# TRACE IDEALS AND THE CENTERS OF ENDOMORPHISM RINGS OF MODULES OVER COMMUTATIVE RINGS

by

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A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics
The University of Utah
August 2016

**The University of Utah Graduate School**

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# ABSTRACT

Let $R$ be a commutative Noetherian ring and $M$ a finitely generated $R$-module. We establish cases in which the centers of $\mathrm{End}_R(M)$ and $\mathrm{End}_R(M^*)$ coincide with the endomorphism ring of the trace ideal of $M$. These observations are exploited to prove results for balanced and rigid modules, as well as modules with $R$-free endomorphism rings. As a consequence, we clarify the relationship between the properties of $M$ and those of its endomorphism ring.

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# ACKNOWLEDGMENTS

Special thanks to my advisor, Srikanth Iyengar, for his enthusiasm, attentiveness and patience. Many thanks to Susan Loepp and my Ph.D. committee members for their helpful comments and guidance in the final year of my Ph.D..

I'd like to thank my parents for their unyielding support, my brother for his palpable confidence in me and my Aunt Hervine Atkinson and Uncle Oscar Lindo for providing invaluable examples of academic excellence.

I also thank the University of Nebraska-Lincoln and the University of Utah, the institutions where I began and finished my Ph.D., and I thank Williams College, where this dissertation was written, for its continued support through the Gaius Charles Bolin Fellowship.

# CHAPTER 1

# INTRODUCTION

Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. The trace ideal of $M$, denoted $\tau_M(R)$, is the ideal $\sum \alpha(M)$ as $\alpha$ ranges over $M^* := \operatorname{Hom}_R(M, R)$ [13, Exercise 95]. We are interested in the connection between the properties of $M$ and those of $\tau_M(R)$. For special classes of modules, the corresponding trace ideals demonstrate remarkable properties. For example, an ideal is the trace ideal of a projective module if and only if it is idempotent and $\tau_M(R) = R$ if and only if every finitely generated $R$-module is a homomorphic image of a direct sum copies of $M$; see [15] and Section 2.1.

This work was motivated by hints in the literature about the relationship between $\tau_M(R)$ and the center of $\operatorname{End}_R(M)$. In [3], Auslander and Goldman show that when $\tau_M(R) = R$, each endomorphism in the center of $\operatorname{End}_R(M)$ is given by multiplication by a unique ring element (see also [13, Exercise 95]). That is, $\mathrm{Z}(\operatorname{End}_R(M)) = R$.

Two of our central results clarify this connection, first in the case when $M$ is reflexive and faithful and second in the case where $\tau_M(R)$ contains a nonzerodivisor; see Theorems 31 and 41. In both cases, $\operatorname{End}_R(\tau_M(R))$ may be identified with a subring of the total ring of quotients; Corollaries 37 and 44.

**Theorem.** *Let $R$ be a Noetherian ring and $M$ a finitely generated $R$-module.*

*(i) If $M$ is reflexive and faithful, then there is an isomorphism of $R$-algebras*

$$\operatorname{End}_R(\tau_M(R)) \cong \mathrm{Z}(\operatorname{End}_R(M))$$

*(ii) If $\tau_M(R)$ contains a nonzerodivisor, then there are isomorphisms of $R$-algebras*

$$\operatorname{End}_R(\tau_M(R)) \cong \operatorname{End}_R(\tau_{M^*}(R)) \cong \mathrm{Z}(\operatorname{End}_R(M^*)).$$

These theorems are established in Chapter 3. The remainder of the paper applies these results in various settings.

Section 4.1 is concerned with modules that have $R$-free endomorphism rings. We extend [22, Theorem 3.1]:

**Theorem.** *Let $R$ be a local Noetherian ring of depth $\leq 1$ and $M$ a finitely generated reflexive $R$-module. If $\operatorname{End}_R(M)$ has a free summand, then so does $M$. Therefore, $\operatorname{End}_R(M)$ is a free $R$-module only if $M$ is a free $R$-module.*

Sections 4.2 and 4.3 apply our results to balanced and rigid modules; these are modules where $\mathrm{Z}(\operatorname{End}_R(M)) = R$ and modules such that $\operatorname{Ext}^1_R(M, M) = 0$, respectively. For example, we prove the result below:

**Theorem.** *Let $R$ be a one-dimensional Gorenstein local ring and $M$ a torsionfree faithful $R$-module. If $M$ is rigid and $\mathrm{Z}(\operatorname{End}_R(M))$ is Gorenstein, then $M$ has a free summand.*

When $M$ is an ideal, this result has been discovered independently, using different techniques, by Huneke, Iyengar and Wiegand in [10]. These results support a conjecture of Huneke and Wiegand ([12, pp. 473-474]) and we further prove that the conjecture holds for trace ideals.

# CHAPTER 2

# TRACE IDEALS

## 2.1 Trace Ideals

In this chapter, we find results about trace ideals with a view to eventually clarifying the relationship between $\mathrm{End}_R(M)$ and $\tau_M(R)$. There are many available sources for otherwise focused discussions of trace ideals. Refer to [15, §2H] for a discussion with a view to Morita Theory, [9] for trace ideals of projective modules and [3, Appendix] for results on trace ideals and their role in detecting the projectivity of modules.

### 2.1.1 The Trace Map

In this section, we collect those observations about trace ideals that are needed in subsequent sections. Throughout, $R$ will be a commutative ring and $M$ a finitely presented $R$-module.

**Definition 1.** The trace ideal of $M$, denoted $\tau_M(R)$, is the ideal $\sum \alpha(M)$ as $\alpha$ ranges over $M^* := \mathrm{Hom}_R(M, R)$.

*Remark* 2. Trace ideals are characterized by the equality

$$\mathrm{End}_R(\tau_M(R)) = \tau_M(R)^*.$$

Set

$$M^* := \mathrm{Hom}_R(M, R),$$

viewed as an $R$-module. The *trace map* is the map

$$\begin{array}{cccc} \vartheta_M\colon & M \otimes_R M^* & \longrightarrow & R \\ & m \otimes \alpha & \longmapsto & \alpha(m), \end{array} \tag{2.1}$$

and the image of the trace map is the trace ideal of $M$ since

$$\begin{aligned}
\operatorname{Im}(\vartheta_M) &= \left\{ \sum_i^n \alpha_i(m_i) | \alpha_i \in M^*, m_i \in M \right\} \\
&= \sum_{\alpha \in M^*} \alpha(M) \\
&= \tau_M(R).
\end{aligned}$$

*Remark* 3. The trace ideal and trace map derived their names from the canonical trace map on matrices given by summing the diagonal entries; see [15, §2H Exercise 28]. When $R$ is a field, a finitely-generated $R$-module $M$ is a vector space and elements of $\operatorname{End}_R(M)$ are given by square matrices with coefficients in $R$. In this case, $\theta_M \in \operatorname{Hom}_R(M \otimes_R M^*, R)$ is the map induced by

$$M \otimes_R M^* \cong \operatorname{End}_R(M) = M_n(R) \xrightarrow{trace} R,$$

where $M \otimes_R M^* \cong \operatorname{End}_R(M)$ because $M$ is a projective $R$-module; see [6, p. 132].

There is a natural *evaluation map*

$$\begin{array}{rccc}
\varepsilon: & M & \longrightarrow & (M^*)^* \\
& m & \mapsto & \{\psi \mapsto \psi(m)\}
\end{array}$$

**Definition 4.** The $R$-module $M$ is said to be *torsionless* if $\varepsilon$ is an injection, and *reflexive* if $\varepsilon$ is an isomorphism.

When $M$ is reflexive, we may use Hom-Tensor adjointness to obtain the isomorphisms:

$$\begin{tikzcd}
\operatorname{Hom}_R(M, M) \arrow[r, "\cong"] \arrow[rr, bend left, "\vartheta"] & \operatorname{Hom}_R(M, M^{**}) \arrow[r, "\cong"] & \operatorname{Hom}_R(M \otimes_R M^*, R) \\
\{f\} \arrow[r, mapsto] & \{m \mapsto \varepsilon(f(m))\} \arrow[r, mapsto] & \{m \otimes \alpha \mapsto \alpha(f(m))\}.
\end{tikzcd} \tag{2.2}$$

Notice, the trace map is the image of the identity endomorphism, that is

$$\theta_M = \theta(\mathrm{id}_M).$$

For ease of notation, we write $\vartheta_f$ for $\vartheta(f)$ given any non-identity $f$ in $\operatorname{Hom}_R(M, M)$, so that

$$\vartheta_f(m \otimes \alpha) := \alpha(f(m)) \tag{2.3}$$

for any $m \otimes \alpha$ in $M \otimes_R M^*$.

One source of trace ideals is ideals of sufficiently high grade. Thanks to Alexandra Seceleanu for communicating the following example, which was observed in [8].

**Example 5.** Let $R$ be a commutative Noetherian ring and $I$ an ideal with

$$\mathrm{grade}(I) \geq 2.$$

Then $\tau_I(R) = I$.

Applying $\mathrm{Hom}_R(?, R)$, to the short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$ yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/I, R) \longrightarrow \mathrm{Hom}_R(R, R) \longrightarrow \mathrm{Hom}_R(I, R) \longrightarrow \mathrm{Ext}^1_R(R/I, R).$$

By [5, Thm 1.2.5 (Rees)],

$$\mathrm{Ext}^1_R(R/I, R) = 0 = \mathrm{Hom}_R(R/I, R)$$

and so all homomorphisms from $I$ to $R$ are given by multiplication by an element of $R$. The images of all such homomorphisms land in $I$; therefore, $\tau_I(R) = I$.

### 2.1.2 Note on Notation: $\tau_M(R)$

Given two $R$-modules $A$ and $B$, consider the trace map of $A$ into $B$.

$$\begin{aligned} A \otimes_R \mathrm{Hom}_R(A, B) &\longrightarrow B \\ a \otimes_R f &\longmapsto f(a). \end{aligned}$$

The image of this pairing, which we denote $\tau(A, B)$, is a function of both $A$ and $B$, and when $B$ is the ring $R$, this image is the trace ideal of $A$ as an $R$-module. Fixing $A$ and given an $R$-homomorphism $f : B_1 \longrightarrow B_2$, then $f$ also maps $\tau(A, B_1)$ to $\tau(A, B_2)$. There is a commutative diagram:

$$\begin{tikzcd} B_1 \arrow[r, "f"] & B_2 \\ \tau(A, B_1) \arrow[u, "\subseteq"] \arrow[r, "f"] & \tau(A, B_2) \arrow[u, "\subseteq"] \end{tikzcd}$$

There is no such natural map between $\tau(A_1, B)$ and $\tau(A_2, B)$ for $R$-modules $A_1$ and $A_2$. That is $\tau(\_, \_)$ is functorial in the second argument but not the first. Some sources, for example [17] and this work, therefore choose the write $\tau(M, R)$ as $\tau_M(R)$, to indicate functoriality in the ring $R$.

### 2.1.3 Calculating the Trace Ideal

We may calculate the trace ideal of a module from the module's presentation matrix. Given a finite presentation for an $R$-module $M$, with presentation matrix $A$, so that

$$R^m \xrightarrow{A} R^n \xrightarrow{\pi} M \longrightarrow 0$$

is exact, suppose $B$ is an $1 \times n$ matrix representing a map from $R^n$ to $R$.

$$\begin{tikzcd}
R^m \arrow[r, "A_{n\times m}"] & R^n \arrow[rr, "B_{1\times n}"] \arrow[dr, "\pi"'] & & R \\
& & M \arrow[ur, "\alpha"'] &
\end{tikzcd}$$

**Lemma 6.** *$B = \alpha \circ \pi$ for some $\alpha \in M^*$ if and only if $BA = 0$*

*Proof.* A map $B : R^n \longrightarrow R$ will factor through $M$ and hence induce a map $M \longrightarrow R$ if $\mathrm{Ker}(B) \supseteq \mathrm{Ker}(\pi) = \mathrm{Im}(A)$, that is, $BA = 0$. Also, any linear functional $\alpha : M \longrightarrow R$ can be precomposed with $\pi$ to obtain a map $B : R^n \longrightarrow R$ whose kernel contains $\mathrm{Ker}(\pi) = \mathrm{Im}(A)$. $\square$

Let $\mathfrak{B}$ be the matrix whose rows generate all such $B$'s. Then

$$\tau_M(R) = I_1(\mathfrak{B});$$

where $I_1(\mathfrak{B})$ is the ideal generated by the $1 \times 1$ minors of $\mathfrak{B}$.

