction, and from these formulas it is obviously symmetric. Also, the constant term is 0 if and only if $i = j$, and the linear term is never 0. So, $\tilde{C}_n$ is a lift of C_n, and the symmetric Kapranov rank of C_n is at most three.

The matrix C_n contains the matrix C_3 as its upper-left 3×3 submatrix, and C_3 has symmetric tropical rank three. So, the symmetric tropical rank of C_n is at least three. As the symmetric tropical rank cannot be greater than the symmetric Kapranov rank, both must be three.

The symmetric Barvinok rank of C_n can be calculated using a proposition from [4], which we cite.

Proposition 2.15 ([4], Proposition 3)**.** *If M is a symmetric matrix and $2m_{i,j} < m_{i,i} + m_{j,j}$ for some i and j, then the symmetric Barvinok rank is infinite.*

So, for $n \geq 2$ the symmetric Barvinok rank of C_n is infinite, which certainly demonstrates the symmetric Barvinok rank can be greater than the symmetric Kapranov rank.

CHAPTER 3

WHEN THE MINORS OF A SYMMETRIC MATRIX FORM A TROPICAL BASIS

With the exception of $r = 4$, which is a special boundary case requiring a more in depth analysis, in this chapter we examine all the cases where the $r \times r$ minors of an $n \times n$ symmetric matrix of variables *do* form a tropical basis. These are the cases $r = 2$, $r = 3$, and $r = n$. The case $r = 4$ is examined in Chapter 4.

Before we prove this, we will want a couple of useful facts:

- If A is a symmetric matrix, and we permute the rows of A by a permutation σ, and the columns of A by the *same* permutation, then the resulting matrix A' will be symmetric, and A' will have the same symmetric tropical and symmetric Kapranov rank as A. We call a permutation of the rows and columns of A by the same permutation a *diagonal permutation*.
- If A is a symmetric matrix, and we tropically multiply row i by a constant c, and tropically multiply column i by the *same* constant, then the resulting matrix A' will be symmetric, and A' will have the same symmetric tropical and symmetric Kapranov rank as A. We call such an operation a *symmetric* scaling of A.

In particular, we will assume without loss of generality that any symmetric matrix A has been symmetrically scaled so that every row/column has 0 as its minimal entry.

3.1 Singular Symmetric Matrices

By definition, a symmetric matrix is singular if it has less than full rank, and it is a fundamental result in linear algebra that this is the case if and only if the matrix has zero determinant.

Theorem 3.1. *The determinant of a symmetric matrix of variables is a tropical basis for the ideal it generates. Equivalently, the $n \times n$ minor of an $n \times n$ symmetric matrix of variables forms a tropical basis.*

Proof. The determinant of a symmetric matrix of variables is a single polynomial, and is therefore a tropical basis by Kapranov's theorem. □

So, an $n \times n$ symmetric matrix has symmetric tropical rank n if and only if it has symmetric Kapranov rank n, which is equivalently stated as $\tilde{S}_{n,n} = S_{n,n}$. If a symmetric matrix is symmetrically tropically singular, it has less than full tropical and Kapranov ranks, and for this symmetric matrix there exists a lift to a symmetric singular matrix over $\tilde{K}$.

3.2 Rank One Symmetric Matrices

The rank one case is straightforward.

Theorem 3.2. *A symmetric matrix has symmetric tropical rank one if and only if it has symmetric Kapranov rank one. Equivalently, the* 2×2 *minors of a symmetric matrix of variables are a tropical basis.*

Proof. As the symmetric tropical rank cannot be greater than the symmetric Kapranov rank, any symmetric matrix with symmetric Kapranov rank one must also have symmetric tropical rank one.

If a symmetric matrix has symmetric tropical rank one, then by Proposition 2.6 it also has standard tropical rank one. This means every column of the matrix is a constant tropical multiple of the first column. If our matrix is of the form:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix},$$

and $\mathbf{a}_i$ represents column i of the matrix A, then $\mathbf{a}_i = c_i \odot \mathbf{a}_1$ for some constant c_i. By assumption A is symmetric, so $a_{i,j} = a_{j,i}$. The matrix A is the tropicalization of the matrix

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{1,1} & \tilde{a}_{1,2} & \cdots & \tilde{a}_{1,n} \\ \tilde{a}_{2,1} & \tilde{a}_{2,2} & \cdots & \tilde{a}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{a}_{n,1} & \tilde{a}_{n,2} & \cdots & \tilde{a}_{n,n} \end{pmatrix},$$

where $\tilde{a}_{i,1} = t^{m_{i,1}}$, and $\tilde{a}_{i,j} = t^{c_j}\tilde{m}_{i,1}$. The matrix $\tilde{A}$ has rank one by construction, and as $a_{i,j} = a_{j,i}$ we have

$$\begin{aligned} \tilde{a}_{i,j} &= t^{c_j}\tilde{a}_{i,1} = t^{c_j + a_{i,1}} = t^{a_{i,j}} = t^{a_{j,i}} \\ &= t^{c_i + a_{j,1}} = t^{c_i}t^{a_{j,1}} = t^{c_i}\tilde{a_{j,1}} = \tilde{a_{j,i}}. \end{aligned}$$

So, $\tilde{A}$ is symmetric, and therefore A has Kapranov rank one. □

Corollary 3.3. *A 3×3 symmetric matrix A has symmetric Kapranov rank two if and only if it has symmetric tropical rank two.*

Proof. If A has symmetric Kapranov rank two, then its symmetric tropical rank cannot be more than two, and by Theorem 3.2 its symmetric tropical rank cannot be one.

If A has symmetric tropical rank two its symmetric Kapranov rank must be at least two, and by Theorem 3.1 its symmetric Kapranov rank cannot be three. □

3.3 Rank Two Symmetric Matrices

In this section we prove that the 3×3 minors of a symmetric $n \times n$ matrix form a tropical basis. We will a few times make the inductive assumption that, for a given natural number n, it is the case that the 3×3 minors of an $m \times m$ symmetric matrix form a tropical basis for $m < n$. The $n = 3$ case from Corollary 3.3 serves as the base. The proof will be built on the foundation of several lemmas. In several places the proof given here uses ideas and modifications of arguments from the corresponding proof in Section 6 of [8].

3.3.1 Matrix Structure

Lemma 3.4. *Let A be a symmetric matrix of symmetric tropical rank two. After possibly a diagonal permutation A has the block structure:*

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C^T & \mathbf{0} \end{pmatrix},$$

where the matrices B_1 and B_2 are symmetric and positive, and the matrix C is non-negative and has no zero columns. Each $\mathbf{0}$ represents a zero matrix of the appropriate size. It is possible that A has no rows/columns with all 0 entries, and so the first row/column blocks of A may be empty. It is also possible that the matrices B_1, B_2 and C may have size zero. The only exceptions being A cannot be a matrix consisting of just one of the positive blocks (B_1 or B_2), nor can A be the zero matrix.

Proof. Our proof follows the proof of Lemma 6.2 in [8], modified appropriately for symmetric matrices. In [8] they prove that if M is a real matrix normalized so that every column has 0 as its minimal entry, then if M has standard tropical rank two it has, after possibly permuting the rows and columns, the block structure:

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & M_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & M_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & M_k \end{pmatrix}.$$

The matices M_i have all positive entries, each $\mathbf{0}$ represents a matrix of zeros of the appropriate size, and the first row and column blocks of M may have size zero. The block structure in the symmetric case is different because we have a modified definition of what it means for a submatrix to be singular, and we are only allowed to make diagonal permutations, not arbitrary permutaitons, of rows and columns.

As defined in [9] the tropical convex hull of a set of real vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is the set of all tropical linear combinations

$$c_1 \odot \mathbf{v}_1 \oplus c_2 \odot \mathbf{v}_2 \oplus \cdots \oplus c_m \odot \mathbf{v}_m \quad \text{where } c_1, \ldots, c_m \in \mathbb{R}.$$

Theorem 4.2 from [8] states that the standard tropical rank of a real matrix is equal to one plus the dimension of the tropical convex hull of its columns. As the standard tropical rank of a matrix is equal to the standard tropical rank of its transpose, the standard tropical rank of a real matrix is also equal to one plus the dimension of the tropical convex hull of its rows.

We construct a matrix A' from A by adjoining the zero vector as the first column:

$$A' := \begin{pmatrix} \mathbf{0} & A \end{pmatrix}.$$

From A' we construct the matrix A^+ by adjoining the zero row as the first row:

$$A^+ := \begin{pmatrix} \mathbf{0} \\ A' \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}.$$

As the matrix A has symmetric tropical rank two, by Corollary 2.7 it must also have standard tropical rank two. Every row of A contains 0 as its minimal entry, and so the tropical convex hull of the columns of A' is equal to the tropical convex hull of the columns of A. Therefore, the standard tropical rank of A' is two. As every column of A' contains zero as its minimal entry the tropical convex hull of the rows of A^+ is equal to the tropical convex hull of the rows of A'. Therefore, the standard tropical rank of A^+ is two.

We derive the asserted block decomposition of A from the claim that any two columns of A^+ have either equal or disjoint cosupports, where the cosupport of a column is the set of positions where it does not have a zero. To prove this, observe that if this were not so A^+ would have the following submatrix, where $+$ denotes a positive entry, ? denotes a nonnegative entry, and the first column of the submatrix is taken from the first column of A^+. (Recall that each column of A contains a zero entry.)

$$\begin{pmatrix} 0 & + & + \\ 0 & 0 & + \\ 0 & ? & 0 \end{pmatrix}$$

This 3×3 matrix is standard tropically nonsingular, which cannot be given A^+ has standard tropical rank two.

If the diagonal entry $a_{i,i}$ of A^+ is positive, then, as A^+ is symmetric, for any entry $a_{j,i}$ with $j \neq i$ if $a_{j,i}$ is positive $a_{i,j}$ is as well, and this means columns i and j have equal cosupports. In particular, $a_{j,j}$ is positive. From this we see that the positive entries of column i, and the positive entries from columns with cosupports equal to column i, form a positive principal submatrix of A^+. After possibly a diagonal permutation, this submatrix is the submatrix B_1 of A^+. If A^+ contains additional positive diagonal entries outside of B_1 then, using identical reasoning, possibly after a diagonal permutation we have the submatrix B_2. There cannot be three positive diagonal blocks, for then we would be able to construct the 3×3 principal submatrix of A:

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

where $a, b, c > 0$. This matrix is not symmetrically tropically singular, and this would contradict that A has symmetric tropical rank two. Note the difference here between the standard and the symmetric case. This 3×3 principal minor is not symmetrically tropically singular, but it is standard tropically singular. This is why in the standard rank two case the number of positive blocks can be arbitrarily large, while in the symmetric rank two case the number is limited to two.

After possibly another diagonal permutation we can arrange the columns and rows of A^+ so that, from left to right, the first columns are the zero columns, followed by the columns that contain B_1, followed by the columns that contain B_2. The remaining columns must all have a 0 entry on the diagonal, and a positive entry $a_{i,j}$ for some $i \neq j$. Row i obviously cannot be a zero row, nor can it intersect B_1 or B_2, and so must be below the submatrix B_2. Denote as A'' the submatrix formed by all columns to the right of B_2, and all rows below B_2.

The submatrix A'' is symmetric, does not contain a zero row/column, and has 0 along its diagonal. In particular its upper-left 1×1 principal submatrix is a zero matrix. Suppose the upper-left $k \times k$ principal submatrix of A'' is a zero matrix. If for some column $\mathbf{a}'_i$

all the terms in $\mathbf{a}'_i$ to the right of this $k \times k$ principal submatrix are 0, then the diagonal permutation that switches indices i and $k+1$ will construct an upper-left $(k+1) \times (k+1)$ principal submatrix that is a zero matrix. We continue this process until no such column $\mathbf{a}'_i$ exists, in which case, given our result about either equal or disjoint cosupports, A'', and therefore A, has our desired block decomposition.

We note finally that A cannot be just a positive block, because that would violate the assumption that the minimum value in every row/column is 0. A also cannot be the zero matrix, for then it would have symmetric tropical rank one. □

Lemma 3.5. *If A is a symmetric matrix normalized so the rows/columns have 0 as their minimal entry, and A^+ is the augmented matrix*

$$A^+ = \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & A \end{pmatrix},$$

then:

1. *If A has symmetric tropical rank two, so does A^+.*
2. *If A has symmetric Kapranov rank two, so does A^+.*

Proof Of Part (1). Suppose A has symmetric tropical rank two. We may assume that, possibly after a diagonal permutation, the matrix A has the block decomposition given in Lemma 3.4. In the proof of Lemma 3.4 we demonstrated that if A has symmetric tropical rank two, then A^+ has standard tropical rank two. By Proposition 2.8 the only way a symmetric matrix can have standard tropical rank two but not symmmetric tropical rank two is if a principal 3×3 submatrix is standard tropically singular but not symmetrically tropically singular. By assumption, A has symmetric tropical rank two, so the only way A^+ could not is if a principal 3×3 submatrix of A^+ involving the initial zero row/column were tropically singular but not symmetrically tropically singular. The possible 3×3 principal submatrices of this type have the forms (where an element not specified as being 0 is positive):

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{i,i} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{i,i} & 0 \\ 0 & 0 & a_{j,j} \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{i,j} \\ 0 & a_{j,i} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & a_{i,i} & a_{i,j} \\ 0 & a_{j,i} & a_{j,j} \end{pmatrix}.$$

Of these possibilities the only one that is not obviously symmetrically tropically singular is the last one. For this possibility, if $a_{i,i} < a_{i,j}$ or $a_{j,j} < a_{i,j}$ then the submatrix is not standard tropically singular, which cannot be. So, assume $a_{i,j} = a_{j,i} < a_{i,i}, a_{j,j}$. If A contains a zero row/column or the submatrix C (from Lemma 3.4) has positive size then, possibly after a diagonal permutation, A must contain the 3×3 matrix under examination as a principal submatrix, and therefore the submatrix must be symmetrically tropically singular. If A consists of two positive blocks and nothing else then A has the following 3×3 submatrix:

$$\begin{pmatrix} a_{i,i} & a_{i,j} & 0 \\ a_{j,i} & a_{j,j} & 0 \\ 0 & 0 & a_{k,k} \end{pmatrix},$$

where $a_{k,k} > 0$. If $a_{i,j} = a_{j,i} < a_{i,i}, a_{j,j}$ then this matrix is not symmetrically tropically singular, which violates our assumption about A. So, A^+ has symmetric tropical rank two. □

Part (2). If A has symmetric Kapranov rank two then there exists a rank two symmetric lift which we will call $\tilde{A}$. From Lemma 3.4 we know A must have two nonzero columns with disjoint cosupports. Denote as $\tilde{\mathbf{a}}_i$ and $\tilde{\mathbf{a}}_j$ the corresponding columns in $\tilde{A}$. If $\lambda, \mu \in \tilde{K}$ have degree zero but are otherwise generic, then the vector

$$\tilde{\mathbf{v}} = \lambda \tilde{\mathbf{a}}_i + \mu \tilde{\mathbf{a}}_j$$

has all degree zero terms. This is because as $\mathbf{a}_i$ and $\mathbf{a}_j$ have disjoint cosupports, the sum

$$v_k = \lambda a_{k,i} + \mu a_{k,j}$$

involves at least one term, $a_{k,i}$ or $a_{k,j}$, of degree zero. If both have degree zero, then λ and μ being generic guarantees we do not have cancellation of leading terms. So, v_k has degree zero.

The matrix formed by adjoining $\tilde{\mathbf{v}}$ to $\tilde{A}$,

$$\tilde{A}' := \begin{pmatrix} \tilde{\mathbf{v}} & \tilde{A} \end{pmatrix},$$

must have rank two. If we augment $\tilde{A}'$ by adding a row formed by the linear combination of rows i and j of $\tilde{A}'$ multiplied by λ and μ, respectively, then as $\tilde{A}$ is symmetric we get the symmetric matrix

$$\tilde{A}^+ := \begin{pmatrix} \tilde{a}_{0,0} & \tilde{\mathbf{v}}^T \\ \tilde{\mathbf{v}} & A \end{pmatrix}.$$

This matrix has rank two. The entry $\tilde{a}_{0,0}$ is:

$$\tilde{a}_{0,0} = \lambda v_i + \lambda v_k = \lambda(\lambda a_{i,i} + \mu a_{i,j}) + \mu(\lambda a_{j,i} + \mu a_{j,j}) = \lambda^2 a_{i,i} + 2\lambda\mu a_{i,j} + \mu^2 a_{j,j}.$$

For the final equality we use $a_{i,j} = a_{j,i}$. At least one of $a_{i,i}, a_{i,j}, a_{j,j}$ has degree zero. As λ, μ are generic we cannot have cancellation of leading terms, and therefore $\tilde{a}_{0,0}$ has degree zero.

So, the above matrix is a rank two symmetric lift of A^+, and therefore A^+ has symmetric Kapranov rank two. □

3.3.2 Kapranov and Tropical Rank

We now state the major theorem of this chapter, which has two implications. The proof of the simpler implication is given first. The proof of the more difficult implication is the subject of the rest of this chapter. An outline of the proof of this more difficult implication is provided below, followed by the complete proof.

Theorem 3.6. *A symmetric matrix A has symmetric tropical rank two if and only if it has symmetric Kapranov rank two.*

Symmetric Kapranov rank two implies symmetric tropical rank two. If A has symmetric Kapranov rank two then by Theorem 3.2 it cannot have symmetric tropical rank one. The symmetric tropical rank cannot be greater than the symmetric Kapranov rank, and so A must have tropical rank two. □

Outline that symmetric tropical rank two implies symmetric Kapranov rank two. Our method of proof for this implication is to first prove some special cases, and then use these special cases to construct our general proof. We may assume that if A has symmetric tropical rank two then it has the block decomposition given by Lemma 3.4. We prove the following special cases:

Case 1 -

Suppose A has the form

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & 0 & \mathbf{0} \\ C^T & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where C is nonnegative and has no zero column. If A has symmetric tropical rank two it has symmetric Kapranov rank two.

Case 2 -

Suppose A has the form

$$\begin{pmatrix} B_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_2 \end{pmatrix},$$

where B_1, B_2 are positive symmetric matrices of positive size. If A has symmetric tropical rank two it has symmetric Kapranov rank two.

Combining these two results, we prove:

Case 3 -

Suppose A has the form

$$\begin{pmatrix} B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C^T & \mathbf{0} \end{pmatrix},$$

where B_1, B_2 are symmetric and positive, C is nonnegative and does not contain a zero column, and either C or both B_1 and B_2 have positive size. If A has symmetric tropical rank two it has symmetric Kapranov rank two.

With this third case proven we will complete the proof of the theorem with a simple induction argument.

The rest of this chapter is devoted to completing the proof sketched by this outline.

□

3.3.3 Supporting Lemmas

We now prove the cases above as a series of lemmas.

Lemma 3.7. *Suppose A is a matrix of the form*

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & 0 & \mathbf{0} \\ C^T & \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where C is nonnegative and has no zero column. If A has symmetric tropical rank two, it has symmetric Kapranov rank two.

Proof. We number the rows and columns of A from $-k$ to l, where $k \times k$ and $l \times l$ are the dimensions of the upper-left and bottom-right zero matrices, respectively. So, the upper-left zero matrix is the submatrix of nonpositive indices, and the bottom-right zero matrix is the submatrix of nonnegative indices. The row and column indexed zero consists of all zeroes. Further, in A the rows and columns in the upper-left zero matrix will be referred to "in reverse." So, the first and second columns of the upper-left zero matrix are indexed 0 and -1 in A. Elements from C or C^T will be represented with an indexed lower-case "c," while other elements will be represented with an indexed lower-case "a."

As C does not contain a zero column we may, possibly after a diagonal permutation, assume the entries $C_{-1,1} = C_{1,-1}$ are positive.

We now construct a symmetric rank two lifting $\tilde{A}$ of A. The upper-right submatrix

$$A_{UR} = \begin{pmatrix} \mathbf{0} & C \\ 0 & \mathbf{0} \end{pmatrix}$$

has (standard) tropical rank two, and so by Theorem 6.5 from [8] there exists a rank two lift $\tilde{A}_{UR}$ of this submatrix.[1] As C does not contain the zero column, the first two columns of $\tilde{A}_{UR}$ must be linearly independent, and every other column of $\tilde{A}_{UR}$ can be written as a linear combination of these first two columns:

$$\lambda_j \mathbf{a}_0 + \mu_j \mathbf{a}_1 = \mathbf{a}_j.$$

The relation

$$\lambda_j a_{0,0} + \mu_j a_{0,1} = a_{0,j}$$

implies the degrees of λ_j and μ_j cannot both be positive, if one has positive degree the other must have degree zero, and if their degrees are both nonpositive they must be equal. If both λ_j and μ_j had negative degrees, then given C does not contain the zero column C would have a negative entry, but this is not allowed as C is nonnegative. If μ_j had positive degree then λ_j would have degree zero, but this cannot be as then C would contain the zero column. So, we must have $deg(\lambda_j) \geq deg(\mu_j) = 0$.

We use this lift $\tilde{A}_{UR}$, and its transpose, for the upper-right and bottom left submatrices of $\tilde{A}$. We must complete the lift with entries $a_{i,j}$ for every i, j with $ij > 0$, such that $deg(a_{i,j}) = 0$, $a_{i,j} = a_{j,i}$, and the entire matrix $\tilde{A}$ has rank two. We begin this task with the 3×3 central minor:

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & c_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ c_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}.$$

We pick $a_{1,1}$ such that $deg(a_{1,1}) = 0$, but otherwise generically. We want this matrix to be singular, and so once $a_{1,1}$ has been picked $a_{-1,-1}$ is determined.

As $a_{1,1}$ is generic, $a_{-1,-1}$ is as well. If $deg(a_{-1,-1}) < 0$, then in order for the above 3×3 matrix to be singular the leading terms in $a_{0,0}a_{1,1} - a_{0,1}a_{1,0}$ would need to cancel, which is impossible if $a_{1,1}$ is generic. If $deg(a_{-1,-1}) > 0$, then as $deg(c_{-1,1}) = deg(c_{1,-1}) > 0$ there would only be a single degree zero term, $a_{-1,0}a_{0,-1}a_{1,1}$, in the determinant of the 3×3 matrix, which would make it impossible for it to be singular. So, $deg(a_{-1,-1}) = 0$.

---

[1] Theorem 6.5 from [8] relies upon Corollary 6.4 from the same paper, and Corollary 6.4 contains an error in its proof. A correction for this error is given in the first appendix of this dissertation.

From here every term $a_{i,1}$ and $a_{i,-1}$, with $i > 1$ or $i < 1$, respectively, is determined by the need for the matrices

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & c_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ c_{i,-1} & a_{i,0} & a_{i,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{i,-1} & a_{i,0} & c_{i,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ c_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

to be, respectively, singular, and that $a_{1,1}$ and $a_{-1,-1}$ are generic ensures all these terms are generic and have degree zero. The remaining entries $i, j > 1$ in the bottom-right zero matrix are determined by the relations:

$$\lambda_j a_{i,0} + \mu_j a_{i,1} = a_{i,j}.$$

As $a_{i,1}$ is generic, $deg(a_{i,j}) = 0$ even if $deg(\lambda_j) = deg(\mu_j)$. The degree zero upper-left entries are determined similarly.

It remains to be proven that our lift is symmetric. We first prove $a_{1,i} = a_{i,1}$, with $i > 1$. We examine the matrices

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & c_{-1,i} \\ a_{0,-1} & a_{0,0} & a_{0,i} \\ c_{1,-1} & a_{1,0} & a_{1,i} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{-1,-1} & a_{-1,0} & c_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ c_{i,-1} & a_{i,0} & a_{i,1} \end{pmatrix}.$$

By construction

$$a_{-1,0} = a_{0,-1}, \quad a_{1,0} = a_{0,1},$$
$$c_{-1,1} = c_{1,-1}, \quad \text{and} \quad c_{-1,i} = c_{i,-1}.$$

So, the formula for the determinant of the first matrix is the same as the formula for the determinant of the second with $a_{1,i}$ replaced by $a_{i,1}$. As both matrices are singular we must have $a_{1,i} = a_{i,1}$.

For the remaining terms verifying symmetry is a straightforward calculation (here $i, j > 1$):

$$\begin{aligned} a_{j,i} &= \lambda_i a_{j,0} + \mu_i a_{j,1} = \lambda_i a_{0,j} + \mu_i a_{1,j} \\ &= \lambda_i(\lambda_j a_{0,0} + \mu_j a_{0,1}) + \mu_i(\lambda_j a_{1,0} + \mu_j a_{1,1}) \\ &= \lambda_j(\lambda_i a_{0,0} + \mu_i a_{1,0}) + \mu_j(\lambda_i a_{0,1} + \mu_i a_{1,1}) \\ &= \lambda_j(\lambda_i a_{0,0} + \mu_i a_{0,1}) + \mu_j(\lambda_i a_{1,0} + \mu_i a_{1,1}) \\ &= \lambda_j a_{0,i} + \mu_j a_{1,i} = \lambda_j a_{i,0} + \mu_j a_{i,1} = a_{i,j}. \end{aligned}$$

The verification of symmetry for $i, j < -1$ is essentially identical. So, we have constructed a symmetric rank two lift $\tilde{A}$ of A, and therefore A has Kapranov rank two. □

Lemma 3.8. *Suppose A is a matrix of the form:*

$$\begin{pmatrix} B_1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & B_2 \end{pmatrix}$$

where B_1 and B_2 are positive symmetric matrices of positive size. If A has symmetric tropical rank two, then it has symmetric Kapranov rank two.

Proof. As in the previous lemma we number the rows and columns from $-k$ to l, where $k \times k$ and $l \times l$ are the dimensions of B_1 and B_2, respectively. Also, as in the previous lemma, we refer to the rows and columns of A "in reverse." Terms from B_1 or B_2 will be represented with an indexed lower-case "b," while all other terms will be represented with an indexed lower-case "a."

By induction we may assume the matrices

$$\begin{pmatrix} B_1 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B_2 \end{pmatrix}$$

have symmetric rank two lifts $\tilde{B}_1$ and $\tilde{B}_2$, respectively, and after possibly scaling we may assume the bottom-right entry of $\tilde{B}_1$ is equal to the top-left entry of $\tilde{B}_2$.

We now construct a symmetric rank two lift $\tilde{A}$ of A. We begin with the lifts $\tilde{B}_1$ and $\tilde{B}_2$, and construct the entries in the upper-right zero matrix.

Like in Lemma 3.7 we start with the 3×3 central minor:

$$\begin{pmatrix} b_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & b_{1,1} \end{pmatrix}.$$

We need this matrix to be singular and symmetric. That we can find degree zero entries $a_{-1,1} = a_{1,-1}$ that make this true follows from applying Kapranov's theorem to the determinant of the matrix

$$\begin{pmatrix} b_{-1,-1} & a_{-1,0} & x \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ x & a_{1,0} & b_{1,1} \end{pmatrix}.$$

Note, we cannot assume $a_{-1,1} = a_{1,-1}$ is generic, but that will not be necessary. Also, note that as the 3×3 central minor is singular, there cannot be cancellation of leading terms for either of its minors:

$$\begin{vmatrix} a_{-1,0} & a_{-1,1} \\ a_{0,0} & a_{0,1} \end{vmatrix}, \quad \text{or} \quad \begin{vmatrix} a_{0,-1} & a_{0,0} \\ a_{1,-1} & a_{1,0} \end{vmatrix}.$$

If in either of these minors we had cancellation of leading terms there would be no way the leading terms could all cancel for the determinant of the entire 3×3 matrix.

