ENUMERATIVE GEOMETRY OF QUOT SCHEMES

by

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ABSTRACT

The main object of study in this thesis is the Grothendieck Quot scheme. Let X be a projective variety over $\mathbb{C}$, let V be a coherent sheaf on X, and let ρ be a cohomology class on X. The Quot scheme $\mathrm{Quot}(V, \rho)$ is a projective scheme that parametrizes coherent sheaf quotients $V \twoheadrightarrow F$ where F has Chern character ρ. Choosing ρ and the Chern character of V to satisfy a certain orthogonality condition, $\mathrm{Quot}(V, \rho)$ is expected to be a finite collection of points. One can ask whether $\mathrm{Quot}(V, \rho)$ is indeed finite when V is general in moduli. If so, then one can try to enumerate the points of $\mathrm{Quot}(V, \rho)$. These counts of points of finite Quot schemes yield interesting formulas and can be used to study strange duality.

When X is a curve, Marian and Oprea proved that general V do produce finite Quot schemes, whose points are counted by the Verlinde formula. We show that these enumerative invariants can be viewed as certain closed invariants inside a weighted topological quantum field theory (TQFT) that encodes the intersection numbers of Schubert varieties on all (not only finite) Quot schemes of general vector bundles on curves. This weighted TQFT contains both the small quantum cohomology of the Grassmannian and a TQFT of Witten that is known to compute the Verlinde numbers.

When X is a del Pezzo surface, even the existence of finite Quot schemes is not known. On $\mathbb{P}^2$, we use exceptional resolutions of sheaves to prove that $\mathrm{Quot}(V, \rho)$ is finite when ρ is the Chern character of an ideal sheaf of points, the orthogonality condition is satisfied, and V is general in moduli. On general del Pezzo surfaces, we use multiple point formulas to compute the expected number of points of Quot schemes that are expected to be finite, where ρ is once again the Chern character of an ideal sheaf of points. The formulas agree with a power series computing Euler characteristics of line bundles on Hilbert schemes of points, thus providing evidence for strange duality on del Pezzo surfaces.

CONTENTS

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CHAPTER 1

INTRODUCTION

How many points lie on the intersection of two general lines in the plane? How many lines intersect four general lines in space? How many conics lie on a general quintic threefold? Questions such as these form the basis of enumerative geometry. The key to formulating such a question is to impose just enough conditions on a class of geometric objects to ensure that the number of such objects is finite. With too few conditions the answer is infinite; with too many conditions the answer is zero. But if the number of conditions is just right, then the answer could be quite interesting.

Sometimes a question in enumerative geometry can be generalized into a whole family of questions. Instead of asking for points on an intersection of lines, one can ask for points on the intersection of a pair of general curves of degrees d and e. When questions are posed in a family, a structure can emerge that explains the answers to all the questions simultaneously and relates them to each other. These structures are the key to acquiring a deeper understanding of the underlying geometry.

1.1 Finite Quot schemes

The enumerative setting for this thesis is as follows. If X is an algebraic variety over $\mathbb{C}$, V is a vector bundle on X, and ρ is a cohomology class on X, there is a projective scheme $\mathrm{Quot}(V, \rho)$ called the **Grothendieck Quot scheme** whose points parametrize coherent sheaf quotients $V \twoheadrightarrow F$ where the Chern character of F is ρ. There is a formula for the expected dimension of $\mathrm{Quot}(V, \rho)$ that depends only on the Chern character of V and on ρ. When this expected dimension is zero, we expect the Quot scheme to be finite and reduced, in which case there is a natural number to associate to the Quot scheme: the number of its points. The naive problem of counting points of finite Quot schemes is surprisingly interesting.

When X is a curve of genus g, counting finite Quot schemes $\mathrm{Quot}(V, \rho)$ in which V has rank $r+s$ and the quotients have rank s yields the formula

$$\#\,\mathrm{Quot}(V,\rho) = \sum_{\substack{T\sqcup U=\{1,\dots,r+s\}\\ |T|=r}} \prod_{\substack{t\in T\\ u\in U}} \left|2\sin\pi\frac{t-u}{r+s}\right|^{g-1}, \tag{1.1}$$

where the expression on the right is the (modified) Verlinde formula that relates to ranks of vector bundles of conformal blocks on moduli spaces of curves. Chapter 3 explains how for fixed r and s but varying genus, these numbers are the traces of $g-1$ powers of a linear operator on the cohomology ring of the Grassmannian $\mathrm{Gr}(r,\mathbb{C}^{r+s})$.

Moving up a dimension, consider the case when X is a del Pezzo surface and ρ_n is the Chern character of an ideal sheaf of n points on X. When V_n is a vector bundle of rank $r+1$ whose first Chern class is represented by a divisor L and whose second Chern class is chosen to make the expected dimension of $\mathrm{Quot}(V_n,\rho_n)$ zero, the results in [BGJ16] suggest the conjectural formula

$$\sum_{n\geq 0} \#\,\mathrm{Quot}(V_n,\rho_n)\, z^n = g_r(z)^{\chi(L)} \cdot f_r(z)^{\frac{1}{2}\chi(\mathcal{O}_X)} \cdot A_r(z)^{L.K_X-\frac{1}{2}K_X^2} \cdot B_r(z)^{K_X^2} \tag{1.2}$$

for particular power series $A_r(Z)$, $B_r(Z)$, $f_r(z)$, $g_r(z)$ in z whose coefficients depend only on r, and $f_r(z)$ and $g_r(z)$ have closed-form expressions

$$f_r(z) = \sum_{k\geq 0} \binom{(1-r^2)(k-1)}{k} z^k; \quad g_r(z) = \sum_{k\geq 0} \frac{1}{1-(r^2-1)k} \binom{1-(r^2-1)k}{k} z^k.$$

The formulas (1.1) and (1.2) do more than just compute the numbers of points of finite Quot schemes. Indeed, they suggest a geometric relationship between different finite Quot schemes. In the case of curves, this is explained in Chapter 3: the finite Quot schemes can be described within a topological quantum field theory, which encodes information about how curves and their Quot schemes change when a smooth curve degenerates to a nodal curve. In the case of surfaces, the right side of (1.2) is based on computations in the complex cobordism ring ([EGL01]), which may be a good setting in which to compare Quot schemes on different del Pezzo surfaces. Since Quot schemes behave well under deformations (there are relative Quot schemes), deformation arguments are a natural way to compare Quot schemes.

These observations suggest the following problems, which could be attempted in much greater generality than the cases described above.

Guiding problems 1.1.1. Let V be a vector bundle on a projective scheme X and ρ be a Chern character on X.

Problem 1: Show that when $\mathrm{Quot}(V, \rho)$ has expected dimension zero, choosing V to be sufficiently general ensures that $\mathrm{Quot}(V, \rho)$ is finite and reduced.

Problem 2: Assuming $\mathrm{Quot}(V, \rho)$ is finite and reduced, compute $\#\mathrm{Quot}(V, \rho)$.

Problem 3: Explain how the numbers $\#\mathrm{Quot}(V, \rho)$ are related using a geometric structure, possibly based on deformation theory.

Chapter 2 provides a brief background in moduli theory with an emphasis on Quot schemes. Chapters 3 and 4 resolve the three guiding problems on curves, where partial answers can be found in the literature but a satisfying geometric picture has not been assembled until now. Chapters 5 and 6 describe the results of [BGJ16] on del Pezzo surfaces, where very little seems to be known. Chapter 5 settles Problem 1 on $\mathbb{P}^2$ when ρ is the Chern character of an ideal sheaf of points. Chapter 6 presents computations on del Pezzo surfaces that provide evidence for the conjectural formula (1.2).

Before getting started, we describe an open problem that provides additional motivation for why finite Quot schemes might be interesting objects to study.

1.2 Strange duality

The following idea for using finite Quot schemes to study strange duality appeared in [MO07], where Marian and Oprea used it to prove strange duality for curves. We will state the strange duality conjecture in the case of del Pezzo surfaces, where the Quot scheme approach seems especially promising ([BGJ16]). For a scattering of strange duality results on a variety of surfaces that use other methods, see [Dan02], [MO09], [MO08], [MO13], [MO14], [Yua12], and [Yua16] .

Let X be a smooth projective del Pezzo surface over $\mathbb{C}$ and let σ and ρ be cohomology classes on X that are orthogonal under the Mukai pairing $\chi(\sigma, \rho)$; this means that whenever E and F are coherent sheaves whose Chern characters are σ and ρ, then

$$\chi(\sigma, \rho) = \chi(E, F) = \sum_{i=0}^{2} \mathrm{ext}^i(E, F) = 0.$$

Let $M(\sigma^\vee)$ and $M(\rho)$ denote the moduli spaces of semistable sheaves with Chern characters $\sigma^\vee$ (dual to σ) and ρ. The point of the orthogonality condition is that every pair of sheaves $(\widehat{E}, F)$ in these moduli spaces is expected to have $h^i(\widehat{E} \otimes F) = 0$ for all $0 \leq i \leq 2$. Thus we expect the jumping locus

$$\Theta \subset M(\sigma^\vee) \times M(\rho)$$

of pairs where $h^0(\widehat{E} \otimes F) = h^1(\widehat{E} \otimes \rho) > 0$ (since $h^2(\widehat{E} \otimes F)$ vanishes by stability) to have the structure of a Cartier divisor.

In cases where Θ is a Cartier divisor, one can show (since X is del Pezzo) that

$$\mathcal{O}(\Theta) = \mathcal{O}(\Theta_\rho) \boxtimes \mathcal{O}(\Theta_{\sigma^\vee}),$$

where Θ_ρ is the divisor on $M(\sigma^\vee)$ obtained by restricting Θ to a general fiber of the projection map $M(\sigma^\vee) \times M(\rho) \twoheadrightarrow M(\rho)$ and $\Theta_{\sigma^\vee}$ is defined similarly. We can think of Θ_ρ as the jumping locus on $M(\sigma^\vee)$ of all $\widehat{E}$ that have cohomology when tensored with a fixed general F in $M(\rho)$. We refer to $\mathcal{O}(\Theta_\rho)$ and $\mathcal{O}(\Theta_{\sigma^\vee})$ as **determinantal line bundles**. The Künneth formula yields the isomorphism

$$H^0\big(\mathcal{O}(\Theta)\big) \simeq H^0\big(\mathcal{O}(\Theta_\rho)\big) \otimes H^0\big(\mathcal{O}(\Theta_{\sigma^\vee})\big)$$

and Θ corresponds to a particular choice of section in $H^0\big(\mathcal{O}(\Theta)\big)$ up to scaling, which induces a pairing

$$\mathrm{SD}_{\sigma,\rho} \colon H^0\big(\mathcal{O}(\Theta_\rho)\big)^* \otimes H^0\big(\mathcal{O}(\Theta_{\sigma^\vee})\big)^* \to \mathbb{C}$$

called the **strange duality pairing**.

Conjecture 1.2.1 (Le Potier's Strange Duality)**.** *$SD_{\sigma,\rho}$ is a perfect pairing.*

The strange duality pairing can be described concretely on pairs of functionals corresponding to sheaves. A sheaf $\widehat{E}$ induces an element of $H^0\big(\mathcal{O}(\Theta_\rho)\big)^*$ defined as evaluation at $[\widehat{E}] \in M(\sigma^\vee)$. Similarly, evaluation at $[F]$ produces an element of $H^0\big(\mathcal{O}(\Theta_{\sigma^\vee})\big)^*$. The strange duality pairing induced by the divisor Θ has the property that

$$\mathrm{SD}_{\sigma,\rho}(\mathrm{ev}_{[\widehat{E}]} \otimes \mathrm{ev}_{[F]}) = \begin{cases} 0 & \text{if } (\widehat{E}, F) \in \Theta; \\ \neq 0 & \text{if } (\widehat{E}, F) \notin \Theta. \end{cases}$$

Since the pairing is only well-defined up to scaling, it make sense that we cannot pin down exact nonzero values.

The connection to finite Quot schemes is as follows. Choose a general vector bundle V with Chern character $\sigma + \rho$. Then $\mathrm{Quot}(V, \rho)$ is expected to be finite and reduced since the tangent space at a point $x_i = [0 \to E_i \to V \to F_i \to 0]$ is $\mathrm{Hom}(E_i, F_i)$, which is expected to vanish since $\chi(E_i, F_i) = 0$ by the orthogonality of σ and ρ. Assume this Quot scheme is indeed finite and reduced and that the E_i are locally free. Then $h^0(E_i^* \otimes F_i) = 0$ for all i since the tangent space at x_i is 0 by assumption. Moreover, $h^0(E_i^* \otimes F_j) \neq 0$ for all $i \neq j$ since vanishing of the composition $E_i \hookrightarrow V \twoheadrightarrow F_j$ would imply that $E_i \hookrightarrow V$ factors through E_j, which would yield an isomorphism between E_i and E_j showing that the points x_i and x_j are the same. In summary, we have shown that $(E_i^*, F_j) \in \Theta$ if and only if $i \neq j$, which implies that the restriction of $\mathrm{SD}_{\sigma,\rho}$ to

$$\mathrm{span}\{\, \mathrm{ev}_{[E_i^*]}\,\} \otimes \mathrm{span}\{\, \mathrm{ev}_{[F_i]}\,\}$$

is a perfect pairing.

Thus the guiding problems 1.1.1 relate to the following outline for proving Conjecture 1.2.1.

Outline for proving strange duality 1.2.2. (a) Compute the dimensions $h^0\big(\mathcal{O}(\Theta_\rho)\big)$ and $h^0\big(\mathcal{O}(\Theta_{e^\vee})\big)$ of the spaces of sections of the determinantal line bundles.

(b) Construct V such that $\mathrm{Quot}(V, \rho)$ is finite and reduced (Problem 1).

(c) Enumerate the points of $\mathrm{Quot}(V, \rho)$ (Problem 2).

In cases where all three steps can be completed and the dimensions in (a) agree and coincide with the computation in (c), strange duality will be proved. Unfortunately, on del Pezzo surfaces (a) has been completed only in very special cases (in particular, when ρ is the Chern character of an ideal sheaf of points). If Quot schemes are to be a useful method for studying strange duality, then Problem 3 may be the key, since a more holistic approach to understanding Quot schemes could provide some insight into the spaces of sections in (a) as well. Ultimately, Problem 3 could provide a geometric perspective for understanding why strange duality should be true.

CHAPTER 2

A TASTE OF MODULI THEORY

We give an introduction to the functorial approach to classification problems in algebraic geometry, followed by a brief description of the moduli spaces that will occupy our attention in later chapters: the Grassmannian, moduli spaces of sheaves, and the omnipresent Quot scheme. Most of the material is standard and can be found in any introduction to moduli theory.

2.1 Classification problems in algebraic geometry

Somewhat surprisingly, many classification problems in algebraic geometry become more tractable when the problem is expanded. Typically, a classification problem specifies a set of algebraic objects and asks whether there is a scheme (called a "moduli space") whose closed points parametrize all the objects in the set. Instead, one can consider the more general problem of classifying families of those objects over arbitrary base schemes S. Assigning the set of all such families over S to the scheme S defines a functor from the category of schemes to the category of sets. The question of the existence of a moduli space parametrizing the chosen objects can then be rephrased in terms of the representability of this functor.

2.1.1 Functor of points

As a warm-up, consider the following classification problem: Given a scheme Y over a field k, is there a moduli space whose closed points parametrize the closed points of Y? The answer, of course, is that the scheme Y parametrizes its own closed points. Thus every scheme can be viewed as a moduli space.

Enlarging this classification problem, consider families of points of Y over a base scheme S. Such a family picks out a closed point of Y for each closed point of S, and these closed points should vary algebraically. You have seen such "families" before: they are simply the algebraic morphisms $S \to Y$. Thus the functor associated with the classification problem of "closed points in Y" is $S \mapsto \mathrm{Hom}(S, Y)$, namely $\mathrm{Hom}(-, Y)$.

Definition 2.1.1. Let Sch/k denote the category of schemes over a field k. Let Y be a scheme over k. The functor $h_Y \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ defined by $X \mapsto \mathrm{Hom}(X, Y)$ is called the **functor of points** of Y.

By the Yoneda lemma, the functor $h \colon \mathrm{Sch}/k \to \mathrm{Functors}\big((\mathrm{Sch}/k)^{\mathrm{op}}, \mathrm{Sets}\big)$ defined by $Y \mapsto h_Y$ is fully faithful, so the scheme Y is determined by its functor of points h_Y. Said another way, no information about Y is lost when we view Y as its functor of points.

We call $\mathrm{Hom}(S, Y)$ the S-**valued points** of Y. Indeed, $\mathrm{Hom}(\mathrm{Spec}\, k, Y)$ is the set of k-valued points of Y.

Definition 2.1.2. Let Y be a scheme over k. A functor $F \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ is **representable by** Y if there is an invertible natural transformation $\eta \colon F \to h_Y$.

Of course, h_Y is representable by Y (taking η to be the identity). Representability is subtler and more interesting when the functor F is defined without reference to a particular scheme Y.

2.1.2 Families

In order to make sense of the passage from classifying objects to classifying families of objects, we need to define notions of "family" for schemes and for sheaves. Most naively, a family of schemes over a base scheme S could be defined as a morphism $f \colon X \to S$, where we think of the fibers of f as the members of the family. This is a badly behaved notion because the fibers of the family can have nothing to do with each other. We could impose a local triviality condition, but this is too restrictive. For instance, we would like the plane curves of a given degree to form a family (parametrized by a projective space), permitting some of the fibers to be singular. Thus we need to put a weaker condition on the morphism, and the condition that has proven to be most useful is flatness.

Definition 2.1.3. Let S be a scheme over k. A **family of schemes over** S is a morphism of schemes $f \colon X \to S$ that is flat. A **family of coherent sheaves on** X **over** S is a coherent sheaf F on $X \times_k S$ that is flat over S.

Before defining what it means for a morphism of schemes (or a coherent sheaf) to be flat, we give a motivating example of a flat limit of schemes (see [EH00] for details). Let Z be a closed subscheme of $\mathbb{A}^n_k \times_k (\mathbb{A}^1_k \setminus 0)$. Then for each closed point $t \in \mathbb{A}^1_k \setminus 0$, the fiber of Z over t under the projection map to $\mathbb{A}^1_k \setminus 0$ is a closed subscheme of $\mathbb{A}^n_k$, hence determines an ideal I_t in the polynomial ring $k[x_1, \ldots, x_n]$. Taking the limit of the ideals I_t as $t \to 0$ produces an ideal I_0 defining a closed subscheme at the point $0 \in \mathbb{A}^1_k$. Extending Z to a closed subscheme of $\mathbb{A}^n_k \times_k \mathbb{A}^1_k$ in this manner ensures that the projection $Z \to \mathbb{A}^1_k$ is flat at 0. The subscheme defined by I_0 is called the flat limit of the subschemes I_t for $t \neq 0$.

Recall that if A is a commutative ring, then an A-module M is called flat if the functor $- \otimes_A M$ is left exact (tensor products are always right exact), which means that given any injective morphism $N \hookrightarrow N'$ of R-modules, the induced map $N \otimes_A M \to N' \otimes_R M$ is still injective.

Definition 2.1.4. Let $f\colon X \to S$ be a morphism and F be a coherent sheaf on X. Then F is **flat over** S **at** $x \in X$ if the stalk F_x is a flat $\mathcal{O}_{S,f(x)}$-module. We say F is **flat over** S if it is flat over S at all $x \in X$. In particular, f is **flat at** $x \in X$ if $\mathcal{O}_X$ is flat over S at x, and f is **flat** if $\mathcal{O}_X$ is flat over S.

It may be surprising that this algebraic definition can give a good geometric notion of what a family should be. The following propositions provide evidence that flatness is a reasonable condition to impose. The first proposition says that the Hilbert polynomial is constant on families of closed subschemes of $\mathbb{P}^n_k$, hence the fibers of the family have the same dimension, degree, and arithmetic genus. The second proposition is a partial generalization that applies to coherent sheaves. All schemes mentioned are assumed to be noetherian.

Proposition 2.1.5 ([Har77])**.** *Suppose S is an integral scheme over k and X is a closed subscheme of $\mathbb{P}^n_k \times_k S$. For all $s \in S$, let $P_s \in \mathbb{Q}[d]$ denote the Hilbert polynomial of the fiber $X_s \subset \mathbb{P}^n_{k(s)}$ of the projection map $f\colon X \to S$. Then f is flat $\iff$ P_s is independent of $s \in S$.*

Proposition 2.1.6 ([Vak])**.** *Suppose $f\colon X \to S$ is a projective morphism, S is irreducible, and F is a coherent sheaf on X that is flat over S. Then $\chi(F|_{f^{-1}(s)})$ is a constant function of $s \in S$.*

2.1.3 Moduli functors

We are ready to describe the general setup for classification problems. Suppose we want to parametrize a certain set $\mathcal{C}$ of objects (such as schemes or coherent sheaves) over the field k. Typically, the set of objects has to carefully chosen (for example, the objects should depend on finitely many parameters) for it to have a chance of being parametrized by a scheme. We then phrase the classification problem in terms of families by defining the **moduli functor**

$$\mathcal{M}_{\mathcal{C}}\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \mathcal{M}_{\mathcal{C}}(S) = \left\{ \begin{array}{c} \text{families of schemes or sheaves over } S \\ \text{whose fibers are objects in the set } \mathcal{C} \end{array} \right\}.$$

Given a morphism $T \to S$, pullback of families defines a function $\mathcal{M}_{\mathcal{C}}(S) \to \mathcal{M}_{\mathcal{C}}(T)$.

An important question is whether $\mathcal{M}_{\mathcal{C}}$ is representable by some scheme $M_{\mathcal{C}}$, namely whether there is an invertible natural transformation $\eta\colon \mathcal{M}_{\mathcal{C}} \to h_{M_{\mathcal{C}}}$. If so, then the k-valued points of $M_{\mathcal{C}}$ are in bijection with the objects in the set $\mathcal{C}$, so $M_{\mathcal{C}}$ parametrizes the objects in $\mathcal{C}$. Moreover, the canonical element $\mathrm{id} \in \mathrm{Hom}(M_{\mathcal{C}}, M_{\mathcal{C}})$ corresponds to a **universal family** $\mathcal{U}_{\mathcal{C}}$ over $M_{\mathcal{C}}$. A family X (or F) over S, which is an element of $\mathcal{M}_{\mathcal{C}}(S)$, corresponds to a morphism $f\colon S \to M_{\mathcal{C}}$, and naturality of η implies that X coincides with $f^*\mathcal{U}_C$. Thus every family is pulled back from the universal family.

2.2 Examples of representable functors

We describe three related moduli problems that work out as nicely as possible. In particular, the moduli functors are representable by projective schemes. Details and proofs can be found in Part 2 of [FGI$^+$05].

2.2.1 Grassmannian

Let V be a vector space over k of dimension $r+s$. Let $\mathcal{C}_{r,V}$ be the set of r-dimensional vector subspaces of V. If S is a scheme, write $V \otimes \mathcal{O}_S$ for the trivial vector bundle on S whose global sections are identified with V. Consider the moduli functor

$$\mathcal{G}_{r,V}\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \{\, \mathcal{E} \subset V \otimes \mathcal{O}_S \mid \mathcal{E} \text{ a vector subbundle of rank } r \,\}.$$

Here no flatness condition is required on $\mathcal{E}$ since all vector bundles are flat over the base. The fiber of a subbundle $\mathcal{E} \subset V \otimes \mathcal{O}_S$ at a point of S is a vector subspace $E \subset V$, so it makes sense to think of $\mathcal{E}$ as a family of vector subspaces of V.

This functor is representable by a nonsingular projective variety $\mathrm{Gr}(r, V)$, called the **Grassmannian**, whose k-valued points parametrize the r-dimensional subspaces of V. The element $\mathrm{id} \in \mathrm{Hom}(\mathrm{Gr}(r, V), \mathrm{Gr}(r, V))$ corresponds to a **universal subsheaf** $\mathcal{S}$ that induces a universal short exact sequence of vector bundles

$$0 \to \mathcal{S} \to V \otimes \mathcal{O}_{\mathrm{Gr}(r,V)} \to \mathcal{Q} \to 0$$

whose sequence of fibers over a point $x = [E \subset V]$ is exactly

$$0 \to E \to V \to F \to 0,$$

where F is the cokernel of $E \subset V$.

We can generalize this construction by replacing the role of the base $\mathrm{Spec}\, k$ by a scheme X over k and replacing the k-vector space V (which is a vector bundle on $\mathrm{Spec}\, k$) by a vector bundle $\mathcal{V}$ on X. Given a scheme S over k, let $\mathcal{V}_S$ denote the pullback of $\mathcal{V}$ under $X \times_k S \to X$. The moduli functor is

$$\mathcal{G}_{r,\mathcal{V}} \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \{\, \mathcal{E} \subset \mathcal{V}_S \mid \mathcal{E} \text{ a vector subbundle of rank } r \text{ on } X \times_k S \,\},$$

which is representable by the **Grassmann bundle** $\mathrm{Gr}(r, \mathcal{V})$ that has a map $\pi \colon \mathrm{Gr}(r, \mathcal{V}) \to X$ whose fiber $\pi^{-1}(p)$ over a closed point $p \in X$ is the Grassmannian $\mathrm{Gr}\big(r, \mathcal{V}(p)\big)$. The universal sequence on $\mathrm{Gr}(r, \mathcal{V})$ is

$$0 \to \mathscr{S} \to \pi^*\mathcal{V} \to \mathscr{Q} \to 0,$$

which restricted to the fiber $\pi^{-1}(p)$ recovers the universal sequence over $\mathrm{Gr}\big(r, \mathcal{V}(p)\big)$.

Remark 2.2.1. Since every vector subspace of V produces a quotient and vice versa, we can identify the Grassmannian $\mathrm{Gr}(r, V)$ of r-dimensional subspaces in V with the Grassmannian $\mathrm{Gr}(V, s)$ of s-dimensional quotients of V. Moreover, there is a duality isomorphism $\mathrm{Gr}(r, V) \simeq \mathrm{Gr}(V^*, r)$. Similar statements can be made for Grassmann bundles.

2.2.2 Hilbert scheme

Let X be a projective scheme over k and let L be an ample line bundle. If F is a coherent sheaf on X, then the Euler characteristic $\chi(F \otimes L^{\otimes d})$ is a polynomial in d, which we call the

Hilbert polynomial of F and denote $P_F(d) \in \mathbb{Q}[d]$. The Hirzebruch-Riemann-Roch formula states that

$$P_F(d) = \chi(F \otimes L^{\otimes d}) = \int_X \operatorname{ch}(F) \cup \operatorname{ch}(L)^d \cup \operatorname{td}(X),$$

where $\operatorname{td}(X)$ is the Todd class of X, so for fixed L the Hilbert polynomial $P_F(d)$ and the Chern character $\operatorname{ch}(F)$ of F determine each other. Given a closed subscheme $Y \subset X$, set $P_Y(d) = P_{\mathcal{O}_Y}(d)$, which we call the **Hilbert polynomial** of Y in X.

Now fix a Hilbert polynomial P and let $\mathcal{C}_{X,P}$ be the set of all closed subschemes of X with Hilbert polynomial P. The corresponding moduli functor is

$$\mathcal{H}ilb_{X,P} \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \left\{ \begin{array}{c} \text{closed subschemes } \mathcal{Y} \subset X \times_k S, \text{ flat over } S, \\ \text{whose fibers have Hilbert polynomial } P \end{array} \right\}.$$

This functor is representable by a projective scheme $\operatorname{Hilb}(X, P)$ called the **Hilbert scheme**, whose closed points parametrize the closed subschemes of X with Hilbert polynomial P. There is a **universal closed subscheme**

$$\mathcal{Y}_P \subset X \times_k \operatorname{Hilb}(X, P)$$

whose fiber over $[Y] \in \operatorname{Hilb}(X, P)$ under the projection map is exactly Y. We describe an example of a Hilbert scheme in §2.4.1.

2.2.3 Quot scheme

The Quot scheme is a generalization of the Grassmannian and the Hilbert scheme. Let X be a projective scheme over k, let V be a coherent sheaf on X, and fix a cohomology class σ on X. Let $\mathcal{C}_{\sigma,V}$ denote the set of coherent subsheaves $E \subset V$ whose Chern character is σ. Given a scheme S over k, let V_S denote the sheaf on $X \times_k S$ obtained from V by fiber product. Consider the **Quot functor**

$$\mathcal{Q}uot_{\sigma,V} \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \left\{ \mathcal{E} \subset V_S \;\middle|\; \begin{array}{c} \mathcal{E} \text{ coherent on } X \times_k S, \text{ fibers of } \mathcal{E} \text{ over } S \\ \text{have Chern character } \sigma \end{array} \right\}.$$

This functor is representable by a projective scheme $\operatorname{Quot}(\sigma, V)$ called the **Grothendieck Quot scheme**. There is a universal sequence

$$0 \to \mathcal{E}_{\sigma,V} \to \pi_X^* V \to \mathcal{F}_{\sigma,V} \to 0$$

over $X \times \operatorname{Quot}(\sigma, V)$ whose fiber over a point $[E \subset V] \in \operatorname{Quot}(\sigma, V)$ is

$$0 \to E \to V \to F \to 0,$$

where F is the cokernel of $E \subset V$. We call $\mathcal{E}_{\sigma,V}$ the **universal subsheaf** and $\mathcal{F}_{\sigma,V}$ the **universal quotient**.

Example 2.2.2. Let $X = \operatorname{Spec} k$. Then the vector bundle V is just a vector space and the Chern character σ is just a number representing a rank, so the Quot scheme coincides with the Grassmannian.

Example 2.2.3. For general X, let $V = \mathcal{O}_X$ and choose σ to be the Chern character of an ideal sheaf of a closed subscheme Y in X. Then the Quot scheme coincides with the Hilbert scheme parametrizing closed subschemes of X with the same Hilbert polynomial as Y (after choosing an ample line bundle L on X).

There is also a relative version of the Quot scheme. Let B be a noetherian scheme, let $\mathcal{X}$ be projective over B, and let $\mathcal{V}$ be a coherent sheaf on $\mathcal{X}$. Unlike Chern characters, Hilbert polynomials in different fibers over B can be directly compared. Let $\mathcal{L}$ be an ample line bundle on $\mathcal{X}$ that restricts to give ample line bundles L_b on each fiber X_b of $\mathcal{X} \to B$, which we use to define Hilbert polynomials on each X_b. Fix a Hilbert polynomial $P \in \mathbb{Q}[d]$. The **relative Quot functor**

$$\mathcal{Q}uot_{P,\mathcal{V},B}\colon (\mathrm{Sch}/B)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \left\{ \mathcal{E} \subset \mathcal{V}_{\mathcal{X}\times_B S} \;\middle|\; \begin{matrix} \mathcal{E} \text{ is flat over } S \text{ and its fibers over } S \\ \text{have Hilbert polynomial } P \end{matrix} \right\}$$

is also representable by a projective scheme $\pi\colon \operatorname{Quot}(P,\mathcal{V},B) \to B$ called the **relative Quot scheme**. Its fibers over B are the Quot schemes $\operatorname{Quot}(P, V_b)$. Let

$$0 \to \mathcal{E}_{P,\mathcal{V},B} \to \pi_{\mathcal{X}}^{*}\mathcal{V} \to \mathcal{F}_{P,\mathcal{V},B} \to 0$$

denote the universal sequence on $\mathcal{X} \times_B \operatorname{Quot}(P,\mathcal{V},B)$. Restricting the base to a closed point $b \in B$ yields the usual universal sequence

$$0 \to \mathcal{E}_{P,V_b} \to \pi_{X_b}^{*} V_b \to \mathcal{F}_{P,V_b} \to 0$$

on $X_b \times_b \operatorname{Quot}(P, V_b)$.

Remark 2.2.4. As in the case of the Grassmannian, we could think of Quot schemes as parametrizing either subsheaves or quotient sheaves (or even short exact sequences). Writing ρ for the Chern character of the quotients, we could replace the notation $\operatorname{Quot}(\sigma, V)$ by $\operatorname{Quot}(V, \rho)$. The former is more convenient in Chapters 3 and 4, but we switch to the latter in Chapters 5 and 6 when we want to emphasize a particular choice of ρ.

2.3 Moduli of sheaves

The three examples in §2.2 are dream cases where the moduli functor is representable. But there are many important moduli problems where representability fails, such as the problem of parametrizing coherent sheaves on a variety.

Let X be a smooth projective variety over k and let ρ be a cohomology class on X. Let $\mathcal{C}'_\rho$ denote the set of coherent sheaves on X whose Chern character is ρ. Consider the moduli functor

$$\mathcal{M}'_\rho \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \left\{ \begin{array}{c} \text{sheaves } \mathcal{F} \text{ on } X \times_k S, \text{ flat over } S, \text{ whose} \\ \text{fibers over } S \text{ have Chern character } \rho \end{array} \right\} / \sim,$$

where two families over S are equivalent if one can be obtained from the other by tensoring by the pullback of a line bundle from S. Despite the apparent similarity to the Quot functor, this functor has no chance of being representable.

The first problem is that the set $\mathcal{C}'_\rho$ is not bounded. There are certain numerical invariants that are bounded on every set of objects that can be parametrized by a scheme of finite type, such as Castelnuovo-Mumford regularity, which are not bounded on objects in the set $\mathcal{C}'_\rho$. For example, on $X = \mathbb{P}^1_k$, the set of rank 2 sheaves of degree 0 contains $\mathcal{O}(-a) \oplus \mathcal{O}(a)$ for all $a \in \mathbb{Z}$ and these sheaves have unbounded regularity.

