POSITIVITY PROPERTIES OF ALGEBRAIC SUBVARIETIES

by

Chung Ching Lau

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

August 2017

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Chung Ching Lau**

has been approved by the following supervisory committee members:

Tommaso de Fernex,	Chair(s)	**20 Apr 2017** Date Approved
Aaron Bertram,	Member	**20 Apr 2017** Date Approved
Christopher Hacon,	Member	**20 Apr 2017** Date Approved
Anurag Singh,	Member	**20 Apr 2017** Date Approved
Kevin Tucker,	Member	**20 Apr 2017** Date Approved

by **Peter Trapa**, Chair/Dean of

the Department/College/School of **Mathematics**

and by **David B. Kieda**, Dean of The Graduate School.

ABSTRACT

In this dissertation, we take a cohomological approach to study positive properties of higher codimensional subvarieties. First, we prove generalizations of Fujita vanishing theorems for q-ample divisors. We then apply them to study positivity of subvarieties with nef normal bundle in the sense of intersection theory. We also study the positivity of the cycle class of an ample curve or a curve with ample normal bundle.

After Ottem's work on ample subschemes, we introduce the notion of a nef subscheme (resp. locally ample subscheme), which generalizes the notion of a subvariety with nef normal bundle (resp. ample normal bundle). We show that restriction of a pseudoeffective (resp. big) divisor to a nef subvariety is pseudoeffective (resp. big). We also show that ampleness, nefness and locally ampleness are transitive properties.

We define the weakly movable cone as the cone generated by the pushforward of cycle classes of nef subvarieties via proper surjective maps. This cone contains the movable cone and shares similar intersection-theoretic properties with it, thanks to the aforementioned properties of nef subvarieties.

On the other hand, we prove that if $Y \subset X$ is an ample subscheme of codimension r and $D|_Y$ is q-ample, then D is $(q+r)$-ample. This is analogous to a result proved by Demailly-Peternell-Schneider.

We also show that the cycle class of an ample curve (resp. locally ample curve) lies in the interior of the movable cone of curves (cone of curves).

For my parents and my wife.

CONTENTS

ACKNOWLEDGEMENTS

I would like to thank my advisor Tommaso de Fernex for initiating my exploration of ample subvarieties. This dissertation would not have existed without his constant support and patience, especially during the time when I struggled.

My research was partially supported by the NSF.

CHAPTER 1

INTRODUCTION

The concept of ampleness of a divisor or a line bundle is central in the subject of algebraic geometry. A line bundle $\mathcal{L}$ on a scheme X is called very ample if the global sections of the line bundle give an embedding into a projective space. A line bundle $\mathcal{L}$ is ample if $\mathcal{L}^{\otimes k}$ for some $k > 0$ is very ample. Ampleness of a divisor plays an important role in intersection theory. For example, Nakai-Moishezon theorem says that a divisor D on a projective variety X is ample if and only if for any closed subvariety $Z \subset X$, $D^{\dim Z} \cdot Z > 0$. Ampleness of a line bundle is also crucial in various vanishing theorems on cohomologies. The Serre vanishing criterion says that a line bundle $\mathcal{L}$ is ample if and only if for any coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$ for $i > 0$ and $m \gg 0$.

Weakening the Serre vanishing condition, a line bundle $\mathcal{L}$ is defined to be *q-ample* if given any coherent sheaf $\mathcal{F}$, there is an m_0 such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q$ and $m > m_0$. After the works of Andreotti-Grauert [2], Sommese [29] and Demailly-Peternell-Schneider [7] on q-ample divisors, Totaro established the basic, yet not elementary, properties of q-ample divisors [30].

In Hartshorne's extensive work [16], he attempted to develop the notion of an ample subvariety, which should give a generalization to higher codimension the concept of an ample divisor. Although a definition was not given in his work, he listed a number of properties that an ample subvariety ought to satisfy, and studied their relationship. We will state two of the properties here. The first one is that an ample subvariety should have ample normal bundle; the second one is that the cohomological dimension of the complement of the subvariety $U := X \backslash Y$ is $r - 1$, where r is the codimension of Y in X. The cohomological dimension of a scheme U, $cd(U)$, is defined to be the largest integer i such that there is a coherent sheaf $\mathcal{F}$ on U with $H^i(U, \mathcal{F}) \neq 0$. Note that U is affine if and only if the cohomological dimension of U equals to 0. Thus, the requirement of $cd(U) = r-1$ generalizes the fact that the complement of an ample divisor is affine. Recently, Ottem discovered what is probably the right notion of an ample subscheme [26]. He defined a subscheme of Y of codimension r of a projective scheme to be *ample* if the exceptional divisor in the blowup of X along Y is $(r-1)$-ample. Note that the exceptional divisor is anti-ample along the fibers over Y, which has dimension at least $(r-1)$. In this sense, $(r-1)$-ampleness is as positive as the exceptional divisor can get. When the subscheme Y is locally complete intersection, this definition holds if and only if both of the properties stated above due to Hartshorne holds. Ottem also showed that if the ground field is of characteristic zero, then the zero locus of a global section of an ample vector bundle is an ample subscheme [26, Proposition 4.5].

In Chapter 3, we prove two generalized versions of Fujita vanishing theorem for q-ample divisors (Theorem 3.3.3 and Proposition 3.3.4). One of the applications of these theorems, stated below, sheds more light on the connection between q-ample divisors and ample subschemes. It says q-ampleness of a divisor may be detected by restricting it to an ample subscheme.

Theorem 1.1. *Let X be a projective scheme of dimension n. Let Y be an ample subscheme of X of codimension r. Suppose $\mathcal{L}$ is a line bundle on X, and that its restriction $\mathcal{L}|_Y$ to Y is q-ample, then $\mathcal{L}$ is $(q+r)$-ample.*

We now move on to study a weaker positivity condition of a subscheme. Given a locally complete intersection subvariety $Y \subset X$ with nef normal bundle, we would like to understand its positivity properties in terms of intersection theory. Fulton and Lazarsfeld [14] gave an answer to this: They showed that if $\dim Y + \dim Z \geq \dim X$, then $\deg_H(Y \cdot Z) \geq 0$. Here H is an ample divisor.

Now let $Y \subset X$ be an arbitrary subscheme of codimension r and let E be the exceptional divisor in $Bl_Y X$. We say that Y is *nef* (resp. *locally ample*) if $(E+\epsilon A)|_E$ is $(r-1)$-ample, (resp. $E|_E$ is $(r-1)$-ample) where is A is an ample divisor and $0 < \epsilon \ll 1$. This definition is inspired by Ottem's definition of an ample subscheme [26]. Assuming that Y is locally complete intersection in X, Y is nef (resp. locally ample) if and only if Y has nef (resp. ample) normal bundle. We show that

Theorem 1.2. *Let $\iota : Y \hookrightarrow X$ be a nef subvariety of codimension r of a projective variety X. Then the natural map $\iota^* : N^1(X)_{\mathbf{R}} \to N^1(Y)_{\mathbf{R}}$ induces $\iota^* : \overline{Eff}^1(X) \to \overline{Eff}^1(Y)$ and $\iota^* : Big(X) \to Big(Y)$.*

When Y is a curve with nef normal sheaf, this is a result of Demailly-Peternell-Schneider [7, Theorem 4.1]. We also show that nefness and ampleness are transitive properties without any assumptions on smoothness, thus generalizing Ottem's result [26, Proposition 6.4].

Theorem 1.3. *Let X be a projective scheme of dimension n. If Y is an ample (resp. nef or locally ample) subscheme of X and Z is an ample (resp. nef or locally ample) subscheme of Y, then Z is ample (resp. nef or locally ample) in X.*

We then study the cycle classes of nef subvarieties. We use this new notion of nef subvarieties to introduce the notion of the weakly movable cone, $\overline{WMov}_d(X)$. We define it as the closure of the convex cone that is generated by pushforward of cycle classes of nef subvarieties of dimension d via proper surjective morphisms. We show that the weakly movable cone shares similar properties to that of the movable cone of d-cycles, $\overline{Mov}_d(X)$.

Theorem 1.4. *Let X be a projective variety of dimension n. For $1 \leq d \leq n-1$,*

1. $\overline{Mov}_d(X) \subseteq \overline{WMov}_d(X)$ *and* $\overline{Mov}_1(X) = \overline{WMov}_1(X)$.
2. $\overline{Eff}^1(X) \cdot \overline{WMov}_d(X) \subseteq \overline{Eff}_{d-1}(X)$.
3. *Let H be a big Cartier divisor, $\alpha \in \overline{WMov}_d(X)$. If $H \cdot \alpha = 0$, then $\alpha = 0$.*
4. $Nef^1(X) \cdot \overline{WMov}_d(X) \subseteq \overline{WMov}_{d-1}(X)$.

Analogous statements of 2, 3 and 4 hold for the movable cone [10, Lemma 3.10]. One can ask whether in general the two cones $\overline{Mov}_d(X)$ and $\overline{WMov}_d(X)$ are the same. This is true if and only if the cycle class of any nef subvariety lies in the movable cone. This question is closely related to the Hartshorne's conjecture A. Hartshorne's conjecture A states that if Y is a smooth subvariety with ample normal bundle of a smooth projective variety X, nY moves in a large algebraic family for n large. A counterexample to this conjecture was given by Fulton and Lazarsfeld [13].

It is unclear what kind of intersection theoretic statements one should expect if we further assume that Y has ample normal bundle. Hartshorne's conjecture B states that if Y and Z are subvarieties with ample normal bundle of a projective variety X and that they have complementary dimension, then $Y \cap Z$ is non-empty. This conjecture is still open. Voisin gave an example of a subvariety with ample normal bundle such that its cycle class lies on the boundary of the pseudoeffective cone of cycles [31]. On the other hand, Ottem showed that the cycle class of a curve with ample normal bundle lies in the interior of the cone of curves [27]. We shall generalize his theorem to locally ample curves in theorem 1.5.

In Chapter 5, assuming $Y \subset X$ is a locally ample subvariety, we study the behaviour of the restriction map $\iota^* : \overline{Eff}^1(X) \to \overline{Eff}^1(Y)$ at the boundary of $\overline{Eff}^1(X)$. We prove that if $\iota^*\eta$ is not big, then the numerical dimension of η, $\kappa_\sigma(\eta)$, is bounded above by that of $\iota^*\eta$, i.e. $\kappa_\sigma(\eta) \leq \kappa_\sigma(\iota^*\eta)$. From this, we deduce that

Theorem 1.5. *The cycle class of an ample curve lies in the interior of the movable cone of curves and the cycle class of a locally ample curve lies in the interior of the cone of curves.*

All schemes in this work are over a field of characteristic 0.

CHAPTER 2

PRELIMINARIES

In this chapter, we will recall some known results on q-ample divisors and ample subschemes; we will also introduce the notions of nef and locally ample subschemes. Some basic facts on the dualizing sheaf and numerical dimension will also be discussed.

2.1 q-ample divisors

In this section, we briefly review some basic facts on q-ample divisors. Let us recall the definition of a q-ample line bundle.

Definition 2.1.1 (q-ample line bundle [7],[30])**.** Let X be a projective scheme. A line bundle $\mathcal{L}$ is *q-ample* if for any coherent sheaf $\mathcal{F}$ on X, there is an m_0 such that

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q$ and $m > m_0$.

Lemma 2.1.2 ([26, Lemma 2.1])**.** *Let X be a projective scheme and fix an ample line bundle $\mathcal{O}(1)$ on X. A line bundle $\mathcal{L}$ is q-ample if and only if for any $l \geq 0$,*

$$H^i(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}(-l)) = 0$$

for $m \gg 0$.

We shall start with the definition of a Koszul-ample line bundle. The details are not very important in this dissertation, but they are included for the sake of completeness. One useful fact is that any large tensor power of an ample line bundle is $2n$-Koszul-ample, where n is the dimension of the underlying projective scheme [4].

Definition 2.1.3 (Koszul-ampleness [30, Section 1])**.** Let X be a projective scheme of dimension n, and that the ring of regular function $\mathcal{O}(X)$ on X is a field (e.g. X is connected and reduced). Let $k = \mathcal{O}(X)$. Given a very ample line bundle $\mathcal{O}_X(1)$, we say that it is *N-Koszul ample* if the homogeneous coordinate ring $A = \bigoplus_j H^0(X, \mathcal{O}_X(j))$ is N-Koszul, i.e. there is a resolution

$$\cdots \to M_1 \to M_0 \to k \to 0$$

where M_i is a free A-module, generated in degree i, where $i \leq N$.

Definition 2.1.4 (q-T-ampleness [30, Definition 6.1])**.** Let X be a projective scheme of dimension n. Suppose the ring of regular functions of X, $\mathcal{O}(X)$ is a field. We fix a $2n$-Koszul-ample line bundle $\mathcal{O}_X(1)$ on X. We say that a line bundle $\mathcal{L}$ is *q-T-ample* if there is a positive integer N, such that

$$H^{q+i}(X, \mathcal{L}^{\otimes N}(-n-i)) = 0,$$

for $0 \leq i \leq n-q$.

Totaro showed that

Theorem 2.1.5 ([30, Theorem 6.3])**.** *Under the same assumptions as in definition 2.1.4, a line bundle is q-ample if and only if it is q-T-ample.*

Even though the q-T-ampleness notion may appear technical, the equivalence is the key result of his paper. It reduces the problem of showing a line bundle being q-ample to checking the vanishing of finitely many cohomology groups. Using the notion of q-T-ampleness, Totaro showed that q-ampleness is Zariski open [30, Theorem 8.1]. We can extend the definition to **R**-Cartier divisors.

Definition 2.1.6 (q-ample **R**-divisors)**.** Let X be a projective scheme. An **R**-Cartier divisor on X is *q-ample* if D is numerically equivalent to $cL + A$ with L a q-ample line bundle, $c \in \mathbf{R}_{>0}$, A an ample **R**-Cartier divisor.

Based on the work of Demailly, Peternell and Schneider, Totaro also proved that

Theorem 2.1.7 ([30, Theorem 8.3])**.** *An integral divisor is q-ample if and only if its associated line bundle is q-ample. The q-ample* **R***-divisors in* $N^1(X)_{\mathbf{R}}$ *define an open cone (but not convex in general) and that the sum of a q-ample* **R***-divisor and a r-ample* **R***-divisor is* $(q+r)$*-ample.*

These facts are nontrivial. We shall use the notion of q-T-ampleness to prove proposition 2.1.9.

We note that $(n-1)$-ampleness admits a pleasant geometric interpretation, which we shall use a few times in this paper.

Theorem 2.1.8 ([30, Theorem 9.1])**.** *Let X be a projective variety of dimension n. A line bundle $\mathcal{L}$ on X is $(n-1)$-ample if and only if $[\mathcal{L}^\vee] \in N^1(X)$ does not lie in the pseudoeffective cone.*

We will need the following result on the positivity of the pullback of a q-ample divisor later.

Proposition 2.1.9 (Pullback of a q-ample divisor)**.** *Let $f : X' \to X$ be a morphism of projective schemes. Let D be a q-ample divisor on X, and let A be a relatively (to f) ample divisor on X'. Then $mf^*D + A$ is q-ample, for $m \gg 0$.*

Proof. First, let us show that it suffices to prove the proposition in the case when both X and X' are irreducible and reduced. Note that a line bundle is q-ample on X' if and only if it is q-ample when restricting to each irreducible component of X' [26, Proposition 2.3.i,ii]. We can now assume X' is integral. Let X_1 be an irreducible component of X that contains the image of X'. The map $X' \to X$ factors through X_1, and $D|_{X_1}$ is again q-ample.

Now we can assume both X and X' are integral. In fact, we shall prove that $mf^*D + A$ is q-T-ample, for $m \gg 0$. In other words, we shall show that for $m \gg 0$, there is a positive integer r, such that

$$H^{q+a}(X', \mathcal{O}_{X'}(r(mf^*D + A)) \otimes \mathcal{O}_{X'}(-n-a)) = 0$$

for $1 \le a \le n-q$. Here $\mathcal{O}_{X'}(1)$ is a $2n$-Koszul-ample line bundle on X', where $n = \dim X'$.

