SEMISTABLE CONJECTURE VIA K-THEORY: INTEGRAL CASE

by

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ABSTRACT

Nizioł proved a p-adic comparison isomorphism of semistable schemes via K-theory. In this paper, we generalize it to integral setting; to make the classical argument work, we need the Gysin map and Grothendieck-Riemann-Roch in the log crystalline setting, and we generalize the method of Berthelot, and Gillet-Messing, respectively.

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CHAPTER 1

INTRODUCTION

Let K be a complete discrete valued field of mixed characteristic $(0,p)$ such that the residue field k of its associated ring of integer V is perfect. Let $X^{\times}$ be a fine and saturated log-smooth proper vertical $(V,\mathbf{N})$ scheme, i.e., the log structure is supported in the special fiber of the canonical morphism $X^{\times} \to (V,\mathbf{N})$; here the log structure is defined by $\mathbf{N} \to O_V, 1 \to \pi$, where π is a uniformizer of V. P-adic comparison siomorphism relates étale cohomology with coefficient $\mathbf{Q}_p$ to (log) crystalline cohomology. In [24], Nizioł gave a K-theoretic proof: under some restriction on dimension and Tate twist, étale cohomology is isomorphic to (higher) algebraic K-theory; on the other hand, on the crystalline side, though $X_{V'}$ (V' is a finite extension of V with fraction field K') may not be reduced, by the theorem of [23], we can blow it up to get a regular model Y. Then since log crystalline cohomology is stable under log blow-up, we also get a morphism from (higher) algebraic K-theory of Y to log crystalline cohomology. Since $K_i(Y) \simeq K_i(Y_{K'}) = K(X_{K'})$ (the first one comes from the localization sequence), we defined a Galois equivariant morphism from étale cohomology to crystalline cohomology. To show it is an isomorphism, notice that by Poincaŕe duality, it suffices to show it is at top degree cohomology.

For the integral case, we can use the argument in [10]: first show that the morphism commutes with both cycle classes, hence so does the trace map, so we can use the compatibility of the trace map to define the left inverse of the morphism we constructed; to show it is also an right inverse, since the cup product is the composite of the Künneth product and diagonal pullback and the morphism we constructed is functorial, it suffices to show that it commutes with diagonal classes, and that is what we show in the beginning of the argument. Nothing essential is changed, but the Poincaré duality and the existence of the cycle class map shall be rechecked.

Since Tate twist is involved in étale cohomology, the above argument can only guarantee the comparison map is an almost isomorphism, namely, isomorphism after inverting the Bott element (see the beginning of Chapter 3). In sum, we have the following theorem:

Theorem 1.1 *If* $p^n \geq 5$, $2b - a \geq max\{2d, 2\}$, $2b - a \geq 3$, *for* $d = 0$ *and* $p = 2$, *and* $p > d + 2b - a + 1$. *For* $0 \leq a \leq b \leq p - 2$, *we have a filtered almost isomorphism*

$$\alpha_{ab}^n : H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes Fil^b(A_{crys,n}) \to Fil^b(H^a(X_n^{\times}/E_n, , \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{crys,n})$$

compatible with Galois and Frobenius actions.

Here X_n means the mod-p^n reduction of X, the filtration of the right-hand side is defined by the filtration of log crystalline cohomology and filtration on $A_{\mathrm{crys},n}$; for the definition of E_n, see Section 2. In the final section, we also try to compare the comparison map with other maps given by Faltings [11], Breuil-Tsuji [8], [28], and recently by Beilinsion and Bhatt; since all the above maps commute with Chern class, that means under the assumption made here, these four morphisms are the same, again up to inverting the Bott element.

In Chapter 2, we recall the basic property of log-crystalline cohomology and syntomic regulator, and in Chapter 3, we formulate the main result and outline the proof; in Chapters 4 and 5, we prove the properties we need in the proof of our main theorem.

CHAPTER 2

PRELIMINARIES

Let V, K be as above. Let K_0 be the fraction field of $W(k)$ and $\bar{K}$ be the (chosen) algebraic closure of K. Let $X_n = X/p^n$.

2.1 Rings of periods

For the purpose of this paper, following [9], we recall the defintions of A_{crys} and sketch the definition of $\widehat{A_{\mathrm{st},\pi}}$. Later we will define a comparison isomorphism using A_{crys} and show how to get a canonical isomorphism using $\widehat{A_{\mathrm{st},\pi}}$, which is the Fontaine-Jannsen conjecture.

Define

$$A_{\mathrm{crys}} = \varprojlim H^0_{\mathrm{cris}}(\bar{V}/p\bar{V}/W_n) \simeq \varprojlim W_n(\bar{V}/p\bar{V})^{\mathrm{DP}},$$

where both inverse limits are induced by the truncation map

$$W_n(\bar{V}/p\bar{V})^{\mathrm{DP}} \to W_{n-1}(\bar{V}/p\bar{V})^{\mathrm{DP}},$$

$$(a_0, a_1, \ldots, a_{n-1}) \to (a_0^p, \ldots, a_{n-2}^p),$$

and DP means divided power envelope with respect to an ideal given by the kernel of the following surjective map

$$\Theta_n : W_n(\bar{V}/p\bar{V})^{\mathrm{DP}} \to \bar{V}/p^n\bar{V}, \quad \theta(a_0, \ldots, a_{n-1}) = \hat{a}_0^{p^n} + p\hat{a}_1^{p^{n-1}} + \cdots + p^{n-1}\hat{a}_{n-1}^p,$$

where $\hat{a}_i$ means any lifting of a_i in $\bar{V}/p^n\bar{V}$. Let $J_n = \ker \Theta_n$ and $J_n^{[i]}$ be its i^{th} divided power. For $0 \leq i \leq p-1$, we define

$$\mathrm{Fil}^i A_{\mathrm{crys}} = \varprojlim \ J_n^{[i]}.$$

From the definition of J_n, we have $\phi(\mathrm{Fil}^i A_{\mathrm{criy}}) \subset p^i A_{\mathrm{crys}}$; in general, we define

$$\mathrm{Fil}^i A_{\mathrm{crys}} = \{x \in A_{\mathrm{crys}} | \phi(x) \in p^i A_{\mathrm{crys}}\}.$$

Let $t = (\xi_n)_n$ be a compatible system of primitive p^n-th roots of unity in $\mathcal{O}_{\bar{K}}$ and $[\xi_n]$ be the Teichmüller representative in $W_n(\mathcal{O}_{\bar{K}}/p)$, so we get an element (which is still denoted by) $t = ([\xi_n])_n$ in A_{crys}; now define $\beta = \log(t)$. It is also an element of A_{crys} since the valuation of $\epsilon_n - 1$ is $\frac{1}{p-1}$; the Taylor expansion of logarithmic function converges.