Every term $a_{i,1}$ with $i < -1$ and $a_{i,-1}$ with $i > 1$ is determined by the need for the matrices

$$\begin{pmatrix} b_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{i,-1} & a_{i,0} & b_{i,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{i,-1} & a_{i,0} & a_{i,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & b_{1,1} \end{pmatrix}$$

to be, respectively, singular. That every such term has degree zero follows from the leading terms of the 2×2 minors discussed above not canceling.

Every column in $\tilde{B}_2$ can be written as a linear combination of the first two:

$$\lambda_j \mathbf{b}_0 + \mu_j \mathbf{b}_1 = \mathbf{b}_j.$$

We use these relations to define the entries $a_{i,j}$ with $i < 0$ and $j > 0$:

$$\lambda_j a_{i,0} + \mu_j a_{i,1} = a_{i,j}.$$

We similarly use the first two columns of $\tilde{B}_1$ to define the terms $a_{i,j}$ with $i > 0, j < 0$. This determines a rank two matrix $\tilde{A}$. We must verify the matrix is symmetric, and is a lift of A.

Suppose $i < 0$. We must verify that all terms $a_{i,j}$ with $j > 1$ have degree zero. We can write column j as a linear combination of columns -1 and 1:

$$\sigma_j \mathbf{a}_{-1} + \rho_j \mathbf{a}_1 = \mathbf{a}_j.$$

As all the terms in row 0 have degree zero, it cannot be that σ_j and ρ_j both have positive degree, and if their degrees were negative they must be equal. If the degrees were negative this would imply elements in $\tilde{B}_2$ with negative degree, which cannot be. If $deg(\rho_j) > 0$ while $deg(\sigma_j) = 0$, then $\tilde{B}_2$ would have a column outside the first where all elements have degree zero, which cannot be. So, we must have $0 = deg(\rho_j) \leq deg(\sigma_j)$. As $a_{i,-1}$ has positive degree and $a_{i,1}$ has degree zero it must be that $a_{i,j}$ has degree zero as well. Identical reasoning gives us that all terms $a_{i,j}$ with $j < -1$ and $i > 0$ also have degree zero.

It remains to be proven that $\tilde{A}$ is symmetric. As $\tilde{B}_1$ and $\tilde{B}_2$ are symmetric, we must only prove $a_{i,j} = a_{j,i}$ when $ij < 0$. Suppose $j > 1$, and examine the two matrices

$$\begin{pmatrix} b_{-1,-1} & a_{-1,0} & a_{-1,j} \\ a_{0,-1} & a_{0,0} & a_{0,j} \\ a_{1,-1} & a_{1,0} & b_{1,j} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} b_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{j,-1} & a_{j,0} & b_{j,1} \end{pmatrix}.$$

By construction

$$a_{-1,0} = a_{0,-1}, \quad a_{0,1} = a_{1,0},$$
$$a_{0,j} = a_{j,0}, \quad \text{and} \quad b_{1,j} = b_{j,1}.$$

As the above matrices are also singular we must have $a_{-1,j} = a_{j,-1}$. The proof that $a_{1,j} = a_{j,1}$ for $j < -1$ is essentially identical. From here verifying symmetry is a calculation:

$$\begin{aligned} a_{j,i} &= \sigma_i a_{j,-1} + \rho_i a_{j,1} = \sigma_i a_{-1,j} + \rho_i a_{1,j} \\ &= \sigma_i(\sigma_j a_{-1,-1} + \rho_j a_{-1,1}) + \mu_i(\sigma_j a_{1,-1} + \rho_j a_{1,1}) \\ &= \sigma_j(\sigma_i a_{-1,-1} + \rho_i a_{1,-1}) + \rho_j(\sigma_i a_{-1,1} + \rho_i a_{1,1}) \\ &= \sigma_j(\sigma_i a_{-1,-1} + \rho_i a_{-1,1}) + \rho_j(\sigma_i a_{1,-1} + \rho_i a_{1,1}) \\ &= \sigma_j a_{-1,i} + \rho_j a_{1,i} = \sigma_j a_{i,-1} + \rho_j a_{i,1} = a_{i,j}. \end{aligned}$$

So, $\tilde{A}$ is a rank two symmetric lift of A, and therefore A has symmetric Kapranov rank two. □

As outlined in Theorem 3.6 above, we combine Lemma 3.7 and Lemma 3.8 in our proof of the next lemma.

Lemma 3.9. *Suppose A has the form*

$$\begin{pmatrix} B_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & C^T & \mathbf{0} \end{pmatrix},$$

where B_1, B_2 are symmetric and positive, C is nonnegative and does not contain a zero column, and either C or both B_1 and B_2 have positive size. If A has symmetric tropical rank two it has symmetric Kapranov rank two.

Proof. If B_1 and B_2 both have size zero, this is Lemma 3.7. If C has size zero, this is Lemma 3.8. So, suppose C has positive size, and at least one of B_1 and B_2 have positive size. The method of proof here is similar to the method used for the previous two lemmas.

By induction we may find a rank two symmetric lift for the upper-left matrix

$$\begin{pmatrix} B_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & 0 \end{pmatrix},$$

and the lower-right matrix

$$\begin{pmatrix} 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & C^T & \mathbf{0} \end{pmatrix}.$$

Call these lifts $\tilde{B}$ and $\tilde{C}$, respectively. After possibly some scaling we may assume the bottom-right entry of $\tilde{B}$ coincides with the top-left entry of $\tilde{C}$.

The lifts $\tilde{B}$ and $\tilde{C}$ will be, respectively, the upper-left and lower-right parts of the lift $\tilde{A}$ we wish to construct. We number the rows and columns of $\tilde{A}$ in a manner similar to Lemmas 3.7 and 3.8, with the $a_{0,0}$ entry being the degree zero entry that must match up for the two lifts. We will refer to any element of A with an indexed lower-case A, and not distinguish among elements in B_1, B_2, C, C^T, or outside these submatrices. We must complete the lift $\tilde{A}$ by finding entries for the terms $a_{i,j}$ with $ij < 0$.

We again start with the 3×3 central submatrix:

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}.$$

We pick $a_{-1,1}$ and $a_{1,-1}$ such that this matrix is singular and $a_{-1,1} = a_{1,-1}$. Every term $a_{i,1}$ for $i < -1$, and $a_{i,-1}$ for $i > 1$, is then determined by the need for the matrices

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{i,-1} & a_{i,0} & a_{i,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{i,-1} & a_{i,0} & a_{i,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

to be, respectively, singular.

Every column of $\tilde{C}$ can be written as a linear combination of the first two:

$$\lambda_j \mathbf{c}_0 + \mu_j \mathbf{c}_1 = \mathbf{c}_j.$$

We use these relations to define the entries $a_{i,j}$ with $i < 0$ and $j > 1$:

$$\lambda_j a_{i,0} + \mu_j a_{i,1} = a_{i,j}.$$

We similarly use the first two columns of $\tilde{B}$ to define the terms $a_{i,j}$ with $i > 0, j < -1$. This determines a rank two matrix $\tilde{A}$. We must verify the matrix is symmetric, and is a lift of A.

We first prove $\tilde{A}$ is symmetric. By construction all terms of the form $a_{i,j}$ with $ij \geq 0$ satisfy $a_{i,j} = a_{j,i}$. Also, by construction $a_{1,-1} = a_{-1,1}$. Using these facts we note the matrices

$$\begin{pmatrix} a_{i,-1} & a_{i,0} & x \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{-1,i} & a_{-1,0} & a_{-1,1} \\ a_{0,i} & a_{0,0} & a_{0,1} \\ x & a_{1,0} & a_{1,1} \end{pmatrix}$$

are transposes. Therefore, $a_{i,1}$, the unique value of x that matrix the matrix on the left singular, is equal to $a_{1,i}$, the unique value of x that makes the matrix on the right singular.

Using these equalities we note the matrices

$$\begin{pmatrix} a_{i,i} & a_{i,0} & x \\ a_{0,i} & a_{0,0} & a_{0,j} \\ a_{1,i} & a_{1,0} & a_{1,j} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{i,i} & a_{i,0} & a_{i,1} \\ a_{0,i} & a_{0,0} & a_{0,1} \\ x & a_{j,0} & a_{j,1} \end{pmatrix}$$

are also transposes. So, $a_{i,j}$, the unique value of x that makes the matrix on the left singular, is equal to $a_{j,i}$, the unique value of x that makes the matrix on the right singular. So, the matrix $\tilde{A}$ is symmetric.

It remains to be proven that each $a_{i,j}$ with $ij < 0$ has degree zero. Suppose $i < 0, j > 0$. That $a_{i,j}$ has degree zero follows because the matrix

$$\begin{pmatrix} a_{i,i} & a_{i,0} & a_{i,j} \\ a_{0,i} & a_{0,0} & a_{0,j} \\ a_{j,i} & a_{j,0} & a_{j,j} \end{pmatrix}$$

is singular, $a_{i,i}$ has positive degree, and all other terms that are not $a_{i,j} = a_{j,i}$ have degree zero. The only way this matrix could possibly be singular is if $a_{i,j}$ has degree zero. As our matrix is symmetric this completes the proof. □

3.3.4 Completed Theorem

We now have all the tools we need to complete the proof of Theorem 3.6.

Symmetric tropical rank two implies symmetric Kapranov rank two. Suppose A is a symmetric matrix with symmetric tropical rank two. We may assume A is in the form given by Lemma 3.4. If A has only one zero row/column then by Lemma 3.9 A has symmetric Kapranov rank two. If A has no zero row/column then the matrix

$$A^+ = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$$

has symmetric tropical rank two by Lemma 3.5, and therefore symmetric Kapranov rank two by Lemma 3.9. If A^+ has symmetric Kapranov rank two, then by eliminating the first row/column from the lift we see A has symmetric Kapranov rank two as well.

If A has more than one zero row/column we may proceed by induction on the number of such columns. In particular, A must have the form

$$A = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A^- \end{pmatrix},$$

where A^- is a symmetric matrix with symmetric tropical rank two, with one fewer zero row/column than A, and therefore by induction A^- has symmetric Kapranov rank two. By Lemma 3.5 A has symmetric Kapranov rank two as well. □

Combining Theorem 3.2 and Theorem 3.6 we see that the $r \times r$ minors of a symmetric $n \times n$ matrix form a tropical basis for $r = 2$ and $r = 3$.

CHAPTER 4

SYMMETRIC TROPICAL RANK THREE

The $r \times r$ minors of an $m \times n$ matrix of variables form a tropical basis if $r = 2, 3$, or $min(m, n)$. They do *not* form a tropical basis if $4 < r < min(m, n)$. The $r = 4$ case is special. The 4×4 minors of an $m \times n$ matrix of variables form a tropical basis if $min(m, n) \leq 6$, but otherwise not.

Tropical rank three is exceptional for symmetric matrices as well. In this chapter we prove that the 4×4 minors of a symmetric 5×5 matrix of variables form a tropical basis, and in proving this develop a method that might generalize to larger matrices. In Chapter 5 we will prove that the 4×4 minors of a symmetric $n \times n$ matrix of variables do *not* form a tropical basis if $n > 12$. Whether the 4×4 minors of an $n \times n$ matrix of variables form a tropical basis for $5 < n < 13$ remains unknown.

Note that throughout this chapter we will frequently be dealing with submatrices of a given matrix. Unless stated otherwise, the columns and rows of a submatrix inherit their labels from the larger matrix. So, if A is a 5×5 matrix the principal submatrix A_{33} has columns and rows labeled sequentially $1, 2, 4, 5$.

4.1 Definitions and Assumptions

Before we get to the meat of the proof we will need to justify a few assumptions we will want to make in order to simplify things.

4.1.1 Symmetric Scaling

We will make frequent use of the facts from Chapter 3 that neither the symmetric tropical rank nor symmetric Kapranov rank of a matrix changes as a result of a symmetric scaling or a diagonal permutation.

Proposition 4.1. *If A is a 5×5 symmetric matrix and σ is a permutation that realizes the tropical determinant, then there exists a matrix A' such that A' can be obtained from A through a sequence of symmetric scalings, every entry in A' is nonnegative, and $a_{i,\sigma(i)} = 0$ for all $1 \leq i \leq n$.*

Proof. Note that within this proof, and only within this proof, if we are talking about the "form" of a matrix a blank entry can have any value, positive or negative.

If $\sigma = id$ then we can form A' by scaling each row/column i by $-a_{i,i}$ to obtain a matrix with the form

$$\begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}.$$

The tropical determinant must be realized by $\sigma = id$, and the matrix must be symmetric, which clearly implies all the off-diagonal elements must be nonnegative.

If σ is a 5-cycle then we can assume without loss of generality that $\sigma = (12345)$. Scale row / column 2 by $-a_{1,2}$, row / column 3 by $-a_{2,3}$, and so on until row / column 5. Next, scale all the rows / columns with odd labels by an amount equal to $-a_{1,5}/2$, and scale all the rows / columns with even labels by an amount equal to $a_{1,5}/2$. The matrix A' obtained from this scaling must have the form

$$\begin{pmatrix} & 0 & & & 0 \\ 0 & & 0 & & \\ & 0 & & 0 & \\ & & 0 & & 0 \\ 0 & & & 0 & \end{pmatrix},$$

and its tropical determinant must be realized by $\sigma = (12345)$, which means its tropical determinant must be 0. If any blank entry, and its symmetric counterpart, above were negative the tropical determinant of the matrix would be negative, which would violate our assumption.

If σ is a 3-cycle we can assume without loss of generality that $\sigma = (123)$. Scale row / column 2 by $-a_{1,2}$, row / column 3 by $-a_{2,3}$. Then, scale rows / columns 1 and 3 by $-a_{1,3}/2$, and row / column 2 by $a_{1,3}/2$. Scale row / column 4 by $-a_{4,4}$, and row / column 5 by $-a_{5,5}$. This scaled matrix will have the form

$$\begin{pmatrix} & 0 & 0 & & \\ 0 & & 0 & & \\ 0 & 0 & & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

and its tropical determinant must be 0. As in the previous example, if any blank entry, and its symmetric counterpart, were negative the matrix would have negative determinant, which would violate our assumption.

If σ is a 3-cycle and a tranposition we can assume without loss of generality that $\sigma = (123)(45)$. We can scale the first three indices exactly as we did in the previous paragraph. If we then scale both rows / columns 4 and 5 by $-a_{4,5}/2$ we construct a matrix A'' of the form

$$\begin{pmatrix} & 0 & 0 & & \\ 0 & & 0 & & \\ 0 & 0 & & & \\ & & & & 0 \\ & & & 0 & \end{pmatrix}.$$

where the determinant is 0. If any entry along the top three diagonal terms were negative the determinant of the matrix would be negative, which is not allowed. If $a''_{4,4} < 0$ then we can scale row / column 4 by $-a''_{4,4}$, and row / column 5 by the opposite amount. This keeps the matrix in the form above, but ensures the lower two diagonal entries are nonnegative. Exactly the same reasoning applies if $a''_{5,5} < 0$. If any other entry were negative we can assume without loss of generality that the minimum entry in the matrix is $a''_{3,4}$ and its symmetric counterpart $a''_{4,3}$. If more than these two entries are negative the matrix would have negative determinant, which cannot be. If we scale row / column 4 by $-a''_{3,4}$, and row / column 5 by the opposite, then the matrix maintains the form above, but with all terms nonnegative. So, we have constructed A'.

Finally, if σ is the product of two transpositions we can assume without loss of generality that $\sigma = (12)(34)$. Scale row / column 1 by $-a_{1,2}/2$ and row / column 2 by $-a_{1,2}/2$. If after this scaling either of the top two diagonal terms are negative we can scale as we did in the previous paragraph to keep the off-diagonal terms 0 and make the diagonal terms nonnegative. The same can be done for the 2×2 block corresponding with the transposition (34). Scale row /column 5 by $-a_{5,5}$ to get the matrix

$$\begin{pmatrix} & 0 & & & \\ 0 & & & & \\ & & & 0 & \\ & & 0 & & \\ & & & & 0 \end{pmatrix}.$$

It is quick to check that, given the determinant of this matrix is 0, any negative terms can be scaled away. □

We will assume without loss of generality that all 5×5 matrices have been symmetrically scaled to satisfy the properties of Proposition 4.1.

4.1.2 The Form Matrix

We will also want to deal with all matrices that have a certain structure, and this structure will be captured by the *form* of the matrix, defined below.

Definition 4.2. A *form matrix* is a matrix in which every entry is either blank, a nonnegative constant, or the symbol $''+''$. A nonnegative matrix A has the form of a form matrix A' if everywhere A' has a constant, A has the same constant, and everywhere A' has a $''+''$, A has a positive entry.

For example, the matrix

$$\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}$$

has any of the following forms:

$$\begin{pmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & \\ 2 & 0 & + \\ & + & 0 \end{pmatrix}, \quad \begin{pmatrix} & + & + \\ + & 0 & + \\ + & + & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix}.$$

It does not, however, have the form

$$\begin{pmatrix} + & + & + \\ + & 0 & + \\ + & + & 0 \end{pmatrix},$$

because it has a 0 as its upper-left entry.

4.2 The Method of Joints

We now define the "method of joints," which will be the primary method by which we prove our theorem.

4.2.1 The Definition of Joints

Definition 4.3. Suppose A is a symmetric matrix, and there are distinct indices i and j (assume without loss of generality $i < j$) such that:

- The principal submatrix A_{ii} is symmetrically tropically singular, and there are distinct minimizing monomials $X_{\sigma_1}, X_{\sigma_2}$, such that the variables in X_{σ_1} involving the index j are not the same as the variables in X_{σ_2} involving the index j.
- The same is true with i and j reversed.
- The submatrix A_{ji} is symmetrically tropically singular, and there are two minimizing monomials X_{τ_1}, X_{τ_2} such that X_{τ_1} contains the variable $X_{i,j}$, while X_{τ_2} does not.

The indices i and j are *joints* of the matrix A. If the submatrix A_{ii} satisfies the first condition above, we say it *satisfies the joint requirement* for joints i and j. Similarly for the submatrix A_{jj}.

For example, consider a matrix A of the form

$$\begin{pmatrix} & 0 & & & \\ 0 & & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix}.$$

We will demonstrate this matrix has joints 4 and 5.

The principal submatrix A_{44} has the form

$$\begin{pmatrix} & 0 & & \\ 0 & & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}.$$

This submatrix is symmetrically tropically singular, with minimizing monomials $X_{1,2}^2 X_{3,3} X_{5,5}$ and $X_{1,2}^2 X_{3,5}^2$. In particular, the only variable in the first monomial involving the index 5 is $X_{5,5}$, while the second monomial contains the variable $X_{3,5}$. So, A_{44} satisfies the joint requirement for joints 4 and 5. Identical reasoning can be applied to the principal submatrix A_{55}.

The submatrix A_{54} has the form

$$\begin{pmatrix} & 0 & & \\ 0 & & & \\ & & 0 & 0 \\ & & 0 & 0 \end{pmatrix}.$$

The submatrix is symmetrically tropically singular, with minimizing monomials $X_{1,2}^2 X_{3,3} X_{4,5}$ and $X_{1,2}^2 X_{3,4} X_{3,5}$. One of these minimizing monomials contains the variable $X_{4,5}$, while the other does not. Therefore, A has joints 4 and 5.

4.2.2 Joints and Kapranov Rank

Our proof that the 4×4 minors of a symmetric 5×5 matrix form a tropical basis is based upon first proving that every symmetric matrix over $\mathbb{R}$ with joints has symmetric Kapranov rank of at most three. We then prove an exceptional case of a 5×5 symmetric matrix over $\mathbb{R}$ that does not have joints, but still has symmetric Kapranov rank three. Finally, we prove that if the 4×4 submatrices of a 5×5 symmetric matrix are all symmetrically tropically singular then either A has joints, or A has the form of the exceptional case.

Proposition 4.4. *If a 5×5 symmetric matrix A has joints, then it has symmetric Kapranov rank of at most three.*

Proof. We will construct a symmetric rank three lift $\tilde{A}$ of A. After possibly a diagonal permutation we may assume A has joints 4 and 5. We define the matrices:

$$X_{55} := \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & X_{1,4} \\ A_{1,2} & A_{2,2} & A_{2,3} & X_{2,4} \\ A_{1,3} & A_{2,3} & A_{3,3} & X_{3,4} \\ X_{1,4} & X_{2,4} & X_{3,4} & X_{4,4} \end{pmatrix},$$

and

$$\tilde{X}_{55} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & x_{1,4} \\ a_{1,2} & a_{2,2} & a_{2,3} & x_{2,4} \\ a_{1,3} & a_{2,3} & a_{3,3} & x_{3,4} \\ x_{1,4} & x_{2,4} & x_{3,4} & x_{4,4} \end{pmatrix},$$

where the $A_{i,j}$ are the same as the corresponding terms in the matrix A, and the $a_{i,j}$ terms are constants in the field $\tilde{K}$ such that $deg(a_{i,j}) = A_{i,j}$, but are otherwise generic. As the $a_{i,j}$ are generic, the tropicalization of the determinant of $\tilde{X}_{55}$ is the tropical determinant of X_{55}.

By Kapranov's theorem if $(A_{1,4}, A_{2,4}, A_{3,4}, A_{4,4})$ is a point on the tropical hypersurface given by the tropical determinant of X_{55}, then there is a lift to a point $(a_{1,4}, a_{2,4}, a_{3,4}, a_{4,4})$ in $\tilde{K}^4$ on the hypersurface given by the determinant of $\tilde{X}_{55}$. This lift gives us a singular 4×4 matrix

$$\tilde{A}_{55} := \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} \end{pmatrix},$$

that tropicalizes to the submatrix A_{55} of A. An identical argument can be used to construct a singular lift of A_{44}

$$\tilde{A}_{44} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,5} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,5} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,5} \\ a_{1,5} & a_{2,5} & a_{3,5} & a_{5,5} \end{pmatrix},$$

where the top-left 3×3 submatrics of $\tilde{A}_{44}$ and $\tilde{A}_{55}$ are identical.

We note that if we multiply the fourth column and the fourth row of $\tilde{A}_{44}$ by the same degree zero generic constant that we will still have a singular symmetric lift of A_{44}. So, we can assume the terms $a_{i,4}$ and $a_{j,5}$ for any $i, j \leq 5$ are generic relative to each other (except for $a_{4,5}$ and $a_{5,4}$, which we have not yet determined, and which must, of course, be equal).

All the entries in a lift of A have now been determined now except $a_{4,5} = a_{5,4}$. To get $a_{4,5}$ we examine the matrices:

$$X_{54} := \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,5} \\ A_{1,2} & A_{2,2} & A_{2,3} & A_{2,5} \\ A_{1,3} & A_{2,3} & A_{3,3} & A_{3,5} \\ A_{1,4} & A_{2,4} & A_{3,4} & X_{4,5} \end{pmatrix},$$

and

$$\tilde{X}_{54} := \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,5} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,5} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,5} \\ a_{1,4} & a_{2,4} & a_{3,4} & x_{4,5} \end{pmatrix}.$$

The determinant of $\tilde{X}_{54}$ is a linear function in the variable $x_{4,5}$, and the tropical determinant of X_{54} is a tropical linear function in the variable $X_{4,5}$. As the terms in the upper-left 3×3 submatrix of $\tilde{X}_{54}$ are generic, and the constant terms in the rightmost column of $\tilde{X}_{54}$ are generic with respect to the constant terms in the bottom row, the tropicalization of the determinant of $\tilde{X}_{54}$ is the determinant of X_{54}.

Again, by Kapranov's theorem, if $A_{4,5}$ is on the tropical hypersurface given by the tropical determinant of X_{54}, then it lifts to a point on the determinant of $\tilde{X}_{54}$. In other words, if the tropical determinant of the matrix

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & A_{1,5} \\ A_{1,2} & A_{2,2} & A_{2,3} & A_{2,5} \\ A_{1,3} & A_{2,3} & A_{3,3} & A_{3,5} \\ A_{1,4} & A_{2,4} & A_{3,4} & A_{4,5} \end{pmatrix}$$

is realized by two minimizing monomials, one involving the variable $X_{4,5}$ and the other not, then there exists a value $a_{4,5} \in \tilde{K}$ that makes the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,5} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,5} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,5} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,5} \end{pmatrix}$$

a singular lift of A_{54}.

The requirements for our three applications of Kapranov's theorem are exactly the requirements that 4 and 5 are joints of A. So, if A has joints 4 and 5 then we have now determined all the elements in a lift of the matrix A:

$$\tilde{A} := \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} \\ a_{1,2} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} & a_{3,5} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & a_{4,5} \\ a_{1,5} & a_{2,5} & a_{3,5} & a_{4,5} & a_{5,5} \end{pmatrix}.$$

It remains to be proven that such a lift has rank three. We do this by first proving there is a linear combination of the first three columns equal to the fourth. As the entries in the upper-left 3×3 submatrix were chosen generically, this submatrix has rank three, and therefore there is a unique set of coefficients $c_1, c_2, c_3 \in \tilde{K}$ such that

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{1,2} \\ a_{1,3} \end{pmatrix} + c_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{2,3} \end{pmatrix} + c_3 \begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \end{pmatrix} = \begin{pmatrix} a_{1,4} \\ a_{2,4} \\ a_{3,4} \end{pmatrix}.$$

That this unique set of coefficients also satisfy

$$c_1 a_{1,4} + c_2 a_{2,4} + c_3 a_{3,4} = a_{4,4},$$

and

$$c_1 a_{1,5} + c_2 a_{2,5} + c_3 a_{3,5} = a_{4,5}$$

follows immediately from the singularity of $\tilde{A}_{45}$ and $\tilde{A}_{55}$, respectively. Identical reasoning proves that there exists a linear combination of the first three columns of $\tilde{A}$ equal to the fifth, using the singularity of $\tilde{A}_{54}$ (which, as it is the transpose of $\tilde{A}_{45}$, follows from the singularity of $\tilde{A}_{45}$) and $\tilde{A}_{44}$. Therefore $\tilde{A}$ is a rank three lift of A, and so A has symmetric Kapranov rank at most three. □

4.3 The Exceptional Case

In our analysis of 5×5 symmetric matrices with symmetric tropical rank three or less, there is one possible form that does not have joints, but which still has symmetric Kapranov rank three.

Proposition 4.5. *If a symmetric tropical matrix A has the form:*

$$\begin{pmatrix} 0 & 0 & + & + & N \\ 0 & 0 & + & + & + \\ + & + & 0 & 0 & P \\ + & + & 0 & 0 & P \\ N & + & P & P & 0 \end{pmatrix},$$

with $N, P > 0$ and $N \otimes P$ less than any element in the 2×2 submatrix determined by rows 1 and 2, and columns 3 and 4, then A has symmetric Kapranov rank three.

Proof. The principal submatrix formed from the columns and rows with indices 1, 3, and 5 has the form

$$\begin{pmatrix} 0 & + & N \\ + & 0 & P \\ N & P & 0 \end{pmatrix}.$$

Any matrix with this form is symmetrically tropically nonsingular, and therefore A must have symmetric tropical rank three. Consequently, its symmetric Kapranov rank must be at least three.