Using the relative ampleness of A, one can find an integer r such that

$$R^j f_*(\mathcal{O}_{X'}(rA) \otimes \mathcal{O}_{X'}(-n-a)) = 0,$$

for $j > 0$ and $1 \le a \le n-q$. The Leray spectral sequence then says

$$\begin{aligned} &H^{q+a}(X', \mathcal{O}_{X'}(r(mf^*D + A)) \otimes \mathcal{O}_{X'}(-n-a)) \\ &\qquad \cong H^{q+a}(X, \mathcal{O}_X(rmD) \otimes f_*(\mathcal{O}_{X'}(rA) \otimes \mathcal{O}_{X'}(-n-a))). \quad (2.1) \end{aligned}$$

The right-hand side group vanishes for all big m, by the q-ampleness of rD. $\square$

2.2 Ample subschemes

In this section, we shall discuss some basic facts on ample subschemes. Let us first review the definition of ample subscheme, given by Ottem:

Definition 2.2.1 (Ample subscheme [26, Definition 3.1])**.** Let X be a projective scheme. Let Y be a closed subscheme of X of codimension r and let $\pi : Bl_Y X \to X$ be the blowup of X with center Y. We say that Y is an *ample subscheme of* X *if the exceptional divisor* E *of* π *is* $(r-1)$*-ample in* $Bl_Y X$.

We shall follow his definition in this paper. An example of an ample subscheme would be the zero locus (of codimension r) of a section of an ample vector bundle of rank r [26, Proposition 4.5]. On the other hand, many good properties listed in Hartshorne's book [16, p.XI] are satisfied under this definition. Before stating some of these properties, we need the definition of *cohomological dimension* of a scheme U: it refers to the number

$$cd(U) := \max\{i \in \mathbb{Z}_{\geq 0} |\, H^i(U, \mathcal{F}) \neq 0, \textit{for some coherent sheaf } \mathcal{F}.\}$$

Theorem 2.2.2. *Let Y be a smooth closed subvariety of a smooth projective variety X.*

1. *Y is ample if and only if its normal bundle is ample and the cohomological dimension of the complement is $r-1$.*

Assume further that Y is an ample subscheme in X. Then

2. *Generalized Lefschetz hyperplane theorem with rational coefficient holds, i.e. $H^i(X, \mathbb{Q}) \to H^i(Y, \mathbb{Q})$ is an isomorphism for $i < \dim Y$ and is an injection for $i = \dim Y$.*

3. *Y is numerically positive, i.e. $Y \cdot Z > 0$ for any effective cycle Z of dimension r.*

4. *$H^i(X, \mathcal{F}) \to H^i(\hat{X}, \hat{\mathcal{F}})$ is an isomorphism for $i < \dim Y$ and is injective for $i = \dim Y$. Here $\hat{X}$ is the formal completion of X along Y, $\mathcal{F}$ is a locally free sheaf on X and $\hat{\mathcal{F}}$ is its restriction to $\hat{X}$.*

Proof. [26, Theorem 5.4], [26, Corollary 5.3] and [16, Chapter III, Theorem 3.4] give 1, 2 and 4, respectively. For 3, since the cohomological dimension of $(X - Y) = r - 1$, Y meets any effective cycle of dimension r. We can then apply the result of Fulton and Lazarsfeld [23, Corollary 8.4.3], which says if Y has ample normal bundle and Y meets Z, where Z is an effective cycle of complementary dimension to that of Y, then $Y \cdot Z > 0$. □

The above list of properties is incomplete; for a more complete picture, c.f. [26].

2.3 Nef and locally ample subschemes

In this section, we shall define the notion of nef and locally ample subschemes. They are generalizations of the notions of subvarieties with nef and ample normal bundle, respectively. We shall study them more closely in later sections. To streamline the arguments, we first make the following definition, which generalizes the notion of a nef divisor.

Definition 2.3.1 (q-almost ample)**.** Let X be a projective scheme, D an $\mathbf{R}$-Cartier divisor on X, A an ample Cartier divisor on X. We say that D is *q-almost ample* if $D + \epsilon A$ is q-ample for $0 < \epsilon \ll 1$.

The definition is clearly independent of the choice of A. Note that D is 0-almost ample if and only if D is nef.

Ottem observed that ampleness of a vector bundle $\mathcal{E}$ can be expressed in terms of q-ampleness of $\mathbb{P}(\mathcal{E}^\vee)$ [26, Proposition 4.1]. We give the straightforward generalization to the case when the vector bundle is nef.

Proposition 2.3.2. *Let $\mathcal{E}$ be a vector bundle of rank r on a projective scheme X. Then $\mathcal{E}$ is ample (resp. nef) if and only if $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$-ample. (resp. $(r-1)$-almost ample.)*

Proof. Let $\pi' : \mathbb{P}(\mathcal{E}^\vee) \to X$ and $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the natural projection maps. Using [17, Exercise III.8.4], we have for $m > 0$,

$$R^j \pi'_* \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-r) \cong \begin{cases} Sym^m \mathcal{E} \otimes \det(\mathcal{E}) \text{ for } j = r-1 \\ 0 \text{ otherwise.} \end{cases}$$

Here we implicitly used the isomorphism $(Sym^m \mathcal{E}^\vee)^\vee \cong Sym^m \mathcal{E}$ which holds when the ground field is of characteristic 0.

Therefore, we have the isomorphisms

$$\begin{aligned} H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-r) \otimes \pi'^*(\mathcal{F} \otimes \det \mathcal{E}^\vee)) &\cong H^i(X, Sym^m \mathcal{E} \otimes \mathcal{F}) \\ &\cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) \otimes \pi^* \mathcal{F}), \quad (2.2) \end{aligned}$$

where $\mathcal{F}$ is locally free on X, $i > 0$ and $m > 0$.

If $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$-ample, then the above observation shows that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is ample. Indeed, any line bundle on $\mathbb{P}(\mathcal{E})$ can be expressed as $\pi^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{E})}(l)$.

Suppose $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$-almost ample. Choose an ample divisor A on X. To show that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef, we want to check that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(k) \otimes \pi^*\mathcal{O}(A)$ is ample for all $k > 0$. By replacing A with a large multiple, we may assume $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1) \otimes \pi^*\mathcal{O}(A)$ is ample. We apply lemma 2.1.2 and fix an $l \geq 0$. Observe that we have the following isomorphisms given by (2.2):

$$\begin{aligned} &H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(mk-l) \otimes \pi^*\mathcal{O}((m-l)A)) \\ &\qquad \cong H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-mk-r+l) \otimes \pi'^*(\mathcal{O}((m-l)A) \otimes \det \mathcal{E}^\vee)), \end{aligned}$$

where $i, m > 0$. The latter term vanishes for $m \gg 0$ since $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-k) \otimes \pi'^*\mathcal{O}(A)$ is $(r-1)$-ample for any $k > 0$. This shows that $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is nef.

Similarly, we may also assume $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(1) \otimes \pi'^*\mathcal{O}(A)$ is ample.

If $\mathcal{E}$ is ample, we fix an $l \geq 0$, we have the following isomorphisms of cohomology groups,

$$\begin{aligned} &H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-m-l) \otimes \pi'^*\mathcal{O}(-lA)) \\ &\qquad \cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(m+l-r) \otimes \pi^*(\mathcal{O}(-lA) \otimes \det \mathcal{E})), \end{aligned}$$

the latter term vanishes for $i > 0$ and $m \gg 0$, which says $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$-ample.

If $\mathcal{E}$ is nef, we fix $l \geq 0$ again, we observe that for any $k > 0$, we have the following isomorphism of cohomology groups,

$$\begin{aligned} &H^{r-1+i}(\mathbb{P}(\mathcal{E}^\vee), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-mk-l) \otimes \pi'^*\mathcal{O}((m-l)A))) \\ &\qquad \cong H^i(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(mk+l-r) \otimes \pi^*(\mathcal{O}((m-l)A) \otimes \det \mathcal{E})), \end{aligned}$$

for $i, m > 0$. The latter term vanishes for $m \gg 0$. This says that $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-k) \otimes \mathcal{O}(A)$ is $(r-1)$-ample for any $k > 0$, which means $\mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(-1)$ is $(r-1)$-almost ample. □

The augmented base locus gives us another measure of how far a divisor is being ample.

Definition 2.3.3 (Augmented base locus [9, Definition 1.2])**.** The augmented base locus of an **R**-divisor D on X is the Zariski-closed subset:

$$\mathbf{B}_+(D) = \bigcap_{D=A+E} SuppE$$

where the intersection is taken over all decompositions $D = A + E$ such that A is ample and E is effective.

Proposition 2.3.4 (Positivity of normal bundle vs. positivity of exceptional divisor)**.** *Let $Y \subset X$ be a subscheme of codimension r. Then the normal bundle of the exceptional divisor E in $Bl_Y X$, $\mathcal{O}_E(E)$ is $(r-1)$-almost ample if and only if $E \subset Bl_Y X$ is $(r-1)$-almost ample.*

Proof. The "if" part of the statment is clear, since restriction of a q-ample divisor to a subscheme is always q-ample. For the "only if" part, observe that $\mathbf{B}_+(E + \epsilon A) \subseteq SuppE$ for $0 < \epsilon \ll 1$, where A is an ample divisor on X. Now we can apply Brown's theorem [6, Theorem 1.1] to $E + \epsilon A$, which says that an **R**-divisor D is q-ample if and only if $D|_{\mathbf{B}_+(D)}$ is q-ample. □

Definition 2.3.5 (Nef and locally ample subschemes)**.** Let Y be a closed subscheme of codimension r of X, a projective scheme, and let E be the exceptional divisor in $Bl_Y X$. Then we say that Y is *nef* (resp. *locally ample*) if $\mathcal{O}_E(E)$ is $(r-1)$-almost ample (resp. $(r-1)$-ample).

Remark 2.3.6*.* Proposition 2.3.4 says that Y is a nef subscheme if and only if E is $(r-1)$-almost ample in $Bl_Y X$. If Y is locally complete intersection in X, then Y is nef (resp. locally ample) if and only if the normal bundle $\mathcal{N}_{Y/X}$ is nef (resp. ample) (proposition 2.3.2). The advantage of making this more general definition, without requiring Y to be locally complete intersection, is to include more subschemes that are apparently "positive", for example, a closed point that is not necessarily nonsingular, or if Y is a smooth subvariety with nef normal bundle, the subscheme of X defined by a power of ideal sheaf of Y is also considered as nef in this definition.

The following proposition is the direct generalization of [26, Proposition 3.4].

Proposition 2.3.7 (Equidimensionality of nef subschemes)**.** *Suppose Y is a nef subscheme of X. Then the restriction of the blowup morphism to E, $\pi|_E : E \to Y$, is equidimensional. In* *particular, Y is pure dimensional.*

Proof. Suppose $Y \subset X$ has codimension r. Let $y \in Y$ be a closed point, we want to show $Z := \pi^{-1}(y)$ is of dimension $(r-1)$. Note that E has dimension $n-1$, where $n = \dim X$. This implies $\dim Z \geq r-1$.

On the other hand, $-E$ is π-ample. In particular, $(-E - \epsilon A)|_Z$ is ample for $1 \gg \epsilon > 0$, where A is an ample divisor on E. We also know that $\mathcal{O}_E(E + \epsilon A)$ is $(r-1)$-ample, for $1 \gg \epsilon > 0$. By theorem 2.1.7, this forces Z to have dimension $(r-1)$. □

Proposition 2.3.8 (Inverse image of nef and locally ample subschemes)**.** *Suppose Y is a nef subscheme of X of codimension r, $p : X' \to X$ a morphism from an equidimensional projective scheme X'. If $p^{-1}(Y)$ has codimension r in X', then $p^{-1}(Y)$ is nef in X'. In particular, if p is equidimensional, $p^{-1}(Y)$ is nef. Moreover, assuming Y is locally ample in X and p is a closed immersion. If $p^{-1}(Y)$ has codimension r in X', then $p^{-1}(Y)$ is locally ample in X'.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} Bl_{p^{-1}(Y)}(X') & \xrightarrow{\tilde{p}} & Bl_Y(X) \\ \downarrow & & \downarrow \\ X' & \xrightarrow[p]{} & X, \end{array}$$

with $\tilde{p}$ induced by the universal property of blowup and $\tilde{p}^*(E) = E'$, where E and E' are exceptional divisors in the respective blowups. We can now apply proposition 2.1.9 to conclude the proof. □

Proposition 2.3.9. *Let Y be an ample (resp. nef or locally ample) subscheme of codimension r of X. Let Z be a closed subscheme of X. If $Y \cap Z$ has codimension r in Z, then $Y \cap Z$ is an ample (resp. nef or locally ample) subscheme of Z.*

Proof. Indeed, we have the following commutative diagram

$$\begin{array}{ccc} Bl_{Y\cap Z}Z & \hookrightarrow & Bl_Y X \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi_X} \\ Z & \hookrightarrow & X. \end{array}$$

The proposition follows from the fact that the exceptional divisor of π_Z is the restriction of the exceptional divisor E of π_X. □

2.4 Dualizing sheaf

We shall need to use the dualizing sheaf to study the σ-dimension of a divisor.

Definition 2.4.1 (Dualizing sheaf [17, p.241])**.** Let X be a projective scheme of dimension n. A *dualizing sheaf* for X is a coherent sheaf ω_X, together with a trace map $t : H^n(X, \omega) \to k$ to the ground field k, such that for any coherent sheaf $\mathcal{F}$ on X, the natural pairing

$$H^n(X, \mathcal{F}) \times Hom(\mathcal{F}, \omega_X) \to H^n(X, \omega_X),$$

followed by t, induces an isomorphism

$$Hom(\mathcal{F}, \omega_X) \cong H^n(X, \mathcal{F})^\vee$$

of k-vector spaces.

Proposition 2.4.2. *[17, Proposition 7.2, 7.5] Let X be a projective scheme of dimension n. Then the dualizing sheaf for X exists and is unique up to unique isomorphism.*

We now show that a dualizing sheaf can be embedded into a sufficiently ample line bundle. The proof can be found in the proof of [30, Theorem 9.1], but we include it here for the sake of convenience.

Lemma 2.4.3 (Embedding a dualizing sheaf into a line bundle)**.** *Let X be a projective variety of dimension n. Given an ample divisor H on X. Then ω_X is torsion-free. Moreover, there is l such that there is an embedding $\omega_X \hookrightarrow \mathcal{O}_X(lH)$.*

Proof. Let us first show that ω_X is torsion-free. Indeed, let $\mathcal{T} \subset \omega_X$ be the torsion subsheaf. Then

$$Hom(\mathcal{T}, \omega_X) \cong H^n(X, \mathcal{T})^\vee = 0.$$

The last equality follows from the fact that $\mathcal{T}$ is supported at a proper closed subscheme of X.

As ω_X is generically a line bundle, $\omega_X^\vee \neq 0$. For l large, there is a nontrivial section $s \in H^0(X, \omega^\vee \otimes \mathcal{O}_X(lH))$. This induces a nontrivial map $\omega_X \to \mathcal{O}_X(lH)$, which has to be an injection, since ω_X is torsion free of rank 1. □

2.5 σ-dimension

We will discuss the notion of σ-dimension, also known as numerical dimension, of an **R**-Cartier **R**-divisor.

Definition 2.5.1 (σ-dimension)**.** Let X be a projective variety. Let $D = \sum a_i C_i$ be an **R**-Cartier **R**-divisor, where $a_i \in \mathbf{R}$ and C_i's are integral Cartier divisor and let H be any integral Cartier divisor. We then define

$$\kappa_\sigma(D, H) = \max\{l \in \mathbb{Z} | \limsup_{t \to \infty} \frac{h^0(X, \mathcal{O}_X(\sum \lfloor t a_i \rfloor C_i + H))}{t^l} > 0\}$$

and

$$\kappa_\sigma(D) := \max_{H \text{ integral Cartier}} \{\kappa_\sigma(D, H)\}.$$

The quantity $\kappa_\sigma(D)$ measures the positivity of an **R**-Cartier **R**-divisor that lies on the boundary of the pseudoeffective cone. However, this definition looks slightly different from the one that appeared in the literature ([25], [24] and [8]), as we do not assume X to be smooth or even normal. We shall prove in proposition (2.5.3) that the definition is well-defined, i.e. independent of the decomposition $D = \sum a_i C_i$; is a numerical invariant and agrees with the usual definition when X is smooth. Nakayama's proof that the σ-dimension is a numerical invariant relies on an Angehrn-Siu type argument, which requires smoothness on X. One can apply resolution of singularities on a singular X and reduce to the case when X is smooth. We shall give a proof that has no assumptions on singularities on X using q-ample divisors.