Seen another way, applying $\mathrm{Hom}_R(?, R)$ to the finite presentation

$$R^m \xrightarrow{A} R^n \xrightarrow{\pi} M \longrightarrow 0$$

yields the exact sequence

$$\begin{tikzcd}
R^t \arrow[rr, "\mathfrak{B}^T"] \arrow[dr] & & (R^n)^* \arrow[r, "A^*"] & (R^m)^* \arrow[r] & \mathrm{Tr}(M) \arrow[r] & 0 \\
& M^* \arrow[ur] & & & &
\end{tikzcd}$$

where $\mathrm{Tr}(M)$ is the Auslander transpose of $M$.

It is clear that $\mathfrak{B}^T$ is a matrix whose columns generate the relations of $A^*$, the transpose of $A$. Also, the images of $\mathfrak{B}^T$ in each copy of $R$ in $(R^n)^*$ together generate the sum of the images of the maps $\alpha \in M^*$. Hence the equality:

$$\tau_M(R) = I_1(\mathfrak{B}^T) = I_1(\mathfrak{B});$$

see, for example, as in [24, Remark 3.3].

**Example 7.** Consider $S = \mathbb{R}[x,y,z]/(xy - z^2)$ and $I = (x,z)$.

A presentation for I is:

$$\begin{tikzcd} S^2 \arrow[r, "\begin{bmatrix} -z & y \\ x & -z \end{bmatrix}"] & S^2 \arrow[rr] \arrow[dr] & & 0 \\ & & (x,z) \arrow[ur] & \end{tikzcd}$$

The (left) nullspace of the presentation matrix is spanned by $\begin{bmatrix} y & z \end{bmatrix}$ and $\begin{bmatrix} z & x \end{bmatrix}$, so

$$\tau_I(R) = (x,y,z)$$

### 2.1.4 Properties of the Trace Ideal

The trace ideal of $M$ is the largest ideal (with respect to inclusion) generated by $M$ in the following sense:

**Definition 8.** Let $M$ and $N$ be $R$-modules. We say $N$ is *generated* by $M$ if $N$ is the homomorphic image of a direct sum of copies of $M$. Said otherwise, there is an exact sequence

$$M^{(\Lambda)} \longrightarrow N \longrightarrow 0$$

for some index set $\Lambda$.

**Definition 9.** An $R$-module $M$ is a *generator* for the category of finitely generated $R$-modules (denoted $R$-mod) if $M$ is finitely generated and generates every finitely generated $R$-module. Equivalently, $M$ is a *generator* if $R$ is a direct summand of $M^n$ for some $n \in N$.

**Example 10.** $M$ generates $\tau_M(R)$ and $M \otimes_R N$ for any $R$-module $N$.

Indeed, from the exact sequence $M \otimes_R M^* \xrightarrow{t} \tau_M(R) \longrightarrow 0$ we may derive another

$$\bigoplus_{i}^{\beta_0(M^*)} M \longrightarrow \tau_M(R) \longrightarrow 0;$$

where $\beta_0(M^*)$ is the minimum number of generators of $M^*$ as an $R$-module.

Similarly, there will be a surjection

$$\bigoplus_{i=0}^{\beta_0(N)} M \longrightarrow M \otimes_R N \longrightarrow 0.$$

In the next proposition, we collect the basic properties of trace ideals needed in this work. Some of these properties are well known, while some are not known to be in print. Proofs are given for lack of a comprehensive reference.

**Proposition 11.** *Let $R$ be a commutative ring and $M$ a finitely presented $R$-module. The following hold:*

*(i)* *If $M$ generates $N$, then $\tau_M(R) \supseteq \tau_N(R)$.*

*(ii)* $\tau_{M\oplus N}(R) = \tau_M(R) + \tau_N(R)$;

*(iii)* $\tau_M(R) = R$ *if and only if $M$ generates all finitely generated $R$-modules. When $R$ is local, this is equivalent to $M$ having a non-zero free summand;*

*(iv)* $I \subseteq \tau_I(R)$ *for ideals $I$, with equality when $I$ is a trace ideal;*

*(v)* $\tau_{M\otimes_R M^*}(R) = \tau_M(R) \subseteq \tau_{M^*}(R)$, *with equality when $M$ is reflexive;*

*(vi)* $\operatorname{End}_R(\tau_M(R)) = \tau_M(R)^*$;

*(vii)* $\operatorname{Ann}_R(\tau_M(R)) = \operatorname{Ann}_R(M)$ *when $M$ is reflexive. When, in addition, $R$ is Noetherian and $M$ is faithful, then $\tau_M(R)$ contains a nonzerodivisor;*

*(viii)* $\tau_M(R)\otimes_R A = \tau_{M\otimes_R A}(A)$ *for any commutative flat $R$-algebra $A$. In particular, taking the trace ideal commutes with localization and completion.*

*Proof.* (i): Any $R$-homomorphism $\alpha \in N^*$ can be precomposed with the surjection

$$M^{(\Lambda)} \xrightarrow{\gamma} N \longrightarrow 0$$

for a given index set $\Lambda$. As $\operatorname{Im}(\alpha\gamma) \supset \operatorname{Im}(\alpha)$, it follows that $\tau_N(R) \subseteq \tau_M(R)$.

(ii): This is a straightforward consequence of the definition of the trace ideal as

$$\tau_M(R) := \sum_{\alpha\in M^*} \alpha(M).$$

(iii): Suppose $M$ is a generator in $R$-mod, the category of finitely generated $R$-modules. Then, in particular, $M$ generates $R$. By (i), $\tau_M(R) \supseteq \tau_R(R) = R$.

If $\tau_M(R) = R$, then by Remark 10, $M$ generates $R$. Since $R$ generates $R$-mod, $M$ too is a generator. This argument also works over noncommutative rings; see [15, Theorem 18.8].

When $R$ is local, if $M = N \oplus R$, then $M$ clearly generates $R$. Now assume $\tau_M(R) = R$. There must exist $\alpha_i \in M^*$ and $m_i \in M$ such that

$$1 = \sum_{i=1}^{n} \alpha_i(m_i).$$

So at least one $\alpha_i(m_i)$ is a unit. For such an $i$, the map $\alpha_i : M \longrightarrow R$ is surjective and thus $M$ has a free summand.

(iv): For an ideal $I \subseteq R$, the inclusion map is an element of $I^*$ and therefore, $I \subseteq \tau_I(R)$. In particular, $\tau_M(R) \subseteq \tau_{\tau_M(R)}(R)$. The reverse inclusion holds because $M$ generates $\tau_M(R)$; see (i).

(v): Recall that $M$ generates $M \otimes_R M^*$ and $M \otimes_R M^*$, in turn, generates $\tau_M(R)$ (see the map $\vartheta_M$ from 2.1), so one has

$$\begin{aligned} \tau_M(R) &\supseteq \tau_{M\otimes_R M^*}(R) && \text{by } (i) \\ &\supseteq \tau_{\tau_M(R)}(R) && \text{by } (i) \\ &= \tau_M(R) && \text{by } (iv) \end{aligned}$$

and $\tau_M(R) = \tau_{M\otimes_R M^*}(R)$.

Given that $M^*$ also generates $M \otimes_R M^*$, one gets

$$\tau_{M^*}(R) \supseteq \tau_{M\otimes_R M^*}(R) = \tau_M(R).$$

When $M$ is reflexive, $M^* \otimes_R M^{**}$ is isomorphic to $M^* \otimes_R M$. By the already established equalities, one gets

$$\begin{aligned} \tau_{M^*}(R) &= \tau_{M^*\otimes_R M^{**}}(R) \\ &= \tau_{M\otimes_R M^*}(R) = \tau_M(R). \end{aligned}$$

(vi): Given $\alpha \in \tau_M(R)^*$, one has

$$\operatorname{Im}(\alpha) \subseteq \tau_{\tau_M(R)}(R) = \tau_M(R).$$

It follows that $\tau_M(R)^* = \operatorname{End}_R(\tau_M(R))$.

(vii): Given $\psi \in \operatorname{Hom}_R(M \otimes_R M^*, R)$, recall that

$$\operatorname{Im}(\psi) \subseteq \tau_{M \otimes_R M^*}(R) = \tau_M(R).$$

This justifies the first of the following equalities,

$$\begin{aligned} \operatorname{Ann}_R(\tau_M(R)) &= \operatorname{Ann}_R(\operatorname{Hom}_R(M \otimes_R M^*, R)) \\ &= \operatorname{Ann}_R(\operatorname{End}_R(M)) && \because M \text{ is reflexive} \\ &= \operatorname{Ann}_R(M) \end{aligned}$$

Assume, in addition, that $R$ is Noetherian and $M$ is faithful. This implies that $\operatorname{Ann}_R(\tau_M(R)) = 0$, and so $\tau_M(R)$ is not contained in any associated prime of $R$. By prime avoidance, $\tau_M(R)$ is not contained in the union of the associated primes of $R$. Therefore, $\tau_M(R)$ must contain a nonzerodivisor.

(viii): Let $\lambda : R \longrightarrow A$ be a homomorphism of commutative rings with $A$ flat over $R$. Since $A$ is flat, the inclusion $\tau_M(R) \subseteq R$ yields the inclusion

$$0 \longrightarrow \tau_M(R) \otimes_R A \longrightarrow R \otimes_R A = A.$$

By [18, Appendix A formula 11; Theorem 7.11] there are isomorphisms

$$\begin{aligned} M \otimes_R M^* \otimes_R A &\cong (M \otimes_R A) \otimes_A (M^* \otimes_R A) \\ &\cong (M \otimes_R A) \otimes_A \operatorname{Hom}_A(M \otimes_R A, R \otimes_R A); \end{aligned}$$

where the second isomorphism holds because $M$ is finitely presented.

Let $a, b \in A$, $\alpha \in M^*$ and $m \in M$. Together with the trace maps of $M$ as an $R$-module and $M \otimes_R A$ as an $A$-module, we obtain the commutative diagram below:

$$
\begin{tikzcd}
(M \otimes_R M^*) \otimes_R A \arrow[rrr, two heads, "\vartheta_M \otimes_R 1_A"] & & & \tau_M(R) \otimes_R A \arrow[dd, hook] \\
& m \otimes_R \alpha \otimes_R ba \arrow[r, mapsto] & \alpha(m) \otimes_R ba \arrow[d, mapsto] & \\
(M \otimes_R A) \otimes_A (M^* \otimes_R A) \arrow[uu, "\cong"] \arrow[dd, "\cong"'] & (m \otimes_R a) \otimes_A (\alpha \otimes_R b) \arrow[u, mapsto] \arrow[d, mapsto] & ab\lambda(\alpha(m)) & A \\
& (m \otimes_R a) \otimes_A b \cdot (\alpha \otimes_R 1_A) \arrow[r, mapsto] & \alpha(m) \otimes_R ab \arrow[u, mapsto] & \\
(M \otimes_R A) \otimes_A \operatorname{Hom}_A(M \otimes_R A, R \otimes_R A) \arrow[rrr, two heads, "\vartheta_{(M \otimes_R A)}"] & & & \tau_{M \otimes_R A}(R \otimes_R A) \arrow[uu, hook]
\end{tikzcd}
$$

This demonstrates the desired equality as subsets of $A$:

$$\tau_M(R) \otimes_R A = \tau_{M \otimes_R A}(A)$$

The left side representing the extension of the trace ideal of the $R$-module $M$, to the ring $A$ and the right side being the trace ideal of the $A$-module $M \otimes_R A$. $\square$

*Remark* 12. An ideal, $I$, is a trace ideal if and only if $\operatorname{End}_R(I) = \operatorname{Hom}_R(I, R)$ as in Proposition 11 (vi).

**Definition 13.** A discrete valuation ring (DVR) is a local principal ideal domain that is not field; see [18, Theorem11.2].