We first note that columns 1, 3, and 5 of A cannot be tropically dependendent, and therefore A must have Kapranov rank at least three.

We augment the matrix A, producing a matrix A' with the form

$$\begin{pmatrix} 0 & 0 & 0 & + & + & N \\ 0 & 0 & 0 & + & + & + \\ 0 & 0 & 0 & P & P & 0 \\ + & + & P & 0 & 0 & P \\ + & + & P & 0 & 0 & P \\ N & + & 0 & P & P & 0 \end{pmatrix},$$

such that $A'_{33} = A$. If A' has a lift $\tilde{A}'$ to a symmetric rank three matrix, then $\tilde{A}'_{33}$ will be a symmetric rank three lift of A. So, it is sufficient to prove that A' has symmetric Kapranov rank three.

The upper-right 4×4 submatrix of A' is tropically singular, and therefore has a lift to a singular 4×4 matrix:

$$\begin{pmatrix} a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} \end{pmatrix}.$$

As $deg(a_{3,4}) = deg(a_{4,3})$ we can multiply the first column of this matrix by a degree zero constant so that $a_{3,4} = a_{4,3}$, and the matrix is still singular. We will use this singular 4×4 matrix with $a_{3,4} = a_{4,3}$ to construct a lift for columns 3 through 6 of A':

$$\begin{pmatrix} a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{3,4} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{3,5} & a_{4,5} & a_{5,5} & a_{5,6} \\ a_{3,6} & a_{4,6} & a_{5,6} & a_{6,6} \end{pmatrix}$$

where $a_{5,5}, a_{5,6}$, and $a_{6,6}$ have not yet been determined. We know there is a linear combination of columns $\mathbf{a}_3, \mathbf{a}_4$, and $\mathbf{a}_6$ (the third, fourth, and sixth columns of $\tilde{A}'$) such that for rows 1 through 4:

$$\alpha\mathbf{a}_3 + \beta\mathbf{a}_4 + \gamma\mathbf{a}_6 = \mathbf{a}_5.$$

If we pick $a_{6,6}$ such that $deg(a_{6,6}) = 0$ but otherwise generically, then this relation uniquely determines $a_{5,6}$ and $a_{5,5}$ in such a way that $deg(a_{5,5}) = 0$ and $deg(a_{5,6}) = P$. We will pick $x_{5,5}, x_{5,6}$, and $x_{6,6}$ such that this is true on all rows, and the values tropicalize appropriately.

Define M to be the minimum element in the 2×2 submatrix of A' determined by rows 1 and 2, and columns 4 and 5. If $deg(\alpha)$ were minimal out of $deg(\alpha), deg(\beta)$, and $deg(\gamma)$, then in order for the linear relation above to hold on the third row we would need either $deg(\alpha) = P$ or $deg(\alpha) = deg(\gamma) < P$. In the first case the linear relation on the fourth row would be impossible. In the second case, given $P \otimes N < M$, the linear relation on the first row would be impossible. So, $deg(\alpha)$ cannot be minimal.

If $deg(\gamma) < deg(\alpha)$ were minimal, then for the linear relation on the third row to work out we would need $deg(\gamma) = P$. This would make the linear relation on the fourth row impossible.

So, $deg(\beta)$ must be uniquely minimal. In order for the linear relation on the fourth row to work out we must have $deg(\beta) = 0$, and in order for the linear relation on the third row to work out we must have $deg(\alpha), deg(\gamma) \geq P$. If $deg(\alpha) = P$, then, again given $P \otimes N < M$, the linear relation on the first row would be impossible. So, $deg(\alpha) > P$.

With $a_{6,6}$ determined the linear relations define $a_{5,6}$ and $a_{5,5}$ as

$$a_{5,6} = \alpha a_{3,6} + \beta a_{4,6} + \gamma a_{6,6},$$
$$a_{5,5} = \alpha a_{3,5} + \beta a_{4,5} + \gamma a_{5,6}.$$

Given the required degrees of α, β, γ, the known degrees of the terms from the lift of the upper-right $4\times$ submatrix of A', and the assumption that $a_{6,6}$ satisfies $deg(a_{6,6}) = 0$ but is otherwise generic, we must have $deg(a_{5,6}) = P$, and $deg(a_{5,5}) = 0$.

What remains is to find values for $x_{1,1}, x_{1,2}, x_{2,2}$ such that the evaluation of the matrix

$$\begin{pmatrix} x_{1,1} & x_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} \\ x_{1,2} & x_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} \\ a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & a_{4,5} & a_{4,6} \\ a_{1,5} & a_{2,5} & a_{3,5} & a_{4,5} & a_{5,5} & a_{5,6} \\ a_{1,6} & a_{2,6} & a_{3,6} & a_{4,6} & a_{5,6} & a_{6,6} \end{pmatrix}$$

has rank three and tropicalizes to A'. If we examine the submatrix formed by columns $1, 3, 4$ and 6,

$$\begin{pmatrix} x_{1,1} & a_{1,3} & a_{1,4} & a_{1,6} \\ x_{1,2} & a_{2,3} & a_{2,4} & a_{2,6} \\ a_{1,3} & a_{3,3} & a_{3,4} & a_{3,6} \\ a_{1,4} & a_{3,4} & a_{4,4} & a_{4,6} \\ a_{1,5} & a_{3,5} & a_{4,5} & a_{5,6} \\ a_{1,6} & a_{3,6} & a_{4,6} & a_{6,6} \end{pmatrix},$$

then we note that, as there is a linear combination of columns $\mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_6$ equal to column $\mathbf{a}_5$ there is linear combination of rows 3 4, and 6 in the above 6×4 matrix equal to row 5. We pick $x_{1,1} = a_{1,1}$ so that the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,3} & a_{1,4} & a_{1,6} \\ a_{1,3} & a_{3,3} & a_{3,4} & a_{3,6} \\ a_{1,4} & a_{3,4} & a_{4,4} & a_{4,6} \\ a_{1,6} & a_{3,6} & a_{4,6} & a_{6,6} \end{pmatrix}$$

is singular. Given the known degrees of the elements in the matrix, and that $a_{6,6}$ is generic, we must have $deg(a_{1,1}) = 0$. We can use an identical method to construct $x_{1,2} = a_{1,2}$ of the appropriate degree. Therefore every row of the above 6×4 matrix can be constructed from rows 3, 4, and 6, and so the matrix has rank three. In particular, this means the third column of $\tilde{A}'$ can be constructed as a linear combination of the first, fourth, and sixth columns.

What remains to be proven is that $x_{2,2}$ can be chosen with the appropriate degree so that the second column of $\tilde{A}'$ can be written as a linear combination of the first, fourth, and sixth columns. To do this we examine the 6×4 submatrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,4} & a_{1,6} \\ a_{1,2} & x_{2,2} & a_{2,4} & a_{2,6} \\ a_{1,3} & a_{2,3} & a_{3,4} & a_{3,6} \\ a_{1,4} & a_{2,4} & a_{4,4} & a_{4,6} \\ a_{1,5} & a_{2,5} & a_{4,5} & a_{5,6} \\ a_{1,6} & a_{2,6} & a_{4,6} & a_{6,6} \end{pmatrix}.$$

We already know rows 3 and 5 of this matrix can be written as a linear combination of rows 1, 4, and 6. To prove this is also true for row 2 we examine the submatrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,4} & a_{1,6} \\ a_{1,2} & x_{2,2} & a_{2,4} & a_{2,6} \\ a_{1,4} & a_{2,4} & a_{4,4} & a_{4,6} \\ a_{1,6} & a_{2,6} & a_{4,6} & a_{6,6} \end{pmatrix},$$

and note that we can pick $x_{2,2} = a_{2,2}$ such that the matrix is singular and $deg(a_{2,2}) = 0$. This means that every row of $\tilde{A}'$ can be written as a linear combination of rows 1, 4, and 6, and therefore $\tilde{A}'$ has a rank three lift.

As A' has a symmetric rank three lift, so does A, and our proof is complete. □

4.4 Searching for Joints

The proof that, with one exception, if every 4×4 submatrix of a 5×5 symmetric matrix is symmetrically tropically singular then the matrix must have joints involves the analysis of a number of cases, and will be broken down into many lemmas.

We will, throughout, assume that A is a symmetric matrix with symmetric tropical rank at most three.

4.4.1 There Must Be a Transposition

Before we go through these cases, we will need an additional fact concerning the permutations that realize the tropical determinant of a symmetrically singular 5×5 matrix.

Lemma 4.6. *If A is a 5×5 symmetrically tropically singular matrix, then there is a permutation with a tranposition in its cycle decomposition realizing the tropical determinant.*

Proof. If σ realizes the tropical determinant of A, then if σ has a 2-cycle in its cycle decomposition there is nothing to prove. If the cycle decomposition of σ has a 4-cycle then by the proof of Proposition 2.10 there must also be a permutation realizing the tropical determinant that is the product of two transpositions. As for the other possibilities, after perhaps a diagonal permutation, the matrix A must have one of the following forms:

$$\textbf{Identity}: \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{pmatrix},$$

$$\textbf{3-cycle}: \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & & 0 & 0 \\ & & 0 & & 0 \\ & & 0 & 0 & \end{pmatrix},$$

$$\textbf{5-cycle}: \begin{pmatrix} & 0 & & & 0 \\ 0 & & 0 & & \\ & 0 & & 0 & \\ & & 0 & & 0 \\ 0 & & & 0 & \end{pmatrix}.$$

If A is symmetrically singular then each of these matrices must have an additional 0 term that is not specified above, and for any of these possibilities an additional 0 term will introduce a permutation realizing the tropical determinant with a cycle decomposition that includes a transposition. □

4.4.2 Not Two Transpositions

After possibly a diagonal permutation, we may assume the upper-left 2×2 submatrix of A has the form:

$$\begin{pmatrix} & 0 \\ 0 & \end{pmatrix},$$

and A has a permutation that realizes the tropical determinant whose disjoint cycle decomposition includes the transposition (12).

Lemma 4.7. *If A has symmetric tropical rank three, and does not have a permutation realizing the determinant whose disjoint cycle decomposition is a product of transpositions, then A has joints.*

Proof. As A must have a permutation realizing the determinant that involves the transposition (12), the only possibilities for this minimizing permutation are (12) and (12)(345), which would give A the form:

$$\begin{pmatrix} & 0 & & & \\ 0 & & & & \\ & & 0 & + & + \\ & & + & 0 & + \\ & & + & + & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} & 0 & & & \\ 0 & & & & \\ & & + & 0 & 0 \\ & & 0 & + & 0 \\ & & 0 & 0 & + \end{pmatrix}.$$

In the first possibility the submatrix A_{12} has the form:

$$\begin{pmatrix} 0 & & & \\ & 0 & + & + \\ & + & 0 & + \\ & + & + & 0 \end{pmatrix}.$$

The submatrix A_{12} must be singular, and so there must be another 0 term in the first row, and a corresponding 0 term in the first column. By corresponding, we mean that if the 0 in the first row of A_{12} is in the ith column, then the 0 in the first column of A_{12} must be in the ith row. Taking this into account, after possibly a diagonal permutation, A will have the form:

$$\begin{pmatrix} & 0 & 0 & & \\ 0 & & 0 & & \\ 0 & 0 & 0 & + & + \\ & & + & 0 & + \\ & & + & + & 0 \end{pmatrix}.$$

The submatrix A_{11} is

$$\begin{pmatrix} & 0 & & \\ 0 & 0 & + & + \\ & + & 0 & + \\ & + & + & 0 \end{pmatrix}.$$

As this submatrix must be symmetrically tropically singular we can see from its form that there must be two permutations realizing the tropical determinant, one (noting A_{11} inherits its indices from A) whose disjoint cycle decomposition contains the transposition (23), and another whose disjoint cycle decomposition does not. The same will be true, mutatis mutandis, of the submatrix A_{22}. From this we can see A has joints 1 and 2.

As for the second possibility, the submatrix A_{12} will have the form:

$$\begin{pmatrix} 0 & & & \\ & + & 0 & 0 \\ & 0 & + & 0 \\ & 0 & 0 & + \end{pmatrix}.$$

A_{12} must be symmetrically tropically singular, and so there must be an additional 0 term in the first row, and an additional 0 term in the first column. Noting this, after possibly a diagonal permutation, the matrix A must have one of the forms:

$$\begin{pmatrix} & 0 & 0 & & \\ 0 & & 0 & & \\ 0 & 0 & + & 0 & 0 \\ & & 0 & + & 0 \\ & & 0 & 0 & + \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} & 0 & 0 & & \\ 0 & & & 0 & \\ 0 & & + & 0 & 0 \\ & 0 & 0 & + & 0 \\ & & 0 & 0 & + \end{pmatrix}.$$

Using essentially identical reasoning as in the first possibility, we find that A will have joints 1 and 2. □

4.4.3 The Case with Five Zeros

So, we may assume A has a permutation that realizes the determinant with a disjoint cycle decomposition that is the product of two transpositions. After possibly a diagonal permutation, we may assume this disjoint cycle decomposition is (12)(34).

Lemma 4.8. *Suppose the matrix A has the form:*

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & + & 0 & \\ & & 0 & + & \\ & & & & 0 \end{pmatrix}.$$

Then A has joints.

Proof. The submatrix A_{55} must be symmetrically tropically singular, and therefore, after possibly a diagonal permutation, it must have the form

$$\begin{pmatrix} + & 0 & & 0 \\ 0 & + & 0 & \\ & 0 & + & 0 \\ 0 & & 0 & + \end{pmatrix}.$$

After another diagonal permutation A_{55} can be arranged to have the form

$$\begin{pmatrix} & & 0 & 0 \\ & & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & & \end{pmatrix}.$$

The matrix A will have the corresponding form

$$\begin{pmatrix} & & 0 & 0 & \\ & & 0 & 0 & \\ 0 & 0 & & & \\ 0 & 0 & & & \\ & & & & 0 \end{pmatrix}.$$

This is a form that will come up as a possibility in other cases, and we will refer to it as *off-diagonal form*. We will complete our lemma by proving that any matrix in off-diagonal form must have joints.

If A has off-diagonal form, the submatrix A_{11} will have the form:

$$\begin{pmatrix} & 0 & 0 & \\ 0 & & & \\ 0 & & & \\ & & & 0 \end{pmatrix}.$$

This submatrix must be symmetrically tropically singular. Denote by M the minimal element in the 2×2 submatrix formed by rows 3 and 4, and columns 3 and 4 (recall A_{11} inherits its indices from A), and denote by N the minimal element in the 2×1 submatrix formed by rows 3 and 4, and column 5. Suppose $M < 2N$. Given A_{11} is symmetrically tropically singular it must, up to a diagonal permutation, have one of the two forms:

$$\begin{pmatrix} & 0 & 0 & \\ 0 & M & & \\ 0 & & M & \\ & & & 0 \end{pmatrix}, \quad \begin{pmatrix} & 0 & 0 & \\ 0 & M & M & \\ 0 & M & & \\ & & & 0 \end{pmatrix}.$$

In either case the submatrix A_{11} satisfies the joint requirement for joints 1 and 3.

If $M = 2N$ then, again given A_{11} is symmetrically tropically singular, it must have, up to a diagonal permutation, one of the three forms:

$$\begin{pmatrix} & 0 & 0 & \\ 0 & 2N & & \\ 0 & & & N \\ & & N & 0 \end{pmatrix}, \quad \begin{pmatrix} & 0 & 0 & \\ 0 & 2N & & N \\ 0 & & & \\ & N & & 0 \end{pmatrix}, \quad \begin{pmatrix} & 0 & 0 & \\ 0 & & 2N & N \\ 0 & 2N & & \\ & N & & 0 \end{pmatrix}.$$

In either case the submatrix A_{11} again satisfies the joint requirement for joints 1 and 3.

Finally, if $M > 2N$ then as A_{11} is symmetrically tropically singular it must have the form:

$$\begin{pmatrix} & 0 & 0 & \\ 0 & & & N \\ 0 & & & N \\ & N & N & 0 \end{pmatrix}.$$

In this case, again, the submatrix A_{11} satisfies the joint requirement for joints 1 and 3.

In each of these six possibilities A_{11} satisfies the joint requirement for joints 1 and 3. An identical analysis can be performed on the submatrix A_{33}, and from this we can get that A has joints 1 and 3. So, any matrix with off-diagonal form has joints. $\square$

4.4.4 The Case with Six Zeros

Lemma 4.9. *Suppose A has the form:*

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & + & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix}.$$

Then A has joints.

Proof. The submatrix A_{55} must be symmetrically tropically singular, and this means either there is a diagonal permutation that will put A in off-diagonal form, in which case we are done, or A_{55} has the form:

$$\begin{pmatrix} + & 0 & 0 & + \\ 0 & + & 0 & + \\ 0 & 0 & + & 0 \\ + & + & 0 & 0 \end{pmatrix}.$$

In this case A_{33} must have the form:

$$\begin{pmatrix} + & 0 & + & \\ 0 & + & + & \\ + & + & 0 & \\ & & & 0 \end{pmatrix}.$$

As A_{33} must be symmetrically tropically singular, it must have one of the following two forms:

$$\begin{pmatrix} + & 0 & + & 0 \\ 0 & + & + & 0 \\ + & + & 0 & \\ 0 & 0 & & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} + & 0 & + & \\ 0 & + & + & \\ + & + & 0 & 0 \\ & & 0 & 0 \end{pmatrix}.$$

In the first possibility A has the form:

$$\begin{pmatrix} + & 0 & 0 & + & 0 \\ 0 & + & 0 & + & 0 \\ 0 & 0 & + & 0 & \\ + & + & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix}.$$

This form has joints 1 and 2. In the second possibility A has the form:

$$\begin{pmatrix} + & 0 & 0 & + & \\ 0 & + & 0 & + & \\ 0 & 0 & + & 0 & \\ + & + & 0 & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}.$$

The submatrix A_{44} has the form:

$$\begin{pmatrix} + & 0 & 0 & \\ 0 & + & 0 & \\ 0 & 0 & + & \\ & & & 0 \end{pmatrix}.$$

This submatrix must be symmetrically tropically singular and therefore, up to a diagonal permutation, must have the form:

$$\begin{pmatrix} + & 0 & 0 & 0 \\ 0 & + & 0 & \\ 0 & 0 & + & \\ 0 & & & 0 \end{pmatrix}.$$

The corresponding form for A is:

$$\begin{pmatrix} + & 0 & 0 & + & 0 \\ 0 & + & 0 & + & \\ 0 & 0 & + & 0 & \\ + & + & 0 & 0 & 0 \\ 0 & & & 0 & 0 \end{pmatrix}.$$

Any matrix of this form has joints 1 and 2. □

4.4.5 The Case with Seven Zeros

Lemma 4.10. *Suppose A has the form:*

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix}.$$

Then A has joints.

Proof. Suppose A has the form

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & 0 & 0 & + \\ & & 0 & 0 & + \\ & & + & + & 0 \end{pmatrix}.$$

The submatrices A_{33} and A_{44} will have the form:

$$\begin{pmatrix} + & 0 & & \\ 0 & + & & \\ & & 0 & + \\ & & + & 0 \end{pmatrix}.$$

For these submatrices to be symmetrically tropically singular they must have, up to a diagonal permutation, one of the two forms:

$$\begin{pmatrix} + & 0 & 0 & \\ 0 & + & 0 & \\ 0 & 0 & 0 & + \\ & & + & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} + & 0 & 0 & \\ 0 & + & & 0 \\ 0 & & 0 & + \\ & 0 & + & 0 \end{pmatrix}.$$

Examining the possibilities and what they imply for the form of A we get that A, possibly after a diagonal permutation, must either have off-diagonal form, in which case we are done, or have one of the following two forms:

$$\begin{pmatrix} + & 0 & 0 & 0 & \\ 0 & + & & & 0 \\ 0 & & 0 & 0 & \\ 0 & & 0 & 0 & \\ & 0 & & & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} + & 0 & & & 0 \\ 0 & + & & & 0 \\ & & 0 & 0 & \\ & & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix}.$$

The first possibility has joints 3 and 4. The second possibility requires more analysis.

Suppose A has the form of our second possibility above. Denote by M the minimal off-diagonal term in A that is not necessarily 0. If M is in the 2×2 submatrix formed by rows 1 and 2, and columns 3 and 4 then, after possibly a diagonal permutation, A will have the form:

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & + & M & & 0 \\ & M & 0 & 0 & \\ & & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix}.$$

Given the submatrix A_{32} must be symmetrically tropically singular, we can deduce that A must have one of the following five forms:

$$\begin{pmatrix} + & 0 & M & & 0 \\ 0 & + & M & & 0 \\ & M & 0 & 0 & \\ & M & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & & M & 0 \\ 0 & + & M & & 0 \\ & M & 0 & 0 & \\ M & & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & + & M & M & 0 \\ & M & 0 & 0 & \\ & M & 0 & 0 & \\ 0 & 0 & & & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & & & 0 \\ 0 & + & M & & 0 \\ & M & 0 & 0 & M \\ & & 0 & 0 & \\ 0 & 0 & M & & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & + & M & & 0 \\ & M & 0 & 0 & \\ & & 0 & 0 & M \\ 0 & 0 & & M & 0 \end{pmatrix}.$$

All these possibilities have joints 2 and 3.

If M is not in that 2×2 submatrix, then, possibly after a diagonal permutation, we may assume $a_{4,5} = M$. As A_{45} must be symmetrically tropically singular we get that A must have the form:

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & + & & & 0 \\ & & 0 & 0 & M \\ & & 0 & 0 & M \\ 0 & 0 & M & M & 0 \end{pmatrix}.$$

This form has joints 4 and 5.

If A has the form:

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix},$$

then, given the submatrix A_{44} must be symmetrically tropically singular, the possible forms of A, up to diagonal permutation, that are distinct from ones we have already examined are:

$$\begin{pmatrix} + & 0 & 0 & & \\ 0 & + & & & 0 \\ 0 & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & 0 & + & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} + & 0 & 0 & & \\ 0 & + & 0 & & \\ 0 & 0 & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix}.$$

The first possibility has joints 3 and 4. In the second possibility we note that the submatrix A_{51} is

$$\begin{pmatrix} 0 & 0 & & \\ + & 0 & & \\ 0 & 0 & 0 & + \\ & 0 & 0 & 0 \end{pmatrix}.$$

This matrix must be symmetrically tropically singular, and therefore, up to diagonal permutation, the matrix A must have one of the forms:

$$\begin{pmatrix} + & 0 & 0 & & \\ 0 & + & 0 & 0 & \\ 0 & 0 & 0 & 0 & + \\ & 0 & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} + & 0 & 0 & & \\ 0 & + & 0 & & 0 \\ 0 & 0 & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & 0 & + & 0 & 0 \end{pmatrix}.$$

Both have joints 3 and 4.

Finally, suppose A has the form

$$\begin{pmatrix} + & 0 & & & \\ 0 & + & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix}.$$

This matrix has the form of the first example matrix we examined, and so has joints 4 and 5. This exhausts all the possible forms of A, given the requirements of the lemma, and we have demonstrated that all these possibilities have joints. □

We combine the results of the last three lemmas as follows.

Lemma 4.11. *Suppose A has a permutation realizing the tropical determinant whose disjoint cycle decomposition is the product of two transpositions, and after a diagonal permutation it can be arranged so this permutation realizing the tropical determinant has cycle-decomposition $(12)(34)$, and the upper-left 2×2 submatrix of A has the form:*

$$\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}.$$

Then A has joints.

Proof. All the possible forms of A that satisfy these requirements are handled by Lemmas 4.8, 4.9, and 4.10. Therefore, A has joints. □

4.4.6 The Case with Eight Zeros

Lemma 4.12. *Suppose A has a permutation realizing the tropical determinant whose disjoint cycle decomposition is the product of two transpositions, and it is possible to find a diagonal permutation such that the permutation realizing the tropical determinant is $(12)(34)$ and the upper-left 2×2 submatrix is:*

$$\begin{pmatrix} + & 0 \\ 0 & 0 \end{pmatrix},$$

while it is impossible to find a diagonal permutation such that the permutation realizing the tropical determinant is $(12)(34)$ and the upper-left 2×2 submatrix is:

$$\begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}.$$

Then A has joints.

Proof. If A has the form:

$$\begin{pmatrix} + & 0 & & & \\ 0 & 0 & & & \\ & & + & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix},$$

then as A_{55} must be singular the only possibility is that A has off-diagonal form.

Suppose A has the form:

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ 0 & & & & 0 \end{pmatrix}.$$

Let M denote the minimal element that is not necessarily 0 and is not $a_{2,5}$ or its symmetric counterpart $a_{5,2}$. If $a_{i,j} = M$ then, given the submatrix A_{ij} must be symmetrically tropically singular, we can derive that, up to a diagonal permutation, A must have one of the following nine forms:

$$\begin{pmatrix} + & 0 & M & & 0 \\ 0 & 0 & M & & \\ M & M & 0 & 0 & \\ & & 0 & 0 & \\ 0 & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & & M & 0 \\ 0 & 0 & M & & \\ & M & 0 & 0 & \\ M & & 0 & 0 & \\ 0 & & & & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & 0 & M & M & \\ & M & 0 & 0 & \\ & M & 0 & 0 & \\ 0 & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & & & 0 \\ 0 & 0 & M & & \\ & M & 0 & 0 & M \\ & & 0 & 0 & \\ 0 & & M & & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & 0 & M & & \\ & M & 0 & 0 & \\ & & 0 & 0 & M \\ 0 & & & M & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & M & M & 0 \\ 0 & 0 & & & \\ M & & 0 & 0 & \\ M & & 0 & 0 & \\ 0 & & & & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & M & & 0 \\ 0 & 0 & & & \\ M & & 0 & 0 & M \\ & & 0 & 0 & \\ 0 & & M & & 0 \end{pmatrix}, \quad \begin{pmatrix} + & 0 & M & & 0 \\ 0 & 0 & & & \\ M & & 0 & 0 & \\ & & 0 & 0 & M \\ 0 & & & M & 0 \end{pmatrix},$$

$$\begin{pmatrix} + & 0 & & & 0 \\ 0 & 0 & & & \\ & & 0 & 0 & M \\ & & 0 & 0 & M \\ 0 & & M & M & 0 \end{pmatrix}.$$

The first five possibilities have joints 2 and 3, possibilities six through eight have joints 1 and 3, and the ninth possibility has joints 4 and 5.

Suppose A has the form:

$$\begin{pmatrix} + & 0 & & & + \\ 0 & 0 & & & \\ & & 0 & 0 & + \\ & & 0 & 0 & + \\ + & & + & + & 0 \end{pmatrix}.$$

Given the submatrices A_{33} and A_{44} must be symmetrically tropically singular, the matrix A must have the form:

$$\begin{pmatrix} + & 0 & 0 & 0 & + \\ 0 & 0 & & & \\ 0 & & 0 & 0 & + \\ 0 & & 0 & 0 & + \\ + & & + & + & 0 \end{pmatrix}.$$

This matrix has joints 3 and 4.