Lemma 2.5.2. *Let X be a projective variety. Let $\mathcal{B} \subset N^1(X)_{\mathbf{R}}$ be a bounded subset. Then there is an integral Cartier divisor H such that for any integral Cartier divisor C with $[C] \in \mathcal{B}$,*

$$H^0(X, \mathcal{O}_X(H - C)) \neq 0.$$

Proof. Let A be an ample divisor on X. Fix a $(2n)$-Koszul-ample line bundle $\mathcal{O}_X(1)$ on X. Let ω_X be the dualizing sheaf of X. There is an embedding $\omega_X \hookrightarrow \mathcal{O}_X(j)$ for some j, and that $\dim Supp(coker(\omega_X \hookrightarrow \mathcal{O}_X(j))) \leq n-1$ by lemma 2.4.3.

One can choose a sufficiently large m such that $\mathcal{O}_X(mA - C) \otimes \mathcal{O}_V(-j-n-1)$ is ample for any integral Cartier divisor C with $[C] \in \mathcal{B}$. In particular, $\mathcal{O}_X(-mA + C) \otimes \mathcal{O}_X(j + n +$

1) is not $(n-1)$-ample. This implies $h^n(X, \mathcal{O}_X(-mA+C) \otimes \mathcal{O}_X(j)) \neq 0$ [30, Theorem 6.3], and $h^0(X, \mathcal{O}_X(mA-C)) = h^n(X, \mathcal{O}_X(-mA+C) \otimes \omega_X) \neq 0$. □

Proposition 2.5.3. *Let X be a projective variety and let D be a pseudoeffective **R**-Cartier **R**-divisor on X. Then*

1. *The definition of $\kappa_\sigma(D)$ does not depend on the decomposition $D = \sum a_i C_i$. In fact, if $D \equiv D'$, then $\kappa_\sigma(D) = \kappa_\sigma(D')$.*

2. *Assuming that X is smooth,*

$$\kappa_\sigma(D) = \max_{H \text{ integral Cartier}} \{\max\{l \in \mathbb{Z} \mid \limsup_{m\to\infty} \frac{h^0(X, \mathcal{O}_X(\lfloor mD \rfloor + H))}{m^l} > 0\}\}.$$

*The right-hand side of this equation is the usual definition of the $\kappa_\sigma(D)$ ([25],[24],[8]). Here we are rounding down D as an **R**-Weil divisor.*

Proof. For (1), suppose $D \equiv D'$, $D = \sum a_i C_i$ and $D' = \sum a'_i C'_i$. By lemma 2.5.2, there is an integral Cartier divisor H' such that $\mathcal{O}_X(H'+C)$ is effective for any integral Cartier $C \equiv \sum r_i C_i + \sum r'_j C'_j$ where $r_i, r'_j \in [-2, 2]$. Given any integral Cartier divisor H, write $\sum \lfloor m a'_i \rfloor C'_i + H + H'$ as

$$\sum \lfloor m a_i \rfloor C_i + H + (\sum \lfloor m a'_i \rfloor C'_i - mD') + (mD - \sum \lfloor m a_i \rfloor C_i) + (mD' - mD) + H'.$$

This implies $h^0(X, \mathcal{O}_X(\sum \lfloor m a_i \rfloor C_i + H)) \leq h^0(X, \mathcal{O}_X(\sum \lfloor m a'_i \rfloor C'_i + H + H'))$. We can reverse the roles of D and D' and conclude (1).

For (2), D is expressed uniquely as $\sum a_i \Gamma_i$, where Γ_i's are prime divisors (which are Cartier by the smoothness assumption), $a_i \in \mathbf{R}$. We have $\lfloor mD \rfloor = \sum \lfloor m a_i \rfloor \Gamma_i$, the equality then follows from (1). □

Thanks to Proposition 2.5.3 (1), we may refer to $\kappa_\sigma(\eta)$, where $\eta \in N^1(X)_\mathbf{R}$, without ambiguity.

Here are some of the basic properties of $\kappa_\sigma(D)$. The proof is essentially the same as the one given in [25, Proposition V.2.7].

Proposition 2.5.4 (Basic properties)**.** *Let X be a projective variety of dimension n and let $\eta \in N^1(X)_\mathbf{R}$ be a pseudoeffective class.*

1. *If* $f : X' \to X$ *is a surjective morphism from a projective variety, then* $\kappa_\sigma(\eta) = \kappa_\sigma(f^*(\eta))$.

2. $0 \leq \kappa_\sigma(\eta) \leq n$.

3. $\kappa_\sigma(\eta) = n$ *if and only if* η *is big.*

CHAPTER 3

GENERALIZED FUJITA VANISHING THEOREMS

In this chapter, we shall prove two generalized versions of Fujita vanishing theorem for q-ample divisors (theorem 3.3.3 and proposition 3.3.4). They will be used repeatedly in the next chapter. Before that, we shall quickly go through the results in section 2 and 3 in Totaro's paper [30]. There Totaro developed on Arapura's idea [3] on using resolution of the diagonal to study Castelnuovo-Mumford regularity of a sheaf. Using these ideas, we shall provide a weak extension of a vanishing theorem for q-ample line bundles proved by Totaro [30, Theorem 6.4] (theorem 3.3.1). From this, we prove a generalization of the Fujita vanishing theorem (theorem 3.3.3) to the q-ample divisors setting. It also generalizes the Fujita-type vanishing theorem that Küronya proved [22, Theorem C].

In this chapter, we assume X to be a projective scheme of dimension n over a field, with the ring of regular functions on X being a field. Furthermore, we fix a $2n$-Koszul-ample line bundle $\mathcal{O}_X(1)$ on X.

3.1 Resolution of the diagonal

We now recall Totaro's result on resolution of the diagonal. The lemma that follows shows how this can be applied towards establishing results on vanishing of cohomologies.

Theorem 3.1.1 (Totaro [30, Theorem 2.1])**.** *On $X \times_k X$, we have the following exact sequence of coherent sheaves:*

$$\mathcal{R}_{2n-1} \boxtimes \mathcal{O}_X(-2n+1) \to \cdots \to \mathcal{R}_1 \boxtimes \mathcal{O}_X(-1) \to \mathcal{R}_0 \boxtimes \mathcal{O}_X \to \mathcal{O}_\Delta \to 0, \tag{3.1}$$

where $\Delta \subset X \times_k X$ is the diagonal. Here all the $\mathcal{R}_i$'s are locally free sheaves on X that can be fit into short exact sequences:

$$0 \to \mathcal{R}_{i+1} \otimes \mathcal{O}_X(-1) \to B_{i+1} \otimes_k \mathcal{O}_X(-1) \to \mathcal{R}_i \to 0, \tag{3.2}$$

where the B_{i+1}'s are k-vector spaces.

Lemma 3.1.2. *[30, Lemma 3.1] Let $\mathcal{E}$ and $\mathcal{F}$ be a locally free sheaf and a coherent sheaf on X, respectively. Suppose that for each pair of integers $0 \leq a \leq 2n - i$ and $b \geq 0$, either $H^b(\mathcal{E} \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(\mathcal{F}(-a)) = 0$. Then $H^i(\mathcal{E} \otimes \mathcal{F}) = 0$.*

Sketch of proof. After tensoring with $\mathcal{E} \boxtimes \mathcal{F}$, the sequence (3.1) remains exact, we now apply Künneth's formula. □

3.2 Partial regularity of a coherent sheaf

Partial regularity is a generalized notion of the Castelnuouvo-Mumford regularity of a coherent sheaf, in the sense that we only focus on studying vanishing of higher cohomology groups.

Definition 3.2.1 (Partial regularity [20, Definition 2.1])**.** Let $\mathcal{G}$ be a coherent sheaf on X and let q be any integer greater than or equal to 0. We say that $\mathcal{G}$ is (m, q)*-regular* if the following holds:

$$H^{q+i}(X, \mathcal{G} \otimes \mathcal{O}_X(m - i)) = 0 \tag{3.3}$$

for all $1 \leq i \leq n - q$.

We set

$$reg^q(\mathcal{F}) = \inf\{m \in \mathbb{Z} \mid \mathcal{F} \text{ is } (m, q)\text{-regular.}\}$$

When $q = 0$, this is just the usual Castelnuovo-Mumford regularity of the sheaf $\mathcal{F}$, relative to $\mathcal{O}_X(1)$. It is clear that $reg^q(\mathcal{F}) \in [-\infty, +\infty)$, by the ampleness of $\mathcal{O}_X(1)$.

Lemma 3.2.2 ([20, Lemma 2.2])**.** *If $\mathcal{F}$ is (m, q)-regular, then $\mathcal{F} \otimes \mathcal{O}_X(1)$ is also (m, q)-regular.*

Lemma 3.2.3 ([30, Lemma 3.3])**.** *If $\mathcal{F}$ is a $(0, q)$-regular coherent sheaf on X, then*

$$H^j(X, \mathcal{F} \otimes \mathcal{R}_i) = 0,$$

for $j > q$ and $i < n + j$. Here, we are referring to the $\mathcal{R}_i$'s that appear in lemma 3.1.2.

We next generalize [30, Theorem 3.4].

Theorem 3.2.4 (Subadditivity of partial regularity)**.** *Let $\mathcal{E}$ and $\mathcal{F}$ be a locally free sheaf and a coherent sheaf on X, respectively, then*

$$reg^q(\mathcal{E} \otimes \mathcal{F}) \leq reg^l(\mathcal{E}) + reg^{q-l}(\mathcal{F})$$

for any $0 \leq l \leq q$.

Proof. Replacing $\mathcal{E}$ and $\mathcal{F}$ by $\mathcal{E} \otimes \mathcal{O}_X(k)$ and $\mathcal{F} \otimes \mathcal{O}_X(k')$, respectively, where k and k' are sufficiently large, we may assume $\mathcal{E}$ and $\mathcal{F}$ are $(0,l)$- and $(0,q-l)$-regular, respectively. We want to show

$$H^{q+i}(X, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{O}_X(-i)) = 0,$$

for $1 \leq i \leq n-q$. We now apply lemma 3.1.2.

Case 1. $b > l$ and $a < n+b$.

By lemma 3.2.3,

$$H^b(X, \mathcal{E} \otimes \mathcal{R}_a) = 0.$$

Case 2. $b > l$ and $n+b \leq a \leq 2n-(q+i)$.

Since $q+i+a-b \geq q+i+n > n$,

$$H^{(q+i)+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(-a-i)) = 0,$$

for dimensional reason.

Case 3. $0 \leq b \leq l$ and $0 \leq a \leq 2n-(q+i)$.

We have $q-b \geq q-l$, and

$$H^{(q-b)+a+i}(X, \mathcal{F} \otimes \mathcal{O}_X(-a-i)) = 0,$$

by $(q-l)$-regularity of $\mathcal{F}$ and lemma 3.2.2. □

3.3 Vanishing theorems

We next prove a vanishing theorem for q-ample divisors, it is an analogue of [30, Theorem 6.4]. This will play a crucial role in proving theorem 3.3.3.

Theorem 3.3.1 (Uniform vanishing)**.** *Let $\mathcal{L}$ be a q-ample line bundle on X. Then for any N, there is an integer m_N, such that, for any coherent sheaf $\mathcal{F}$ on X with $reg^{q'}(\mathcal{F}) \leq N$,*

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) = 0$$

for $i > q + q'$ and $m > m_N$.

Proof. Fix an integer i such that $q + q' < i \leq n$, by lemma 3.1.2, it is enough to show that there is an M, depending only on the choice of N, but not the coherent sheaf $\mathcal{F}$, such that for $m > M$, $0 \leq a \leq 2n - i$ and $b \geq 0$, either $H^b(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N) \otimes \mathcal{R}_a) = 0$ or $H^{i+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(N - a))$. Here $\mathcal{F}$ is any coherent sheaf with $reg^{q'}(\mathcal{F}) \leq N$.

Case 1. $b > q$ and $0 \leq a < n + b$.

Using the q-ampleness of $\mathcal{L}$, there is an m_N, such that we have

$$H^{q+j}(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N - j)) = 0$$

for all $1 \leq j \leq n - q$ and $m > m_N$, i.e. $\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N)$ is $(0, q)$-regular for all $m > m_N$. Now lemma 3.2.3 says

$$H^b(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-N) \otimes \mathcal{R}_a) = 0$$

for all $m > m_N$, $b > q$ and $a < n + b$.

Case 2. $b > q$ and $n + b \leq a \leq 2n - i$.

We have $i + a - b \geq i + n > n$, and

$$H^{i+a-b}(X, \mathcal{F} \otimes \mathcal{O}_X(N - a)) = 0$$

for dimensional reason.

Case 3. $0 \leq b \leq q$ and $0 \leq a \leq 2n - i$.

We have $i - b > q'$, and $H^{(i-b)+a}(X, \mathcal{F} \otimes \mathcal{O}_X(N - a)) = 0$ by the partial regularity assumption of $\mathcal{F}$ and lemma 3.2.2. This proves the theorem. □

Lemma 3.3.2. *There is an N such that $reg^0(\mathcal{P}) \leq N$ for any nef line bundle $\mathcal{P}$ on X.*

Proof. By the Fujita vanishing theorem, there is an N such that

$$H^a(X, \mathcal{O}_X(N - a) \otimes \mathcal{P}) = 0$$

for $a > 0$ and any nef line bundle $\mathcal{P}$. □

We prove a Fujita-type vanishing theorem for q-ample divisors. It is a generalization of the Fujita-type vanishing theorem that Küronya proved in [22, Theorem C], thanks to the fact that a divisor D is q-ample if and only if its restriction to its augmented base locus $D|_{\mathbf{B}_+(D)}$ is q-ample [6]. Note that we do not assume $\mathcal{O}(Z)$ is a field in the following.

Theorem 3.3.3 (Fujita-type vanishing theorem for q-ample divisors)**.** *Let Z be a projective scheme of dimension n. Let $\mathcal{L}_j$ be q_j-ample line bundles on Z, $1 \leq j \leq k$ and let $\mathcal{F}$ be a coherent sheaf on Z. Then for any $(k-1)$-tuple $(M_2, \cdots, M_k) \in \mathbb{Z}^{k-1}$, there is an M_1, such that*

$$H^i(Z, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}) = 0$$

for $i > \sum_{j=1}^{k} q_j$, $m_j \geq M_j$, where $1 \leq j \leq k$, and any nef line bundle $\mathcal{P}$ on Z.

Proof. We can assume that Z is connected. It suffices to prove the lemma assuming that Z is also reduced. Indeed, let $\mathcal{N}$ be the nilradical ideal sheaf of Z, and chase through the following exact sequence:

$$\begin{aligned} 0 \to \mathcal{N}^{e+1} \cdot \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \\ \to \mathcal{N}^{e} \cdot \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \\ \to (\mathcal{N}^{e} \cdot \mathcal{F} / \mathcal{N}^{e+1} \cdot \mathcal{F}) \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P} \to 0. \end{aligned}$$

Note that $(\mathcal{N}^{e} \cdot \mathcal{F} / \mathcal{N}^{e+1} \cdot \mathcal{F}) \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}$ is a coherent sheaf on Z_{red}, and that $\mathcal{N}^e = 0$ for $e \gg 0$.

Since $\mathcal{L}_j$ is q_j-ample,

$$H^{q_i+a}(Z, \mathcal{L}_j^{\otimes m_j} \otimes \mathcal{O}(-a)) = 0$$

for $m_j \gg 0$ and $1 \leq a \leq n - q_i$. This says $reg^{q_j}(\mathcal{L}_j^{\otimes m_j}) \leq 0$ for all $m_j \gg 0$. Therefore, there are N_j such that $reg^{q_j}(\mathcal{L}_j^{\otimes m_j}) \leq N_j$ for all $m_j \geq M_j$. We apply theorem 3.2.4 and lemma 3.3.2 to see that $reg^{\sum_{i=2}^{k} q_j}(\mathcal{F} \otimes \mathcal{L}_2^{\otimes m_2} \otimes \cdots \otimes \mathcal{L}_k^{\otimes m_k} \otimes \mathcal{P}) \leq reg^0(\mathcal{F}) + \sum_{j=2}^{k} N_j + N$ for all $m_j \geq M_j$, where $2 \leq j \leq k$ and N is the one mentioned in lemma 3.3.2. Now, we may apply theorem 3.3.1 to get the desired result. □

Suppose we are only interested in the vanishing of the top cohomology group, we may relax the assumption in theorem 3.3.3 a bit. We shall use this to prove theorem 4.3.1.