**Proposition 14.** *Let $R$ be a Noetherian domain. If $\mathfrak{p} \in Spec(R)$ of grade one such that $R_\mathfrak{p}$ is not a Discrete Valuation Ring, then $\tau_\mathfrak{p}(R) = \mathfrak{p}$*

*Proof.* We know $\tau_\mathfrak{p}(R) \supseteq \mathfrak{p}$ by Proposition 11 (iv). Since $R_\mathfrak{p}$ is a domain, $\mathfrak{p}R_\mathfrak{p}$ is indecomposable. Therefore, if $\mathfrak{p}R_\mathfrak{p}$ has an $R_\mathfrak{p}$-free summand, then $\mathfrak{p}R_\mathfrak{p} \cong R_\mathfrak{p}$. However, $\mathfrak{p}R_\mathfrak{p} \ncong R_\mathfrak{p}$ because $R_\mathfrak{p}$ is not a DVR. By Proposition 11 (viii), trace ideals behave as expected under localization and so:

$$
\begin{aligned}
(\tau_\mathfrak{p}(R))_\mathfrak{p} &= \tau_{\mathfrak{p}R_\mathfrak{p}}(R_\mathfrak{p}) \\
&= \mathfrak{p}R_\mathfrak{p} & \because \mathfrak{p}R_\mathfrak{p} \ncong R_\mathfrak{p}
\end{aligned}
$$

It follows that $\tau_\mathfrak{p}(R) \subseteq \mathfrak{p}$. We conclude that $\tau_\mathfrak{p}(R) = \mathfrak{p}$. $\square$

**Proposition 15.** *If $I$ is a trace ideal, then $\sqrt{I}$ is a trace ideal.*

*Proof.* Given $x \in \sqrt{I}$, there exists an $n \in \mathbb{N}$ such that $x^n \in I$. Take $f \in \operatorname{Hom}_R(\sqrt{I}, R)$. To prove the proposition, it is enough to show that $f \in \operatorname{End}_R(\sqrt{I})$. For $k \geq 2$, if $(f(x))^{k-1} = f^{k-1}(x^{k-1})$, then

$$\begin{aligned}
(f(x))^k &= (f(x))^{k-1} \cdot f(x) \\
&= f^{k-1}(x^{k-1}) \cdot f(x) \\
&= f^{k-1}(f(x) \cdot x^{k-1}) \\
&= f^{k-1}(f(x \cdot x^{k-1})) \\
&= f^{k-1}(f(x^k)) \\
&= f^k(x^k).
\end{aligned}$$

By induction, $(f(x))^n = f^n(x^n)$. Recall $I \subseteq \sqrt{I}$. The map $f$ restricted to $I$ is in $\operatorname{Hom}_R(I, R)$ and since $I$ is a trace ideal, $\operatorname{Hom}_R(I, R) = \operatorname{End}_R(I)$. It follows that $(f(x))^n = f^n(x^n) \in I$, that is to say $f(x) \in \sqrt{I}$. □

**Lemma 16.** *Suppose $I, J \subseteq R$ are ideals with $I$ a trace ideal and $I \subseteq J$. Then for any $\alpha \in \operatorname{Hom}_R(J, R)$, and $a_k \in I^k$*

$$\alpha^k(a_k) \in I^k.$$

*Proof.* Note $\alpha|_I \in \operatorname{Hom}_R(I, R) = \operatorname{End}_R(I)$. Given any $a_k \in I^k$, we may write $a_k = i_1 \cdots i_k$, for some $i_j \in I$. Therefore,

$$\begin{aligned}
\alpha^k(a_k) &= \alpha^k(i_1 \cdots i_k) \\
&= \alpha^{k-1}(\alpha(i_1 \cdots i_k)) \\
&= \alpha^{k-1}(\alpha(i_1) i_2 \cdots i_k)) \\
&= \alpha(i_1)\alpha^{k-1}(i_2 \cdots i_k) \\
&= \cdots \\
&= \alpha(i_1) \cdots \alpha(i_k) \in I^k.
\end{aligned}$$

□

**Proposition 17.** *If $I$ is a trace ideal, then its integral closure, $\bar{I}$, is a trace ideal.*

*Proof.* Given $r \in \bar{I}$, there exists $a_i \in I^i$ and $n \in \mathbb{N}$ such that

$$r^n + a_1 r^{n-1} + \cdots a_{n-1} r + a_n = 0.$$

For an $\alpha \in \mathrm{Hom}_R(\bar{I}, R)$, notice:

$$\begin{aligned}(\alpha(r))^n + \alpha(a_1)(\alpha(r))^{n-1} + \cdots \alpha^{n-1}(a_{n-1})(\alpha(r)) + \alpha^n(a_n)\\ = \alpha^n(r^n) + \alpha(a_1)\alpha^{n-1}(r^{n-1}) + \cdots \alpha^{n-1}(a_{n-1})\alpha(r) + \alpha^n(a_n)\\ = \alpha^n(r^n) + \alpha^{n-1}(\alpha(a_1)r^{n-1}) + \cdots \alpha(\alpha^{n-1}(a_{n-1})r) + \alpha^n(a_n)\\ = \alpha^n(r^n) + \alpha^{n-1}(\alpha(a_1r^{n-1})) + \cdots \alpha(\alpha^{n-1}(a_{n-1}r)) + \alpha^n(a_n)\\ = \alpha^n(r^n) + \alpha^n(a_1r^{n-1})) + \cdots \alpha^n(a_{n-1}r) + \alpha^n(a_n)\\ = \alpha^n(r^n + a_1r^{n-1} + \cdots a_{n-1}r + a_n) = 0\end{aligned}$$

where $\alpha^i(r^i) = (\alpha(r))^i$ by a similar argument as was used in Proposition 15. Since $\alpha^k(a_k) \in I^k$ by Lemma 16, it follows that $\alpha(r) \in \bar{I}$ and therefore, $\alpha \in \mathrm{End}_R(\bar{I})$. Thus, $\bar{I}$ is a trace ideal. □

**Proposition 18.** *Suppose $M$ is a finitely generated $R$-module such that* $\mathrm{Ext}^1_R(M, R) = 0$ *(for example, a maximal Cohen-Macaulay module over a Gorenstein ring). If $x \in \tau_M(R)$ is regular on $M$ and $R$, then $\tau_{M/xM}(R/(x)) = \tau_M(R)/xR$*

*Proof.* Consider the exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/(x) \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(M, ?)$, one gets the exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(M, R) \xrightarrow{x} \mathrm{Hom}_R(M, R) \longrightarrow \mathrm{Hom}_R(M, R/(x)) \longrightarrow \mathrm{Ext}^1_R(M, R) = 0.$$

That is,

$$\begin{aligned}\mathrm{Hom}_R(M, R)/x\mathrm{Hom}_R(M, R) &\cong \mathrm{Hom}_R(M, R/(x))\\ &\cong \mathrm{Hom}_{R/(x)}(M/xM, R/(x)),\end{aligned}$$

the second isomorphism following from [18, §18 Lemma 2]. The claim follows. □

# CHAPTER 3

# CENTERS OF ENDOMORPHISM RINGS

There are many results linking the properties of $\mathrm{End}_R(M)$ to those of $M$. Consider, for example [22, Theorem 3.1]:

*Theorem.* Let $R$ be a one-dimensional Gorenstein ring. Then $\mathrm{End}_R(M)$ is projective if and only if $M$ is projective.

See also [4, Theorem 4.4] and [2, Theorem 1.3].

One conceit of this work is that the relationship between a module and its endomorphism ring is sometimes captured by the center of the endomorphism ring, in part, because the center of the endomorphism is related to the trace ideal and the properties of the trace ideal are intimately linked to the structure of $M$. This chapter seeks to characterize the center of $\mathrm{End}_R(M)$ in terms of $\tau_M(R)$.

One should note that there is general interest in the the centers of endomorphism rings across many fields of Mathematics. Thus, the center $\mathrm{End}_R(M)$ has been fruitfully characterized in many other ways. For example, centers of endomorphism rings are sometimes studied in Representation Theory under the name "double-centralizer". Under some hypotheses, we may use $\mathrm{add}(M)$-approximations to identify this double-centralizer with a subspace of $M$; see Theorem 22.

**Definition 19.** Set $E := \mathrm{End}_R(M)$. The module $\mathrm{End}_E(M)$ is called the *Double Centralizer of* $M$.

**Definition 20.** Let $A$ be an algebra and let $\mathfrak{C}$ be a subcategory of $A$-mod. Let $M$ be an $A$-module. Then a homomorphism $f : M \longrightarrow C$ is called a left $\mathfrak{C}$-approximation of $M$ if and only if $C$ is an object of $\mathfrak{C}$ and the induced morphism

$$\mathrm{Hom}_A(f, D) : \mathrm{Hom}_A(C, D) \longrightarrow \mathrm{Hom}_A(M, D)$$

is an epimorphism for all objects $D$ in $\mathfrak{C}$.

**Definition 21.** Let $M$ be an $R$-module. We write add$(M)$ for the subcategory of $R$-Mod consisting of direct summands of finite direct sums of copies of $M$.

Now suppose that $R$ is a commutative Noetherian domain and let $A$ be an finitely generated and free associative $R$-algebra with unit. Let $M$ be a finitely generated $A$-module. The following result is [14, Theorem 2.7]:

**Theorem 22.** *Suppose there exists an injective left add(M)-approximation*

$$0 \longrightarrow A \xrightarrow{\delta} M.$$

*Denote by $B$ the centralizer algebra $\mathrm{End}_A(A_M)$ and by $C$ the double centralizer $\mathrm{End}_B(M_B)$. Then $C$ can be identified (as an $A$-module) with a subspace of $M$ as follows:*

$$C \cong \bigcap_{\{f\in B|f(A)=0\}} \mathrm{Ker}(f)$$

## 3.1 $\mathrm{Z}(\mathrm{End}_R(M))$ and the Trace Ideal

Let $R$ be a commutative ring and $M$ a finitely generated $R$-module. In what follows, $\mathrm{Z}(R)$ denotes the center of $R$ and $Q(R)$ denotes the total ring of quotients. We write $\beta_R(M)$ for the minimal number of generators of $M$ as an $R$-module.

In Theorem 31, the first main result of this chapter, we construct an $R$-algebra isomorphism, $\sigma$, between $\mathrm{End}_R(\tau_M(R))$ and $\mathrm{Z}(\mathrm{End}_R(M))$ when $M$ is reflexive and faithful. We call on the properties of the trace ideal from Section 2.1 as well as results established over noncommutative rings by Suzuki in [21]. For ease of reference, we include a proof of Suzuki's result below.

### 3.1.1 Reflexive Modules

If $M$ is reflexive, there is a monomorphism from $\mathrm{End}_R(\tau_M(R))$ to $\mathrm{End}_R(M)$. Indeed, one may apply $\mathrm{Hom}_R(\,?\,, R)$ to the exact sequence

$$M \otimes_R M^* \xrightarrow{\vartheta_M} \tau_M(R) \longrightarrow 0, \tag{3.1}$$

to obtain the top row of the diagram below.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(\tau_M(R), R) \arrow[r, "\vartheta_M^*"] \arrow[d, "11(vi)"', "="] & \mathrm{Hom}_R(M \otimes_R M^*, R) \arrow[d, "\cong"', "\vartheta^{-1} \text{ in } 2.2"] \\
& \mathrm{End}_R(\tau_M(R)) \arrow[r, hook, dashed, "\sigma"] & \mathrm{End}_R(M)
\end{tikzcd} \tag{3.2}$$

First we will show that the induced map $\sigma$ is an $R$-algebra monomorphism with its image in the center of $\mathrm{End}_R(M)$. Then we prove that when $M$ is also faithful, $\sigma$ is an isomorphism onto the center of $\mathrm{End}_R(M)$.

In what follows, we identify each $\alpha \in M^*$ with the map $\alpha : M \longrightarrow \tau_M(R)$.

**Lemma 23.** *Let $M$ be a reflexive $R$-module. The image of $\sigma$ in (3.2) is the set*

$$\{f \in \mathrm{End}_R(M) : \exists!\, \tilde{f} \text{ s.t. the square below commutes } \forall\ \alpha \in M^*\}$$

$$\begin{tikzcd}
M \arrow[r, "f"] \arrow[d, "\alpha"] & M \arrow[d, "\alpha"] \\
\tau_M(R) \arrow[r, dashed, "\tilde{f}"] & \tau_M(R)
\end{tikzcd}$$

*Proof.* Let $B$ denote the set defined in the statement.