If A has the form:

$$\begin{pmatrix} + & 0 & & & + \\ 0 & 0 & & & \\ & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ + & & + & 0 & 0 \end{pmatrix},$$

then, given A_{44} must be symmetrically tropically singular, A must have the form:

$$\begin{pmatrix} + & 0 & 0 & & + \\ 0 & 0 & & & \\ 0 & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ + & & + & 0 & 0 \end{pmatrix},$$

which, after a diagonal permutation, is a form analyzed earlier in the lemma.

Finally, if A has the form:

$$\begin{pmatrix} + & 0 & & & + \\ 0 & 0 & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ + & & 0 & 0 & 0 \end{pmatrix},$$

then it is of the form of the example matrix we first analyzed in this chapter, and A has joints 4 and 5. This exhausts all the possibilities, and the lemma is proven. $\square$

4.4.7 The Case with Nine Zeros

Up to diagonal permutation the only form we have yet to consider is:

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix}.$$

Denote the minimal term in the 2×2 submatrix formed by rows $1, 2$ and columns $3, 4$ as M, the minimal term in the 2×1 submatrix formed by rows $1, 2$ and column 5 as N, and the minimal term in the 2×1 submatrix formed by rows $3, 4$ and column 5 as P.

If either $N = 0$ or $P = 0$ we can, after a diagonal permutation, assume A has the form:

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \end{pmatrix}.$$

Lemma 4.13. *If A has the form:*

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \end{pmatrix},$$

then A has joints.

Proof. If A has the form:

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & & 0 & 0 & 0 \end{pmatrix},$$

then A has the form of the first example analyzed in this chapter, and therefore has joints 4 and 5.

Suppose A has the form:

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix},$$

and denote by M the minimal term in A that is not necessarily 0 and is not the term $a_{3,5}$ or its symmetric counterpart $a_{5,3}$. Then, given the submatrix A_{42} has the form:

$$\begin{pmatrix} 0 & & & \\ 0 & & & \\ & 0 & 0 & + \\ & + & 0 & 0 \end{pmatrix},$$

and must be symmetrically tropically singular, A must have, up to diagonal permutation, one of the following six forms:

$$\begin{pmatrix} 0 & 0 & & M & \\ 0 & 0 & M & & \\ & M & 0 & 0 & + \\ M & & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & M & & \\ 0 & 0 & M & & \\ M & M & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & M & M & \\ & M & 0 & 0 & + \\ & M & 0 & 0 & 0 \\ & & + & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & & & M \\ 0 & 0 & M & & \\ & M & 0 & 0 & + \\ & & 0 & 0 & 0 \\ M & & + & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & M & & M \\ & M & 0 & 0 & + \\ & & 0 & 0 & 0 \\ & M & + & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & & & M \\ 0 & 0 & & & M \\ & & 0 & 0 & + \\ & & 0 & 0 & 0 \\ M & M & + & 0 & 0 \end{pmatrix}.$$

The first five possibilities have joints 2 and 3. The final possibility has joints 1 and 5. □

Lemma 4.14. *If A has the form:*

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & & & \\ & & 0 & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix},$$

with M, N, P defined above, if $M \leq N \otimes P$ then A has joints.

Proof. After possibly a diagonal permutation we may assume A has the form:

$$\begin{pmatrix} 0 & 0 & & & \\ 0 & 0 & M & & \\ & M & 0 & 0 & \\ & & 0 & 0 & \\ & & & & 0 \end{pmatrix}.$$

The submatrix A_{32} has the form

$$\begin{pmatrix} 0 & & & \\ 0 & M & & \\ & 0 & 0 & \\ & & & 0 \end{pmatrix}$$

and must be symmetrically tropically singular. Therefore, A_{32} must have two distinct permutations realizing the tropical determinant, one involving the M term and the other not. Therefore A has joints 2 and 3. □

The final possibility, if $N, P > 0$ and $N \otimes P < M$, is, up to diagonal permutation, the only form that does not have joints. It is the exceptional form handled by Proposition 4.5. The results of these many lemmas can now be summarized.

Proposition 4.15. *If A is a 5×5 symmetric matrix with symmetric tropical rank three, and if A does not have exceptional form, then A has joints.*

Proof. All the possible forms for A, up to diagonal permutation, are proven to have joints by Lemmas 4.11, 4.12, 4.13, and 4.14. □

4.5 The 4×4 Minors of a 5×5 Symmetric Matrix

We now have all we need to prove the major theorem of this chapter.

Theorem 4.16. *The 4×4 minors of a 5×5 symmetric matrix form a tropical basis.*

Proof. This is an immediate consequence of Propositions 4.4, 4.5, and 4.15. □

I conjecture that a generalization of the techniques used in this chapter can be used to prove the 4×4 minors of an $n \times n$ symmetric matrix form a tropical basis for $n \leq 12$.

Conjecture 4.17. The 4×4 minors of a $n \times n$ symmetric matrix are a tropical basis for $n \leq 12$.

It is definitely not the case that the 4×4 minors of a symmetric $n \times n$ matrix are a tropical basis for $n > 12$, as we will see in the next chapter.

CHAPTER 5

WHEN THE MINORS OF A SYMMETRIC MATRIX DO NOT FORM A TROPICAL BASIS

In this chapter we prove that the $k \times k$ minors of an $n \times n$ symmetric matrix do *not* form a tropical basis if $4 < k < n$. Nor do they form a tropical basis if $k = 4$ and $n > 12$.

We also prove that, for standard matrices, if the prevariety given by the $k \times k$ minors of an $m \times n$ matrix is not equal to the variety given by the minors, then the prevariety has greater dimension than the variety. We prove that same for symmetric matrices with $k > 4$.

All statements about tropical ranks and symmetric tropical ranks for specific matrices, and in fact all specific computational claims of any sort, made in this section can be verified using Maple code available online: http://www.math.utah.edu/~zwick/Dissertation/. The Maple code used to verify the specific examples from this chapter is given in Appendix B of this dissertation.

5.1 The Foundational Examples

The examination of when the minors of a standard matrix do not form a tropical basis begins with a couple of fundamental examples. The same is true in the symmetric case.

5.1.1 Rank Three

In [8] the authors proved that the cocircuit matrix of the Fano matroid,

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

has tropical rank three but Kapranov rank four. If we permute the rows of this matrix with the permutation given by the disjoint cycle decomposition (27)(36)(45) we get the symmetric matrix [5]

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

As explained in Chapter 2, this symmetric matrix has standard tropical rank three, but symmetric tropical rank four, and is therefore *not* an example of a matrix with symmetric tropical rank three but greater symmetric Kapranov rank.

This matrix can, however, be used to construct a symmetric matrix with symmetric tropical rank three, but greater symmetric Kapranov rank:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The upper-right, and bottom-left, 7×7 submatrices of the above 13×13 symmetric matrix are the symmetric version of the cocircuit matrix of the Fano matroid. This 13×13 matrix has symmetric tropical rank three. If it had symmetric Kapranov rank three then its upper-right 7×7 submatrix would have standard Kapranov rank three, and this is impossible.

5.1.2 Rank Four

In [18] the matrix

$$\begin{pmatrix} 0 & 0 & 4 & 4 & 4 & 4 \\ 0 & 0 & 2 & 4 & 1 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 1 & 4 & 0 & 0 \end{pmatrix},$$

was shown to have tropical rank four but Kapranov rank five. If we permute the rows of this matrix with the permutation (135)(246), and the columns with the permutation

(16)(25)(34), we get the symmetric matrix

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 1 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 1 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{pmatrix}.$$

This symmetric 6×6 matrix has symmetric tropical rank four, and, as its Kapranov rank is five, its symmetric Kapranov rank is at least five. Applying Theorem 3.1 we see its symmetric Kapranov rank is exactly five. So, it is a 6×6 symmetric matrix with different symmetric tropical and symmetric Kapranov ranks.

5.2 Dimension Growth of Determinantal Prevarieties

If a basis for an ideal is not a tropical basis, a natural question to ask is whether the corresponding tropical prevariety has greater dimension than the corresponding tropical variety. In the context of determinantal varieties, using the notation from Chapter 2, this question is whether when the containment

$$\tilde{T}_{m,n,r} \subseteq T_{m,n,r},$$

is proper, the inequality

$$dim(\tilde{T}_{m,n,r}) \leq dim(T_{m,n,r}),$$

is strict. For symmetric matrices we can ask the analogous question, namely, whether when the containment

$$\tilde{S}_{n,r} \subseteq S_{n,r}$$

is proper the inequality

$$dim(\tilde{S}_{n,r}) \leq dim(S_{n,r})$$

is strict.

In this chapter we prove the answer for standard matrices is yes, and for symmetric matrices the answer is yes for all cases outside rank three. For rank three symmetric matrices, I suspect, but do not prove, the answer is no. Note that this answer in the case of standard matrices seems to be known to the mathematical community [6], but I am unaware of a source for a proof outside this dissertation.

The proofs for the standard and the symmetric cases are similar, and so will be given in parallel. The proofs are inductive, and will rely upon applying preliminary lemmas to specific base cases. We first prove these lemmas, then examine the base cases, and finally prove the main theorems. We begin, in this section, with the lemmas.

Our first lemma concerns tropical linear combinations of tropically linearly independent columns, and could be viewed as a corollary of Theorem 4.2 from [8].

Lemma 5.1. *If A is an $r \times r$ tropically nonsingular matrix and the permutation $\sigma \in S_r$ realizes the tropical determinant, then there exist constants $c_1, \ldots, c_r$ such that*

$$c_{\sigma(i)} \odot a_{i,\sigma(i)} \leq c_j \odot a_{i,j};$$

for all $i, j \leq r$, with equality if and only if $\sigma(i) = j$.

Proof. Denote the columns of A by $\mathbf{a}_1, \ldots, \mathbf{a}_r$. As the tropical rank of A is r, by Theorem 4.2 of [8] the dimension of the tropical convex hull of the columns of A is r.[1] In particular, if we choose $c_1, \ldots, c_r$ such that

$$c_1 \odot \mathbf{a}_1 \oplus c_2 \odot \mathbf{a}_2 \oplus \cdots \oplus c_r \odot \mathbf{a}_r$$

is in the interior of the tropical convex hull, then any small modification of a coefficient c_i must change the corresponding point in the convex hull. This requires that there exists a permutation $\rho \in S_r$ such that $c_i + a_{\rho(i),i} \leq c_k + a_{\rho(i),k}$ for all $k \leq r$, with equality if and only if $i = k$. The sum of these $a_{\rho(i),i}$ terms must be the determinant, and our lemma is proved with $\sigma = \rho^{-1}$. □

5.2.1 The Standard Case

We now present, in both the standard and symmetric cases, how given a matrix A with tropical or symmetric tropical rank r, we can construct larger matrices from A with desired tropical or symmetric tropical ranks. We begin with the standard case.

Lemma 5.2. *Suppose A is an $m \times n$ matrix with tropical rank r. Construct the $m \times (n+1)$ matrix A' from A by appending to A a column formed as a tropical linear combination of columns from A. The matrix A' has tropical rank r. If we construct the $(m+1) \times n$ matrix A'' from A by appending to A a row formed as a tropical linear combination of rows from A, then the matrix A'' has tropical rank r as well.*

---

[1] Note that we view the tropical convex hull as a subset of $\mathbb{R}^r$, and not of $\mathbb{TP}^{r-1}$, which is the reason the dimension is r here and not $r - 1$.

Proof. As column $n+1$ of A' is a tropical linear combination of the columns of A, the tropical convex hull of the columns of A' is the same as the tropical convex hull of the columns of A, and therefore by Theorem 4.2 from [8] the two matrices have the same tropical rank. An identical argument, mutatis mutandis, proves A'' has tropical rank r. □

Lemma 5.3. *Suppose A is an $m \times n$ matrix with tropical rank r. Construct the $(m+1) \times (n+1)$ matrix A' from A by choosing a number P that is greater than any entry of A, a number M that is less than any entry of A, and defining*

$$A' = \left(\begin{array}{ccc|c} & & & P \\ & A & & \vdots \\ & & & P \\ \hline P & \cdots & P & M \end{array}\right).$$

The matrix A' has tropical rank $r+1$.

Proof. As A has tropical rank r there is an $r \times r$ submatrix of A that is tropically nonsingular. Let $a_1, \ldots, a_r$ denote the rows of A that define this submatrix, $b_1, \ldots, b_r$ denote the columns of A that define this submatrix, and D denote the submatrix's tropical determinant. The tropical determinant of the $(r+1) \times (r+1)$ submatrix of A' defined by the rows $a_1, \ldots, a_r, a_{m+1}$, and the columns $b_1, \ldots, b_r, b_{n+1}$ must, given the definitions of P and M, be equal to $D \odot M$, and the submatrix must be nonsingular. So, the tropical rank of A' must be at least $r+1$.

Take any $(r+2) \times (r+2)$ submatrix of A'. If it is a submatrix of A then, as A has tropical rank r, it must be singular. If the submatrix is formed from row $m+1$ of A', but not column $n+1$, then we can see it must be tropically singular by taking a row expansion along the submatrix's bottom row, and noting that every $(r+1) \times (r+1)$ submatrix of A is tropically singular. Similarly, if the submatrix is formed from column $n+1$ of A', but not row $m+1$, the submatrix must be tropically singular. Finally, if the submatrix is formed from row $m+1$ and column $n+1$ then, given the definitions of P and M, every tropical product of terms that equals the tropical determinant must involve the term $a_{m+1,n+1} = M$, and singularity of the $(r+2) \times (r+2)$ submatrix follows from the fact that every $(r+1) \times (r+1)$ submatrix of A is tropically singular. So, the tropical rank of A' is at most $r+1$, and combining this with the result from the previous paragraph we see the tropical rank equals $r+1$. □

5.2.2 The Symmetric Case

The corresponding lemmas for symmetric matrices are similar. However, for the symmetric version of Lemma 5.2 we do not have a corresponding convenient reference like Theorem 4.2 from [8], and consequently the proof is much longer and more involved.

Lemma 5.4. *Suppose A is an $n \times n$ symmetric matrix with symmetric tropical rank r. Construct the $n \times (n+1)$ matrix A' from A by appending to the right of A a column formed as a tropical linear combination of columns from A. So, if $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ are the columns of A and $\boldsymbol{a}'_{n+1}$ is column $n+1$ of A', then*

$$\boldsymbol{a}'_{n+1} = c_{i_1} \odot \boldsymbol{a}_{i_1} \oplus c_{i_2} \odot \boldsymbol{a}_{i_2} \oplus \cdots \oplus c_{i_k} \odot \boldsymbol{a}_{i_k}.$$

Construct the $(n+1) \times (n+1)$ matrix A'' from A' by appending to the bottom of A' a row formed as a tropical linear combination of rows from A' in the same manner. So, if $\boldsymbol{a}'_1, \ldots, \boldsymbol{a}'_{n+1}$ are the rows of A' and $\boldsymbol{a}''_{n+1}$ is column $n+1$ of A'', then

$$\boldsymbol{a}''_{n+1} = c_{i_1} \cdot \boldsymbol{a}'_{i_1} \oplus c_{i_2} \cdot \boldsymbol{a}'_{i_2} \oplus \cdots \oplus c_{i_k} \odot \boldsymbol{a}'_{i_k}.$$

The matrix A'' is symmetric, and has symmetric tropical rank r.

Proof. The entry $a''_{j,n+1}$ of A'', where $j < n+1$, is a tropical linear combination of elements from row j of A:

$$a''_{j,n+1} = c_{k_1} \odot a_{j,k_1} \oplus c_{k_2} \odot a_{j,k_2} \oplus \cdots \oplus c_{k_l} \odot a_{j,k_l}.$$

The entry $a''_{n+1,j}$ is similarly a tropical linear combination of elements from column j of A:

$$a''_{n+1,j} = c_{k_1} \odot a_{k_1,j} \oplus c_{k_2} \odot a_{k_2,j} \oplus \cdots \oplus c_{k_l} \odot a_{k_l,j}.$$

As A is symmetric we see immediately that $a''_{j,n+1} = a''_{n+1,j}$, and therefore A'' is also symmetric.

Suppose M is an $(r+1) \times (r+1)$ submatrix of A'' that inherits its row and column indices from A''. Denote the row indices of M in ascending order by $i_1, i_2, \ldots, i_{r+1}$, and the column indices in ascending order by $j_1, j_2, \ldots, j_{r+1}$. Denote by $\mathbf{m}_j$ the column vector formed by rows $i_1, \ldots, i_{r+1}$ and column j of A''. So,

$$M = \begin{pmatrix} \mathbf{m}_{j_1} & \mathbf{m}_{j_2} & \cdots & \mathbf{m}_{j_{r+1}} \end{pmatrix}.$$

If M does not have a column $n+1$ or row $n+1$ then M corresponds with a submatrix of A. In this case as A has symmetric tropical rank r, M must be symmetrically tropically singular.

Suppose M has a column $n+1$, but no row $n+1$. There exists a bijection σ from the column indices of M to its row indices such that

$$\begin{aligned} tropdet(M) &= m_{\sigma(j_1),j_1} \odot m_{\sigma(j_2),j_2} \odot \cdots \odot m_{\sigma(j_r),j_r} \odot m_{\sigma(n+1),n+1} \\ &= a''_{\sigma(j_1),j_1} \odot a''_{\sigma(j_2),j_2} \odot \cdots \odot a''_{\sigma(j_r),j_r} \odot a''_{\sigma(n+1),n+1}. \end{aligned}$$

Note that this bijection σ is not necessarily unique.

We know from the construction of A'' that $a''_{\sigma(n+1),n+1} = c_{k_i} \odot a''_{\sigma(n+1),k_i}$ for some index $k_i < n+1$. Using this information, define the matrix

$$M' = \begin{pmatrix} \mathbf{m}_{j_1} & \mathbf{m}_{j_2} & \cdots & \mathbf{m}_{j_r} & c_{k_i} \odot \mathbf{m}_{k_i} \end{pmatrix}.$$

Index the rows and columns of M' with the same indices as M. The matrices M and M' differ only in their rightmost column, and $m_{i,n+1} \leq m'_{i,n+1}$ for all entries in their respective rightmost columns. Therefore, $tropdet(M) \leq tropdet(M')$, and it follows immediately that

$$\begin{aligned} tropdet(M) &= m_{\sigma(j_1),j_1} \odot m_{\sigma(j_2),j_2} \odot \cdots \odot m_{\sigma(j_r),j_r} \odot m_{\sigma(n+1),n+1} \\ &= a''_{\sigma(j_1),j_1} \odot a''_{\sigma(j_2),j_2} \odot \cdots \odot a''_{\sigma(j_r),j_r} \odot a''_{\sigma(n+1),n+1} \\ &= m'_{\sigma(j_1),j_1} \odot m'_{\sigma(j_2),j_2} \odot \cdots \odot m'_{\sigma(j_r),j_r} \odot m'_{\sigma(n+1),n+1} = tropdet(M'). \end{aligned}$$

So, $tropdet(M) = tropdet(M')$, and if τ is a bijection from $\{j_1, \ldots, j_{r+1}\}$ to $\{i_1, \ldots, i_{r+1}\}$ such that

$$tropdet(M') = m'_{\tau(j_1),j_1} \odot m'_{\tau(j_2),j_2} \odot \cdots \odot m'_{\tau(j_r),j_r} \odot m'_{\tau(j_{r+1}),j_{r+1}},$$

then

$$tropdet(M) = m_{\tau(j_1),j_1} \odot m_{\tau(j_2),j_2} \odot \cdots \odot m_{\tau(j_r),j_r} \odot m_{\tau(j_{r+1}),j_{r+1}}.$$

Suppose $k_i \in \{j_1, \ldots, j_r\}$. In this case one of the columns of M' is a tropical multiple of another, and by Proposition 2.9 there are two distinct bijections σ_1 and σ_2 such that

$$tropdet(M') = m'_{\sigma_1(j_1),j_1} \odot \cdots \odot m'_{\sigma_1(j_r),j_r} = m'_{\sigma_2(j_1),j_1} \odot \cdots \odot m'_{\sigma_2(j_{r+1}),j_{r+1}}$$

and the monomials

$$X_1 = X_{\sigma_1(j_1),j_1} \odot \cdots \odot X_{\sigma_1(j_{r+1}),j_{r+1}}, \quad \text{and} \quad X_2 = X_{\sigma_2(j_1),j_1} \odot \cdots \odot X_{\sigma_2(j_{r+1}),j_{r+1}}$$

are distinct even given the relation $X_{i,j} = X_{j,i}$. The monomials X_1 and X_2 must both be minimizing monomials for the submatrix M, and therefore this submatrix is symmetrically tropically singular.

If $k_i \notin \{j_1, \ldots, j_r\}$ then suppose $j_q < k_i < j_{q+1}$. Take the submatrix of A'' given by

$$M'' = \begin{pmatrix} \mathbf{m}_{j_1} & \mathbf{m}_{j_2} & \cdots & \mathbf{m}_{j_q} & \mathbf{m}_{k_i} & \mathbf{m}_{j_{q+1}} & \cdots & \mathbf{m}_{j_r} \end{pmatrix}$$

where M'' inherits its row and column indices from A''. Any bijection

$$\sigma'' : \{j_1, \ldots, j_q, k_i, j_{q+1}, \ldots, j_r\} \to \{i_1, i_2, \ldots, i_{r+1}\}$$

such that

$$tropdet(M'') = m''_{\sigma''(j_1),j_1} \odot \cdots \odot m''_{\sigma''(j_q),j_q} \odot m''_{\sigma''(k_i),k_i} \odot m''_{\sigma''(j_{q+1}),j_{q+1}} \odot \cdots \odot m''_{\sigma''(j_r),j_r}$$

corresponds with a bijection σ' from $\{j_1, \ldots, j_r, j_{r+1}\}$ to $\{i_1, \ldots, i_r, i_{r+1}\}$ where $\sigma'(j_p) = \sigma''(j_p)$ for $p < r+1$, $\sigma'(j_{r+1}) = \sigma''(k_i)$, and

$$tropdet(M') = m'_{\sigma'(j_1),j_1} \odot \cdots \odot m'_{\sigma'(j_{r+1}),j_{r+1}}.$$

The submatrix M'' corresponds with an $(r+1) \times (r+1)$ submatrix of A, and therefore is symmetrically tropically singular. This implies there are two distinct bijections from $\{j_1, \ldots, j_{r+1}\}$ to $\{i_1, \ldots, i_{r+1}\}$, both of which define the tropical determinant of M' in the way σ' did above, and which define two monomials that are distinct even under the equivalence $X_{i,j} = X_{j,i}$. These monomials must be minimizing monomials for the submatrix M, and therefore M is symmetrically tropically singular.

Identical reasoning applies if M has a row $n+1$, but not a column $n+1$.

If M has both a row $n+1$ and a column $n+1$ then we may define M' exactly as we did above, and if $k_i \in \{j_1, \ldots, j_r\}$ then the proof goes through without modification. So, suppose $k_i \notin \{j_1, \ldots, j_r\}$. In this case the proof above still goes through without modification, if we just note that M'' corresponds with an $(r+1) \times (r+1)$ submatrix of A'' with a row $n+1$, but not a column $n+1$, and is therefore symmetrically tropically singular.

So, every $(r+1) \times (r+1)$ submatrix of A'' is symmetrically tropically singular, and therefore A'' has symmetric tropical rank at most r. As A has symmetric tropical rank r there is an $r \times r$ submatrix of A that is symmetrically tropically nonsingular, and there will be a corresponding submatrix in A''. So, A'' has symmetric tropical rank r. □

Corollary 5.5. *If the $r \times r$ minors of an $n \times n$ symmetric matrix of variables are not a tropical basis, then the $r \times r$ minors of an $(n+1) \times (n+1)$ symmetric matrix of variables are not a tropical basis.*

Proof. That the $r \times r$ minors of an $n \times n$ symmetric matrix of variables are not a tropical basis is equivalent to the existence of an $n \times n$ symmetric matrix with symmetric tropical rank $r - 1$, but greater symmetric Kapranov rank. If A is such a matrix, then, by Lemma 5.4, there exists an $(n+1) \times (n+1)$ matrix A' with symmetric tropical rank $r-1$ containing A as a principal submatrix. If A' had symmetric Kapranov rank $r - 1$ so would A, and so the symmetric Kapranov rank of A' must be greater than $r - 1$. This implies the $r \times r$ minors of an $(n + 1) \times (n + 1)$ symmetric matrix of variables are not a tropical basis. □

In particular, based on our earlier 13×13 example, we can conclude the 4×4 minors of an $n \times n$ symmetric matrix of variables are *not* a tropical basis when $n \geq 13$.

Lemma 5.6. *Suppose A is an $n \times n$ symmetric matrix with symmetric tropical rank r. Construct the $(n + 1) \times (n + 1)$ matrix A''' from A by choosing a number P that is greater than any entry of A, a number M that is less than any entry of A, and defining*

$$A''' = \left(\begin{array}{ccc|c} & & & P \\ & A & & \vdots \\ & & & P \\ \hline P & \cdots & P & M \end{array}\right).$$

The matrix A''' is symmetric and has symmetric tropical rank $r + 1$.

Proof. As A is symmetric A''' is obviously symmetric.

The proof that A''' has tropical rank $r + 1$ goes exactly the same as the proof of Lemma 5.3, replacing all the pertinent definitions by their symmetric counterparts. □

5.2.3 Dimension Growth for Standard Matrices

We now prove the lemmas at the heart of this chapter. All concern how the dimensions of the determinantal tropical prevarities grow when the size of the matrix is increased. We begin with general matrices, and then turn to symmetric matrices.

Lemma 5.7. $(dim(T_{m,n+1,r}) - dim(T_{m,n,r})) \geq r - 1$, *and* $(dim(T_{m+1,n,r}) - dim(T_{m,n,r})) \geq r - 1$.

Proof. Suppose A is an $m \times n$ matrix of tropical rank $r-1$. Permuting the rows and columns of a matrix does not change the tropical rank, and so we may assume that the upper-left $(r - 1) \times (r - 1)$ submatrix of A is tropically nonsingular, and its determinant is realized by the tropical product of the diagonal terms (the classical trace).

Using A, define an $m \times (n+1)$ matrix A' by appending to A a tropical linear combination of the first $r-1$ columns of A. By Lemma 5.1 we can pick the coefficients $c_1, \ldots, c_{r-1}$ for this linear combination such that $c_i \odot a_{ii} < c_j \odot a_{ij}$ for all $i, j \leq r-1$ with $i \neq j$. By Lemma 5.2 this matrix A' will have tropical rank $r-1$.

Viewing A as a point in $\mathbb{R}^{m \times n}$ we define $T_{A,\epsilon}$ to be the intersection of $T_{m,n,r}$ with $B_{A,\epsilon}$, an ϵ-ball centered at A:

$$T_{A,\epsilon} = T_{m,n,r} \cap B_{A,\epsilon}.$$

For ϵ sufficiently small every matrix in $T_{A,\epsilon}$ will, like A, have a nonsingular $(r-1) \times (r-1)$ upper-left submatrix with determinant given by the tropical product of the diagonal terms. Similarly, for sufficiently small ϵ, we can use the coefficients $c_1, \ldots, c_{r-1}$ to define a matrix $B' \in T_{m,n+1,r}$ for any matrix $B \in T_{A,\epsilon}$, such that $c_i \odot b_{ii} < c_j \odot b_{ij}$ for all $i, j \leq r-1$ with $i \neq j$. This defines an embedding of $T_{A,\epsilon}$ into $T_{m,n+1,r}$. Call this embedding $T'_{A,\epsilon}$.