Proposition 3.3.4. *Let Z be a projective scheme of dimension n. Let $\mathcal{L}_1$ and $\mathcal{L}_i$ be line bundles on Z that are q_1-ample and q_i-almost ample, respectively, where $2 \leq i \leq k$ and $\sum_{i=1}^{k} q_i \leq n-1$. Then for any coherent sheaf $\mathcal{F}$ on Z and any $(k-1)$-tuple $(M_i)_{2\leq i\leq k} \in \mathbb{Z}^{k-1}$, there is an M_1 such that,*

$$H^n(Z, \mathcal{F} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes m_i}) = 0$$

for $m_i \geq M_i$.

Proof. Let us first reduce to the case where Z is integral. Indeed, argue as in the proof of theorem 3.3.3, we may assume Z is reduced. Suppose $Z = \bigcup_{i=1}^{k} Z_i$, where Z_i are the irreducible components of Z. Let $\mathcal{I}$ be the ideal sheaf of $Z_1 \subset Z$. Consider the short exact sequence

$$0 \to \mathcal{I} \cdot \mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathcal{I} \cdot \mathcal{F} \to 0.$$

Note that $\mathcal{I} \cdot \mathcal{F}$ and $\mathcal{F}/\mathcal{I} \cdot \mathcal{F}$ are supported on $\bigcup_{i=2}^{k} Z_i$ and Z_1, respectively. We then tensor the above short exact sequence with $\mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes m_i}$ and induct on the number of irreducible components of Z. Therefore, we may assume that Z is irreducible as well.

Now we assume Z is a projective variety. We can find a surjection $\oplus \mathcal{O}_Z(a) \twoheadrightarrow \mathcal{F}$, where $\mathcal{O}_Z(1)$ is an ample line bundle on Z. Thus it suffices to prove the case when $\mathcal{F}$ is a line bundle $\mathcal{M}$. Let ω_Z be the dualizing sheaf of Z [17, III.7]. We have

$$H^n(Z, \mathcal{M} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes m_i}) \cong H^0(Z, \mathcal{M}^\vee \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes -m_i} \otimes \omega_Z)^\vee$$

We can embed $\omega_Z \hookrightarrow \mathcal{O}(j)$ [30, Proof of Theorem 9.1]. This reduces to proving the vanishing of $H^0(Z, \mathcal{M}^\vee \otimes \mathcal{O}(j) \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes -m_i})$. We may find an M_1 such that $\mathcal{L}_1^{\otimes m_1} \otimes \bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes M_i} \otimes \mathcal{M} \otimes \mathcal{O}(-j)$ is q_1-ample for $m_1 \geq M_1$, by theorem 2.1.7. By theorem 2.1.7 again, $\bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes m_i} \otimes \mathcal{L}_1^{\otimes m_1} \otimes \mathcal{M} \otimes \mathcal{O}(-j)$ is $(n-1)$-ample for $m_i \geq M_i$ and $m_1 \geq M_1$. By theorem 2.1.8, $\bigotimes_{i=2}^{k} \mathcal{L}_i^{\otimes -m_i} \otimes \mathcal{L}_1^{\otimes -m_1} \otimes \mathcal{M}^\vee \otimes \mathcal{O}(j)$ is not pseudoeffective for $m_i \geq M_i$. Therefore, it cannot have any global sections. □

CHAPTER 4

AMPLE, NEF AND LOCALLY AMPLE SUBVARIETIES

In this chapter, we shall show how the generalized Fujita vanishing theorems for q-ample divisors (theorem 3.3.3 and proposition 3.3.4) can be applied to study positivity of higher codimensional subvarieties.

4.1 Transitivity properties

Using theorem 3.3.3, we shall deduce that the notions of ample, nef and locally ample subschemes are transitive properties, respectively.

4.1.1 Ample case

The following theorem generalizes the transitivity property of ample subschemes [26, Proposition 6.4] in the sense that we do not require Y (resp. Z) to be lci in X (resp. Y). This gives further evidence that Ottem's definition of an ample subscheme is a natural one.

Theorem 4.1.1 (Transitivity of ample subschemes)**.** *Let $Y \subset X$ be an ample subscheme of codimension r_1, $Z \subset Y$ be an ample subscheme of codimension r_2. Then $Z \subset X$ is also an ample subscheme of codimension $r_1 + r_2$.*

Before we can prove the theorem, we need two lemmas:

Lemma 4.1.2. *Let X be a projective scheme and let Y be a closed subscheme of X of codimension r. Suppose the blowup of X along Y, $\pi : Bl_Y X \to X$, has fiber dimension at most $r - 1$. If a line bundle $\mathcal{L}$ on X is q-ample on Y, and*

$$H^i(Bl_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$$

for any $l \geq 0$, $i > q + r$ and $m \gg 0$, then $\mathcal{L}$ is $(q + r)$-ample. Here $\mathcal{O}_X(1)$ is an ample line bundle on X.

Proof. Applying the Leray spectral sequence, we have

$$E_2^{p,s} = H^p(X, R^s\pi_*\mathcal{O}_{Bl_Y X} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \Rightarrow H^{p+s}(Bl_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))).$$

Since the fiber dimension of π is at most $r-1$ (proposition), $R^s\pi_*\mathcal{O}_{Bl_Y X} = 0$ and $E_2^{p,s} = 0$ for $s > r-1$.

For $s > 0$, $R^s\pi_*\mathcal{O}_{Bl_Y X}$ is a coherent sheaf on Y. Indeed, this follows by considering the long exact sequence

$$\cdots \to R^s\pi_*\mathcal{O}_{Bl_Y X}(-jE) \to R^s\pi_*\mathcal{O}_{Bl_Y X}((-j+1)E) \to R^s\pi_*\mathcal{O}_E((-j+1)E) \to \cdots,$$

where E is the exceptional divisor, and the fact that $R^s\pi_*\mathcal{O}_{\tilde{X}}(-jE) = 0$ for $j \gg 0$, since $-E$ is π-ample.

By the q-ampleness of $\mathcal{L}|_Y$, we have $E_2^{p,s} = H^p(X, R^s\pi_*\mathcal{O}_{Bl_Y X} \otimes \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) = 0$ for $p > q$, $s > 0$ and $m \gg 0$.

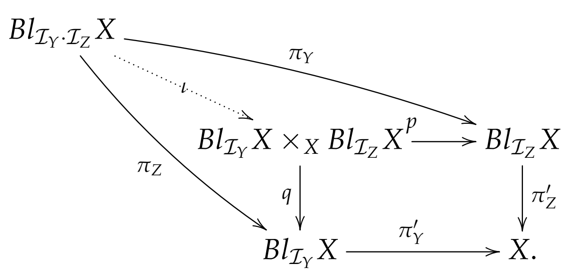
These two vanishing results imply that $E_h^{p-h,h-1} = E_2^{p-h,h-1} = 0$ for $h \geq 2$, $p > q+r$ and $m \gg 0$.

By the hypothesis,

$$E_\infty^{p,0} = H^p(Bl_Y X, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$$

for $p > q+r$ and $m \gg 0$. Hence we arrive at the desired vanishing $E_2^{p,0} = H^p(X, \mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) = 0$ for $p > q+r$ and $m \gg 0$. □

Lemma 4.1.3. *Under the same hypothesis as in the theorem, we have the following commutative diagram.*



Here π'_Z (resp. π'_Y) is the blowup of X along $\mathcal{I}_Z$ (resp. $\mathcal{I}_Y$), with exceptional divisor E'_Z (resp. E'_Y); π_Z and π_Y are blowups along the ideal sheaves $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y} X}$ and $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(E'_Z)$, with exceptional divisor E_Z and E_Y, respectively. The composition $\pi_Y \circ \pi'_Z = \pi_Z \circ \pi'_Y$ is the blowup map of X along $\mathcal{I}_Y \cdot \mathcal{I}_Z$. The square in the above diagram is a fiber diagram, with ι induced by the maps π_Z and π_Y. Moreover,

1. $\pi_Y^* E_Z' = E_Z$ *and* $\pi_Z^* E_Y' = E_Y + E_Z$.

2. ι *is a closed immersion.*

Proof. First, let us check that the blowup of X along $\mathcal{I}_Y \cdot \mathcal{I}_Z$ factors through the maps π_Y' and π_Z'. By the universal property of blowup, it suffices to check that the inverse image ideal sheaves $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ and $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ are invertible. Let $\mathcal{J}$ be the inverse of $(\mathcal{I}_Y \cdot \mathcal{I}_Z) \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$, i.e. the fractional ideal sheaf such that $(\mathcal{I}_Y \cdot \mathcal{I}_Z) \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cdot \mathcal{J} = \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$. We check locally that $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is invertible. Let $\mathfrak{a}$ and $\mathfrak{b}$ be the stalk of $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ and $(\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}) \cdot \mathcal{J}$ at a scheme-theoretic point $x \in Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$, respectively, and let $R = \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, x}$ be the local ring at x. Since $\mathfrak{a} \cdot \mathfrak{b} = R$, we may write $\sum_i a_i b_i = 1$, where $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{b}$. Note that each $a_i b_i \in R$, so there must be some j such that $a_j b_j$ is a unit. Let $u = (a_j b_j)^{-1}$. Let $f : R \to \mathfrak{a}$ be the R-module homomorphism that sends $r \mapsto r a_i$. We shall see that f is an isomorphism. For any $a \in \mathfrak{a}$, we can write $a = (a b_j u) a_j$. Note that $(a b_j u) \in R$. Thus, f is onto. Suppose there is an $r \in R$ such that $f(r) = r a_j = 0$. Then $r = r(a_j b_j u) = 0$. Therefore, f is injective. We conclude that $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is locally free of rank 1, hence is invertible. Applying a similar argument, we see that $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is also invertible. This gives us the maps π_Z and π_Y.

Next, let us check that $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(E_Z')$ is an ideal sheaf. Indeed, we have the inclusion $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \subset \mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \cong \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-E_Z')$. We then tensor the terms in the inclusion by $\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(E_Z')$ to see that $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(E_Z') \subset \mathcal{O}_{Bl_{\mathcal{I}_Z} X}$. Applying the universal property of blowup again, we see that $\pi_Z : Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X \to Bl_{\mathcal{I}_Y} X$ and $\pi_Y : Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X \to Bl_{\mathcal{I}_Z} X$ are the same as the blowup of $Bl_{\mathcal{I}_Y} X$ and $Bl_{\mathcal{I}_Z} X$ along $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y} X}$ and $\mathcal{I}_Y \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \otimes \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(E_Z')$.

For 1, note that $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X} \cong \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-E_Z')$. Therefore, we have the surjection $\pi_Y^* \mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-E_Z') \twoheadrightarrow (\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X}) \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cong \mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X} \cong \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}(-E_Z)$. This is also an injection, since the pullback of a local generator of $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Z} X}$ to $Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ is not a zero divisor, thanks to the fact that $\mathcal{I}_Z \cdot \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}$ is invertible. A similar argument leads to the second statement in 1.

For 2, let W be the scheme-theoretic image of $Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X$ under ι. It suffices to show that $\mathcal{I}_Y \cdot \mathcal{O}_W$ (resp. $\mathcal{I}_Z \cdot \mathcal{O}_W$) is invertible. Note that the natural surjection $q^* \mathcal{O}_{Bl_{\mathcal{I}_Y} X}(-E_Y') \to \mathcal{I}_Y \cdot \mathcal{O}_W$ is injective if and only if the pullback of a local generator of $\mathcal{O}_{Bl_{\mathcal{I}_Y} X}(-E_Y')$ is not a zero divisor, which follows from the fact that the natural map $\mathcal{O}_W \to \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z}} X$ is an injection [1, Lemma 28.6.3]. We can use the same argument to show that $\mathcal{I}_Z \cdot \mathcal{O}_W$ is also invertible. □

Proof of theorem 4.1.1. Note that π_Y has fiber dimension at most $r_1 - 1$. This follows from 2 of lemma 4.1.3 and the fact that π'_Y has fiber dimension at most $r_1 - 1$ (proposition 4.1.1).

Let $\tilde{Y}$ be the strict transform of Y in $Bl_{\mathcal{I}_Z} X$. Since Z is an ample subscheme of Y, $E'_Z|_{\tilde{Y}}$ is $(r_2 - 1)$-ample.

By lemma 4.1.2, it suffices to prove that given any $l \in \mathbb{Z}_{\geq 0}$,

$$H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}_{Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X}(mE_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) = 0$$

for $i > r_1 + r_2 - 1$ and $m \gg 0$. Here H is an ample divisor on $Bl_{\mathcal{I}_Z} X$. We fix an $l \in \mathbb{Z}_{\geq 0}$ from now on.

Claim 1. $(E_Z - \delta E_Y)|_{E_Y}$ is $(r_2 - 1)$-ample for $0 < \delta \ll 1$.

Proof of claim. Since $-E_Y$ is π_Y-ample, $(\pi_Y^* E'_Z - \delta E_Y)|_{E_Y} = (E_Z - \delta E_Y)|_{E_Y}$ is $(r_2 - 1)$-ample, for $0 < \delta \ll 1$, by proposition 2.1.9. □

Claim 2. $E_Z + E_Y - \epsilon E_Z$ is $(r_1 - 1)$-ample for $0 < \epsilon \ll 1$.

Proof of claim. Indeed, $E_Z + E_Y = \pi_Y^* E'_Y$ and E'_Y is $(r_1 - 1)$-ample by ampleness of $Y \subset X$. Note that $-E_Z$ is π_Y-ample. The claim then follows from proposition 2.1.9. □

By the above claims, we may choose a big enough $k \in \mathbb{Z}$ such that $(kE_Z - E_Y)|_{E_Y}$ is $(r_2 - 1)$-ample and $kE_Z + (k+1)E_Y$ is $(r_1 - 1)$-ample.

Write

$$m_1 E_Y + m_2 E_Z = \lambda_1 (kE_Z - E_Y) + \lambda_2 (kE_2 + (k+1)E_Y) + j_1 E_Y + j_2 E_Z,$$

where $\lambda_2 = \lfloor \frac{m_1 + \lfloor \frac{m_2}{k} \rfloor}{k+2} \rfloor$; $\lambda_1 = \lfloor \frac{m_2}{k} \rfloor - \lambda_2$; $j_1 = ((m_1 + \lfloor \frac{m_2}{k} \rfloor) \bmod (k+2))$ and $j_2 = (m_2 \bmod k)$. Note that $0 \leq j_1 < k+2$ and $0 \leq j_2 < k$. The precise formulae for λ_1 and λ_2 are not very important. The plan is to choose a big m_2, then let m_1 increases. As m_1 grows, λ_1 decreases and λ_2 increases. We then use the positivity of $(kE_Z - E_Y)|_{E_Y}$ and $kE_Z + (k+1)E_Y$ to prove the required vanishing statement.

Since $kE_Z + (k+1)E_Y$ is $(r_1 - 1)$-ample, we may find Λ_2 such that

$$H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\lambda_2(kE_2 + (k+1)E_Y) + j_1 E_Y + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) = 0 \tag{4.1}$$

for $i > r_1 - 1$, $\lambda_2 \geq \Lambda_2$, $0 \leq j_1 < k+2$ and $0 \leq j_2 < k$.

Applying theorem 3.3.3 to the scheme E_Y, there is an Λ_2' such that

$$H^i(E_Y, \mathcal{O}_{E_Y}(\lambda_1(kE_Z - E_Y) + \lambda_2(kE_2 + (k+1)E_Y) + j_1 E_Y + j_2 E_Z) \otimes \pi_Y^* \mathcal{O}_{Bl_{\mathcal{I}_Z}}(-lH)) = 0$$

for $i > (r_2 - 1) + (r_1 - 1)$, $\lambda_1 \geq 0$, $\lambda_2 \geq \Lambda_2'$, $0 \leq j_1 < k+2$ and $0 \leq j_2 < k$. This implies

$$\begin{aligned} &H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z + m_1 E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) \\ &\qquad \cong H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z + (m_1+1)E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) \end{aligned} \tag{4.2}$$

for $i > r_1 + r_2 - 1$, $0 < m_1 + 1 < (k+1)\lfloor \frac{m_2}{k} \rfloor + k + 2$ and $\lfloor \frac{m_1 + 1 + \lfloor \frac{m_2}{k} \rfloor}{k+2} \rfloor \geq \Lambda_2'$.