Fix $m \otimes \alpha$ in $M \otimes_R M^*$. If $f \in B$, there exists a unique $\tilde{f} \in \mathrm{End}_R(\tau_M(R))$ such as that in (2.3) and (2.1):

$$\begin{aligned}
\vartheta_M^*(\tilde{f})(m \otimes \alpha) &= \tilde{f} \circ \vartheta_M(m \otimes \alpha) \\
&= \tilde{f}\alpha(m) \\
&= \alpha(f(m)) \\
&= \vartheta_f(m \otimes \alpha).
\end{aligned}$$

It follows that $\sigma(\tilde{f}) = \vartheta^{-1}(\vartheta_f) = f$ and so $f \in \mathrm{Im}(\sigma)$.

On the other hand, say $\tilde{f} \in \mathrm{End}_R(\tau_M(R))$. Then

$$\vartheta_M^*(\tilde{f}) \in \mathrm{Hom}_R(M \otimes_R M^*, R) \xrightarrow[\cong]{\vartheta^{-1}} \mathrm{Hom}_R(M, M).$$

In particular, there exists an $f \in \mathrm{End}_R(M)$ such that, in the notation established above, $\tilde{f} \circ \vartheta_M = \vartheta_M^*(\tilde{f}) = \vartheta_f$. One gets

$$\tilde{f}(\alpha(m)) = \tilde{f} \circ \vartheta_M(m \otimes \alpha) = \vartheta_f(m \otimes \alpha) = \alpha(f(m))$$

for all $m \otimes \alpha \in M \otimes_R M^*$. That is, the square in the definition of $B$ commutes.

Say $\alpha(g(m)) = \tilde{f}(\alpha(m)) = \alpha(f(m))$ for some other $g \in \mathrm{Hom}_R(M, M)$ and all $m \otimes_R \alpha \in M \otimes_R M^*$. Then $\alpha(f(m) - g(m)) = 0$ for all $\alpha \in M^*$ and $m \in M$. Since $M$ is torsionless, $f(m) = g(m)$ for all $m \in M$. Thus $\tilde{f}$ is unique to $f$. It follows that $\mathrm{Im}(\sigma) \subseteq B$ $\square$

**Proposition 24.** *Let $M$ be a reflexive $R$-module. The map $\sigma$ in (3.2) is a monomorphism of $R$-algebras with* $\mathrm{Im}(\sigma) \subseteq \mathrm{Z}(\mathrm{End}_R(M))$.

*Proof.* Take $f \in \mathrm{Im}(\sigma)$, $g \in \mathrm{End}_R(M)$, $m \in M$ and $\alpha \in M^*$. Since $\alpha g \in M^*$ and $g(m) \in M$, by Lemma 23, there exists $\tilde{f} \in \mathrm{End}_R(\tau_M(R))$ such that

$$(\alpha g)(f(m)) = \tilde{f}(\alpha g(m)) = (\alpha f)(g(m)).$$

This shows $\alpha(gf(m) - fg(m)) = 0$ for all $\alpha \in M^*$ and $m \in M$. As $M$ is torsionless, it follows that $gf(m) = fg(m)$ for all $m \in M$.

Further, $\sigma$ is an $R$-algebra homomorphism; for $\tilde{f}, \tilde{g} \in \mathrm{End}_R(\tau_M(R))$:

$$\tilde{f}\tilde{g} \circ \vartheta_M(m \otimes \alpha) = \tilde{f}\tilde{g}(\alpha(m)) = \tilde{f}\alpha(g(m)) = \alpha(fg(m)) = \vartheta_{fg}(m \otimes \alpha)$$

That is, $\sigma(\tilde{f}\tilde{g}) = fg = \sigma(\tilde{f})\sigma(\tilde{g})$. $\square$

**Corollary 25.** *If $R$ is a commutative ring and $M$ is a reflexive $R$-module, then* $\mathrm{End}_R(\tau_M(R))$ *is commutative.* $\square$

*Remark* 26. Consider an ideal $I$ with $\mathrm{grade}(I) \geq 2$. As a consequence of Example 5, the ring of endomorphisms, $\mathrm{End}_R(I)$, is $R$. However, there do exist ideals for which $\mathrm{End}_R(I)$ is not commutative. For example, given a field $k$, the ideal $(x, y)$ in the ring $k[x, y]/(xy, y^2)$ has a noncommutative endomorphism ring. To see this, consider the $f, g \in \mathrm{End}_R(I)$ with

$$\begin{array}{lcl} f(x) = x & \& & f(y) = 0 \\ g(x) = y & \& & g(y) = 0. \end{array}$$

Here $fg \neq gf$.

Given that ideals are typically not reflexive, the preceding corollary and remark suggest the following:

*Question* 1. What are the necessary and sufficient conditions for an ideal to be the trace ideal of a reflexive module?

*Remark* 27. The trace ideals of reflexive modules have not been characterized. It is evident, however, that not all ideals can be trace ideals of reflexive modules. By Corollary 25, $\mathrm{End}_R(\tau_M(R))$ is always commutative; however, every ring whose total ring of quotients is not quasi-Frobenius contains ideals with noncommutative endomorphism rings [1] (see also [23] and [7] Theorem 1.1).

### 3.1.2 Reflexive Faithful Modules

When $M$ is faithful in addition to being reflexive, we may construct

$$\sigma^{-1} : \mathrm{Z}(\mathrm{End}_R(M)) \longrightarrow \mathrm{End}_R(\tau_M(R)).$$

The definition below and the result that follows are from Suzuki [21]. For ease of reference, we include a proof of the theorem.

**Definition 28.** Let $M$ and $U$ be $R$-modules. We say $M$ is $U$-torsionless if for every non-zero $m \in M$, there exists $\alpha \in \mathrm{Hom}_R(M, U)$ with $\alpha(m) \neq 0$.

**Theorem 29.** *Let $M$ and $U$ be $R$-modules. If $M$ generates $U$ and $U$ is $M$-torsionless, there exists a homomorphism of rings,*

$$\sigma' : \mathrm{Z}(\mathrm{End}_R(M)) \longrightarrow \mathrm{Z}(\mathrm{End}_R(U))$$

*such that $\sigma'$ is a monomorphism whenever $M$ is $U$-torsionless.*

*Proof.* Since $M$ generates $U$, every element $u \in U$ may be written

$$u = \sum \phi_i(m_i)$$

for some $m_i \in M$ and $\phi_i \in \mathrm{Hom}_R(M, U)$.

Given $q \in Q := \mathrm{Z}(\mathrm{End}_R(M))$, define $\bar{q} \in \bar{Q} := \mathrm{Z}(\mathrm{End}_R(U))$ thusly:

$$\bar{q}(u) = \bar{q}\left(\sum \phi_i(m_i)\right) = \sum \phi_i(q(m_i)).$$

We now define a map

$$\sigma' : \mathrm{Z}(\mathrm{End}_R(M)) \longrightarrow \mathrm{Z}(\mathrm{End}_R(U))$$

by $q \mapsto \bar{q}$.

We must establish that $\sigma'$ is a well-defined ring homomorphism and further that

$$\mathrm{Im}(\sigma') \subseteq \mathrm{Z}(\mathrm{End}_R(U)).$$

*$\sigma'$ is well-defined:*

Given two representations of $u \in U$

$$\sum \phi_i'(m_i') = u = \sum \phi_i(m_i),$$

to show $\sigma'$ is well-defined is to show $\bar{q}\left(\sum \phi_i'(m_i') - \sum \phi_i(m_i)\right) = 0$. It is enough to show that for arbitrary $\phi_i \in \mathrm{Hom}_R(M, U)$, $m_i \in M$ and $q \in \mathrm{Z}(\mathrm{End}_R(M))$,

$$\sum \phi_i(m_i) = 0 \text{ implies} \sum \phi_i(q(m_i)) = 0.$$

Assume $\sum \phi_i(m_i) = 0$. Take any $d \in \mathrm{Hom}_R(U, M)$ and note that $d\phi_i \in \mathrm{End}_R(M)$ and so commutes with $q$. It follows that:

$$\begin{aligned} 0 = qd\left(\sum \phi_i(m_i)\right) &= \sum qd\left(\phi_i(m_i)\right) \\ &= \sum d\phi_i(q(m_i)) \\ &= d\left(\sum \phi_i(q(m_i))\right). \end{aligned}$$

Recall, $U$ is $M$-torsionless by assumption. Since $d\left(\sum \phi_i(q(m_i))\right) = 0$ for any $d \in \mathrm{Hom}_R(U, M)$, it must be that $\sum \phi_i(q(m_i)) = 0$.

It is clear that $\sigma'$ is an $R$-module homomorphism.

*Further, $\sigma'$ is a ring homomorphism:*

$$\begin{aligned} \overline{q_1 q_2}\left(\sum \phi_i(m_i)\right) &= \sum \phi_i(q_1 q_2(m_i)) \\ &= \bar{q}_1\left(\sum \phi_i(q_2(m_i))\right) \\ &= \bar{q}_1 \bar{q}_2\left(\sum (\phi_i(m_i))\right). \end{aligned}$$

That is, given $q_i \in Q$,

$$\sigma'(q_1 q_2) = \sigma'(q_1)\sigma'(q_2).$$

*When $M$ is $U$-torsionless, $\sigma'$ is a monomorphism:*

Given any non-zero $q \in Q$, there exists $m \in M$ with $q(m) \neq 0$. Since $M$ is $U$-torsionless, there also exists $\phi \in \mathrm{Hom}_R(M, U)$ with $\phi(q(m)) \neq 0$. That is, $\bar{q}(\phi(m)) \neq 0$ implying $\bar{q} \neq 0$ and therefore $\sigma'$ is a monomorphism.

*The image of $\sigma'$ is a subset of* $\mathrm{Z}(\mathrm{End}_R(U))$*:*

Finally, notice $\bar{q} \in \bar{Q}$ since for any $f \in \mathrm{End}_R(U)$ and $u \in U$ :

$$\begin{aligned}
\bar{q}(f(u)) &= \bar{q}\left(f\left(\sum \phi_i(m_i)\right)\right) \\
&= \bar{q}\sum f\phi_i(m_i) \\
&:= \sum f\phi_i(q(m_i)) \\
&= f\left(\sum \phi_i(q(m_i))\right) \\
&:= f\bar{q}\left(\sum \phi_i(m_i)\right) = f\bar{q}(u)
\end{aligned}$$

□

**Corollary 30.** *If an $R$-module $M$ is torsionless and faithful, there is a monomorphism of $R$-algebras*

$$\mathrm{Z}(\mathrm{End}_R(M)) \longrightarrow \mathrm{End}_R(\tau_M(R)).$$

*Proof.* Recall that $M$ generates $\tau_M(R)$.

For each map $\alpha \in M^*$, we have $\mathrm{Im}(\alpha) \subseteq \tau_M(R)$. Therefore, $M$ torsionless implies that $M$ is $\tau_M(R)$-torsionless. Also, $M$ faithful implies

$$\tau_M(R) \cap \mathrm{Ann}_R(M) = 0.$$

So for all $t \in \tau_M(R)$, there exists $m \in M$ with $tm \neq 0$. The maps

$$\begin{array}{ccc} R & \longrightarrow & R \\ 1 & \mapsto & m \end{array}$$

restricted to $\tau_M(R)$ demonstrate that $\tau_M(R)$ is $M$-torsionless.