Tropically multiplying a column of a matrix by a real number does not change the tropical rank. So, for any matrix $B' \in T_{m,n+1,r}$ we can multiply the first $r-1$ columns by constants $c_1, \ldots, c_{r-1}$ and obtain another point in $T_{m,n+1,r}$. In this way we construct an $(r-1)$-dimensional linear subspace of $T_{m,n+1,r}$. Call this linear subspace $L'_{B'}$. Suppose $B' \in T'_{A,\epsilon}$, and so B' is the image of a matrix $B \in T_{A,\epsilon}$ under our embedding. The intersection $L'_{B'} \cap T'_{A,\epsilon}$ is just the point B'. To see this, suppose there were another point, $C' \in L'_{B'} \cap T'_{A,\epsilon}$. This matrix C' would have to be the image of a matrix $C \in T_{A,\epsilon}$ under our embedding, and C would be given by tropically multiplying the first $(r-1)$ columns of B by the appropriate real numbers. The final column of C' would have to be the same as the final column of B', but this would imply the first $(r-1)$ diagonal entries of C are the same as the first $(r-1)$ diagonal entries of B, which would imply all the real number tropical multiples are 0, which would mean $B = C$, and so $B' = C'$.

From this we get $(dim(T_{m,n+1,r}) - dim(T_{m,n,r})) \geq r-1$, and using identical reasoning we can get $(dim(T_{m+1,n,r}) - dim(T_{m,n,r})) \geq r-1$. □

Lemma 5.8. $(dim(T_{m+1,n+1,r+1}) - dim(T_{m,n,r})) \geq m+n+1$.

Proof. For $A \in T_{m,n,r}$ we define $T_{A,\epsilon}$ in exactly the same manner as the previous lemma. By Lemma 5.3, for any matrix $B \in T_{A,\epsilon}$ there is a matrix $B'' \in T_{m+1,n+1,r+1}$ defined by

$$B'' = \left(\begin{array}{ccc|c} & & & P \\ & B & & P \\ & & & P \\ \hline P & P & P & M \end{array}\right)$$

where P is larger than any entry in A, and M is smaller than any entry in A. For ϵ sufficiently small, this defines an embedding of $T_{A,\epsilon}$ into $T_{m+1,n+1,r+1}$, where the values of P and M are the same for every matrix in the image of the embedding. Call this embedding $T''_{A,\epsilon}$.

As noted in the previous lemma, tropically multiplying a column or row of a matrix by a real number does not change its tropical rank. So, for any matrix $B'' \in T''_{A,\epsilon}$ there is a $m+n+1$ dimensional subspace of $T_{m+1,n+1,r+1}$ formed by tropically multiplying the rows and columns of B'' by real numbers (It is not an $m+n+2$ dimensional subspace because adding the same number to all the columns, and then subtracting that number from all the rows, leaves the matrix unchanged). Call this subspace $L''_{B''}$. The intersection $L''_{B''} \cap T''_{A,\epsilon}$ is just the matrix B''. We can see this by noting that for every element of $T''_{A,\epsilon}$ the right column and bottom row are the same, and the only element of $L''_{B''}$ with this given right column and bottom row is the matrix B''. □

5.2.4 Dimension Growth for Symmetric Matrices

Lemmas 5.7 and 5.8 both focus on a neighborhood of a matrix $A \in T_{m,n,r}$. For the symmetric version of Lemma 5.7 we will require that our matrix $A \in S_{n,r}$ not only have a symmetrically tropically nonsingular $r \times r$ submatrix, but a *tropically nonsingular* $r \times r$ submatrix.

Lemma 5.9. *Suppose $A \in S_{n,r}$, and A has an $(r-1) \times (r-1)$ submatrix that is tropically nonsingular (not just symmetrically tropically nonsingular). Viewing A as a point in $\mathbb{R}^{\binom{n}{2}}$ define $S_{A,\epsilon}$ to be the intersection of $S_{n,r}$ with $B_{A,\epsilon}$, an ϵ-ball centered at A:*

$$S_{A,\epsilon} = S_{n,r} \cap B_{A,\epsilon}.$$

For ϵ sufficiently small we have the relation $(dim(S_{n+1,r}) - dim(S_{A,\epsilon})) \geq r-1$.

Proof. The matrix A has an $(r-1) \times (r-1)$ submatrix that is tropically nonsingular. This submatrix is formed by the row indices $\{i_1, \ldots, i_{r-1}\}$ and the column indices $\{j_1, \ldots, j_{r-1}\}$. By Lemma 5.1 there exists a bijection σ from the row indices to the column indices of this submatrix, and coefficients $c_{\sigma(i_1)}, \ldots, c_{\sigma(i_{r-1})}$ such that, for all $k, l \leq r-1$,

$$c_{\sigma(i_k)} \odot a_{i_k,\sigma(i_k)} \leq c_{j_l} \odot a_{i_k,j_l},$$

with equality if and only if $\sigma(i_k) = j_l$.

Construct the matrix A' by appending to the right of A the column defined by

$$\mathbf{a}'_{n+1} = c_{\sigma(i_1)} \odot \mathbf{a}_{\sigma(i_1)} \oplus \cdots \oplus c_{\sigma(i_{r-1})} \odot \mathbf{a}_{\sigma(i_{r-1})},$$

and construct the matrix A'' by appending to the bottom of A' the row defined as a linear combination of rows from A' in the same manner. By Lemma 5.4, the matrix A'' is symmetric and has symmetric tropical rank $r-1$.

For ϵ sufficiently small every matrix in $S_{A,\epsilon}$ will, like A, have a tropically nonsingular $(r-1)\times(r-1)$ submatrix with row indices $\{i_1,\ldots,i_{r-1}\}$ and column indices $\{j_1,\ldots,j_{r-1}\}$. Furthermore, again for ϵ sufficiently small, we can use the coefficients $c_{\sigma(i_1)},\ldots,c_{\sigma(i_{r-1})}$ to define a matrix $B''\in S_{n+1,r}$ for any matrix $B\in S_{A,\epsilon}$. This defines an embedding of $S_{A,\epsilon}$ into $S_{n+1,r}$. Call this embedding $S''_{A,\epsilon}$.

If we tropically multiply both row i and column i of a symmetric matrix by a real number c, then the matrix formed is still symmetric, and has the same symmetric tropical rank as the original matrix. So, for any matrix $B''\in S_{n+1,r}$ we can tropically multiply rows $j_1,j_2,\ldots,j_{r-1}$ by constants $d_{j_1},d_{j_2},\ldots,d_{j_{r-1}}$, and columns $j_1,j_2,\ldots,j_{r-1}$ by the same constants to obtain another point in $S_{n+1,r}$. In this way we construct an $(r-1)$-dimensional linear subspace of $S_{n+1,r}$. Call this linear subspace $L''_{B''}$.

Suppose $B''\in S''_{A,\epsilon}$, and so B'' is the image of a matrix $B\in S_{A,\epsilon}$ under our embedding. The intersection $L''_{B''}\cap S''_{A,\epsilon}$ is just the point B''. To see this, suppose there were another point $C''\in L''_{B''}\cap S''_{A,\epsilon}$. This matrix C'' would be the image of a matrix $C\in S_{A,\epsilon}$, and C would be given by tropically multiplying the rows $j_1,j_2,\ldots,j_{r-1}$ and the columns $j_1,j_2,\ldots,j_{r-1}$ of B by the constants $d_{j_1},\ldots,d_{j_{r-1}}$. The $(i_k,n+1)$ term of the image of C will be

$$(c_{\sigma(i_k)}\odot a_{i_k,\sigma(i_k)})\odot d_{\sigma(i_k)}\odot d_{i_k}$$

if $i_k\in\{j_1,\ldots,j_{r-1}\}$, and

$$(c_{\sigma(i_k)}\odot a_{i_k,\sigma(i_k)})\odot d_{\sigma(i_k)}$$

if not. The $(i_k,n+1)$ term of C' will be

$$(c_{\sigma(i_k)}\odot a_{i_k,\sigma(i_k)})\odot d_{i_k}$$

if $i_k\in\{j_1,\ldots,j_{r-1}\}$, and

$$(c_{\sigma(i_k)}\odot a_{i_k,\sigma(i_k)})$$

if not. In either case, for these terms to be equal we must have $d_{\sigma(i_k)}=0$, and as this must be true for all row indices i_k, and as σ is a bijection from the row indices to the column indices, we have $d_{j_1}=d_{j_2}=\cdots=d_{j_{r-1}}=0$. So, $C=B$, and therefore $C'=B'$.

From this we get $(dim(S_{n+1,r})-dim(S_{A,\epsilon}))\geq r-1$, and our lemma is proven. □

The symmetric version of Lemma 5.8 is very similar to its general counterpart.

Lemma 5.10. $(dim(S_{n+1,r+1}) - dim(S_{n,r})) \geq n+1$.

Proof. For $A \in S_{n,r}$ we define $S_{A,\epsilon}$ in exactly the same manner as in Lemma 5.9. By Lemma 5.6, for any matrix $A \in S_{n,r}$ there is a matrix $A''' \in S_{n+1,r+1}$ defined by

$$A''' = \left(\begin{array}{ccc|c} & & & P \\ & A & & P \\ & & & P \\ \hline P & P & P & M \end{array}\right)$$

where P is larger than any entry in A, and M is smaller than any entry in A. For ϵ sufficiently small, this defines an embedding of $S_{A,\epsilon}$ into $S_{n+1,r+1}$, where the same values of P and M are used for each matrix in the image of this embedding. Call this embedding $S'''_{A,\epsilon}$.

As noted in Lemma 5.9, tropically multiplying a column and row with the same index by a real number does not change the symmetric tropical rank of a matrix. So, for any matrix $B''' \in S'''_{A,\epsilon}$ there is a $n+1$ dimensional subspace of $S_{n+1,r+1}$ formed by tropically multiplying the rows and columns of B''' with the same indices by real numbers. Call this subspace $L'''_{B'''}$. The intersection $L'''_{B'''} \cap S'''_{A,\epsilon}$ is just the matrix B'''. We can see this by noting that for every element of $S'''_{A,\epsilon}$ the right column and bottom row are the same, and the only element of $L'''_{B'''}$ with this given right column and bottom row is the matrix B'''. □

5.3 The Base Cases

In this section we will use the foundational examples from Section 5.1 of this chapter to construct the base cases for our dimension inequalities.

5.3.1 The Standard Case

We now have all the lemmas required to prove the inductive parts of our theorems. We simply require the base cases. To prove these, we note that if A is an $n \times n$ singular matrix, with permutations that realize the tropical determinant $\sigma_1, \sigma_2, \ldots, \sigma_k$, then A, viewed as a point in $\mathbb{R}^{n \times n}$, will be on the linear space determined by the linear equations

$$\begin{aligned} x_{1,\sigma_1(1)} + x_{2,\sigma_1(2)} + \cdots + x_{n,\sigma_1(n)} &= x_{1,\sigma_2(1)} + x_{2,\sigma_2(2)} + \cdots + x_{n,\sigma_1(n)}, \\ x_{1,\sigma_1(1)} + x_{2,\sigma_1(2)} + \cdots + x_{n,\sigma_1(n)} &= x_{1,\sigma_3(1)} + x_{2,\sigma_3(2)} + \cdots + x_{n,\sigma_3(n)}, \\ &\vdots \\ x_{1,\sigma_1(1)} + x_{2,\sigma_1(2)} + \cdots + x_{n,\sigma_1(n)} &= x_{1,\sigma_k(1)} + x_{2,\sigma_k(2)} + \cdots + x_{n,\sigma_k(n)}. \end{aligned}$$

If we intersect this linear space with a sufficiently small ϵ-ball in $\mathbb{R}^{n\times n}$ centered at A, every point in this intersection will correspond with a matrix having the same minimizing permutations as A. The dimension of this intersection will be the dimension of the linear space.

For example, the singular matrix

$$Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

will be on the linear space defined by the linear equation

$$x_{1,1} + x_{2,2} + x_{3,3} = x_{1,2} + x_{2,1} + x_{3,3}.$$

Any matrix on this linear space within a sufficiently small ϵ-ball around Q will also be singular, and will have the same minimizing permutations as Q. Similarly, the singular matrix

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

will be on the linear space defined by the linear equations

$$x_{1,1} + x_{2,3} + x_{3,2} = x_{1,2} + x_{2,3} + x_{3,1},$$
$$x_{1,1} + x_{2,3} + x_{3,2} = x_{1,3} + x_{2,1} + x_{3,2}.$$

Any matrix on this linear space within a sufficiently small ϵ-ball around R will also be singular, and will have the same three minimizing permutations as R.

Extending this idea, if B is an $m\times n$ matrix with tropical rank $r-1$, then for every $r\times r$ submatrix the permutations realizing the tropical determinant determine a linear space, and the intersection of the linear spaces determined by all the $r\times r$ submatrices is again a linear space. If we intersect the linear space determined by all $r\times r$ submatrices with a sufficiently small ϵ-ball in $\mathbb{R}^{m\times n}$ centered at B, then every point in this intersection will correspond with an $m\times n$ matrix with tropical rank $r-1$, for which every $r\times r$ submatrix has the same minimizing permutations as the corresponding submatrix in B. In particular, the dimension of this intersection will be the dimension of the linear space determined by all $r\times r$ submatrices, and the dimension of this linear space cannot be greater than the dimension of the tropical prevariety $T_{m,n,r}$.

Along these lines we examine the matrix

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 2 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{pmatrix},$$

the symmetric version of the matrix from [18]. The minimizing permutations for each 5×5 submatrix determine the linear equations:

$$x_{2,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,2} + x_{3,5} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{2,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,2} + x_{3,6} + x_{4,5} + x_{5,3} + x_{6,4};$$
$$x_{2,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,2} + x_{3,6} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{2,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,1} + x_{3,5} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{2,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,1} + x_{3,6} + x_{4,5} + x_{5,3} + x_{6,4};$$
$$x_{2,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,1} + x_{3,6} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{2,2} + x_{3,5} + x_{4,6} + x_{5,1} + x_{6,4} = x_{2,2} + x_{3,6} + x_{4,5} + x_{5,1} + x_{6,3};$$
$$x_{2,2} + x_{3,5} + x_{4,6} + x_{5,1} + x_{6,3} = x_{2,2} + x_{3,6} + x_{4,5} + x_{5,1} + x_{6,3};$$
$$x_{2,2} + x_{3,1} + x_{4,6} + x_{5,3} + x_{6,4} = x_{2,2} + x_{3,1} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{2,2} + x_{3,1} + x_{4,5} + x_{5,3} + x_{6,4} = x_{2,2} + x_{3,1} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{1,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,5} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,6} + x_{4,5} + x_{5,3} + x_{6,4};$$
$$x_{1,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,6} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{1,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,1} + x_{3,5} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,1} + x_{3,6} + x_{4,5} + x_{5,3} + x_{6,4};$$
$$x_{1,1} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,1} + x_{3,6} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{1,2} + x_{3,5} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,6} + x_{4,5} + x_{5,1} + x_{6,4};$$
$$x_{1,2} + x_{3,5} + x_{4,6} + x_{5,1} + x_{6,3} = x_{1,2} + x_{3,6} + x_{4,5} + x_{5,1} + x_{6,3};$$
$$x_{1,2} + x_{3,1} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,1} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,2} + x_{3,1} + x_{4,5} + x_{5,3} + x_{6,4} = x_{1,2} + x_{3,1} + x_{4,5} + x_{5,4} + x_{6,3};$$
$$x_{1,3} + x_{2,2} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,3} + x_{2,2} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,3} + x_{2,2} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,3} + x_{2,1} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,1} + x_{2,2} + x_{4,6} + x_{5,5} + x_{6,4} = x_{1,2} + x_{2,1} + x_{4,6} + x_{5,5} + x_{6,4};$$
$$x_{1,1} + x_{2,2} + x_{4,6} + x_{5,5} + x_{6,3} = x_{1,2} + x_{2,1} + x_{4,6} + x_{5,5} + x_{6,3};$$
$$x_{1,1} + x_{2,2} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,1} + x_{2,2} + x_{4,6} + x_{5,4} + x_{6,3};$$
$$x_{1,1} + x_{2,2} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{2,1} + x_{4,6} + x_{5,3} + x_{6,4};$$
$$x_{1,1} + x_{2,2} + x_{4,6} + x_{5,3} + x_{6,4} = x_{1,2} + x_{2,1} + x_{4,6} + x_{5,4} + x_{6,3};$$

$$x_{1,1}+x_{2,2}+x_{4,5}+x_{5,3}+x_{6,4}=x_{1,1}+x_{2,2}+x_{4,5}+x_{5,4}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{4,5}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{4,5}+x_{5,3}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{4,5}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{4,5}+x_{5,4}+x_{6,3};$$
$$x_{1,3}+x_{2,2}+x_{3,6}+x_{5,3}+x_{6,4}=x_{1,5}+x_{2,2}+x_{3,6}+x_{5,4}+x_{6,3};$$
$$x_{1,5}+x_{2,1}+x_{3,6}+x_{5,3}+x_{6,4}=x_{1,5}+x_{2,1}+x_{3,6}+x_{5,4}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,6}+x_{5,5}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,6}+x_{5,5}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,6}+x_{5,5}+x_{6,3}=x_{1,2}+x_{2,1}+x_{3,6}+x_{5,5}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,6}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,6}+x_{5,3}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,6}+x_{5,3}+x_{6,4}=x_{1,1}+x_{2,2}+x_{3,6}+x_{5,4}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,6}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,6}+x_{5,4}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,5}+x_{5,3}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{5,3}+x_{6,4}=x_{1,1}+x_{2,2}+x_{3,5}+x_{5,4}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{5,3}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,5}+x_{5,4}+x_{6,3};$$
$$x_{1,3}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,4}=x_{1,3}+x_{2,2}+x_{3,6}+x_{4,5}+x_{6,4};$$
$$x_{1,3}+x_{2,1}+x_{3,5}+x_{4,6}+x_{6,4}=x_{1,3}+x_{2,1}+x_{3,6}+x_{4,5}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,4}=x_{1,1}+x_{2,2}+x_{3,6}+x_{4,5}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,5}+x_{4,6}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,6}+x_{4,5}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,3}=x_{1,1}+x_{2,2}+x_{3,6}+x_{4,5}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,3}=x_{1,2}+x_{2,1}+x_{3,5}+x_{4,6}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{6,3}=x_{1,2}+x_{2,1}+x_{3,6}+x_{4,5}+x_{6,3};$$
$$x_{1,1}+x_{2,2}+x_{3,3}+x_{4,6}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,3}+x_{4,6}+x_{6,4};$$
$$x_{1,1}+x_{2,2}+x_{3,3}+x_{4,5}+x_{6,4}=x_{1,2}+x_{2,1}+x_{3,3}+x_{4,5}+x_{6,4};$$
$$x_{1,3}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,4}=x_{1,3}+x_{2,2}+x_{3,6}+x_{4,5}+x_{5,4};$$
$$x_{1,3}+x_{2,1}+x_{3,5}+x_{4,6}+x_{5,4}=x_{1,3}+x_{2,1}+x_{3,6}+x_{4,5}+x_{5,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,4}=x_{1,2}+x_{2,1}+x_{3,5}+x_{4,6}+x_{5,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,4}=x_{1,1}+x_{2,2}+x_{3,6}+x_{4,5}+x_{5,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,4}=x_{1,2}+x_{2,1}+x_{3,6}+x_{4,5}+x_{5,4};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,3}=x_{1,2}+x_{2,1}+x_{3,5}+x_{4,6}+x_{5,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,3}=x_{1,1}+x_{2,2}+x_{3,6}+x_{4,5}+x_{5,3};$$
$$x_{1,1}+x_{2,2}+x_{3,5}+x_{4,6}+x_{5,3}=x_{1,2}+x_{2,1}+x_{3,6}+x_{4,5}+x_{5,3};$$
$$x_{1,1}+x_{2,2}+x_{3,3}+x_{4,6}+x_{5,4}=x_{1,2}+x_{2,1}+x_{3,3}+x_{4,6}+x_{5,4};$$
$$x_{1,1}+x_{2,2}+x_{3,3}+x_{4,5}+x_{5,4}=x_{1,2}+x_{2,1}+x_{3,3}+x_{4,5}+x_{5,4}.$$

The linear space determined by these linear equations has dimension 33. The linear equations coming from the 4×4 submatrices of the matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

the symmetric version of the cocircuit matrix of the Fano matroid from [8], are too numerous to be practical to list, but the linear space they determine has dimension 34.

5.3.2 The Symmetric Case

For symmetric matrices we can apply the same analysis. The only difference is the relation $x_{i,j} = x_{j,i}$ on the variables, and that the space of $n \times n$ symmetric matrices is, consequently, equivalent to $\mathbb{R}^{\binom{n}{2}}$. For the 6×6 symmetric matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

we get the linear equations:

$$x_{2,2} + 2x_{3,5} + 2x_{4,6} = x_{2,2} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{2,2} + 2x_{3,5} + 2x_{4,6} = x_{2,2} + 2x_{3,6} + 2x_{4,5};$$
$$x_{1,2} + 2x_{3,5} + 2x_{4,6} = x_{1,2} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,2} + 2x_{3,5} + 2x_{4,6} = x_{1,2} + 2x_{3,6} + 2x_{4,5};$$
$$x_{1,5} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,5} + x_{2,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,5} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,5} + x_{2,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,3} + x_{2,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,3} + x_{2,2} + x_{3,6} + 2x_{4,5};$$
$$x_{1,2} + 2x_{3,5} + 2x_{4,6} = x_{1,2} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,2} + 2x_{3,5} + 2x_{4,6} = x_{1,2} + 2x_{3,6} + 2x_{4,5};$$
$$x_{1,1} + 2x_{3,5} + 2x_{4,6} = x_{1,1} + x_{3,5} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + 2x_{3,5} + 2x_{4,6} = x_{1,1} + 2x_{3,6} + 2x_{4,5};$$
$$x_{1,2} + 2x_{3,5} + 2x_{4,6} = x_{1,2} + x_{1,5} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,2} + x_{1,5} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,2} + x_{1,5} + 2x_{3,6} + x_{4,5};$$
$$x_{1,2} + x_{1,3} + x_{3,5} + 2x_{4,6} = x_{1,2} + x_{1,3} + x_{3,6} + x_{4,5} + x_{4,6};$$

$$x_{1,2} + x_{1,3} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,2} + x_{1,3} + x_{3,6} + 2x_{4,5};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,3} + x_{2,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,2} + x_{1,3} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + 2x_{4,6} + x_{5,5} = 2x_{1,2} + 2x_{4,6} + x_{5,5};$$
$$x_{1,1} + x_{2,2} + x_{3,6} + x_{4,6} + x_{5,5} = 2x_{1,2} + x_{3,6} + x_{4,6} + x_{5,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,1} + x_{2,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = 2x_{1,2} + x_{3,5} + 2x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = 2x_{1,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,1} + x_{2,2} + x_{3,6} + 2x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,6} + 2x_{4,5};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,5} + x_{2,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,2} + x_{1,5} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,2} + x_{1,5} + 2x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,6} + x_{4,6} + x_{5,5} = 2x_{1,2} + x_{3,6} + x_{4,6} + x_{5,5};$$
$$x_{1,1} + x_{2,2} + 2x_{3,6} + x_{5,5} = 2x_{1,2} + 2x_{3,6} + x_{5,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{3,6} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,1} + x_{2,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = 2x_{1,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = 2x_{1,2} + 2x_{3,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{3,6} + x_{4,5};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,3} + x_{2,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,2} + x_{1,3} + x_{3,5} + 2x_{4,6} = x_{1,2} + x_{1,3} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = x_{1,1} + x_{2,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = 2x_{1,2} + x_{3,5} + 2x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + 2x_{4,6} = 2x_{1,2} + x_{3,6} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = x_{1,1} + x_{2,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{3,6} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,6} = 2x_{1,2} + 2x_{3,6} + x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,3} + 2x_{4,6} = 2x_{1,2} + x_{3,3} + 2x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,3} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,3} + x_{4,5} + x_{4,6};$$
$$x_{1,3} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,3} + x_{2,2} + x_{3,6} + 2x_{4,5};$$
$$x_{1,2} + x_{1,3} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,2} + x_{1,3} + x_{3,6} + 2x_{4,5};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{4,5} + x_{4,6};$$
$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = x_{1,1} + x_{2,2} + x_{3,6} + 2x_{4,5};$$

$$x_{1,1} + x_{2,2} + x_{3,5} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,6} + 2x_{4,5};$$

$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = 2x_{1,2} + 2x_{3,5} + x_{4,6};$$

$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = x_{1,1} + x_{2,2} + x_{3,5} + x_{3,6} + x_{4,5};$$

$$x_{1,1} + x_{2,2} + 2x_{3,5} + x_{4,6} = 2x_{1,2} + x_{3,5} + x_{3,6} + x_{4,5};$$

$$x_{1,1} + x_{2,2} + x_{3,3} + x_{4,5} + x_{4,6} = 2x_{1,2} + x_{3,3} + x_{4,5} + x_{4,6};$$

$$x_{1,1} + x_{2,2} + x_{3,3} + 2x_{4,5} = 2x_{1,2} + x_{3,3} + 2x_{4,5}.$$

The linear space determined by these linear equations has dimension 19. Note also that the principal 4×4 submatrix with row/column indices $\{2, 3, 4, 5\}$ is tropically singular, and not just symmetrically tropically singular. So, Lemma 5.9 can be applied.

We can modify the 13×13 example from earlier in this chapter to get the matrix

$$\begin{pmatrix}
-5 & -5 & -5 & -5 & -5 & -5 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
-5 & -5 & -5 & -5 & -5 & -5 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
-5 & -5 & -5 & -5 & -5 & -5 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
-5 & -5 & -5 & -5 & -5 & -5 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
-5 & -5 & -5 & -5 & -5 & -5 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
-5 & -5 & -5 & -5 & -5 & -5 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & -10 & -10 & -10 & -10 & -10 & -10 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & -10 & -10 & -10 & -10 & -10 & -10 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & -10 & -10 & -10 & -10 & -10 & -10
\end{pmatrix}.$$

This matrix has symmetric tropical rank three. The linear space determined by the 4×4 submatrices has at least dimension 36. One way of seeing this is that any of the 21 terms equal to 1 in the upper-right (and, symmetrically, bottom-left) 7×7 submatrix can be modified slightly, all the terms in the upper-left or bottom-right 6×6 submatrices can be modified slightly by the same amount, and any of the 13 row/column pairs can modified slightly by the same amount, all without affecting the symmetric tropical rank. This gives us $21 + 2 + 13 = 36$ dimensions. I believe this is the local dimension of $S_{13,4}$ around this matrix, and it is the same as the dimension of $\tilde{S}_{13,4}$. So, I suspect that the dimension of the tropical prevariety $S_{13,4}$ is equal to the dimension of the tropical variety $\tilde{S}_{13,4}$, even though the first set properly contains the second. However, this has not yet been proven.

5.4 The Dimension Theorems

We now have everything we need to prove the main theorems of this chapter.

5.4.1 The Standard Case

We now have everything we need in order to prove the main theorems of this chapter. The first theorem is about standard matrices, their determinantal varieties, and the dimensions of the associated tropical varieties and tropical prevarieties.