Choose some big M_2 such that $\lfloor \frac{\lfloor \frac{M_2}{k} \rfloor}{k+2} \rfloor \geq \max\{\Lambda_2, \Lambda_2'\}$. Applying (4.2) repeatedly, we have for $m_2 > M_2$,

$$\begin{aligned} &H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) \\ &\qquad \cong H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(m_2 E_Z + (k+1)\lfloor \frac{m_2}{k} \rfloor E_Y) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))) \end{aligned} \tag{4.3}$$

for $i > r_1 + r_2 - 1$. The above cohomology group can be rewritten as

$$H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\lfloor \frac{m_2}{k} \rfloor (kE_Z + (k+1)E_Y) + (m_2 - k\lfloor \frac{m_2}{k} \rfloor)E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-lH))),$$

which is 0 by (4.1). □

4.1.2 Nef and locally ample case

We then prove the analogue of theorem 4.1.1 for nef subschemes. The idea of the proof is essentially the same, although we have to use the full statement of theorem 3.3.3 by allowing a nef term, as well as take extra care with the variables.

Theorem 4.1.4 (Transitivity of nef subschemes). *Let $Y \subset X$ be a nef subscheme of codimension r_1, $Z \subset Y$ be a nef subscheme of codimension r_2. Then $Z \subset X$ is also a nef subscheme of codimension $r_1 + r_2$.*

Proof. Lemma 4.1.3 still holds under the hypothesis of the theorem. We shall use the same notation as in lemma 4.1.3. Since $-E_Z'$ and $-E_Y'$ is π_Z'-ample and π_Y'-ample, respectively, we may choose an ample divisor A' on X such that $\pi_Z'^*A' - E_Z'$ and $\pi_Y'^*A' - E_Y'$ are ample. Let $A = \pi_Y^*\pi_Z'^*A'$ be the pullback of A to $Bl_{\mathcal{I}_Y\cdot\mathcal{I}_Z}X$, note that A is nef.

Note that we can write $kE_Z + A$ as $\pi_Y^*((k+1)E_Z' + (\pi_Z'^*A' - E_Z'))$. By lemma 4.1.2, it suffices to prove that given any $l \geq 0$, for $k \gg 0$

$$H^i(Bl_{\mathcal{I}_Y\cdot\mathcal{I}_Z}X, \mathcal{O}(m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z}X}(-l))) = 0$$

for $i > r_1 + r_2 - 1$ and $m_2 \gg 0$. We fix l and k from this point on.

Note that $F_1' := (E_Z + \frac{1}{3k}\pi_Y^*(\pi_Z'^*A' - E_Z') - \frac{1}{k_1}E_Y)$ is (r_2-1)-ample when restricted to E_Y and $F_2' := E_Z + E_Y + \frac{1}{3k}\pi_Z^*(\pi_Y'^*A' - E_Y') - \frac{1}{k_1}E_Z$ is (r_1-1)-ample for $k_1 \gg 0$. We fix such a k_1. Let $\alpha = 3kk_1 - k_1$ and $\beta = 3kk_1 - k_1 - 3k$. Let $F_1 = 3kk_1\beta F_1'$ and $F_2 = 3kk_1\alpha F_2'$. They are both integral divisors. In fact, $F_1 = \beta(\alpha E_Z - 3kE_Y + k_1A)$ and $F_2 = \alpha(\beta E_Z + \alpha E_Y + k_1A)$.

Write

$$m_1E_Y + m_2(kE_Z + A) = \lambda_1F_1 + \lambda_2F_2 + \lambda_3A + j_1E_Y + j_2E_Z,$$

where $\lambda_2 = \lfloor\frac{m_1+3\beta k\lfloor\frac{m_2k}{\alpha\beta}\rfloor}{\alpha^2+3\beta k}\rfloor$; $\lambda_1 = \lfloor\frac{m_2k}{\alpha\beta}\rfloor - \lambda_2$; $\lambda_3 = m_2 - \lambda_1\beta k_1 - \lambda_2\alpha k_1$; $j_1 = ((m_1 + 3\beta k\lfloor\frac{m_2k}{\alpha\beta}\rfloor) \bmod (\alpha^2 + 3\beta k))$ and $j_2 = (m_2k \bmod \alpha\beta)$. Note that if $0 \leq m_1 \leq \alpha^2\lfloor\frac{m_2k}{\alpha\beta}\rfloor$, then $\lambda_1 \geq 0$ and $\lambda_3 \geq 0$.

Since F_2 is (r_1-1)-ample and A is nef, there is a Λ_2 such that

$$H^i(Bl_{\mathcal{I}_Y\cdot\mathcal{I}_Z}X, \mathcal{O}(\lambda_2F_2 + \lambda_3A + j_2E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z}X}(-l))) = 0 \tag{4.4}$$

for $i > r_1 - 1$, $\lambda_2 > \Lambda_2$, $\lambda_3 \geq 0$ and $0 \leq j_2 < \alpha\beta$.

Since $F_1|_{E_Y}$ is (r_2-1)-ample, F_2 is (r_1-1)-ample and A is nef, there is a Λ_2' such that

$$H^i(E_Y, \mathcal{O}_{E_Y}(\lambda_1F_1 + \lambda_2F_2 + \lambda_3A + j_1E_Y + j_2E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z}X}(-l))) = 0$$

for $i > (r_2-1) + (r_1-1)$, $\lambda_2 > \Lambda_2'$, $\lambda_1 \geq 0$, $\lambda_3 \geq 0$, $0 \leq j_1 < \alpha^2 + 3\beta k$ and $0 \leq j_2 < \alpha\beta$. This implies if $\lfloor\frac{m_1+3\beta k\lfloor\frac{m_2k}{\alpha\beta}\rfloor}{\alpha^2+3\beta k}\rfloor > \Lambda_2'$,

$$\begin{aligned} &H^i(Bl_{\mathcal{I}_Y\cdot\mathcal{I}_Z}X, \mathcal{O}(m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z}X}(-l))) \\ &\qquad \cong H^i(Bl_{\mathcal{I}_Y\cdot\mathcal{I}_Z}X, \mathcal{O}(\alpha^2\lfloor\frac{m_2k}{\alpha\beta}\rfloor E_Y + m_2(kE_Z + A)) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z}X}(-l))) \end{aligned}$$

for $i > r_1 + r_2 - 1$.

The above cohomology groups can be rewritten as

$$H^i(Bl_{\mathcal{I}_Y \cdot \mathcal{I}_Z} X, \mathcal{O}(\lfloor \frac{m_2 k}{\alpha\beta} \rfloor F_2 + \lambda_3 A + j_2 E_Z) \otimes \pi_Y^*(\mathcal{O}_{Bl_{\mathcal{I}_Z} X}(-l)))$$

where $\lambda_3 \geq 0$ and $0 \leq j_2 < \alpha\beta$. By (4.4), the above cohomology groups vanish for $m_2 \gg 0$. □

The analogous statement holds for locally ample subschemes. The proof is essentially the same, therefore we shall omit it.

Theorem 4.1.5 (Transitivity of locally ample subschemes)**.** *Let Y be a locally ample subscheme of X of codimension r_1 and let Z be a locally ample subscheme of Y of codimension r_2. Then Z is a locally ample subscheme of X of codimension $r_1 + r_2$.*

The following corollary says that intersection of 2 ample (resp. nef or locally ample) subschemes is ample (resp. nef or locally ample), respectively, assuming the intersection has the desired codimension. It is the generalization of [26, Proposition 6.3], in the sense that we do not assume that X is smooth and the subschemes are lci in X.

Corollary 4.1.6 (Intersection of ample, nef and locally ample subschemes)**.** *If Y and Z are both ample (resp. nef or locally ample) subschemes of X, of codimension r and s, respectively, and $Y \cap Z$ has codimension $r + s$ in X, then $Y \cap Z$ is an ample (resp. nef or locally ample) subscheme of X.*

Proof. By proposition 2.3.9, $Y \cap Z$ is an ample (resp. nef or locally ample) subscheme of Z. We now conclude using the transitivity property of ample (resp. nef or locally ample) subschemes (theorem 4.1.1, theorem 4.1.4 or theorem 4.1.5, respectively). □

4.2 Positivity upon restriction

If a line bundle is ample after restricting to an ample subscheme, it is reasonable to expect the line bundle to exhibit some positivity features. The following theorem demonstrates a nice interplay between ample subschemes and q-ample divisors. The proof again uses one of the generalized Fujita vanishing theorems (theorem 3.3.3).

Theorem 4.2.1. *Let X be a projective scheme of dimension n. Let Y be an ample subscheme of X of codimension r. Suppose $\mathcal{L}$ is a line bundle on X, and that its restriction $\mathcal{L}|_Y$ to Y is q-ample. Then $\mathcal{L}$ is $(q+r)$-ample.*

Proof. We fix an ample line bundle $\mathcal{O}_X(1)$ on X. Let $\pi : \tilde{X} \to X$ be the blowup of X along Y.

Step 1. Pass to the blowup.

By lemma 4.1.2, it suffices to prove that

$$H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0 \tag{4.5}$$

for $i > q+r$ and $m \gg 0$.

Step 2. Pass to the exceptional divisor.

We claim that it is enough to show that there is an m_0 such that

$$H^i(E, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE)) = 0 \tag{4.6}$$

for $i > r+q-1$, $m > m_0$ and $k \geq 1$. Here E is the exceptional divisor on the blowup $\tilde{X}$.

Indeed, let us consider the short exact sequence:

$$\begin{aligned} 0 \to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E) \to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE) \\ \to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \to 0. \end{aligned}$$

By looking at the long exact sequence of cohomology groups induced from the above short exact sequence and using the hypothesis (4.6), we observe that

$$H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E)) \cong H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)) \tag{4.7}$$

for $i > r+q$, $m > m_0$ and $k \geq 1$.

Since E is $(r-1)$-ample, for any fixed m,

$$H^i(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)) = 0$$

for $k \gg 0$ and $i > r-1$. Together with the isomorphisms in (4.7), we have the desired vanishing result (4.5).

Step 3. Rewrite the line bundles of interest in (4.6) in terms of q- and $(r-1)$- ample line bundles.

Note that since $-E$ is π-ample, there is an $N > 0$ such that $\pi^*(\mathcal{L}^{\otimes N}) \otimes \mathcal{O}_{\tilde{X}}(-E)$ is q-ample, by proposition 2.1.9. We can replace $\mathcal{L}$ by $\mathcal{L}^{\otimes N}$ and assume that $\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E)$ is q-ample. We now rewrite the line bundle on E in (4.6):

$$\pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE)) \cong \pi^*(\mathcal{O}_X(-l)) \otimes \mathcal{O}_E((k+m)E) \otimes (\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E))^{\otimes m}$$

with the second term $\mathcal{O}_E((k+m)E)$ on the right-hand side being an $(r-1)$-ample line bundle, and the third term $(\pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E))^{\otimes m}$ being an q-ample line bundle.

We now apply theorem 3.3.3 with $\mathcal{L}_1 := \pi^*(\mathcal{L}) \otimes \mathcal{O}_E(-E)$, $\mathcal{L}_2 = \mathcal{O}_E(E)$ and $M_2 = 1$ to conclude. □

This theorem can be compared to a result by Demailly-Peternell-Schneider [7, Theorem 3.4]. Given a chain of codimension 1 subvarieties $Y_{n-r} \subset Y_{n-r+1} \subset \cdots \subset Y_{n-1} \subset Y_n = X$, such that for $n-r \leq i \leq n-1$, there exists an ample divisor Z_i in the normalization of Y_{i+1}, with Y_i being the image of Z_i under the normalization map. Assuming Totaro's results on q-ample divisors, they showed a posteriori if $\mathcal{L}|_{Y_{n-r}}$ is ample, then $\mathcal{L}$ is r-ample, .

One may ask whether we have a converse to theorem 4.2.1, i.e. given an r-ample line bundle $\mathcal{L}$ on a projective scheme X, there is a codimension r ample subscheme Y, such that $\mathcal{L}|_Y$ is ample. Demailly, Peternell and Schneider gave a counter-example to this in [7, Example 5.6]:

Example 4.2.2. Let S be a general quartic surface in $\mathbb{P}^3$. Let $X = \mathbb{P}(\Omega^1_S)$. They showed that $-K_X$ is 1-ample, and yet for any ample divisor Y in X, $(-K_X)^2 \cdot Y < 0$, thus $-K_X$ cannot be ample when it is restricted to any ample divisor.

For the reader's convenience, we shall include the proof of $-K_X$ being 1-ample in example [7]. In fact, it might be worthwhile to extract from the argument of [7, Example 5.6] the following general property.

Proposition 4.2.3. *Let*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{L} \to 0 \tag{4.8}$$

be a short exact sequence of vector bundles on a projective scheme X. We assume $\mathcal{E}$ to be a q-ample vector bundle of rank r, $\mathcal{E}'$ is of rank $(r-1)$ and $\mathcal{L}$ is of rank 1. *Then $\mathcal{E}'$ is $(q+1)$-ample.*

Proof. We first dualize (4.8), then take symmetric product, and dualize again. This will give us the following short exact sequence

$$0 \to Sym^k\mathcal{E}' \to Sym^k\mathcal{E} \to Sym^{k-1}\mathcal{E} \otimes \mathcal{L} \to 0.$$

Fix an ample line bundle $\mathcal{O}_X(1)$ on X, and tensor the above short exact sequence with $\mathcal{O}_X(-l)$, for $l \geq 0$. Note that $H^i(X, Sym^k\mathcal{E} \otimes \mathcal{O}_X(-l)) = H^i(X, Sym^{k-1}\mathcal{E} \otimes \mathcal{L} \otimes \mathcal{O}_X(-l)) = 0$, for $i > q$ and $k \gg 0$. Hence $H^i(X, Sym^k\mathcal{E}' \otimes \mathcal{O}_X(-l)) = 0$, for $i > q+1$ and $k \gg 0$. □

Going back to the example 4.2.2, note that $\Omega^1_S \cong \mathcal{T}_S$, where $\mathcal{T}_S$ is the tangent sheaf of S. We have the following short exact sequence of locally free sheaves on S.

$$0 \to \mathcal{T}_S \to \mathcal{T}_{\mathbb{P}^3}|_S \to \mathcal{O}_S(S) \to 0.$$

The tangent bundle of a projective space is ample, therefore the tangent bundle of S is 1-ample by the lemma. Since $\mathcal{O}_X(-K_X) \cong \mathcal{O}_{\mathbb{P}(\Omega^1_S)}(2)$, $-K_X$ is 1-ample. It is not ample since the tangent bundle of S is not ample (S is a K3-surface).

Remark 4.2.4. Interestingly, we note that

$$H^2(X, K_X - K_X) \cong H^2(S, \mathcal{O}_S) \neq 0,$$

Hence, Kodaira-type vanishing theorem fails for $-K_X$, which is 1-ample. Ottem also gave a counterexample to Koadaira-type vanishing theorem for q-ample divisors [26, Chapter 9].

Example 4.2.5. One may also ask if we can relax the positivity assumption on Y in theorem 4.2.1. For example, if we only assume that the normal bundle of Y is ample, we shall see the conclusion of the theorem does not hold in general. Let us start with a smooth ample subvariety $Y \subset X$ of a smooth projective variety. We blowup a closed point p in $X \setminus Y$. Observe that the normal bundle of $Y \subset Bl_pX$ is still ample. Let $E \cong \mathbb{P}^{n-1}$ be the exceptional divisor, and let A be an ample divisor on $Bl_p(X)$. Then $E + \epsilon A$ is not $(n-2)$-ample, for $0 < \epsilon \ll 1$, since it is anti-ample when restricted to the exceptional divisor. But $(E + \epsilon A)|_Y = \epsilon A|_Y$ is ample.

On the other hand, as we shall see in the following section, a small yet interesting part of the theorem still holds if we assume Y is a nef subvariety.

4.3 Restriction pseudoeffective divisors

There are not many results regarding the positivity of subvariety with nef normal bundle, in terms of intersection theory. Here are two such results the author is aware of.