The desired monomorphism exists by Theorem 29. □

**Theorem 31.** *Let $R$ be a commutative Noetherian ring. If $M$ is a finitely generated, faithful and reflexive $R$-module, then the map*

$$\sigma : \mathrm{End}_R(\tau_M(R)) \longrightarrow \mathrm{Z}(\mathrm{End}_R(M)),$$

*in (3.2), is an isomorphism of $R$-algebras.*

*Proof.* By Proposition 24, $\sigma : \mathrm{End}_R(\tau_M(R)) \hookrightarrow \mathrm{Z}(\mathrm{End}_R(M))$ is a monomorphism of $R$-algebras; in particular $\mathrm{End}_R(\tau_M(R))$ is commutative. Now by Corollary 30, there exists a monomorphism

$$\sigma' : \mathrm{Z}(\mathrm{End}_R(M)) \hookrightarrow \mathrm{Z}(\mathrm{End}_R(\tau_M(R))) = \mathrm{End}_R(\tau_M(R))$$

via $\{f \mapsto f'\}$ where $f'$ is defined as follows: for any $x \in \tau_M(R)$, there exists $\alpha_i \in M^*$ and $m_i \in M$ with $x = \sum_{i=1}^{n} \alpha_i(m_i)$; set

$$f'(x) = f'\left(\sum_{i=1}^{n} \alpha_i(m_i)\right) := \sum_{i=1}^{n} \alpha_i(f(m_i)).$$

Notice

$$\begin{aligned} f' \circ \vartheta_M \left(\sum_{i=1}^{n} m_i \otimes \alpha_i\right) &= f'\left(\sum_{i=1}^{n} \alpha_i(m_i)\right) \\ &= \sum_{i=1}^{n} \alpha_i(f(m_i)) \\ &= \vartheta_f\left(\sum_{i=1}^{n} m_i \otimes \alpha_i\right) \end{aligned}$$

So $\sigma(f') = \vartheta^{-1}(\vartheta_f) = f$ and for a given $f \in \mathrm{Z}(\mathrm{End}_R(M))$,

$$\sigma\sigma'(f) = \sigma(f') = f.$$

Since both $\sigma$ and $\sigma'$ are ring monomorphisms, they are inverse isomorphisms. $\square$

The assumptions on $M$ in Theorem 31 can be relaxed given additional assumptions on $\tau_M(R)$; the following lemma, for example, is an extension of Exercise 95 in [13] and is proved, in a noncommutative setting, in [3, Theorem A.2. (g)].

**Lemma 32.** *If $\tau_M(R) = R$, then $\mathrm{Z}(\mathrm{End}_R(M)) \cong R$.*

*Proof.* Since $M$ generates $\tau_M(R) = R$, there exists an $n \in \mathbb{N}$ and an $R$-module $N$ such that $M^n \cong N \oplus R$. It follows that $M^n$ is faithful and therefore, the map $R \longrightarrow \mathrm{Z}(\mathrm{End}_R(M^n))$

sending $r \in R$ to multiplication by $r$ is an injection. Consider the endomorphism, $\pi$, which projects $M^n$ onto $R$, that is $\pi(n+r) = r$ for any $n+r \in N \oplus R$. If $f \in \mathrm{Z}(\mathrm{End}_R(M^n))$, then

$$\begin{aligned} f(r) &= f \circ \pi(n+r) \\ &= \pi \circ f(n+r) \in R. \end{aligned}$$

It follows that $f$ restricted to $R \subseteq M^n$ is an element of $\mathrm{End}_R(R)$ and so is given by multiplication by some $x \in R$. Now for any $m \in N$, take the endomorphism $\gamma_m : N \oplus R \longrightarrow N \oplus R$ such that $\gamma_m(n+r) = rm$. One gets

$$\begin{aligned} f(m) &= f \circ \gamma_m(1) \\ &= \gamma_m \circ f(1) \\ &= \gamma_m(x) \\ &= x\gamma_m(1) = xm \end{aligned}$$

Altogether, every $f \in \mathrm{Z}(\mathrm{End}_R(M^n))$ is given by multiplication by some element $x$ in $R$. That is to say $\mathrm{Z}(\mathrm{End}_R(M^n)) = R$. It follows that $\mathrm{Z}(\mathrm{End}_R(M)) \cong R$ because $\mathrm{End}_R(M)$ and $\mathrm{End}_R(M^n)$ are morita equivalent rings; see [16, Corollary A.2] and [15, 18.42]. $\square$

*Remark* 33. In general, the converse is not true; Example 5 shows that $\mathrm{End}_R(I) = R$ for all ideals with $\mathrm{grade}(I) \geq 2$. We use Theorem 31 to prove the converse for reflexive modules over local Noetherian rings of depth less than or equal to one; see Proposition 52.

**Definition 34.** Let $M$ be an $R$ module. We say $M$ is *torsionfree* provided that given any nonzerodivisor $r \in R$ and $m \in M$, if $rm = 0$, then $m = 0$.

The following result is well-known; see Exercise 4.31 in [17]:

**Lemma 35.** *Let $I \subset R$ be an ideal that contains a nonzerodivisor. Then* $\mathrm{Hom}_R(I, R)$ *may be identified with*

$$\left\{ \frac{a}{b} \in Q(R) | \frac{a}{b} \cdot I \subseteq R \right\}.$$

*where the multiplication $a/b \cdot I$ is multiplication in $Q(R)$.*

*Proof.* Let $x \in I$ be a nonzerodivisor. For $\alpha$ in $\mathrm{Hom}_R(I, R)$, the fraction $\alpha(x)/x$ is in $Q(R)$. So for any $i \in I$

$$x\left(\alpha(i) - \frac{\alpha(x)}{x} \cdot i\right) = x\alpha(i) - \alpha(x)i = 0$$

Since $x$ is a nonzerodivisor in $Q(R)$, one gets $\alpha(i) = (\alpha(x)/x) \cdot i$. □

*Remark* 36. Recall $\tau_M(R)^* = \mathrm{End}_R(\tau_M(R))$. As a consequence of Lemma 35, $\mathrm{End}_R(\tau_M(R))$ is commutative whenever $\tau_M(R)$ contains a nonzerodivisor. This is the case, for example, when the $R$-module $M$ is finitely generated, reflexive and faithful; see Proposition 11 (vii).

When $\tau_M(R)$ contains a nonzerdivisor, we henceforth identify $\mathrm{End}_R(\tau_M(R))$ with

$$\left\{\frac{a}{b} \in Q(R) | \frac{a}{b}\tau_M(R) \subseteq \tau_M(R)\right\}.$$

*Corollary* 37. *Let $R$ be a Noetherian ring. If $M$ is a finitely generated, faithful, reflexive $R$-module, then*

$$\mathrm{Z}(\mathrm{End}_R(M)) = \mathrm{End}_R(\tau_M(R))$$

*as subsets of $Q(R)$, where equality is understood in the following sense: there is a bijection between the sets given by $\sigma$ from Theorem 31. Further, for any $m$ in $M$ and $a/b \in \mathrm{End}_R(\tau_M(R))$, there exists an $n \in M$ such that $am = bn$ and $\sigma(a/b)(m) = n$.*

*Proof.* By the hypotheses on $M$ and Remark 36, we identify $\mathrm{End}_R(\tau_M(R))$ with

$$\left\{\frac{a}{b} \in Q(R) | \frac{a}{b}\tau_M(R) \subseteq \tau_M(R)\right\}.$$

Theorem 31 provides a bijection between $\mathrm{End}_R(\tau_M(R))$ and $\mathrm{Z}(\mathrm{End}_R(M))$, so that a given $f \in \mathrm{Z}(\mathrm{End}_R(M))$ is $\sigma(a/b)$ for some unique $a/b \in \mathrm{End}_R(\tau_M(R))$. One gets that $M$ is an $\mathrm{End}_R(\tau_M(R))$-module via the action

$$\frac{a}{b} \cdot m := \sigma\left(\frac{a}{b}\right)(m).$$

By definition, $\sigma(1) = id_M$. Therefore,

$$\begin{aligned} b\left(\frac{a}{b} \cdot m\right) &= b\sigma\left(\frac{a}{b}\right)(m) \\ &= a\sigma(1)(m) \\ &= a(\mathrm{id}_M(m)) \\ &= am. \end{aligned}$$

It follows that $am = bn$ for some $n \in M$. Since $b$ is a nonzerdivisor and $M$ is torsionfree [5, Exercise 1.4.20 (a)],

$$b\left(\frac{a}{b} \cdot m - n\right) = 0$$

implies $\sigma(a/b)(m) = n$. □

### 3.1.3 When $\tau_M(R)$ Contains a Nonzerodivisor

When $\tau_M(R)$ contains a nonzerdivisor but $M$ is not necessarily reflexive and faithful (for example any module over a domain) we shall prove a result similar to Theorem 31, this time relating $\mathrm{End}_R(\tau_M(R))$ to $\mathrm{Z}(\mathrm{End}_R(M^*))$.

Note, for each $\alpha \in M^*$, one has $\mathrm{Im}(\alpha) \subseteq \tau_M(R)$. Thus $M^*$ is always a left $\mathrm{End}_R(\tau_M(R))$-module via postcomposition:

$$M \xrightarrow{\alpha} \tau_M(R) \xrightarrow{h} \tau_M(R) \subseteq R.$$

In particular, the surjection (3.1) yields a second commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_R(\tau_M(R), R) \arrow[r, "\vartheta_M^*"] \arrow[d, "11(vi)"', "="] & \mathrm{Hom}_R(M \otimes_R M^*, R) \arrow[d, "\cong"'] \\ & \mathrm{End}_R(\tau_M(R)) \arrow[r, hook, dashed, "\rho"] & \mathrm{End}_R(M^*) \end{tikzcd} \tag{3.3}$$

the isomorphism on the right following from Hom-Tensor adjointness. The induced map $\rho$ sends $h \in \mathrm{End}_R(\tau_M(R))$ to postcomposition by $h$. To illustrate, for all $m$ in $M$, there exists some $\psi_g$ in $\mathrm{Hom}_R(M \otimes_R M^*, R)$ and $g$ in $\mathrm{End}_R(M^*)$ such that

$$(h\alpha)m = h(\alpha(m)) = h\vartheta_M(\alpha \otimes m) = \psi_g(\alpha \otimes m) = g(\alpha)m.$$

That is, $h\alpha = g(\alpha)$.

Again we claim this injection, $\rho$, is an $R$-algebra isomorphism from $\mathrm{End}_R(\tau_M(R))$ onto the center of $\mathrm{End}_R(M^*)$.

**Lemma 38.** *For a finitely generated $R$-module $M$, the $R$-dual, $M^*$, is both torsionfree and torsionless. If $\tau_{M^*}(R)$ contains a nonzerodivisor, $M^*$ is also faithful.*

*Proof.* For all $r \in R$ and nonzero $\alpha \in M^*$, $r\alpha = 0$ implies $r\,\mathrm{Im}(\alpha) = 0$. When $r$ is a nonzerodivisor, $r\,\mathrm{Im}(\alpha) \neq 0$. It follows that $M^*$ must be torsionfree.

We show that $M^*$ is torsionless by showing that the natural map $\iota : M^* \longrightarrow M^{***}$ given by $\alpha \mapsto ev_\alpha$ is an injection. We prove $\iota$ is an injection by showing it splits.

Recall, there is also the natural map $\psi : M \longrightarrow M^{**}$ given by $\psi(m) = ev_m$. Applying $\operatorname{Hom}_R(?, R)$, one obtains the desired splitting map

$$\begin{aligned} \psi^* : M^{***} &\longrightarrow M^* \\ \Phi &\longmapsto \Phi \circ \psi. \end{aligned}$$

Indeed, for $\alpha \in M^*$, observe the composition $\psi^* \circ \iota(\alpha) = \psi^*(ev_\alpha) = ev_\alpha \circ \psi$. The image of $\alpha$ is therefore the map

$$\begin{array}{ccccc} M & \xrightarrow{\psi} & M^{**} & \xrightarrow{ev_\alpha} & R \\ m & \longmapsto & ev_m & \longmapsto & ev_\alpha(ev_m). \end{array}$$

where $ev_\alpha(ev_m) = ev_m(\alpha) = \alpha(m)$. That is, the composition $\psi^* \circ \iota$ is identity on $M^*$.