This first problem can be fixed by requiring that the sheaves be semistable. Choose a line bundle L on X. A torsion-free sheaf F on X is **semistable** if, for every proper subsheaf $E \subsetneq F$,

$$p_E(d) \le p_F(d) \text{ for all } d \gg 0.$$

Here $p_F(d)$ denotes the reduced Hilbert polynomial of F, which is obtained by normalizing the usual Hilbert polynomial so that its top-degree coefficient is one. If the inequality of reduced Hilbert polynomials is strict for all proper subsheaves, then F is **stable**. Requiring sheaves in $\mathcal{C}'_\rho$ to be semistable yields a new set $\mathcal{C}_\rho$ that is bounded. The improved moduli functor is

$$\mathcal{M}_\rho \colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}, \quad S \mapsto \left\{ \begin{array}{c} \text{semistable sheaves } \mathcal{F} \text{ on } X \times_k S, \text{ flat over } S, \\ \text{whose fibers over } S \text{ have Chern character } \rho \end{array} \right\} / \sim .$$

The second problem is that semistable sheaves may have automorphisms in addition to the homotheties. For instance, $\mathcal{O}^r_{\mathbb{P}^1_k}$ has automorphism group $\mathrm{GL}(r,k)$. The presence of interesting automorphisms in the objects being parametrized usually spoils any chance of the moduli functor being representable. Intuitively, the idea is that a trivial family $\pi_S^* E$ on $S \times [0,1]$ can be made into a nontrivial family on $S \times S^1$ by using an automorphism of E to glue the copies of E on $S \times 0$ and $S \times 1$ (similar to a mapping torus). If the moduli functor were representable, then this family should be the pullback of the universal family on the moduli space M under the induced morphism $S^1 \to M$. Since the fibers of the family are constant, so is the morphism to M, hence the pullback should be a trivial bundle, which it is not.

Giving up on the notion of representability, one can still ask whether there is a moduli space that parametrizes semistable sheaves in a weaker sense.

Definition 2.3.1. Let Y be a scheme over k. A functor $F\colon (\mathrm{Sch}/k)^{\mathrm{op}} \to \mathrm{Sets}$ is **coarsely representable** by Y if there is a natural transformation $\eta\colon F \to h_Y$ such that given any other scheme Z and natural transformation $\nu\colon F \to h_Z$, there is a unique morphism $\phi\colon Y \to Z$ such that $\phi_* \circ \eta = \nu$.

Indeed, there is a projective scheme $M(\rho)$, called the **moduli space of sheaves**, that coarsely represents the moduli functor $\mathcal{M}_\rho$. But coarse representability is not as nice as representability. In general, not only is there no universal family, but there is not even a bijection between sheaves in the set $\mathcal{C}_\rho$ and the closed points of $M(\rho)$. The closed points of $M(\rho)$ parametrize "S-equivalence classes" of semistable sheaves: every semistable sheaf F has a Jordan-Hölder filtration

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = F$$

in which the factors F_k/F_{k-1} are all stable, and two sheaves are S-equivalent if they have the same factors (up to reordering).

2.4 Details on Hilbert schemes and Quot schemes

We take a closer look at particularly nice Hilbert schemes on surfaces and summarize the basic deformation theory of Quot schemes.

2.4.1 Hilbert schemes of points on surfaces

One moduli space that plays an especially important role in Chapter 6 is the Hilbert scheme of points on a smooth projective surface S. For $n > 0$, let P be the constant function n. Then $\mathrm{Hilb}(S,P)$, which is often denoted $S^{[n]}$ and is called the **Hilbert scheme of points** on S, parametrizes zero-dimensional subschemes of S of length n. Generically, these subschemes are direct sums of one-dimensional skyscraper sheaves supported at n distinct points, but these points are allowed to collide, in which case nonreduced structure remembers tangency information about how the points came together. In particular, subschemes of length two supported at a point p correspond to tangent directions at p. By a theorem in [Fog68], $S^{[n]}$ is an irreducible smooth projective variety of dimension $2n$. This dimension makes sense since each of the n points is chosen on the two-dimensional surface. Note that when $n = 1$, $S^{[1]}$ is naturally isomorphic to S.

There is a **Hilbert-Chow morphism** $S^{[n]} \to S^{(n)}$ to the symmetric product of S that forgets the nonreduced structure and remembers only multiplicities. The symmetric product is singular for $n > 1$ and the Hilbert-Chow morphism is a resolution of singularities. Let B denote the exceptional divisor, which contains the nonreduced subschemes. One can show that $\frac{B}{2}$ is also a Cartier divisor.

The Picard group of $S^{[n]}$ for $n > 1$ has the simple description

$$\mathrm{Pic}(S^{[n]}) \simeq \mathrm{Pic}(S) \oplus \mathbb{Z} \cdot \tfrac{B}{2}.$$

The map $\mathrm{Pic}(S) \to \mathrm{Pic}(S^{[n]})$ can be described as follows. Given a line bundle L on S, there is a line bundle $\otimes_{i=1}^{n} \pi_i^* L$ on the product S^n, where $\pi_i \colon S^n \to S$ is the ith projection. This line bundle carries an action of the symmetric group, so it descends to give a line bundle on $S^{(n)}$, which can be pulled back to a line bundle L_n on $S^{[n]}$ under the Hilbert-Chow morphism. On effective divisors D on S, the induced divisor D_n on $S^{[n]}$ consists of all subschemes whose support intersects D.

In principle, the following beautiful formula can be used to compute the Euler characteristic of any line bundle on any Hilbert scheme of points for any surface S.

Theorem 2.4.1 ([EGL01] Theorem 5.3)**.** *For any surface S,*

$$\sum_{n \geq 0} \chi\left(\mathcal{O}_{S^{[n]}}(L_n - r\tfrac{B}{2})\right) z^n = g_r(z)^{\chi(L)} \cdot f_r(z)^{\frac{1}{2}\chi(\mathcal{O}_S)} \cdot A_r(z)^{L.K_S - \frac{1}{2}K_S^2} \cdot B_r(z)^{K_S^2}, \tag{2.1}$$

where $A_r(Z)$, $B_r(Z)$, $f_r(z)$, $g_r(z)$ are power series in z depending only on r, and

$$f_r(z) = \sum_{k \geq 0} \binom{(1-r^2)(k-1)}{k} z^k; \quad g_r(z) = \sum_{k \geq 0} \frac{1}{1-(r^2-1)k} \binom{1-(r^2-1)k}{k} z^k.$$

For details of how to compute the $A_r(z)$ and $B_r(z)$ series, see [Joh16], which also provides the explicit formulas to order 11.

Remark 2.4.2. We have already seen the right side of (2.1) appear in the conjectural formula (1.2), so an explanation is necessary. $S^{[n]}$ can be viewed as the moduli space of sheaves parametrizing ideal sheaves of n points on S. When L is sufficiently ample (to be precise, $(n-1)r$-very ample; see [BGJ16]), then $\chi(\mathcal{O}_{S^{[n]}}(L_n - r\frac{B}{2})) = h^0(\mathcal{O}_{S^{[n]}}(L_n - r\frac{B}{2}))$, so the left side of (2.1) computes dimensions of spaces of global sections of determinantal line bundles on $S^{[n]}$. Theorem 6.1.1 provides evidence that when S is del Pezzo and $n \leq 7$, the numbers of points of finite quot schemes agree with these dimensions. If this could be proven, then strange duality would be resolved (in these cases) by the method described in §1.2.

We give some examples of computations of Euler characteristics using the formula. In the case $n = 1$, the formula is unnecessary since $S^{[1]} = S$ and the Euler characteristic is just $\chi(L)$. In the case $n = 2$, the formula yields

$$\chi(\mathcal{O}_{S^{[2]}}(L_2 - r\tfrac{B}{2})) = \binom{\chi(L)}{2} - (r^2-1)\chi(L) - \binom{r+1}{3} L.K_S - \binom{r+1}{4} K_S^2 + \tfrac{1}{2}\binom{r^2}{2}. \quad (2.2)$$

When $n = 3$ and $r = 2$, the formula is

$$\chi(\mathcal{O}_{S^{[3]}}(L_3 - B)) = \binom{\chi(L)}{3} - \chi(L)(3\chi(L) + L.K_S - 21) + 9L.K_S + K_S^2 - 28. \quad (2.3)$$

2.4.2 Deformation theory of Quot schemes

As in §2.2.3, let X be a projective scheme, V be a vector bundle on X, σ be a Chern character, and $\mathrm{Quot}(\sigma, V)$ be the Quot scheme. We can think of closed points of $\mathrm{Quot}(\sigma, V)$ as short exact sequences $x = [0 \to E \to V \to F \to 0]$, where E has Chern character σ and F has Chern character $\rho = \mathrm{ch}(V) - \sigma$ by additivity.

The Zariski tangent space to $\mathrm{Quot}(\sigma, V)$ at x is $\mathrm{Hom}(E, F)$. As a result, the **expected dimension** of $\mathrm{Quot}(\sigma, V)$ is

$$\chi(\sigma, \rho) = \chi(E, F) = \sum_{i=0}^{\dim X} (-1)^i \mathrm{ext}^i(E, F)$$

since at general points x the higher $\mathrm{ext}^i(E, F)$ should vanish. The obstructions to deforming a short exact sequence x lie in $\mathrm{Ext}^1(E, F)$. If the higher $\mathrm{ext}^i(E, F)$ do all vanish, then the tangent space has the expected dimension and there are no obstructions to deforming x, so x is a smooth point of $\mathrm{Quot}(\sigma, V)$. In general, Quot schemes have (singular) points where $\mathrm{ext}^1(E, F)$ is nonzero and $\hom(E, F)$ jumps above the expected dimension. The spaces $\mathrm{Ext}^i(E, F)$ contain higher-order obstructions.

The following theorem about the relative Quot scheme is a powerful tool that can be used to study deformations of quotients. It could be stated with a more general vector bundle rather than the trivial bundle. A version for Hilbert schemes is Theorem 2.15 of [Kol96].

Theorem 2.4.3 ([Kol96])**.** *Let $\mathcal{X}$ be a scheme that is projective and flat over B. Let X_b denote the fiber of $\mathcal{X}$ over $b \in B$. Let $x = [E \subset \mathcal{O}^r_{X_b}]$ be a point in $\mathrm{Quot}(P, \mathcal{O}^r_{X_b})$. Assume B is equidimensional at b. Then the dimension of every irreducible component of $\mathrm{Quot}(P, \mathcal{O}^r_{\mathcal{X}}, B)$ at x is at least*

$$\hom_{X_b}(\mathcal{E}, \mathcal{F}) - \mathrm{ext}^1_{X_b}(\mathcal{E}, \mathcal{F}) + \dim_b B.$$

If equality holds, then $\mathrm{Quot}(P, \mathcal{O}^r_{\mathcal{X}}, B) \to B$ is lci at x; in particular, it is flat at x.

Remarkably, the theorem implies that if a particular point of a Quot scheme has the right dimension, then that point deforms in every relative Quot scheme containing that Quot scheme as a fiber.

Corollary 2.4.4. *Let X_0 be a scheme over k. If the dimension of an irreducible component of $\mathrm{Quot}(P, \mathcal{O}^{r+s}_{X_0})$ at some point x is equal to $\hom_{X_0}(\mathcal{E}, \mathcal{F}) - \mathrm{ext}^1_{X_0}(\mathcal{E}, \mathcal{F})$, then x deforms nicely in* every *relative quot scheme $\mathrm{Quot}(P, \mathcal{O}^{r+s}_{\mathcal{X}}, B)$ such that $\mathcal{X}|_b = X_0$ for some $b \in B$ at which B is equidimensional.*

CHAPTER 3

QUOT SCHEMES ON CURVES

We explain geometric structures (topological quantum field theories) that can be used to study the enumerative geometry of finite Quot schemes on curves in the context of all Quot schemes of general vector bundles on curves. We use many facts about Quot schemes without providing justification; these are proved in Chapter 4.

3.1 Motivation

On a curve C, we can give satisfactory answers to both guiding questions about Quot schemes. Let g be the genus of C and let $\sigma = (r, -e)$ and $\rho = (s, f)$ be orthogonal Chern characters. Then

$$\chi(\sigma, \rho) = rf + se - rs(g-1) = 0.$$

In particular, let $e = r(g-1)$ and $f = 0$.

Theorem 3.1.1 ([MO07]). *Let $\sigma = (r, -r(g-1))$. There exists a ("general") vector bundle V of rank $r+s$ and degree $-r(g-1)$ such that $\mathrm{Quot}(\sigma, V)$ is finite and reduced and its points are counted by the (modified) Verlinde formula*

$$V_g^{r,s} = \sum_{\substack{T \sqcup U = \{1,\dots,r+s\} \\ |T| = r}} \prod_{\substack{t \in T \\ u \in U}} \left| 2 \sin \pi \frac{t-u}{r+s} \right|^{g-1}.$$

The Verlinde numbers $V_g^{r,s}$ are related to the ranks of vector bundles of conformal blocks on moduli spaces of curves, so they are integers, which is by no means obvious from the formula.

Example 3.1.2 ($g = 1$). A general semistable vector bundle of degree 0 on a curve of genus 1 is of the form $V = \oplus_{i=1}^{r+s} L_i$, where the L_i are distinct degree 0 line bundles. Then every subsheaf of V with rank r and degree 0 is of the form $L_{i_1} \oplus \cdots \oplus L_{i_r}$. Thus the Quot scheme parametrizes choices of r of the $r+s$ line bundles, so $\#\mathrm{Quot}\big((r,0), V\big) = \binom{r+s}{r}$.

Indeed, since $g-1=0$, the exponents in the Verlinde formula are 0, hence $V_1^{r,s}$ equals the number of partitions $T \sqcup U = \{1,\dots,r+s\}$, which is $\binom{r+s}{r}$. Note the importance of V being general, since for example taking $V = \mathcal{O}_C^{r+s}$, the Quot scheme would parametrize subsheaves $\mathcal{O}_C^r \subset \mathcal{O}_C^{r+s}$, hence it would be isomorphic to $\mathrm{Gr}(r, \mathbb{C}^{r+s})$.

Proof sketch. Choose L sufficiently ample of degree ℓ and construct V as a general elementary modification

$$0 \to V \otimes L^* \to \mathcal{O}_C^{r+s} \to \bigoplus_{i=1}^{r(g-1)+(r+s)\ell} \mathbb{C}_{q_i} \to 0,$$

which is determined by distinct points q_i and for each i a general quotient $\pi_i \colon \mathbb{C}^{r+s} \twoheadrightarrow \mathbb{C}$. This induces an embedding

$$\mathrm{Quot}(\sigma, V) \hookrightarrow \mathrm{Quot}(\sigma', \mathcal{O}_C^{r+s}), \quad [E \subset V] \mapsto [E \otimes L^* \subset V \otimes L^* \subset \mathcal{O}_C^{r+s}],$$

and the image is cut out by the "Schubert varieties" $\overline{W}_{1^r}(q_i)$ on the Quot scheme consisting of those $[E' \subset \mathcal{O}_C^{r+s}]$ such that the compositions $E' \hookrightarrow \mathcal{O}_C^{r+s} \twoheadrightarrow \mathbb{C}_{q_i}$ are 0, namely the image of the fiber of E' at q_i is contained in $\ker \pi_i$. Since the elementary modification is general, so are the Schubert varieties. By a theorem of Bertram, for all σ' having degree sufficiently negative, intersections of general Schubert varieties on $\mathrm{Quot}(\sigma', \mathcal{O}_C^{r+s})$ are proper (they have the expected codimension) and top intersections are reduced. Since $\dim \mathrm{Quot}(\sigma', \mathcal{O}_C^{r+s}) = r(r+s)\ell + r^2(g-1)$ and each $\overline{W}_{1^r}(q_i)$ has codimension r, we see that $\mathrm{Quot}(\sigma', V)$ is finite and reduced. Moreover, the intersection of the Schubert varieties can be computed in the cohomology of the Quot scheme using the Vafa-Intriligator formula, and applying trigonometric identities yields the Verlinde formula. □

The proof features a number of ideas that we use later in this chapter and prove in the next chapter:

(a) We can embed Quot schemes of arbitrary vector bundles inside Quot schemes of trivial bundles. If the vector bundle is general, the image will inherit good properties.

(b) We can define Schubert varieties $\overline{W}_{\vec{a}}(q)$ and do intersection theory on Quot schemes.

(c) Schubert varieties of type $\overline{W}_{1^r}$ will appear whenever we lower the degree of a vector bundle by elementary modifications.

Using these ideas, we will show that finite Quot schemes are computed by a weighted topological quantum field theory (TQFT) that encodes information about all Quot schemes of general vector bundles on curves. In particular, the TQFT explains why the cardinality of a finite Quot scheme should be a sum of $(g-1)$-powers by computing it as the trace of the $(g-1)$-power of a linear operator on the cohomology ring $H^*\big(\mathrm{Gr}(r,\mathbb{C}^{r+s}),\mathbb{C}\big)$.

3.2 Schubert calculus

Let $r, s \geq 1$ be integers and set $V = \mathbb{C}^{r+s}$. The Grassmannian is an rs-dimensional compact complex manifold that can be described as a set by

$$\mathrm{Gr}(r,V) = \{\, E \subset V \mid E \text{ a vector subspace of } V \text{ of dimension } r \,\}.$$

It has the structure of a projective variety by the Plücker embedding

$$\mathrm{Gr}(r,V) \hookrightarrow \mathbb{P}(\wedge^r V), \quad [E \subset V] \mapsto [\wedge^r E \subset \wedge^r V]$$

into the projective space of lines in $\wedge^r V$. For convenience, write $G = \mathrm{Gr}(r,V)$.

Let $0 = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{r+s} = V$ be a full flag in V, which we denote by $V_\bullet$. Let $\vec{a} = (a_1, \ldots, a_r)$ satisfy $s \geq a_1 \geq \cdots \geq a_r \geq 0$, which we call a partition. Define the **Schubert variety**

$$W_{\vec{a}}(V_\bullet) = \{\, [E \subset V] \in G \mid \dim(E \cap V_{s+i-a_i}) \geq i \text{ for all } 1 \leq i \leq r \,\},$$

which has complex codimension in G equal to the size $|\vec{a}| = \sum a_i$ of the partition. Let $\sigma_{\vec{a}} = [W_{\vec{a}}(V_\bullet)]$ denote the associated **Schubert cycle** in the cohomology ring $H^*(G,\mathbb{C})$; we drop $V_\bullet$ from the notation because the cohomology class of a Schubert variety does not depend on the flag. Intuitively, one can think of the cohomology ring as a complex vector space generated by classes of subvarieties up to equivalence, graded by codimension of the subvarieties, and equipped with the cup product $\cup$, which is defined on classes of subvarieties Y and Z that intersect transversally as intersection: $[Y] \cup [Z] = [Y \pitchfork Z]$. The set

$$\{\, \sigma_{\vec{a}} \mid \vec{a} \text{ a partition as above} \,\}$$

is an additive basis of $H^*(G,\mathbb{C})$.

Example 3.2.1 ($r = 1$)**.** An additive basis for the cohomology ring of $\mathrm{Gr}(1, V) = \mathbb{P}(V) = \mathbb{P}^s$ is given by $1, h, h^2, \ldots, h^s$, where h is the class of a hyperplane, h^s is the class [pt] of a point, and $h^k = \sigma_k$ for all k.

Schubert calculus is concerned with computing cup products with respect to the basis of Schubert cycles. We will need to understand the **Poincaré pairing**

$$\langle \sigma_{\vec{a}}, \sigma_{\vec{b}} \rangle = \text{coefficient of [pt] in } \sigma_{\vec{a}} \cup \sigma_{\vec{b}},$$

which is equal to the number of intersection points of Schubert varieties $W_{\vec{a}}(V_\bullet)$ and $W_{\vec{a}}(V'_\bullet)$ or 0 if this number is not finite; the flags $V_\bullet, V'_\bullet$ should be chosen to be general with respect to each other (namely $V_i \cap V'_{r+s-j}$ should be $(i-j)$-dimensional for all $0 \leq j \leq i \leq r+s$) to ensure a proper (of the expected codimension) intersection.

We extend the Poincaré pairing to all cohomology classes and to more than two arguments (by taking the cup product of all the arguments). We can see the Poincaré duality in the following lemma.

Lemma 3.2.2. *For all partitions $\vec{a}$ and $\vec{b}$ such that $|\vec{a}| + |\vec{b}| = rs$,*

$$\langle \sigma_{\vec{a}}, \sigma_{\vec{b}} \rangle = \begin{cases} 1 & \text{if } \vec{b} = \vec{a}^c; \\ 0 & \text{otherwise} \end{cases}$$

where $\vec{a}^c = (s - a_r, s - a_{r-1}, \ldots, s - a_1)$.

Proof. Let $V_\bullet$ and $V'_\bullet$ be general flags and suppose $[E \subset V] \in W_{\vec{a}}(V_\bullet) \cap W_{\vec{b}}(V'_\bullet)$. Then for all $1 \leq i \leq r$, the conditions

$$\dim(E \cap V_{s+i-a_i}) \geq i \quad \text{and} \quad \dim(E \cap V'_{s+(r+1-i)-b_{r+1-i}}) \geq r+1-i$$

imply that $E \cap V_{s+i-a_i}$ and $E \cap V'_{s+(r+1-i)-b_{r+1-i}}$ cannot be disjoint since E is r-dimensional. Since the flags are general, $V_{s+i-a_i} \cap V'_{s+(r+1-i)-b_{r+1-i}}$ has dimension $a_i + b_{r+1-i} - s - 1$, and the only way this can be positive for all i subject to the constraint $|\vec{a}| + |\vec{b}| = rs$ is if $a_i + b_{r+1-i} = s$ for all i, namely $\vec{b} = \vec{a}^c$. In this case, E is uniquely determined as the span of these one-dimensional intersections. □

Since the basis of Schubert cycles is orthonormal under the Poincaré pairing, we can write $\sigma_{\vec{a}_1} \cup \cdots \cup \sigma_{\vec{a}_N}$ as

$$\sigma_{\vec{a}_1} \cup \cdots \cup \sigma_{\vec{a}_N} = \sum_{\vec{c}} \langle \sigma_{\vec{a}_1}, \ldots, \sigma_{\vec{a}_N}, \sigma_{\vec{c}} \rangle \, \sigma_{\vec{c}^c}.$$

Since $\cup$ is associative (intersection is associative), the complete Schubert calculus is determined by the triple pairings of Schubert cycles arising in the cup products

$$\sigma_{\vec{a}} \cup \sigma_{\vec{b}} = \sum_{\vec{c}} \langle \sigma_{\vec{a}}, \sigma_{\vec{b}}, \sigma_{\vec{c}} \rangle \, \sigma_{\vec{c}^c}.$$

There are classical formulas of Giambelli and Pieri that allow for the mechanical computation of the Schubert calculus.

3.3 Quantum Schubert calculus

We define a quantum deformation of the cup product by introducing a new integer parameter $e \geq 0$. There is a smooth quasiprojective variety $\mathrm{Mor}_e(\mathbb{P}^1, G)$ of dimension $(r+s)e + rs$ whose points can be described as

$$\mathrm{Mor}_e(\mathbb{P}^1, G) = \{\, \phi \colon \mathbb{P}^1 \to G \text{ holomorphic} \mid \deg \phi = e \,\}.$$

Here the degree of ϕ is measured relative to $\mathcal{O}_G(1)$ from the Plücker embedding, namely by intersecting the image of the composition $\mathbb{P}^1 \to G \to \mathbb{P}(\wedge^r \mathbb{C}^{r+s})$ with a general hyperplane in $\mathbb{P}(\wedge^r \mathbb{C}^{r+s})$ and counting points.

Given a choice of general points $p_1, \ldots, p_N \in \mathbb{P}^1$ and general Schubert varieties $W_{\vec{a}_1}$, $\ldots$, $W_{\vec{a}_N}$, define the **Gromov-Witten number** $\langle W_{\vec{a}_1}, \ldots, W_{\vec{a}_N} \rangle_e$ as

$$\#\{\, \phi \in \mathrm{Mor}_e(\mathbb{P}^1, G) \mid \phi(p_i) \in W_{\vec{a}_i} \text{ for all } 1 \leq i \leq N \,\}$$

when this number is finite and zero otherwise. Requiring $\phi(p_i) \in W_{\vec{a}_i}$ is a codimension-$|\vec{a}_i|$ condition in $\mathrm{Mor}_e(\mathbb{P}^1, G)$, hence a necessary condition for $\langle W_{\vec{a}_1}, \ldots, W_{\vec{a}_N} \rangle_e$ to be nonzero is that the total size of the partitions matches the dimension of $\mathrm{Mor}_e(\mathbb{P}^1, G)$, namely $\sum |\vec{a}_i| = (r+s)e + rs$.

As the notation suggests, the Gromov-Witten numbers do not depend on the choice of points or the general full flags used to define the Schubert varieties. We prove this in §4.1.2 by defining them as intersections of generalized Schubert varieties on Quot schemes that compactify $\mathrm{Mor}_e(\mathbb{P}^1, G)$. Intersection theory works better on compact spaces, so the strategy is to compactify $\mathrm{Mor}_e(\mathbb{P}^1, G)$, define the intersection theory on the compactification, and show that there is no contribution from the boundary.

More generally, we make the following constructions. If V is a vector bundle on a curve C and e is an integer, the Quot scheme $Q_{e,V} = \mathrm{Quot}\big((r, -e), V\big)$ parametrizes coherent subsheaves $E \subset V$ that have rank r and degree $-e$. Since V is a vector bundle, E is necessarily locally free, but the cokernel may have torsion. Letting $U_{e,V} \subset Q_{e,V}$ denote the open set where the cokernel has no torsion and choosing a point $p \in C$, there is an evaluation morphism

$$\mathrm{ev}_p \colon U_{e,V} \to G, \quad [E \subset V] \mapsto [E(p) \subset V(p)].$$

Choosing a full flag $V_\bullet \subset V(p)$, we can define the Schubert variety $\overline{W}_{\vec{a}}(p, V_\bullet) \subset Q_{e,V}$ to be the closure of the preimage $\mathrm{ev}_p^{-1}(W_{\vec{a}}(V_\bullet))$. In the case when $V = \mathcal{O}_{\mathbb{P}^1}^{r+s}$, the space of morphisms $\mathrm{Mor}_e(\mathbb{P}^1, G) = U_{e,\mathcal{O}_{\mathbb{P}^1}^{r+s}}$ is a dense open subscheme in $Q_{e,\mathcal{O}_{\mathbb{P}^1}^{r+s}}$ and the Schubert variety $\overline{W}_{\vec{a}}(p, V_\bullet)$ is the closure of the locus of maps $\phi \in \mathrm{Mor}_e(\mathbb{P}^1, G)$ such that $\phi(p) \in W_{\vec{a}}(V_\bullet)$.

Letting $\bar{\sigma}_{\vec{a}}$ denote the cohomology class of $\overline{W}_{\vec{a}}(p, V_\bullet)$ in $Q_{e,V}$, which does not depend on p or on $V_\bullet$, define

$$\int_{Q_{e,V}} \bar{\sigma}_{\vec{a}_1} \cup \cdots \cup \bar{\sigma}_{\vec{a}_N} = \# \left(\overline{W}_{\vec{a}_1}(p_1, V_\bullet^{(1)}) \cap \cdots \cap \overline{W}_{\vec{a}_N}(p_N, V_\bullet^{(N)}) \right),$$

where the points p_i are distinct and the flags $V_\bullet^{(i)}$ are general. In the particular case when $V = \mathcal{O}_{\mathbb{P}^1}^{r+s}$, these integrals define the Gromov-Witten numbers and show that they are independent of the choices of points or flags.

The following example is the case when $\mathrm{Mor}_e(\mathbb{P}^1, G)$ is already compact and carries the theory of Schubert cycles we described in §3.2.

Example 3.3.1 ($e = 0$)**.** $\mathrm{Mor}_0(\mathbb{P}^1, G) \simeq G$ since the holomorphic maps ϕ of degree 0 are the constant maps. Under this identification, the condition $\phi(p_i) \in W_{\vec{a}_i}$ translates to $\phi \in W_{\vec{a}_i}$ for all i, hence the Gromov-Witten numbers are just counting points in the intersection of the Schubert varieties, so $\langle W_{\vec{a}_1}, \ldots, W_{\vec{a}_N} \rangle_0 = \langle \sigma_{\vec{a}_1}, \ldots, \sigma_{\vec{a}_N} \rangle$.

Example 3.3.2. $\langle W_{\vec{a}_1}, W_{\vec{a}_2} \rangle_e = 0$ for all $e > 0$. This is because there is a one-dimensional family of automorphisms of $\mathbb{P}^1$ that fixes two points, hence the number of morphisms of degree $e > 0$ satisfying conditions imposed at only two points is infinite.

Assuming for now that the Gromov-Witten numbers depend only on e and on the partitions $\vec{a}_i$, we can construct a "small quantum deformation" of the cup product called the **quantum product**, which is defined by

$$\sigma_{\vec{a}_1} * \cdots * \sigma_{\vec{a}_N} = \sum_{e \geq 0} q^e \left(\sum_{\vec{b}} \langle W_{\vec{a}_1}, \ldots, W_{\vec{a}_N}, W_{\vec{b}} \rangle_e \, \sigma_{\vec{b}^c} \right).$$

By Example 3.3.1, the $e = 0$ term is $\sigma_{\vec{a}_1} \cup \cdots \cup \sigma_{\vec{a}_N}$. Here q is a formal variable of degree $r+s$ to make the product homogeneous, which can be dropped if desired since the degree of each term depends on e. The sum is finite since $e \gg 0$ (namely $(r+s)e + rs > (N+1)rs$) makes it impossible for an intersection of $N+1$ Schubert varieties in $Q_{e,\mathcal{O}^{r+s}_{\mathbb{P}^1}}$ to be zero-dimensional.

The quantum product is clearly commutative, but it takes a good deal of work to show it is associative ([RT95], [KM97]). Since $*$ is associative, all Gromov-Witten numbers (and hence all quantum products) are determined by the three-point numbers that arise in products of the form

$$\sigma_{\vec{a}} * \sigma_{\vec{b}} = \sum_{e \geq 0} q^e \left(\sum_{\vec{c}} \langle W_{\vec{a}}, W_{\vec{b}}, W_{\vec{c}} \rangle_e \, \sigma_{\vec{c}^c} \right).$$

Equipping the vector space $H^*(G, \mathbb{C})[q]$ with the quantum product instead of the cup product yields the (small) **quantum cohomology ring** $QH^*(G)$. If we also consider the Poincaré pairing, which is compatible with the quantum product in the sense that

$$\langle \sigma_{\vec{a}} * \sigma_{\vec{b}}, \sigma_{\vec{c}} \rangle = \sum_{e \geq 0} \langle W_{\vec{a}}, W_{\vec{b}}, W_{\vec{c}} \rangle_e = \langle \sigma_{\vec{a}}, \sigma_{\vec{b}} * \sigma_{\vec{c}} \rangle,$$

then $QH^*(G)$ has the structure of a Frobenius algebra. A geometric way to visualize this structure is as a topological quantum field theory, which we discuss in the next section.

3.4 Topological quantum field theories

The terse description of topological quantum field theories in this section is based on a much nicer exposition with motivation and pictures in [Cav05].

A (two-dimensional) **topological quantum field theory** (TQFT) is a functor of tensor categories

$$F \colon \text{2Cob} \to \text{Vect}_{\mathbb{C}}.$$

The category 2Cob is composed of

(a) objects: finite disjoint unions of oriented circles;

(b) morphisms: equivalence classes of oriented cobordisms, which are oriented topological surfaces with oriented boundary circles;

(c) composition: concatenation of cobordisms by gluing boundary circles on one surface to boundary circles of opposite orientation on another surface;

(d) tensor structure: disjoint union.

Let H denote the image $F(S^1)$, which is a vector space. Since F is a functor of tensor categories, the image of n disjoint copies of S^1 is $H^{\otimes n}$ and the image of the empty union of circles is the base field $\mathbb{C}$. The genus g surface $\Sigma(g)^n_m$ with m boundary circles of one orientation and n boundary circles of the other orientation gets mapped to a linear transformation $F(g)^n_m \colon H^{\otimes n} \to H^{\otimes m}$. Thus each surface with boundary specifies an algebraic structure on the tensor powers of H and these algebraic structures must satisfy a large number of relations coming from composition of cobordisms.

Example 3.4.1. (a) The cylinder $\Sigma(0)^1_1$ is the identity under concatenation of cobordisms, hence $F(0)^1_1 \colon H \to H$ is the identity map.

(b) The closed torus $\Sigma(1)^0_0$ can be obtained by gluing the ends of the cylinder. In $\mathrm{Vect}_{\mathbb{C}}$, gluing the ends corresponds to taking the trace, so $F(1)^0_0 \colon \mathbb{C} \to \mathbb{C}$ is the map $1 \mapsto \dim H$.

(c) The pair of pants $\Sigma(0)^1_2$ defines a product $F(0)^1_2 \colon H \otimes H \to H$ on H.

(d) The cap $\Sigma(0)^1_0$ composed with the pair of pants produces the cylinder, so the map $F(0)^1_0 \colon \mathbb{C} \to H$ picks out the multiplicative identity in H.

(e) The macaroni $\Sigma(0)^0_2$ corresponds to a pairing $F(0)^0_2 \colon H \otimes H \to \mathbb{C}$.

(f) Powers of $\Sigma(1)^1_1$ can be used to produce every $\Sigma(g)^1_1$. We call $F(1)^1_1$ the genus-addition operator.

(g) We think of $F(g)^0_0$, the closed invariants of the TQFT, as complex numbers. Since $\Sigma(g)^0_0$ can be obtained by concatenating $g-1$ copies of $\Sigma(1)^1_1$ and gluing the ends, we see that $F(g)^0_0$ can be computed as the trace of the $g-1$ power of the genus-addition operator.

By decomposing surfaces into pairs of pants, we can write every morphism as a composition of genus 0 surfaces with ≤ 3 boundary circles. In fact, F is determined by $F(0)^1_2$, which defines a product $H \otimes H \to H$, and $F(0)^0_2$, which defines a pairing $H \otimes H \to \mathbb{C}$. The key idea is that the pairing determines an isomorphism $\phi\colon H \simeq H^*$, which can be used to "flip" the linear maps in the TQFT: for instance, $F(0)^1_2$ determines a map $H \to H \otimes H^*$, which via ϕ can be identified with a map $H \to H \otimes H$ that coincides with $F(0)^2_1$. Moreover, the composition relations on the TQFT imply that this product and pairing must satisfy the compatibility condition of a Frobenius algebra. Thus a TQFT determines a Frobenius algebra and vice versa.

Example 3.4.2 (Quantum cohomology TQFT)**.** Equipping the small quantum cohomology ring $QH^*(G)$ with the Poincaré pairing defines a Frobenius algebra, which determines a TQFT that we will call the **quantum cohomology TQFT**.