Now we turn to $\widehat{A_{\text{st},\pi}}$. Let E be the p-adic completion of

$$W < u >= \{\Sigma_{i=0}^{n} w_i u^i / i! | w_i \in W := W(\bar{k}), n \in \mathbf{N}\}.$$

(In [9] it is denoted by S). Define the Frobenius on E by $\phi(\Sigma w_i u^i / i!) = \Sigma \phi(w_i) u^{pi} / i!$ and define $\text{Fil}^i E$ = p-adic completion of the ideal generated by $\{(u - \pi)^j / j!, j \geq i\}$. Again, we have $\phi(\text{Fil}^i E) \subset p^i E$. Now define the monodromy operation as a W-linear derivation by $N(u) = -u$. The log structure of E is given by $\mathbf{N} \to E, 1 \to u$.

Now define

$$\widehat{A_{\text{st},\pi}} = \varprojlim H^0_{\text{crys}}((\bar{V}/p\bar{V})/(E/p^n E)) \simeq \varprojlim W_n(\bar{V}/p\bar{V})^{\text{DP}} < X_\pi >,$$

where X_π is the chosen parameter and the last isomorphism can be found in [26] Lemma 1.6.5 (it is denoted by $v_\beta - 1$ in [26]). Now choose a compatible system $\{\pi_n\}$ such that $\pi_n^p = \pi_{n-1}$. This system defines an element in $\varprojlim W_n(\bar{V}/p\bar{V})$ and we denote $[\underline{\pi}]$ as the corresponding image in $\varprojlim W_n(\bar{V}/p\bar{V})^{\text{DP}} \simeq A_{crys}$. Then we have an identification of $\widehat{A_{\text{st},\pi}}$ with the p-adic completion of $A_{crys} < X_\pi >$ and $u = [\underline{\pi}](1 + X_\pi)^{-1}$.

Now define $\phi(X_\pi) = (1 + X_\pi)^p - 1$ and $N(X_\pi) = X_\pi + 1$ and define

$$\text{Fil}^i \widehat{A_{\text{st},\pi}} = \{\Sigma_{j=0}^{\infty} a_j \frac{X_\pi^j}{j!} | a_j \in \text{Fil}^{i-j} A_{\text{crys}}, a_j \to 0\}.$$

Again we will have $\phi(\text{Fil}^i \widehat{A_{\text{st},\pi}}) \subset p^i \widehat{A_{\text{st},\pi}}$.

For Galois action, by definition, there is a continuous action of G_{K_0} on A_{crys} that preserves the filtration and commutes with Frobenius; for $g \in G_{K_0}$, now we extend the Galois action on $\widehat{A_{\text{st},\pi}}$ by defining $g(X_\pi) = [\epsilon(g)] X_\pi + [\epsilon(g)] - 1$, where $\epsilon : G_{K_0} \to \varinjlim \mu_{p^n}(\bar{K})$ is the continuous 1-cocycle determined by the choice of compatible system of p^n-th roots of π. This action preserves Frobenius and monodromy action.

We denote $A_{crys,n}$ by A_{crys}/p^n, and similarly, for a fixed π, we set $\widehat{A_{\text{st},n}} = \widehat{A_{\text{st},\pi}}/p^n$.

2.2 Log-syntomic cohomology

Here we recall the definition of the complex $\mathcal{S}_n(r)_{X^\times}$ for fine and saturated log smooth scheme $X^\times$ in [24].

First assume there exists an exact closed immersion $X_n^\times \hookrightarrow Z_n^\times$ such that $Z_n^\times$ is log smooth over $W_n(k)$ (whose underlying scheme is $W_n(k)$ associated with trivial log structure) and the Frobenius map on $W_n(k)$ extends to $Z_n^\times$. Denote $D_n^\times$ as the associated DP-envelope and by abuse of notation, we still denote by ϕ the Frobenius extending from $Z_n^{\times}$. Denote $\mathcal{J}_{D_n}^{[i]}$, the ideal of $\mathcal{O}_{D_n}$ generated by its $j-$th power of ideal of the DP-thickening, for $j \geq i$. Since D_n is flat over $W_n(k)$ by construction, $\mathcal{J}_{D_n}$ is flat over $W_n(k)$; by [24] Section 2.1, for $0 \leq r \leq p-1$, there exists an unique $\phi_r : \mathcal{J}_{D_n} \to \mathcal{O}_{D_n}$ such that the following diagram is commutative :

$$\begin{array}{ccc} \mathcal{J}_{D_{n+r}} & \xrightarrow{\phi} & \mathcal{O}_{D_{n+r}} \\ \downarrow & & \downarrow{\scriptstyle p^r} \\ \mathcal{J}_{D_n} & \xrightarrow{\phi_r} & \mathcal{O}_{D_n}. \end{array}$$

Now define

$$\mathcal{S}_n(r)_{X^\times} = \ \mathrm{Cone}(1-\phi_r : \mathcal{J}_{D_n}^{[r-\cdot]} \otimes_{O_{Z_n}} \omega^{\cdot}_{Z_n/W_n(k)} \to O_{D_n} \otimes_{O_{Z_n}} \omega^{\cdot}_{Z_n/W_n(k)})[-1]$$

In general, we can choose an étale affine covering such that in each cover, we have such log smooth lifting extending Frobenius. Then they form a double complex; the log syntomic complex is defined as the cone of the associated double complexes.