In Fulton-Lazarsfeld's work [14] (see also [23, Theorem 8.4.1]), they proved that if Y is a closed, lci subvariety of a projective variety X and the normal bundle of Y is nef, then for any closed subscheme $Z \subset X$ with $\dim Y + \dim Z \geq \dim X$, $\deg_H(Y \cdot Z) \geq 0$. (Here H is an ample divisor on X.) On the other hand, it is not hard to show that if Y has globally generated normal bundle, then restriction of any effective cycle to Y is either effective or 0 [12, Theorem 12.1.a)].

We show that the restriction of a pseudoeffective divisor to a nef subvariety is still pseudoeffective.

Theorem 4.3.1. *Let Y be a nef subvariety of codimension r of a projective variety X. Then*

$$\iota^* \overline{Eff}^1(X) \subseteq \overline{Eff}^1(Y)$$

and

$$\iota^* Big(X) \subseteq Big(Y).$$

Here $\iota : Y \hookrightarrow X$ is the inclusion map, $\iota^* : N^1(X)_{\mathbf{R}} \to N^1(Y)_{\mathbf{R}}$ is the induced map on the Néron-Severi group with $\mathbf{R}$-coefficients and $\overline{Eff}^1(X)$ (resp. $Big(X)$) is the cone of pseudoeffective (resp. big) $\mathbf{R}$-Cartier divisors.

Remark 4.3.2. Before proving the theorem, let us point out it is rather straightforward to obtain the conclusion under the stronger assumptions in theorem 4.2.1 and the added assumption that X and Y are integral. Let D be a pseudoeffective divisor on X, i.e. $-D$ is not $(n-1)$-ample (theorem 2.1.8). Suppose on the contrary $D|_Y$ is not pseudoeffective. Then $-D|_Y$ is $(n-r-1)$-ample. This gives a contradiction to theorem 4.2.1.

Proof of theorem 4.3.1. A divisor is big if and only if it can be written as the sum of a pseudoeffective divisor and an ample divisor. Therefore, we can focus on the pseudoeffective case. We shall follow the steps in the proof of theorem 4.2.1 closely. Recall that a Cartier divisor D is $(n-1)$-ample if and only if $-D$ is not pseudoeffective (theorem 2.1.8). Given a line bundle $\mathcal{L}$ on X such that $\mathcal{L}|_Y$ is $(n-r-1)$-ample, we need to show $\mathcal{L}$ is $(n-1)$-ample. Fix an ample line bundle $\mathcal{O}_X(1)$ on X.

Step 1 (Pass to the blowup). It suffices to show for any $l \geq 0$, there is an m_0 such that $H^n(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l))) = 0$ for $m \geq m_0$.

This is true by lemma 4.1.2.

We now fix l.

Step 2 (Pass to the exceptional divisor). It is enough to show that there is an m_0 such that

$$H^{n-1}(E, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE)) = 0$$

for $m \geq m_0$ and $k \geq 1$.

We just have to repeat the argument in step 2 in the proof of theorem 4.2.1, i.e. consider the long exact sequence of cohomologies associated to

$$0 \to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}((k-1)E) \to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)$$
$$\to \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \to 0.$$

Also note that for a fixed m,

$$H^n(\tilde{X}, \pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_{\tilde{X}}(kE)) = 0$$

for $k \gg 0$. Indeed, E is $(n-1)$-ample ($-E$ is not pseudoeffective!).

Step 3 (Rewrite in terms of an $(n-r-1)$-ample line bundle and an $(r-1)$-almost ample line bundle).

Replacing $\mathcal{L}$ with $\mathcal{L}^{\otimes N}$ for N large enough, we may assume $\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E)$ is $(n-r-1)$-ample, by proposition 2.1.9. Now we can write

$$\pi^*(\mathcal{L}^{\otimes m} \otimes \mathcal{O}_X(-l)) \otimes \mathcal{O}_E(kE) \cong \pi^*\mathcal{O}_X(-l) \otimes (\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E))^{\otimes m} \otimes \mathcal{O}_E((k+m)E).$$

By proposition 3.3.4, there is an m_0 such that

$$H^{n-1}(E, \pi^*\mathcal{O}_X(-l) \otimes (\pi^*\mathcal{L} \otimes \mathcal{O}_E(-E))^{\otimes m} \otimes \mathcal{O}_E((k+m)E)) = 0$$

for $k \geq 1$ and $m \geq m_0$. This proves the theorem.

□

Remark 4.3.3. Suppose the conclusion of theorem 4.3.1 holds, the normal bundle of Y is not necessarily nef. Take a 3-fold with Picard number 1 that contains a rational curve C with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. The condition $D \cdot C > 0$ for any pseudoeffective divisor D is obvious due to the Picard number 1 condition on the 3-fold. This example is taken from Ottem's paper [27, Example 1.2.vii].

Boucksom, Demailly, Păun and Peternell showed that the dual cone of the pseudoeffective cone is the cone of movable curves [5]. Hence we have the equivalent statement:

Corollary 4.3.4. *With the same assumptions as in theorem 4.3.1, the map on the numerical equivalence classes of* 1*-cycles,* $\iota_* : N_1(Y) \to N_1(X)$*, induces* $\iota_* : \overline{Mov}_1(Y) \to \overline{Mov}_1(X)$*, where* $\overline{Mov}_1(Y)$ *and* $\overline{Mov}_1(X)$ *are the cones of movable curves in* Y *and* X*, respectively.*

We apply the adjunction formula to get

Corollary 4.3.5. *If both* X *and* Y *are non-singular,* Y *has nef normal bundle and* K_X *is pseudoeffective, then* K_Y *is also pseudoeffective. If* K_X *is big, then* K_Y *is also big.*

Remark 4.3.6. The first assertion in the above corollary follows also from [5] and the theory of deformation of rational curve. More specifically, Boucksom-Demailly-Păun-Peterenell showed that on a smooth projective variety Z, K_Z is pseudoeffective if and only if Z is not uniruled. If Y is uniruled, take a smooth rational curve C that covers Y. By considering the short exact sequence of normal bundles on C, we see that the normal bundle of C in X is nef. Thus, X is uniruled.

4.4 Weakly movable cone

We shall define and study the weakly movable cone. In this section, we assume the ground field k is algebraically closed and of characteristic zero. On a smooth projective variety, we know that the movable cone of divisors is the smallest closed convex cone that contains all the pushforwards of nef divisors from X_π, where $\pi : X_\pi \to X$ is projective and birational. With this in mind, we define the weakly movable cone as the closure of the cone that is generated by pushforward of cycles of nef subvariety via generically finite morphism. We find that it contains the movable cone and satisfies some desirable intersection theoretic properties.

First, let us recall the definition of a family of effective cycles. We shall follow Fulger-Lehmann's definition [10].

Definition 4.4.1 (Family of effective cycles)**.** Let X be a projective variety over k. A *family of effective d-cycles* on X with $\mathbb{Z}$-coefficient, $(g : U \to W)$, consists of a closed reduced subscheme $SuppU$ of $W \times_k X$, where W is a variety over k; a coefficient $a_i \in \mathbb{Z}_{>0}$ for each irreducible component U_i of $SuppU$; and the projection morphisms $g_i : U_i \to W$ is proper and flat of relative dimension d.

Over a closed point $w \in W$, $g_i^{-1}(w)$ is a closed subscheme of X. Its fundamental cycle $[g_i^{-1}(w)]$ is a d-cycle of X. We define the *cycle theoretic fiber* over w to be $\sum a_i[g_i^{-1}(w)]$.

We say that the family of effective d-cycles is *irreducible* if $SuppU$ is irreducible.

Remark 4.4.2. Kollár's definition [21, Definition I.3.11] of a *well-defined family of d-dimensional proper algebraic cycles* is more general. By [21, Lemma I.3.14], given an effective, well-defined family of proper algebraic cycles of a projective variety X over a variety W (both are over k), there is a proper surjective morphism $W' \to W$ from a variety W' such that there is a family of effective cycles (in the sense of Fulger-Lehmann) over W' that "preserves" the cycle theoretic fibers over the closed points of the original family. Therefore for our purpose, it is enough to use Fulger-Lehmann's definition.

Definition 4.4.3 (Strictly movable cycles [10, Definition 3.1])**.** We say that a family of effective d-cycles of X $(g : U \to W)$ is *strictly movable* if each of the irreducible components U_i of $SuppU$ dominates X via the second projection.

We say that an effective d-cycle of X (with $\mathbb{Z}$-coefficient) is *strictly movable* if it is the cycle theoretic fiber over a closed point of a strictly movable family of d-cycles on X.

We define the *movable cone of d-cycles $\overline{Mov}_d(X) \subset N_d(X)$ to be the closure of the convex cone generated by strictly movable d-cycles.*

Proposition 4.4.4. *The movable cone of d-cycles is the closure of the convex cone generated by irreducible, strictly movable d-cycles.*

Proof. Suppose $\sum a_i Z_i$ is the cycle theoretic fiber over a closed point of a family of strictly movable d-cycles $(g : U \to W)$ with irreducible components U_i. It suffices to show that Z_i

is algebraically equivalent to a sum of irreducible strictly movable d-cycles. If the generic fiber of $p_i : U_i \to W$ is geometrically integral, then the fiber over a general closed point is also (geometrically) integral [15, Théorème 9.7.7], and we are done.

Suppose the generic fiber of p_i is not geometrically integral. Let η_W be the generic point of W, let $\overline{k(\eta_W)}$ be the algebraic closure of $k(\eta_W)$ and let $U'_{ij} \subset X_{\overline{k(\eta_W)}}$ be the irreducible components of $Spec\overline{k(\eta_W)} \times_{Speck(\eta_W)} U_i$. We may take a finite field extension $k(\eta_W) \subset K$, such that the generators of the ideal sheaves of U'_{ij} are defined over K. Then all the irreducible components of $SpecK \times_{Speck(\eta_W)} U_i$ are geometrically integral. These components dominate the generic fiber of p_i. Take a variety V with function field K such that the map $SpecK \to Speck(\eta_W)$ extends to $V \to W$. By generic flatness, we may replace V by a smaller open set and assume that each irreducible components U_{ij} of $V \times_W U_i$ is flat over V. Note that all U_{ij} dominates U_i, hence also X. Thus, each U_{ij} is a strictly movable family of d-cycles of X over V (with coefficient 1), and the cycle theoretic fiber over a general closed point of V is (geometrically) integral, by [15, Théorème 9.7.7] again. Then Z_i is algebraically equivalent to the sum of the cycle theoretic fibers of U_{ij}'s, with $\mathbb{Z}$-coefficient, over a general closed point of V. □

Proposition 4.4.5. *An irreducible, strictly movable cycle can be realized as the pushforward of a multiple of the cycle class of a nef subvariety via a proper, surjective morphism, up to numerical equivalence.*

Proof. From the proof of proposition 4.4.4, we may assume the irreducible, strictly movable cycle is the cycle theoretic fiber over a closed point of an irreducible, strictly movable family of $(g : U \to W)$, with the fiber of $g' : SuppU \to W$ over a general closed point of W integral. Using the argument in [10, Remark 2.13] or [21, Proposition I.3.14], we may assume W is projective. We note that a closed point $w \in W$ is nef, hence $g'^{-1}(w)$ is also nef, by proposition 2.3.8, and that $g'^{-1}(w)$ is integral if w is general. □

Definition 4.4.6 (Weakly movable cone)**.** Let X be a projective variety over k. We define the *weakly movable cone* $\overline{WMov}_d(X) \in N_d(X)$ to be the closure of the convex cone generated by $\pi_*[Z]$, where $\pi : Y \to X$ is proper, surjective morphism from a projective variety and Z is a nef subvariety of dimension d in Y.

We shall compare the movable cone and the weakly movable cone.

Proposition 4.4.7. *Let X be a projective variety over k. We have*

$$\overline{Mov}_d(X) \subseteq \overline{WMov}_d(X).$$

In particular, $\overline{WMov}_d(X)$ is a full dimensional cone in $N_d(X)$.

Proof. This follows from proposition 4.4.5 and [10, Proposition 3.8]. □

The following proposition is an analogue of the first statement of [10, Lemma 3.6].

Proposition 4.4.8. *Let X' and X be projective variety over k. Suppose $h : X' \to X$ is a proper surjective morphism. Then $h_*\overline{WMov}_d(X') \subseteq \overline{WMov}_d(X)$.*

Proof. It follows from the definition of the weakly movable cone. □

The following theorem is an analogue of [10, Lemma 3.10].

Theorem 4.4.9. *Let X be a projective variety over k and let $\alpha \in \overline{WMov}_d(X)$. Then*

1. *If $\beta \in \overline{Eff}^1(X)$, then $\beta \cdot \alpha \in \overline{Eff}_{d-1}(X)$.*
2. *Let H be a big Cartier divisor. If $H \cdot \alpha = 0$, then $\alpha = 0$.*
3. *If $\beta \in Nef^1(X)$, then $\beta \cdot \alpha \in \overline{WMov}_{d-1}(X)$.*

Proof. For (1), we may assume $\alpha = \pi_*[Z]$, where $\pi : Y \to X$ is a proper, surjective map and Z a nef subvariety of Y. By projection formula, we have $\beta \cdot \pi_*[Z] = \pi_*(\pi^*\beta \cdot [Z])$. We know that $\pi^*\beta$ is pseudoeffective. By theorem 4.3.1, $\pi^*\beta \cdot [Z] \in \overline{Eff}_{d-1}(Y)$. Since $\pi_*\overline{Eff}_{d-1}(Y) \subseteq \overline{Eff}_{d-1}(X)$, we have $\beta \cdot \pi_*[Z] \in \overline{Eff}_{d-1}(X)$.

For (2), we follow Fulger-Lehmann's argument [10, Proof of Lemma 3.10]. We write $H = A + E$, where A is ample and E is effective. By (1), $A \cdot \alpha, E \cdot \alpha \in \overline{Eff}_{d-1}(X)$. In particular, $H \cdot \alpha = 0$ implies $A \cdot \alpha = 0$ [11, Corollary 3.8], which can only happen when $\alpha = 0$ [11, Corollary 3.16].

For (3), we may again assume $\alpha = \pi_*[Z]$, where $\pi : Y \to X$ is a proper, surjective map and Z a nef subvariety of Y. We also assume $d \geq 2$, otherwise the result already follows from (1). Note that $\pi_*\overline{WMov}_{d-1}(Y) \subseteq \overline{WMov}_{d-1}(X)$ by the definition of weakly movable cone. It suffices to show that $H \cdot [Z] \in \overline{WMov}_{d-1}(Y)$, where H is a very ample divisor on Y. We may assume that $H \cap Z$ is of dimension $d-1$ and is integral [19, Corollaire 6.11]. By corollary 4.1.6, $H \cap Z$ is a nef subvariety in Y. □

Proposition 4.4.10. *Let X be a projective variety of dimension n over k. Then*

$$\overline{WMov}_1(X) = \overline{Mov}_1(X)$$

Proof. Let $\pi : Y \to X$ be a proper, surjective map, $Z \subset Y$ be a nef subvariety of dimension 1. To show that $\pi_*[Z] \in \overline{Mov}_1(X)$, it suffices to show that $D \cdot \pi_*[Z] = \pi^* D \cdot [Z] \geq 0$ for any pseudoeffective divisor on X, since the dual cone of $\overline{Mov}_1(X)$ is the cone of pseudoeffective divisors [5]. This follows from theorem 4.3.1. □

Let us recall Hartshorne's conjecture A:

Conjecture 4.4.11 ([16, Conjecture 4.4])**.** *Let X be a smooth projective variety, and let Y be a smooth subvariety with ample normal bundle. Then $n[Y]$ moves in a large algebraic family for $n \gg 0$.*

This was disproved by Fulton and Lazarsfeld. They constructed an ample rank 2 vector bundle on $\mathbb{P}^2$, such that any multiple of the zero section in the total space of the vector bundle does not move.

In view of proposition 4.4.7, theorem 4.4.9 and proposition 4.4.10, it seems reasonable for us to ask the following

Question 1. Let X be a projective variety of dimension n. Do we have

$$\overline{WMov}_d(X) = \overline{Mov}_d(X),$$

for $1 \leq d \leq n-1$?