There are several advantages to interpreting Frobenius algebras as TQFTs. First, the TQFTs we consider in the next sections are related to algebraic curves, with the cobordisms encoding information about curves with marked points. The gluing of cobordisms reflects what happens to that information when the curve degenerates to a nodal curve. Second, the TQFT emphasizes different features of the data than the Frobenius algebra does. In particular, the closed invariants of a TQFT seem obscure from the Frobenius algebra point of view, but they often encode interesting enumerative data, which the TQFT is set up to compute: $F(g)^0_0$ is the trace of the $(g-1)$-power of the genus-addition operator. If the genus-addition operator can be diagonalized, then we call the TQFT **semisimple**, and if its eigenvalues $\lambda_1, \ldots, \lambda_n$ can be computed one immediately gets the formula

$$F(g)^0_0 = \sum_{i=1}^{n} \lambda_i^{g-1}.$$

Example 3.4.3 (Preview of Witten's TQFT)**.** Equipping $H^*(G, \mathbb{C})$ with the quantum product and a different pairing

$$\langle \sigma_{\vec{a}}, \sigma_{\vec{b}} \rangle_{\text{Witten}} = \langle \sigma_{\vec{a}} * \sigma_{\vec{b}}, [\text{pt}] \rangle,$$

produces a different Frobenius algebra. Amazingly, the closed invariants $F(g)^0_0$ of the TQFT are the Verlinde numbers $V_g^{r,s}$ introduced in (1.1)! Thus the Verlinde numbers can be computed by taking the trace of the $g-1$ power of the genus addition operator (whose eigenvalues are the products of sines in the Verlinde formula). We describe this TQFT more thoroughly in the next section.

It can be useful to define variations of a TQFT to allow for richer data to be encoded. In particular, the morphisms in a **weighted TQFT** can be indexed by $F(g|d)^n_m$, where d is a weight that is additive under composition. Restricting to the morphisms with $d = 0$ then defines a usual TQFT. The key extra information in a weighted TQFT are the cylinders with weight ± 1, which can be used to obtain any $F(g|d)^n_m$ from the weight-0 linear maps $F(g|0)^n_m$. Thus the data encoded by a weighted TQFT are equivalent to a Frobenius algebra equipped with an invertible operator.

Example 3.4.4 (Preview of the weighted TQFT)**.** We can form a weighted TQFT containing the quantum cohomology TQFT as the $d = 0$ slice and Witten's TQFT as a "diagonal" slice. Let the $d = 0$ morphisms be exactly as in the quantum cohomology TQFT, and add the data of the invertible operator $*\sigma_{1^r}$, which we view as the cylinder with weight -1, and its inverse $*\sigma_s$, which we view as the cylinder of weight 1. We show in §3.6 that the linear map $F(g|{-r}(g{-}1+n))^n_m$ in this weighted TQFT coincides with $F(g)^n_m$ from Witten's TQFT. In particular, the Verlinde numbers occur in the weighted TQFT as the closed invariants $F(g|{-r}(g{-}1))^0_0$.

3.5 Witten's TQFT

We summarize the TQFT studied by Witten ([Wit95]) that was mentioned in Example 3.4.3. We follow a description of this TQFT in standard mathematical language by Marian and Oprea ([MO10]).

Let C be a curve of genus g. Set $Q_{e,C} = \operatorname{Quot}\left((r,-e), \mathcal{O}_C^{r+s}\right)$ and let $Q_C = \bigsqcup_e Q_{e,C}$. Witten's TQFT can be defined by specifying the coefficients of the linear maps $F(g)^n_m$ in the Schubert basis as integrals on Q_C. These integrals do not depend on the choice of smooth curve. When $g > 0$, the Quot schemes $Q_{e,C}$ are only of the expected dimension for $e \gg 0$ (recall Example 3.1.2), so one has to integrate against virtual fundamental classes $[Q_C]^{\mathrm{vir}}$ that behave as if the Quot scheme had the expected dimension for all e. Define

$$F(g)^n_m \colon \sigma_{\vec{a}_1} \otimes \cdots \otimes \sigma_{\vec{a}_m} \mapsto \sum_{\vec{b}_1,\ldots,\vec{b}_n} \left(\int_{[Q_C]^{\mathrm{vir}}} \bar{\sigma}_{\underline{\vec{a}}} \cup \bar{\sigma}_{\underline{\vec{b}}} \cup \bar{\sigma}_{1^r}^{r(g+n)+s} \right) \sigma_{\vec{b}_1^c} \otimes \cdots \otimes \sigma_{\vec{b}_n^c},$$

where we use the notation $\bar{\sigma}_{\underline{\vec{a}}} = \bar{\sigma}_{\vec{a}_1} \cup \cdots \cup \bar{\sigma}_{\vec{a}_m}$ and similarly $\bar{\sigma}_{\underline{\vec{b}}} = \bar{\sigma}_{\vec{b}_1} \cup \cdots \cup \bar{\sigma}_{\vec{b}_n}$.

One can show that the class $\bar{\sigma}_{1^r}^{r+s}$ simply shifts the degree of the Quot scheme on which the integration is taking place, which yields the identity

$$\int_{Q_{e,\mathbb{P}^1}} \bar{\sigma}_{\underline{\vec{a}}} \cup \bar{\sigma}_{1^r}^{r+s} = \int_{Q_{e-r,\mathbb{P}^1}} \bar{\sigma}_{\underline{\vec{a}}}.$$

Recalling that the three-point Gromov-Witten numbers can be written as

$$\int_{Q_{e,\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} \cup \bar{\sigma}_{\vec{c}} = \langle W_{\vec{a}}, W_{\vec{b}}, W_{\vec{c}} \rangle_e,$$

we see that the product

$$F(0)_2^1 \colon \sigma_{\vec{a}} \otimes \sigma_{\vec{b}} \mapsto \sum_{\vec{c}} \left(\int_{Q_{\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} \cup \bar{\sigma}_{\vec{c}} \cup \bar{\sigma}_{1^r}^{r+s} \right) \bar{\sigma}_{\vec{c}^c}$$

is the quantum product. Note that the formal variable q has been dropped from the notation. Moreover, the pairing

$$F(0)_2^0 \colon \sigma_{\vec{a}} \otimes \sigma_{\vec{b}} \mapsto \int_{Q_{\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} \cup \bar{\sigma}_{1^r}^{s}$$

is different from the Poincaré pairing, and the closed invariants are

$$F(g)_0^0 = \int_{[Q_C]^{\mathrm{vir}}} \bar{\sigma}_{1^r}^{rg+s} = V_g^{r,s},$$

as can be computed with the Vafa-Intriligator formula. This TQFT is not ideal for a few reasons:

(1) As mentioned, the Quot schemes $Q_{C,e}$ are not necessarily of the expected dimension, so one has to integrate against virtual fundamental classes.

(2) The powers of $\bar{\sigma}_{1^r}$ look unnatural.

The weighted TQFT introduced in the next section will resolve (1) by replacing integrals on Quot schemes of trivial bundles by integrals on Quot schemes of general vector bundles, which always have the right dimension. Moreover, it will explain the special role of $\bar{\sigma}_{1^r}$ by viewing it as the weight-lowering operator corresponding to decreasing the degree of these general vector bundles.

3.6 Weighted TQFT

Let C be a curve of genus g, let $d \in \mathbb{Z}$, and let V be a general vector bundle on C of rank $r+s$ and degree d. For each e, set $Q_{e,V} = \mathrm{Quot}\big((r,-e),V\big)$ and $Q_V = \bigsqcup_e Q_{e,V}$. As in the previous section, the notation $Q_{e,C}$ and Q_C is used when $V = \mathcal{O}_C^{r+s}$. Define

$$F(g|d)^n_m \colon \sigma_{\vec{a}_1} \otimes \cdots \otimes \sigma_{\vec{a}_m} \mapsto \sum_{\vec{b}_1,\dots,\vec{b}_n} \left(\int_{Q_V} \bar{\sigma}_{\underline{\vec{a}}} \cup \bar{\sigma}_{\underline{\vec{b}}} \right) \sigma_{\vec{b}_1^c} \otimes \cdots \otimes \sigma_{\vec{b}_n^c},$$

where we do not need to use virtual classes because the $Q_{e,V}$ are all of the expected dimension. The degree of V is recorded as an additive weight on the morphisms in the weighted TQFT. Crucially, these integrals depend only on the numerical invariants and not on the choice of C or V in moduli.

Remark 3.6.1. Because the $Q_{e,V}$ all have the expected dimension, we can interpret the maps $F(g|d)^n_m$ geometrically as follows. Given distinct points $p_1, \dots, p_N$ on C, there are rational maps

$$\mathrm{ev}_{p_1,\dots,p_N} \colon \bigsqcup_e Q_{e,V} \dashrightarrow G^N.$$

Let $\eta_{g,d,N} \in H^*(G,\mathbb{C})^{\otimes N}$ denote the sum of the pushforwards of the fundamental classes of the $Q_{e,V}$. Then the coefficient of $\sigma_{\vec{a}_1^c} \otimes \cdots \otimes \sigma_{\vec{a}_N^c}$ equals $\int_{Q_V} \bar{\sigma}_{\vec{a}_1} \cup \cdots \cup \bar{\sigma}_{\vec{a}_N}$. The class $\eta_{g,d,N}$ induces a linear map $\mathbb{C} \to H^*(G,\mathbb{C})^{\otimes N}$ defined by $1 \mapsto \eta_{g,d,N}$. Choosing a partition $N = m+n$, dualizing the first m copies of $H^*(G,\mathbb{C})$ yields a linear map $(H^*(G,\mathbb{C})^*)^{\otimes m} \to H^*(G,\mathbb{C})^{\otimes n}$, and Poincaré duality yields an isomorphism between $H^*(G,\mathbb{C})$ and its dual, hence the class $\eta_{g,d,m+n}$ determines a linear map

$$F(g|d)^n_m \colon H^*(G,\mathbb{C})^{\otimes m} \to H^*(G,\mathbb{C})^{\otimes n}$$

that is exactly the map defined above.

Theorem 3.6.2. *The maps $F(g|d)^n_m$ are the morphisms of a weighted topological quantum field theory that contains the quantum cohomology TQFT and Witten's TQFT.*

We sketch the proof at the end of the section and give a full proof in the next chapter. First, we explain why the slice $d = 0$ of the weighted TQFT is exactly the quantum cohomology TQFT. The general degree zero vector bundle on $\mathbb{P}^1$ is the trivial bundle, so the map

$$F(0|0)_2^1 \colon \sigma_{\vec{a}} \otimes \sigma_{\vec{b}} \mapsto \sum_{\vec{c}} \left(\int_{Q_{\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} \cup \bar{\sigma}_{\vec{c}} \right) \sigma_{\vec{c}^c}$$

is the quantum product. The pairing

$$F(0|0)_2^0 \colon \sigma_{\vec{a}} \otimes \sigma_{\vec{b}} \mapsto \int_{Q_{\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}}$$

is just the Poincaré pairing because the two-point Gromov-Witten numbers vanish for $e > 0$ (there are infinitely many automorphisms of $\mathbb{P}^1$ fixing two points).

Second, we compute the degree-lowering operator $F(0|-1)_1^1$. Consider a general elementary modification $0 \to V \to \mathcal{O}_{\mathbb{P}^1}^{r+s} \to \mathbb{C}_q \to 0$. The image of the embedding $Q_{e,V} \hookrightarrow Q_{e,\mathbb{P}^1}$ is exactly $\overline{W}_{1^r}(q)$. Thus

$$\int_{Q_V} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} = \int_{Q_{\mathbb{P}^1}} \bar{\sigma}_{\vec{a}} \cup \bar{\sigma}_{\vec{b}} \cup \bar{\sigma}_{1^r} = \sum_{e \geq 0} \langle W_{\vec{a}}, W_{\vec{b}}, W_{1^r} \rangle_e,$$

so the operator defined by these integrals is

$$\sigma_{\vec{a}} \mapsto \sum_{e \geq 0} \sum_{\vec{b}} \langle W_{\vec{a}}, W_{\vec{b}}, W_{1^r} \rangle_e \, \sigma_{\vec{b}^c} = \sigma_{\vec{a}} * \sigma_{1^r},$$

namely the degree-lowering operator is $*\sigma_{1^r}$.

Remark 3.6.3. The degree-lowering operator is invertible in the sense that $\sigma_{1^r} * \sigma_s = q$, so $*q^{-1}\sigma_s$ is the degree-raising operator (we can formally adjoin an inverse of q). Moreover, the quantum powers of σ_{1^r} are

$$\sigma_{1^r}^k = \begin{cases} \sigma_{k^r} & \text{for } 1 \leq k \leq s \\ q^{k-s} \sigma_{s^{r+s-k}} & \text{for } s \leq k \leq r+s \end{cases}.$$

In particular, $\sigma_{1^r}^s = [\mathrm{pt}]$ and $\sigma_{1^r}^{r+s} = q^r$ (which can be interpreted as σ_{1^r} having finite order). Thus the higher powers are the same modulo $r+s$ up to a power of q. This makes sense since lowering the degree of the vector bundle V by $r+s$ produces a vector bundle with the same degree as a twist of V by a line bundle of degree -1. These computations follow from a quantum analog of the Pieri formula ([Ber97]).

Using the degree-lowering operator, we can remove a power $\sigma_{1^r}^k$ from an integral if we lower the degree d of the vector bundle by k. Thus $F(g)_m^n$ in Witten's TQFT coincides with

$F(g|-r(g+n)-s)^n_m$ in the weighted TQFT, which we renormalize to $F(g|-r(g-1+n))^n_m$. For example, the Verlinde numbers occur as

$$V_g^{r,s} = F(g)^0_0 = F(g|-r(g-1))^0_0,$$

which reflects the fact that for an appropriate choice of e, the Quot scheme $Q_{e,V}$ is finite, reduced, and contains $V_g^{r,s}$ points when V is a general vector bundle of degree $-r(g-1)$ on a curve C of genus g. As we saw in §3.1, the appropriate choice in this case is $e = r(g-1)$.

A simple computation (for instance, computing $F(0|0)^1_2 \circ F(0|0)^2_1$) shows that the genus-addition operator $F(1|0)^1_1$ is $* \sum_{\vec{b}} \sigma_{\vec{b}} * \sigma_{\vec{b}^c}$ (quantum product by the quantum diagonal). Thus we can express Witten's genus-addition operator as

$$F(1)^1_1 = * \left(\sum_{\vec{b}} \sigma_{\vec{b}} * \sigma_{\vec{b}^c} \right) * \sigma^r_{1^r},$$

which is the operator whose eigenvectors are the products of sines in the Verlinde formula. We can compute $V_g^{r,s}$ by taking the trace of the $g-1$ power of this operator. Another way to compute $F(g|-r(g-1))^0_0$ is by composing g genus addition operators, $r(g-1)$ degree-lowering operators, and capping both ends. This yields the formula

$$V_g^{r,s} = \text{coefficient of [pt] in } \left(\sum_{\vec{b}} \sigma_{\vec{b}} * \sigma_{\vec{b}^c} \right)^g * \sigma_{1^r}^{r(g-1)}$$

expressing the Verlinde numbers as coefficients in quantum products.

The proof that the $F(g|d)^n_m$ satisfy the composition relations is the ultimate goal of Chapter 4 and will require a large amount of technical work. The idea, however, is simple.

Sketch of proof of theorem. Since any composition of morphisms can be viewed as gluing one pair of boundary circles at a time, it suffices to prove the relations for gluing one pair of boundary circles.

The idea is that given a smooth curve C' of genus g and a vector bundle V' of rank $r+s$ and degree d, there is a degeneration of C' into a reducible nodal curve C with two smooth components C_1 and C_2 glued at points p_1 and p_2 to produce a simple node. The genera g_1 and g_2 of the components sum to g. The vector bundle V' also degenerates, producing a vector bundle V on C that can be described by gluing vector bundles V_i on C_i by an isomorphism $V_1(p_1) \simeq V_2(p_2)$ of their fibers over the node. The degrees d_i of the V_i satisfy $d_1 + d_2 = d$, which reflects the additivity of the weight in the TQFT under composition. The good behavior of the Quot scheme under deformation guarantees that the intersections numbers on $Q_{e,V'}$ agree with the intersection numbers on $Q_{e,V}$, namely

$$\int_{Q_{e,V'}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2} = \int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2},$$

where we think of the Schubert cycles $\bar{\sigma}_{\underline{\vec{a}}_i}$ on $Q_{e,V'}$ as being based at points that degenerate to lie on C_i (this subtlety will turn out to be unnecessary).

The remaining work is to compute the integral on the nodal curve C in terms of intersection numbers on its components C_i. The Quot scheme on C can be described by gluing subsheaves (or quotients) on the C_i along their fibers at the points p_i. Roughly, the isomorphism $V_1(p_1) \simeq V_2(p_2)$ allows us to think of the codomain of

$$\mathrm{ev}_{p_1,p_2} \colon Q_{e_1,V_1} \times Q_{e_2,V_2} \dashrightarrow G \times G$$

as two copies of the same Grassmannian, and the pairs of quotients that can be glued are the preimage Δ_{e_1,e_2} of the diagonal. In fact, Δ_{e_1,e_2} embeds as a dense open set in a component of $Q_{e,V}$, and the various partitions $e = e_1 + e_2$ of e correspond to the top-dimensional components of $Q_{e,V}$. Recalling that the cohomology class of the diagonal in $G \times G$ is $\sum_{\vec{b}} \sigma_{\vec{b}} \otimes \sigma_{\vec{b}^c}$, we thus obtain the formula

$$\int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2} = \sum_{e_1+e_2=e} \sum_{\vec{b}} \left(\int_{Q_{e_1,V_1}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\vec{b}} \right) \left(\int_{Q_{e_2,V_2}} \bar{\sigma}_{\underline{\vec{a}}_2} \cup \bar{\sigma}_{\vec{b}^c} \right).$$

Combining this formula with the above equality of integrals and summing over all e produces the gluing relation corresponding to identifying boundary circles on two morphisms in the weighted TQFT. □

CHAPTER 4

TECHNICAL STUDY OF QUOT SCHEMES ON CURVES

Quot schemes of trivial bundles on $\mathbb{P}^1$ have two very nice properties: they are of the expected dimension, and intersections of general Schubert varieties are proper. We show that these properties extend to Quot schemes of trivial bundles on arbitrary curves when the degree of the subsheaves is sufficiently negative. By embedding Quot schemes of general vector bundles in Quot schemes of trivial bundles, we deduce that Quot schemes of general vector bundles always have the same two nice properties. This justifies the definition of the maps $F(g|d)^n_m$ in the weighted TQFT of the previous chapter. In order to prove that the $F(g|d)^n_m$ satisfy the composition relations of a TQFT, we study Quot schemes of reducible nodal curves, show that they also have the two nice properties, and prove that their intersection numbers can be computed from intersection numbers on the components.

4.1 Schubert varieties on Quot schemes

We saw in Example 3.3.1 that the $e = 0$ Gromov-Witten numbers can be interpreted as intersection numbers in the cohomology ring of $G = \mathrm{Gr}(r, \mathbb{C}^{r+s})$, hence they are independent of the chosen points and the chosen general flags. The plan for $e > 0$ is similar: use the compactification $\mathcal{Q}_{e,\mathbb{P}^1} = \mathrm{Quot}\big((r,-e), \mathcal{O}^{r+s}_{\mathbb{P}^1}\big)$ of $\mathrm{Mor}_e(\mathbb{P}^1, G)$ to carry out intersection theory, and argue that the boundary does not contribute to top intersections, hence top intersections can be interpreted as counts of maps in $\mathrm{Mor}_e(\mathbb{P}^1, G)$ satisfying certain Schubert conditions.

4.1.1 Motivation

By the universal property of the Grassmannian, pulling back the universal subbundle on G under a morphism $\phi\colon \mathbb{P}^1 \to G$ defines a bijection

$$\mathrm{Mor}_e(\mathbb{P}^1, G) \to \{\, E \subset \mathcal{O}_{\mathbb{P}^1}^{r+s} \mid E \text{ is a rank } r \text{ vector subbundle of degree } -e \,\}.$$

The Quot scheme $Q_{e,\mathbb{P}^1}$ naturally generalizes the right side by allowing $E \subset \mathcal{O}_{\mathbb{P}^1}^{r+s}$ to be a locally free subsheaf rather than a subbundle, namely the inclusion map may drop rank at finitely many points of $\mathbb{P}^1$. Dropping rank is a closed condition in the Quot scheme, which is irreducible ([BDW96]), so $\mathrm{Mor}_e(\mathbb{P}^1, G) \subset Q_{e,\mathbb{P}^1}$ is a dense open subscheme.

Since subsheaves $E \subset \mathcal{O}_{\mathbb{P}^1}^{r+s}$ are locally free, they split as direct sums of line bundles, which must all have degree ≤ 0 since $\mathcal{O}_{\mathbb{P}^1}^{r+s}$ is semistable. Similarly, the quotient sheaf F splits as a direct sum of line bundles of degree ≥ 0 and torsion sheaves supported at points. It follows that $\mathrm{ext}^1(E, F) = 0$ for all $[0 \to E \to \mathcal{O}_{\mathbb{P}^1}^{r+s} \to F \to 0] \in Q_{e,\mathbb{P}^1}$, hence $Q_{e,\mathbb{P}^1}$ is smooth and of the expected dimension $\chi(E, F) = (r+s)e + rs$. Thus $\mathrm{Mor}_e(\mathbb{P}^1, G)$ is also smooth and of dimension $(r+s)e + rs$.

Given a point $p \in \mathbb{P}^1$, there is an evaluation map

$$\mathrm{ev}_p \colon \mathrm{Mor}_e(\mathbb{P}^1, G) \to G, \quad \phi \mapsto \phi(p).$$

Taking the preimage of a Schubert variety $W_{\vec{a}}(V_\bullet)$ under ev_p and taking the closure in $Q_{e,\mathbb{P}^1}$ produces a closed subscheme of $Q_{e,\mathbb{P}^1}$ that we will denote $\overline{W}_{\vec{a}}(V_\bullet)$ and call a "Schubert variety" on the Quot scheme.

By definition, $\overline{W}_{\vec{a}}(V_\bullet)$ is the closure in $Q_{e,\mathbb{P}^1}$ of the set of $\phi \in \mathrm{Mor}_e(\mathbb{P}^1, G)$ satisfying $\phi(p) \in W_{\vec{a}}(V_\bullet) \subset G$. In the case when $|\vec{a}_1| + \cdots + |\vec{a}_N| = \dim Q_{e,\mathbb{P}^1} = (r+s)e + rs$, we will see that the intersection of corresponding general Schubert varieties is contained within the dense open set $\mathrm{Mor}_e(\mathbb{P}^1, G)$. These intersections can be computed in cohomology of $Q_{e,\mathbb{P}^1}$ by integrating the cohomology classes $\bar{\sigma}_{\vec{a}}$ corresponding to the Schubert varieties $\overline{W}_{\vec{a}}(V_\bullet)$, namely

$$\langle W_{\vec{a}_1}, \ldots, W_{\vec{a}_N} \rangle_e = \int_{Q_{e,\mathbb{P}^1}} \bar{\sigma}_{\vec{a}_1} \cup \cdots \cup \bar{\sigma}_{\vec{a}_N}.$$

We will see that since the $\bar{\sigma}_{\vec{a}}$ are independent of the choice of point p and the flag $V_\bullet$, so are the Gromov-Witten numbers.

4.1.2 General definition

We now define Schubert varieties more carefully and more generally for arbitrary Quot schemes. Let V be a vector bundle of rank $r+s$ on a curve C, let $Q_{e,V} = \mathrm{Quot}\big((r, -e), V\big)$, and let $0 \to \mathcal{E} \to \pi^* V \to \mathcal{F} \to 0$ denote the universal sequence on $C \times Q_{e,V}$. Choose a point $p \in C$, a full flag $V_\bullet$ of the fiber $V(p)$, and a partition $\vec{a}$. The $V_i \subset V(p)$ have cokernels $V(p) \twoheadrightarrow V^i$. Restricting the universal sequence to $p \times Q_{e,V} \simeq Q_{e,V}$ yields an exact sequence

$$0 \to \mathcal{E}|_{p \times Q_{e,V}} \to V(p) \otimes \mathcal{O} \to \mathcal{F}|_{p \times Q_{e,V}} \to 0$$

on the Quot scheme.

Definition 4.1.1. The **Schubert variety** $\overline{W}_{\vec{a}}(p, V_\bullet) \subset Q_{e,V}$ is the intersection for all $1 \leq i \leq r$ of the degeneracy loci where the compositions

$$\mathcal{E}|_{p \times Q_{e,V}} \to V(p) \otimes \mathcal{O} \twoheadrightarrow V^{s+i-a_i} \otimes \mathcal{O}$$

have kernel of dimension $\geq i$.

In the case when $\overline{W}_{\vec{a}}(p, V_\bullet)$ has pure codimension $|\vec{a}|$ for all p and $V_\bullet$, the following lemma guarantees that the Schubert cycle $\bar{\sigma}_{\vec{a}} = [\overline{W}_{\vec{a}}(p, V_\bullet)]$ in the cohomology of the Quot scheme is independent of p and $V_\bullet$.

Lemma 4.1.2. *Suppose $\overline{W}_{\vec{a}}(p, V_\bullet)$ has pure codimension $|\vec{a}|$ for all $p \in \mathbb{C}$ and flags $V_\bullet$. Then the cohomology class $\bar{\sigma}_{\vec{a}}$ of $\overline{W}_{\vec{a}}(p, V_\bullet)$ in $Q_{e,V}$ is independent of p and $V_\bullet$.*

Proof. Since the $\overline{W}_{\vec{a}}(p, V_\bullet)$ have pure codimension $|\vec{a}|$, we can apply a theorem of Kempf-Laksov ([KL74]) to the maps $\mathcal{E}|_{p \times Q_{e,V}} \to V(p) \otimes \mathcal{O} \to V^i \otimes \mathcal{O}$ to express the cohomology class of the degeneracy locus as a determinantal formula involving only the Chern classes of $\mathcal{E}|_{p \times Q_{e,V}}$ (and of the flag, but these are trivial). But since $\mathcal{E}$ is a vector bundle over $C \times Q_{e,V}$, the Chern classes of the restrictions of $\mathcal{E}$ over $p \in C$ are independent of p by Proposition 2.1.6. □

The lemma is a first hint that we need Schubert varieties and their intersections to have the right codimension if we hope to get a useful intersection theory on the Quot scheme. Indeed, if intersections on $Q_{e,V}$ are well-behaved, then it makes sense to define analogs of the Gromov-Witten numbers as

$$\int_{Q_{e,V}} \bar{\sigma}_{\vec{a}_1} \cup \cdots \cup \bar{\sigma}_{\vec{a}_N}$$

for $|\vec{a}_1| + \cdots + |\vec{a}_N| = \dim Q_{e,V}$.

Definition 4.1.3. An intersection of Schubert varieties on $Q_{e,V}$ is **proper** if it is empty or of pure codimension equal to the total size of the partitions. An intersection of Schubert varieties on $Q_{e,V}$ is a **top intersection** if the total size of the partitions equals $\dim Q_{e,V}$.

On part of the Quot scheme, there is no difficulty in proving that Schubert varieties intersect properly. Let $U_{e,V} \subset Q_{e,V}$ denote the open (but possibly empty) locus where the quotients are torsion-free (which generalizes $\mathrm{Mor}_e(\mathbb{P}^1, G) \subset Q_{e,\mathbb{P}^1}$). Then for each $p \in C$, there is a morphism

$$\mathrm{ev}_p \colon U_{e,V} \to \mathrm{Gr}\big(r, V(p)\big), \quad [E \subset V] \mapsto [E(p) \subset V(p)]$$

induced by $\mathcal{E}|_{p \times U_{e,V}} \hookrightarrow V(p) \otimes \mathcal{O}$ on $p \times U_{e,V} \simeq U_{e,V}$. The presence of morphisms to the Grassmannian makes it easy to control the behavior of intersections of Schubert varieties. Given distinct points $p_1, \ldots, p_N$, the following lemma guarantees that the intersection $\mathrm{ev}_{p_1}^{-1}(W_{\vec{a}_1}) \cap \cdots \cap \mathrm{ev}_{p_N}^{-1}(W_{\vec{a}_N})$ is proper on each component of $U_{e,V}$.

Lemma 4.1.4. *Let X be an irreducible scheme with morphisms $f_i \colon X \to G$ for all $1 \leq i \leq N$. Then for all choices of partitions $\vec{a}_1, \ldots, \vec{a}_N$ and Schubert varieties $W_{\vec{a}_1}, \ldots, W_{\vec{a}_N}$ defined using general flags, the preimage $f_1^{-1}(W_{\vec{a}_1}) \cap \cdots \cap f_N^{-1}(W_{\vec{a}_N})$ is empty or has pure codimension $A = |\vec{a}_1| + \cdots + |\vec{a}_N|$ in X.*

Proof. Induction on N. The base case $N = 1$ is Kleiman's theorem ([Kle74]). For the inductive step, assume $X' = f_1^{-1}(W_{\vec{a}_1}) \cap \cdots \cap f_{N-1}^{-1}(W_{\vec{a}_{N-1}})$ has pure codimension $A - |\vec{a}_N|$ in X. Restricting f_N to each component Y of X', the base case guarantees that $f_N|_Y^{-1}(W_{\vec{a}_N})$ is empty or has pure codimension $|\vec{a}_N|$ in Y, which completes the proof. □

Thus the challenge in proving that Schubert varieties intersect properly is to understand how they intersect the boundary $U_{e,V}^c$ of the Quot scheme. In the next section, we explain how the boundary is the image of Grassmann bundles defined over smaller Quot schemes $Q_{e-\ell,V}$. This recursive description will allow us to study Schubert intersections on the boundary in cases where we have control over the dimension of $Q_{e-\ell,V}$ for all $0 \leq \ell \leq r$.

4.2 Structure theory of the Quot scheme

The structure theory described in this section holds for arbitrary vector bundles V, but we only apply it in cases where the Quot schemes have the expected dimension. Since V

is fixed throughout the section, we drop it from the notation.

4.2.1 Recursive structure of the boundary

Let V be a vector bundle on a smooth curve C, let $Q_e = \mathrm{Quot}\big((r,-e),V\big)$, and let $\mathcal{E}_e$ denote the universal subsheaf, which is a vector bundle on $C \times Q_e$. Given $p \in C$ and an integer $1 \le \ell \le r$, the fiber of the Grassmann bundle $\mathrm{Gr}(\mathcal{E}_e,\ell)$ over a point $(p, x = [0 \to E \to V \to F \to 0]) \in C \times Q_e$ parametrizes quotients $\mathcal{E}_e|_{(p,x)} = E(p) \twoheadrightarrow \mathbb{C}^\ell$. By composing $E \twoheadrightarrow E(p) \twoheadrightarrow \mathbb{C}^\ell$, these quotients induce elementary modifications $0 \to E' \to E \to \mathbb{C}^\ell_p \to 0$ in which $\deg E' = \deg E - \ell$. Since $E' \hookrightarrow E \hookrightarrow V$, there is a short exact sequence $0 \to E' \to V \to F' \to 0$, and the assignment $[E(p) \twoheadrightarrow \mathbb{C}^\ell] \mapsto [0 \to E' \to V \to F' \to 0]$ defines a set-theoretic map

$$\begin{array}{ccc} \mathrm{Gr}(\mathcal{E}_e,\ell) & \xrightarrow{\beta_\ell} & Q_{e+\ell} \\ {\scriptstyle \pi_\ell}\downarrow & & \\ C \times Q_e & & \end{array}$$

Let $0 \to \mathcal{S} \to \pi_\ell^*\mathcal{E}_e \to \mathcal{Q} \to 0$ denote the tautological sequence of vector bundles on $\mathrm{Gr}(\mathcal{E}_e,\ell)$, which is equipped with a map $\pi_\ell^*\mathcal{E}_e \to \pi_C^* V$ from the tautological sequence on Q_e. Let $U_e \subset Q_e$ denote the open (possibly empty) subscheme where the cokernel is torsion-free.

Proposition 4.2.1 ([Ber97]). *In the setting above,*

(a) β_ℓ *is a morphism of schemes.*

(b) $\mathrm{im}(\beta_\ell)$ *is closed in* $Q_{e+\ell}$ *and contains exactly those points* $x \in Q_{e+\ell}$ *such that the universal quotient* $\mathcal{F}_{e+\ell}$ *has rank* $\ge s+\ell$ *at* $(p,x) \in C \times Q_{e+\ell}$ *for some* $p \in C$.

(c) The restriction of β_ℓ *to* $\pi_\ell^{-1}(C \times U_e)$ *is an embedding.*

(d) Let $p \in C$ *and let* $\overline{W}_{\vec{a}}(p)$ *denote a Schubert variety in* $Q_{e+\ell}$ *defined using a full flag* $V_\bullet$ *in* $V(p)$. *Then*

$$\beta_\ell^{-1}(\overline{W}_{\vec{a}}(p)) = \pi_\ell^{-1}(C \times \overline{W}_{\vec{a}}(p)) \cup \hat{W}_{a_{\ell+1},\dots,a_r}(p),$$

where $\hat{W}_{b_1,\dots,b_{r-\ell}}(p)$ *is the degeneracy locus inside* $\pi_\ell^{-1}(p \times Q_e)$ *where the kernel of* $\mathcal{S} \to V^{s+\ell+j-b_j} \otimes \mathcal{O}$ *has rank* $\ge j$ *for all* $1 \le j \le r-\ell$.

Remark 4.2.2. One can think of $\hat{W}_{\vec{b}}(p) \subset \pi_\ell^{-1}(p \times Q_e)$, where $\vec{b}$ has length $r-\ell$, as follows. On the open locus $\pi_\ell^{-1}(p \times U_e)$, the elementary modifications yield $0 \to E' \to V \to F' \to 0$, where $E'(p) \to V(p)$ has rank exactly $r-\ell$. Thus we get a map $\pi_\ell^{-1}(p \times U_e) \to \mathrm{Gr}\big(r-\ell, V(p)\big)$, and pulling back the Schubert variety $W_{\vec{b}}(V_\bullet)$ and taking its closure yields $\hat{W}_{\vec{b}}(p)$.

Part (d) of the proposition will be a critical tool for studying intersections of Schubert varieties on Quot schemes. In particular, the fact that the varieties $\hat{W}_{\vec{b}}(p)$ based at different points p are disjoint proves the following corollary.