As in [28] p. 542, for $0 \leq r, r', r+r' \leq p-1$, one can define product map

$$\mathcal{S}_n(r)_{X^\times} \otimes \mathcal{S}_n(r')_{X^\times} \to \mathcal{S}_n(r+r')_{X^\times},$$

and notice that that the natural map $\mathcal{S}_n(r)_X \to \mathcal{S}_n(r)_{X^\times}$ (the former one is a scheme equipped with trivial log structure), which is compatible with such product structure.

2.3 Basic property of log crystalline cohomology

Now, assume $X^\times$ is vertical log smooth and universally saturated over $(V, \mathbf{N})$; we want to formulate some properties of its log-crystalline cohomology over E_n here.

Proposition 2.1 *Künneth formula holds on $H^i(X_n^\times/E_n, \mathcal{O}_{X_n/E_n})$, i.e., if we have Y_n, which is vertical log smooth and universally saturated over $(V, \mathbf{N})$, then we have the canonical isomorphism:*

$$H^k((X_n \times Y_n)^\times/E_n, \mathcal{O}_{(X_n\times Y_n)^\times/E_n}) \simeq \oplus_{i+j=k} H^i(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \otimes H^i(Y_n^\times/E_n, \mathcal{O}_{X_n/E_n}).$$

Proof : The proof essentially is the same as in [3] V Proposition 4.1.7.

It suffices to define the Künneth isomorphism étale locally and make sure of the compatibility. Thus we assume $X_n^\times$ admits a lifting $Z_n^\times$ log smooth over E_n. Here we may assume Z_n affine, and since $X_n^\times$ is log smooth over $(V_n, \mathbf{N})$, the cohomology is equal to cohomology of de Rham complex $\omega^{\cdot}_{Z_n/E_n}$. Assume there is another log smooth scheme of $Y_n^{\times}$ with local lifting $Z_n'^{\times}$. In this case, the assertion is reduced to :

$$\omega^1_{Z_n/E_n} \otimes \mathcal{O}_{Z'_n} + \mathcal{O}_{Z'_n} \otimes \omega^1_{Z_n/E_n} = \omega^1_{Z_n \otimes_{E_n} Z'_n/E_n}.$$

Since $M_{Z'_n \otimes_{E_n} Z'_n} = M_{Z_n} \otimes_{M_{E_n}} 1 + 1 \otimes_{M_{E_n}} M_{Z'_n}$ (by assumption on X_n, the log structure after fiber product is still saturated, so we do not need to worry about saturation) and the compatibility holds, the assertion follows.

Proposition 2.2 *Poincaré duality holds on* $H^i(X_n^{\times}/E_n, \mathcal{O}_{X_n/E_n})$.

Here we mainly follow the method of [3] and [27]. In fact, it is already proved in [11] p. 249; the idea is as follows: though E_n is not noetherian, we can first work on $A_n := W_n[t]/t^s$, where s is large enough such that we have a ring homomorphism $A_n \to E_n$, and the log structure is defined on t. Assume locally on X_n that we have a log smooth lifting X'_n on A_n, then $X'_n \otimes_{A_n} E_n$ is a log smooth lifting on E_n and locally the associated de Rham complex is isomorphic to $\omega^{\cdot}_{X'_n/A_n} \otimes_{A_n} E_n$, and the reader will see the construction of the trace map can be deduced from the A_n case. First, we prove a lemma, which can help us to compute log crystalline cohomology, using only Zariski topology:

Lemma 2.3 *For log smooth and separated of finite type* $X_n^{\times}$ *over* $(V_n, \mathbf{N})$*, the log crystalline cohomology* $H^i(X_n^{\times}/E_n, \mathcal{J}^{[m]}_{X_n/E_n})$ *is invariant under log blow-up.*

Proof : Denote the log scheme after log blow-up by X'_n. Choose an affine covering $(U_{n,i})_{i\in I}$ on X_n such that it admits a log smooth lifting on E_n; let $U'_{n,i}$ be the inverse image of $U_{n,i}$ in X'_n; by the construction of Proj, it also admits smooth lifting. So the crystalline cohomology is the cohomology associated to the double complex formed by the Cěch complex corresponding to $(U_{n,i})_{i\in I}$ and $(U'_{n,i})_{i\in I}$. In particular, we have a natural map corresponding to these two double complexes. We can form the spectral sequence associated to these double complexes, which are both regular (i.e., fix p, q, $d^r_{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ is the zero map for r large enough since the double complex is in the first quadrant). It suffices to show the isomorphism on the E_1 term.

Now consider the single term $\mathcal{C}(U^{\text{lift}}_{n,i}, \mathcal{J}^{[n-j]}_{U^{\text{lift}}_{n,i}} \otimes_{\mathcal{O}_{U^{\text{lift}}_{n,i}/E_n}} \omega^j_{U^{\text{lift}}_{n,i}/E_n})$. Here $U^{\text{lift}}_{n,i}/E_n$ means a smooth lifting of $U_{n,i}/E_n$. Notice that we have a canonical morphism

$$\mathcal{J}^{[n-j]}_{U^{\text{lift}}_{n,i}} \otimes_{\mathcal{O}_{U^{\text{lift}}_{n,i}/E_n}} \omega^j_{U^{\text{lift}}_{n,i}/E_n} \to Rf_*(\mathcal{J}'^{[n-j]}_{U'^{\text{lift}}_{n,i}} \otimes_{\mathcal{O}_{U'^{\text{lift}}_{n,i}/E_n}} \omega^j_{U'^{\text{lift}}_{n,i}/E_n})$$