Theorem 5.11. *If $m_1, \ldots, m_s$ are the $r \times r$ minors of an $m \times n$ matrix of variables, and $I_{m,n,r} = (m_1, \ldots, m_s)$ is the corresponding determinantal ideal, then the minors are a tropical basis if and only if the dimension of the tropical variety $\mathcal{T}(\boldsymbol{V}(I_{m,n,r}))$ is equal to the dimension of the corresponding tropical prevariety $\cap_{i=1}^{s} \boldsymbol{V}(\mathcal{T}(m_i))$.*

Proof. Denote by $M_{m,n,r}$ the affine determinantal variety of $m \times n$ matrices with rank less than r. It is a standard result in algebraic geometry ([11] Proposition 12.2, for example) that the dimension of $M_{m,n,r}$ is $(m+n-r+1)(r-1)$. It was proven in [3], or earlier in [2], that the tropical variety $\tilde{T}_{m,n,r}$ is a pure polyhedral fan with dimension equal to that of $M_{m,n,r}$.

Using these formulas and our results from Section 5.3 we compute

$$dim(\tilde{T}_{6,6,5}) = (6+6-5+1)(5-1) = 32 < 33 \leq dim(T_{6,6,5}),$$

and

$$dim(\tilde{T}_{7,7,4}) = (7+7-4+1)(4-1) = 33 < 34 \leq dim(T_{7,7,4}).$$

Again, using these formulas we get

$$dim(\tilde{T}_{m+1,n,r}) - dim(\tilde{T}_{m,n,r}) = (m+n-r+2)(r-1) - (m+n-r+1)(r-1) = r-1,$$

similarly,

$$dim(\tilde{T}_{m,n+1,r}) - dim(\tilde{T}_{m,n,r}) = (m+n-r+2)(r-1) - (m+n-r+1)(r-1) = r-1,$$

and,

$$dim(\tilde{T}_{m+1,n+1,r+1}) - dim(\tilde{T}_{m,n,r}) - (m+n-r+2)r - (m+n-r+1)(r-1) = m+n+1.$$

These, combined with Lemmas 5.7 and 5.8, prove that if $dim(\tilde{T}_{m,n,r}) < dim(T_{m,n,r})$ then

$$dim(\tilde{T}_{m+1,n,r}) < dim(T_{m+1,n,r}),$$
$$dim(\tilde{T}_{m,n+1,r}) < dim(T_{m,n+1,r}),$$
$$dim(\tilde{T}_{m+1,n+1,r+1}) < dim(T_{m+1,n+1,r+1}).$$

From these results we may conclude $dim(\tilde{T}_{m,n,r}) < dim(T_{m,n,r})$ when $min(m,n) = 6$ and $r = 5$, or when $min(m,n) > 6$ and $4 \leq r < min(m,n)$. This covers all cases where the $r \times r$ minors do not form a tropical basis. □

5.4.2 The Symmetric Case

We have a similar theorem for symmetric matrices.

Theorem 5.12. *If $m_1, \ldots, m_s$ are the $r \times r$ minors of an $n \times n$ symmetric matrix of variables, and $J_{n,r} = (m_1, \ldots, m_s)$ is the corresponding determinantal ideal, then for $4 < r < n$ the dimension of the tropical variety $\mathcal{T}(\boldsymbol{V}(J_{n,r}))$ is less than the dimension of the corresponding tropical prevariety $\cap_{i=1}^{s} \boldsymbol{V}(\mathcal{T}(m_i))$.*

Proof. Denote by $Q_{n,r}$ the affine determinantal variety of symmetric $n \times n$ matrcies of rank less than r. It is a standard result in algebraic geometry ([11] Chapter 22, Page 299) that the dimension of $Q_{n,r}$ is $(2nr - 2n + 3r - r^2 - 2)/2$. As in the previous theorem, the tropical variety $\tilde{S}_{n,r}$ is a pure polyhedral fan with dimension equal to that of $Q_{n,r}$.

Using these formulas and our results from Section 5.3 we compute

$$dim(\tilde{S}_{6,5}) = (60 - 12 + 15 - 25 - 2)/2 = 18 < 19 \leq dim(S_{6,5}).$$

Again, using these formulas we get

$$\begin{gathered} dim(\tilde{S}_{n+1,r}) - dim(\tilde{S}_{n,r}) \\ = \frac{2(n+1)r - 2(n+1) + 3r - 2 - r^2}{2} - \frac{2nr - 2n + 3r - 2 - r^2}{2} = r - 1, \end{gathered}$$

and,

$$\begin{gathered} dim(\tilde{S}_{n+1,r+1}) - dim(\tilde{S}_{n,r}) \\ = \frac{2(n+1)(r+1) - 2(n+1) + 3(r+1) - 2 - (r+1)^2}{2} - \frac{2nr - 2n + 3r - 2 - r^2}{2} \\ = n + 1. \end{gathered}$$

These, combined with Lemmas 5.9 and 5.10, prove that if $dim(\tilde{S}_{n,r}) < dim(S_{n,r})$, then

$$dim(\tilde{S}_{n+1,r}) < dim(S_{n+1,r}),$$

and

$$dim(\tilde{S}_{n+1,r+1}) < dim(S_{n+1,r+1}).$$

From these results we may conclude $dim(\tilde{S}_{n,r}) < dim(S_{n,r})$ when $4 < r < n$. □

If the dimension of the tropical variety defined by a set of polynomials is smaller than the dimension of the tropical prevariety defined by those polynomials, then obviously the polynomials do not form a tropical basis. Combining Theorem 5.12 with the result discussed after Corollary 5.5 we arrive at the following theorem.

Theorem 5.13. *The $r \times r$ minors of an $n \times n$ symmetric matrix are* not *a tropical basis for $r = 4$ and $n \geq 13$, or for $4 < r < n$.*

CHAPTER 6

TROPICAL QUADRICS

In classical algebraic geometry a quadric is a hypersurface in $\mathbb{P}^{n-1}$ defined by a homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of degree two

$$f = a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n + a_{22}x_2^2 + a_{23}x_2x_3 + \cdots + a_{nn}x_n^2.$$

For each such polynomial there is a corresponding symmetric matrix, A, defined by the relations

$$f = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \cdots & \frac{1}{2}a_{1n} \\ \frac{1}{2}a_{12} & a_{22} & \cdots & \frac{1}{2}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}a_{1n} & \frac{1}{2}a_{2n} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{x}^T A\mathbf{x}.$$

The hypersurface $\mathbf{V}(f)$ is singular if and only if the corresponding symmetric matrix is singular, and the rank of a quadric is defined to be the rank of the corresponding symmetric matrix.

In this chapter we touch upon the tropical version of quadrics. As compared to proving theorems, this chapter is designed more to illustrate and explore a potential area where the ideas about symmetric tropical matrices developed in this dissertation can be applied, and to suggest ideas for future theorems.

6.1 Determinantal Profiles and Dual Complexes

A tropical quadric is a hypersurface in $\mathbb{TP}^{n-1}$ defined by a homogeneous tropical polynomial of degree two, and like its classical counterpart a tropical quadric will also have a corresponding symmetric tropical matrix (without the annoying $\frac{1}{2}$ multiples on the off-diagonal entries). We can require without loss of generality that the terms of the matrix satisfy $a_{ij} \otimes a_{ji} = a_{ij}^2 \leq a_{ii} \otimes a_{jj}$. As in the classical case, a tropical quadric will be nonsingular if and only if its corresponding symmetric matrix is symmetrically nonsingular.

In regard to the symmetric matrix that corresponds with a tropical quadric, we will not only be interested in the cycle classes (recall the terminology from Chapter 2) that realize the determinant of the symmetric matrix, but also the cycle classes that realize the determinant of its principal submatrices. For a symmetric matrix A, let $A[i_1, i_2, \ldots, i_k]$ represent the principal submatrix formed by the rows and columns with indices $i_1, i_2, \ldots, i_k$. The permutations realizing the determinant for this submatrix will be expressed as permutations of the indices $\{i_1, i_2, \ldots, i_k\}$. As an example, for the matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the permutation realizing the determinant of the submatrix

$$C[2,3] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is (23).

For an $n \times n$ symmetric matrix fix an ordering on all the principal submatrices of size at least 2×2, such that larger submatrices come before smaller submatrices in the ordering. For example; $C[1,2,3], C[1,2], C[1,3], C[2,3]$. For any $n \times n$ symmetric matrix with a given principal submatrix ordering, the *determinantal profile* of the matrix is an ordered list in which the elements in the list correspond with the principal submatrices in the submatrix ordering, and each element in the list contains the cycle classes that realize the determinant of the corresponding principal submatrix. We display a cycle class using a disjoint cycle decomposition of a representative permutaiton from the class. Using the principal submatrix ordering given above, the matrix C has the determinantal profile

$$\{\{(123)\}, \{(12)\}, \{(13)\}, \{(23)\}\}.$$

Using the same principal submatrix ordering (which we will use for all 3×3 symmetric matrices), the determinantal profile of the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is

$$\{\{(13), (23), (123)\}, \{id, (12)\}, \{(13)\}, \{(23)\}\}.$$

Two determinantal profiles are *conjugate* if one can be obtained from the other by a relabeling of the n indices. A determinantal profile is *totally nonsingular* if each element in the list contains only one cycle class. In other words, the determinantal profile of a symmetric matrix is totally nonsingular if and only if the symmetric matrix, and each of its principal submatrices, is symmetrically nonsingular.

For any tropical hypersurface there is a corresponding polyhedral complex called the *dual complex*. If F is a tropical polynomial defining the hypersurface, then for every monomial $X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m}$ in F there is a corresponding integral point $(a_1, a_2, \cdots, a_m) \in \mathbb{N}^m$, and the facets of the dual complex are formed from the convex hulls of the points corresponding with monomials that can be simultaneously minimized. We say two tropical hypersurfaces are *combinatorially equivalent* if their corresponding dual complexes are combinatorially equivalent, and *combinatorially distinct* otherwise.

6.2 Exploring Tropical Conics

For tropical conics, which are tropical quadrics in $\mathbb{TP}^2$, there are 20 possible dual complexes, 7 of which are combinatorially distinct. These are all pictured below, with a representative tropical conic superimposed upon the dual complex, a tropical polynomial defining each conic, the corresponding symmetric matrix, and the determinantal profile of each symmetric matrix. A *type* of tropical conic is the set of conics with a given dual complex.

First, there are four types of tropical conic with corresponding symmetric matrices that are totally nonsingular. These types fall into two combinatorially distinct classes. The first class has a single type. This type is illustrated in Figure 6.1.

The second class has three combinatorially equivalent types. These are illustrated in Figure 6.2.

There are six combinatorially equivalent types of tropical conic with corresponding symmetric matrices that are symmetrically nonsingular, but have one symmetrically singular 2×2 principal submatrix. These are illustrated in Figure 6.3.

There are three types of combinatorially equivalent tropical conic with corresponding symmetric matrices that are symmetrically nonsingular, but have two symmetrically singu-

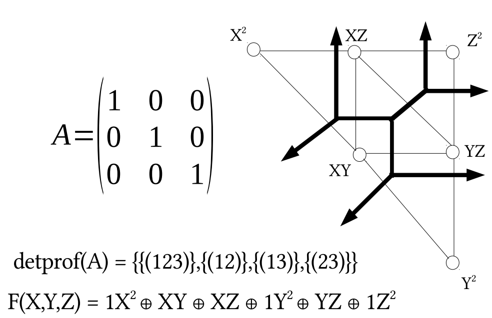


Figure 6.1. Combinatorially distinct class of tropical conics 1

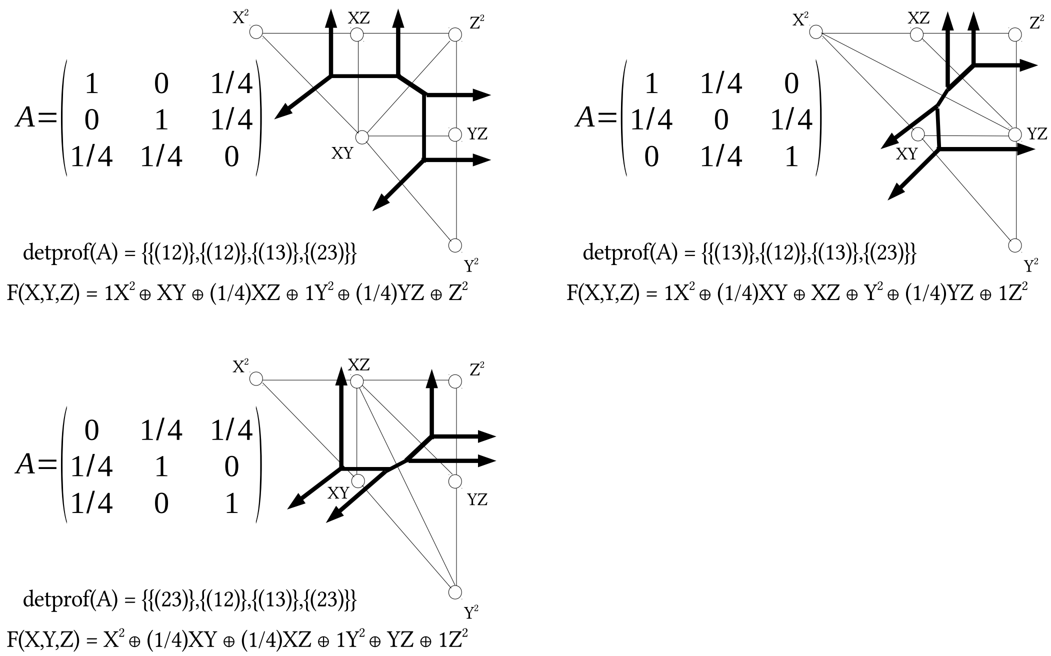


Figure 6.2. Combinatorially distinct class of tropical conics 2

lar 2×2 principal submatrices. These are illustrated in Figure 6.4.

There are three types of combinatorially equivalent tropical conic with corresponding symmetric matrices that are symmetrically singular, but have no symmetrically singular 2×2 principal submatrices. These are all pairs of tropical lines intersecting at a point. These are illustrated in Figure 6.5.

There are three types of combinatorially equivalent tropical conic with corresponding symmetric matrices that are symmetrically singular, and have one symmetrically singular 2×2 principal submatrix. These are all pairs of tropical lines intersecting at a ray. These are illustrated in Figure 6.6.

Finally, there is the type of tropical conic with corresponding symmetric matrix that is symmetrically singular, and all 2×2 principal submatrices are symmetrically singular as well. This is the double line. It is illustrated in Figure 6.7.

There is a bijection between the dual complexes of the conics and the determinantal profiles of the symmetric matrices, and two complexes are combinatorially equivalent if and only if their corresponding determinantal profiles are conjugate. It would be interesting to see if this relation held for general conics.

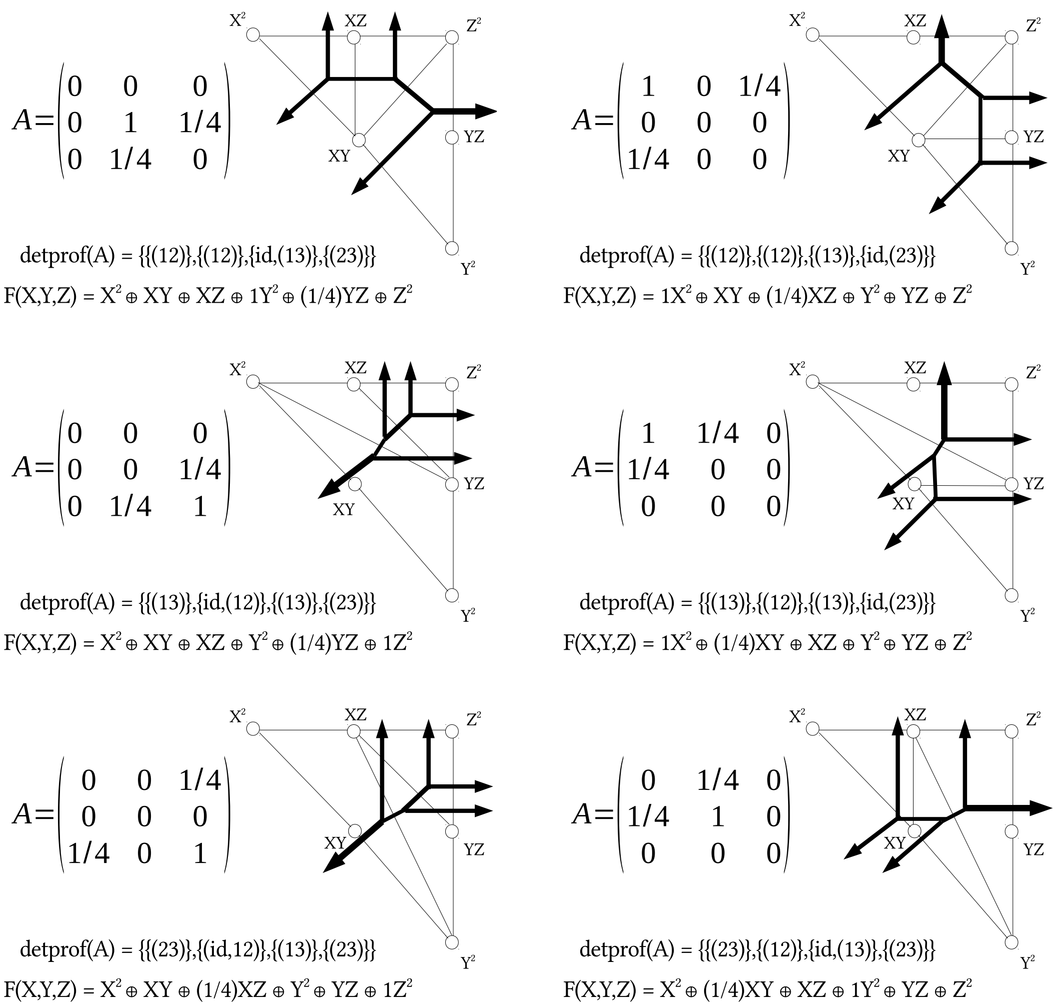
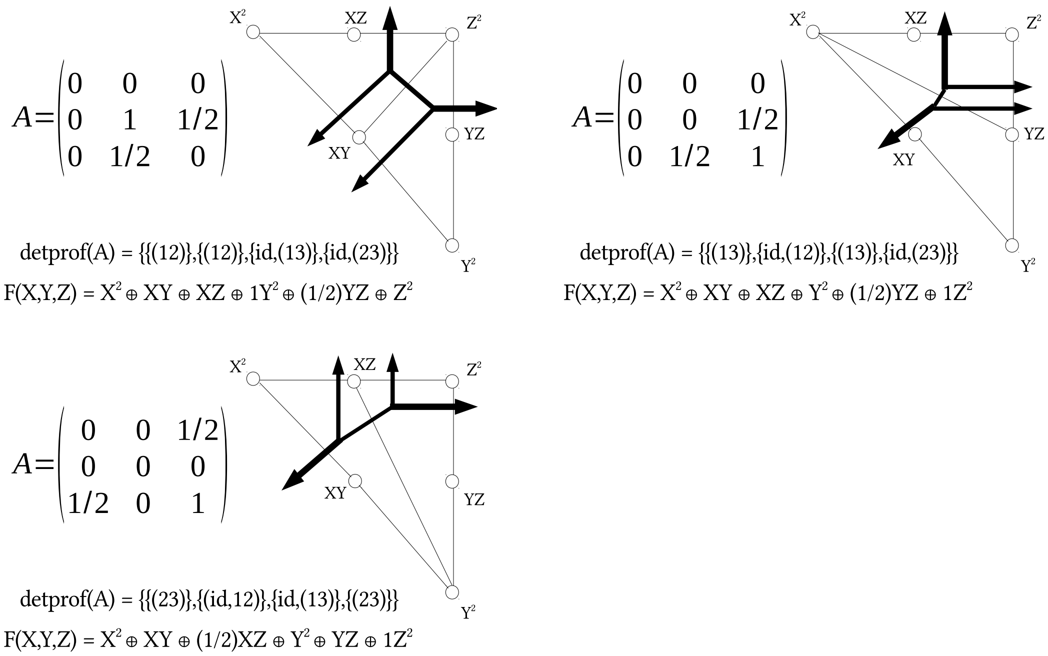


Figure 6.3. Combinatorially distinct class of tropical conics 3



$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 1/2\\ 0 & 1/2 & 0\end{pmatrix}$$

detprof(A) = {{(12)},{(12)},{id,(13)},{id,(23)}}

$F(X,Y,Z) = X^2 \oplus XY \oplus XZ \oplus 1Y^2 \oplus (1/2)YZ \oplus Z^2$

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 1/2\\ 0 & 1/2 & 1\end{pmatrix}$$

detprof(A) = {{(13)},{id,(12)},{(13)},{id,(23)}}

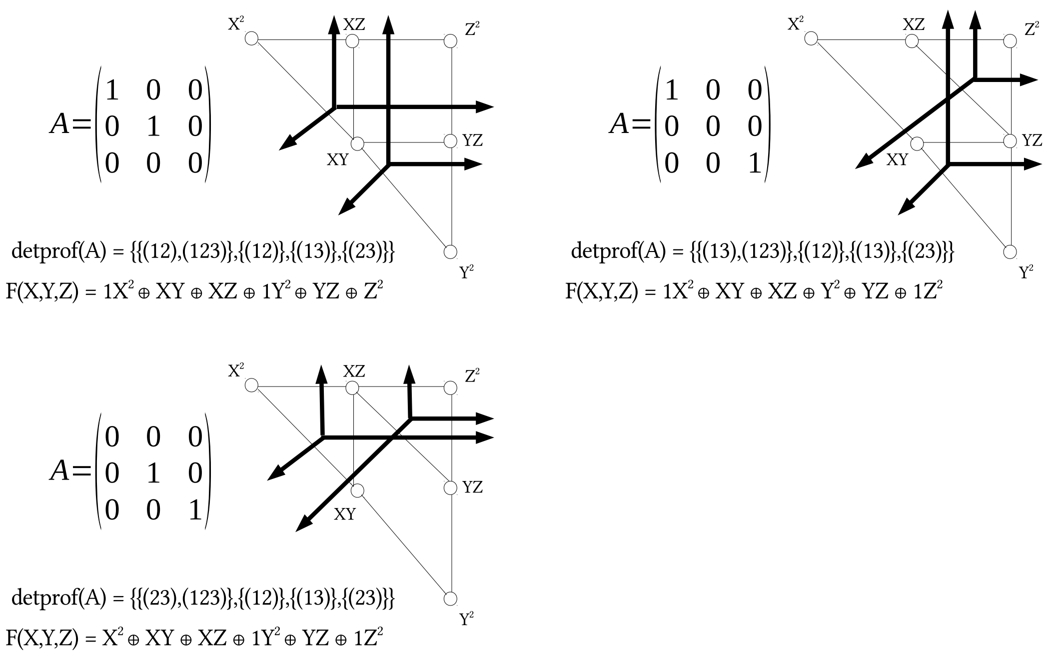
$F(X,Y,Z) = X^2 \oplus XY \oplus XZ \oplus Y^2 \oplus (1/2)YZ \oplus 1Z^2$

$$A=\begin{pmatrix}0 & 0 & 1/2\\ 0 & 0 & 0\\ 1/2 & 0 & 1\end{pmatrix}$$

detprof(A) = {{(23)},{(id,12)},{id,(13)},{(23)}}

$F(X,Y,Z) = X^2 \oplus XY \oplus (1/2)XZ \oplus Y^2 \oplus YZ \oplus 1Z^2$

Figure 6.4. Combinatorially distinct class of tropical conics 4



$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

detprof(A) = {{(12),(123)},{(12)},{(13)},{(23)}}

$F(X,Y,Z) = 1X^2 \oplus XY \oplus XZ \oplus 1Y^2 \oplus YZ \oplus Z^2$

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

detprof(A) = {{(13),(123)},{(12)},{(13)},{(23)}}

$F(X,Y,Z) = 1X^2 \oplus XY \oplus XZ \oplus Y^2 \oplus YZ \oplus 1Z^2$

$$A=\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

detprof(A) = {{(23),(123)},{(12)},{(13)},{(23)}}

$F(X,Y,Z) = X^2 \oplus XY \oplus XZ \oplus 1Y^2 \oplus YZ \oplus 1Z^2$

Figure 6.5. Combinatorially distinct class of tropical conics 5

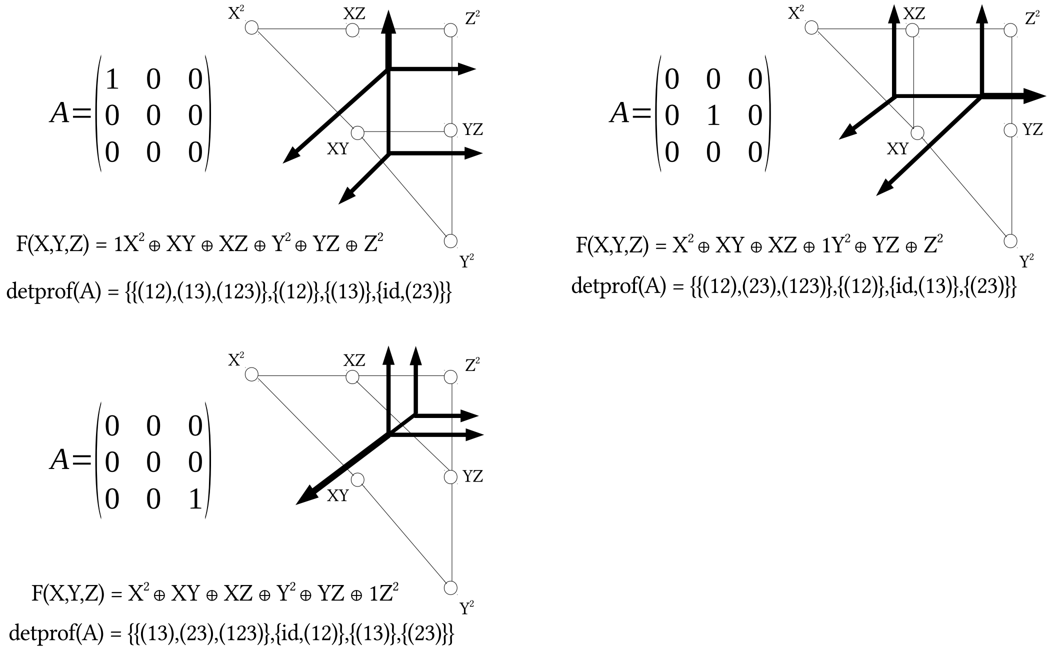


Figure 6.6. Combinatorially distinct class of tropical conics 6

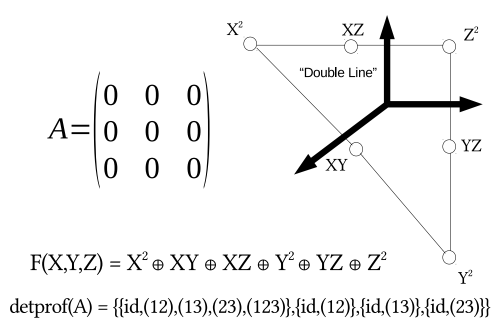


Figure 6.7. Combinatorially distinct class of tropical conics 7

CHAPTER 7

FURTHER QUESTIONS

In this concluding chapter we summarize and state a number of questions about symmetric tropical matrices that remain open, and are potential avenues for further work and study.