Corollary 4.2.3. *In the setting above, let $\overline{W}_{\vec{a}_1}(p_1), \ldots, \overline{W}_{\vec{a}_N}(p_N)$ be Schubert varieties in $Q_{e+\ell}$ defined at distinct points p_i, and let W denote their intersection. Then, up to reindexing the $\vec{a}_i$, $\beta_\ell^{-1}(W)$ is a union of intersections of the following types:*

Type 1: $\pi_\ell^{-1}\left(C \times \left(\overline{W}_{\vec{a}_1}(p_1) \cap \cdots \cap \overline{W}_{\vec{a}_N}(p_N)\right)\right)$;

Type 2: $\pi_\ell^{-1}\left(p_N \times \left(\overline{W}_{\vec{a}_1}(p_1) \cap \cdots \cap \overline{W}_{\vec{a}_{N-1}}(p_{N-1})\right)\right) \cap \hat{W}_{(\vec{a}_N)_{\ell+1},\ldots(\vec{a}_N)_r}$.

The proposition is stated for $C = \mathbb{P}^1$ and V the trivial vector bundle in [Ber97]. We sketch the proof, which generalizes without requiring modification.

Proof of Proposition 4.2.1. (a): To see β_ℓ is a morphism, we need to construct the elementary modifications described above as a family over $C \times \mathrm{Gr}(\mathcal{E}_e, \ell)$. The universal sequence

$$0 \to \mathcal{S} \to \pi_\ell^* \mathcal{E}_e \to \mathcal{Q} \to 0$$

on $\mathrm{Gr}(\mathcal{E}_e, \ell)$ gives simultaneous quotients of all the fibers, so $\mathcal{S}$ has rank $r-\ell$, which is not what we want. Instead, consider the diagram

$$\begin{array}{ccc} & & C \times \mathrm{Gr}(\mathcal{E}_e, \ell) \\ & & \Big\downarrow \pi = \pi_C \times \pi_\ell \\ C \times Q_e & \xrightarrow[\Delta_C \times \mathrm{id}]{} & C \times C \times Q_e \end{array}$$

where $\Delta_C \colon C \to C \times C$ is the diagonal embedding. Pushing forward $\mathcal{Q}$ along the natural map $\mathrm{Gr}(\mathcal{E}_e, \ell) \to C \times \mathrm{Gr}(\mathcal{E}_e, \ell)$ defines a sheaf $\mathcal{Q}_\Delta$. Pushing forward the tautological subsheaf on $C \times Q_e$ along $\Delta_C \times \mathrm{id}$ yields a sheaf $\mathcal{E}_{e,\Delta}$. Let $\pi_{1,3}\colon C \times C \times Q_e \to C \times Q_e$ denote projection onto the first and third factors. The composition of the surjective maps

$$\phi\colon \pi^*\pi_{1,3}^*\,\mathcal{E}_e \twoheadrightarrow \pi^*\mathcal{E}_{e,\Delta} \twoheadrightarrow \mathcal{Q}_\Delta,$$

when restricted to the slice $C \times (p, [E \subset V], E(p) \twoheadrightarrow \mathbb{C}^\ell) \simeq C$, yield

$$E \twoheadrightarrow E(p) \twoheadrightarrow \mathbb{C}_p^\ell.$$

Thus the kernel $\mathcal{E}'$ of ϕ assembles the elementary modifications, and the inclusion $\mathcal{E}' \to \pi^*\pi_{1,3}^*\,\mathcal{E}_e \to \pi^*\pi_{1,3}^*\pi_C^*V$ induces the map β_ℓ to $Q_{e+\ell}$.

(b): Since $\mathrm{Gr}(\mathcal{E}_e, \ell)$ is projective, the image of β_ℓ is closed in $Q_{e+\ell}$. This image consists of all sequences $0 \to E' \to V \to F' \to 0$ such that there is an inclusion $\mathbb{C}_p^\ell \hookrightarrow F'$ for some $p \in C$. To see this, given such F', pass to the cokernel F, which induces a sequence $0 \to E \to V \to F \to 0$; by the snake lemma, the cokernel of the natural map $E' \to E$ is $\mathbb{C}_p^\ell$.

(c): This preimage may not be unique if the subsheaf $\mathbb{C}_p^\ell \to F'$ is not unique (this will happen when there are such inclusions at multiple points p or when the torsion at p has rank $> \ell$). However, letting β'_ℓ denote the restriction of β_ℓ to $\pi^{-1}(C \times U_e)$, β'_ℓ is an embedding, with inverse map $\mathrm{im}(\beta'_\ell) \to \mathrm{Gr}(\mathcal{E}_e, \ell)$ obtained as follows. Restricting the universal map $\pi_C^*V^* \to \mathcal{E}_{e+\ell}^*$ on $C \times Q_{e+\ell}$ to $C \times \mathrm{im}(\beta'_\ell)$ yields a cokernel N whose support is the image of a section $\alpha\colon \mathrm{im}(\beta'_\ell) \to C \times \mathrm{im}(\beta'_\ell)$. The kernel $\mathcal{K}^*$ of $\mathcal{E}_{e+\ell}^* \twoheadrightarrow N$ is a vector bundle and the cokernel N' of $\mathcal{E}_{e+\ell} \to \mathcal{K}$ has the same support as N. The inclusion $\mathcal{K} \hookrightarrow \pi_C^*V$ restricted to $\mathrm{im}(\alpha)$, which we identify with $\mathrm{im}(\beta'_\ell)$, yields a map $\mathrm{im}(\beta'_\ell) \to Q_e$. Combining this with the projection to C defines a map $\gamma\colon \mathrm{im}(\beta'_\ell) \to C \times Q_e$ under which $\mathcal{K}$ is the pullback of $\mathcal{E}_e$. Since α is an isomorphism onto its image, we can view N' as a vector bundle of rank ℓ on $\mathrm{im}(\beta'_\ell)$. Under this identification, the quotient $\mathcal{K} = \gamma^*\mathcal{E}_e \twoheadrightarrow N'$ corresponds to a map $\gamma'\colon \mathrm{im}(\beta'_\ell) \to \mathrm{Gr}(\mathcal{E}_e, \ell)$ by the universal property of $\mathrm{Gr}(\mathcal{E}_e, \ell)$. To see that γ' is the inverse of β'_ℓ, we note that these maps commute with the maps to the base $C \times Q_e$, that the maps are set-theoretically inverses, and that the fibers of $\mathrm{Gr}(\mathcal{E}_e, \ell)$ over $C \times Q_e$ are smooth.

(d): Given a point $p \in C$ and a flag $V_\bullet$ in $V(p)$, recall that the Schubert variety $\overline{W}_{\vec{a}}(p)$ in $Q_{e+\ell}$ is defined as the degeneracy locus in $Q_{e+\ell} \simeq p \times Q_{e+\ell}$ where the kernels of the compositions $\mathcal{E}_{e+\ell}|_{p \times Q_{e+\ell}} \to V(p) \otimes \mathcal{O} \twoheadrightarrow V^{s+i-a_i} \otimes \mathcal{O}$ have dimension $\geq i$. We can compute the pullback of the degeneracy locus as the degeneracy locus of the pullback. Consider the composition

$$\alpha_i \colon \beta_\ell^* \mathcal{E}_{e+\ell}|_{p \times Q_{e+\ell}} \xrightarrow{\delta} \pi_\ell^* \mathcal{E}_e \to V(p) \otimes \mathcal{O} \to V^{s+i-a_i} \otimes \mathcal{O}$$

of vector bundles on $\mathrm{Gr}(\mathcal{E}_e, \ell)$. The preimage $\beta_\ell^{-1}\big(\overline{W}_{\vec{a}}(p)\big)$ is the degeneracy locus where α_i has kernel of dimension $\geq i$. On $\pi_\ell^{-1}((C \setminus p) \times Q_e)$, the map δ is an isomorphism, so one component of the preimage of the Schubert variety is the preimage $\pi_\ell^{-1}(C \times \overline{W}_{\vec{a}}(p))$ of the Schubert variety on the base. On $\pi_\ell^{-1}(p \times Q_e)$, the map δ factors through $\mathcal{S}$, so α_i has kernel of dimension $\geq i$ at a point if and only if the composition $\mathcal{S} \to V(p) \otimes \mathcal{O} \to V^{s+i-a_i} \otimes \mathcal{O}$ has kernel of dimension $\geq i - \ell$ at that point. The latter condition is trivial for $i \leq \ell$, and the change of variables $j = i - \ell$ gives the stated result. □

4.2.2 Stratification

The recursive nature of the boundary of the Quot scheme described in the previous section can be used to obtain stratifications both of the Quot scheme and of the Grassmann bundles. This will be particularly useful for studying intersections of Schubert varieties.

As in the previous section, let V be a vector bundle and set $Q_e = \mathrm{Quot}\big((r, -e), V\big)$. Fix $p \in C$ and let $\mathrm{Gr}(\ell, p)$ denote the locus $\pi_\ell^{-1}(p \times Q_{e-\ell}) \subset \mathrm{Gr}(\mathcal{E}_{e-\ell}, \ell)$ parametrizing elementary modifications of the universal subsheaf at the fixed point p. As before, there are maps

$$\begin{array}{ccc} \mathrm{Gr}(\ell, p) & \xrightarrow{\beta_{\ell,p}} & Q_e \\ {\scriptstyle \pi_{\ell,p}} \big\downarrow & & \\ Q_{e-\ell} & & \end{array}$$

and there is a universal sequence $0 \to \mathcal{S}_{\ell,p} \to \pi_{\ell,p}^* \mathcal{E}_{e-\ell}|_{p \times Q_{e-\ell}} \to \mathcal{Q}_{\ell,p} \to 0$ on $\mathrm{Gr}(\ell, p)$. Let $U_{\ell,p} \subset \mathrm{Gr}(\ell, p)$ denote the open subscheme where the map $\mathcal{S}_{\ell,p} \to V(p) \otimes \mathcal{O}$ is injective. Then $\beta_{\ell,p}$ is injective on $U_{\ell,p}$ and its image $\beta_{\ell,p}(U_{\ell,p})$ is the locally-closed locus in Q_e where the kernel $[E \subset V]$ has rank exactly $r - \ell$ at p. This locus is important, so we introduce the notation $Z_{\ell,p} = \beta_{\ell,p}(U_{\ell,p})$. Let $U_{\ell,p}^c$ denote the complement of $U_{\ell,p}$ in $\mathrm{Gr}(\ell, p)$. Then $\beta_{\ell,p}(U_{\ell,p}^c)$ is the locus of $[E \subset V]$ that have rank $\leq r - \ell - 1$ at p, which coincides with the image of $\beta_{\ell+1,p}$. We thus get a stratification

$$\beta_{\ell,p}(U_{\ell,p}^c) = \bigsqcup_{\ell < \ell' \leq r} Z_{\ell',p} \subset Q_e$$

as well as a stratification

$$U^c_{\ell,p} = \bigsqcup_{\ell<\ell'\leq r} \beta^{-1}_{\ell,p}(Z_{\ell',p}) \subset \mathrm{Gr}(\ell, p).$$

At a point $x = [0 \to E \to V \to F \to 0]$ in $Z_{\ell',p}$, the torsion subsheaf T of F has fiber of dimension exactly ℓ' at p, hence there is a canonical subsheaf $\mathbb{C}^{\ell'}_p \subset T \subset F$ and every map $\mathbb{C}_p \to F$ factors through $\mathbb{C}^{\ell'}_p$. The fiber of $\beta_{\ell,p}$ over x consists of all $[0 \to E' \to V \to F' \to 0]$ such that there is an elementary modification $E' \twoheadrightarrow \mathbb{C}^{\ell}_p$ yielding E as the kernel. By the snake lemma, such modifications correspond to maps $\mathbb{C}^{\ell}_p \hookrightarrow F$ (whose cokernel is F'). Thus the fiber over x is isomorphic to $\mathrm{Gr}(\ell, \mathbb{C}^{\ell'})$.

In the case when the Quot schemes $Q_{e-\ell}$ have the expected dimension for all $0 \leq \ell \leq r$, we can compute the codimensions of the loci in these stratifications, obtaining

$$\dim Q_e - \dim Z_{\ell,p} = \ell(s+\ell), \tag{4.1}$$

$$\dim \mathrm{Gr}(\ell, p) - \dim \beta^{-1}_{\ell,p}(Z_{\ell',p}) = (\ell' - \ell)(s + \ell'). \tag{4.2}$$

These equations are useful for computing the codimension of intersections of Schubert varieties with the boundary of the Quot scheme.

4.3 Quot schemes of trivial bundles

We prove that Quot schemes of trivial bundles have nice properties when the degree of the subsheaf is sufficiently negative: they are all of the expected dimension and intersections of general Schubert varieties are all proper. We use the notation $Q_{e,C} = \mathrm{Quot}\big((r, -e), \mathcal{O}^{r+s}_C\big)$, namely the vector bundle V is always assumed to be trivial. When the curve is clear from context, we may even write just Q_e.

On $\mathbb{P}^1$, we have already observed that every Quot scheme $Q_{e,\mathbb{P}^1}$ is irreducible, smooth, of the expected dimension $(r+s)e + rs$, and contains $\mathrm{Mor}_e(\mathbb{P}^1, G)$ as a dense open subscheme. There is a theorem of Bertram ensuring that the intersection theory on the Quot scheme is as nice as possible.

Theorem 4.3.1 ([Ber97]). *Intersections W of general Schubert varieties in $Q_{e,\mathbb{P}^1}$ are proper, and $W \cap \mathrm{Mor}_e(\mathbb{P}^1, G)$ is dense in W. Top intersections of general Schubert varieties are finite, reduced, and contained in $\mathrm{Mor}_e(\mathbb{P}^1, G)$.*

As the following example illustrates, the properness and denseness claims in Bertram's theorem can fail for curves of positive genus when e is small but positive.

Example 4.3.2. Let C be a curve of genus 1. As usual $Q_0 = G$, which has dimension rs rather than the expected dimension 0. However, Schubert varieties do intersect properly in G. But this is not the case for Q_1 when for instance $r = 1$. Degree-one line bundles L on C have only a single section, hence are not globally generated, so the maps $L^* \to \mathcal{O}_C^{r+s}$ must drop rank at a point. Thus every quotient in Q_1 has torsion, hence must be of the form $\mathcal{O}^s \oplus \mathbb{C}_p$ for some $p \in C$. Thus the map β in the diagram

$$\begin{array}{ccc} \mathrm{Gr}(1) & \xrightarrow{\beta} & Q_1 \\ {\scriptstyle \pi}\downarrow & & \\ C \times Q_0 & & \end{array}$$

is an isomorphism. By chance, Q_1 has the expected dimension $s+1$ (since $rs + r = r + s$ when $s = 1$). However, the Schubert variety $\overline{W}_s(p)$ splits into two pieces on $\mathrm{Gr}(1)$. The first is $\pi^{-1}(C \times \overline{W}_s(p))$, which has the correct codimension s in $\mathrm{Gr}(1)$, but the second is the entire preimage $\pi^{-1}(p \times Q_0)$, which has only codimension 1 in $\mathrm{Gr}(1)$.

Despite the disheartening example, Quot schemes of trivial bundles on curves of arbitrary genus are well behaved when e is sufficiently large. First of all, there is an e_0 such that for all $e \geq e_0$, $Q_{e,C}$ is irreducible, generically reduced, of the expected dimension $(r+s)e - rs$, and contains $\mathrm{Mor}_e(C, G)$ as a dense open subscheme ([BDW96]). In fact, we can modify the proof of Bertram's theorem for $\mathbb{P}^1$ to show that for all $e \gg 0$, intersections of general Schubert varieties in $Q_{e,C}$ are proper and that top intersections are finite and reduced (the statement for top intersections was proved in [Ber94]). For this, we will need to control the failure of Schubert varieties to be proper on Quot schemes for smaller e. We will use the following terminology.

Definition 4.3.3. Given a morphism of schemes $\beta \colon Y \to Q_{e,C}$, an intersection W of Schubert varieties corresponding to partitions of total size A has **failure** ν in Y if $\beta^{-1}(W)$ has codimension $A - \nu$ in Y.

The proof will use induction on the degree of the subsheaves as well as the stratification from the previous section.

Proposition 4.3.4. *Let C be a curve of genus g. Let e_0 be an integer such that $Q_e = Q_{e,C}$ has the expected dimension $(r+s)e - rs(g-1)$ for all $e \geq e_0$. Let $\nu \geq 0$ be the maximum failure over all Schubert intersections on Q_{e_0+i} for all $0 \leq i \leq r$. Then for all $t \geq 0$, the maximum failure of Schubert intersections on Q_{e_0+r+t} is either 0 or $\leq \nu - t$.*

Proof. Induction on t. The base case $t = 0$ is trivial by definition of ν. Suppose $t > 0$ and that W is a Schubert intersection on Q_{e_0+r+t} with failure $\nu' > 0$. We show that $\nu' \leq \nu - t$ by proving appropriate inequalities on a stratification of Q_e.

As usual, the Schubert intersections are proper on the (dense) open locus of morphisms, hence have failure 0 on that locus. The complement of this locus is the image of

$$\begin{array}{ccc} \mathrm{Gr}(1) & \xrightarrow{\beta} & Q_{e_0+r+t} \\ \downarrow{\scriptstyle \pi} & & \\ C \times Q_{e_0+r+t-1}. & & \end{array}$$

Since e_0 is sufficiently large, the image of β has codimension s in Q_{e_0+r+t}, so it suffices to prove that the preimage of W has failure $\leq \nu - t + s$ in $\mathrm{Gr}(1)$. Recall from Corollary 4.2.3 that the preimage of W yields two types of intersections in $\mathrm{Gr}(1)$. For Type 1, the inductive hypothesis guarantees that W has failure at most $\nu - t + 1$ on $Q_{e_0+r+t-1}$, which is sufficient.

For Type 2, one of the Schubert varieties is based at p and we consider the preimage $\mathrm{Gr}(1,p) = \pi^{-1}(p \times Q_{e_0+r+t-1})$, which has codimension one in $\mathrm{Gr}(1)$, so it suffices to show the failure of W on $\mathrm{Gr}(1,p)$ is $\leq \nu - t + s + 1$. As explained in §4.2.2, we can further stratify the image

$$\beta(\mathrm{Gr}(1,p)) = \bigsqcup_{1 \leq \ell \leq r} Z_{\ell,p},$$

where $Z_{\ell,p}$ is the image of the open set $U_{\ell,p} \subset \mathrm{Gr}(\ell,p)$ consisting of elementary modifications yielding a subsheaf that drops rank by exactly ℓ at p. Recall that since $Q_{e_0+r+t-\ell}$ has the expected dimension for all $\ell \leq r$, the preimages $\beta^{-1}(Z_{\ell,p})$ stratify $\mathrm{Gr}(1,p)$ and have codimension $(\ell-1)(s+\ell)$ in $\mathrm{Gr}(1,p)$ by (4.2). Thus it suffices to prove that the failure of W on each $U_{\ell,p}$ is

$$\leq \nu - t + s + 1 + (\ell-1)(s+\ell) = \nu - t + 1 + \ell s + \ell(\ell-1). \tag{4.3}$$

There are structure maps $U_{\ell,p} \to Q_{e_0+r+t-\ell}$ and by induction the failure of the Schubert varieties not based at p is $\leq \nu - \max(t-\ell, 0)$. By Proposition 4.2.1 (d), the additional

Schubert variety $\overline{W}_{\vec{a}}(p)$ is the preimage of $W_{a_{\ell+1},\dots,a_r}$ under the map $U_{\ell,p} \to G_\ell$, yielding failure $\sum_{i=1}^{\ell} a_i \leq \ell s$. Thus the failure on $U_{\ell,p}$ is

$$\leq \nu - \max(t-\ell,0) + \ell s,$$

which together with the inequalities $(\ell-1)^2 \geq 0$ and $\max(t-\ell,0)-(t-\ell) \geq 0$ implies the inequality (4.3). □

Corollary 4.3.5. *For $e \gg 0$, $Q_{e,C}$ has the expected dimension $(r+s)e - rs(g-1)$ and all intersections W of Schubert varieties are proper. Moreover, $W \cap \mathrm{Mor}_e(C,G)$ is dense in W, and top intersections are finite, reduced, and contained in $\mathrm{Mor}_e(C,G)$.*

Proof. The properness claim follows from the proposition by taking $e \geq e_0 + r + \nu$. For the denseness statement, note that since $Q_{e,C}$ is irreducible and the codimension of W cannot be greater than its total size A since it is a degeneracy locus, it suffices to show that W actually has failure $< s$ in $\mathrm{Gr}(1)$. We can achieve this by taking $e \geq e_0 + r + \nu$ and copying the proof of the proposition, replacing the $\nu - t$ appearing in the inductive estimates on the failure with zero. For the statement about top intersections, the only part left to be shown is reducedness. But since W is contained in $\mathrm{Mor}_e(C,G)$ and $Q_{e,C}$ is generically reduced, the general fibers of the evaluation map to G^N (where N is the number of Schubert varieties) are finite and hence reduced, so the intersection with general Schubert varieties will avoid the branch locus. □

To end the section, we present two important results about Quot schemes of trivial bundles.

Proposition 4.3.6. *(a) The intersection numbers on $Q_{e,C}$ can be computed as an integral of Chern classes of the universal subsheaf.*

(b) The intersection numbers on $Q_{e,C}$ do not depend on the choice of smooth curve C of genus g.

Proof. (a): The cohomology classes of the dual of the universal subbundle on the Grassmannian (which are σ_{1^k} for $1 \leq k \leq r$) generate the cohomology ring of the Grassmannian. The same formulas express $\bar{\sigma}_{\vec{a}}$ in terms of the Chern classes of the dual of the universal subbundle on the Quot scheme.

(b): This is Proposition 1.5 in [Ber94]. Since we will make a similar argument for Proposition 4.6.1, we will just summarize the key steps in the proof. Consider a family of smooth curves of fixed genus over a base curve B. The relative Quot scheme $Q \to B$ of the family contains the Quot schemes of trivial bundles over each of the curves in the family as its fibers over B. The intersection numbers of each Quot scheme can be computed in terms of Chern classes of its universal subbundle, and flatness of Q implies that these products of Chern classes are independent of the base point (by expressing them in terms of Chern classes of the universal subbundle of the relative Quot scheme). □

4.4 Quot schemes of very general vector bundles

In the previous section, we showed that intersections of Schubert varieties are proper on Quot schemes of trivial bundles if the subsheaves have sufficiently negative degree. In order to get similar results for Quot schemes of general vector bundles V, we embed them in Quot schemes of trivial bundles (which are typically far from general) by using elementary modifications. Amazingly, we conclude that for a very general vector bundle, *all* Quot schemes have the expected dimension and proper Schubert intersections.

We begin with some terminology. Let C be a curve of genus g. We will say that a vector bundle V on C is **stable** if

$$\begin{cases} V \text{ is balanced} & \text{if } g = 0; \\ V \text{ is semistable} & \text{if } g = 1; \\ V \text{ is stable} & \text{if } g \geq 2. \end{cases}$$

A vector bundle on $\mathbb{P}^1$ is **balanced** if its splitting $V = \oplus_{i=1}^{\mathrm{rk}(V)} \mathcal{O}_{\mathbb{P}^1}(d_i)$ has the property that $|d_i - d_j| \leq 1$ for all $1 \leq i, j \leq \mathrm{rk}(V)$. When we call the vector bundle **general**, we mean that it is stable in the above sense and does not lie on a finite collection of proper closed subvarieties of the moduli space when $g \geq 1$. Even stronger, we say the vector bundle is **very general** if it does not lie on a countable collection of proper closed subvarieties of the moduli space.

We begin by showing that on $\mathbb{P}^1$, stability (balancedness) of vector bundles is preserved by general elementary modifications.

Lemma 4.4.1. *Suppose a vector bundle V on $\mathbb{P}^1$ of positive rank is balanced. Then for any $p \in \mathbb{P}^1$, the kernels of general elementary modifications $V \twoheadrightarrow \mathbb{C}_p$ are balanced.*

Proof. Since V is balanced, $V \simeq \mathcal{O}(d)^a \oplus \mathcal{O}(d+1)^b$ for some $d \in \mathbb{Z}$, $a \geq 0$, and $b > 0$. A general elementary modification $V \twoheadrightarrow \mathbb{C}_p$ induces a surjection $\mathcal{O}(d+1) \twoheadrightarrow \mathbb{C}_p$ on one of the summands of V of type $\mathcal{O}(d+1)$, yielding a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}(d) & \longrightarrow & \mathcal{O}(d+1) & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & \mathbb{C}_p & \longrightarrow & 0 \end{array}$$

The exact sequence of cokernels implies that the cokernels of the first two vertical maps are isomorphic, hence there is an exact sequence $\mathcal{O}(d) \hookrightarrow V' \twoheadrightarrow \mathcal{O}(d)^a \oplus \mathcal{O}(d+1)^{b-1}$. But every such extension splits, so $V' \simeq \mathcal{O}(d)^{a+1} \oplus \mathcal{O}(d+1)^{b-1}$, which is balanced. □

Next, we prove the key result allows us to relate Quot schemes of general vector bundles to Quot schemes of trivial bundles.

Proposition 4.4.2. *Let L of degree ℓ be a sufficiently ample line bundle such that $V^* \otimes L$ is globally generated and has vanishing higher cohomology for all $V \in M(r+s, d)$. Then general V and kernels of general elementary modifications $\mathcal{O}_C^{r+s} \twoheadrightarrow \oplus_{i=1}^{d+(r+s)\ell} \mathbb{C}_{q_i}$ coincide.*

Proof. Note that it is possible to find such L because the Castelnuovo-Mumford regularity is bounded on $M(r+s, d)$. Let $N = (r+s)\ell - d$. The moduli space parametrizing all elementary modifications is $\mathrm{Quot}\big(\mathcal{O}_C^{r+s}, (0, N)\big)$, which is irreducible, smooth, and of the expected dimension $N(r+s)$ since $\mathrm{ext}^1(E, F) = 0$ whenever E is locally free and F is torsion. The universal sequence $0 \to \mathcal{E} \to \mathcal{O}^{r+s} \to \mathcal{F} \to 0$ on $C \times \mathrm{Quot}\big(\mathcal{O}_C^{r+s}, (0, N)\big)$ assembles the elementary modifications. Given any stable vector bundle V, choosing $r+s$ general sections of $V^* \otimes L$ yields a sequence

$$0 \to V \otimes L^* \to \mathcal{O}_C^{r+s} \to T \to 0,$$

where T is torsion of length N. In particular, this sequence occurs in the universal family over the Quot scheme. Since stability is an open condition in families, this guarantees that general elementary modifications parametrized by the universal sequence are stable, so by the universal property of the moduli space of sheaves, there is a dominant rational map

$$\mathrm{Quot}\left(\mathcal{O}_C^{r+s}, (0, N)\right) \dashrightarrow M(r+s, d),$$

which completes the proof. □

Given a vector bundle V on a curve C of genus g, let $Q_{e,V} = \mathrm{Quot}\big((r,-e),V\big)$.

Proposition 4.4.3. *Let V be very general of rank $r+s$ and degree d. Then for all e:*

(a) $Q_{e,V}$ is equidimensional of the expected dimension $(r+s)e + rd - rs(g-1)$;

(b) $Q_{e,V}$ is generically reduced and the subscheme $U_{e,V}$ of torsion-free quotients in $Q_{e,V}$ is open and dense;

(c) intersections W of general Schubert varieties in $Q_{e,V}$ are proper, $W \cap U_{e,V}$ is dense in W, and top intersections are finite, reduced, and contained in $U_{e,V}$.

Proof. For each e, choose a line bundle L_e of degree ℓ_e sufficiently ample such that $e + r\ell_e$ is sufficiently large to ensure $Q_{e+r\ell_e}$ has the properties in Corollary 4.3.5 and also that $V^* \otimes L_e$ is globally generated with vanishing higher cohomology for all $V \in M(r+s,d)$. General elementary modifications

$$\mathcal{O}_C^{r+s} \twoheadrightarrow \bigoplus_{i=1}^{(r+s)\ell_e - d} \mathbb{C}_{q_i}$$

produce kernels which, when twisted by L_e^*, are general in $M(r+s,d)$. Thus choosing V very general ensures that it fits into sequences

$$0 \to V \otimes L_e^* \to \mathcal{O}_C^{r+s} \to \bigoplus_{i=1}^{(r+s)\ell_e - d} \mathbb{C}_{q_i} \to 0$$

for all e. Now, there are embeddings

$$Q_{e,V} \hookrightarrow Q_{e+r\ell_e,C}, \quad [E \subset V] \mapsto [E \otimes L_e^* \subset V \otimes L_e^* \subset \mathcal{O}_C^{r+s}]$$

and the image consists of those subsheaves of $\mathcal{O}_C^{r+s}$ whose map to the skyscraper sheaf in the elementary modification is zero, namely the image is an intersection

$$\bigcap_{i=1}^{(r+s)\ell_e - d} \overline{W}_{1^r}(q_i).$$

Since the elementary modification is general, so are the $\overline{W}_{1^r}(q_i)$. Since $Q_{e+r\ell_e,C}$ has the expected dimension and proper Schubert intersections,

$$\dim Q_{e,V} = \dim Q_{e+r\ell_e,C} - ((r+s)\ell_e - d)r = (r+s)e + rd - rs(g-1),$$

which is the expected dimension of $Q_{e,V}$ and proves (a).

For (b), note that quotients in $Q_{e,V}$ (twisted by L^*) are obtained from quotients in $Q_{e+r\ell_e,C}$ by elementary modification along the same $\bigoplus \mathbb{C}_{q_i}$. If the latter quotients are torsion-free, so are the former, which proves that the intersection of $U_{e+r\ell_e,C}$ with the image of the embedding is contained in $U_{e,V}$. Now we get (b) from the same properties for $Q_{e+r\ell_e,C}$ and the fact that the intersection of the $\overline{W}_{1^r}(q_i)$ with $U_{e+r\ell_e,C}$ is dense in the image of the embedding.

For (c), we note that any intersection of Schubert varieties on $Q_{e,V}$ at points other than the q_i can be expressed as the same intersection on $Q_{e+r\ell_e,C}$ together with the additional Schubert varieties $\overline{W}_{1^r}(q_i)$, and since the latter intersection is proper, so is the former. The statement about top intersections also follows immediately from the same statement for $Q_{e,V}$. □

The embedding in the proof of the proposition allows us to compute integrals on Quot schemes of general vector bundles on Quot schemes of trivial bundles.

Corollary 4.4.4. *Let V be very general of rank $r+s$ and degree d. Then for all e and all $\ell \gg 0$,*

$$\int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}} = \int_{Q_{e+r\ell,C}} \bar{\sigma}_{\underline{\vec{a}}} \cup \bar{\sigma}_{1^r}^{(r+s)\ell-d}.$$

4.5 Quot schemes on nodal curves

To prove the $F(g|d)^n_m$ satisfy the relations of a weighted TQFT, we relate Quot schemes on smooth curves to Quot schemes on nodal curves, where the nodal curve is obtained by degeneration. The first important observation is that Schubert varieties in Quot schemes over a nodal curve can be defined at smooth points p in exactly the same way as they were defined over smooth curves. However, the cohomology class of a Schubert variety now depends on which component of the curve contains p.

Throughout this section, let C be a reducible nodal curve with two smooth components C_1 and C_2 of genus g_1 and g_2 meeting at a simple node $\nu \in C$. Let $\iota_i \colon C_i \hookrightarrow C$ denote the embeddings and let $p_i \in C_i$ denote the points $\iota_i^{-1}(\nu)$ lying over the node.

4.5.1 Sheaves on nodal curves

We review some facts about sheaves on reducible nodal curves described in [Ses82]. If E is a rank r torsion-free sheaf on C, then its stalk at the node ν is of the form $E_\nu \simeq \mathcal{O}_\nu^{r-a} \oplus m_\nu^a$

for some $0 \leq a \leq r$, where m_ν is the ideal sheaf of ν.

Definition 4.5.1. Let E be torsion-free sheaf of rank r on the nodal curve C. If $E|_\nu \simeq \mathcal{O}_\nu^{r-a} \oplus m_\nu^a$, then we say E is a**-defective**.

The following proposition lists some relationships between torsion-free sheaves on C and locally free sheaves on the components of C.

Proposition 4.5.2. *(a) Let E be rank r and a-defective on C. Then $\iota_i^* E \simeq E_i \oplus \mathbb{C}_{p_i}^a$, where the E_i are vector bundles of rank r on C_i and $\deg E_1 + \deg E_2 = \deg E - a$.*

(b) If E_i are rank r vector bundles on C_i, then $\iota_{1*}E_1 \oplus \iota_{2*}E_2$ is r-defective on C and $\deg(\iota_{1*}E_1 \oplus \iota_{2*}E_2) = \deg E_1 + \deg E_2 + r$.

(c) Using the notation in (a), there is a canonical short exact sequence

$$0 \to E \to \iota_{1*}E_1 \oplus \iota_{2*}E_2 \to E(\nu)/(\iota_{1*}\mathbb{C}_{p_1}^a \oplus \iota_{2*}\mathbb{C}_{p_2}^a) \to 0$$

in which each map $\iota_{i*}E_i \to E(\nu)/(\iota_{1*}\mathbb{C}_{p_1}^a \oplus \iota_{2*}\mathbb{C}_{p_2}^a)$ is surjective. The quotient in the sequence is isomorphic to $\mathbb{C}_\nu^{r-a}$.

Proof. (a): We argue as in [Ses82]. Let L be a line bundle on C with a section $\mathcal{O}_C \to L$ that does not vanish at ν, namely the cokernel is supported away from ν. Pulling back along the ι_i yields sections $\mathcal{O}_{C_i} \to \iota_i^* L$ whose cokernels partition the original cokernel. Thus $\deg \iota_1^* L + \deg \iota_2^* L = \deg L$. Moreover, a local analysis shows that $\iota_i^* m_\nu = m_{p_i} \oplus \mathbb{C}_{p_i}$, so $\deg m_\nu = \deg m_{p_1} + \deg m_{p_2} + 1$. These facts can be used to deduce the claim.

(b): The key observation is $\iota_{1*}m_{p_1} \oplus \iota_{2*}m_{p_2} = m_\nu$. The degree statement is true even if E_i are general coherent sheaves because the length of torsion is preserved by ι_{i*}.