(where we denote f as the blow-up map) which is an isomorphism since by [20] Theorem 11.3, the canonical map $\mathcal{O}_{U^{\text{lift}}_{n,i}} \to Rf_*(\mathcal{O}_{U'^{\text{lift}}_{n,i}})$ is an isomorphism. Hence, the same assertion holds for $J_{U^{\text{lift}}_{n,i}}$ since it is a DP-ideal of $\mathcal{O}_{U^{\text{lift}}_{n,i}}$; and from the isomorphism $\mathcal{O}_{U^{\text{lift}}_{n,i}} \simeq Rf_*(\mathcal{O}_{U'^{\text{lift}}_{n,i}})$, we get $Lf^*\mathcal{O}_{U^{\text{lift}}_{n,i}} \simeq \mathcal{O}_{U'^{\text{lift}}_{n,i}}$, so we have $f^*\omega^j_{U^{\text{lift}}_{n,i}/E_n} \simeq \omega^j_{U'^{\text{lift}}_{n,i}/E_n}$, and hence,

$$Rf_*\omega^j_{U'^{\text{lift}}_{n,i}/E_n} \simeq Rf_* \circ Lf^*\omega^j_{U^{\text{lift}}_{n,i}/E_n} \simeq \omega^j_{U^{\text{lift}}_{n,i}/E_n} \otimes^L Rf_*(\mathcal{O}_{U'^{\text{lift}}_{n,i}}) \simeq \omega^j_{U^{\text{lift}}_{n,i}/E_n}$$

from the projection formula; hence, each term of these double complexes are the same. In general, this argument works on every term in the double complex; hence the proof.

Thus we reduce to the vertical semistable case. The following argument follows from [27] with some modifications since our base is E_n; on the other hand, while in [27] Tsuji works on étale site, here, first we use Lemma 2.3 to reduce the problem to Zariski topology and illustrate the main idea. In fact, it is not necessary to do so; one of the reasons we decided to work on Zariski topology is to simplify notation. The other reason is, as discussed in Chapter 4, when we begin to construct the cycle class map, the argument presented here strongly relies on the fundamental local isomorphism in [16] III Proposition 7.2, which is Zariski in nature (again, it also works for étale topology. For example, we can first choose an étale covering such that the scheme and the corresponding cycle admits smooth lifting on each open covering, and we first work on the de Rham complex of each smooth lifting, and then after checking the compatibility of the construction, we will get an extended version of Proposition 4.12 and Proposition 4.13; hence, the cycle class). The process follows from [3]; we will construct a trace morphism using residue complex; while using residue complex we can reduce to local case, then locally X_n admits a smooth lifting over E_n; to prove the vanishing of the residue map whose source coming from the class of lower codimension, we follow from the method of [27], Proposition 3.1.

Since X_n is Cohen-Macaulay, locally around a closed point x we can find a smooth lifting $f' : X'^{\times}_n \to A_n$, (following [11], p. 249, define $A_n = V_n[t]/t^s$, for s large enough such that it maps to E_n) such that the X'_n is also Cohen-Macaulay. Moreover, the smooth locus of f' contains all codimension 1 points of X'_n. As in [20] Theorem 11.2, we have $f'^!A_n = \omega^d_{X'_n/A_n}[d]$, which is the dualizing complex of X'_n (see [20] (11.2) or [27] Theorem 2.21).

Thus as in [16] p. 344 Lemma 4.4, we get a morphism of A_n sheaves $f'_*f'^!A_n \to A_n$. Tensoring with E_n and taking the cohomology, we get a morphism

$$\mathcal{H}^d(X'_n, \omega^d_{X'_n/E_n}) \overset{\mathrm{Tr}_{f',x}}{\to} E_n.$$

The goal is that we want to construct a trace map

$$H^{2n}_{\text{crys}}(X^{\times}_n/E_n, \mathcal{O}_{X_n/E_n}) \to \mathcal{O}_{E_n}.$$

First, we show that the composition map

$$\mathcal{H}^d(X'_n, \omega^{d-1}_{X'_n/E_n}) \xrightarrow{d} \mathcal{H}^d(X'_n, \omega^{d}_{X'_n/E_n}) \overset{\mathrm{Tr}_{f',x}}{\to} E_n.$$

is zero map. Taking the Cousin complex of $\omega^{d-1}_{X'_n/E_n}$ and $\omega^{d}_{X'_n/E_n}$, it suffices to show:

Lemma 2.4 *For all closed points x of X_n. The composite homomorphism*

$$\mathcal{H}^d_x(\omega^{d-1}_{X'_n/E_n}) \xrightarrow{d} \mathcal{H}^d_x(\omega^{d}_{X'_n/E_n}) \xrightarrow{Tr_{f',x}} E_n$$

is zero.

Proof : Here we slightly modify Tsuji's proof [27]. Consider the exact closed immersion $E_n \hookrightarrow E_n[u]$ induced by the map $u \to t$ (and the log structure of $A_n[u]$ is given by $\mathbf{N} \to A_n[u], 1 \to u$. Since the question is local, we may assume X'_n admits a log-smooth lifting $X''^{\times}_n$, and we denoted the closed immersion by $i : X'^{\times}_n \hookrightarrow X''^{\times}_n$. On the other hand, consider the morphism $E_n[u] \to E^{\circ}_n$ (the log scheme whose underlying scheme is E_n with trivial log structure) induced by the trivial map $E_n[u] \to E_n$, then X''_n is (classically) smooth over E_n. (We deform $W_n(k)[x, y]/xy - p$ to $W_n(k)[x, y, w]/(xy - w)$). Now we hope an analogue of Berthelot's Proposition holds ([3] VII Proposition 1.2.6), namely the composition map

$$\mathcal{H}^{d+1}_{x''}(\Omega^{d}_{X''_n/E_n}) \xrightarrow{d} \mathcal{H}^{d+1}_{x''}(\Omega^{d+1}_{X''_n/E_n}) \xrightarrow{\mathrm{Tr}_{f''}} E_n$$

is zero, where x'' is defined by $\{$the regular sequence defining $x, u - t\}$.