If $\tau_{M^*}(R)$ contains a nonzerodivisor, $x \in R$ then there exists a $\Phi_i \in M^{**}$ and $\alpha_i \in M^*$ with $\sum_{i=1}^{n} \Phi_i(\alpha_i) = x$ for some $n \in \mathbb{N}$. If $a \in \operatorname{Ann}_R(M^*)$, then

$$\begin{aligned} a \cdot x &= a \cdot \left( \sum_{i=1}^{n} \Phi_i(\alpha_i) \right) \\ &= \sum_{i=1}^{n} \Phi_i(a \cdot \alpha_i) \\ &= 0. \end{aligned}$$

It follows that $M^*$ is faithful. $\square$

**Proposition 39.** *When $\tau_M(R)$ contains a nonzerdivisor, the map $\rho$ in (3.3) is a monomorphism of $R$-algebras with*

$$\operatorname{Im}(\rho) \subseteq \operatorname{Z}(\operatorname{End}_R(M^*)).$$

*Proof.* Recall the identification from Lemma 35. Taking $a/b$ in $\operatorname{End}_R(\tau_M(R))$, a map $\alpha$ in $M^*$ and an endomorphism $g$ in $\operatorname{End}_R(M^*)$, we have:

$$b \left( \frac{a}{b} g(\alpha) - g\left(\frac{a}{b}\alpha\right) \right) = 0.$$

Since $b$ is a nonzerodivisor and $M^*$ is torsionfree, we conclude

$$\frac{a}{b} \cdot g(\alpha) = g\left(\frac{a}{b} \cdot \alpha\right).$$

Recall the injection $\rho : \mathrm{End}_R(\tau_M(R)) \hookrightarrow \mathrm{End}_R(M^*)$ constructed in (3.3) sends $a/b$ in $\mathrm{End}_R(\tau_M(R))$ to postcomposition by $a/b$. Since

$$\frac{a}{b}g(\alpha) = g\left(\frac{a}{b}\alpha\right),$$

for all $g \in \mathrm{End}_R(M^*)$ and $a/b \in \mathrm{End}_R(\tau_M(R))$, the image of $\rho$ is in the center of $\mathrm{End}_R(M^*)$.

It is clear that

$$\rho\left(\frac{a}{b}\cdot\frac{a'}{b'}\right)(\alpha) = \frac{a}{b}\cdot\frac{a'}{b'}(\alpha) = \rho\left(\frac{a}{b}\right)\left(\frac{a'}{b'}\alpha\right) = \rho\left(\frac{a}{b}\right)\rho\left(\frac{a'}{b'}\right)(\alpha)$$

and therefore, $\rho$ is a monomorphism of $R$-algebra. □

**Lemma 40.** *Let $M$ be an $R$-module and $I \subseteq J$ ideals. If $I$ contains a nonzerodivisor and $M$ is $J$-torsionless, then $M$ is $I$-torsionless.*

*Proof.* This follows from the fact that for all nonzero $m \in M$, there exists $\alpha \in \mathrm{Hom}_R(M, J)$ with $\alpha(m) \neq 0$ and therefore, for any nonzerodivisor $x \in I$, the product $x\alpha(m) \neq 0$ with $x\alpha \in \mathrm{Hom}_R(M, I)$. We use this observation in the proof of the following result. □

**Theorem 41.** *Let $R$ be a Noetherian ring. Suppose $M$ is a finitely generated $R$-module such that $\tau_M(R)$ contains a nonzerodivisor. Then the map*

$$\rho : \mathrm{End}_R(\tau_M(R)) \longrightarrow \mathrm{Z}(\mathrm{End}_R(M^*)).$$

*in (3.3) is an $R$-algebra isomorphism.*

*Proof.* Recall, $M^*$ is $\tau_M(R)$-torsionless, faithful and generates $\tau_M(R)$; see Lemmas 38 & 40 and Example 10. By Corollary 30, there is an injection

$$\sigma' : \mathrm{Z}(\mathrm{End}_R(M^*)) \hookrightarrow \mathrm{End}_R(\tau_M(R)).$$

$$q \mapsto \bar{q}$$

By Proposition 39, there is another injection

$$\rho : \mathrm{End}_R(\tau_M(R)) \hookrightarrow \mathrm{Z}(\mathrm{End}_R(M^*))$$

which sends $a/b \in \mathrm{End}_R(\tau_M(R))$ to postcomposition by $a/b$.

These maps are inverses. Indeed, given $t \in \tau_M(R)$, there exist $\varphi_i \in \operatorname{Hom}_R(M^*, \tau_M(R))$ and $\alpha_i \in M^*$ such that

$$t = \sum_i \varphi_i(\alpha_i).$$

For $a/b \in \operatorname{End}_R(\tau_M(R))$,

$$\begin{aligned}
\sigma'\left(\rho\left(\frac{a}{b}\right)\right) &= \overline{\rho\left(\frac{a}{b}\right)}(t) \\
&= \overline{\rho\left(\frac{a}{b}\right)}\left(\sum_i \varphi_i(\alpha_i)\right) \\
&= \sum_i \varphi_i\left(\rho\left(\frac{a}{b}\right)\alpha_i\right) \\
&= \sum_i \varphi_i\left(\frac{a}{b}\alpha_i\right) \\
&= \frac{a}{b}\left(\sum_i \varphi_i\left(\alpha_i\right)\right) = \frac{a}{b}(t);
\end{aligned}$$

the penultimate equality following from $\tau_M(R)$ being torsionfree. That is,

$$b \cdot \left[\frac{a}{b}\left(\sum_i \varphi_i\left(\alpha_i\right)\right) - \sum_i \varphi_i\left(\frac{a}{b}\alpha_i\right)\right] = 0$$

implies $\sum_i \varphi_i\left(a/b \cdot \alpha_i\right) = a/b \cdot \left(\sum_i \varphi_i\left(\alpha_i\right)\right)$. We conclude that $\sigma'\rho\left(a/b\right) = a/b$, that is, the composition of the injections is identity. $\square$

*Remark* 42. When $M$ is reflexive and faithful, $\tau_M(R)$ contains a nonzerodivisor and $\tau_{M^*}(R) = \tau_M(R)$. By Theorem 41

$$\mathrm{Z}(\operatorname{End}_R(M^{**})) \cong \operatorname{End}_R(\tau_{M^*}(R)).$$

In other words,

$$\mathrm{Z}(\operatorname{End}_R(M)) \cong \operatorname{End}_R(\tau_M(R)).$$

This is another proof of Theorem 31.

**Corollary 43.** *Let $R$ be a commutative Noetherian ring. Suppose $M$ is a finitely generated $R$-module such that $\tau_M(R)$ contains a nonzerodivisor. Then*

$$\operatorname{End}_R(\tau_{M^*}(R)) \cong \operatorname{End}_R(\tau_M(R)) \cong \mathrm{Z}(\operatorname{End}_R(M^*))$$

*Proof.* Recall $\tau_{M^*}(R) \supseteq \tau_M(R)$. Since $\tau_M(R)$ contains a nonzerodivisor, the module $\tau_{M^*}(R)/\tau_M(R)$ is torsion and therefore, the inclusion $\tau_M(R) \subseteq \tau_{M^*}(R)$ yields an inclusion

$$\mathrm{Hom}_R(\tau_{M^*}(R), R) \subseteq \mathrm{Hom}_R(\tau_M(R), R)$$

as subsets of $Q(R)$; see Remark 36. That is,

$$\mathrm{End}_R(\tau_{M^*}(R)) \subseteq \mathrm{End}_R(\tau_M(R)).$$

Note, $M^*$ is torsionless and faithful, and also generates $\tau_{M^*}(R)$; see Lemmas 38 & 40 and Example 10. Therefore, by Corollary 30 and Remark 36, there is an injection

$$\mathrm{Z}(\mathrm{End}_R(M^*)) \hookrightarrow \mathrm{Z}(\mathrm{End}_R(\tau_{M^*}(R))) = \mathrm{End}_R(\tau_{M^*}(R)).$$

By Theorem 41, there is now a sequence of monomorphisms whose composition is the identity on map $\mathrm{End}_R(\tau_{M^*}(R))$:

$$\mathrm{End}_R(\tau_{M^*}(R)) \overset{i}{\hookrightarrow} \mathrm{End}_R(\tau_M(R)) \overset{\rho}{\hookrightarrow} \mathrm{Z}(\mathrm{End}_R(M^*)) \overset{\sigma'}{\hookrightarrow} \mathrm{End}_R(\tau_{M^*}(R)). \tag{3.4}$$

Indeed, given $a/b \in \mathrm{End}_R(\tau_{M^*}(R))$, one has that $i(a/b)$ is the restriction of multiplication by $a/b$ on $\tau_M(R)$. For $\alpha \in M^*$, the homomorphism $\rho(a/b)(\alpha)$ is the map $a/b \cdot \alpha$ in $M^*$, that is $\rho(a/b)$ is postcomposition by $a/b$. Lastly, in a similar argument as in the proof of Theorem 41, for $t \in \tau_{M^*}(R)$, there exists $\alpha_i \in M^*$ and $\Phi_i \in \mathrm{Hom}_R(M, R)$ with

$$t = \sum_{i=1}^{n} \Phi_i(\alpha_i).$$

Therefore,

$$\begin{aligned}
\sigma'\left(\frac{a}{b}\right)(t) &= \sigma'\left(\frac{a}{b}\right)\left(\sum_{i=1}^{n} \Phi_i(\alpha_i)\right) \\
&= \sum_{i=1}^{n} \Phi_i\left(\frac{a}{b} \cdot \alpha_i\right) \\
&= \frac{a}{b} \cdot \sum_{i=1}^{n} \Phi_i(\cdot \alpha_i) \\
&= \frac{a}{b} \cdot t
\end{aligned}$$

The penultimate equality following from $\tau_{M^*}(R)$ being torsionfree. Altogether, $\sigma' \circ \rho \circ i$ is the identity map on $\mathrm{End}_R(\tau_{M^*}(R))$. Since all the maps are monomorphisms, it follows that they are all isomorphisms. □

**Corollary 44.** *Let $R$ be a commutative Noetherian ring. Suppose $M$ is a finitely generated $R$-module such that $\tau_M(R)$ contains a nonzerodivisor. Then the modules* $\mathrm{End}_R(\tau_M(R))$, $\mathrm{End}_R(\tau_{M^*}(R))$ *and* $\mathrm{Z}(\mathrm{End}_R(M^*))$ *may be identified as $R$-submodules of the total ring of quotients $Q(R)$, and then one has*

$$\mathrm{Z}(\mathrm{End}_R(M^*)) = \mathrm{End}_R(\tau_M(R)) = \mathrm{End}_R(\tau_{M^*}(R))$$

*as subsets of $Q(R)$.*

*Proof.* As in Corollary 37, this identification is an easy consequence of the definition of the maps established in Corollary 43, and Lemma 35.

Indeed, the isomorphism $i$ from Corollary 43 establishes the equality $\mathrm{End}_R(\tau_M(R)) = \mathrm{End}_R(\tau_{M^*}(R))$ as subrings of $Q(R)$. Since we have also shown that $\rho(a/b)(\alpha)$ is the map $a/b \cdot \alpha$, one gets that $\mathrm{End}_R(\tau_M(R)) = \mathrm{Z}(\mathrm{End}_R(M^*))$. $\square$

# CHAPTER 4

# APPLICATIONS

In this chapter, we use results from Chapter 3 to relate the properties of $\mathrm{End}_R(M)$ to those of $M$ in various contexts.

## 4.1 $R$-free $\mathrm{End}_R(M)$

In this section, we concentrate on when $\mathrm{End}_R(M)$ is a free $R$-module.

The following result is well-known; for example, it is used in the proof of [4, Proposition 7.2].

**Lemma 45.** *Let $R$ be a local commutative Noetherian ring of depth $\leq 1$ and $I \subseteq R$ an ideal. If $I^* = R$, then $I = R$.*

*Proof.* Applying $\mathrm{Hom}_R(?, R)$, to

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(R/I, R) \longrightarrow \mathrm{Hom}_R(R, R) \xrightarrow{i} \mathrm{Hom}_R(I, R) \longrightarrow \mathrm{Ext}^1_R(R/I, R) \longrightarrow 0.$$

Recall $\mathrm{Hom}_R(R, R) = R$. For $r \in \mathrm{Hom}_R(R, R)$, the map $i(r) \in \mathrm{Hom}_R(I, R)$ is the restriction of multiplication by $r$ to $I$. By assumption $I^* = R$, hence $i$ is an isomorphism, and

$$\mathrm{Hom}_R(R/I, R) = 0 = \mathrm{Ext}^1_R(R/I, R).$$

By [5, Thm 1.2.5], this implies grade $\tau_M(R) \geq 2$, which is not possible given depth $R \leq 1$. We conclude that $R/I = 0$, that is, I = R. □

**Theorem 46.** *Let $R$ be a local Noetherian ring of depth $\leq 1$ and $M$ a finitely generated reflexive $R$-module. If $\mathrm{End}_R(M)$ has a free summand, then so does $M$. Therefore, $\mathrm{End}_R(M)$ is a free $R$-module only if $M$ is a free $R$-module.*

*Proof.* Since $\mathrm{End}_R(M)$ has a free summand, $\mathrm{End}_R(M)$ is faithful. Therefore, $M$ is faithful in addition to being reflexive, and hence $\tau_M(R)$ contains a nonzerodivisor by Proposition 11 (v),(vii). Then, as $R$-submodules of $Q(R)$

$$\begin{aligned}
\tau_M(R)^* &= \mathrm{End}_R(\tau_M(R)) && \text{by Proposition } 11(vi) \\
&= \mathrm{End}_R(\tau_{M\otimes_R M^*}(R)) && \because \tau_M(R) = \tau_{M\otimes_R M^*}(R) \\
&= \mathrm{End}_R(\tau_{(M\otimes_R M^*)^*}(R)) && \text{by Corollary 44} \\
&= \mathrm{End}_R(\tau_{\mathrm{End}_R(M)}(R)) && \because M \text{ is reflexive} \\
&= R && \because \tau_{\mathrm{End}_R(M)}(R) = R.
\end{aligned}$$

Since depth $R \le 1$, Lemma 45 applies and yields

$$\tau_M(R) = R,$$

and hence $M$ has a free summand.