(c) As is clear on the level of modules, there are canonical maps $E \to \iota_{1*}\iota_1^* E \oplus \iota_{2*}\iota_2^* E$ that are isomorphisms away from ν. The functors $\iota_{i*}\iota_i^*$ and these canonical maps yield a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & \iota_{1*}\iota_1^* E \oplus \iota_{2*}\iota_2^* E & \longrightarrow & \mathbb{C}_\nu^{r+a} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (\iota_{i*}\iota_i^* f)} & & \downarrow & & \\
0 & \longrightarrow & E(\nu) & \longrightarrow & \iota_{1*}\iota_1^* E(\nu) \oplus \iota_{2*}\iota_2^* E(\nu) & \longrightarrow & E(\nu) & \longrightarrow & 0
\end{array}$$

in which the first map in the first row is an inclusion since it is an isomorphism away from ν and E is torsion-free. We identify the cokernel in the first row by its degree. The first map in the bottom row is diagonal inclusion $E(\nu) \to E(\nu) \oplus E(\nu)$. The right vertical map must be a surjection, hence an isomorphism. Removing torsion in the pullbacks yields a map $E \to \iota_{1*}E_1 \oplus \iota_{2*}E_2$ that is also injective (for degree reasons), which produces the desired sequence. The surjectivity claim follows from the commutativity of the right side of the diagram and the fact that in the bottom row, each $E(\nu)$-summand in the middle surjects onto the quotient. $\square$

Let V be a vector bundle of rank $r+s$ on C. Letting V_i denote $\iota_i^* V$, the sequence (c) in the proposition is

$$0 \to V \to \iota_{1*}V_1 \oplus \iota_{2*}V_2 \to V(\nu) \to 0.$$

The maps $\iota_{i*}V_i \twoheadrightarrow V(\nu)$ are push-forwards of the quotients $V_i \twoheadrightarrow V_i(p_i)$. If we think of the sequence in reverse, starting with V_1 and V_2, then we are constructing a vector bundle V on the nodal curve by gluing V_1 and V_2 along the node via a choice of isomorphism $V_1(p_1) \simeq V_2(p_2)$ of their fibers over the node. We can summarize this as a bijection

$$\left\{ \begin{array}{c} \text{vector bundles } V \text{ on } C \text{ of} \\ \text{rank } r+s \text{ and degree } d \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{vector bundles } V_i \text{ on } C_i \text{ of rank } r+s \text{ and} \\ \text{degree } d_i \text{ satisfying } d_1+d_2=d, \text{ together} \\ \text{with an isomorphism } V_1(p_1) \simeq V_2(p_2) \end{array} \right\}.$$

The map from left to right is $V \mapsto (\iota_1^*V, \iota_2^*V)$ together with the canonical isomorphisms $(\iota_1^*V)(p_1) \simeq V(\nu) \simeq (\iota_2^*V)(p_2)$ coming from the fact that fibers are defined as a pullback, hence are preserved under pullback. The map from right to left is $(V_1, V_2) \mapsto \ker(\iota_{1*}V_1 \oplus \iota_{2*}V_2 \twoheadrightarrow \iota_{2*}V_2(p_2))$, where the map $\iota_{1*}V_1 \to \iota_{2*}V_2(p_2)$ is the composition of the natural surjection onto the fiber composed with the isomorphism $V_1(p_1) \simeq V_2(p_2)$.

Given a short exact sequence $0 \to E \to V \to F \to 0$ in $Q_{e,V}$, where E is torsion-free but F may have torsion, the natural pull-push maps as in (c) in the proposition fit in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & 0 \end{array}$$

in which the rows are exact, $F_i = \iota_i^* F$, $E_i = \iota_i^* E/\text{torsion}$, and the push-forward notation is suppressed in the second row. The snake lemma long exact sequence is of the form

$$0 \to \mathbb{C}_\nu^b \to \mathbb{C}_\nu^{r-a} \to \mathbb{C}_\nu^{r+s} \to \mathbb{C}_\nu^{s+a+b} \to 0$$

for some $a \geq 0$ and $0 \leq b \leq r - a$ that can be interpreted as follows: E is a-defective and the rank of $E(\nu) \to V(\nu)$ is $r - a - b$. Note that $0 \to E_i \to V_i \to F_i \to 0$ are short exact sequences on C_i and that $\deg E_1 + \deg E_2 = \deg E - a$.

4.5.2 Structure of Quot schemes on nodal curves

To control intersections of Schubert varieties on $Q_{e,V} = \mathrm{Quot}\big((r, -e), V\big)$ over the nodal curve C, we relate $Q_{e,V}$ to the Quot schemes of the components C_i.

For all $0 \leq a \leq r$, let $Z_{a,\nu,e}$ denote the locally-closed subscheme in $Q_{e,V}$ consisting of sequences $0 \to E \to V \to F \to 0$ where E is a-defective. As in §4.2.2, let Z_{a,p_i} denote the locally-closed subscheme in $Q_{e_i,V_i} = \mathrm{Quot}\big((r, -e_i), V_i\big)$ where the subsheaf drops rank by exactly a at p_i. Setting $G_a = \mathrm{Gr}\big(r - a, V_1(p_1)\big)$ (which is identified with $\mathrm{Gr}(r - a, V_2(p_2))$, there is an evaluation map

$$Z_{a,p_1} \times Z_{a,p_2} \to G_a \times G_a$$

and we let Δ_a denote the preimage of the diagonal.

Proposition 4.5.3. *Let $a \geq 0$, $e_1 + e_2 = e + a$, and $0 < b \leq r - a$.*

(a) There is an embedding

$$\phi_a \colon \Delta_a \hookrightarrow Z_{a,\nu,e}$$

whose image is exactly those $[E \subset V]$ where E is a-defective, the map $E(\nu) \to V(\nu)$ has rank exactly $r - a$, and the pullbacks $E_i = (\iota_i^* E)/\text{torsion}$ have degree e_i.

(b) Let $\mathcal{E}|_\nu$ denote the universal subsheaf restricted to $\nu \times Z_{a+b,\nu,e-b}$. Let $U_b \subset \mathrm{Gr}(\mathcal{E}|_\nu, b)$ denote the open subscheme of elementary modifications that produce an a-defective kernel. There is a map

$$\begin{array}{ccc} U_b & \xrightarrow{\beta_{a,b}} & Z_{a,\nu,e} \\ \downarrow & & \\ Z_{a+b,\nu,e-b} & & \end{array}$$

whose image contains all $0 \to E \to V \to F \to 0$ where E is a-defective and $E(\nu) \to V(\nu)$ has rank $r - a - b$.

(c) For each a, as b and the partition $e_1 + e_2$ of $e + a$ vary, the images of the maps ϕ_a and $\beta_{a,b}$ cover $Z_{a,\nu,e}$.

Proof. (a): Set-theoretically, define ϕ_a as follows. A point in Δ_a is a pair of subsheaves $E_i \subset V_i$ whose maps on fibers at p_i have rank $r - a$ and the images are equal under the identification $V_1(p_1) \simeq V_2(p_2)$. Thus there is a canonical sequence

$$0 \to E \to \iota_{1*}E_1 \oplus \iota_{2*}E_2 \to \mathbb{C}_\nu^{r-a} \to 0$$

producing a sheaf E on C that is a-defective. Moreover, the image of $E(\nu) \to V(\nu)$ coincides with the image of $E_i(p_i) \to V_i(p_i)$ after identifying $V_i(p_i)$ with $V(\nu)$.

To construct this morphism algebraically, we construct the same sequence in families. First, we construct vector bundles $\widehat{\mathcal{E}_i}$ on $C \times \Delta_a$ as follows: restrict the universal subsheaves $\mathcal{E}_i \subset \pi_{C_i}^* V_i$ on $C_i \times Q_{e_i,V_i}$ to $C_i \times Z_{a,p_i}$, push forward under the inclusion $\iota_i \times \mathrm{id}$ to get a sheaf on $C \times Z_{a,p_i}$, pull back to $C \times Z_{a,p_1} \times Z_{a,p_2}$ under the projection map, and finally restrict to $C \times \Delta_a$. There is a morphism

$$\widehat{\mathcal{E}_1} \oplus \widehat{\mathcal{E}_2} \to \pi_C^* V_1(p_1)$$

whose kernel $\mathcal{E}$ is a flat family of a-defective sheaves on C of degree e, hence yielding a map to $Q_{e,V}$ whose image is contained in $Z_{a,\nu,e}$.

Similarly, to get the inverse, we pull back the universal subsheaf $\mathcal{E} \to \pi^* V$ on $C \times \mathrm{im}(\phi_a)$ to $C_i \times \mathrm{im}(\phi_a)$ and remove the torsion in $\iota_i^* \mathcal{E}$. Then $\iota_i^* \mathcal{E}/\text{torsion}$ is a flat family of sheaves of degree e_i, hence defines a map $\mathrm{im}(\phi_a) \to Z_{a,p_i}$. The image of the product of these maps is contained in Δ_a since the restrictions to p_i of the inclusions $\iota_i^* \mathcal{E}/\text{torsion} \hookrightarrow \pi^* V_i$ have image coinciding with the image of $\mathcal{E}|_\nu \to \pi^* V(\nu)$ under the identification $V_i(p_i) \simeq V(\nu)$.

(b): The open subscheme U_b of the Grassmann bundle parametrizes elementary modifications $0 \to E \to E' \to \mathbb{C}_\nu^b \to 0$. Since E' is $(a+b)$-defective and the elementary modification is general, a local computation shows that E, the image under $\beta_{a,b}$, is a-defective. Moreover, $E(\nu) \to V(\nu)$ factors through $E'(\nu) \to V(\nu)$, which has rank $\leq r - a - b$.

Conversely, starting with $0 \to E \to V \to F \to 0$ in which E is a-defective and $E(\nu) \to V(\nu)$ has rank exactly $r - a - b$, by the discussion at the end of §4.5.1, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E & \longrightarrow & V & \longrightarrow & F & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & 0
\end{array}$$

and the snake lemma long exact sequence is $0 \to \mathbb{C}^b_\nu \to \mathbb{C}^{r-a}_\nu \to \mathbb{C}^{r+s}_\nu \to \mathbb{C}^{s+a+b}_\nu \to 0$. The rank of the map of fibers $E_i(p_i) \to V_i(p_i)$ is only $r-a-b$, so we can pass to the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 \oplus E_2 & \longrightarrow & V_1 \oplus V_2 & \longrightarrow & F_1 \oplus F_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C}^{r-a-b}_\nu & \longrightarrow & \mathbb{C}^{r+s}_\nu & \longrightarrow & \mathbb{C}^{s+a+b}_\nu & \longrightarrow & 0 \end{array}$$

and in the short exact sequence of kernels $0 \to E' \to V \to F' \to 0$ the sheaf E' is $(a+b)$-defective. Moreover, there is an induced short exact sequence $0 \to E \to E' \to \mathbb{C}^b_\nu \to 0$, so $0 \to E \to V \to F \to 0$ is in the image of $\beta_{a,b}$.

Now we construct $\beta_{a,b}$. The restriction of the universal sequence to $C \times Z_{a+b,\nu,e-b}$ yields $0 \to \mathcal{E} \to \pi_C^* V \to \mathcal{F} \to 0$ in which each fiber of $\mathcal{E}$ over $Z_{a+b,\nu,e-b}$ is $(a+b)$-defective, hence $\mathcal{E}|_\nu$ is a vector bundle of rank $r+a+b$ on $Z_{a+b,\nu,e-b}$. Pushing forward the universal sequence $0 \to \mathcal{S} \to \pi^*\mathcal{E}|_\nu \to \mathcal{Q} \to 0$ on $\mathrm{Gr}(\mathcal{E}|_\nu, b)$ under the inclusion $\mathrm{Gr}(\mathcal{E}|_\nu, b) \to C \times \mathrm{Gr}(\mathcal{E}|_\nu, b)$ at the point ν, we get a map $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{Q}$ on $C \times \mathrm{Gr}(\mathcal{E}|_\nu, b)$ whose kernel $\mathcal{K}$ is a flat family of degree e sheaves on C. The embedding $\mathcal{K} \hookrightarrow \pi^*\mathcal{E} \hookrightarrow \pi^* V$ induces the map $\beta_{a,b} \colon \mathrm{Gr}(\mathcal{E}|_\nu, b) \to Q_{e,V}$. Restricting to U_b ensures that the image is contained in $Z_{a,\nu,e}$.

(c): Let $x = [0 \to E \to V \to F \to 0]$ be a short exact sequence in which E is a-defective and $E(\nu) \to V(\nu)$ has rank $r-a-b$. If $b > 0$, then x is in the image of ϕ_a (for $e_i = \deg E_i$). If $b > 0$, then x is in the image of $\beta_{a,b}$. □

Remark 4.5.4. In the case when the V_i are very general and the isomorphism $V_1(p_1) \simeq V_2(p_2)$ defining V is sufficiently general to ensure that Δ_0 always has the expected dimension, it follows from (a) that for fixed e, each partition $e = e_1 + e_2$ for which Δ_a is nonempty yields a component of $Q_{e,V}$ of the expected dimension. By (c) and a dimension count, we see that any other component of $Q_{e,V}$ must have smaller dimension. Thus, although, $Q_{e,V}$ is in general not irreducible, each of its top-dimensional components is the pullback of the diagonal in a product $U_{e_1,V_1} \times U_{e_2,V_2}$ for some partition $e_1 + e_2 = e$. In particular, when V is trivial, we see that each component of $\mathrm{Mor}_e(C, G)$ is the pullback of the diagonal in $\mathrm{Mor}_{e_1}(C_1, G) \times \mathrm{Mor}_{e_2}(C_2, G)$, namely a morphism $C \to G$ of degree e corresponds to a pair of morphisms $C_i \to G$ of degrees summing to e such that the points p_i have the same image in G.

4.5.3 Intersections of Schubert varieties are proper

Using the maps described in the previous section, we study intersections of Schubert varieties in $Q_{e,V}$ by controlling their preimages. We use induction for the maps β_b, so the main tool we still need is a properness statement for intersections of Schubert varieties in $\Delta_a \subset Z_{a,p_1} \times Z_{a,p_2}$. For the following proposition, since $0 \le a \le r$ and the p_i are fixed, we drop a, p_1, and p_2 from the notation, but we need to keep track of the degrees e_i.

Proposition 4.5.5. *Suppose $\Delta \subset Z_{e_1} \times Z_{e_2}$ is empty or has pure codimension $(r-a)(s+a)$ for all e_1, e_2. Then intersections W of general Schubert varieties on $Q_{e_1,V_1} \times Q_{e_2,V_2}$ are proper on Δ. Moreover, if $\mathcal{U} \subset Z_{e_1} \times Z_{e_2}$ denotes the open locus where the quotients are torsion-free away from p_1 and p_2, then $W \cap \Delta \cap \mathcal{U}$ is dense in $W \cap \Delta$. In particular, top intersections are finite, reduced, and contained in $\mathcal{U} \cap \Delta$.*

Proof. Let W be the intersection of $\overline{W}_{\vec{a}_1}(q_1), \ldots, \overline{W}_{\vec{a}_k}(q_k)$ and $\overline{W}_{\vec{b}_1}(q'_1), \ldots, \overline{W}_{\vec{b}_m}(q'_m)$, where $q_1, \ldots, q_k \in (C_1 \setminus p)$ and $q'_1, \ldots, q'_m \in (C_2 \setminus q)$ are distinct points. Set $A = \sum_{i=1}^{k} |\vec{a}_i| + \sum_{j=1}^{m} |\vec{b}_i|$. Let $\mathcal{B}$ denote the complement of $\mathcal{U}$ in $Z_{e_1} \times Z_{e_2}$. Since the Schubert varieties on Δ are still defined as degeneracy loci of the restriction of the relevant bundles on $Z_{e_1} \times Z_{e_2}$, their codimension in Δ cannot be strictly larger than A. Thus it suffices to prove that

(a) $W \cap \Delta \cap \mathcal{U}$ is empty or has pure codimension A in Δ;

(b) $W \cap \Delta \cap \mathcal{B}$ is empty or has codimension $> A$ in Δ.

(a): As usual, there is an evaluation morphism $\mathrm{ev}_{q_1,\ldots,q_k,q'_1,\ldots,q'_m} \colon \mathcal{U} \to G^{k+m}$. Restricting the domain to $\Delta \cap \mathcal{U}$ and using Lemma 4.1.4 gives the result.

(b): The boundary $\mathcal{B}$ is the image of the map

$$\beta_{1,0} \sqcup \beta_{0,1} \colon \big(\mathrm{Gr}(1) \times Z_{e_2}\big) \sqcup \big(Z_{e_1} \times \mathrm{Gr}(1)\big) \to Z_{e_1} \times Z_{e_2},$$

where the definition of Grassmann bundles over $Z_{e_1-\ell}$ is as before, except that we restrict the elementary modification to a point of $C_1 \setminus p_1$, thus obtaining a recursive structure on the $Z_{e_1-\ell}$ for varying ℓ (and similar for Z_{e_2-1}).

We pull back Δ under $\beta_{1,0}$ and $\beta_{0,1}$ to get closed subvarieties $\Delta_{1,0}$ and $\Delta_{0,1}$ in the Grassmann bundles. As usual, it suffices to prove that the pullback of W has codimension

$$> A - (\dim \Delta - \dim \Delta_{1,0})$$

in $\Delta_{1,0}$ (then by symmetry, we will also get codimension $> A - (\dim \Delta - \dim \Delta_{0,1})$ in $\Delta_{0,1}$). Since the elementary modifications in $\mathrm{Gr}(1)$ occur at points other than p_1, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Gr}(1) \times Z_{e_2} & \xrightarrow{\beta_{1,0}} & Z_{e_1} \times Z_{e_2} \\ {\scriptstyle \pi_{1,0}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_{p_1,p_2}} \\ (C \setminus p) \times Z_{e_1-1} \times Z_{e_2} & \xrightarrow[\mathrm{ev}'_{p_1,p_2}]{} & G_a \times G_a \end{array}$$

so $\Delta_{1,0}$ can be obtained by pulling back the diagonal under ev'_{p_1,p_2} and then $\pi_{1,0}$. By assumption, the pullback under ev'_{p_1,p_2} is empty or has codimension $(r-a)(s+a)$, hence we get that same codimension in the Grassmann bundle. Thus

$$\dim \Delta - \dim \Delta_{1,0} = \dim(Z_{e_1} \times Z_{e_2}) - \dim(\mathrm{Gr}(1) \times Z_{e_2}) = s.$$

We prove that the intersection $W \cap \Delta_{1,0}$ is empty or has codimension $> A - s$ in $\Delta_{1,0}$ by induction on $e_1 + e_2$. The base case is trivial since $Z_{e_1} \times Z_{e_2}$ is empty when $e_1 + e_2$ is sufficiently small. For the inductive step, we may assume there is at least one Schubert variety since otherwise there is nothing to prove. We can immediately deal with the case $k = 0$ and $\ell > 0$ since by the inductive hypothesis the Schubert varieties impose the right codimension on $Z_{e_1-1} \times Z_{e_2}$, hence also on $\mathrm{Gr}(1) \times Z_{e_2}$.

The remaining case is $k > 0$. As usual, the intersections on $\mathrm{Gr}(1)$ come in two types. Type 1 reduces to an intersection on the base Z_{e_1-1}, where we are done by induction. Up to relabeling of the q_i, each intersection of Type 2 is supported on $\mathrm{Gr}(1, q_k) = \pi_1^{-1}(q_k \times Z_{e_1-1})$ and includes the Schubert variety $\hat{W}_{(\vec{a}_k)_{\ell+1},\dots,(\vec{a}_k)_r}(q_k)$. As described in §4.2.2, there is a stratification

$$\mathrm{Gr}(1, q_k) = \bigsqcup_{1 \le \ell \le r} \beta_{1,q_k}^{-1}(Z_{\ell,q_k}),$$

where U_{ℓ,q_k} is the open subscheme of the Grassmann bundle $\mathrm{Gr}(\ell, q_k)$ over $Z_{e_1-\ell}$ and $Z_{\ell,q_k} = \beta_{\ell,q_k}(U_{\ell,q_k}) \subset Z_{e_1}$. The pullback of Δ (which we also write as Δ) intersects each $U_{\ell,q_k} \times Z_{e_2}$ properly since it is proper on the base $Z_{e_1-\ell} \times Z_2$. The preimage of β_{1,q_k} over Z_{ℓ,q_k} has fibers of dimension $\ell - 1$. There are evaluation maps $\mathrm{ev}_{q_k} \colon U_{\ell,q_k} \to \mathrm{Gr}\big(r - \ell, V_1(q_k)\big)$ yielding maps

$$U_{\ell,p_k} \times Z_{e_2} \to \mathrm{Gr}\big(r - \ell, V_1(q_k)\big).$$

The inductive assumption ensures that the intersection of Δ and the Schubert varieties not based at q_k is proper in $Z_{e_1-\ell} \times Z_{e_2}$, hence also in $U_{\ell,q_k} \times Z_{e_2}$. A general choice of $W_{\vec{a}_k}$ in G ensures that each $W_{(\vec{a}_k)_{\ell+1},\ldots,(\vec{a}_k)_r}$ is general in $\mathrm{Gr}\big(r-\ell, V_1(q_k)\big)$, so the Schubert varieties impose codimension $\geq A - \ell s$ in $\Delta \cap (U_{\ell,q_k} \times Z_{e_2})$, hence also in $\Delta \cap (\beta^{-1}_{1,q_k}(Z_{\ell,q_k}) \times Z_{e_2})$. Thus by (4.2), the preimage under β_{1,q_k} of the intersection of Δ and the Schubert varieties has codimension

$$\geq A - \ell s + \big(\dim \mathrm{Gr}(1, q_k) - \dim \mathrm{Gr}(\ell, q_k)\big) - (\ell - 1) = A - s + \ell(\ell - 1)$$

in $\Delta \cap (\mathrm{Gr}(1, q_k) \times Z_{e_2})$, and passing to $\Delta \cap (\mathrm{Gr}(1) \times Z_2)$ yields one additional codimension. The largest such locus is obtained when $\ell = 1$, but this still has codimension $A - s + 1$. This completes the case $k > 0$. □

With this technical tool in hand, we can prove our result. Recall that V is determined by specifying an isomorphism $V_1(p_1) \simeq V_2(p_2)$. We call V **very general** if the V_i are very general (as in §4.4) and this isomorphism is very general. This guarantees that the properness assumption on Δ in the previous proposition is satisfied.

Corollary 4.5.6. *Suppose V is very general. Then for all $e \in \mathbb{Z}$, intersections of general Schubert varieties are proper in each component of $Q_{e,V}$. Moreover, top intersections are finite, reduced, and contained in $U_{e,V}$.*

Proof. We cover $Q_{e,V}$ with the images of maps ϕ_a and $\beta_{a,b}$ from Proposition 4.5.3 for all $0 \leq a \leq r$, $0 < b \leq r - a$, and $e_1 + e_2 = e + a$. It suffices to show the preimages of Schubert varieties in $Q_{e,V}$ are proper in the domain of these maps. For ϕ_a this follows from the previous proposition. For the maps $\beta_{a,b}$, we can perform the intersection on the base, where it is proper by induction on e.

Note that the locus where the subsheaves are locally free is the image of the maps of type ϕ_0, which yield the top-dimensional components. Dimension counts show that the boundary does not contribute to top intersections, hence we inherit good properties of top intersections from the same properties on the U_{e_i,V_i}. □

Corollary 4.5.7. *Suppose V is very general. Let $\bar{\sigma}_{\vec{\underline{a}}_i}$ denote a cup product of Schubert cycles based at points on the component C_i. Then for all e,*

$$\int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2} = \sum_{e_1+e_2=e} \sum_{\vec{b}} \left(\int_{Q_{e,V_1}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\vec{b}} \right) \left(\int_{Q_{e,V_2}} \bar{\sigma}_{\underline{\vec{a}}_2} \cup \bar{\sigma}_{\vec{b}^c} \right).$$

Proof. By the previous corollary, top intersections are contained in $U_{e,V}$. Since the components of $U_{e,V}$ are isomorphic to the images $\phi_0\big((U_{e_1,V_1} \times U_{e_2,V_2}) \cap \Delta_0\big)$ ranging over all partitions $e_1 + e_2 = e$, we can pull back the Schubert cycles and compute the intersection on each $(U_{e_1,V_1} \times U_{e_2,V_2}) \cap \Delta_0$. Since the class of the diagonal in $G \times G$ is $\sum_{\vec{b}} \sigma_{\vec{b}} \otimes \sigma_{\vec{b}^c}$, the closure of the class of Δ_0 in $Q_{e_1,V_1} \times Q_{e_2,V_2}$ is $\sum_{\vec{b}} \bar{\sigma}_{\vec{b}} \otimes \bar{\sigma}_{\vec{b}^c}$. Now the formula follows by pairing the class of Δ_0 with the Schubert cycles in cohomology. □

Since the V_i are very general, we can express the integrals on the smooth components in the corollary as integrals on Quot schemes of trivial bundles for very large e. We then apply the corollary again to get an integral on $Q_{e,C}$ (though the corollary is only stated for V very general, it also holds for sufficiently large e when the V_i are trivial bundles since the Q_{e_i,C_i} have the right properties and the gluing of the trivial bundles can still be chosen very general). Thus, as in the case of smooth curves, all integrals on Quot schemes of very general vector bundles can be expressed as integrals on Quot schemes of trivial bundles.

Corollary 4.5.8. *In the setting of the previous corollary, let* $\deg V_i = -d_i$ *with* $d_1 + d_2 = \deg V$. *Then letting* $\ell = \ell_1 + \ell_2$ *for* ℓ_i *sufficiently large,*

$$\int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2} = \int_{Q_{e+r\ell,C}} \left(\bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{1^r}^{d_1+(r+s)\ell_1}\right) \cup \left(\bar{\sigma}_{\underline{\vec{a}}_2} \cup \bar{\sigma}_{1^r}^{d_2+(r+s)\ell_2}\right).$$

4.6 Proof of the weighted TQFT relations

Given the formula in Corollary 4.5.7 relating intersection numbers on nodal curves and their components, the last step is to relate intersection numbers on nodal curves with intersection numbers on smooth curves of the same genus. Let C be a nodal curve of arithmetic genus g and C' denote a smooth curve of genus g obtained by smoothing C. Let $Q_{e,C} = \mathrm{Quot}\big((r,-e), \mathcal{O}_C^{r+s}\big)$ and $Q_{e,C'} = \mathrm{Quot}\big((r,-e), \mathcal{O}_{C'}^{r+s}\big)$.

Proposition 4.6.1. *For all* $e \gg 0$, *the intersection numbers on* $Q_{e,C}$ *agree with the intersection numbers on* $Q_{e,C'}$.

Proof. The proof is similar to Proposition 1.5 in [Ber94]. Let $\mathcal{C}$ be a family over a base curve B smoothing C, where $\mathcal{O}_{\mathcal{C}}^{r+s}$ is the natural deformation of $\mathcal{O}_C^{r+s}$. Consider the relative Quot scheme $\pi\colon Q = \operatorname{Quot}\big((r,-e), \mathcal{O}_{\mathcal{C}}^{r+s}, B\big) \to B$, whose fibers over $b \in B$ are Q_{e,C_b} and which is projective over B. If π is not already flat at the central fiber $b_0 \in B$, then there must be associated points of Q supported at b_0; these arise either from nonreducedness or from small components of $Q_{e,C}$ because the top-dimensional components are all of the expected dimension and hence deform in all families by Corollary 2.4.4. In any case, we can replace the central fiber by its flat limit without affecting the top-dimensional components.

Now π is flat and we want to apply Lemma 1.6 of [Ber94]. After restriction and base change, we can find a section σ of $\mathcal{C}$ near each $b \in B$ and restrict the universal subsheaf (which is defined on $\mathcal{C} \times_B Q$) to $\sigma \times_B Q$. The lemma implies that top intersections of Chern classes of the universal subbundles in the fibers are independent of the base point, hence the intersection numbers on the Quot schemes are independent of the base point by Proposition 4.3.6. On the central fiber these intersection numbers compute the Gromov-Witten numbers on the nodal curve (because the lower-dimensional components do not interfere with top intersections of Schubert varieties), and the intersection numbers on nearby smooth curves yield the usual Gromov-Witten numbers. □

Now let V and V' be very general vector bundles on C and C' of the same rank and degree. Write $Q_{e,V} = \operatorname{Quot}\big((r,-e), V\big)$ and $Q_{e,V'} = \operatorname{Quot}\big((r,-e), V'\big)$.

Corollary 4.6.2. *$Q_{e,V}$ and $Q_{e,V'}$ have the same intersection numbers, namely*

$$\int_{Q_{e,V}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2} = \int_{Q_{e,V'}} \bar{\sigma}_{\underline{\vec{a}}_1} \cup \bar{\sigma}_{\underline{\vec{a}}_2}.$$

Thus the intersection numbers on the nodal curve do not depend on how the Schubert cycles are distributed across the components.

Proof. This was just proved when V and V' are trivial. Since the intersection numbers on very general vector bundles can be computed as intersection numbers on Quot schemes of trivial bundles (Corollary 4.4.4 for smooth curves and Corollary 4.5.8 for nodal curves), the corollary follows immediately. □

Combining this corollary with Corollary 4.5.7 proves the degeneration relations for composing morphisms in the weighted TQFT along one boundary circle. This completes the proof of Theorem 3.6.2.

CHAPTER 5

EXISTENCE OF FINITE QUOT SCHEMES ON $\mathbb{P}^2$

In this chapter, we transition to the surface $\mathbb{P}^2$ and consider vector bundles V on $\mathbb{P}^2$ whose Chern character v is chosen such that the expected dimension of the Quot scheme $\mathrm{Quot}\big(V,(1,0,-n)\big)$ is zero. The invariants of the quotients being parametrized are the Chern character of an ideal sheaf of n points. We prove that when V is a general stable vector bundle of sufficiently negative degree on $\mathbb{P}^2$, the Quot scheme $\mathrm{Quot}\big(V,(1,0,-n)\big)$ is finite, reduced, and each quotient is an ideal sheaf of a reduced subscheme. The results of this chapter are based on joint work with Aaron Bertram and Drew Johnson in [BGJ16].

5.1 Statement of main result

On $\mathbb{P}^2$, the powers of the class of a hyperplane form a basis of the cohomology ring, so we write Chern characters of coherent sheaves as triples of numbers corresponding to the rank, the degree, and the second Chern character.

We consider short exact sequences $0 \to E \to V \to F \to 0$ of sheaves on $\mathbb{P}^2$ in which $\rho = \mathrm{ch}(F) = (1,0,-n)$ are the invariants of ideal sheaves $\mathcal{I}_Z$ of n points. The moduli space $M(\rho)$ parametrizing ideas sheaves of n points is isomorphic to the Hilbert scheme of points $(\mathbb{P}^2)^{[n]}$. Letting $\sigma = \mathrm{ch}(E)$ and solving $\chi(\sigma,\rho) = 0$ for σ using the Hirzebruch-Riemann-Roch formula, we get

$$\sigma = (r, -\lambda, (n-1)r - \tfrac{3}{2}\lambda)$$

for some r and λ. In this notation,

$$v = \mathrm{ch}(V) = \sigma + \rho = (r+1, -\lambda, (n-1)r - n - \tfrac{3}{2}\lambda)$$

is the Chern character of the vector bundles V for which $\mathrm{Quot}(V,\rho)$ has expected dimension zero.

Theorem 5.1.1 ([BGJ16]). *Suppose $n \geq 1$, $r \geq 2$, and $\lambda \gg 0$. Let V be a general stable vector bundle on $\mathbb{P}^2$ with $\mathrm{ch}(V) = v$ as above. Then $\mathrm{Quot}\big(V, (1, 0, -n)\big)$ is finite and reduced, and each quotient is an ideal sheaf of a reduced subscheme.*

The proof of the theorem will use the fact that the duals of general stable vector bundles V with $\mathrm{ch}(V) = v$ have resolutions of the form

$$0 \to \mathcal{O}(-2)^C \to \mathcal{O}(-1)^B \oplus \mathcal{O}^A \to V^* \to 0.$$

Working with resolution spaces instead of moduli of stable sheaves allows us to sidestep questions of stability that arise when studying these Quot schemes, such as whether the general quotient $V \twoheadrightarrow \mathcal{I}_Z$ has a stable kernel. We also use resolutions of this form to deduce the following statement of general interest about when general stable bundles on $\mathbb{P}^2$ are globally generated, which we could not find in the literature.

Proposition 5.1.2 ([BGJ16]). *Let $\xi = (r, \lambda, d)$ be a Chern character on $\mathbb{P}^2$ such that $r \geq 1$, $\lambda \geq 0$, and $\chi(\xi) \geq r + 2$. Then general sheaves in $M(\xi)$ are globally generated.*

5.2 Background on resolutions

Consider short exact sequences

$$0 \to E \to V \to \mathcal{I}_Z \to 0$$

of sheaves on $\mathbb{P}^2$, where Z is a zero-dimensional subscheme of $\mathbb{P}^2$ of length n, $e = \mathrm{ch}(E) = (r, -\lambda, (n-1)r - \frac{3}{2}\lambda)$, and thus $v = \mathrm{ch}(V) = (r+1, -\lambda, (n-1)r - n - \frac{3}{2}\lambda)$. It will often be convenient to consider the dual long exact sequence

$$0 \to \mathcal{O} \to V^* \to E^* \to \mathcal{O}_Z \to 0$$

in which the section $\mathcal{O} \to V^*$ vanishes along Z, so the cokernel fails to be locally free along Z. We write $\sigma^\vee$ and $v^\vee$ for the dual invariants. We assume $r \geq 2$, $n \geq 1$, and λ sufficiently large relative to r and n so that the moduli space $M(v)$ is positive dimensional of the expected dimension, as guaranteed by

Theorem 5.2.1 ([LP97]). *There exists a positive dimensional moduli space $M(\xi)$ if and only if $\chi(\xi)$ and $c_1(\xi)$ are integral and $\Delta(\xi) \geq \delta(\mu(\xi))$. In this case $M(\xi)$ is a normal, irreducible, factorial projective variety of dimension $1 - \chi(\xi, \xi)$.*

The discriminant $\Delta(\xi)$ in the theorem is defined by the formula

$$\Delta(\xi) = \tfrac{1}{2}\mu(\xi)^2 - \mathrm{ch}_2(\xi)/r(\xi),$$

where r is the rank and $\mu(\xi) = \mathrm{c}_1(\xi)/r(\xi)$ is the slope, while the function δ has a complicated fractal-like structure but is bounded above by 1, so checking $\Delta(v) \geq 1$ is sufficient for getting a nice moduli space.