To see this, it suffices to show the analogue of [3] VII Lemma 1.2.5, i.e., denote $t_1, \ldots, t_{d+1}$ the lifting of regular system of parameter of $\mathcal{O}_{X''_n,x''}/m\mathcal{O}_{X''_n,x''}$ in $\mathcal{O}_{X''_n,x''}$, where m is the maximal ideal of E_n, so we have an isomorphism $\varinjlim_k \Omega^{d+1}_{X''_n/E_n,x''}/(t^k_1, \ldots, t^k_{d+1}) \simeq \mathcal{H}^{d+1}_{x''}(\Omega^{d+1}_{X''_n/E_n})$; here we denote $\Omega^{d+1}_{X''_n/E_n}$ as the classical, not log differential then for any $w \in \Omega^{d+1}_{X''_n/E_n,x''}$ under this isomorphism, we have

$$\mathrm{Tr}_{f'',x''}(w/t^k_1 \cdot \cdots \cdot t^k_{d+1}) = \mathrm{Res}\begin{bmatrix} w \\ t^k_1, \cdots, t^k_{d+1} \end{bmatrix}.$$

Denote $j_k : Z_k \hookrightarrow X''_{n,x}$ the closed subscheme of $\mathcal{O}_{X''_n x''}$ with defining ideal $t^k_1, \cdots, t^k_{d+1}$, then $\mathrm{Res}\begin{bmatrix} w \\ t^k_1, \cdots, t^k_{d+1} \end{bmatrix}$ is the composition map

$$f''_*(\Omega^{d+1}_{X''_n/E_n,x''} \otimes_{\mathcal{O}_{X''_n x''}} J_k) \to \underline{\mathrm{Hom}}_{E_n}(\mathcal{O}_{Z_k}, E_n) \to E_n,$$

where $\omega_k := (\wedge^{d+1}(t^k_1, \ldots, t^k_{d+1})/(t^k_1, \ldots, t^k_{d+1})^2)^{\vee}$. Namely, it is a linear extension of [16] III 8 from A_n to E_n. So it suffices to show that the argument in [3] VII Lemma 1.2.5

is compatible with extension from A_n to E_n, the answer is positive since the argument is attributed to the following commutative diagram:

$$\begin{array}{ccc} f''_*\underline{\mathrm{Hom}}_{\mathcal{O}_{X''_n,x''}}(\mathcal{O}_{Z_k},\mathcal{H}^{d+1}_{x''}(\Omega^{d+1}_{X''_n/E_n})) & \xrightarrow{\simeq} & f''_*(\Omega^{d+1}_{X''_n/E_n,x''}\otimes_{\mathcal{O}_{X''_nx''}}\omega_k) \\ \big\downarrow & & \big\downarrow \mathrm{Tr}_{Z_k} \\ f''_*\mathcal{H}^{d+1}_{x''}(\Omega^{d+1}_{X''_n/E_n}) & \xrightarrow{\mathrm{Tr}_{f'',x''}} & E_n \end{array}$$

The isomorphism in the upper horizontal arrows is valid for any smooth morphism such that it is regular on the special fiber, and the trace morphism in [16] is defined on the complex level, so the trace map in the E_n setting can be defined as linearly extension from A_n to E_n; finally, the commutativity of the above diagram comes from the transitivity of trace map, which is still true after linearly extending to E_n, so we still have the above commutative diagram (though we did not try to define $f''^{\triangle}$ in this case), hence the claim.

Go back to the argument; without loss of generality, we may assume $t_{d+1} = u - t$; now denote t'_i the reduction of t_i in $\mathcal{O}_{X'_n,x}$. On the other hand, Zariski locally $\omega^d_{X'_n/E_n} \simeq \Omega^{d+1}_{X''_n/E_n} \otimes \check{\mathcal{N}}_{X'_n/X''_n}$, (the map is defined as $\omega \to (dt \wedge \omega) \otimes (t-u)^\vee$). Again

$$\mathcal{H}^d_x(\omega^d_{X'_n/E_n}) \simeq \varinjlim \omega^d_{X'_n/E_n}/t_1'^{k_1}\cdots t_d'^{k_d},$$

where $k_1 \cdots k_d$ runs through all positive integers and the map

$$\mathcal{H}^d_x(\omega^d_{X'_n/E_n}) \simeq \mathcal{H}^d_{x''}(\Omega^{d+1}_{X''_n} \otimes \check{\mathcal{N}}_{X'_n/X''_n})$$

$$\to \mathcal{H}^d_{x''}(\mathrm{Hom}_{\mathcal{O}_{X''_n}}(\mathcal{O}_{X'_n}, \mathcal{H}^{d+1}_{x''}(\omega^{d+1}_{X''_n}[d+1]))) \to \mathcal{H}^{d+1}_{x''}(\Omega^{d+1}_{X''_n}[d+1])$$

is given by $\frac{\omega}{t'_1\cdots t'_d} \to \frac{\omega}{(t-u)t_1\cdots t_d}$; denote this map by $\mathrm{Tr}_{i,x}$; if we define the above map in this way, then we have the transitivity of trace map, namely $\mathrm{Tr}_{f'',x} \circ \mathrm{Tr}_{i,x} = \mathrm{Tr}_{f',x}$. By direct computation ([27] p. 26), we see :

$$d(\frac{\omega}{t'_1\cdots t'_d}) = \frac{d\omega}{t'_1\cdots t'_d} - \Sigma_{1\le i\le d}\frac{dt'_i\wedge\omega}{t'_1\cdots t'^2_i\cdots t'_d},$$

$$\frac{dt\wedge\omega}{(t-u)t'_1\cdots t'_d} = \frac{dt\wedge d\omega}{(u-t)t'_1\cdots t'_d} - \Sigma_{1\le i\le d}\frac{dt\wedge dt'_i\wedge\omega}{(u-t)t'_1\cdots t'^2_i\cdots t'_d}.$$

So the latter is the image of the former, by transitivity of trace map, so we reduce to the smooth case, i.e., [3] VII Proposition 1.2.6, hence the proof.