If the answer is yes, the cycle class of any nef subvariety of X will lie in the movable cone. The key point in the question is that we only consider the cycle classes up to numerical equivalence; the movable cone is also defined to be the closure of the cone generated by movable cycles. This seems to be one of the weakest possible ways of stating the conjecture that relates positivity of the normal bundle of subvarieties and their movability. However, it is possible that the two cones are different in general.

One might want to study the closure of the convex cone generated by the cycle class of nef subvarieties of dimension d (in $N_d(X)$) instead. We now give an example where it is not of full dimension, when $d = \dim X - 1$.

Lemma 4.4.12 ([26, Corollary 3.4])**.** *Let X be a normal projective variety over k. Let $Y \subset X$ be a nef subscheme of codimension 1. Then Y is a (nef) Cartier divisor.*

Proof. Let $\pi : Bl_Y X \to X$ be the blowup of X along Y, with exceptional divisor E. Then $\pi|_E : E \to Y$ is equidimensional of relative dimension 0, by proposition 4.1.1. Therefore, π is quasi-finite. A proper and quasi-finite morphism is finite, so π is finite and birational, with X normal. This implies that π is in fact an isomorphism. □

Let X be a projective variety of dimension n over k. By [12, Example 19.3.3], the natural map $N^1(X) \xrightarrow{\cdot[X]} N_{n-1}(X)$ is injective. Fulger-Lehmann gave an example [11, Example 2.7] where $N^1(X) \xhookrightarrow{\cdot[X]} N_{n-1}(X)$ is not surjective. We may assume that X is normal in their example. By the above lemma, the closure of the convex cone generated by the cycle class of nef subschemes of codimension 1 lies in the subspace $N^1(X) \subsetneq N_{n-1}(X)$, hence is not full dimensional.

CHAPTER 5

NUMERICAL DIMENSION AND LOCALLY AMPLE CURVES

We showed that the restriction of a pseudoeffective divisor to a nef subvariety is pseudoeffective (theorem 4.3.1). In this chapter, we shall study how the numerical dimension of the classes on the boundary of $\overline{Eff}^1(X)$ behave under the restriction $\iota^* : \overline{Eff}^1(X) \to \overline{Eff}^1(Y)$, assuming Y is locally ample.

Nakayama showed that if H is a smooth ample divisor of a smooth projective variety X and $\eta \in N^1(X)_{\mathbf{R}}$ is not big, then $\kappa_\sigma(\eta) \le \kappa_\sigma(\eta|_H)$ [25, Proposition 2.7(5)]. On the other hand, Ottem showed that if X is a smooth projective variety, Y is a locally complete intersection subvariety with ample normal bundle and $\eta \in N^1(X)_{\mathbf{R}}$ satisfies $\eta|_Y = 0$, then $\kappa_\sigma(\eta) = 0$ [27, Theorem 1]. This was a conjecture due to Peternell [28, Conjecture 4.12]. The following theorem generalizes both of the above results.

Theorem 5.1. *Let $\iota : Y \hookrightarrow X$ be a locally ample subvariety of codimension r of a projective variety X. If $\eta \in N^1(X)_{\mathbf{R}}$ is a pseudoeffective class such that $\eta|_Y$ is not big, then $\kappa_\sigma(\eta) \le \kappa_\sigma(\eta|_Y)$.*

From this, we deduce the following result (see theorem 5.3.5). Let Y be a locally ample subvariety of X and let $f : X \to Z$ be a morphism from X to a projective variety Z. If $\dim f(Y) < \dim Y$, then $f|_Y : Y \to Z$ is surjective, i.e. $f(Y) = Z$.

One can regard these results as evidence that it is natural to study the notion of locally ample subvariety.

We now turn our focus to the main application of theorem 5.1.

It seems interesting to ask how the positivity of the normal bundle of a subvariety influences the positivity of the underlying cycle class of the subvariety. The divisor case is well-known. For example, ample divisors generate an open cone in $N^1(X)_{\mathbb{R}}$, called the ample cone. The closure of the ample cone is dual to the closure of the cone generated by curves in X (Kleiman). Furthermore, an effective Cartier divisor with ample normal bundle is big [16, Theorem III.4.2]. In this paper, we want to see whether similar properties hold for curves. Boucksom, Demailly, Păun and Peternell [5] showed that the closure of the cone of effective divisors in $N^1(X)_{\mathbb{R}}$, called the pseudoeffective cone, is dual to the closure of the cone generated by strongly movable curves, called the movable cone of curves. Using this result, one can show that the cycle class of a nef curve (in particular a curve with nef normal bundle) lies in the movable cone of curves ([7, Theorem 4.1], theorem 4.3.1). By analogy to the divisor case, it is natural to pose the following question: given a locally ample (resp. ample) curve, does the cycle class of the curve lie in the *interior* of the cone of curves (resp. movable cone of curves)? In this dissertation, we give a positive answer to this question.

Theorem 5.2. *Let X be a projective variety and let Y be a locally ample curve in X. Then $[Y] \in N_1(X)_{\mathbf{R}}$ is big, i.e. it lies in the interior of the cone of curves. Furthermore, if Y meets all prime divisors of X, e.g. Y is ample, then $[Y]$ lies in the interior of the movable cone of curves.*

Following an observation of Peternell [28, Conjecture 4.1], Ottem already deduced that the cycle class of a locally complete intersection curve with ample normal bundle in a smooth projective variety lies in the interior of the cone of curves ([27, Theorem 2]). Indeed, if $\eta \in N^1(X)_{\mathbf{R}}$ is nef and $\eta|_Y = 0$, then the conjecture says $\kappa_\sigma(\eta) = 0$, which forces $\eta = 0$. Theorem 5.2 improves upon Ottem's result by removing the assumptions that X is smooth and Y is locally complete intersection. Our proof is different from Ottem's in the sense that the theory of q-ample divisors is used here. The use of q-ample divisors also enables us to obtain the more general statement.

5.1 Numerical dominance

In this section, we prove a basic fact on Nakayama's notion of numerical dominance (proposition 5.1.3), which will streamline the argument in the proof of theorem 5.1.

Let us first start by stating the definition of numerical dominance.

Definition 5.1.1. [25, Definition 2.12] Given two classes $\eta_1, \eta_2 \in N^1(X)_{\mathbf{R}}$. We say that η_1 *numerically dominates* η_2 if for any ample divisor A and for any $b \in \mathbf{R}$ there are $t_1, t_2 > b$ such that $t_1\eta_1 - t_2\eta_2 + A$ is pseudoeffective.

We say that a class $\eta \in N^1(X)_{\mathbf{R}}$ numerically dominates a closed subvariety Y of X if on the blowup $\pi : Bl_Y X \to X$, $\pi^*\eta$ numerically dominates the exceptional divisor E.

Lemma 5.1.2. *Let X be a projective variety and let $\eta_1, \eta_2 \in N^1(X)_{\mathbf{R}}$. Then η_1 numerically dominates η_2 if there exists an ample divisor A such that for any $b \in \mathbf{R}$ there are $t_1, t_2 > b$ such that $t_1\eta_1 - t_2\eta_2 + A$ is pseudoeffective.*

Proof. Suppose the hypothesis in the lemma holds. Given an ample divisor A', choose a large enough integer a such that $aA' - A$ is pseudoeffective. Given $b > 0$, take $t_1, t_2 > ab$ such that $t_1\eta_1 - t_2\eta_2 + A$ is pseudoeffective. Then $\frac{t_1}{a}\eta_1 - \frac{t_2}{a}\eta_2 + A' = \frac{1}{a}(t_1\eta_1 - t_2\eta_2 + A) + (A' - \frac{1}{a}A)$ is pseudoeffective. ☐

Let us relate the negation of numerical dominance and vanishing of the top cohomology group.

Proposition 5.1.3. *Let X be a projective variety of dimension n and let Y be a subvariety of X. Let E be the exceptional divisor on $\tilde{X} := Bl_Y X$, the blowup of X along Y. Let D be a pseudoeffective $\mathbf{R}$-Cartier $\mathbf{R}$-divisor on X, written as $\sum a_i C_i$, where $a_i \in \mathbf{R}$ and C_i's are integral Cartier divisors. Fix a $2n$-Koszul-ample line bundle $\mathcal{O}(H)$ on $\tilde{X}$.*

If there is some ample integral Cartier divisor A such that $A - (n+1)H$ and $A - (n+1)H + eE - \sum c_i C_i$ are ample for $e, c_i \in [0,1]$ on $\tilde{X}$ and there is some $b \in \mathbf{R}$ such that

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - A)) = 0$$

for all $t \in (b, +\infty)$ and for all integer $k > b$, then D does not numerically dominate Y.

On the other hand, if D does not numerically dominate Y, then for any divisor B, there is $b \in \mathbf{R}$ such that

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - B)) = 0$$

for all $t \in (b, +\infty)$ and for all integer $k > b$.

Proof. For the first statement, by the hypothesis,

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - A + (n+1)H) \otimes \mathcal{O}_{\tilde{X}}(-(n+1)H)) = 0$$

for $k, t > b$, $k \in \mathbb{Z}$. By theorem 2.1.5, $kE - \sum \lfloor ta_i \rfloor \pi^* C_i - A + (n+1)H$ is $(n-1)$-ample for $k, t > b$, $k \in \mathbb{Z}$. For $t_1, t_2 > b$, we can write $t_2 E - t_1 \pi^* D - (A - (n+1)H) =$

$(\lfloor t_2 \rfloor E - \lfloor t_1 \rfloor \pi^* D - \epsilon(A - (n+1)H)) + ((1-\epsilon)(A - (n+1)H) + \{t_2\}E - \sum\{t_1 a_i\}\pi^* C_i)$ and observe that the first term is $(n-1)$-ample and the second term is ample for $0 < \epsilon \ll 1$. It follows that $t_2 E - t_1 \pi^* D - (A - (n+1)H)$ is $(n-1)$-ample for $t_1, t_2 > b$. Thus, $t_1 \pi^* D - t_2 E + (A - (n+1)H)$ is not pseudoeffective for $t_1, t_2 > b$. This proves the first assertion.

For the second statement, for sufficiently large l, we can embed $\omega_{\tilde{X}} \hookrightarrow \mathcal{O}(lH)$. We may also assume that $B + lH$ is ample. By lemma 5.1.2, there is a b such that $t_1 \pi^* D - t_2 E + B + lH$ is not pseudoeffective for $t_1, t_2 > b$. Thus, for $k, t > b$ and $k \in \mathbb{Z}$,

$$\begin{aligned} h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \textstyle\sum \lfloor ta_i \rfloor \pi^* C_i - B)) &= h^0(\tilde{X}, \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\textstyle\sum \lfloor ta_i \rfloor \pi^* C_i - kE + B)) && \text{(Duality)} \\ &\leq h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\textstyle\sum \lfloor ta_i \rfloor \pi^* C_i - kE + B + lH)) && (\omega_{\tilde{X}} \hookrightarrow \mathcal{O}(lH)) \\ &= 0. \end{aligned}$$

□

5.2 Proof of theorem 5.1

We are now ready to demonstrate how the notion of numerical dominance comes into the picture.

Proposition 5.2.1. *Let X be a projective variety of dimension n, let Y be a locally ample subvariety of codimension r of X and let $\eta \in N^1(X)_{\mathbf{R}}$ be a pseudoeffective class such that $\eta|_Y$ is not big. Then η does not numerically dominate Y.*

Proof. Let $\tilde{X}$ be the blowup of X along Y, with exceptional divisor E. We fix a Koszul-ample line bundle $\mathcal{O}_{\tilde{X}}(H)$. Take $D = \sum a_i C_i$ to be an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor such that its class equals to η. Here $a_i \in \mathbf{R}$ and C_i's are integral Cartier divisors. We fix an integer $l > n+1$ such that $(l - (n+1))H + eE - \sum c_i C_i$ is ample for any $e, c_i \in [0,1]$.

We would like to prove that for any coherent sheaf $\mathcal{F}$ on E, there is k_0 such that

$$h^{n-1}(E, \mathcal{F} \otimes \mathcal{O}_E(kE - \textstyle\sum \lfloor ta_i \rfloor \pi^* C_i - lH)) = 0 \tag{5.1}$$

for $k \geq k_0$ and $t \geq 0$. It is enough to prove that for the vanishing of cohomology groups on each of the irreducible components of E. In other words, letting E' be an irreducible component of E, it suffices to prove that there is k_0' such that $h^{n-1}(E', \mathcal{F} \otimes \mathcal{O}_{E'}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i -$

$lH)) = 0$ for $k \geq k_0'$ and $t \geq 0$. As there is a surjection $\oplus \mathcal{O}(B) \twoheadrightarrow \mathcal{F}$, where $\mathcal{O}(B)$ is a line bundle, it suffices to prove the vanishing assuming $\mathcal{F}$ is a line bundle $\mathcal{O}(B)$. By duality,

$$h^{n-1}(E', \mathcal{O}_{E'}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i + B - lH)) = h^0(E', \omega_{E'} \otimes \mathcal{O}_{E'}(-kE + \sum \lfloor ta_i \rfloor \pi^* C_i - B + lH)),$$

where $\omega_{E'}$ is the dualizing sheaf of E'. We may embed $\omega_{E'} \hookrightarrow \mathcal{O}_{E'}(jH)$ for some j by lemma 2.4.3. It suffices to prove that there is k_0' such that

$$h^0(E', \mathcal{O}_{E'}(-kE + \sum \lfloor ta_i \rfloor \pi^* C_i - B + (l+j)H)) = 0 \tag{5.2}$$

for $k \geq k_0'$ and $t \geq 0$.

As $D|_Y$ is not big, $-D|_Y$ is $(n-r-1)$-almost ample. By proposition 2.1.9, $\pi^*(-D)|_E$ is also $(n-r-1)$-almost ample. Since $\mathcal{O}_E(E)$ is $(r-1)$-ample, we may take k_0' such that $(kE + \sum e_i \pi^* C_i + B - (l+j)H)|_{E'}$ is $(r-1)$-ample for $k \geq k_0'$ and $e_i \in [0,1]$, thanks to the openness of the $(r-1)$-ample cone (theorem 2.1.7). Thus, for $k \geq k_0'$ and $t \geq 0$,

$$(kE - \sum \lfloor ta_i \rfloor \pi^* C_i + B - (l+j)H)|_{E'} = ((kE + \sum \{ta_i\} \pi^* C_i + B - (l+j)H) + \pi^*(-tD))|_{E'}$$

is $(r-1) + (n-r-1) = (n-2)$-ample, by theorem 2.1.7. Now we have (5.2) by [30, Theorem 9.1], hence also (5.1).

If we fix t and take k large enough, then $h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - lH))) = 0$, since E is $(n-1)$-ample. We tensor the short exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(kE) \to \mathcal{O}_{\tilde{X}}((k+1)E) \to \mathcal{O}_E((k+1)E) \to 0 \tag{5.3}$$

by $\mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - lH)$, and consider its associated long exact sequence of cohomologies. We apply (5.1), letting $\mathcal{F}$ to be the structure sheaf $\mathcal{O}_E$, there is k_0 such that $h^{n-1}(E, \mathcal{O}_E(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - lH)) = 0$ for $k \geq k_0$ and $t \geq 0$. Therefore,

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - lH)) = 0$$

for $k \geq k_0$ and $t \geq 0$. We may now conclude the proof by applying proposition 5.1.3. □

Proposition 5.2.2. *Let X be a projective variety and let Y be a subvariety of X. Let D be a pseudoeffective* **R***-Cartier* **R***-divisor such that D does not numerically dominate Y. Let $\pi : \tilde{X} \to X$ be the blowup of X along Y, with exceptional divisor E. Suppose $\pi|_E : E \to Y$ is an equidimensional morphism. Then $\kappa_\sigma(D) \leq \kappa_\sigma(D|_Y)$.*

Proof. We use the same notations as in the proof of the preceding proposition. By proposition 2.5.4, $\kappa_\sigma(D) = \kappa_\sigma(\pi^* D)$. It is enough to look at the growth (in t) of $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\sum \lfloor ta_i \rfloor \pi^* C_i + b_1 H))$, for a large enough integer b_1. Since $\omega_{\tilde{X}}$ is generically a line bundle, the natural map $\mathcal{O}_{\tilde{X}} \to \omega_{\tilde{X}}^\vee \otimes \omega_{\tilde{X}}$ is an injection. We have the inequality

$$\begin{aligned} h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\sum \lfloor ta_i \rfloor \pi^* C_i + b_1 H)) &\leq h^0(\tilde{X}, \omega_{\tilde{X}}^\vee \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(\sum \lfloor ta_i \rfloor \pi^* C_i + b_1 H)) \\ &= h^n(\tilde{X}, \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - b_1 H)). \end{aligned}$$

There is some surjection $\oplus^N \mathcal{O}_{\tilde{X}}(-b_2 H) \twoheadrightarrow \omega_{\tilde{X}}$. Therefore,

$$h^n(\tilde{X}, \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - b_1 H)) \leq N \cdot h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H))$$

By proposition 5.2.1 and proposition 5.1.3, there is k_0 such that

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H)) = 0$$

for $k \geq k_0$ and $t \geq k_0$. Tensoring the short exact sequence 5.3 by $\mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H)$ and considering the associated long exact sequence of cohomologies, we have

$$h^n(\tilde{X}, \mathcal{O}_{\tilde{X}}(-\sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H)) \leq \sum_{k=1}^{k_0} h^{n-1}(E, \mathcal{O}_E(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H))$$

for $t \geq k_0$.