As we will prove later in Proposition 5.5.1, if ξ is a Chern character on $\mathbb{P}^2$ satisfying certain inequalities, then general stable sheaves G in $M(\xi)$ have resolutions of the form

$$0 \to \mathcal{O}(-2)^{\gamma} \to \mathcal{O}(-1)^{\beta} \oplus \mathcal{O}^{\alpha} \to G \to 0, \tag{†}$$

where $\alpha, \beta, \gamma \geq 0$ are uniquely determined by ξ. These resolutions will play a critical role in our study of general vector bundles, and we will refer to them as (†)-resolutions. In particular, applying Proposition 5.5.1 to the invariants $\sigma^\vee$ and $v^\vee$ yields

Proposition 5.2.2. *Assume $(n-1)r < \frac{3-\sqrt{5}}{2}\lambda$. Then a general sheaf E^* in $M(\sigma^\vee)$ has a (†)-resolution*

$$0 \to \mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \oplus \mathcal{O}^a \to E^* \to 0$$

and a general sheaf V^* in $M(v^\vee)$ has a (†)-resolution

$$0 \to \mathcal{O}(-2)^{c+n} \to \mathcal{O}(-1)^{b+2n} \oplus \mathcal{O}^{a+1-n} \to V^* \to 0,$$

where

$$a = nr, \quad b = \lambda - 2(n-1)r, \quad c = \lambda - (n-1)r.$$

Conversely, cokernels of general maps $\mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \oplus \mathcal{O}^a$ and $\mathcal{O}(-2)^{c+n} \to \mathcal{O}(-1)^{b+2n} \oplus \mathcal{O}^{a+1-n}$ are semistable.

Remark 5.2.3. Throughout this section, a, b, c will be as in the proposition, and $A = a + 1 - n$, $B = b + 2n$, $C = c + n$ will be used to simplify notation. Since general E^* and V^* in moduli are locally free, we can dualize the (†)-resolutions in the proposition to get resolutions $0 \to E \to \mathcal{O}^a \oplus \mathcal{O}(1)^b \to \mathcal{O}(2)^c \to 0$ and $0 \to V \to \mathcal{O}^A \oplus \mathcal{O}(1)^B \to \mathcal{O}(2)^C \to 0$ for general $E \in M(\sigma)$ and $V \in M(v)$.

A more general way to describe globally generated vector bundles of rank $r+1$ is to consider the inclusion of r general sections, whose cokernel will have rank one and will jump in rank on a closed subscheme representing the second Chern class. Reversing the process, we can start with $\mathcal{O}^r$ and a general rank one sheaf with the right Chern character and consider general extensions. In our case, recalling that the first Chern class agrees with the first Chern character and the second Chern class satisfies $c_2 = \frac{1}{2}c_1^2 - \mathrm{ch}_2$, we compute

$$c_2(V^*) = \tfrac{1}{2}(-c_1(V))^2 - \mathrm{ch}_2(V) = \binom{\lambda+2}{2} - (n-1)(r-1).$$

It follows from Proposition 5.2.2 that this description of general vector bundles also produces general sheaves in moduli.

Corollary 5.2.4. *General sheaves in $M(v^\vee)$ coincide with general extensions*

$$0 \to \mathcal{O}^r \to V^* \to \mathcal{O}(\lambda) \otimes \mathcal{I}_W \to 0,$$

where $W \subset \mathbb{P}^2$ is a general zero-dimensional subscheme of length $\binom{\lambda+2}{2} - (n-1)(r-1)$. If $(n-1)(r-1) \geq 3$, then general sheaves in $M(v^\vee)$ are globally generated.

Proposition 5.1.2 is a more general result about global generation of stable sheaves on $\mathbb{P}^2$.

Proof of Corollary 5.2.4. Since general V^* in $M(v^\vee)$ have a general (†)-resolution, the cokernel of a general map $\mathcal{O}^r \to V^*$ is also the cokernel of a general map

$$\mathcal{O}(-2)^C \to \mathcal{O}(-1)^B \oplus \mathcal{O}^{A-r},$$

so it is torsion-free and thus of the form $\mathcal{O}(\lambda) \otimes \mathcal{I}_W$. The fact that W is general follows from a construction of these resolutions in families, yielding a dominant rational map to the Hilbert scheme. To see that general extensions yield stable sheaves we use the fact that stability is an open condition in families. Since $\mathcal{O}(\lambda) \otimes \mathcal{I}_W$ is globally generated when it has ≥ 3 sections, so is V^*. □

It will be convenient to have a criterion for detecting when a given coherent sheaf on $\mathbb{P}^2$ has a (†)-resolution. The following proposition applies to coherent sheaves with arbitrary Chern classes that may not be stable or locally free.

Proposition 5.2.5. *A coherent sheaf G has a (†)-resolution if and only if $h^0(G(-1)) = h^1(G) = h^2(G(-1)) = 0$ and $\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \otimes H^0(G) \to H^0(G(1))$ is injective.*

Proof. The ($\Longrightarrow$) direction follows from the fact that line bundles on $\mathbb{P}^2$ have no first cohomology and the observation that $\mathrm{Hom}(\mathcal{O}(-1),\mathcal{O}) \otimes H^0(\mathcal{O}^\alpha) \to H^0(\mathcal{O}(1)^\alpha)$ is an isomorphism for all α.

For ($\Longleftarrow$), set $\alpha = h^0(G)$ and $\beta = h^0(G(1)) - 3\alpha$. The vanishing $h^1(G) = h^2(G(-1)) = 0$ implies that G has Castelnuovo-Mumford regularity ≤ 1, so $G(1)$ is globally generated. Starting with the surjection $\mathcal{O}^{3\alpha+\beta} \twoheadrightarrow G(1)$, observe that 3α of these sections factor through $\mathcal{O}(1)$, and conclude that there is a (non-canonical) surjection $\mathcal{O}^\beta \oplus \mathcal{O}(1)^\alpha \twoheadrightarrow G(1)$, which we twist to get a short exact sequence

$$0 \to K \to \mathcal{O}(-1)^\beta \oplus \mathcal{O}^\alpha \xrightarrow{f} G \to 0.$$

Since line bundles have no first cohomology and f induces an isomorphism on global sections, $h^1(K) = 0$ and hence $h^1(K(n)) = 0$ for all $n \geq 0$. By assumption $h^0(G(-1)) = 0$, so $h^0(G(n)) = 0$ for all $n < 0$, which implies $h^1(K(n)) = 0$ for all $n < 0$. Since all twists of K have vanishing first cohomology, K must be a direct sum of line bundles by the splitting criterion of Horrocks ([OSS80]). K cannot contain any $\mathcal{O}$ or $\mathcal{O}(-1)$-summands by construction, nor can it contain $\mathcal{O}(-n)$-summands for $n \geq 3$ since $h^1(G) = 0$, so $K \simeq \mathcal{O}(-2)^\gamma$ for some γ. $\square$

5.3 Setup for proof of theorem

Instead of working with the moduli space $M(v)$, it will be more convenient to work with the resolution space. We define $R(v)$ to be the open subset of the projective space $\mathbb{P}^N = P\big(\mathrm{Hom}(\mathcal{O}^A \oplus \mathcal{O}(1)^B, \mathcal{O}(2)^C)\big)$ consisting of surjective morphisms (whose kernels thus have Chern character v). $R(v)$ has dimension $N = 6AC + 3BC - 1$ and the subset of resolutions of stable sheaves is open and dense by Proposition 5.2.2. A useful feature of $R(v)$ is that it has a universal family $\mathcal{V}$ over $R(v) \times \mathbb{P}^2$ defined as the kernel of a morphism of vector bundles

$$q^*\mathcal{O}_{R(v)}(-1) \otimes p^*(\mathcal{O}_{\mathbb{P}^2}^A \oplus \mathcal{O}_{\mathbb{P}^2}(1)^B) \to p^*\mathcal{O}_{\mathbb{P}^2}(2)^C,$$

where $R(v) \xleftarrow{q} R(v) \times \mathbb{P}^2 \xrightarrow{p} \mathbb{P}^2$ are the projections. The morphism is defined by the general matrix of linear and quadratic forms in the coordinates of $\mathbb{P}^2$, where the projective coordinates of $R(v)$ parametrize the coefficients of the linear and quadratic forms.

Since we are interested in $\mathcal{I}_Z$ quotients of V, we consider commutative diagrams

$$\begin{array}{ccc} \mathcal{O}^A \oplus \mathcal{O}(1)^B & \xrightarrow{f} & \mathcal{O}(2)^C \\ \pi \downarrow & & \downarrow g \\ \mathcal{O} & \xrightarrow[\pi_Z]{} & \mathcal{O}_Z \end{array} \tag{5.1}$$

in which f need not be surjective, g is necessarily surjective, Z varies in $(\mathbb{P}^2)^{[n]}$, π is induced by a rank one quotient $\mathcal{O}^A \twoheadrightarrow \mathcal{O}$, and π_Z is the canonical map.

In the case when f is surjective, we claim that these commutative diagrams are in bijection with the maps $V \to \mathcal{I}_Z$. The induced map on kernels of the rows in the diagram is of the form $V \to \mathcal{I}_Z$. Conversely, the map $V \to \mathcal{O}^A$ can be identified with $V \to \operatorname{Hom}(V, \mathcal{O})^* \otimes \mathcal{O}$, so given any map $V \to \mathcal{I}_Z$ (possibly non-surjective), the composition $V \to \mathcal{I}_Z \to \mathcal{O}$ factors through $V \to \mathcal{O}^A$ and yields a diagram (5.1).

To globalize the above diagrams, consider the vector bundle $\mathcal{E} = q_*(\mathcal{O}_{\mathcal{Z}} \otimes p^*\mathcal{O}(-2))$ on $(\mathbb{P}^2)^{[n]}$, where $\mathcal{Z} \subset (\mathbb{P}^2)^{[n]} \times \mathbb{P}^2$ is the universal subscheme and we abuse notation by again writing q, p for the projections. The fiber of $\mathcal{E}$ at Z is $H^0(\mathcal{O}_Z \otimes p^*\mathcal{O}(-2)) \cong \operatorname{Hom}(\mathcal{O}(2), \mathcal{O}_Z)$ by Cohomology and Base Change ([Har77]). Consider the incidence variety

$$I_{r,\lambda,n} = \{\, (f, g, \pi) \mid g \circ f = \pi_Z \circ \pi \text{ in } P\big(\operatorname{Hom}(\mathcal{O}^A \oplus \mathcal{O}(1)^B, \mathcal{O}_Z)\big) \,\}$$

contained in $\mathbb{P}^N \times P(\mathcal{E}^C) \times \mathbb{P}^{A-1}$, which we will usually view as a family over $\mathbb{P}^N$. We will prove

Proposition 5.3.1. *Let $\lambda \gg 0$. Then*

(a) $I_{r,\lambda,n}$ has a unique component of dimension N and any other components have strictly smaller dimension;

(b) For $\lambda \gg 0$, general sheaves E in $M(\sigma)$ and $\mathcal{I}_Z$ in $(\mathbb{P}^2)^{[n]}$ satisfy $\hom(E, \mathcal{I}_Z) = 0$.

(c) There is a resolution f such that $V = \ker f$ has an $\mathcal{I}_Z$ quotient that is an isolated point in $\operatorname{Quot}\big(V, (1, 0, -n)\big)$*;*

(d) For an open set $U \subset R(v)$, the restriction $I_{r,\lambda,n}|_U$ coincides with the relative Quot scheme $\operatorname{Quot}\big(\mathcal{V}|_U, (1, 0, -n), U\big)$*, whose fibers over U are finite, reduced, and consist of quotients $V \twoheadrightarrow \mathcal{I}_Z$ for which Z is a collection of n distinct points that are general in $\mathbb{P}^2$.*

Part (d) of the proposition immediately implies Theorem 5.1.1 since choosing a general resolution in U, which corresponds to a general stable vector bundle V, the fiber of the relative Quot scheme is $\mathrm{Quot}\big(V,(1,0,-n)\big)$.

5.4 Proof of existence of finite Quot schemes

In this section we prove the four parts of Proposition 5.3.1.

5.4.1 Proof of (a)

We stratify the fiber $P(\mathrm{Hom}(\mathcal{O}(2)^C,\mathcal{O}_Z))$ of $P(\mathcal{E}^C)$ over $Z \in (\mathbb{P}^2)^{[n]}$ as a union of varieties over which we can control the dimension of $I_{r,\lambda,n}$. Let

$$W_k \subset \mathrm{Hom}(\mathcal{O}(2)^C,\mathcal{O}_Z)$$

be the locally-closed subscheme of maps g that factor through a map $\mathcal{O}(2)^C \to \mathcal{O}(2)^k$ but not through a map $\mathcal{O}(2)^C \to \mathcal{O}(2)^{k-1}$. Since $\mathrm{Hom}(\mathcal{O}(2),\mathcal{O}_Z) = n$, we can write

$$P(\mathrm{Hom}(\mathcal{O}(2)^C,\mathcal{O}_Z)) = P(W_1) \sqcup \cdots \sqcup P(W_n).$$

The codimension of W_k in $\mathrm{Hom}(\mathcal{O}(2)^C,\mathcal{O}_Z)$ is $(n-k)(C-k)$, which is computed by adding the dimensions of the Grassmannian $\mathrm{Gr}(\mathbb{C}^C,k)$ and $\mathrm{Hom}(\mathcal{O}(2)^k,\mathcal{O}_Z)$.

We compute the dimension of $I_{r,\lambda,n}$ over these strata by describing the fibers. For each fixed pair $(g,\pi) \in P(W_k) \times \mathbb{P}^{A-1}$ over Z, there is an exact sequence

$$\mathrm{Hom}(\mathcal{O}^A \oplus \mathcal{O}(1)^B,\mathcal{O}(2)^C) \xrightarrow{g_*} \mathrm{Hom}(\mathcal{O}^A \oplus \mathcal{O}(1)^B,\mathcal{O}_Z) \to \mathrm{Ext}^1(\mathcal{O}^A \oplus \mathcal{O}(1)^B,\ker g) \to 0,$$

and the fiber in $I_{r,\lambda,n}$ over (g,π) is the projectivization of the preimage under g_* of $\pi_Z \circ \pi$. We observe that $\mathrm{Ext}^1(\mathcal{O},\ker g) = h^1(\ker g)$ measures the failure of $H^0(\mathcal{O}(2)^C) \to H^0(\mathcal{O}_Z)$ to be surjective and $\mathrm{Ext}^1(\mathcal{O}(1),\ker g) = h^1(\ker g(-1))$ measures the failure of the map $H^0(\mathcal{O}(1)^C) \to H^0(\mathcal{O}_Z)$, which is induced by $g(-1)$, to be surjective.

In the case $k = n$, both of these maps of global sections are surjective. Since W_n has codimension 0, the dimension of $P(W_n) \times \mathbb{P}^{A-1}$ is $nC + (A-1)$. The codimension of the fibers of g_* in $\mathrm{Hom}(\mathcal{O}^A \oplus \mathcal{O}(1)^B,\mathcal{O}(2)^C)$ is $\hom(\mathcal{O}^A \oplus \mathcal{O}(1)^B,\mathcal{O}_Z) = n(A+B)$, which is equal to $nC + (A-1) + 2n$. Letting Z vary in $(\mathbb{P}^2)^{[n]}$, we see that the dimension of $I_{r,\lambda,n}$ is equal to $N = \dim R(v)$ since the following lemma guarantees that the jump in dimension of the fibers of $I_{r,\lambda,n}$ over W_k for $k < n$ is strictly less than the codimension of W_k in $\mathrm{Hom}(\mathcal{O}(2)^C,\mathcal{O}_Z)$.

In fact, this dimension count will complete the proof of (a) as follows. For every N-dimensional component of $I_{r,\lambda,n}$, the fiber over a general point in $P(\mathcal{E}^C)$ must contain a nonempty open set in the fiber of $I_{r,\lambda,n}$ for dimension reasons. But the fibers of $I_{r,\lambda,n}$ are projective spaces, so the intersection of two nonempty open sets contains a nonempty open set, hence there can be only one component.

Lemma 5.4.1. *Assume $\lambda \gg 0$ and $1 \leq k < n$. Then for all $g \in W_k$,*

$$\operatorname{Ext}^1(\mathcal{O}^A \oplus \mathcal{O}(1)^B, \ker(g)) < \operatorname{codim} W_k = (n-k)(C-k).$$

Proof. We think of $H^0(\mathcal{O}(1))$ as the space of linear forms on $\mathbb{P}^2$. Fixing $\ell \in H^0(\mathcal{O}(1))$ that does not vanish on any points in the support of Z, we can identify every map $\mathcal{O}(2) \to \mathcal{O}_Z$ with a global section of $\mathcal{O}_Z$ by multiplying by ℓ^2. Thus $g \in W_k$ determines a k-plane $H_\ell \subset H^0(\mathcal{O}_Z)$, and the image of the global section map $H^0(\mathcal{O}(1)^C) \to H^0(\mathcal{O}_Z)$ is obtained by multiplying H_ℓ by the space of rational functions $H^0(\mathcal{O}(1))/\ell$ and hence contains H_ℓ.

We claim that the rank of $H^0(\mathcal{O}(1)^C) \to H^0(\mathcal{O}_Z)$ is $\geq k+1$. If not, then the image must be exactly H_ℓ. But then the image of every map $H^0(\mathcal{O}(d)^C) \to H^0(\mathcal{O}_Z)$ obtained by twisting g is also H_ℓ, which contradicts the fact that $H^0(\mathcal{O}(d)^C) \to H^0(\mathcal{O}_Z)$ is surjective for $d \gg 0$ by Serre vanishing applied to $\ker g$. By the same argument, the rank of $H^0(\mathcal{O}(2)^C) \to H^0(\mathcal{O}_Z)$ is $\geq k+1$ as well.

Thus the left side of the proposed inequality is

$$A \cdot \operatorname{Ext}^1(\mathcal{O}, \ker(g)) + B \cdot \operatorname{Ext}^1(\mathcal{O}(1), \ker(g)) \leq (n-k-1)(A+B),$$

so since $A+B-C = r+1$, it suffices to show $(n-k)(1+r+k) < A+B$, which is achieved by choosing λ sufficiently large. □

5.4.2 Proof of (b)

Theorem 5.2.1 implies that $M(\sigma)$ is non-empty for $\lambda \gg 0$. If E is locally free, then $\hom(E, \mathcal{I}_Z) = h^0(E^* \otimes \mathcal{I}_Z)$, and dualizing locally free sheaves yields an isomorphism between dense open subschemes of $M(\sigma)$ and $M(\sigma^\vee)$. Thus it suffices to prove that for $\lambda \gg 0$, general $\widehat{E} \in M(\sigma^\vee)$ and $\mathcal{I}_Z \in (\mathbb{P}^2)^{[n]}$ satisfy $h^0(\widehat{E} \otimes \mathcal{I}_Z) = 0$.

We use induction on λ. We will emphasize the dependence of σ on λ by writing $\sigma = \sigma_\lambda$. For each λ, it suffices to construct a single pair $(\widehat{E}, \mathcal{I}_Z)$ such that $h^0(\widehat{E} \otimes \mathcal{I}_Z) = 0$, where

$\hat{E}$ is in $M(\sigma_\lambda^\vee)$ or in $R(\sigma_\lambda^\vee)$ (the space of (†)-resolutions). This is because the vanishing $h^0(\widehat{E} \otimes I_Z) = 0$ is open in families by Cohomology and Base Change ([Har77]).

For a base case, let $\lambda = rk$ for k minimal such that $\chi(\mathcal{O}_{\mathbb{P}^2}(k)) \geq n$. Choose Z general of length n and Z' such that $|Z'| + |Z| = \chi(\mathcal{O}(k))$ and $Z' \cup Z$ is not contained on a curve of degree k. Then $\widehat{E} = \mathcal{I}_{Z'}(k)^r$ is in $M(\sigma_\lambda^\vee)$ and $h^0(\widehat{E} \otimes \mathcal{I}_Z) = 0$ since $h^0(\mathcal{I}_{Z' \cup Z}(k)) = 0$ by our choice of Z'.

For the inductive step $\lambda \implies \lambda + 1$, we may assume there exist general $E_\lambda^* \in R(\sigma_\lambda^\vee)$ and $\mathcal{I}_Z \in (\mathbb{P}^2)^{[n]}$ satisfying $h^0(E_\lambda^* \otimes \mathcal{I}_Z) = 0$. Since E_λ^* is general, it is locally free with dual E_λ. Let ℓ be a general line in $\mathbb{P}^2$, for which $E_\lambda|_\ell$ splits into a direct sum of line bundles, at least two of which have negative degree by the Grauert-Mülich Theorem ([HL10]). Choosing a surjection $E_\lambda|_\ell \twoheadrightarrow \mathcal{O}_\ell(2)$ yields an elementary modification

$$0 \to E_{\lambda+1} \to E_\lambda \to \mathcal{O}_\ell(2) \to 0,$$

where restricting the sequence to ℓ produces the kernel $0 \to \mathcal{O}_\ell(1) \to E_{\lambda+1}|_\ell \to E_\lambda|_\ell$. Exactness in the middle of the sequence

$$0 = \operatorname{Hom}(E_\lambda, \mathcal{I}_Z) \to \operatorname{Hom}(E_{\lambda+1}, \mathcal{I}_Z) \to \operatorname{Ext}^1(\mathcal{O}_\ell(2), \mathcal{I}_Z) \simeq H^1(\mathcal{O}_\ell(-1)) = 0$$

yields the vanishing $h^0(E_{\lambda+1}^* \otimes \mathcal{I}_Z) = \hom(E_{\lambda+1}, \mathcal{I}_Z) = 0$.

Consider the dual sequence of sheaves

$$0 \to E_\lambda^* \to E_{\lambda+1}^* \to \mathcal{O}_\ell(-1) \to 0.$$

Since E_λ^* has a (†)-resolution, $\mathcal{O}_\ell(-1)$ has no cohomology, and $\mathcal{O}_\ell(-2)$ has only cohomology in degree 1, Proposition 5.2.5 ensures that $E_{\lambda+1}^*$ has a (†)-resolution. This completes the proof.

5.4.3 Proof of (c)

We prove (c) in Proposition 5.3.1 by constructing a vector bundle V^* with a resolution and an appropriate section, and then dualizing.

Let E be a general vector bundle in $M(\sigma)$ and $\mathcal{I}_Z$ in $(\mathbb{P}^2)^{[n]}$ be general, which ensures that $\operatorname{Hom}(E, \mathcal{I}_Z) = 0$ by Proposition 5.3.1 (b). Choose a general surjection $E^* \to \mathcal{O}_Z$ that induces a surjection on global sections. Let J be the kernel, which fails to be locally free along Z. We will show that a general extension $0 \to \mathcal{O} \to V^* \to J \to 0$ produces V^* that is locally free and has a (†)-resolution. This will complete the argument because dualizing the sequence defining V^* yields the short exact sequence $0 \to E \to V \to \mathcal{I}_Z \to 0$, which is an isolated point of $\mathrm{Quot}\,(V, (1, 0, -n))$ since $\mathrm{Hom}(E, \mathcal{I}_Z) = 0$, and V has a resolution since V^* does.

Lemma 5.4.2. *(1) V^* is locally free.*

(2) J has a (†)-resolution.

(3) V^* has a (†)- resolution.

Proof. (1): If an extension $\widehat{V}$ of J by $\mathcal{O}$ fails to be locally free along a subscheme $W \subset Z$, whose residual we denote $W' \subset Z$, then the inclusion $\widehat{V} \to \widehat{V}^{\vee\vee}$ yields a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \widehat{V} & \longrightarrow & J & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O} & \longrightarrow & \widehat{V}^{\vee\vee} & \longrightarrow & J' & \longrightarrow & 0 \end{array}$$

in which the extension in the top row is pulled back from the extension in the bottom row. Here J' is a subsheaf of E^* with cokernel $\mathcal{O}_{W'}$. Since

$$\mathrm{ext}^1(J', \mathcal{O}) = h^1(J'(-3)) = h^1(E^*(-3)) + |W'| < h^1(E) + n = \mathrm{ext}^1(J, \mathcal{O})$$

and there are only finitely many such J' (corresponding to finitely many subschemes W' of Z), the total locus in $\mathrm{Ext}^1(J, \mathcal{O})$ of all extensions pulled back from a J' has codimension ≥ 1. Avoiding this locus yields V^* locally free.

(2): We will use the fact that E^* has a (†)-resolution (since it is general in moduli) and the criterion provided in Proposition 5.2.5. Consider the sequence

$$0 \to J \to E^* \to \mathcal{O}_Z \to 0$$

defining J. Since $H^0(E^*) \to H^0(\mathcal{O}_Z)$ is surjective by construction, $h^1(J) = 0$. The other properties for J required in Proposition 5.2.5 follow immediately from the same properties for E^*.

(3): Pulling back the extension $0 \to \mathcal{O} \to V^* \to J \to 0$ using the map to J in the (†)-resolution of J yields a (†)-resolution of V^* that looks like the resolution for J with one additional $\mathcal{O}$. □

5.4.4 Proof of (d)

To analyze the points of $\mathrm{Quot}\big(V,(1,0,-n)\big)$, we need to know what kinds of quotients can arise. We can classify coherent sheaves on $\mathbb{P}^2$ with Chern character $(1,0,-n)$ into four types.

Lemma 5.4.3. *Let F be a coherent sheaf with $\mathrm{ch}(F) = (1,0,-n)$ for $n \in \mathbb{Z}_{\geq 0}$. Then F must be one of the following:*

(1) $\mathcal{I}_Z$, where $|Z| = n$ and Z is reduced;

(2) $\mathcal{I}_Z$, where $|Z| = n$ but Z is not reduced;

(3) An extension $0 \to \mathcal{O}_{Z'} \to F \to \mathcal{I}_Z \to 0$, where $|Z| > n$ and $|Z'| = |Z| - n$;

(4) An extension $0 \to \mathcal{O}_C(D) \oplus \mathcal{O}_Z \to F \to \mathcal{O}_S(-C) \otimes \mathcal{I}_{Z'} \to 0$, where C is a curve, D is a divisor on C, and $|Z| = n - |Z'| - \deg D$.

Proof. Let T be the torsion subsheaf of F, which fits in an exact sequence $0 \to T \to F \to Q \to 0$. Since Q is torsion free of rank one, it must be a line bundle tensored by an ideal sheaf of points. When T is empty or supported on points, $c_1(Q) = 0$, so Q is an ideal sheaf of points, which yields cases (1), (2), and (3). If T is supported on a curve of class C, then $c_1(Q) = -C$, so $Q = \mathcal{O}_S(-C) \otimes I_{Z'}$, yielding case (4). □

To prove (d), we deform the isolated quotient

$$0 \to E \to V \to \mathcal{I}_Z \to 0$$

constructed above to conclude that general resolutions in $R(v)$ have isolated $\mathcal{I}_Z$ quotients.

Let $\mathcal{V}$ be the universal bundle over $R(v)$ and consider the relative Quot scheme $Q = \mathrm{Quot}\big(\mathcal{V},(1,0,-n),R(v)\big)$ over $R(v)$, which has the quotient $x = [V \twoheadrightarrow \mathcal{I}_Z]$ as an isolated point in the fiber over $[V]$. Theorem 2.4.3 ensures that the dimension of the component of Q containing x is at least

$$\hom(E,\mathcal{I}_Z) - \mathrm{ext}^1(E,\mathcal{I}_Z) + \dim R(v) = \dim R(v).$$

Since the fiber dimension at x is 0, upper semicontinuity implies that the fiber dimension is generically 0, so there is an open set $\mathcal{U} \subset Q$ consisting entirely of points at which the relative Zariski tangent space is zero-dimensional, namely the map to $R(v)$ is étale.

We now show that $\mathcal{U}$ consists entirely of $\mathcal{I}_Z$ quotients by ruling out the other possible sheaves with Chern character $(1, 0, -n)$ listed in Lemma 5.4.3. If the cokernel F in $0 \to E \to V \to F \to 0$ has zero-dimensional torsion, then $\hom(E, F) > 0$. If F has torsion along a curve, then V has a map to a line bundle of negative degree, which is impossible since V has a (†)-resolution. Thus the quotients in $\mathcal{U}$ are all ideal sheaves.

But since every map from sheaves V with resolutions to ideal sheaves $\mathcal{I}_Z$ occurs in $I_{r,\lambda,n}$, this yields an injective morphism $\iota\colon \mathcal{U} \hookrightarrow I_{r,\lambda,n}$. Set-theoretically, the map is obtained by extending a quotient $V \twoheadrightarrow \mathcal{I}_Z$ to a diagram (5.1). We check below that this map is algebraic. The image has dimension $\dim R(v)$, hence must be contained in the unique component of $I_{r,\lambda,n}$ of this dimension. The complement $I_{r,\lambda,n} \setminus \operatorname{im}(\iota)$ is thus of dimension $< \dim R(v)$. Then $U \subset R(v)$, the complement of the image of $I_{r,\lambda,n} \setminus \operatorname{im}(\iota) \to R(v)$, is open in $R(v)$. By construction, the fibers of $I_{r,\lambda,n}$ over U are fully contained in the image of $\mathcal{U}$.

We claim that the composition of the inclusions $I_{r,\lambda,n}|_U \hookrightarrow \mathcal{U} \hookrightarrow Q|_U$ is an isomorphism. We need to rule out quotients in $Q|_U$ at which the map to $R(v)$ is not étale, which we can do by showing that every such quotient yields a (possibly nonsurjective) map to $V \to \mathcal{I}_Z$, which is impossible since the full fibers of $I_{r,\lambda,n}$ are contained in $\mathcal{U}$. A nonétale point in Q must be either a quotient $0 \to E \to V \to \mathcal{I}_Z \to 0$ for which $\hom(E, \mathcal{I}_Z) > 0$ or a quotient $V \twoheadrightarrow F$ where F is of type (3) or (4) in Lemma 5.4.3. We have already ruled out type (4) quotients since V has a resolution. The type (3) quotients yield quotients $V \twoheadrightarrow F \twoheadrightarrow \mathcal{I}_W$, where $|W| > n$, and every choice of length-n subscheme $Z \subset W$ gives rise to an inclusion $\mathcal{I}_W \hookrightarrow \mathcal{I}_Z$ and hence a nonsurjective map $V \to \mathcal{I}_Z$. Thus the map $I_{r,\lambda,n}|_U \to Q|_U$ is surjective, hence an isomorphism.

Since $I_{r,\lambda,n}|_U = Q|_U \to U$ is étale, the fibers are finite and reduced. To ensure that all Z occurring as quotients $V \twoheadrightarrow \mathcal{I}_Z$ consist of n general distinct points, we can restrict $I_{r,\lambda,n}$ to any special locus in $(\mathbb{P}^2)^{[n]}$, take the image in $R(v)$, and shrink U further to avoid this image.

To complete the proof of Proposition 5.3.1, we construct the morphism $\mathcal{U} \hookrightarrow I_{r,\lambda,n}$ algebraically by compiling the diagrams (5.1) into a family. Over $\mathcal{U} \times \mathbb{P}^2$, the universal quotient $\mathcal{V} \to \mathcal{F}$, where $\mathcal{V}$ is pulled back from $R(v)$, is a family of sheaves with Chern character $(1, 0, -n)$. Thus there is a map $\phi\colon \mathcal{U} \to (\mathbb{P}^2)^{[n]}$ such that $\mathcal{F}$, up to a twist by a line bundle L on $\mathcal{U}$, is the pullback of $\mathcal{I}_{\mathcal{Z}}$. We will construct a surjective morphism α yielding a commutative diagram

$$\begin{array}{ccccc}
\mathcal{V} & \longrightarrow & \mathcal{O}_{R(v)}(-1) \otimes p^*(\mathcal{O}_{\mathbb{P}^2}^A \oplus \mathcal{O}_{\mathbb{P}^2}(1)^B) & \longrightarrow & p^*\mathcal{O}_{\mathbb{P}^2}(2)^C \\
\downarrow & & \downarrow \alpha & & \downarrow \beta \\
L \otimes (\phi \times \mathrm{id})^* \mathcal{I}_{\mathcal{Z}} & \longrightarrow & L \otimes \mathcal{O}_{\mathcal{U} \times \mathbb{P}^2} & \longrightarrow & L \otimes (\phi \times \mathrm{id})^* \mathcal{O}_{\mathcal{Z}}
\end{array}$$

on $\mathcal{U} \times \mathbb{P}^2$, where we have omitted some of the pull backs from the notation. We construct α locally. First, trivialize $\mathcal{O}_{R(v)}(-1)$ and L on open sets $\mathcal{U}_i \subset \mathcal{U}$ such that $\mathcal{O}_{\mathcal{U}_i}$ has no higher cohomology. On $\mathcal{U}_i \times \mathbb{P}^2$ the left side of the diagram is

$$\begin{array}{ccc}
\mathcal{V}|_{\mathcal{U}_i} & \longrightarrow & p^*\mathcal{O}_{\mathbb{P}^2}^A \oplus p^*\mathcal{O}_{\mathbb{P}^2}(1)^B \\
\downarrow & & \downarrow \alpha_i \\
(\phi|_{\mathcal{U}_i} \times \mathrm{id})^* \mathcal{I}_{\mathcal{Z}} & \longrightarrow & \mathcal{O}_{\mathcal{U}_i \times \mathbb{P}^2}
\end{array}$$

and α_i is the unique preimage of the composition $[\mathcal{V}|_{\mathcal{U}_i} \to \mathcal{O}_{\mathcal{U}_i \times \mathbb{P}^2}]$ under the isomorphism

$$\mathrm{Hom}(p^*(\mathcal{O}^A \oplus \mathcal{O}(1)^B), \mathcal{O}) \to \mathrm{Hom}(\mathcal{V}, \mathcal{O})$$

coming from the long exact sequence obtained from the functor $\mathrm{Hom}(-, \mathcal{O})$. The vanishing $\hom(p^*\mathcal{O}(2)^C, \mathcal{O}) = \mathrm{ext}^1(p^*\mathcal{O}(2)^C, \mathcal{O}) = 0$ in this long exact sequence is guaranteed by the Künneth formula and the vanishing of the higher cohomology of $\mathcal{O}_{\mathcal{U}_i}$. These α_i are surjective since they are nonzero and they glue together into a map α since they agree on fibers over points in $\mathcal{U}$. We get β as the induced map on cokernels.