For the rest it is exactly the same as in Tsuji's argument [27]; what we need to check is that the argument is compatible with extension from A_n to E_n; it is sketched as follows: notice that $\mathcal{H}^i_x(\mathcal{O}_{X_n/E_n})$ is a log crystal, in particular, we have

$$R^d f_{X_n/E_n,*}(\mathcal{H}^d_x(\mathcal{O}_{X_n/E_n})) \simeq H^d_x(\omega^{\cdot}_{X'_n/E_n}),$$

so after the above lemma, we can define the residue map

$$\mathrm{Res}_{f,x} : R^d f_{X_n/E_n,*}(\mathcal{H}^d_x(\mathcal{O}_{X_n/E_n})) \to E_n$$

as the composition map of the above isomorphism and the trace map; it is independent of the smooth lifting we choose; (notice the objects below are all crystals on log crystalline site)

$$\mathcal{O}_{X_n/E_n} \to \mathcal{H}^0_{X^0_n/X^1_n}(\mathcal{O}_{X_n/E_n}) \to$$

$$\mathcal{H}^1_{X^1_n/X^2_n}(\mathcal{O}_{X_n/E_n}) \to \cdots \mathcal{H}^i_{X^i_n/X^{i+1}_n}(\mathcal{O}_{X_n/E_n}) \to \cdots$$

where we denote X^i_n the point of codimension i of X_n. First by [27] Lemma 8.7, we have the isomorphism

$$\bigoplus_{x\in X^i_n/X^{i+1}_n} R^j f_{X_n/E_n,*}(\mathcal{H}^i_x(\mathcal{O}_{X_n/E_n})) \simeq R^j f_{X_n/E_n,*}(\mathcal{H}^i_{X^i_n/X^{i+1}_n}(\mathcal{O}_{X_n/E_n})).$$

That means what we need to prove is for for $z \in X^{d-1}_n/X^d_n$, $x \in X^d_n \cap \{\bar{z}\}$, the composition map

$$R^d f_{X_n/E_n,*}(\mathcal{H}^{d-1}_z(\mathcal{O}_{X_n/E_n})) \to R^d f_{X_n/E_n,*}(\mathcal{H}^d_x(\mathcal{O}_{X_n/E_n})) \overset{\mathrm{Tr}_{f,x}}{\to} \mathcal{O}_{E_n}$$

is zero, where the last map is induced from the surjective map

$$H^d_{x'}(\omega^d_{X'_n/E_n}) \to H^{2d}_{x'}(\omega^{\cdot}_{X'_n/E_n})$$

and Lemma 2.4. If this claim is proved, consider the spectral sequence

$$E^{i,j}_1 = R^j f_{X_n/E_n,*}(\mathcal{H}^i_{X^i_n/X^{i+1}_n}(\mathcal{O}_{X_n/E_n})) \Rightarrow R^{i+j} f_{X_n/E_n,*}(\mathcal{O}_{x_n/E_n})$$

and since

$$R^j f_{X_n/E_n,*}(\mathcal{H}^i_x(\mathcal{O}_{X_n/E_n})) = 0, \text{ for } \ j > d,$$

we have an exact sequence

$$R^d f_{X_n/E_n,*}\mathcal{H}^d_{X^{d-1}_n/X^d_n}(\mathcal{O}_{X_n/E_n}) \to R^d f_{X_n/E_n,*}\mathcal{H}^d_{X^d_n}(\mathcal{O}_{X_n/E_n})$$

$$\to R^{2d} f_{X_n/E_n,*}(\mathcal{O}_{X_n/E_n}) \to 0.$$

Combined with residue theorem, we get a trace map as desired ([3] VII Theorem 1.4.6). Now back to the claim, the composition map is equivalent to

$$H^{d-1}_{z'}(\omega^d_{X'_n/E_n}) \xrightarrow{\delta} H^d_{z'}(\omega^d_{X'_n/E_n}) \overset{\mathrm{Res}_{f,x}}{\to} E_n.$$

On the other hand, denote $Z := \{\bar{z}\}$ and $u_k : W'_k \hookrightarrow X'_n$ the k-th infinitesimal neighborhood of Z in X'_n with the log structure induced from X'_n; denote $h'_k : W'_k \to E_n$ the canonical map, then we can consider the residue map $h'_{k*}h'^{\triangle}_k(E_n)_x \to E_n$, which is the linear extension of residue map in [16] to E_n. Denote $h'_{k*}h'^{\triangle}_k(E_n)^i$ the i-th degree of the complex $h'_{k*}h'^{\triangle}_k(E_n)$; by the transivisity of the trace map, we have the following commutative diagram

$$\begin{array}{ccccc} h'_{k*}h'^{\triangle}_k(E_n)^{-1} & \xrightarrow{\delta} & h'_{k*}h'^{\triangle}_k(E_n)^0_x & \xrightarrow{\mathrm{Res}_{h'_k,x}} & E_n \\ \downarrow{\scriptstyle \mathrm{Tr}_{u'_k}} & & \downarrow{\scriptstyle \mathrm{Tr}_{u'_k}} & & \downarrow{\scriptstyle =} \\ H^{d-1}_{z'}(\omega^d_{X'_n/E_n}) & \xrightarrow{\delta} & H^d_{z'}(\omega^d_{X'_n/E_n}) & \xrightarrow{\mathrm{Res}_{f,x}} & E_n \end{array}$$

Now by [27] Corollary 8.13 ([3] VII Corollary 1.3.7, where the flatness of f is used), there exists W_k over E_n such that for any local lifting X'_n and W'_k be as the above, we have the E_n isomorphism $W_k|U \simeq W'_k$, where $U := X'_n \times_{\mathrm{Spec}E_n} \mathrm{Spec}W_n$, the local neighborhood of z. That means globally, we have the map

$$h_{k*}h^{\triangle}_k(E_n)^{-1} \xrightarrow{\delta} h_{k*}h^{\triangle}_k(E_n)^0_x \overset{\mathrm{Res}_{h_k,x}}{\to} E_n,$$

which is compatible with the commutative diagram above, now notice that

$$h'_{k*}h'^{\triangle}_k(E_n)^{-1} \simeq \mathrm{Hom}_{\mathcal{O}_{X'_n}}(\mathcal{O}_{Z_k}, i'_{z*}H^{d-1}_z(\omega^{d-1}_{X'_n/E_n})),$$

where $i'_z : z \to X'_n$ is the canonical map so that means for any element $b \in H^{d-1}_{z'}(\omega^d_{X'_n/E_n})$, it will lie in the image of $h'_{k*}h'^{\triangle}_k(E_n)^{-1}$, for k large enough, hence in $h_{k*}h^{\triangle}_k(E_n)^{-1}$; then the assertion follows from the composition map

$$h_{k*}h^{\triangle}_k(E_n)^{-1} \xrightarrow{\delta} h_{k*}h^{\triangle}_k(E_n)^0 \overset{\mathrm{Res}_{h_k,x}}{\to} E_n,$$

(notice we did not localize at x) is 0, since the trace map is the morphism of complexes and the target concentrate at the 0-th degree.