7.1 Tropical Bases for Symmetric Matrices

The question of when the $r \times r$ minors of an $m \times n$ standard matrix form a tropical basis was answered by a series of results given in [8], [7], [18], and [19], and can be summarized by the following theorem.

Theorem 7.1 (Theorem 1.11 from [19])**.** *The $r \times r$ minors of an $m \times n$ standard matrix form a tropical basis if and only if at least one of the following conditions hold:*

1. $r \leq 3$;
2. $r = min(m, n)$;
3. $r = 4$ *and* $min(m, n) \leq 6$.

This theorem is summarized in Table 7.1.

For symmetric matrices, combining the results from Theorems 3.1, 3.2, 3.6, 4.16, and 5.13 we obtain the following theorem.

Table 7.1. Do the $r \times r$ minors of an $m \times n$ standard matrix form a tropical basis?

$r, min(m,n)$	2	3	4	5	6	7	8
2	yes	yes	yes	yes	yes	yes	yes
3		yes	yes	yes	yes	yes	yes
4			yes	yes	yes	no	no
5				yes	no	no	no
6					yes	no	no
7						yes	no
8							yes

Theorem 7.2. *The $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis if at least one of the following conditions hold:*

1. $r \leq 3$;
2. $r = n$;
3. $r = 4$ *and* $n = 5$.

The $r \times r$ minors of an $n \times n$ symmetric matrix do not *form a tropical basis if $r = 4$ and $n \geq 13$, or if $4 < r < n$.*

This theorem is summarized in Table 7.2.

As can be seen from Table 7.2, the fundamental question posed in [7] of when the $r \times r$ minors of a symmetric $n \times n$ matrix form a tropical basis still lacks a complete answer. As with standard matrices, $r = 4$ is a special, transition case, and for symmetric matrices it is still not completely understood. This question is probably the most substantial one that remains to be answered about symmetric tropical matrices.

Question 7.3. Do the 4×4 minors of an $n \times n$ symmetric matrix form a tropical basis for $5 < n < 13$?

Note that a negative answer for any n in this range would imply a negative answer for all greater n, and, consequently, an affirmative answer for any n in this range would imply an affirmative answer for all smaller n. In particular, an affirmative answer for $n = 12$ would completely answer the question. As given in Conjecture 4.17, I suspect that the "method

Table 7.2. Do the $r \times r$ minors of an $n \times n$ symmetric matrix form a tropical basis?

r, n	2	3	4	5	6	7	8	9	10	11	12	13	14
2	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
3		yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes	yes
4			yes	yes	?	?	?	?	?	?	?	no	no
5				yes	no	no	no	no	no	no	no	no	no
6					yes	no	no	no	no	no	no	no	no
7						yes	no	no	no	no	no	no	no
8							yes	no	no	no	no	no	no
9								yes	no	no	no	no	no
10									yes	no	no	no	no
11										yes	no	no	no
12											yes	no	no
13												yes	no
14													yes

of joints" from Chapter 4 can be generalized and used to answer the question affirmatively up to $n = 12$.

7.2 The Dimensions of Determinantal Prevarieties

In Chapter 5 we proved for standard matrices that the tropical prevariety determined by the $r \times r$ minors of an $m \times n$ matrix are a tropical basis if and only if the dimension of the tropical prevariety is equal to the dimension of the tropical variety determined by the same minors.

In the situations where the prevariety has dimension greater than the variety we only provided a lower bound on the dimension of the prevariety. We did not actually calculate it. These exact dimensions remain unknown.

Question 7.4. When the $r \times r$ minors of an $m \times n$ matrix are not a tropical basis what is the dimension of the tropical prevariety they determine?

For symmetric matrices we can ask a similar question.

Question 7.5. For $r > 4$ when the $r \times r$ minors of an $n \times n$ symmetric matrix are not a tropical basis what is the dimension of the tropical prevariety they determine?

For symmetric matrices of tropical symmetric rank three whether the dimension of the prevariety exceeds the dimension of the variety remains unknown.

Question 7.6. For $n > 12$ does the dimension of the tropical prevariety determined by the 4×4 minors of an $n \times n$ matrix exceed the dimension of the tropical variety determined by the minors?

For reasons given in Chapter 5 I suspect the answer to this question is no, for at least $n = 13$.

7.3 Dual Complexes of Tropical Quadrics

In Chapter 6 we explored tropical conics, and discovered that there is a correspondence between the dual complex of a tropical conic and the determinantal profile of the symmetric matrix corresponding with the conic. It is natural to wonder whether the results we found for conics generalize to all quadrics.

Question 7.7. Is the dual complex of a tropical quadric completely determined by the determinantal profile of the symmetric matrix corresponding with the quadric? If so, how many combinatorially distinct dual complexes are there for quadrics of a given dimension?

I suspect the answer to the first part of this question is affirmative, and a proof can be obtained by appropriately modifying the proof of the following theorem from [9].

Theorem 7.8 (Theorem 1 from [9]). *The combinatorial types of tropical complexes generated by a set of r vertices in $\mathbb{TP}^{n-1}$ are in natural bijection with the regular polyhedral subdivisions of the product of two simplices $\Delta_{n-1} \times \Delta_{r-1}$.*

It would also be worthwhile to verify that defining a tropical quadric to be singular if its corresponding symmetric matrix is symmetrically singular aligns with the definition of a singular tropical hypersurface given by Dickenstein and Tabera [10].

7.4 Shellability of Symmetric Rank Two Matrices

Marwig and Yu proved in [14] that the space of tropically collinear points is shellable. This is equivalent to the statement that the space of matrices with tropical rank two is shellable. We can ask a similar question for symmetric matrices.

Question 7.9. Is the space of symmetric matrices with symmetric tropical rank two shellable?

I suspect the answer is affirmative, and that this can be proven by modifying the argument given in [14].

7.5 Computing and Comparing Symmetric Ranks

We saw in Chapter 2 that a symmetric matrix can have different tropical rank and symmetric tropical rank. This leads naturally to the following question.

Question 7.10. Is the symmetric tropical rank of a symmetric matrix bounded by its standard tropical rank?

The same question can be asked for Kapranov rank. I suspect the answer is no.

There are also some questions that have been answered for standard tropical matrices that can be asked again for symmetric tropical matrices.

Question 7.11. Can the symmetric tropical rank of a symmetric matrix be computed in polynomial time?

In [1] it was proven that calculating standard tropical rank of a zero-one matrix is NP-complete, and I think it is very likely the same is true for symmetric tropical rank, which would make no the answer to this question.

Question 7.12. Is the symmetric Kapranov rank of a symmetric matrix bounded by the symmetric tropical rank?

In [12] it was proven that Kapranov rank is not bounded by tropical rank, and that determining the Kapranov rank of a tropical matrix is NP-hard over any infinite field. I suspect the same is true for symmetric Kapranov and symmetric tropical rank.

7.6 Other Special Matrices

We can ask about when the $r \times r$ minors of a special matrix form a tropical basis for all sorts of special matrices outside symmetric ones. For example, in [7] in addition to asking the tropical basis question about the minors of a symmetric matrix they ask the same question, which we repeat, about the minors of a Hankel (also known as catalecticant) matrix.

Question 7.13. When do the $r \times r$ minors of an $n \times n$ Hankel matrix form a tropical basis?

We can even more generally ask the above question for the matrices which define secant varieties of rational normal curves.

So, as can be seen, many questions still exist about symmetric tropical matrices, and there remains work to be done.

APPENDIX A

CORRECTION TO A PROOF IN "ON THE RANK OF A TROPICAL MATRIX" BY DEVELIN, SANTOS, AND STURMFELS

In their paper "On The Rank of a Tropical Matrix" Develin, Santos, and Sturmfels [8] prove the following lemma, where $\tilde{K}$ is the field of all formal power series $c_1 t^{a_1} + c_2 t^{a_2} + \cdots$ where the a_i can be real numbers.

Lemma A.1 (Lemma 6.3 from [8])**.** *Let A be a nonnegative matrix with no zero column and suppose that the smallest entry in A occurs most frequently in the first column. Let $\tilde{A}$ be the matrix*

$$\begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & A \end{pmatrix}$$

obtained by adjoining a row and a column of zeroes. If $\tilde{A}$ has Kapranov rank two, then $\tilde{A}$ has a rank-2 lift $F \in \tilde{K}^{d \times n}$ in which every 2×2 submatrix is nonsingular and the i-th column can be written as a lienar combination $\lambda_i u_1 + \mu_i u_2$ of the first two columns u_1 and u_2, with $deg(\lambda_i) \geq deg(u_i) = 0$.

This lemma is then used in what I believe is an incorrect proof of the following corollary.

Corollary A.2 (Corollay 6.4 from [8])**.** *Let A and B be nonnegative matrices. Assume that the two matrices*

$$\tilde{A} := \begin{pmatrix} A & \boldsymbol{0} \\ \boldsymbol{0} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} := \begin{pmatrix} 0 & \boldsymbol{0} \\ \boldsymbol{0} & B \end{pmatrix}$$

have Kapranov rank two. Then, the matrix

$$M := \begin{pmatrix} A & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & 0 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & B \end{pmatrix}$$

has Kapranov rank two as well.

The proof given below is the proof from [8], where the only modification is the correction of a few typos.

Proof. We may assume that neither A nor B has a zero column. Hence Lemma 6.3 applies to both of them. We number the rows of M from $-k$ to k' and its columns from $-l$ to l', where $k \times l$ and $k' \times l'$ are the dimensions of A and B, respectively. In this way, A (respectively, B) is the submatrix of negative (respectively, positive) indices. The row and column indexed zero consists of all zeroes. To further exhibit the symmetry between A and B the columns and rows in $\tilde{A}$ will be referrred to "in reverse." That is to say, the first and second columns of it are the ones indexed 0 and -1 in M.

We now construct a lifting $F = (a_{i,j}) \in \tilde{K}^{d\times n}$ of M. We assume that we are given rank two lifts of $\tilde{A}$ and $\tilde{B}$ which satisfy the conditions of the previous lemma. Furthermore, we assume that the lift of the entry $(0, 0)$ is the same in both, which can be achieved by scaling the first row in one of them.

We use exactly those lifts of $\tilde{A}$ and $\tilde{B}$ for the upper-left and bottom-right corner submatrices of M. Our task is to complete that with an entry $a_{i,j}$ for every i, j with $ij < 0$, such that $deg(a_{i,j}) = 0$ and the whole matrix still has rank two. We claim that it suffices to choose the entry $a_{-1,1}$ of degree zero and sufficiently generic. That this choice fixes the rest of the matrix is easy to see: The entry $a_{1,-1}$ is fixed by the fact that the 3×3 submatrix

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

needs to have rank two. All other entries $a_{i,-1}$ and $a_{i,1}$ are fixed by the fact that the entries $a_{i,-1}, a_{i,0}$ and $a_{i,1}$ (two of which come from either $\tilde{A}$ or $\tilde{B}$) must satisfy the same dependence as the three columns of the submatrix above. For each $j = -l, \dots, -2$ (respectively, $j = 2, \dots, l'$), let λ_j and μ_j be the coefficients in the expression of the j-th column of $\tilde{A}$ (respectively, of $\tilde{B}$) as $\lambda_j u_0 + \mu_j u_{-1}$ (respectively, $\lambda_j u_0 + \mu_j u_1$). Then $a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,-1}$ (respectively, $a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,1}$).

What remains to be shown is that if $a_{-1,1}$ is of degree zero and sufficiently generic, all the new entries are of degree zero too. For this, observe that if $j \in \{-l', \dots, -2\}$ then $a_{i,j}$ is of degree zero as long as the coefficient of degree zero in $a_{i,-1}$ is different from the degree zero coefficients in the quotient $-\lambda_j a_{i,0}/\mu_j$ (here, we are using the assumption that $deg(\lambda_j) \geq deg(\mu_j) = 0$). The same is true for $j \in \{2, \dots, l\}$, with $a_{i,1}$ instead of $a_{i,-1}$. In terms of the choice of $a_{-1,1}$, this translates to the following determinant having nonzero coefficient in degree zero:

$$\begin{pmatrix} a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \\ -\lambda_j a_{i,0}/\mu_j & a_{i,0} & a_{i,1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{i,-1} & a_{i,0} & -\lambda_j a_{i,0}/\mu_j \\ a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \end{pmatrix},$$

respectively, for $j \in \{-l', \ldots, -2\}$ or $j \in \{2, \ldots, l\}$. That $a_{-1,1}$ and $a_{1,-1}$ sufficiently generic imply nonsingularity of these matrices follows from the fact that the following 2×2 submatrices come from the given lifts of $\tilde{A}$ and $\tilde{B}$, hence they are nonsingular:

$$\begin{pmatrix} a_{i,-1} & a_{i,0} \\ a_{0,-1} & a_{0,0} \end{pmatrix}, \quad \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{i,0} & a_{i,1} \end{pmatrix}.$$

□

I believe this proof is incorrect. Precisely, the argument from the last paragraph is based upon the fact that the given 2×2 submatrices are nonsingular. However, what is required is not just that the submatrices are nonsingular, but that the coefficient of the degree zero term in the determinant of these submatrices is nonzero. This is a more powerful assumption than that these submatrices are nonsingular, and it is an assumption that is not justified based upon Lemma 6.3 from [8] or that lemma's proof.

The proof of the corollary can be salvaged, however, in the following way.

Assume $j \in \{-l', \ldots, -2\}$. First, note that the degree zero coefficients of the 2×2 minor

$$\begin{vmatrix} a_{0,0} & a_{0,1} \\ a_{i,0} & a_{i,1} \end{vmatrix}$$

cannot vanish if $deg(a_{i,1}) > 0$. So, suppose $\deg(a_{i,1}) = 0$. Denote the leading term of $a_{i,j}$ by $c_{i,j}$, and the leading terms of λ_j and μ_j by, respectively, p_j and q_j. We must prove that the following matrices

$$\begin{pmatrix} c_{0,-1} & c_{0,0} & c_{0,1} \\ c_{1,-1} & c_{1,0} & 0 \\ -p_j c_{i,0}/q_j & c_{i,0} & c_{i,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} c_{0,0} & c_{0,1} \\ c_{i,0} & c_{i,1} \end{pmatrix}$$

cannot both be singular. Suppose otherwise. Then the singularity of the 2×2 matrix implies

$$c_{i,0} = \frac{c_{0,0} c_{i,1}}{c_{0,1}}.$$

Singularity of the 3×3 matrix is the relation

$$c_{i,0}(c_{1,-1}c_{0,1} + p_j c_{1,0} c_{0,1}/q_j) = c_{i,1}(c_{0,0}c_{1,-1} - c_{0,-1}c_{1,0}).$$

If we plug the first relation into the second we get:

$$c_{i,1}c_{0,0}c_{1,-1} + p_j c_{0,0} c_{i,1} c_{1,0}/q_j = c_{i,1}c_{0,0}c_{1,-1} - c_{i,1}c_{0,-1}c_{1,0}.$$

With a little algebra this becomes:

$$p_j c_{0,0} + q_j c_{0,-1} = 0.$$

Given $a_{0,j} = \lambda_j a_{0,0} + \mu_j a_{0,-1}$ the above equality would imply $deg(a_{0,j}) > 0$, which is not true. Exactly the same proof, mutatis mutandis, applies for $j \in \{2, \ldots, l\}$.

Note that this modification to the proof of Corollary 6.4 from [8] no longer requires Lemma 6.3 from [8], and would therefore not be a corollary.

The proof of Corollary 6.4 from [8] can also be modified in the following way, which again no longer requires Lemma 6.3 from [8], and would therefore not be a corollary.

Proposition A.3. *Let A and B be nonnegative matrices, neither of which contain the zero column. Assume that the two matrices*

$$\tilde{A} := \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} := \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix}$$

have Kapranov rank two. Then the matrix

$$M := \begin{pmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B \end{pmatrix}$$

has Kapranov rank two as well.

Proof. We number the rows of M from $-k$ to k' and the columns from $-l$ to l', where $k \times l$ and $k' \times l'$ are the dimensions of A and B, respectively. In this way, A (respectively, B) is the submatrix of negative (respectively, positive) indices. The row and column indexed zero consists of all zeros. To further exhibit the symmetry between A and B the columns and rows in $\tilde{A}$ will be referred to "in reverse." That is to say, the first and second columns of $\tilde{A}$ are the ones indexed 0 and -1 in M.

We now construct a lifting $F = (a_{i,j}) \in \tilde{K}^{d \times n}$ of M. As $\tilde{A}$ and $\tilde{B}$ have Kapranov rank two, they have rank two lifts $\tilde{A}$ and $\tilde{B}$. We assume that the lift of the entry $(0, 0)$ is the same in both, which can be achieved by scaling the first row in one of them. In fact, by scaling both we may assume this entry is 1 if we wish. Also, possibly after permuting some rows we can assume $deg(a_{-1,-1}), deg(a_{1,1}) > 0$.

We use the lifts $\tilde{A}$ and $\tilde{B}$ for the upper-left and bottom-right corner submatrices of F. Our task is to complete this lift with entries $a_{i,j}$ for every i, j with $ij < 0$, such that $deg(a_{i,j}) = 0$ and the whole matrix still has rank two. I claim that it suffices to choose the entry $a_{-1,1}$ of degree zero and sufficiently generic. That this choice fixes the rest of the matrix is easy to see. The entry $a_{1,-1}$ is fixed by the requirement that the central 3×3 submatrix

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

must have rank two. If $deg(a_{1,-1}) > 0$, then the above matrix would be nonsingular, so we must have $deg(a_{1,-1}) = 0$, and as $a_{-1,1}$ is generic, so is $a_{1,-1}$.

All other entries of the form $a_{i,-1}$ and $a_{i,1}$ are fixed by, respectively, the requirement that the matrices

$$\begin{pmatrix} a_{-1,-1} & a_{-1,0} & a_{-1,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{i,-1} & a_{i,0} & a_{i,1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{i,-1} & a_{i,0} & a_{i,1} \\ a_{0,-1} & a_{0,0} & a_{0,1} \\ a_{1,-1} & a_{1,0} & a_{1,1} \end{pmatrix}$$

are singular. If $deg(a_{i,-1}) > 0$ for $i > 1$, then the leading terms of the determinant

$$\begin{vmatrix} a_{-1,0} & a_{-1,1} \\ a_{0,0} & a_{0,1} \end{vmatrix}$$

must cancel, which cannot be as $a_{-1,1}$ is generic. So, $deg(a_{i,-1}) = 0$. As $a_{-1,1}$ is generic, so is $a_{i,-1}$. An identical argument proves $deg(a_{i,1}) = 0$ and $a_{i,1}$ is generic for $i < -1$.

For each $j = -l, \ldots, -2$ (respectively, $j = 2, \ldots, l'$), let λ_j and μ_j be the coefficients in the expression of the j-th column of $\tilde{A}$ (respectively, of $\tilde{B}$) as

$$\mathbf{u}_j = \lambda_j \mathbf{u}_0 + \mu_j \mathbf{u}_{-1} \text{ (respectively } \mathbf{u}_j = \lambda_j \mathbf{u}_0 + \mu_j \mathbf{u}_1).$$

If the degrees of λ_j and μ_j are different, then their minimum must be zero in order to get a degree zero element in the first entry of column j. However, then $deg(\mu_j) > deg(\lambda_j) = 0$ is impossible, because it would make the i-th column of A (respectively, B) all zero. Hence $deg(\lambda_j) > deg(\mu_j) = 0$. If the degrees are equal, then they are nonpositive in order to get degree zero for the first entry $\mathbf{u}_j$. They cannot be equal and negative, or otherwise entries of positive degree in $\mathbf{u}_{-1}$ (repsectively, $\mathbf{u}_1$) would produce entries of negative degree in $\mathbf{u}_j$. Hence $deg(\lambda_j) = deg(\mu_j) = 0$ in this case.

For $j \in \{-l', \ldots, -2\}$ and $i > 0$ we define

$$a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,-1},$$

and for $j \in \{2, \ldots, l\}$ and $i < 0$ we define

$$a_{i,j} = \lambda_j a_{i,0} + \mu_j a_{i,1}.$$

We must prove that every $a_{i,j}$ defined in this way has degree zero. For $j \in \{-l', \ldots, -2\}$ and $i > 0$ both $a_{i,0}$ and $a_{i,-1}$ have degree zero. If $deg(\lambda_j) > deg(\mu_j)$, then obviously $deg(a_{i,j}) = 0$. If $deg(\lambda_j) = deg(\mu_j) = 0$, then $deg(a_{i,j}) = 0$ follows from the leading term of $a_{i,-1}$ being generic. An identical argument applies for $a_{i,j}$ with $j \in \{2, \ldots, l\}$ and $i < 0$. $\square$

APPENDIX B

MAPLE CODE USED TO PERFORM THE COMPUTATIONS IN CHAPTER 5

This appendix describes the Maple code used to generate the computational results from Chapter 5 of the dissertation. The appendix also contains the text of the relevant Maple worksheets. You can download the actual worksheets, along with the Maple library archive files, from the website: `http://www.math.utah.edu/~zwick/Dissertation/`

B.1 Rank Calculations

Maple procedures for calculating the tropical rank and symmetric tropical rank of a real matrix are available from the Maple package `TropLinAlg`, which is contained in the Maple library archive file `TropLinAlg.mla`. The procedures from this package that are used for calculating tropical rank and symmetric tropical rank are, respectively, *tropicalrank* and *symtropicalrank*.

The Maple worksheet `TropicalRankCalculations.mw` first identifies the directory containing the `TropLinAlg.mla` file, then loads the package `TropLinAlg`. The worksheet then uses the relevant procedures from the package to verify that the symmetric version of the cocircuit matrix of the Fano matroid,

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

has tropical rank three, but symmetric tropical rank four.

The worksheet then computes that the symmetric version of the 6×6 matrix discovered by Shitov,

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 1 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 1 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{pmatrix},$$

has both tropical and symmetric tropical rank four.

Finally, the worksheet verifies that the 13×13 matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} -5 & -5 & -5 & -5 & -5 & -5 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -5 & -5 & -5 & -5 & -5 & -5 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ -5 & -5 & -5 & -5 & -5 & -5 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ -5 & -5 & -5 & -5 & -5 & -5 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ -5 & -5 & -5 & -5 & -5 & -5 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ -5 & -5 & -5 & -5 & -5 & -5 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & -10 & -10 & -10 & -10 & -10 & -10 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & -10 & -10 & -10 & -10 & -10 & -10 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & -10 & -10 & -10 & -10 & -10 & -10 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & -10 & -10 & -10 & -10 & -10 & -10 \end{pmatrix}$$

both have symmetric tropical rank three.

The text from the Maple worksheet `TropicalRankCalculations.mw` is given beginning on the next page, followed by the text from the Maple worksheet `TropicalRankModule.mw`, which is used to create the package `TropLinAlg` and generate the Maple library archive file `TropLinAlg.mla`.

#First, we must append the directory containing the TropLinAlg.mla Maple archive to the libname variable. Note that this directory will be wherever the TropLinAlg.mla file resides on your computer, and so will almost certainly not have the name "/u/ma/zwick/Dissertation/Maple/" used below.

```
libname := "/u/ma/zwick/Dissertation/Maple/", libname;
"/u/ma/zwick/Dissertation/Maple/", "/usr/local/sys/maple/mapleV17/lib",        (1)
    "/usr/local/sys/maple/mapleV17/toolbox/NAG/lib", "."
```

#We now load the TropLinAlg package, which contains the procedures tropicalrank and symtropicalrank.

```
with(TropLinAlg);
                    [symtropicalrank, tropicalrank]                          (2)
```

#We can verify that the matrix C3 has tropical rank two, but symmetric tropical rank three.

```
C3 := Matrix([[1, 0, 0], [0, 1, 0], [0, 0, 1]]);
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

```
tropicalrank(C3);
                              2                                              (4)
symtropicalrank(C3);
                              3                                              (5)
```

#We can also verify that the symmetric cocircuit matrix of the Fano matroid has tropical rank three, but symmetric tropical rank four.