Since α vanishes on $p^*\mathcal{O}_{\mathbb{P}^2}(1)^B$, it induces a map $\mathcal{U} \to \mathbb{P}^{A-1}$. To lift the map $\phi\colon \mathcal{U} \to (\mathbb{P}^2)^{[n]}$ to $P(\mathcal{E}^C)$, we push forward β by q. Taking the product of these maps with the projection $\mathcal{U} \to R(v)$ yields a map $\mathcal{U} \to R(v) \times P(\mathcal{E}^C) \times \mathbb{P}^{A-1}$ that coincides with our set-theoretic description on closed points. Since $\mathcal{U}$ is smooth, this guarantees that the image is contained in $I_{r,\lambda,n}$, which completes the argument.

5.5 Resolutions and global generation

We let ξ be a Chern character on $\mathbb{P}^2$ satisfying some mild inequalities and prove that general stable sheaves in $M(\xi)$ have particularly nice resolutions, which implies Proposition 5.2.2 as a particular case. As a corollary, we deduce Proposition 5.1.2 guaranteeing that general stable sheaves with large enough Euler characteristic are globally generated. We ignore the assumptions on r, λ, a, b, c made previously in this chapter.

Proposition 5.5.1. *Let $\xi = (r, \lambda, d)$ be a Chern character on $\mathbb{P}^2$ such that*

$$r \geq 1, \quad \lambda \geq 0, \quad \chi(\xi) \geq 0, \quad \text{and} \quad d < -\tfrac{\sqrt{5}}{2}\lambda.$$

Then the general sheaf G in $M(\xi)$ has a resolution of the form

$$0 \to \mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \oplus \mathcal{O}^a \to G \to 0,$$

where

$$a = \chi(\xi) = r + \tfrac{3}{2}\lambda + d; \qquad b = -2(\lambda + d); \qquad c = -\tfrac{1}{2}\lambda - d.$$

Conversely, cokernels of general maps $\mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \oplus \mathcal{O}^a$ are stable sheaves in $M(\xi)$.

Proposition 5.5.1 follows from a more general result in [CHW14] about resolutions of general sheaves in $M(\xi)$ by triads of exceptional vector bundles. We begin by recalling some basic facts, following [CHW14]. A stable vector bundle E on $\mathbb{P}^2$ is an **exceptional bundle** if $\mathrm{Ext}^1(E, E) = 0$, in which case we call the slope α of E an **exceptional slope** and write $E = E_\alpha$ since E is the unique exceptional bundle with slope α. All integers are exceptional slopes since the line bundles $\mathcal{O}(n)$ are exceptional bundles.

Let r_α be the rank of E_α, let $\Delta_\alpha = \frac{1}{2}(1 - \frac{1}{r_\alpha^2})$ be the discriminant of E_α, and let $\xi_\alpha = \mathrm{ch}(E_\alpha)$. The set of exceptional slopes $\mathcal{E}$ is in bijection with the dyadic integers via a function $\varepsilon\colon \mathbb{Z}[\frac{1}{2}] \to \mathcal{E}$ defined inductively by $\varepsilon(n) = n$ for $n \in \mathbb{Z}$ and by setting

$$\varepsilon\left(\tfrac{2p+1}{2^{q+1}}\right) = \varepsilon\left(\tfrac{p}{2^q}\right).\varepsilon\left(\tfrac{p+1}{2^q}\right), \tag{5.2}$$

where the product operation on exceptional slopes is defined by $\alpha.\beta = \frac{\alpha+\beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3+\alpha-\beta}$. Since each dyadic integer can be written uniquely as $\frac{2p+1}{2^{q+1}}$, the p and q in (5.2) are uniquely determined, and we call (5.2) the **standard decomposition** of the exceptional slope $\varepsilon\left(\frac{2p+1}{2^{q+1}}\right)$.

To find the right triad for resolving general sheaves in $M(\xi)$, one needs the **corresponding exceptional slope** γ of ξ. This is obtained by computing

$$\mu_0 = -\tfrac{3}{2} - \mu + \sqrt{\tfrac{5}{4} + \mu^2 - \tfrac{2d}{r}},$$

where $\mu = \lambda/r$ is the slope of ξ, and then γ is the unique exceptional slope satisfying $|\mu_0 - \gamma| < x_\gamma$, where $x_\gamma = \frac{3}{2} - \sqrt{\frac{9}{4} - \frac{1}{r_\gamma^2}}$. Then the resolution is described by

Proposition 5.5.2 ([CHW14])**.** *Let ξ be a Chern character, let γ be the corresponding exceptional slope to ξ, and let $\gamma = \alpha.\beta$ be the standard decomposition of γ. Then:*

(1) If $\chi(\xi \cup \xi_\gamma) \geq 0$, then the general $G \in M(\xi)$ has a resolution

$$0 \to E_{-\alpha-3}^{m_1} \to E_{-\beta}^{m_2} \oplus E_{-\gamma}^{m_3} \to G \to 0,$$

where $m_1 = -\chi(G \otimes E_\alpha)$, $m_2 = -\chi(G \otimes E_{\alpha.\gamma})$, $m_3 = \chi(G \otimes E_\gamma)$.

(2) If $\chi(\xi \cup \xi_\gamma) \leq 0$, then the general $G \in M(\xi)$ has a resolution

$$0 \to E_{-\alpha-3}^{m_1} \oplus E_{-\gamma-3}^{m_3} \to E_{-\beta}^{m_2} \to G \to 0,$$

where $m_1 = \chi(G \otimes E_{\gamma.\beta})$, $m_2 = \chi(G \otimes E_\beta)$, $m_3 = -\chi(G \otimes E_\gamma)$.

Our proposition follows from the special case when $\chi(\xi) \geq 0$ and $\gamma = 0$.

Proof of Proposition 5.5.1. We first claim that γ, the corresponding exceptional slope to ξ, is 0. For $\lambda \geq 0$, the inequality $|\mu_0| < x_0 = \frac{3-\sqrt{5}}{2}$ is equivalent to the pair of inequalities

$$-(3 - \tfrac{\sqrt{5}}{2})\lambda - \tfrac{9-3\sqrt{5}}{2}r < d < -\tfrac{\sqrt{5}}{2}\lambda.$$

The left inequality is guaranteed by $\chi(\xi) \geq 0$, which yields $d \geq -\frac{3}{2}\lambda - r$, while the right inequality is a hypothesis.

Since $\chi(\xi) \geq 0$, the resolution of general sheaves G in $M(\xi)$ is thus of the form

$$0 \to \mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \oplus \mathcal{O}^a \to G \to 0.$$

Applying χ, we see that $a = \chi(\xi) = r + \frac{3}{2}\lambda + d$. Additivity on Chern characters yields the relations $a + b - c = r$, $-b + 2c = \lambda$, and $\frac{1}{2}b - 2c = d$, which can be solved to get $b = -2(\lambda + d)$, $c = -\frac{1}{2}\lambda - d$. This proves the first part of the proposition.

The converse follows from a dimension count. The dimension of such resolutions is the dimension of $\mathrm{Hom}(\mathcal{O}(-2)^c, \mathcal{O}(-1)^b \oplus \mathcal{O}^a)$, minus the dimensions of automorphisms of $\mathcal{O}(-2)^c$ and $\mathcal{O}(-1)^b \oplus \mathcal{O}^a$, plus 1 because we are accounting for scalars twice. The result is exactly

$$3bc + 6ac - c^2 - b^2 - a^2 - 3ab + 1 = \lambda^2 - 2rd - r^2 + 1 = \dim M(\xi),$$

so the general resolution produces the general vector bundle in $M(\xi)$. □

We now prove Proposition 5.1.2, which states that a general stable sheaf G of positive rank on $\mathbb{P}^2$ is globally generated when $\chi(G) \geq \mathrm{rk}(G) + 2$. The rank 1 case is an analysis of line bundles and ideal sheaves, and the rank 2 case was known to Le Potier ([LP80]).

Proof of Proposition 5.1.2. The hardest case is when γ, the corresponding exceptional slope to ξ, is 0. Let G be a general sheaf in $M(\xi)$, which has a resolution given by Proposition 5.5.1. The snake lemma applied to the commutative diagram

$$\begin{array}{ccccccccc} & & & & \mathcal{O}^a & \xrightarrow{\simeq} & H^0(G)\otimes\mathcal{O} & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{O}(-2)^c & \longrightarrow & \mathcal{O}(-1)^b\oplus\mathcal{O}^a & \longrightarrow & G & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ & & \mathcal{O}(-2)^c & \longrightarrow & \mathcal{O}(-1)^b & \longrightarrow & F & & \end{array}$$

yields an exact sequence $\mathcal{O}(-2)^c \to \mathcal{O}(-1)^b \to F \to 0$ in which the first map is general if G is general. The assumption that $\chi(\xi) \geq r+2$ is equivalent to $c \geq b+2$, which guarantees that general maps $\mathcal{O}(-2)^c \to \mathcal{O}(-1)^b$ are surjective. Thus $F = 0$, so G is globally generated.

Next, we rule out the case $\gamma > 0$. As in the proof of the Proposition 5.5.1, our assumption on $\chi(\xi)$ (even $\chi(\xi) \geq 0$ suffices) ensures that $\mu_0 < \frac{3-\sqrt{5}}{2}$. Thus $\gamma \leq 0$.

The last case is $\gamma < 0$. In this case $0 \leq -\beta < -\gamma$, so the resolution for general G given by Proposition 5.5.2 expresses G as a quotient of exceptional bundles with nonnegative slope. The following lemma guarantees that such exceptional bundles are globally generated, hence so is G. □

Lemma 5.5.3. *Let $\gamma \geq 0$ be an exceptional slope. Then E_γ is globally generated.*

Proof. If $\gamma = n \geq 0$ is an integer, then $E_\gamma = \mathcal{O}_{\mathbb{P}^2}(n)$ is globally generated. For $\gamma > 0$ a noninteger, we use induction on q, where $p \geq 0$ and $q \geq 0$ are the unique integers such that $\gamma = \epsilon(\frac{2p+1}{2^{q+1}})$. Let $\gamma = \alpha.\beta$ be the standard decomposition of γ. By the inductive assumption, E_α is globally generated. Since we can write $\alpha = \epsilon(\frac{2p}{2^{q+1}})$, Theorem 2 of [Dre86] implies that $E_\alpha \otimes \operatorname{Hom}(E_\alpha, E_\gamma) \to E_\gamma$ is surjective. Since E_α is globally generated, so is E_γ. □

CHAPTER 6

USING MULTIPLE POINT FORMULAS TO ENUMERATE QUOT SCHEMES

In Chapter 5, we gave a partial affirmative answer to Guiding Problem 1 by exhibiting a large class of Quot schemes on $\mathbb{P}^2$ that are finite and reduced. Guiding Problem 2 asks whether we can enumerate their points. We show how multiple point formulas can be used to count the points of finite Quot schemes on del Pezzo surfaces when the quotients are ideal sheaves of points. In cases where the Quot scheme is indeed finite and reduced (as on $\mathbb{P}^2$) and where certain genericity conditions are satisfied that ensure the multiple point formula is counting actual multiple points (we are only able to prove this in very special cases), the enumeration is successful. In cases where we are not able to check these genericity conditions, we are still able to show that the resulting *expected* count of the points of the Quot scheme agrees with (2.1). This chapter is also based on joint work with Aaron Bertram and Drew Johnson in [BGJ16].

6.1 Enumerating Quot schemes on del Pezzo surfaces

On a general surface S, we write the Chern character as a triple: a rank, a divisor class, and the coefficient of the point class in the second Chern character. Since the enumeration problem is motivated by strange duality, we will once again let $\rho = (1, 0, -n)$ denote the Chern character of an ideal sheaf of n points. This will provide access to the formula (2.1) for computing dimensions of the spaces of sections of determinantal line bundles on $S^{[n]}$.

We make a further restriction and require that S be a del Pezzo surface, namely $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a blow up of $\mathbb{P}^2$ at ≤ 8 general points. This is because we hope to count the points of finite Quot schemes $\mathrm{Quot}(V, \rho)$, where the Chern character σ of the subsheaves E should satisfy $\chi(\sigma, \rho) = 0$. In general, this orthogonality will imply that $\hom(E, F) = \mathrm{ext}^1(E, F) = \mathrm{ext}^2(E, F) = 0$ when $\mathrm{ch}\, E = \sigma$ and $\mathrm{ch}\, F = \rho$. But V is supposed to be an extension of F by E for some F and E, and as we have seen, we will need V to be general if we hope to get a finite Quot scheme. Thus we want $\operatorname{ext}^1(F,E) > 0$. By Serre duality, $\operatorname{ext}^1(F,E) = \operatorname{ext}^1(E, F\otimes\omega_S)$, which given the vanishing $\operatorname{ext}^1(E,F) = 0$ will only be nonzero if ω_S is negative. Thus it makes sense to require ω_S to be antiample, which brings us to the del Pezzo condition.

Consider short exact sequences

$$0 \to E \to V \to \mathcal{I}_Z \to 0$$

in which $\chi(E,\mathcal{I}_Z) = 0$, so the expected dimension of $\operatorname{Quot}\big(V,(1,0,-n)\big)$ is zero. We can parametrize Chern characters σ that are orthogonal to $\rho = (1,0,-n)$ as

$$\sigma = \big(r, -L, (n-1)r + \tfrac{1}{2}L.K_S\big).$$

The Chern character of V is then

$$v = \operatorname{ch} V = \sigma + \rho = \big(r+1, -L, (n-1)r - n + \tfrac{1}{2}L.K_S\big),$$

which can be used to compute

$$\chi(v^\vee) = n(r-1)+1.$$

Theorem 6.1.1. *Fix $1 \le n \le 7$ and $r \ge 2$ in the setting above. For L sufficiently ample, the dimension $h^0\big(\mathcal{O}_{S^{[n]}}(\Theta_{\sigma^\vee})\big)$ agrees with the* expected *number of points of* $\operatorname{Quot}\big(V,(1,0,-n)\big)$ *for a vector bundle V on S of class v.*

The word *expected* is used for two reasons. First, we do not know in this generality whether the Quot scheme is actually finite. Second, the multiple point formulas are topological in nature and thus are only guaranteed to count actual multiple points if the map under consideration satisfies certain topological genericity conditions, which we cannot check for the algebraic maps we construct.

6.2 The case $n = 1$

Theorem 6.1.1 is easy to prove when $n = 1$. In this case, $\rho = (1,0,-1)$ and $\sigma = (r, -L, \frac{1}{2}L.K_S)$. Suppose L has vanishing higher cohomology, so that $h^0(L) = \chi(L)$. Let W be a general collection of $|W| = \chi(L)$ points on S. Let V^* be a general extension

$$0 \to \mathcal{O}^r \to V^* \to L\otimes\mathcal{I}_W \to 0.$$

Then V^* is locally free and the Chern character of its dual V is $\sigma + \rho$. Moreover, V^* has vanishing higher cohomology and since $h^0(V^*) = \chi(V^*) = r$, all the sections of V^* occur in the map $\mathcal{O}^r \to V^*$ in the extension. These sections have full rank away from W and drop to rank exactly $r-1$ on W, hence for each $p \in W$, there is a unique section (up to scaling) that vanishes at p. We will prove that if the extension defining V^* is general, then p is the only zero of that section. Thus $\#\mathrm{Quot}(V, \rho) = \chi(L)$.

A section $\mathcal{O} \to V^*$ vanishing at ≥ 2 points induces a commutative diagram

$$\begin{array}{ccccccccc}
 & & \mathcal{O} & = & \mathcal{O} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \mathcal{O}^r & \to & V^* & \to & L \otimes \mathcal{I}_W & \to & 0 \\
 & & \downarrow \beta & & \downarrow & & \| & & \\
0 & \to & \mathcal{O}^{r-1} & \to & \widehat{V} & \to & L \otimes \mathcal{I}_W & \to & 0
\end{array}$$

in which $\widehat{V}$ fails to be locally free at two points. By the following lemma, the extension producing $\widehat{V}$ occurs in a locus Z_2 of codimension $2(r-1)$ in $\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^{r-1})$. But there is a short exact sequence

$$0 \to \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}) \to \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^r) \xrightarrow{\beta_*} \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^{r-1}) \to 0,$$

hence V^* lies in the preimage $\beta_*^{-1}(Z_2)$, which also has codimension $2(r-1)$ in $\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^r)$. As the choice of $\beta \colon \mathcal{O}^r \twoheadrightarrow \mathcal{O}^{r-1}$ varies (in $\mathbb{P}^{r-1}$), these preimages $\beta_*^{-1}(Z_2)$ sweep out a locus of codimension $\geq 2(r-1) - (r-1) = r - 1 > 0$ in $\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^r)$, so choosing the extension defining V^* to avoid this locus guarantees that V^* has no sections vanishing at ≥ 2 points.

Lemma 6.2.1. *Let L be very ample, $r, k \geq 0$, and W be a collection of $|W| \geq h^0(L \otimes \omega_S) + k$ general points in S. Then the locus of extensions in $\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}^r)$ producing a sheaf that is not locally free at $\geq k$ points has codimension exactly kr.*

Proof. An extension $\widehat{V}$ that is not locally free on a subscheme $W' \subset W$ of length k induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}^r & \to & \widehat{V} & \to & L \otimes \mathcal{I}_W & \to & 0 \\
 & & \| & & \downarrow & & \downarrow \alpha_{W'} & & \\
0 & \to & \mathcal{O}^r & \to & \widehat{V}' & \to & L \otimes \mathcal{I}_{W \setminus W'} & \to & 0
\end{array}$$

The extension producing $\widehat{V}$ is the pullback of the extension producing $\widehat{V}'$, so it is in the image of

$$\alpha_{W'}^*\colon \mathrm{Ext}^1(L\otimes \mathcal{I}_{W\setminus p}, \mathcal{O}^r) \to \mathrm{Ext}^1(L\otimes \mathcal{I}_W, \mathcal{O}^r).$$

This map is injective since the cokernel of $\alpha_{W'}$ is $\mathcal{O}_{W'}$ and $\mathrm{ext}^1(\mathcal{O}_{W'}, \mathcal{O}^r) = 0$. Since

$$\mathrm{ext}^2(L\otimes \mathcal{I}_{W\setminus W'}, \mathcal{O}^r) = \hom(\mathcal{O}^r, L\otimes \omega_S \otimes \mathcal{I}_{W\setminus W'}) = 0$$

because $|W \setminus W'| \geq h^0(L\otimes\omega_S)$ by assumption, the cokernel of $\alpha_{W'}^*$ is $\mathrm{Ext}^2(\mathcal{O}_{W'}, \mathcal{O}^r)$, which has dimension kr. Thus the image of $\alpha_{W'}^*$ has codimension kr in $\mathrm{Ext}^1(L\otimes\mathcal{I}_W, \mathcal{O}^r)$. Choosing an extension that avoids the images of $\alpha_{W'}^*$ for each of the finitely many choices $W' \subset W$ with $|W'| = k$ produces a sheaf that fails to be locally free at $\leq k-1$ points. □

6.3 Three classical cases

We prove Theorem 6.1.1 in each of the special cases when $(n,r) \in \{\,(2,2),(2,3),(3,2)\,\}$ by constructing a general vector bundle V, computing the double or triple points of a particular morphism, and checking that the result agrees with the formula for $\chi\big(\mathcal{O}_{S^{[n]}}(L_n - \frac{r}{2}B)\big)$ in §2.4.1.

6.3.1 Double points of an immersed plane curve, $(n,r) = (2,2)$

Let L be an ample line bundle on S satisfying $-L.K_S \geq 4$. Let W be a general collection of $|W| = \chi(L) - 1$ points in S, such that the unique curve C of class L containing W is smooth and W is general on C. Let V^* be a general extension

$$0 \to \mathcal{O}_S^3 \to V^* \to L\otimes\mathcal{O}_C(-W) \to 0.$$

Then V^* is locally free, has rank 3, and its 3 sections drop to rank 2 on C. Thus we get a morphism

$$f\colon C \longrightarrow P\left(H^0(V^*)\right) = \mathbb{P}^2$$

which sends each point $p \in C$ to the unique (up to scaling) section of V^* vanishing at p.

We claim that this morphism is a general projection of the embedding defined by the line bundle $\mathcal{O}_C(W)$. (The condition $-L.K_S \geq 4$ ensures that W is very ample on C since it is general of degree $\chi(L) - 1 \geq \chi(L+K_S) + 3 = g_C + 3$.) To identify the sections of V^*

that vanish at points, we restrict the sequence defining V^* to C. This identifies the sections that vanish as the kernel

$$0 \to \mathcal{O}_C(-W) \to H^0(V^*) \otimes \mathcal{O}_C \to V^*|_C \to L|_C \otimes \mathcal{O}_C(-W) \to 0,$$

so f is induced by the morphism $H^0(V^*)^* \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C(W)$. To ensure that V^* does not contain $\mathcal{O}_S$-summands, and since automorphisms of $\mathcal{O}_S^3$ do not affect the isomorphism class of the extension V^*, we should choose the extension V^* as a point of $\mathrm{Gr}\big(3, \mathrm{Ext}^1(L \otimes \mathcal{O}_C(-W), \mathcal{O}_S)\big)$, which is naturally isomorphic to $\mathrm{Gr}\big(3, H^0(\mathcal{O}_C(W))\big)$ parametrizing choices of three sections of $\mathcal{O}_C(W)$. Choosing V^* to be a general extension ensures that the sections of $H^0(\mathcal{O}_C(W))$ are general, which proves the claim.

Thus we see that $f(C)$ is a plane curve with only simple nodes. The preimage $f^{-1}(s)$ of a closed point s is equal to the vanishing locus of s as a section of V^*, and since $n = 2$, we want to count sections that vanish at exactly two points, which correspond to the nodes of $f(C)$. Since degree is preserved by projection, $f(C)$ has degree $|W| = \chi(L) - 1 = c_2(V^*)$ and its normalization C has genus $\frac{1}{2}L.(L + K_S) + 1$ by adjunction. Thus the number of nodes is the difference

$$\binom{c_2(V^*) - 1}{2} - \big(\tfrac{1}{2}L.(L + K_S) + 1\big),$$

which agrees with the formula for $\chi\big(\mathcal{O}_{S^{[2]}}(L_2 - B)\big)$ obtained by setting $r = 2$ in (2.2).

6.3.2 Double points of a blowup of S immersed in $\mathbb{P}^4$, $(n, r) = (2, 3)$

Let L be a very ample line bundle on S satisfying $-L.K_S \geq 5$ and an additional positivity condition that we will state below. Choose W' to be a collection of $\chi(L + K_S)$ general points on S not contained on any curve of class $L + K_S$, which also impose independent conditions on curves of class L. Let C, C' be general smooth curves of class L containing W' and intersecting transversally, and let $W \subset C \cap C'$ with $|W| = \chi(L) - 2$ be the residual of W'. Then W is not contained on a curve of class $L + K_S$ and imposes independent conditions on curves of class L by Cayley-Bacharach ([TV00]).

Let V^* be a general extension

$$0 \to \mathcal{O}_S^3 \to V^* \to L \otimes \mathcal{I}_W \to 0,$$

chosen as a point of $\mathrm{Gr}\big(3, \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big)$ to ensure that V^* has no $\mathcal{O}_S$-summands. This Grassmannian is nonempty for $-L.K_S \geq 5$. Then V^* is rank 4 with 5 sections, so there is a section vanishing at each point of S, and in fact the sections of V^* drop to rank 3 on W'. We can separate these additional sections by passing to $X = \mathrm{Bl}_{W'}S \xrightarrow{\pi} S$, which yields a morphism

$$f\colon X = \mathrm{Bl}_{W'}S \longrightarrow P\left(H^0(V^*)\right) = \mathbb{P}^4$$

sending each $p \in X$ to the unique section of V^* vanishing at p.

More precisely, f is a general projection to $\mathbb{P}^4$ of the embedding determined by the line bundle $\mathcal{L} = \pi^* L(-\sum E_i)$, where E_i are the exceptional divisors of the blowup π. To see that $\mathcal{L}$ is very ample on X, we use a criterion of Ballico-Coppens (Theorem 0.1 in [BC97]). First, we view X as a blow-up $\tilde{\pi}\colon X \to \mathbb{P}^2$ with exceptional divisors E_i from blowing up W' and F_i from the del Pezzo surface S. (The case $S = \mathbb{P}^1 \times \mathbb{P}^1$ can be handled by a similar argument since blowing up a point of $\mathbb{P}^1 \times \mathbb{P}^1$ yields the blow-up of $\mathbb{P}^2$ at two points.) Write

$$\mathcal{L} = \tilde{\pi}^*(\mathcal{O}_{\mathbb{P}^2}(d)) \otimes \mathcal{O}_X\big(-\textstyle\sum m_i F_i - \sum E_i\big).$$

To apply the criterion, we note that $\mathcal{O}_{\mathbb{P}^2}(1)$ is very ample and that $m_i + m_j \leq d-1$ (this is property (C1) since the blown-up points are general) since L is very ample on S ([DR96]), so the last condition we need to check to guarantee that $\mathcal{L}$ is very ample is

$$h^1(\mathbb{P}^2, \mathcal{I}_{\mathbf{m}} \otimes \mathcal{O}_{\mathbb{P}^2}(d-1)) = 0,$$

where $\mathbf{m} = \sum m_i P_i + \sum Q_i$ is the weighted sum of the points $P_i, Q_i \in \mathbb{P}^2$ corresponding to the exceptional divisors F_i, E_i. Since all the blown up points are general, this last condition holds if we assume that

$$\binom{d+1}{2} - \sum \binom{m_i+1}{2} - |W'| = 2d - 1 - \sum m_i \geq 0,$$

which is the positivity hypothesis on L we mentioned above.

To see that f is a general projection to $\mathbb{P}^4$ of the embedding determined by $\mathcal{L}$, note that the kernel L^* in the exact sequence

$$0 \to L^* \to H^0(V^*) \otimes \mathcal{O}_S \to V^* \to \mathcal{O}_{W'} \to 0$$

fails to identify the additional sections of V^* that vanish along W', but this is corrected by pulling back $H^0(V^*) \otimes \mathcal{O}_S \to V^* \to \mathcal{O}_{W'}$ to X, which yields a new kernel

$$0 \to \mathcal{L}^* \to H^0(V^*) \otimes \mathcal{O}_X \to \pi^* V^* \to \mathcal{O}_{\sqcup E_i} \to 0,$$

where $\mathcal{L}$ is defined as above. We can think of f as the induced morphism $P(\mathcal{L}^*) \to P(H^0(V^*))$, which is the composition of the morphism $X \to \mathbb{P}H^0(\mathcal{L})$ and the projection onto the image of the induced inclusion $H^0(V^*)^* \hookrightarrow H^0(\mathcal{L})$, which by construction contains the span of C, C' viewed as sections of $\mathcal{L}$. For fixed L and W, assigning this image to each extension V^* gives an isomorphism

$$\mathrm{Gr}\big(3, \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big) \simeq \mathrm{Gr}\big(3, H^0(\mathcal{L})/\mathrm{span}(C, C')\big),$$

and these Grassmannians are nonempty by the condition $-L.K_S \geq 5$. Thus a general choice of V^* yields a choice of 3 general sections of $\mathcal{L}$ in addition to C, C'. Since C, C' are general curves containing W', the sections of a general V^* yield 5 general sections of $\mathcal{L}$, namely the projection in the definition of f is general. See §6.4.2 for a more detailed argument in a similar situation.

Thus $f(X)$ is an immersed surface in $\mathbb{P}^4$ with ordinary double points. The number of ordinary double points of an immersion can be computed using the following theorem, which is known as the Herbert-Ronga formula.

Theorem 6.3.1 ([Kle81] Theorem 5.8)**.** *Let X and Y be smooth varieties, let $k \geq 2$, and let $f\colon X \to Y$ be practically k-generic of codimension $\ell \geq 1$. Suppose f is an immersion. Then the k-fold point class in X is*

$$x_k = f^* f_* x_{k-1} - (k-1)\, c_\ell\, x_{k-1} \ \in A^{(k-1)\ell}(X),$$

where $c_\ell = c_\ell(f^* T_Y / T_X)$ and $x_1 = [X]$ is the fundamental class.

In [Kle81], Kleiman constructs "derived maps" $f_{k-1}\colon X_k \to X_{k-1}$ inductively ($f_0 = f$), where X_k is the residual to the diagonal subscheme in the fibered product of two copies of f_{k-2}, and f_{k-1} is defined to be the second projection. The k-fold point class x_k is $(f_1 \circ \cdots \circ f_{k-1})_*[X_k]$. In general, x_k can be thought of as the closure of the locus of points p in X at which the fiber of $f(p)$ has length k. The **practically k-generic** assumption is that f_{j-1} is an lci of codimension ℓ for all $j \leq k$, which guarantees that each x_j has the expected codimension $(j-1)\ell$.

For our application, we consider the Herbert-Ronga double point formula

$$x_2 = f^* f_*[X] - c_\ell \ \in A^\ell(X),$$

which holds even if f is not an immersion ([Kle81] Theorem 5.6). We want to count certain sections of V^*, namely the double points of f in $Y = \mathbb{P}^4$, so we push forward the Herbert-Ronga formula, dividing by 2 to account for the fact that every double point has two preimages. The resulting formula is

Corollary 6.3.2 (Double point formula). *Let $f\colon X \to Y$ be a morphism of smooth varieties. Assume f is practically 2-generic and has codimension $\ell = \dim Y - \dim X \geq 1$. Then the double point class in Y is*

$$y_2 = \tfrac{1}{2}\left((f_*[X])^2 - f_*c_\ell\right) \in A^{2\ell}(X). \tag{6.1}$$

In our application, the first derived map f_1 is the second projection to X from the finite locus $X_2 = \{(p,q)\colon f(p) = f(q) \text{ and } p \neq q\} \subset X \times X \setminus \Delta$, so f is practically 2-generic and the double point formula correctly computes the number of double points on $f(X)$. To compute $c_2 = c_2(f^*T_{\mathbb{P}^4}/T_X)$, we use the total Chern classes $c(T_{\mathbb{P}^4}) = (1+H)^5$ and $c(T_X) = 1 - K_X + (12\rho - K_X^2)$, where ρ is the class of a point and $K_X = K_S + \sum E_i$, as well as the identity $f^*H = L - \sum E_i$. After some simplification, we get

$$c_2 = [(1 + (L - \textstyle\sum E_i))^5(1 + K_X + 2(K_X^2 - 6))]_2 = 5L.K_S + 2K_S^2 + 10L^2 - 7|W'| - 12,$$

and substituting $|W'| = L^2 - c_2(V^*)$ yields

$$y_2 = \tfrac{1}{2}\left(c_2(V^*)^2 - 7c_2(V^*) - 5L.K_S - 2K_S^2 - 3L^2 + 12\right)$$

which matches the formula for $\chi\left(\mathcal{O}_{S^{[2]}}(L_2 - 3\frac{B}{2})\right)$ in (2.2) when we set $r = 3$ and make the further substitutions $c_2(V^*) = \chi(L) - 2$ and $L^2 = 2\chi(L) + L.K_S - 2$.

Remark 6.3.3. We could also use the double point formula to recover the count in the above case $(n,r) = (2,2)$ of a degree $c_2(V^*)$ immersion $f\colon C \to \mathbb{P}^2$ of a smooth curve of genus $\frac{1}{2}L.(L+K_S)$. In this case $c(T_{\mathbb{P}^2}) = (1+H)^3$ and $c(T_C) = 1 - K_C$, so

$$c_1 = c_1(f^*T_{\mathbb{P}^2}/T_C) = [f^*(1+3H)(1+K_C)]_1 = 3c_2(V^*) + L.(L+K_S)$$

and therefore

$$y_2 = \tfrac{1}{2}\big(c_2(V^*)^2 - (3c_2(V^*) + L.(L+K_S))\big),$$

which agrees with the previous computation.

6.3.3 Triple points of a nonimmersed blowup of S in $\mathbb{P}^3$, $(n,r) = (3,2)$

As in the case $(n,r) = (2,3)$, choose sufficiently positive L (with the same conditions except that $-L.K_S \geq 4$ suffices in this case), general W' of length $|W'| = \chi(L + K_S)$, and smooth transversal curves C, C' of class L containing W'. Let W be the residual to W' in $C \cap C'$. We let V^* be a general extension $0 \to \mathcal{O}_S^{\oplus 2} \to V^* \to L \otimes \mathcal{I}_W \to 0$, which has rank 3 and 4 sections that drop to rank 2 on W'. As before we get a morphism

$$f\colon X = \mathrm{Bl}_{W'}S \longrightarrow P\left(H^0(V^*)\right) = \mathbb{P}^3$$

sending $p \in X$ to the unique section of V^* vanishing at p, and once again f is a general projection to $\mathbb{P}^3$ of the embedding of X determined by the line bundle $\pi^*L(-\sum E_i)$, where E_i are the exceptional divisors of the blow-up.

Since f is a general projection to $\mathbb{P}^3$ of an embedded surface, we can give an explicit description of its singularities, following [CF11] and [MP97]. The singular points of the image $f(X)$ form an irreducible curve C_0 (the double point locus) containing finitely many ordinary triple points (which are three-branch nodes of the curve) and finitely many pinch points (which are smooth points of the curve, but at which the derivative of f drops rank by 1, so f is not an immersion). The preimage $C_1 := f^{-1}(C_0) \subset X$ is a curve and $f|_{C_1}\colon C_1 \to C_0$ is generically two-to-one. The pinch points on C_0 are branch points of $f|_{C_1}$, over which C_1 is smooth. The only singularities of C_1 are triples of simple nodes lying over each triple point of C_0.

Since $n = 3$, we want to compute the number of these triple points. This can be done using Kleiman's triple point formula.

Theorem 6.3.4 ([Kle81] Theorem 5.9)**.** *If $f\colon X \to Y$ is practically 3-generic of codimension ℓ between smooth varieties, then the triple point class in X is*

$$x_3 = f^*f_*x_2 - 2c_\ell x_2 + \left(\sum_{k=1}^{\ell} 2^k c_{\ell-k}c_{\ell+k}\right) \in A^{2\ell}(X).$$

To obtain a corresponding formula on Y, we substitute the double point formula for x_2, push forward to Y, divide by 3, and use the projection formula.