Finally, the trace map is compatible with base change, namely we have a commutative diagram for $m \le n$ ([3], VII Proposition 1.4.8)

$$\begin{array}{ccc} R^{2d} f_{X_n/E_n,*}(\mathcal{O}_{X_n/E_n}) \otimes_{E_n} E_m & \longrightarrow & R^{2d} f_{X_m/E_m,*}(\mathcal{O}_{X_m/E_m}) \\ & \searrow^{\mathrm{Tr}_{f_n} \otimes \mathrm{Id}} \quad \swarrow_{\mathrm{Tr}_{f_m}} & \\ & \mathcal{O}_{X_m/E_m} & \end{array}$$

Back to the proof of Poincaré duality, for $n = 1$, let the Frobenius map F act on t (recall $A_1 = k[t]/t^s$) trivially, then as a $\mathcal{O}_{X_1''^{(p)}}$ module (base change w.r.t. Frobenius map), we still have Cartier isomorphism

$$\omega^{\cdot}_{X_1''^{(p)}/E_1} \simeq \mathbb{H}^*(\omega^{\cdot}_{X_1''/E_1}).$$

Here both sides satisfy Küneeth formula and by cohomological descent, it suffices to treat the case $X_1'' = E_1[v]$, which is true by direct computation.

That means when $n = 1$, the trace map is an isomorphism. By Nakayama lemma, the trace map is an isomorphism for general n. Now we consider pairings. Since $RHom$ commutes with tensor product (for perfect complexes, which is true in this case by taking Cěch complex), by Nakayama lemma again, we reduce to the case $n = 1$, which is true again by Cartier isomorphisms. Hence we establish Poincaré duality.

2.4 Syntomic regulators

In this section, we briefly recall the construction of syntomic Chern class in [22] Section 2.2. All we need to do is to replace the rational definition to an integral one. Also we work on syntomic cohomology $\mathcal{S}_n(i)_X$, i.e., with trivial log structure and then the log syntomic regulator is induced by the natural map $\mathcal{S}_n(i)_X \to \mathcal{S}_n(i)_{X^\times}$; while writing $\mathcal{S}_n(i)$, we view it as a functor from schemes over $W_n(k)$ to abelian groups.

Consider the group scheme BGL_m over $W_n(k)$, and consider the projective bundle associated to it. As in [14] Definition 2.3, now we consider the tautological divisor on it, denoted by ξ. Since BGL_m is smooth over $W_n(k)$, $\mathcal{S}_n(i)_{BGL_m}$ has Dold-Thom isomorphism ([16] (4.2)). That means we can consider the universal Chern classs as the coefficient of the following equation in the projective bundle:

$$\xi^n + p^*(C_1)\xi^{n-1} + \cdots + p^*(C_n) = 0,$$

where p is the natural projection from the projective bundle to BGL_m. To compute the cohomology of $\mathcal{S}_n(i)$, we assume there is a functorial way to get acyclic resolutions (for the syntomic side we run through all affine covering which admits smooth lifting). Denote B_kGL_m as the k-th level of the simplicial scheme BGL_m, so $H^{2i}(BGL_m/W_n(k), \mathcal{S}_n(i)_{BGL_m})$ can be computed as the double complex whose (k,l) term is $\mathrm{Hom}(B_kGL_n, \mathcal{S}_n(i)^l_{GL_m})$, where $\mathcal{S}_n(i)^l_{GL_n}$ is the l-th acyclic resolution of $\mathcal{S}_n(i)_{GL_n}$.

Since for each $W_n(k)$ morphism X_n to BGL_m is equivalent to a compatible family of $W_n(k)$ morphism X_n to B_kGL_m, and through this map we have a natural $W_n(k)$ morphism $\mathcal{S}_n(i)^l_{GL_m}$ to $\mathcal{S}_n(i)^l_X$; since $Hom_{W_n(k)}(X_n, B_kGL_m) = B_kGL_m(\mathcal{O}_{X_n})$, we get a compatible family of morphisms

$$\mathrm{Hom}(B_kGL_m, \mathcal{S}_n(i)_X) \to \mathrm{Hom}(X_n, \mathcal{S}_n(i)_X).$$

Denote $H^{2i}(X_n, \mathcal{GL}(\mathcal{O}_{X_n}), \mathcal{S}_n(i)_X)$ by the right derived functor of $\mathcal{S}_n(i)^{\mathcal{GL}(\mathcal{O}_{X_n})}_{X_n}$, so we get a map

$$H^{2i}(BGL_m/W_n(k), \mathcal{S}_n(i)_{BGL_m}) \to H^{2i}(X_n, \mathcal{GL}(\mathcal{O}_{X_n}), \mathcal{S}_n(i)_X).$$

Thus we get universal classes induced by universal Chern classes through the above morphism; let C_i be the map which composes the above map with the natural map $B\mathcal{GL}_n(\mathcal{O}_X) \to B\mathcal{GL}_n(\mathcal{O}_{X_n})$, it induces a map of simplicial sheaves

$$C_i : B\mathcal{GL}_n(\mathcal{O}_X) \to \mathcal{K}(2i, \mathcal{S}_n(i)_X).$$