```
F7 := Matrix([[1, 1, 0, 1, 0, 0, 0], [1, 0, 1, 0, 0, 0, 1], [0, 1, 0, 0, 0, 1, 1], [1, 0, 0, 0, 1, 1,
    0], [0, 0, 0, 1, 1, 0, 1], [0, 0, 1, 1, 0, 1, 0], [0, 1, 1, 0, 1, 0, 0]]);
```

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \qquad (6)$$

```
tropicalrank(F7);
                              3                                              (7)
```

symtropicalrank(*F7*);

4 **(8)**

#Next, we verify that the symmetric version of the 6x6 matrix discovered by Shitov has both tropical and symmetric tropical rank four.

S6 := *Matrix*([[0, 0, 2, 4, 2, 4], [0, 0, 4, 4, 4, 4], [2, 4, 2, 4, 0, 0], [4, 4, 4, 4, 0, 0], [2, 4, 0, 0, 2, 4], [4, 4, 0, 0, 4, 4]]);

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 2 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{bmatrix} \qquad \textbf{(9)}$$

tropicalrank(*S6*);

4 **(10)**

symtropicalrank(*S6*);

4 **(11)**

#Finally, we verify that the two 13 x 13 symmetric matrices below both have symmetric tropical rank three.

M13 := *Matrix*([[0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1], [0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1], [0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 0, 1, 0], [1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0], [1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0], [1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0], [0, 1, 1, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]]);

13 x 13 Matrix
Data Type: anything
Storage: rectangular
Order: Fortran_order

(12)

N13 := *Matrix*([[-5, -5, -5, -5, -5, -5, 1, 1, 0, 1, 0, 0, 0], [-5, -5, -5, -5, -5, -5, 1, 0, 1, 0, 0, 0, 1], [-5, -5, -5, -5, -5, -5, 0, 1, 0, 0, 0, 1, 1], [-5, -5, -5, -5, -5, -5, 1, 0, 0, 0, 1, 1, 0], [-5, -5, -5, -5, -5, -5, 0, 0, 0, 1, 1, 0, 1], [-5, -5, -5, -5, -5, -5, 0, 0, 1, 1, 0, 1, 0], [1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 0, 0], [1, 0, 1, 0, 0, 0, 1, -10, -10, -10, -10, -10, -10], [0, 1, 0, 0, 0, 1, 1, -10, -10, -10, -10, -10, -10], [1, 0, 0, 0, 1, 1, 0, -10, -10, -10, -10, -10, -10], [0, 0, 0, 1, 1, 0, 1, -10, -10, -10, -10, -10, -10], [0, 0, 1, 1, 0, 1, 0, -10, -10, -10, -10, -10, -10], [0, 1, 1, 0, 1, 0, 0, -10, -10, -10, -10, -10, -10]]);

$$\begin{bmatrix} \text{13 x 13 Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad (13)$$

#The symmetric tropical ranks of these matrices could be verified with the symtropicalrank command. However, this computation takes a very, very long time. It's much more efficient to use the rankcheck parameter. (This saves computing that the matrices don't have symmetric tropical ranks 13 through 5.)

symtropicalrank(*M13*, 3);

TRUE **(14)**

symtropicalrank(*M13*, 4); *#This procedure takes a while to execute.*

FALSE **(15)**

symtropicalrank(*N13*, 3);

TRUE **(16)**

symtropicalrank(*N13*, 4); *#This procedure also takes a while to execute.*

FALSE **(17)**

```
TropLinAlg :=module( )
description "A package for calculating the tropical rank and symmetric tropical rank
    of a real matrix."
option package
export tropicalrank, symtropicalrank;

    #This procedure calculates the tropical rank of an input Matrix A. If the optional
    positive integer term "rankcheck" is passed, the procedure returns a boolean TRUE
    or FALSE value depending on if the matrix has at least the tropical rank set by
    "rankcheck". If no "rankcheck" term is passed, the procedure returns the tropical
    rank of the Matrix A.

tropicalrank := proc(A :: Matrix, rankcheck :: nonnegint := 0)
    # Make sure the input is a "Matrix" and not a "matrix". Maple will complain
    otherwise. The "rankcheck" term is optional, and if specified the procedure checks if
    the matrix has at least the tropical rank set by "rankcheck".

  local numrows, numcolumns, n, r, possiblerows, possiblecolumns, rowchoice,
    columnchoice, columnperm, minsum, minperm, sig, localsum, m :
  description "This procedure calculates the tropical rank of a matrix." :

  numrows := ArrayTools:-Size(A)[1] :
  numcolumns := ArrayTools:-Size(A)[2] :
  n := min(numrows, numcolumns) :

  if rankcheck = 1 then return TRUE : elif rankcheck > n then return FALSE : end if:
    #Checks the extreme values of "rankcheck".

  if rankcheck > 0 then r := rankcheck : else r := n : end if:
    # If rankcheck has been specified, then the procedure checks if the matrix has at
    least that tropical rank. If not, the procedure checks if the matrix has tropical rank
    r, beginning with the maximum possible rank n, and if not, r is decremented and
    the process repeats.

  while r > 1 do # The minimum tropical rank of a real matrix is 1.

     possiblerows := combinat:-choose(numrows, r) :
    # This creates a list of all possible choices of r rows from the matrix A.
     possiblecolumns := combinat:-choose(numcolumns, r) :
    # Creates a list of all possible choices of r columns from the matrix A.
     columnperm := combinat:-permute(r) :
    # Creates a list of all possible permutations of {1,2,...,r}.

     for rowchoice from 1 to nops(possiblerows) do
        for columnchoice from 1 to nops(possiblecolumns) do

        # This procedure checks all r x r minors of the matrix A.

        minsum := infinity : # Keeps track of the minimum sum over any permutation.
```

```
minperm := [ ] : # Keeps track of the minimizing permutations.

for sig from 1 to nops(columnperm) do # Checks all possible permutations.

   localsum := 0 : # Keeps track of the sum for the given permutation.

   for m from 1 to r do # Calculates the sum for the given permutation.

      localsum := localsum
+
A[possiblerows[rowchoice][m]][possiblecolumns[columnchoice][columnperm[sig][
m]]] :

   end do:


# If the given permutation is minimizing over all the permutations checked so far, it
provides the new minsum and

# resets minperm. If the permutation is equal to the minimum over all the
permutations checked so far, minsum is
   # not changed, but minperm is increased.

   if localsum < minsum then minsum := localsum: minperm
:= [columnperm[sig]] :
   elif localsum = minsum then minperm := [op(minperm), columnperm[sig]] :
   end if:
 end do:
 if nops(minperm) = 1 then if rankcheck > 0 then return TRUE: else return r:
end if: end if:
# If a permutation is uniquely minimizing, then the tropical rank is determined. If
rankcheck has been specified, then the matrix has at least the specified rank. If not,
then the matrix has rank k.

 end do: end do:

 if rankcheck > 0 then return FALSE: else r := r − 1 : end if:
# If every r x r minor is singular, and rankcheck has been specified, then the matrix
does not have at least the specified rank. If rankcheck has not been specified, then
the (r−1) x (r−1) minors must be checked.

end do:

return 1 : # If every 2 x 2 minor is singular, the matrix has tropical rank 1.

end proc;
```

#This procedure calculates the symmetric tropical rank of an input Matrix A. If the optional positive integer term "rankcheck" is passed, the procedure returns a

boolean TRUE or FALSE value depending on if the matrix has at least the symmetric tropical rank set by "rankcheck". If no "rankcheck" term is passed, the procedure returns the symmetric tropical rank of the Matrix A.

```
symtropicalrank := proc(A :: Matrix, rankcheck :: nonnegint := 0)
    # Make sure the input is a "Matrix" and not a "matrix". Maple will complain
    otherwise. The "rankcheck" term is optional, and if specified the procedure checks if
    the matrix has at least the symmetric tropical rank set by "rankcheck".

  local n, r, possiblerows, possiblecolumns, rowchoice, columnchoice, columnperm,
    minsum, minperm, sig, localsum, m, q, s, t_index, samemonomial, allsamemonomial:
  description "Calculates the symmetric tropical rank of a symmetric matrix.":

   if Student[NumericalAnalysis]:-IsMatrixShape(A, symmetric) = false then
    error "Matrix must be symmetric": end if; # Checks if the matrix is symmetric.

   n := ArrayTools:-Size(A)[1]: # Calculates the maximum rank of the matrix.

   if rankcheck = 1 then return TRUE: elif rankcheck > n then return FALSE: end if:
    # Checks the extreme values of "rankcheck".

   if rankcheck > 0 then r := rankcheck: else r := n: end if:
    # If rankcheck has been specified, then the procedure checks if the matrix has at
    least that symmetric tropical rank. If not, the procedure checks if the matrix has
    symmetric tropical rank r, beginning with the maximum possible rank n, and, if not,
    r is decremented and the process repeats.

   while r > 1 do

     possiblerows := combinat:-choose(n, r):
    # Creates a list of all possible choices of r rows from the matrix A.
     possiblecolumns := combinat:-choose(n, r):
    # Creates a list of all possible choices of r columns from the matrix A.

     for rowchoice from 1 to nops(possiblerows) do
        for columnchoice from 1 to nops(possiblecolumns) do

        columnperm := combinat:-permute(possiblecolumns[columnchoice]):
    # Creates a list of all permutations of the selected columns.

        minsum := infinity: # Keeps track of the minimum sum over any permutation.
        minperm := [ ]: # Keeps track of the minimizing permutations.

        for sig from 1 to nops(columnperm) do
    # Runs through all the possible column permutations.

           localsum := 0: # Keeps track of the sum for the given permutation.

           for m from 1 to r do # Calculates the sum for the given permutation.
```

```
            localsum := localsum
+ A[possiblerows[rowchoice][m]][columnperm[sig][m]] :

        end do;


# If the given permutation is minimizing over all the permutations checked so far, it
provides the new minsum and

# resets minperm. If the permutation is equal to the minimum over all the
permutations checked so far, minsum is
        # not changed, but minperm is increased.

        if localsum < minsum then minsum := localsum: minperm
:= [columnperm[sig]] :
        elif localsum = minsum then minperm := [op(minperm), columnperm[sig]] :
        end if;
      end do;

      if nops(minperm) = 1 then if rankcheck > 0 then return TRUE: else return r:
end if;

# If there's only one minimizing permutation, the minor is definitely nonsingular.
So, if rankcheck has been set, the matrix has at least symmetric tropical rank equal
to rankcheck, and if rankcheck has not been set, the matrix has symmetric tropical
rank r.


# If there's more than one minimizing permutation, we need to check if the
corresponding monomials are distinct.

      else allsamemonomial := true:
#Initially assumes that every minimizing permutations represents the same
monomial.
          for q from 2 to nops(minperm) do
             samemonomial := true:
#Initially assumes permutation 1 and permutation q represent the same monomial.
             for s from 1 to r do

# If permutation 1 involves element A[s][t] and permutation q does not, we must
check if permutation q involves
                 # the element A[t][s].
                   if minperm[1][s] ≠ minperm[q][s] then
                        t_index := ListTools:-Search(minperm[1][s],
possiblerows[rowchoice]);
                        if t_index = 0 then samemonomial := false:
#If row t is not an option, the monomials are distinct.

#If row t is an option, we need to check whether permutation r involves the element
```

```
A[t][s]
                    elif ListTools:-Search(minperm[q][t_index],
possiblerows[rowchoice]) ≠ s then samemonomial := false:
                    end if;
                end if;
            end do;
            # If two monomials are different, they're not all the same.
            if samemonomial = false then allsamemonomial := false: end if;
        end do;
        if allsamemonomial = true then if rankcheck > 0 then return TRUE: else
return r: end if; end if;
# If all the monomials are the same, it's symmetrically nonsingular.
    end if;

  end do; end do;

  if rankcheck > 0 then return FALSE: else r := r − 1 : end if;
# If all r x r minors are symmetrically tropically singular, and rankcheck has been
specified, then the matrix does not have at least the specified symmetric tropical
rank. If rankcheck has not been specified, we check the (r−1) x (r−1) minors.

 end do;

 return 1 :
# If all 2 x 2 minors are symmetrically tropically singular, the symmetric matrix has
symmetric tropical rank one.

end proc;

end module;
module( )                                                                    (1)
    option package;
    export tropicalrank, symtropicalrank;
    description
    "A package for calculating the tropical rank and symmetric tropical rank of
    a real matrix.";

end module
savelib('TropLinAlg', "/u/ma/zwick/Dissertation/Maple/TropLinAlg.mla")
```

B.2 Dimension Lower Bounds Calculations

A Maple procedure, *localdimensionlowerbounds*, for calculating a lower bound on the local dimension of a neighborhood of a point in $T_{m,n,r}$ is available from the Maple package `TropLinAlgLocalDimensionLowerBounds`. A similar Maple procedure, *symmetriclocaldimensionlowerbounds*, for calculating a lower bound on the local dimension of a neighborhood of a point in $S_{n,r}$ is also available from the same package. This package is contained in the Maple library archive file `TropLinAlgLocalDimensionLowerBounds.mla`.

The Maple worksheet `LocalDimensionLowerBoundsCalculations.mw` first identifies the directory containing the `TropLinAlgLocalDimensionLowerBounds.mla` file, then loads the package `TropLinAlgLocalDimensionLowerBounds`. The worksheet then uses the relevant procedures from the package to verify that the lower bounds on the local dimensions of $T_{7,7,4}$, $T_{6,6,5}$, and $S_{6,5}$ are 34, 33, and 19, respectively.

For $T_{7,7,4}$ the symmetric version of the cocircuit matrix of the Fano matroid,

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix},$$

is used.

For both $T_{6,6,5}$ and $S_{6,5}$ the symmetric version of the 6×6 matrix discovered by Shitov,

$$\begin{pmatrix} 0 & 0 & 2 & 4 & 1 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 1 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{pmatrix},$$

is used.

The text from the Maple worksheet `LocalDimensionLowerBoundsCalculations.mw` is given beginning on the next page, followed by the text from the Maple worksheet `TropLinAlgLocalDimensionLowerBoundsModule.mw`, which is used to create the package `TropLinAlgLocalDimensionLowerBounds` and generate the file Maple library archive file `TropLinAlgLocalDimensionLowerBounds.mla`.

#First, we must append the directory containing the TropLinAlgLocalDimensionLowerBounds.mla Maple archive to the libname variable. Note that this directory will be wherever the TropLinAlgLocalDimensionLowerBounds.mla file resides on your computer, and so will almost certainly not have the name "/u/ma/zwick/Dissertation/Maple/" used below.

libname := "/u/ma/zwick/Dissertation/Maple/", *libname*;

"/u/ma/zwick/Dissertation/Maple/", "/usr/local/sys/maple/mapleV17/lib", "/usr/local/sys/maple/mapleV17/toolbox/NAG/lib", "." **(1)**

#We now load the TropLinAlgLocalDimensionLowerBounds package, which contains the procedures localdimensionlowerbounds and symmetriclocaldimensionlowerbounds.

with(*TropLinAlgLocalDimensionLowerBounds*);

[*localdimensionlowerbounds*, *symmetriclocaldimensionlowerbounds*] **(2)**

#We can use this procedure to get a lower bound on the local dimension of the cocircuit matrix of the Fano matroid.

F7 := *Matrix*([[1, 1, 0, 1, 0, 0, 0], [1, 0, 1, 0, 0, 0, 1], [0, 1, 0, 0, 0, 1, 1], [1, 0, 0, 0, 1, 1, 0], [0, 0, 0, 1, 1, 0, 1], [0, 0, 1, 1, 0, 1, 0], [0, 1, 1, 0, 1, 0, 0]]);

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix} \quad \textbf{(3)}$$

localdimensionlowerbounds(*F7*, 4); *#WARNING – This procedure takes a while to finish.*

34 **(4)**

#We can also use this procedure to get a lower bound on the local dimension of the symmetric version of the 6 x 6 matrix discovered by Shitov. Note that this is viewing the matrix as a standard matrix, and not a symmetric matrix.

S6 := *Matrix*([[0, 0, 2, 4, 2, 4], [0, 0, 4, 4, 4, 4], [2, 4, 2, 4, 0, 0], [4, 4, 4, 4, 0, 0], [2, 4, 0, 0, 2, 4], [4, 4, 0, 0, 4, 4]]);

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 2 & 4 \\ 0 & 0 & 4 & 4 & 4 & 4 \\ 2 & 4 & 2 & 4 & 0 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 \\ 2 & 4 & 0 & 0 & 2 & 4 \\ 4 & 4 & 0 & 0 & 4 & 4 \end{bmatrix} \quad \textbf{(5)}$$

localdimensionlowerbounds(*S6*, 5);

$$33 \quad \textbf{(6)}$$

#Finally, we can calculate a lower bound on the local dimension of the symmetric version of the matrix discovered by Shitov, viewing the matrix as a symmetric matrix.

symmetriclocaldimensionlowerbounds(*S6*, 5);

$$19 \quad \textbf{(7)}$$

```
TropLinAlgLocalDimensionLowerBounds :=module( )

  description "A package for calculating local dimensions of tropical determinantal
  prevarieties of matrices and symmetric matrices."
option package
export localdimensionlowerbounds, symmetriclocaldimensionlowerbounds;

  # This procedure calculates a lower bound on the dimension of the tropical
  prevariety of m x n matrices with tropical rank less than r around a matrix A within
  this prevariety. This tropical prevariety is denoted T(m,n,r).

localdimensionlowerbounds := proc(A :: Matrix, r :: posint)
  # Make sure the input is a "Matrix" and not a "matrix". Maple will complain
  otherwise.

  local numrows, numcolumns, possiblerows, possiblecolumns, columnperm, B, C, b,
  nulldim, rowchoice, columnchoice, minsum, minperm, sig, localsum, i, m :

  description "Calculates a lower bound on the dimension of T(m,n,r) around a point
  A within this set." :

  numrows := ArrayTools:-Size(A)[1] :
  numcolumns := ArrayTools:-Size(A)[2] :

  possiblerows := combinat:-choose(numrows, r) :
  # This creates a list of all possible choices of r rows from the matrix A.
  possiblecolumns := combinat:-choose(numcolumns, r) :
  # Creates a list of all possible choices of r columns from the matrix A.
  columnperm := combinat:-permute(r) :
  # Creates a list of all possible permutations of {1,2,...,r}.
  B := Matrix(1, numrows·numcolumns) :
  #Matrix used to store the linear equations that decrease the dimension of the linear
  space.
  C := Matrix(1, numrows·numcolumns) :
  #Test matrix used to see if a new linear equation decreases the dimension.
  b := Matrix(1, numrows·numcolumns) :
  #Vector that stores a linear relation given by a permutation that realizes a tropical
  determinant.
  nulldim := numrows·numcolumns :
  #The initial value of the lower bound, set to as large as the dimension could possibly
  be.

   for rowchoice from 1 to nops(possiblerows) do
     for columnchoice from 1 to nops(possiblecolumns) do

       # This procedure checks all r x r minors of the matrix A.

       minsum := infinity : # Keeps track of the minimum sum over any permutation.
       minperm := [ ] :
```

```
# Keeps track of the permutations realizing the tropical determinant.

    for sig from 1 to nops(columnperm) do # Checks all possible permutation.

        localsum := 0 : # Keeps track of the sum for the given permutation.

        for m from 1 to r do # Calculates the sum for the given permutation.

            localsum := localsum
+
A[possiblerows[rowchoice][m]][possiblecolumns[columnchoice][columnperm[sig][
m]]] :

        end do:


# If the given permutation has the smallest local sum over all the permutations
checked so far, it provides the new

# minsum and resets minperm. If the localsum is equal to the minimum over all the
permutations checked so far,
        # minsum is not changed, but minperm is increased.

        if localsum < minsum then minsum := localsum: minperm
:= [columnperm[sig]] :
        elif localsum = minsum then minperm := [op(minperm), columnperm[sig]] :
        end if:
    end do:

    if nops(minperm) < 2 then
error "The matrix has a nonsingular submatrix of the specfied size" end if:


# If there's only a single permutation that realizes the tropical determinant for some
r x r submatrix, then the matrix A
       # is not a point of T(m,n,r), and therefore the procedure returns an error.

        for i from 2 to nops(minperm) do


#Creates a vector representing the new linear equations introduced by each
permutation that realizes the tropical determinant.

            for m from 1 to r do
              b[1, (possiblerows[rowchoice][m] - 1)·numcolumns
 + possiblecolumns[columnchoice][minperm[1][m]]] := b[1,
(possiblerows[rowchoice][m]-1)·numcolumns
 + possiblecolumns[columnchoice][minperm[1][m]]] + 1 :
```

```
b[1, (possiblerows[rowchoice][m] − 1)·numcolumns
 + possiblecolumns[columnchoice][minperm[i][m]]] := b[1,
(possiblerows[rowchoice][m]-1)·numcolumns
 + possiblecolumns[columnchoice][minperm[i][m]]]-1 :
          end do:


#Checks to see if the new linear equation decreases the local dimension lower
bound.

          C(ArrayTools:-Size(B)[1] + 1, 1 ..) := ⟨b⟩ :
          if (LinearAlgebra:-ColumnDimension(C) − LinearAlgebra:-Rank(C))
< nulldim then B := LinearAlgebra:-Copy(C) : nulldim := (LinearAlgebra:-
ColumnDimension(C) − LinearAlgebra:-Rank(C)) : else C := LinearAlgebra:-
Copy(B) : end if:
          b := Matrix(1, numrows·numcolumns) :

      end do:

  end do: end do:

 return nulldim;

end proc;

 # This procedure calculates a lower bound on the dimension of the tropical
 prevariety of n x n symmetric matrices with tropical rank less than r around a
 matrix A within this prevariety. This tropical prevariety is denoted S(n,r). The lower
 bound is explained in Section 5.3 of the dissertation.

symmetriclocaldimensionlowerbounds := proc(A :: Matrix, r :: posint)
 # Make sure the input is a "Matrix" and not a "matrix". Maple will complain
 otherwise.

local n, subsets, perms, B, C, b, nulldim, rowchoice, columnchoice, minsum, minperm,
 sig, localsum, m, notinlist :

 description "Calculates a lower bound on the dimension of S(n,r) around a point
 within this set." :

if Student[NumericalAnalysis]:-IsMatrixShape(A, symmetric) = false then
 error "Matrix must be symmetric" : end if;

n := ArrayTools:-Size(A)[1] :

subsets := combinat:-choose(n, r) :
 # This creates a list of all possible size r subsets from the n indices.
perms := combinat:-permute(r) :
 # Creates a list of all possible permutations of {1,2,...,r}.
```

$B := Matrix\left(1, \frac{(n\cdot(n+1))}{2}\right):$

#Matrix used to store the linear equations that decrease the dimension of the linear space.

$C := Matrix\left(1, \frac{(n\cdot(n+1))}{2}\right):$

#Test matrix used to see if a new linear equation decreases the dimension.

$b := Matrix\left(1, \frac{(n\cdot(n+1))}{2}\right):$

#Vector that stores a linear relation given by a permutation that realizes a tropical determinant.

$nulldim := \left(\frac{n\cdot(n+1)}{2}\right):$

#The initial value of the lower bound, set to as large as the dimension could possibly be.

for *rowchoice* **from** 1 **to** *nops*(*subsets*) **do**
for *columnchoice* **from** 1 **to** *nops*(*subsets*) **do**

 # This procedure checks all r x r minors of the matrix A.

minsum := *infinity* : *# Keeps track of the minimum sum over any permutation.*
minperm := [ ] : *# Keeps track of the minimizing permutations.*

for *sig* **from** 1 **to** *nops*(*perms*) **do** *# Checks all possible permutation.*

localsum := 0 : *# Keeps track of the sum for the given permutation.*

for *m* **from** 1 **to** *r* **do** *# Calculates the sum for the given permutation.*

localsum := *localsum*
+ *A*[*subsets*[*rowchoice*][*m*]][*subsets*[*columnchoice*][*perms*[*sig*][*m*]]] :

end do:

 # This rather scary looking set of commands creates a vector representing the linear equation determined by a minimizing
 # permutation, taking into account the symmetry of the respective variables.

if *localsum* ≤ *minsum* **then**
for *m* **from** 1 **to** *r* **do**
if *subsets*[*rowchoice*][*m*] < *subsets*[*columnchoice*][*perms*[*sig*][*m*]]
then

$b\Big[1, \Big(n\cdot(subsets[rowchoice][m]-1)$

$-\frac{(subsets[rowchoice][m]-1)\cdot(subsets[rowchoice][m]-2)}{2}$

```
+ ((subsets[columnchoice][perms[sig][m]] − subsets[rowchoice][m]) + 1))]
:= b[1, (n·(subsets[rowchoice][m] − 1)
− (subsets[rowchoice][m] − 1)·(subsets[rowchoice][m] − 2) / 2
+ ((subsets[columnchoice][perms[sig][m]] − subsets[rowchoice][m]) + 1))] + 1 :
        else
      b[1, (n·(subsets[columnchoice][perms[sig][m]] − 1)
− 1/2 ((subsets[columnchoice][perms[sig][m]] − 1)
·(subsets[columnchoice][perms[sig][m]]-2)) + ((subsets[rowchoice][m]
− subsets[columnchoice][perms[sig][m]]) + 1))] := b[1, (n
·(subsets[columnchoice][perms[sig][m]] − 1)
− 1/2 ((subsets[columnchoice][perms[sig][m]] − 1)
·(subsets[columnchoice][perms[sig][m]]-2)) + ((subsets[rowchoice][m]
− subsets[columnchoice][perms[sig][m]]) + 1))] + 1 :
        end if:
      end do:
      if localsum < minsum then
        minsum := localsum:
        minperm := [b]:
      else
        notinlist := true:
        for m from 1 to nops(minperm) do
          if LinearAlgebra:-Equal(b, minperm[m]) then notinlist := false: end
if:
        end do:
        if notinlist then minperm := [op(minperm), b]: end if:
      end if:
      b := Matrix(1, (n·(n+1))/2) :
    end if:
  end do:

  if nops(minperm) < 2 then
error "The matrix has a nonsingular submatrix of the specfied size" end if:
```

```
# This set of commands determines if a new linear equation actually modifies the dimension lower bound.

    for m from 2 to nops(minperm) do
        b := minperm[1] - minperm[m]:
        C(ArrayTools:-Size(B)[1] + 1, 1 ..) := <b> :
        b := Matrix(1, (n·(n+1))/2) :
        if (LinearAlgebra:-ColumnDimension(C) - LinearAlgebra:-Rank(C))
< nulldim then B := LinearAlgebra:-Copy(C) : nulldim := (LinearAlgebra:-
ColumnDimension(C) - LinearAlgebra:-Rank(C)) : else C := LinearAlgebra:-
Copy(B) : end if:

    end do:
  end do: end do:

  return nulldim;

end proc;

 end module;
module( )                                                              (1)
    option package;
    export localdimensionlowerbounds, symmetriclocaldimensionlowerbounds;
    description
    "A package for calculating local dimensions of tropical determinantal
    prevarieties of matrices and symmetric matrices.";

end module
savelib('TropLinAlgLocalDimensionLowerBounds',
    "/u/ma/zwick/Dissertation/Maple/TropLinAlgLocalDimensionLowerBounds.mla")
```

REFERENCES

[1] Matthew Borman, *The ranks of a tropical matrix*, Undergraduate Thesis, Reed College, May 2007.

[2] Robert Bieri and J.R.J. Groves, *The geometry of the set of characters induced by valuations*, J. Reine Agnew. Math. **347** (1984), 168-195.

[3] T. Bogart, A. Jensen, D. Speyer, B. Sturmfels, and R. Thomas, *Computing tropical varieties*, J. Symbolic Comput. **42** (2007), nos. 1-2, 54-73.

[4] Dustin Cartwright and Melody Chan, *Three notions of tropical rank for symmetric matrices*, Combinatoria. **32** (2012), no. 1, 55-84.

[5] Melody Chan, *Personal correspondence.*

[6] Melody Chan, *Research statement.* http://www.math.harvard.edu/~mtchan/.

[7] Melody Chan, Anders Jensen, and Elena Rubei, *The* 4×4 *minors of a* $5 \times n$ *matrix are a tropical basis*, Linear Algebra and its Applications. **435** (2011), 1590-1611.

[8] Mike Develin, Francisco Santos, and Bernd Sturmfels, *On the rank of a tropical matrix*, Discrete and Computational Geometry, (E. Goodman, J.Pach and E.Welzl, eds.), MSRI Publicaitons, Cambridge Univ. Press, 2005.

[9] Mike Develin and Bernd Sturmfels, *Tropical convexivity*, Documenta Mathematica. **9** (2004), 1-27.

[10] Alicia Dickenstein and Luis F. Tabera, *Singular tropical hypersurfaces*, Discrete and Computational Geometry. **47** (2012), no. 2, 430-453.

[11] Joe Harris, *Algebraic geometry: a first course*, Graudate Texts in Mathematics, vol. 133, Springer, 1992.

[12] K.H. Kim and F.W. Roush, *Kapranov rank vs. tropical rank*, Proceedings of the American Mathematical Society. **134** (2006), no. 9, 2487-2494.

[13] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, 2014, http://homepages.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.html.

[14] Hannah Markwig and Josephine Yu, *The space of tropically collilnear points is shellable*, Collectanea Mathematica. **60** (2009), no. 1, 63-77.

[15] Isaac Newton, *The correspondence of Sir Isaac Newton, vol II (1676-1687)*, pp. 20-42 and 110-163, Cambridge University Press, 1960.

[16] Brian Osserman, *Personal correspondence.*

[17] Victor Puiseux, *Recherches sur les fonctions algébriques*, Journal de Mathématiques Pures et Appliquées. **15** (1850), 365-480.

[18] Yaroslav Shitov, *Example of a 6-by-6 matrix with different tropical and Kapranov ranks*, (Russian, with English version available at arXiv:1012.5507v1.), Vestnik Moskov. Univ. Ser. 1. **5** (2011), 58-61.

[19] Yaroslav Shitov, *When do the r-by-r minors of a matrix form a tropical basis?*, Journal of Combinatorial Theory, Series A. **120** (2013), 1166-1201.

[20] Bernd Sturmfels, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics, vol. 97, American Mathematical Society, Providence, RI and CBMS, Washington, DC, 2002.

[21] C.T.C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, vol. 63, Cambridge University Press, 2004.