Note that the restriction of $\pi : \tilde{X} \to X$ to the exceptional divisor $\pi|_E : E \to Y$ is an equidimensional morphism, with fiber dimension equal to $r - 1$. Thus, $R^d(\pi|_E)_* \mathcal{O}_E(kE - (b_1 + b_2) H) = 0$ for $d > r - 1$. Note also that $\dim Y = n - r$, which implies that $h^d(Y, \mathcal{F}) = 0$ for $d > n - r$ and for any coherent sheaf $\mathcal{F}$ on Y. We now apply Leray spectral sequence and the above remarks to see that for $1 \leq k \leq k_0$,

$$\begin{aligned} &h^{n-1}(E, \mathcal{O}_E(kE - \sum \lfloor ta_i \rfloor \pi^* C_i - (b_1 + b_2) H)) \\ &= h^{n-r}(Y, (R^{r-1}(\pi|_E)_* \mathcal{O}_E(kE - (b_1 + b_2) H)) \otimes \mathcal{O}_Y(-\lfloor ta_i \rfloor C_i)) \\ &= h^0(Y, \omega_Y \otimes (R^{r-1}(\pi|_E)_* \mathcal{O}_E(kE - (b_1 + b_2) H))^\vee \otimes \mathcal{O}_Y(\lfloor ta_i \rfloor C_i)) \qquad \text{(Duality)} \end{aligned}$$

Since $(R^{r-1}(\pi|_E)_* \mathcal{O}_E(kE - (b_1 + b_2) H))^\vee$ is reflexive [18, Corollary 1.2] and by lemma 2.4.3, for sufficiently large l, there is an embedding $\omega_Y \otimes (R^{r-1}(\pi|_E)_* \mathcal{O}_E(kE - (b_1 + b_2) H))^\vee \hookrightarrow \oplus^{N_k} \mathcal{O}_Y(lH)$ for $1 \leq k \leq k_0$. We can conclude that $h^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\sum \lfloor ta_i \rfloor \pi^* C_i + b_1 H)) \leq N \cdot (\sum_{k=1}^{k_0} N_k) \cdot h^0(Y, \mathcal{O}_Y(\lfloor ta_i \rfloor C_i + lH))$ for $t \gg 0$. This proves the proposition. □

Proof of theorem 5.1. Combine proposition 5.2.1 and 5.2.2 and note that if Y is locally ample, then $E \to Y$ is equidimensional (proposition 4.1.1). □

5.3 Applications of theorem 5.1

We give two applications of theorem 5.1. The first one is on positivity of cycle classes of locally ample and ample curves (theorem 5.2); the second one concerns the fact that locally ample subvarieties cannot be contracted (theorem 5.3.5).

5.3.1 Cycle classes of locally ample curves

Peternell conjectured that if Y is a smooth curve with ample normal bundle in a smooth projective variety X and $\eta \in N^1(X)$ is a pseudoeffective class with $\eta|_Y = 0$, then $\kappa_\sigma(\eta) = 0$ [28, Conjecture 4.12]. Ottem later showed that the conjecture is indeed true [27, Theorem 1]. From there, Peternell observed that the cycle class of a smooth curve with ample normal bundle lies in the interior of the cone of curves ([28, Conjecture 4.1], [27, Theorem 2]). Indeed, if $\eta \in N^1(X)_{\mathbf{R}}$ is nef and $\eta|_Y = 0$, the conjecture says $\kappa_\sigma(\eta) = 0$. But this forces $\eta = 0$. We are able to generalize this result by removing any restrictions on smoothness on X and Y.

Proposition 5.3.1. *[27] Let X be a projective variety. Let $\eta \in N^1(X)_{\mathbf{R}}$ be a pseudoeffective class. If $\kappa_\sigma(\eta) = 0$ and η is nef, then $\eta = 0$.*

Proof. It follows from the argument on [27, p. 5]. We include the proof here for the sake of completeness.

Let H be an ample divisor of X. Note that if we can prove that $\eta|_H = 0$, it would imply $\eta = 0$. By induction on dimension of X, it suffices to show that $\kappa_\sigma(\eta|_H) = 0$. Let $D = \sum a_i C_i$ be a pseudoeffective $\mathbf{R}$-Cartier $\mathbf{R}$-divisor such that the numerical class of D is η. Here $a_i \in \mathbf{R}$ and C_i's are integral Cartier divisors. By Fujita vanishing theorem, there is a k_1 such that for $k \geq k_0$,

$$H^1(X, \mathcal{O}_X(kH + N) = 0,$$

for any nef divisor N. Take a sufficiently large k_1 such that $k_1 H - \sum e_i C_i$ is ample, for any $e_i \in [0, 1]$. For $t \geq 0$, $k_1 H + \sum \lfloor t a_i \rfloor C_i = tD + (k_1 H - \sum \{t a_i\} C_i)$ is nef. Thus,

$$H^1(X, \mathcal{O}_X(kH + \sum \lfloor t a_i \rfloor D) = 0$$

for $k \geq k_0 + k_1$. Therefore, we have the surjection

$$H^0(X, \mathcal{O}_X(\sum \rfloor ta_i \lfloor D + kH) \twoheadrightarrow H^0(H, \mathcal{O}_H(\sum \rfloor ta_i \lfloor D + kH)$$

for $k \geq k_0 + k_1$ and $t \geq 0$. Hence $\kappa_\sigma(\eta|_H) = 0$. □

The following theorem generalizes the first half of the main theorem in Ottem's paper [27, Theorem 2].

Theorem 5.3.2. *Let X be a projective variety. Let Y be a locally ample subvariety of dimension 1 of X. Then the cycle class of Y in $N_1(X)_{\mathbf{R}}$ is big, i.e. it lies in the interior of the cone of curves, $\overline{\mathrm{NE}}(X)$.*

Proof. Suppose there is some nef class $\eta \in N^1(X)_{\mathbf{R}}$ such that $\eta|_Y = 0$. By theorem 5.1, $\kappa_\sigma(\eta) = 0$. We then apply proposition 5.3.1 to conclude that $\eta = 0$. □

We shall need the following proposition which shows that a pseudoeffective class $\eta \in N^1(X)_{\mathbf{R}}$ on a smooth projective variety with $\kappa_\sigma(\eta) = 0$ is in fact "effective".

Proposition 5.3.3. *[25, Proposition V.2.7] Let X be a smooth projective variety. Let $\eta \in N^1(X)_{\mathbf{R}}$ be a pseudoeffective class. If $\kappa_\sigma(\eta) = 0$, then there is an $\mathbf{R}$-Cartier $\mathbf{R}$-divisor $\sum a_i C_i$, where $a_i \in \mathbf{R}_{>0}$ and C_i are prime divisors, such that its numerical class in $N^1(X)_{\mathbf{R}}$ equals to η.*

We are now ready to show that the cycle class of an ample curve lies in the interior of the movable cone of curves. This strengthens the second half of [27, Theorem 2].

Theorem 5.3.4. *Let X be a projective variety and let Y be a locally ample curve in X. Suppose Y meets all prime divisors of X. Then the cycle class $[Y]$ lies in the interior of the movable cone of curves. In particular, the cycle class of an ample subvariety of dimension 1 lies in the interior of the movable cone of curves.*

Proof. Note that the second statement follows from the first. Indeed, if Y is an ample curve in X, then $H^{n-1}(X \backslash Y, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ on $X \backslash Y$ [26, Proposition 5.1]. In particular, $X \backslash Y$ cannot contain any prime divisor.

Let $\pi : \tilde{X} \to X$ be the blowup of X along Y, let $X' \xrightarrow{f'} \tilde{X} = Bl_Y X$ be a resolution of singularities on $\tilde{X}$ and let $f = \pi \circ f'$ be the composition. The famous result in [5] says that the dual cone of the movable cone of curves is the pseudoeffective cone. We can apply theorem 4.3.1 to see that $[Y]$ lies in the movable cone of curves. It suffices to show that for any pseudoeffective class $\eta \in N^1(X)_{\mathbf{R}}$ such that $\eta \cdot [Y] = 0$, then $\eta = 0$.

Theorem 5.1 says that $\kappa_\sigma(f^*\eta) = \kappa_\sigma(\eta) = 0$. As $f^*\eta$ is pseudoeffective, it is equal to the class of an effective **R**-Cartier **R** divisor $\sum b_i B_i$ where $b_i > 0$ and B_i's are prime divisors by proposition 5.3.3.

Suppose $\bigcup Supp(B_i) \cap f^{-1}(Y) = \varnothing$. By the projection formula, $[\eta] \equiv \sum b_i f_*[B_i]$ in $N_{n-1}(X)$. But $\bigcup Supp f(B_i) \cap Y = \varnothing$ and the hypothesis imply all B_i's are exceptional. Thus $[\eta] = 0$ in $N_{n-1}(X)$ and $\eta = 0$ by [11, Example 2.7].

We may assume $\bigcup Supp(B_i) \cap f^{-1}(Y) \neq \varnothing$. Applying the negativity lemma to $\sum b_i B_i$ (note that $-\sum b_i B_i$ is clearly f-nef), for any closed point $p \in f(\bigcup Supp(B_i))$, $f^{-1}(p) \subset \bigcup Supp(B_i)$. Take a curve $C' \subset f^{-1}(Y)$ such that $f(C') = Y$. By the previous remark, $C' \cap \bigcup Supp(B_i) \neq \varnothing$. On the other hand, $\sum b_i B_i \cdot [C'] = f^*\eta \cdot [C'] = \deg(\kappa(C) : \kappa(Y))\eta \cdot [Y] = 0$. Therefore, $C' \subset \bigcup Supp(B_i)$ and $f^{-1}(Y) \subset \bigcup Supp(B_i)$. Thus, $f'^*(\pi^*\eta - \epsilon E)$ is pseudoeffective for some small $\epsilon > 0$. But proposition 5.2.1 says that η does not dominate Y numerically. This gives a contradiction. □

5.3.2 Locally ample subvarieties cannot be contracted

In this subsection, we show that, as a consequence of theorem 5.1, a locally ample subvariety cannot be contracted.

Theorem 5.3.5. *Let X be a projective variety and let Y be a locally ample subvariety of X. Suppose $f : X \to Z$ is a morphism from X to a projective variety Z. Then if $\dim f(Y) < \dim Y$, then $f|_Y : Y \to Z$ is surjective, i.e. $f(Y) = Z$.*

Proof. Let A be an ample divisor on Z. Then $\dim f(Y) = \kappa_\sigma(A|_{f(Y)}) = \kappa_\sigma(f^*(A)|_Y) < \dim Y$. Note that $f^*(A)|_Y$ is not big. By theorem 5.1,

$$\kappa_\sigma(f^*(A)) \leq \kappa_\sigma(f^*A|_Y).$$

But $\kappa_\sigma(f^*(A)) = \dim Z$. This forces the equality $\dim Z = \dim f(Y)$. □

Remark 5.3.6. The special case of theorem 5.3.5, where Y is contracted to a point, is observed by Ottem by an elementary argument [27, Proof of Lemma 12].

REFERENCES

[1] *Stacks project*, 2016.

[2] A. Andreotti and H. Grauert, *Théorèmes de finitude pour la cohomologie des espaces complexes*, Bulletin de la Société mathématique de France, 79 (1962), pp. 193–259.

[3] D. Arapura, *Partial regularity and amplitude*, American Journal of Mathematics, 128 (2006), pp. 1025–1056.

[4] J. Backelin, *On the rates of growth of the homologies of veronese subrings*, in Lecture Notes in Mathematics, Springer Nature, 1986, pp. 79–100.

[5] S. Boucksom, J.-P. Demailly, M. Păun, and T. Peternell, *The pseudo-effective cone of a compact khler manifold and varieties of negative kodaira dimension*, Journal of Algebraic Geometry, 22 (2012), pp. 201–248.

[6] M. V. Brown, *Big q-ample line bundles*, Compositio Mathematica, 148 (2012), pp. 790–798.

[7] J.-P. Demailly, T. Peternell, and M. Schneider, *Holomorphic line bundles with partially vanishing cohomology*, Israel Mathematical Conference Proceedings, 9 (1996), pp. 165–198.

[8] T. Eckl, *Numerical analogues of the kodaira dimension and the abundance conjecture*, Manuscripta Mathematica, 150 (2016), pp. 337–356.

[9] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye, and M. Popa, *Asymptotic invariants of base loci*, Annales de l'institut Fourier, 56 (2006), pp. 1701–1734.

[10] M. Fulger and B. Lehmann, *Zariski decompositions of numerical cycle classes*, Journal of Algebraic Geometry, 26 (2016), pp. 43–106.

[11] ——, *Positive cones of dual cycle classes*, Algebraic Geometry, 4 (2017), pp. 1–28.

[12] W. Fulton, *Intersection Theory*, Springer Nature, 1998.

[13] W. Fulton and R. Lazarsfeld, *Positivity and excess intersection*, in Enumerative Geometry and Classical Algebraic Geometry, Springer Nature, 1982, pp. 97–105.

[14] ——, *Positive polynomials for ample vector bundles*, The Annals of Mathematics, 118 (1983), p. 35.

[15] A. Grothendieck and J. Dieudonné, *éléments de géométrie algébrique. iv. étude locale des schémas et des morphismes de schémas. iii*, Publications mathématiques de l'IHÉS, 28 (1966), pp. 5–248.

[16] R. Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Springer Berlin Heidelberg, 1970.

[17] ——, *Algebraic Geometry*, Springer New York, 1977.

[18] ——, *Stable reflexive sheaves*, Mathematische Annalen, 254 (1980), pp. 121–176.

[19] J. Jouanolou, *Theoremes de Bertini et Applications (Progress in Mathematics) (French Edition)*, Birkhuser, 1983.

[20] D. S. Keeler, *Ample filters of invertible sheaves*, Journal of Algebra, 259 (2003), pp. 243–283.

[21] J. Kollár, *Rational Curves on Algebraic Varieties*, Springer Nature, 1996.

[22] A. Küronya, *Positivity on subvarieties and vanishing of higher cohomology*, Annales de l'institut Fourier, 63 (2013), pp. 1717–1737.

[23] R. Lazarsfeld, *Positivity in algebraic geometry. I and II*, Springer-Verlag, Berlin, 2004.

[24] B. Lehmann, *Comparing numerical dimensions*, Algebra & Number Theory, 7 (2013), pp. 1065–1100.

[25] N. Nakayama, *Zariski-decomposition and Abundance*, The Mathematical Society of Japan, 2004.

[26] J. C. Ottem, *Ample subvarieties and q-ample divisors*, Advances in Mathematics, 229 (2012), pp. 2868–2887.

[27] ——, *On subvarieties with ample normal bundle*, Journal of the European Mathematical Society, 18 (2016), pp. 2459–2468.

[28] T. Peternell, *Compact subvarieties with ample normal bundles, algebraicity, and cones of cycles*, The Michigan Mathematical Journal, 61 (2012), pp. 875–889.

[29] A. J. Sommese, *Submanifolds of abelian varieties*, Mathematische Annalen, 233 (1978), pp. 229–256.

[30] B. Totaro, *Line bundles with partially vanishing cohomology*, Journal of the European Mathematical Society, 15 (2013), pp. 731–754.

[31] C. Voisin, *Coniveau 2 complete intersections and effective cones*, Geometric and Functional Analysis, 19 (2009), pp. 1494–1513.