Corollary 6.3.5 (Triple point formula)**.** *Let $f\colon X \to Y$ be a practically 3-generic codimension ℓ morphism of smooth varieties. Then the triple point class in Y is*

$$y_3 = \frac{1}{6}\left((f_*[X])^3 - 3(f_*c_\ell)(f_*[X]) + 2f_*(c_\ell^2) + \sum_{k=1}^{\ell} 2^k f_*(c_{\ell-k}c_{\ell+k})\right) \in A^{3\ell}(Y). \tag{6.2}$$

In our setting, the first derived map $f_1\colon X_2 \to X_1 = X$ is the normalization of C_1, and the set of closed points in X_2 lying over the nodes of C_1 is exactly the image of the second derived map $f_2\colon X_3 \to X_2$. Thus f is practically 3-generic, so the triple point formula applies. Note that there are three nodes over each triple point of C_0, and two points in X_3 over each of these nodes, which explains the factor of $\frac{1}{6}$ in the formula.

Letting ρ denote the point class in $A^2(X)$, we compute the total Chern class

$$c(f^*T_{\mathbb{P}^3}/T_X) = (1 + (L - \sum E_i))^4(1 + K_X + 2(K_X^2 - 6\rho)),$$

from which we extract $c_1 = 4L + K_S - 3\sum E_i$ and $c_2 = 6L^2 + 4L.K_S + 2K_S^2 - 4|W'| - 12$ after substituting $K_X = K_S + \sum E_i$. To compute y_3 we need to know how a divisor D on X pushes forward under f, but this is easy since f_*D in $\mathbb{P}^3$ is determined by its degree $(f_*D).H = D.f^*H = D.(L - \sum E_i)$. In particular $f_*c_1 = 4L^2 + L.K_S - 3|W'|$, so

$$\begin{aligned} y_3 &= \frac{1}{6}\left[c_2(V^*)^3 - 3c_2(V^*)f_*(c_1) + 2f_*(c_1^2) + 2f_*(c_2)\right] \\ &= \frac{1}{6}\left[c_2(V^*)^3 - 3c_2(V^*)(4L^2 + L.K_S - 3|W'|) + 44L^2 + 24L.K_S + 6K_S^2 \right. \\ &\qquad \left. -26|W'| - 24\right], \end{aligned}$$

which agrees with the formula (2.3) for $\chi\big(\mathcal{O}_{S^{[3]}}(L_3 - B)\big)$ when we substitute $c_2(V^*) = \chi(L) - 2$, $|W'| = \chi(L) + L.K_S$, and $L^2 = 2\chi(L) + L.K_S - 2$.

6.4 General case

We prove Theorem 6.1.1 in the general case, namely when $n, r \geq 2$, $(n,r) \notin \{(2,2), (2,3), (3,2)\}$, and $n \leq 7$. To do so, we choose a general globally generated vector bundle V^* with the appropriate invariants and collect the sections of V^* that vanish at points as the kernel $0 \to G \to H^0(V^*) \otimes \mathcal{O}_S \to V^* \to 0$. The sections of V^* that vanish at n points correspond to the n-fold points of the natural map $f\colon P(G) \to P\big(H^0(V^*)\big)$, which on closed points is just $(p,s) \mapsto s$, where $p \in S$ and s is a section of V^* vanishing at p. We compute the number of n-fold points of f using multiple point formulas, which are only known in sufficient generality (f has corank 2) up to $n = 7$. Our computer code (which can be found in [Joh16]) checks that these computations agree with the value of $\chi\big(\mathcal{O}_{S^{[n]}}(L_n - r\frac{B}{2})\big)$ obtained from the power series (2.1).

6.4.1 Choosing V

Let L be an ample line bundle on S. Let W be a collection of $|W| = \chi(L) - (n-1)(r-1)$ general points on S. Define V^* as an extension

$$0 \to \mathcal{O}_S^r \to V^* \to L \otimes \mathcal{I}_W \to 0$$

corresponding to a general point in $\mathrm{Gr}\big(r, \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big)$, which is nonempty if and only if $-L.K_S \geq n(r-1)+1$. The Grassmannian is the natural extension space since we are mainly interested in the isomorphism class of the middle object in the extension. More precisely, the isomorphism

$$\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S^r) \to \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S)^r$$

defined by pushing forward extensions along the r projection maps $p_i \colon \mathcal{O}_S^r \to \mathcal{O}_S$ is $\mathrm{GL}(r, \mathbb{C})$-equivariant, where the action on the left is by pushing forward along automorphisms of $\mathcal{O}_S^r$ (which has the effect of precomposing the map $\mathcal{O}_S^r \to V^*$ by the inverse automorphism) and the action on the right is the natural action on the r summands. Removing the locus where the action is not free (which corresponds on the left to extensions that have $\mathcal{O}_S$-summands and on the right to linearly dependent r-tuples) and passing to the quotient yields the Grassmannian above.

Proposition 6.4.1. *Let $n, r \geq 2$ and suppose $-L.K_S \geq n(r-1)+1$ and L is $N = \max\big\{(n-1)(r-1), 3\big\}$-very ample. Let W be general of length $\chi(L) - (n-1)(r-1)$. Then the extensions $0 \to \mathcal{O}_S^r \to V^* \to L \otimes \mathcal{I}_W \to 0$ parametrized by general points of $\mathrm{Gr}\big(r, \mathrm{Ext}(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big)$ satisfy*

(a) $\mathrm{ch}(V^*) = (r+1, L, (n-1)(r-1) - 1 + \frac{1}{2}LK_S)$;

(b) *V^* is globally generated* $\iff$ $(n, r) \notin \{(2,2), (2,3), (3,2)\}$;

(c) $h^1(V^*) = h^2(V^*) = 0$ *and* $h^0(V^*) = \chi(V^*) = n(r-1)+1$;

(d) *V^* is locally free;*

(e) $h^0(V) = 0$;

(f) *no section of V^* vanishes along a curve.*

Proof. Part (a) is additivity of the Chern character. Part (b) follows from the fact that V^* is globally generated if and only if $L \otimes \mathcal{I}_W$ is globally generated. Part (c) is obtained from the cohomology long exact sequence and the fact that vanishing on W imposes independent conditions on curves of class L. Part (d) holds since the Cayley-Bacharach property ([HL10] Theorem 5.1.1) is satisfied for the pair $(L \otimes \omega_S, W)$ as long as $-L.K_S \geq (n-1)(r-1)+1$. For (e) one can check the equivalent assertion $\hom(V^*, \mathcal{O}_S) = 0$ by applying the functor $\mathrm{Hom}(-, \mathcal{O}_S)$ and observing that $\mathrm{Hom}(\mathcal{O}_S^r, \mathcal{O}_S) \to \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O})$ is injective since its image yields the point in $\mathrm{Gr}\big(r, \mathrm{Ext}(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big)$ corresponding to the extension.

If (f) fails, then there is a nonzero effective divisor D such that $V^*(-D)$ has a section. Tensoring the sequence defining V^* by $\mathcal{O}(-D)$, we see that $L(-D) \otimes \mathcal{I}_W$ must have a section. Since W is general, we can rule out the existence of such a section by proving that $h^0(L(-D)) \leq |W|$ for all $D > 0$, and it suffices to consider only minimal effective D, namely all D in the basis $\mathcal{B}_{\mathrm{eff}}$ of the effective cone in $\mathrm{Pic}(S)$ described in [BP04] (Corollary 3.3). A brute force check reveals that

$$D.D' \leq 3 \quad \text{and} \quad \chi(-D) = \begin{cases} 1 & \text{if } D = -K_{S_8} \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } D, D' \in \mathcal{B}_{\mathrm{eff}}.$$

Since L is N-very ample, $(L-D).D' \geq N-3 \geq 0$ for all $D' \in \mathcal{B}_{\mathrm{eff}}$, so $L-D$ is nef, which implies that its higher cohomology vanishes ([Knu03]). By Riemann-Roch,

$$h^0(L(-D)) = h^0(L) - L.D + \chi(-D) - 1 \leq \chi(L) - N \leq |W|,$$

which completes the proof. $\square$

6.4.2 Projective bundle

Assume V^* is chosen as in the previous proposition with $(n,r) \notin \{(2,2),(2,3),(3,2)\}$ ensuring that V^* is globally generated. The snake lemma yields a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & G & = & G & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_S^r & \to & H^0(V^*) \otimes \mathcal{O}_S & \to & H^0(L \otimes \mathcal{I}_W) \otimes \mathcal{O}_S & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_S^r & \to & V^* & \to & L \otimes \mathcal{I}_W & \to & 0
\end{array}$$

in which the two vertical sequences defining the kernel G are exact. G is locally free, has no cohomology, and its dual fits in the short exact sequence $0 \to V \to H^0(V^*)^* \otimes \mathcal{O}_S \to G^* \to$

0. Since $h^0(V) = 0$, the diagram yields inclusions $H^0(L \otimes \mathcal{I}_W)^* \hookrightarrow H^0(V^*)^* \hookrightarrow H^0(G^*)$, which make explicit the natural isomorphism

$$\mathrm{Gr}\big(r, \mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S)\big) \cong \mathrm{Gr}(r, H^0(G^*)/H^0(L \otimes \mathcal{I}_W)^*)$$

coming from the isomorphism $\mathrm{Ext}^1(L \otimes \mathcal{I}_W, \mathcal{O}_S) \cong H^0(G^*)/H^0(L \otimes \mathcal{I}_W)^*$ induced by the cohomology long exact sequence associated to the right vertical sequence in the diagram above tensored by ω_S. Thus generic choices of extensions V^* correspond to generic subspaces of $H^0(G^*)$ containing $H^0(L \otimes \mathcal{I}_W)^*$.

In fact, we now show that the image of $H^0(L \otimes \mathcal{I}_W)^* \hookrightarrow H^0(G^*)$ is a general subspace of $H^0(G^*)$ for general W, which implies that the subspace $H^0(V^*)^* \hookrightarrow H^0(G^*)$ is general when V^* is general. Since G^* is globally generated, $N = (n-1)(r-1)$ general sections of G^* yield an exact sequence

$$0 \to L^* \to \mathcal{O}_S^N \to G^* \to \mathcal{O}_W \to 0$$

whose dual sequence

$$0 \to G \to \mathcal{O}_S^N \to L \otimes \mathcal{I}_W \to 0$$

shows that the N general sections are dual to the sections of some $L \otimes \mathcal{I}_W$, and W is general since G has no cohomology.

The fibers of G parametrize the sections of V^* that vanish at points, and we can compile them into a map whose n-fold points exactly correspond to sections of V^* vanishing at n points.

Proposition 6.4.2. *There is a map $f\colon X = P(G) \to P\big(H^0(V^*)\big) = Y$ described in two ways as*

(1) the composition $P(G) \hookrightarrow P\big(H^0(V^*)\big) \times S \twoheadrightarrow P\big(H^0(V^*)\big)$ of the projectivization of $G \to H^0(V^*) \otimes \mathcal{O}_S$ and projection to the first factor;

(2) the composition $P(G) \to \mathbb{P}H^0(G^*) \dashrightarrow Y$ of the map induced by $\mathcal{O}_{P(G)}(1)$ and a general projection onto a projective space of dimension $n(r-1)$;

which has the following properties:

(a) f is linear inclusion on fibers of $\pi\colon X \to S$;

(b) the image of f spans Y;

(c) $f^*\big(\mathcal{O}_Y(1)\big) = \mathcal{O}_X(1)$;

(d) *For every* $s \in H^0(V^*)$, *the inverse image* $f^{-1}(s)$ *viewed in* $\{s\} \times S \simeq S$ *using the inclusion* $P(G) \hookrightarrow P\big(H^0(V^*)\big) \times S$ *is equal to the scheme-theoretic vanishing locus of s as a section of* V^*.

Proof. Part (a) follows from description (1) since both maps preserve fibers of the projective bundles. Part (b) is clear from (2) and can be checked from (1) using $h^0(V) = 0$. Part (c) is clear from (2). For (d), note that the fiber of π over p coincides with the fiber $G(p)$, whose image in Y is the sections of V^* vanishing at p. The scheme inverse image $f^{-1}(s)$ identifies all the points p at which s vanishes and supplies the appropriate scheme structure. □

Part (d) of the proposition identifies the sections of V^* vanishing at n points as the n-fold points of the map f. To count these n-fold points, we need an n-fold point formula. Unfortunately, since the n-fold point loci are zero-dimensional, Kleiman's n-fold point formulas ([Kle81]) will only work if f has corank 1, namely if its derivative drops rank by at most 1. By an expected codimension computation ([Kaz03]), f should have corank 1 when

$$n = 2, 3 \text{ and any } r; \qquad n = 4 \text{ and } r \leq 4; \qquad n = 5, 6 \text{ and } r = 2;$$

and one can check that the resulting computations using Kleiman's n-fold point formulas agree with $\chi\left(\mathcal{O}_{S^{[n]}}(L_n - r\frac{B}{2})\right)$. In these cases, we expect but have not been able to prove that Ran's results on general projections ([Ran15a], [Ran15b]) should guarantee that Kleiman's formulas are counting only ordinary multiple points. But in general f has corank 2: although f is a linear inclusion on the projective fibers of $P(G)$, the derivative of f can and will vanish in both directions coming from the base S.

Since the algebraic multiple point theory does not seem to cover corank 2 maps, we pass to the topological theory. There we can view (d) of Proposition 6.4.2 as a dictionary between vanishing loci of sections and all multisingularities, which we now explain.

6.4.3 Multisingularities

The following brief introduction to multisingularities is based on the more detailed discussion in [MR10].

Let $f\colon X \to Y$ be a holomorphic map of complex manifolds of dimensions $m = \dim X$ and $n = \dim Y$ such that $\ell = \dim Y - \dim X \geq 1$. The germ of f at each point $p \in X$ is a map $f_p\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ defined by n power series $f_1, \ldots, f_n \in \mathbb{C}[\![x_1, \ldots, x_m]\!]$ with no constant term. The **local algebra** of f at p is defined to be $Q_{f,p} = \mathbb{C}[\![x_1, \ldots, x_m]\!]/(f_1, \ldots, f_n)$ and its isomorphism class characterizes the **contact singularity** of f at p. We will use the notation α to denote a general contact singularity and Q_α to denote the corresponding isomorphism class of local algebras. We will consider only singularities α for which Q_α is finite-dimensional over $\mathbb{C}$, which are called finite singularities. We say a singularity α has **corank** r if Q_α can by minimally generated as an algebra by r generators, which is equivalent to the derivative dropping rank by r. The map f has corank r if all singularities of f have corank $\leq r$.

The unique corank 0 singularity, denoted A_0, has $Q_{A_0} \simeq \mathbb{C}$. A_0 singularities are points at which f is an immersion, so f has corank 0 if and only if it is an immersion. The corank 1 singularities, often called **Morin singularities**, are denoted $A_1, A_2, \ldots$ and have $Q_{A_i} \simeq \mathbb{C}[t]/(t^{i+1})$. The classification of corank ≥ 2 singularities becomes complicated.

Example 6.4.3. The normalization $\operatorname{Spec}\mathbb{C}[t] \to \operatorname{Spec}\mathbb{C}[x,y]/(y^2 - x^3)$ defined by $x \mapsto t^2$, $y \mapsto t^3$ of the cuspidal plane cubic curve has an A_1 singularity above the singular point since $\mathbb{C}[\![t]\!]/(t^2, t^3) \simeq \mathbb{C}[t]/(t^2)$.

The types of singularities give a stratification of X but not a stratification of Y since a point of Y may have preimages with different singularities.

Definition 6.4.4. The map $f\colon X \to Y$ has a **multisingularity** of type $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ at a point $p_1 \in X$ if f has singularity α_1 at p_1 and if the other preimages $p_2, \ldots, p_k$ of $f(p_1)$ have singularities $\alpha_2, \ldots, \alpha_k$. We use $Q_{\underline{\alpha}}$ to denote the list of local algebras $(Q_{\alpha_1}, \ldots, Q_{\alpha_k})$ and we define the **length** of $\underline{\alpha}$ to be $\sum_{i=1}^k \dim_{\mathbb{C}} Q_{\alpha_i}$.

We will be most interested in multisingularities of type $\underline{\alpha} = (A_0, \ldots, A_0) = A_0^k$, which we call **$k$-fold points** of f since they correspond to points $q \in Y$ such that the preimage of q under f is $\{p_1, \ldots, p_k\}$ and f is an immersion at each p_i.

Example 6.4.5. The normalization $\operatorname{Spec}\mathbb{C}[t] \to \operatorname{Spec}\mathbb{C}[x,y]/\big(y^2 - x^2(x+1)\big)$ defined by

$x \mapsto t^2 - 1$, $y \mapsto t(t^2 - 1)$ of the nodal plane cubic curve has a double point above the node.

We can stratify X and Y into multisingularity types. If $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ denotes a multisingularity, then the locus

$$Y_{\underline{\alpha}} = \left\{ q \in Y \colon \begin{array}{c} q \text{ has exactly } k \text{ preimages } p_1, \ldots, p_k \\ \text{and } f \text{ has singularity } \alpha_i \text{ at } p_i \end{array} \right\}$$

of all points in Y over which f has multisingularity $\underline{\alpha}$ is the image of the locus

$$X_{\underline{\alpha}} = \left\{ p_1 \in X \colon \begin{array}{c} f(p_1) \text{ has exactly } k \text{ preimages } p_1, \ldots, p_k \\ \text{and } f \text{ has singularity } \alpha_i \text{ at } p_i \end{array} \right\}.$$

The $Y_{\underline{\alpha}}$ stratify Y and the $X_{\underline{\alpha}}$ are a refinement of the stratification of X into singularity types. We let $x_{\underline{\alpha}} \in H^*(X, \mathbb{C})$ and $y_{\underline{\alpha}} \in H^*(Y, \mathbb{C})$ denote the Poincaré-dual cohomology classes of the closures of $X_{\underline{\alpha}}$ and $Y_{\underline{\alpha}}$, with multiplicities $\# \operatorname{Aut}(\alpha_2, \ldots, \alpha_k)$ and $\# \operatorname{Aut}(\underline{\alpha})$, respectively. The multiplicities are chosen to ensure $f_* x_{\underline{\alpha}} = y_{\underline{\alpha}}$.

We focus on the loci of k-fold points, which we abbreviate as X_k and Y_k. We use x_k to denote the cohomology class of the closure of X_k with multiplicity $\#\operatorname{Aut}(A_0^{k-1}) = (k-1)!$, so $x_k = x_{A_0^k}$, but we break from convention by writing y_k for the closure of Y_k, without any scaling, since the unscaled class will have a direct geometric interpretation. With this normalization, $f_* x_k = k!\, y_k = y_{A_0^k}$.

Remark 6.4.6. As we will see below, there are formulas for computing the classes $x_{\underline{\alpha}}$ and $y_{\underline{\alpha}}$ for certain multisingularities $\underline{\alpha}$. These formulas are only valid when f is admissible ([MR10] §2.4) . Roughly speaking, there is an infinite-dimensional classifying space M containing a submanifold $M_{\underline{\alpha}}$ for each multisingularity $\underline{\alpha}$. The codimension $\operatorname{codim} \underline{\alpha}$ of $M_{\underline{\alpha}}$ in M is finite. A map $f \colon X \to Y$ induces a map $k_f \colon Y \to M$ such that the locus $Y_{\underline{\alpha}}$ of points in Y over which f has multisingularity $\underline{\alpha}$ is the preimage of $M_{\underline{\alpha}}$ under k_f. We say that f is **admissible** if k_f is transversal to each $M_{\underline{\alpha}}$. In particular, admissibility implies that each $Y_{\underline{\alpha}}$ occurs in the expected codimension $\operatorname{codim} \underline{\alpha}$, but the converse is false. Since there is no known algebraic formulation of admissibility, it is nearly impossible to check that an algebraic map is admissible, so it is common practice in the literature to assume admissibility when the map is constructed geometrically.

Example 6.4.7. The codimension (in the codomain Y) of an A_i singularity is $\ell + i(\ell+1)$. The codimension of a multisingularity $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is $\operatorname{codim} \underline{\alpha} = \sum_{i=1}^{k} \operatorname{codim} \alpha_i$. In particular, one can see that among all multisingularities of length n, the multiple point locus A_0^n has the smallest codimension $n\ell$.

The word "expected" in Theorem 6.1.1 reflects the assumption that the map $f\colon P(G) \to P(H^0(V^*))$ constructed in §6.4.2 is admissible. To make this assumption seem plausible, we note that V is chosen using a general point in a Grassmannian, and f is a general projection from the map determined by $\mathcal{O}_{P(G)}(1)$. One can show that on $\mathbb{P}^2$ this map is an embedding (using Proposition 5.5.2), and there is a general expectation that general projections of smooth projective varieties $X \subset \mathbb{P}^N$ should have only expected singularities (this is a classical problem; some recent papers in this direction are [Ran15b], [Ran15a], [GP13], and the references listed in those papers). In our setting, such a "general projection conjecture" could take the following form:

Conjecture 6.4.8. *Let S be a smooth projective surface, let G be a very ample vector bundle on S, and suppose $f\colon P(G) \to \mathbb{P}^m$ is a general projection of the embedding defined by $\mathcal{O}_{P(G)}(1)$. Then f is admissible.*

Returning to our discussion of multisingularities, we note that when $f\colon X \to Y$ is a map of smooth varieties with finite fibers, a multisingularity of type $\underline{\alpha}$ of f over a point $q \in Y$ can be viewed as a closed subscheme $\operatorname{Spec} Q_{\underline{\alpha}} = \bigsqcup \operatorname{Spec} Q_{\alpha_i} \subset X$ supported on the preimage of y, which exactly agrees with the scheme-theoretic fiber $f^{-1}(q)$. Intuitively, nonreducedness at a point p in the fiber corresponds to vanishing of derivatives and higher order derivatives of f at p, which is encoded by $Q_{f,p}$.

For the map $f\colon X = P(G) \to P\big(H^0(V^*)\big) = Y$ from Proposition 6.4.2, we can describe $Y_{\underline{\alpha}}$ as the locus of points of Y at which the fiber of f is isomorphic to $\operatorname{Spec} Q_{\underline{\alpha}}$. By Proposition 6.4.2 (d), these fibers are the vanishing loci of the sections parametrized by Y, so $Y_{\underline{\alpha}}$ consists of exactly those sections of V^* whose vanishing locus is isomorphic to $\operatorname{Spec} Q_{\underline{\alpha}}$. In particular, the locus of k-fold points Y_k is in bijection with the sections of V^* that vanish at exactly k distinct points.

Proposition 6.4.9. *Define V^* be as in Proposition 6.4.1 and $f\colon X = P(G) \to P\big(H^0(V^*)\big) = Y$*

as in Proposition 6.4.2, so $\dim X = (n-1)(r-1)$, $\dim Y = n(r-1)$, *and* $\ell = \operatorname{codim} f = r-1$. *Assume f is admissible. Then there is a bijection between the closed points of* $\operatorname{Quot}(V,(1,0,-n))$ *and the n-fold point locus Y_n, which is finite. In particular, all closed points of* $\operatorname{Quot}(V,(1,0,-n))$ *are ideal sheaf quotients of reduced zero-dimensional subschemes.*

Since the expected codimension of the n-fold point locus is $n\ell = \dim Y$, the admissibility condition guarantees that Y_n is a finite set. Because of the dictionary between multisingularities and vanishing loci of sections of V^*, which in turn correspond to quotients of V, it suffices to prove that the only quotients of V with Chern character $(1,0,-n)$ are ideal sheaves $\mathcal{I}_Z$, where Z consists of n distinct points. We do this by ruling out all other possibilities, which were described in Lemma 5.4.3.

Proof of Proposition 6.4.9. We show that the only quotients with Chern character $(1,0,-n)$ that can occur are of type (1) in Lemma 5.4.3, which are in bijection with Y_n. Quotients of type (4) do not occur since V^* has no sections that vanish on curves. Quotients of type (2) and (3) yield multisingularities of length $\geq n$ that are not n-fold points, so they occur in codimension $> \dim Y$, namely not at all. □

Remark 6.4.10. For the application to strange duality, it would be ideal to define a natural scheme structure on the n-fold point locus Y_n (which would be reduced when f is admissible) and extend the bijection in Proposition 6.4.9 to a scheme-theoretic isomorphism $Y_n \simeq \operatorname{Quot}(V,(1,0,-n))$ that takes into account nonreduced structure.

6.4.4 General multiple point formulas

In order to compute the number of n-fold points of $f\colon X \to Y$ for $n \leq 7$, which by Proposition 6.4.9 will count the number of closed points in $\operatorname{Quot}(V,(1,0,-n))$, we use a formula that computes the Poincaré dual cohomology class y_n of the n-fold point locus Y_n as a polynomial in the Chern classes c_i of the virtual normal sheaf f^*T_Y/T_X.

Let X and Y be complex manifolds and let $f\colon X \to Y$ be a holomorphic map of codimension $\ell = \dim Y - \dim X > 0$. In §6.4.3, we described the locus $X_{\underline{\alpha}}$ of multisingularities of type $\underline{\alpha}$ in X and its image $Y_{\underline{\alpha}}$ in Y. We let $x_{\underline{\alpha}} \in H^*(X,\mathbb{C})$ and $y_{\underline{\alpha}} \in H^*(Y,\mathbb{C})$ denote the Poincaré-dual cohomology classes of the closures of these loci, with multiplicities $\#\operatorname{Aut}(\alpha_2,\ldots,\alpha_k)$ and $\#\operatorname{Aut}(\underline{\alpha})$, respectively. The multiplicities ensure that $f_* x_{\underline{\alpha}} = y_{\underline{\alpha}}$.

Kazaryan discovered a general form for multisingularity formulas that compute $x_{\underline{\alpha}}$ and $y_{\underline{\alpha}}$ ([Kaz03] Theorem 3.2). The key ingredient in these formulas is the residual polynomial $R_{\underline{\alpha}}(\ell)$ of $\underline{\alpha}$, which is a universal polynomial in the Chern classes of the virtual normal sheaf of f that depends only on the codimension ℓ of f. If $\underline{\alpha} = (\alpha)$ is a monosingularity, then $R_{\underline{\alpha}}(\ell)$ agrees with the Thom polynomial and computes the class $x_{\underline{\alpha}}$ when f is admissible. For general $\underline{\alpha}$ and f admissible, there is an iterative formula

$$x_{\underline{\alpha}} = R_{\underline{\alpha}}(\ell) + \sum_{1 \in J \subsetneq \{1,\ldots,r\}} R_{\underline{\alpha}_J}(\ell) f^*(y_{\underline{\alpha}_{\bar{J}}}) \ \in H^*(X, \mathbb{C}), \tag{6.3}$$

where $\underline{\alpha}_J$ is the sub-tuple of $\underline{\alpha}$ defined by J, and $\bar{J}$ is the complement of J in $\{1, \ldots, r\}$. There are two main obstructions to the implementation of this formula. First, the formula is only valid if f is admissible, which we discussed in Remark 6.4.6. Second, very few residual polynomials are known (see [MR10] for a summary).

In the case of the k-fold point multisingularity $\underline{\alpha} = A_0^k$, the $R_{A_0^k}(\ell)$ are known for $k \leq 7$ by a result of Marangell and Rimányi.

Theorem 6.4.11 ([MR10] Theorem 5.1)**.** *For $i \leq 6$, $R_{A_0^{i+1}}(\ell) = (-1)^i\, i!\, R_{A_i}(\ell - 1)$.*

Here $R_{A_i}(\ell)$ are the Thom polynomials of the Morin singularities introduced in §6.4.3, which can be computed by an algorithm in [BS12]. The steps of the algorithm and the resulting $R_{A_i}(\ell)$ can be found in §2.7.3 and Appendix B of [Joh16] in the special case when $c_{\ell+i} = 0$ for $i > 4$, which will be shown to hold in our setting in §6.4.5. In [MR10], Marangell and Rimányi combine their theorem with Kazaryan's formula and a computation of $R_{A_3}(\ell)$ to obtain a general quadruple point formula. Following their approach, we obtain a formula that generalizes the cohomological versions of the Herbert-Ronga double point formula (6.1) and Kleiman's triple point formula (6.2) as well as the general quadruple point formula.

Proposition 6.4.12. *Assume $f \colon X \to Y$ is an admissible map of codimension ℓ between complex manifolds. Then for $k \leq 7$,*

$$y_k = \frac{1}{k} \sum_{i=0}^{k-1} (-1)^i f_*\big(R_{A_i}(\ell - 1)\big)\, y_{k-1-i} \ \in H^{2k\ell}(Y, \mathbb{C}). \tag{6.4}$$

Proof. Setting $y_0 = [Y]$ for convenience, Kazaryan's formula (6.3) yields

$$x_{A_0^k} = \sum_{i=0}^{k-1} \binom{k-1}{i} R_{A_0^{i+1}}(\ell)\, f^*(y_{A_0^{k-1-i}}).$$

Pushing forward by f_* and using the projection formula, we get

$$y_{A_0^k} = \sum_{i=0}^{k-1} \binom{k-1}{i} f_*(R_{A_0^{i+1}}(\ell))\, y_{A_0^{k-1-i}}.$$

Now we use $y_{A_0^k} = k!\, y_k$ and Theorem 6.4.11 to deduce (6.4). □

6.4.5 Computation of multiple point classes

We complete the proof of Theorem 6.1.1 in the cases $n, r \geq 2$, $n \leq 7$, and $(n, r) \notin \{(2,2), (2,3), (3,2)\}$ by computing the number of ideal sheaf quotients of V using multiple point formulas and checking that the result matches the formula for $\chi\big(\mathcal{O}_{S^{[n]}}(L_n - r\frac{B}{2})\big)$ in (2.1).

The following ingredients are needed to compute $\#\mathrm{Quot}\big(V, (1, 0, -n)\big)$:

(1) a map $f\colon X \to Y$ of codimension $\ell = r - 1$ whose locus of n-fold points Y_n is in bijection with the sections of V^* vanishing at n points;

(2) an iterative formula for the cohomology class y_n of Y_n for $n \leq 7$, which counts the n-fold points if f is admissible;

(3) the residual polynomials $R_{A_i}(\ell - 1)$ that appear in the formula, assuming the vanishing $c_{\ell+i} = 0$ for $i > 4$;

(4) the relative Chern classes c_i that appear in the $R_{A_i}(\ell - 1)$;

(5) identities for computing push forwards of products of the c_i.

We have not yet explicitly described the last two items, so we do that now. The Chern classes $c_i = c_i(f^*T_Y/T_X)$ of the virtual normal sheaf of f are

$$\begin{aligned} c_i = \binom{r+1}{i}\xi^i &+ \left[\binom{r}{i-1}L + \binom{r+1}{i-1}K_S\right]\xi^{i-1} \\ &+ \left[\binom{r-1}{i-2}c_2(V^*) + \binom{r}{i-2}LK_S + \binom{r+1}{i-2}2(K_S^2 - 6\rho)\right]\xi^{i-2}, \end{aligned}$$

where ξ is the divisor class associated to $\mathcal{O}_X(1)$, ρ is the pullback of the point class p on S under $X \to S$, and L, K_S, and $c_2(V^*)$ denote the pullbacks of these classes from S. This formula is obtained by an elementary computation on the projective bundle $X = P(G)$ using the fact that Y is a projective space. First, we note that $f^*T_Y = (1+\xi)^{n(r-1)+1}$. Second, the sequences $0 \to T_{X/S} \to T_X \to \pi^*T_S \to 0$ and the relative Euler sequence $0 \to \mathcal{O}_X \to \pi^*G \otimes \mathcal{O}_X(1) \to T_{X/S} \to 0$, together with $c(T_S) = 1 - K_S + (12\chi(\mathcal{O}_S)p - K_S^2)$ yield $c(T_X) = c(\pi^*G \otimes \mathcal{O}_X(1))(1 - K_S + (12\rho - K_S^2))$. Third, we compute $c(\pi^*G \otimes \mathcal{O}_X(1))$ using 3.2.3b of [Ful98]. Finally, we extract the degree i part of the appropriate product to get the formula for c_i.

The push forward identities are simple since classes are determined by their degree on the projective space Y. Let H denote the hyperplane class on Y. Let δ be the divisor class on X obtained by pulling back a divisor class d on S. Then the projection formula yields

$$f_*(\xi^k) = ([S].c_2(V^*))H^{r-1+k}; \quad f_*(\delta\xi^k) = (d.L)H^{r+k}; \quad f_*(\rho\xi^k) = H^{r+1+k}.$$

Appendix A of [Joh16] contains computer code in Sage that computes (6.4) for $n \le 7$ using these ingredients. The results for $n \le 3$, with the substitution $L^2 = 2\chi(L) + L.K_S - 2$ to reduce the length of the output, are

$$\begin{aligned}
y_1 &= \chi(L);\\
y_2 &= \tfrac{1}{2}\chi(L)^2 + \chi(L)\big(-r^2+\tfrac{1}{2}\big) + K_S^2\big(-\tfrac{r^4}{24}+\tfrac{r^3}{12}+\tfrac{r^2}{24}-\tfrac{r}{12}\big) + L.K_S\big(-\tfrac{r^3}{6}+\tfrac{r}{6}\big)\\
&\quad + \tfrac{r^4}{4} - \tfrac{r^2}{4};\\
y_3 &= \tfrac{1}{6}\chi(L)^3 + \chi(L)^2\big(-r^2+\tfrac{1}{2}\big) + \chi(L)\big(\tfrac{7r^4}{4}-\tfrac{7r^2}{4}+\tfrac{1}{3}\big)\\
&\quad + \chi(L)K_S^2\big(-\tfrac{r^4}{24}+\tfrac{r^3}{12}+\tfrac{r^2}{24}-\tfrac{r}{12}\big) + \chi(L)L.K_S\big(-\tfrac{r^3}{6}+\tfrac{r}{6}\big)\\
&\quad + K_S^2\big(\tfrac{97r^6}{720}-\tfrac{17r^5}{80}-\tfrac{31r^4}{144}+\tfrac{5r^3}{16}+\tfrac{29r^2}{360}-\tfrac{r}{10}\big) + L.K_S\big(\tfrac{17r^5}{40}-\tfrac{5r^3}{8}+\tfrac{r}{5}\big)\\
&\quad - \tfrac{2r^6}{3} + r^4 - \tfrac{r^2}{3}.
\end{aligned}$$

These formulas for y_n match $\chi\big(\mathcal{O}_{S^{[n]}}(L_n - r\tfrac{B}{2})\big)$, as do the formulas for $4 \le n \le 7$, which can be found in Appendix B of [Joh16]. This computation completes the proof of Theorem 6.1.1.

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