We write $M = N \oplus R$ for some $R$-module $N$. If $\mathrm{End}_R(M)$ is a free $R$-module, then for some $n \in \mathbb{N}$

$$R^n \cong \mathrm{End}_R(M) \cong \mathrm{Hom}_R(N \oplus R, M) \cong M \oplus \mathrm{Hom}_R(N, M).$$

As a direct summand of a free module, $M$ is projective over a local ring and therefore free. □

**Definition 47.** Let $R$ be a Noetherian local ring. An $R$-module $M$ is called *maximal Cohen-Macaulay* (MCM) provided the depth of $M$ is equal to the Krull dimension of $R$.

**Definition 48.** Let $R$ be a Noetherian local ring. Then $R$ is called Gorenstein provided $\mathrm{Ext}^i_R(k, R) = 0$ for some $i > \dim R$, where $k$ is the residue field of $R$; see [18, Theorem 18.1].

A Noetherian ring, $R$, is called Gorenstein provided its localization at every maximal ideal is a Gorenstein local ring.

The next result is [22, Theorem 3.1]. We provide a new proof.

**Corollary 49.** *Let $R$ be a one-dimensional Gorenstein ring and $M$ a finitely generated $R$-module. If $\mathrm{End}_R(M)$ is projective, then the $R$-module $M$ is projective.*

*Proof.* We may assume $R$ is local and therefore $\operatorname{End}_R(M)$ is free. Since

$$\operatorname{Ass}(\operatorname{End}_R(M)) = \operatorname{Supp}(M) \cap \operatorname{Ass}(M) = \operatorname{Ass}(M),$$

the maximal ideal is not an associated prime of $M$. It follows that $M$ is MCM and being MCM over a Gorenstein ring, $M$ is also reflexive; see [22, Corollary 2.3]. Then by Theorem 46, $M$ is free. $\square$

**Proposition 50.** *Let $R$ be a local Noetherian ring and $M$ a finitely generated $R$-module such that both $M$ and $M \otimes_R M^*$ are reflexive. If $\operatorname{End}_R(M)$ has a free summand, then so does $M$. Therefore, $\operatorname{End}_R(M)$ is a free $R$-module only if $M$ is a free $R$-module.*

*Proof.* Assume $\operatorname{End}_R(M)$ has a free summand. By Proposition 11 (v), one has

$$\begin{aligned} \tau_M(R) &= \tau_{M\otimes_R M^*}(R) \\ &= \tau_{(M\otimes_R M^*)^*}(R) \\ &= \tau_{\operatorname{End}_R(M)}(R) && \because M \text{ is reflexive} \\ &= R \end{aligned}$$

and therefore, $M$ has a free summand; Proposition 11 (iii).

If, in addition, $\operatorname{End}_R(M)$ is a free $R$-module, then as in the proof of Theorem 46, we write $M = N \oplus R$ for some $R$-module $N$. Then for some $n \in \mathbb{N}$

$$R^n \cong \operatorname{End}_R(M) \cong \operatorname{Hom}_R(N \oplus R, M) \cong M \oplus \operatorname{Hom}_R(N, M).$$

As a direct summand of a free module, $M$ is projective over a local ring and therefore free.

$\square$

## 4.2 Balanced Modules

In this section, let $R$ be a commutative Noetherian ring.

**Definition 51.** Let $R \longrightarrow \mathrm{Z}(\operatorname{End}_R(M))$ be the natural map from $R$ to the center of $\operatorname{End}_R(M)$ where $r \in R$ is sent to multiplication by $r$. The $R$-module $M$ is *balanced* when this map is an isomorphism.

In this section, we discuss the implications of balancedness for reflexive modules when $R$ has depth less than or equal to one. Over rings of arbitrary depth, we establish a type of purity theorem for balancedness; this property need only be checked at primes of grade less than or equal to one.

The following proposition approaches a converse to Lemma 32. The proof of the proposition adapts ideas from Vasconcelos [23].

**Proposition 52.** *Let $R$ be a local ring of depth $\leq 1$ and $M$ a finitely-generated reflexive $R$-module. Then $M$ is balanced if and only if $M$ has a free summand.*

*Proof.* If $M$ has a free summand, then $M$ is balanced by Lemma 32.

Now, suppose that $M$ is balanced. Then $M$ is also faithful since:

$$\begin{aligned} \operatorname{Ann}_R(M) &= \operatorname{Ann}_R(\operatorname{End}_R(M)) \\ &\subseteq \operatorname{Ann}_R(\mathrm{Z}(\operatorname{End}_R(M))) \\ &= \operatorname{Ann}_R(R) = 0. \end{aligned}$$

By Corollary 37, $\operatorname{End}_R(\tau_M(R)) = \mathrm{Z}(\operatorname{End}_R(M)) = R$. That is to say, $\tau_M(R)^* = R$. By Lemma 45, we conclude that $\tau_M(R) = R$. Therefore, $M$ has a free summand; see Proposition 11 (iii). $\square$

The following is a corollary of Theorem 46 and Proposition 52.

**Corollary 53.** *Let $R$ be a local ring of depth $\leq 1$ and $M$ a finitely generated reflexive $R$-module. The following are equivalent:*

*(i)* $\operatorname{End}_R(M)$ *has a free summand;*

*(ii) $M$ has a free summand;*

*(iii) $M$ is balanced.* $\square$

**Definition 54.** We say a property of an $R$-module $M$ holds *in codimension $n$* if $M_{\mathfrak{p}}$ holds the property for all $\mathfrak{p} \in \operatorname{Spec} R$ such that grade $\mathfrak{p} \leq n$.

Consider the following result, which is Proposition 2.1 in [23]:

**Proposition 55.** *Let $M$ be a finitely generated, torsionless, faithful $R$-module. Then if $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free for each prime ideal $\mathfrak{p}$ with grade $\mathfrak{p}R_{\mathfrak{p}} \leq 1$ (as $R_{\mathfrak{p}}$ ideal), then $M$ is balanced.* □

Given that if $M$ has a free summand then $M$ is balanced (Remark 32), the following proposition extends Proposition 55.

**Proposition 56.** *Let $M$ be a finitely generated, torsionless and faithful $R$-module. If $M$ has a free summand in codimension one, then $M$ is balanced.*

*Proof.* Trace ideals behave well under localization by Proposition 11 (viii). Thus, by hypothesis, $\tau_M(R)_{\mathfrak{p}}$ is free for all prime ideals $\mathfrak{p}$ with depth $R_{\mathfrak{p}} \leq 1$.

There are injections

$$R \hookrightarrow \mathrm{Z}(\mathrm{End}_R(M)) \hookrightarrow \mathrm{End}_R(\tau_M(R)); \tag{4.1}$$

the first one holds because $M$ is faithful, the second because $M$ is also torsionless; see Corollary 30. Note, an $r \in R$ is sent to multiplication by $r$ under both of these maps.

Set $C := \mathrm{End}_R(\tau_M(R))$ and consider the exact sequence induced by the compositions of injections in (4.1):

$$0 \longrightarrow R \longrightarrow C \longrightarrow X \longrightarrow 0. \tag{4.2}$$

Notice $X_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ with grade $\mathfrak{p} \leq 1$. Therefore, $X$ has a non-zero annihilator, say $I$, with grade $I \geq 2$. Apply $\mathrm{Hom}_R(R/I, ?)$ to (4.2) to get an exact sequence

$$0 \to \mathrm{Hom}_R(R/I, R) \to \mathrm{Hom}_R(R/I, C) \to \mathrm{Hom}_R(R/I, X) \to \mathrm{Ext}^1_R(R/I, R).$$

First, $\mathrm{Ext}^1_R(R/I, R) = 0$ by [5, Thm 1.2.5]. Second, as an ideal $\tau_M(R)$ is torsionfree, hence $C$ is torsionfree. Recall, also, that since $I$ contains a nonzerodivisor, $R/I$ is torsion. Therefore, $\mathrm{Hom}_R(R/I, C) = 0$. Together, these force $\mathrm{Hom}_R(R/I, X) = 0$. However, as $I = \mathrm{Ann}_R(X)$, this implies $X = 0$. That is to say, the inclusion $R \hookrightarrow \mathrm{End}_R(\tau_M(R))$ is an equality.

Then (4.1) reads:

$$R \hookrightarrow \mathrm{Z}(\mathrm{End}_R(M)) \hookrightarrow R,$$

and since both the first injection and the composition send $r \in R$ to multiplication by $r$:

$$\mathrm{Z}(\mathrm{End}_R(M)) = R.$$

□

The following is a corollary of Propositions 52 and 56.

**Corollary 57.** *Let $M$ be a finitely generated, reflexive and faithful $R$-module. The following are equivalent:*

*(i) $M$ is balanced in codimension one;*

*(ii) $M$ is balanced.* □

## 4.3 One-Dimensional Gorenstein Rings

In this section, we apply the results of the previous sections to rigid modules over Gorenstein rings of dimension one.

**Definition 58.** An $R$-module, $M$, is called *rigid* if $\operatorname{Ext}^1_R(M, M) = 0$.

*Remark* 59. Let $R$ be a one-dimensional Gorenstein local ring. If $M$ is maximal Cohen-Macaulay then $\operatorname{Ext}^1_R(M, M) = 0$ implies $M \otimes_R M^*$ is maximal Cohen-Macaulay; see [11, Theorem 5.9]. The converse holds when $\operatorname{Ext}^1_R(M, M)$ has finite length, for example, when $M$ is free on the punctured spectrum.

**Lemma 60.** *Let $R$ be a commutative ring. Given a ring $S$ such that $R \subseteq S \subseteq Q(R)$ and $S$-modules $M$ and $N$, if $M \otimes_R N$ is $R$-torsionfree, then $M \otimes_R N = M \otimes_S N$.*

*Proof.* For any $a/b \in S$, with $b$ a nonzerodivisor, $m \in M$ and $n \in N$

$$b\left(m\frac{a}{b} \otimes_R n - m \otimes_R \frac{a}{b}n\right) = 0$$

implies $(m \cdot a/b) \otimes_R n = m \otimes_R (a/b \cdot n)$ since $M \otimes_R M^*$ is torison free. The desired result follows. □

**Lemma 61.** *Let $R$ be a commutative ring. Given a ring $S$ such that $R \subseteq S \subseteq Q(R)$ and $S$-modules $M$ and $N$, such that $N$ is torsionfree as an $R$-module, then* $\operatorname{Hom}_R(M, N) = \operatorname{Hom}_S(M, N)$.

*Proof.* Since $R \subseteq S$, one gets $\operatorname{Hom}_R(M, N) \supseteq \operatorname{Hom}_S(M, N)$.

For each $f \in \operatorname{Hom}_R(M, N)$, $m \in M$ and $a/b \in S \subseteq Q(R)$

$$b\left(f\left(\frac{a}{b}m\right) - \frac{a}{b}f(m)\right) = 0.$$

Since $N$ is torsionfree, $f\,(a/b \cdot m) = a/b \cdot f(m)$ and therefore $f \in \mathrm{Hom}_S(M,N)$. $\square$

**Theorem 62.** *Let $R$ be a one-dimensional Gorenstein local ring and $M$ a finitely-generated, torsionfree and faithful $R$-module. If $M$ is rigid and $\mathrm{Z}(\mathrm{End}_R(M))$ is Gorenstein, then $M$ has a free summand.*

*Proof.* The hypotheses are stable under completion. Indeed, let $\widehat{R}$ be the completion of $R$ with respect to its maximal ideal. Then $\widehat{R}$ is a one-dimensional Gorenstein local ring. Over such a ring, the properties torsionfree, maximal Cohen-Macaulay, torsionless and reflexive are pairwise equivalent for finitely generated modules; see [22, Corollary 2.3, Theorem A.1,], [5, Exercise 1.4.19], and [4, Theorem 6.2]. Since depth is preserved under completions, $\widehat{M}$ is MCM and therefore torsionfree over $\widehat{R}$.

As $M$ is faithful, there is an injection $R \hookrightarrow \mathrm{End}_R(M)$. Tensoring with $\widehat{R}$, one gets

$$\widehat{R} \hookrightarrow \mathrm{End}_R(M) \otimes_R \widehat{R} \cong \mathrm{End}_{\widehat{R}}(\widehat{M}),$$

and so $\mathrm{Ann}_{\widehat{R}} \widehat{M} = \mathrm{Ann}_{\widehat{R}} \mathrm{End}_{\widehat{R}}(\widehat{M}) = 0$. Therefore, $\widehat{M}$ is faithful.

Since $M$ is finitely generated,

$$0 = \mathrm{Ext}^1_R(M,M) \otimes_R \widehat{R} \cong \mathrm{Ext}^1_{\widehat{R}}(\widehat{M},\widehat{M}).$$

That is to say, $\widehat{M}$ is rigid.

Note $M$ and $\widehat{M}$ are reflexive and faithful over $R$ and $\widehat{R}$, respectively. Therefore, by Theorem 31 and Proposition 11 (viii),

$$\begin{aligned} \mathrm{Z}(\mathrm{End}_{\widehat{R}}(\widehat{M})) &\cong \mathrm{End}_{\widehat{R}}(\tau_{\widehat{M}}(\widehat{R})) \\ &\cong \mathrm{End}_R(\tau_M(R)) \otimes_R \widehat{R} \\ &\cong \mathrm{Z}(\mathrm{End}_R(M)) \otimes_R \widehat{R} \end{aligned}$$

as $R$-algebras. Because $\tau_M(R)$ is finitely generated, the extension $R \subseteq \mathrm{End}_R(\tau_M(R))$ is finite and therefore integral. Let $\mathfrak{m}$ be the maximal ideal in $R$. Because the Jacobson radical of $\mathrm{End}_R(\tau_M(R))$ is cofinal with $\mathfrak{m}\,\mathrm{End}_R(\tau_M(R))$, the module $\mathrm{End}_R(\tau_M(R)) \otimes_R \widehat{R}$ is the completion of $\mathrm{End}_R(\tau_M(R))$. It follows that $\mathrm{Z}(\mathrm{End}_{\widehat{R}}(\widehat{M}))$ is Gorenstein; see Definition 48.

Moreover, if $\tau_{\widehat{M}}(\widehat{R}) = \widehat{R}$, then $\tau_M(R) = R$ by Proposition 11 (viii). Equivalently, if $\widehat{M}$ has an $\widehat{R}$-free summand, then $M$ has an $R$-free summand. Thus we may assume that $R$ is complete.

We write $C$ for $\operatorname{End}_R(\tau_M(R))$ and note that $R \subseteq C$ is a finite extension and hence $C$ is a semilocal complete one-dimensional Gorenstein ring. As such, $C \cong \oplus_{i=1}^n C_i$ where $C_i$ are complete local Gorenstein rings; see [5, Thm 8.15]. Thus

$$\begin{aligned}
\tau_M(R) &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(\tau_M(R), R), R) \\
&\cong \operatorname{Hom}_R(C, R) \\
&\cong \operatorname{Hom}_R(\oplus_{i=1}^n C_i, R) \\
&\cong \oplus_{i=1}^n \operatorname{Hom}_R(C_i, R) \\
&\cong \oplus_{i=1}^n C_i \\
&\cong C.
\end{aligned}$$

The first isomorphism follows from the reflexivity of $\tau_M(R)$ and $\operatorname{Hom}_R(C_i, R) \cong C_i$ by [5, Thm 3.3.7].

Let $t = \beta_R(M^*)$. Since $C \subseteq Q(R)$, Lemma 61 applies and the following $R$-homomorphisms are also $C$-homomorphisms:

$$M^t \twoheadrightarrow \tau_M(R) \xrightarrow{\cong} C.$$

This is a surjective map from the $C$-module $M^t$ onto $C$. It follows that $M^t \cong N \oplus C$ for some $C$-module $N$. One has

$$\operatorname{Ext}^1_R(C, C) \subseteq \operatorname{Ext}^1_R(M^t, M^t) = 0$$

implying $C \otimes_R \operatorname{Hom}_R(C, R)$ is torsionfree; see Remark 59. Using Lemma 60, as $C$-modules, and therefore also as $R$-modules, one has :

$$C \otimes_R C^* = C \otimes_C C^* \cong C^*.$$

Note that, $\beta_R(C \otimes_R C^*) = \beta_R(C)\beta_R(C^*)$. The isomorphism $C \otimes_R C^* \cong C^*$ implies that $C$ is a cyclic $R$-module. Moreover, $R \subseteq C$ and $C$ is $R$-torsionfree and therefore, $C = R$. Since $C = \tau_M(R)^*$ and $\tau_M(R)$ is reflexive, one has $\tau_M(R) \cong R$ and therefore, $\tau_M(R) = R$ by Proposition 11 (i) and (iv). It follows that $M$ has a free summand. $\square$

**Corollary 63.** *Let $R$ be a $d$-dimensional local ring that is Gorenstein in codimension one. Let $M$ be a finitely generated torsionfree faithful $R$-module. If $M$ is rigid and* $\mathrm{Z}(\mathrm{End}_R(M))$ *is Gorenstein in codimension one, then $M$ is balanced.*

*Proof.* This is a consequence of Theorem 62 and Corollary 57. □

*Remark* 64. The ring $\mathrm{Z}(\mathrm{End}_R(M))$ being Gorenstein does not, by itself, imply that $M$ has a free summand. For example, suppose $R$ is a one-dimensional commutative domain with a finitely generated integral closure (for example, $R$ is complete) such that every ideal of $R$ is two-generated. Then every ring between $R$ and its integral closure is Gorenstein. In particular, $\mathrm{End}_R(I)$ is Gorenstein for each ideal $I$; see [4, Section 7], [19].

However, over a one-dimensional Gorenstein local domain with $M$ a torsionfree module, it is conjectured that rigidity (equivalently $M \otimes_R M^*$ is torsionfree) is sufficient to ensure $M$ is free:

**Conjecture 65.** *(Huneke and Wiegand [12, pp. 473-474]) Let $R$ be a Gorenstein local domain of dimension one and $M$ a nonzero finitely generated torsionfree $R$-module, that is not free. Then $M \otimes_R M^*$ has a nonzero torsion submodule.*

**Proposition 66.** *Conjecture 65 is true for any ideal isomorphic to a trace ideal.*

*Proof.* It is enough to prove the proposition for trace ideals. We prove the contrapositive. Writing $C$ for $\mathrm{End}_R(\tau_M(R))$, if $\tau_M(R) \otimes_R \tau_M(R)^*$ is torsionfree, then

$$\tau_M(R) \otimes_R \tau_M(R)^* = \tau_M(R) \otimes_C C \cong \tau_M(R).$$

The final map is also an $R$-isomorphism, implying $\tau_M(R)^*$ is cyclic over $R$. Since $\tau_M(R)^*$ is torsionfree and $R$ is a domain, we have $\tau_M(R)^* \cong R$. Finally, since $\tau_M(R)$ is reflexive (an ideal over a one-dimensional Gorenstein ring), $\tau_M(R) \cong R$ implying $\tau_M(R) = R$ by Proposition 11 (i) and (iv). □

*Question* 2. Over a commutative Noetherian ring of depth one, which ideals are isomorphic to a trace ideal?

For a local ring $(R, \mathfrak{m}, k)$ that is not a DVR, certainly the isomorphism classes containing $R$ and $\mathfrak{m}$ contain trace ideals: $R$ and $\mathfrak{m}$. However, not all isomorphism classes of ideals do.

**Example 67.** For a field $k$, consider the ring

$$R = k[x, y, z]/(y^2 - xz, x^2y - z^2, x^3 - yz) \cong k[t^3, t^4, t^5].$$

The ideal $I = (x, y)$ is not isomorphic to a trace ideal. For, if the isomorphism class of $I$ contained a trace ideal, then $I \cong \tau_I(R)$. However,

$$\tau_{(x,y)}(R) = (x, y, z).$$

Note, $R$ is a one-dimensional Cohen Macaulay domain which is not Gorenstein.

*Remark* 68. We have seen that under the hypotheses of Theorem 62, $\mathrm{Z}(\mathrm{End}_R(M))$ Gorenstein implies $\mathrm{Z}(\mathrm{End}_R(M)) = R$ and this implies $M$ has a free summand. To prove Conjecture 65, it is left to show that for reflexive modules over a one-dimensional Gorenstein ring, rigidity implies $\mathrm{Z}(\mathrm{End}_R(M))$ is Gorenstein, or more directly, that over a one-dimensional Gorenstein domain, rigid implies balanced.

This investigation naturally leads to the the following question:

*Question* 3. Suppose $R$ is a commutative Noetherian ring. What are the necessary and sufficient conditions for a rigid module to be balanced?

**Example 69.** Let $R = k[[x, y]]/(xy)$ and $M = R/(x)$. Recall a projective resolution of $M$ over $R$ is

$$\cdots \xrightarrow{x} R \xrightarrow{y} R \xrightarrow{x} R \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(?, R/(x))$ yields the complex

$$0 \longrightarrow \mathrm{Hom}_R(R, R/(x)) \xrightarrow{x} \mathrm{Hom}_R(R, R/(x)) \xrightarrow{y} \mathrm{Hom}_R(R, R/(x)) \longrightarrow \cdots.$$

Since multiplication by $y$ is an injective map on $R/(x)$, one gets

$$\mathrm{Ext}^1_R(R/(x), R/(x)) = 0.$$

So $M$ is rigid, but not balanced since $\mathrm{Hom}_R(R/(x), R/(x)) \cong R/(x)$.

**Definition 70.** There is a natural map from $R$ to the double centralizer of $M$,

$$R \longrightarrow \mathrm{End}_E(M)$$

given by sending $r \in R$ to multiplication by $r$. A module $M$ is said to have the *Double Centralizer Property* (DCP) when this map is a surjection.

*Remark* 71. If $M$ is faithful, the map $R \longrightarrow \mathrm{End}_E(M)$ is always an injection and therefore, having the DCP is equivalent to being balanced; see definition 51.

**Lemma 72.** *[20, Lemma 2] Let $R$ be a ring (not necessarily commutative) and $M_R$ a right $R$-module. Assume that there exists the $R$-exact sequence*

$$0 \longrightarrow R_R \xrightarrow{\delta} M_R$$

*such that $M_R$ is generated by $\delta(1)$ as an $\mathrm{End}_R(M)$-module. Then the following are equivalent*

1. *$M_R$ has the Double Centralizer Property*
2. *$M_R/\delta(R) \hookrightarrow \prod M_R$* $\square$

*Remark* 73. Suppose $R$ is a commutative ring. If $\{x_1, \ldots, x_n\}$ generate $M$ as an $R$-module, there is an injection

$$0 \longrightarrow R \xrightarrow{\delta} M^n$$

where $\delta(1) = (x_1, \ldots, x_n)$. This map fulfills the hypotheses of the lemma; see [14, Theorems 2.7, 2.8].

Recall, $\tau_{M^n}(R) = \tau_M(R)$. So when $M$ is a reflexive faithful $R$-module,

$$\mathrm{Z}(\mathrm{End}_R(M)) = \mathrm{Z}(\mathrm{End}_R(M^n)),$$

and we may replace $M$ by $M^n$. Given $R$ injects into $M$ and Lemma 72, proving Conjecture 65 is equivalent to showing $M/\delta(R)$ is torsionfree.

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