The right-hand side is the Dold-Puppe functor on the syntomic complex. Now for $l \geq 0$, the map proceeds as in the diagram in [14] p. 229

$$c^{\text{syn}}_{il} : K_j(X) \to H^{-l}(X, \mathbf{Z} \times \mathbf{Z}_\infty B\mathcal{GL}(\mathcal{O}_X)) \xrightarrow{C_i}$$

$$H^{-l}(X, \mathbf{Z}_\infty \mathcal{K}(2i, \mathcal{S}_n(i)_X)) \simeq H^{2i-l}(X, \mathcal{S}_n(i)_X).$$

Here, $H^{-l}(X, \quad) := \pi_l(R\Gamma(X, \quad))$, the first map comes from a natural map in [14] Proposition 2.15, $\mathbf{Z}_\infty$ denotes the completion functor of Bousfield and Kan, and the final isomorphism comes from the weak equivalence on $\mathbf{Z}_\infty\mathcal{K}(2i, \mathcal{S}_n(i)_X)$ and $\mathcal{K}(2i, \Gamma(i)_X)$, hence induce the same homotopy.

For K-theory with coefficients, for $l \geq 2$, denote P^n be the l-dimensional mod p^n Moore space, which only exists for $l \geq 2$. The Chern classes are now defined as the composition

$$\begin{aligned} K_l(X, \mathbf{Z}/p^n) &\to H^{-l}(X, \mathbf{Z} \times \mathbf{Z}_\infty(B.\mathcal{GL}(O_X)), \mathbf{Z}/p^n) \\ &\to H^{-l}(X, \mathbf{Z}_\infty(B.\mathcal{GL}(O_X)), \mathbf{Z}/p^n) \\ &\xrightarrow{C_i} H^{-l}(X, \mathcal{K}(2i, \mathcal{S}_n(i)_X), \mathbf{Z}/p^n) \\ &\xrightarrow{f} H^{2i-l}(X, \mathcal{S}_n(i)_X), \end{aligned}$$

where f is defined as the composition

$$\begin{aligned} H^{-l}(X,\mathcal{K}(2i,\mathcal{S}_n(i)_X),\mathbf{Z}/p^n) &= \pi^{-l}(X,\mathcal{K}(2i,\mathcal{S}_n(i)_X),\mathbf{Z}/p^n) \\ &\xrightarrow{h_l} H_l(\mathcal{S}_n(i)_X[2i]) \\ &= H^{2i-l}(X,\mathcal{S}_n(r)_X(i)), \end{aligned}$$

where h_l is the Hurewicz morphism.

Lemma 2.5 *The syntomic Chern classes have the following properties:*

(1) c_{il}^{syn} for $j>0$ is a group homomorphism;

(2) $\bar{c}_{il}^{syn}$ for $j\geq 2$ is a group homomorphism unless $l=2$ and $p=2$;

(3) $\bar{c}_{il}^{syn}$ are compatible with the reduction maps $\mathcal{S}_n(i)_X\to\mathcal{S}_m(i)_X$, $n\geq m$; moreover, if X is regular, one can consider the following γ-filtration

$$F_\gamma^k K_0(X)=\begin{cases} K_0(X) & \text{if } k\leq 0, \\ <\gamma_{i_1}(x_1)\cdots\gamma_{i_t}(x_t)|\varepsilon(x_1)=\cdots=\varepsilon(x_1)=0, i_1+\cdots+i_t\geq k> & \text{if } k>0, \end{cases}$$

and we have the following:

(4) Let p be odd, or let $p=2, n\geq 2$, and $l,q\neq 2$. If $\alpha\in K_l(X;\mathbf{Z}/p^n)$ and $\alpha'\in K_q(X;\mathbf{Z}/p^n)$, then

$$\bar{c}_{il}^{syn}(\alpha\alpha')=-\Sigma_{r+s=i}\frac{(i-1)!}{(r-1)!(s-1)!}\bar{c}_{rm}^{syn}(\alpha)\bar{c}_{sq}^{syn}(\alpha')$$

assuming that $m,q\geq 2, m+q=l, 2i\geq l, i\geq 0, p\neq 2$.

(5) If $\alpha\in F_\gamma^l K_0(X), l\neq 0$, and $\alpha'\in F_\gamma^k K_q(X;\mathbf{Z}/p^n), q\geq 2$, are such that $\bar{c}_{il}^{syn}(\alpha')=0$ for $l\neq k$, then

$$\bar{c}_{l+k,q}^{syn}(\alpha\alpha')=-\frac{(l+k-1)!}{(l-1)!(k-1)!}\bar{c}_{l0}^{syn}(\alpha)\bar{c}_{kq}^{syn}(\alpha').$$

(6) The above multiplication formulas hold also for $p=2, n\geq 4, q=2$, and α' such that $\partial\alpha'\in K_1(X)\in V^*$.

(7) The integral Chern-class maps c_{i0}^{syn} restrict to zero on $F_\gamma^{i+1}K_l(X;\mathbf{Z}/p^n)$, $l\geq 2$, unless $l=2, p=2$.

Proof : See [22] Lemma 2.1.

For X_K, similarly one can also construct

$$c_{il}^{\grave{e}t}: K_l(X_K)\to H^{2i-l}(X_K,\mathbf{Z}/p^n(i)),\quad \bar{c}_{il}^{\grave{e}t}: K_l(X_K;\mathbf{Z}/p^n)\to H_{\grave{e}t}^{2i-l}(X_K,\mathbf{Z}/p^n)$$

and has similar properties as the above lemma.

CHAPTER 3

COMPARISON THEOREM

Now we can establish our comparison morphism: we recall some facts from [24].

First, we recall the definition of Bott elements. For a scheme Y whose global section contains a primitive p^n-th root of unity, for $p^n > 2$, there are compatible functorial Bott element homomorphisms

$$\beta_Y : \mu_{p^n}(Y) \to K_2(Y; \mathbf{Z}/p^n).$$

Explicitly, denote ζ_n by a fixed primitive p^n-th root of unity, denote β_0 the image of ζ_n under the following isomorphism:

$$\pi_2(B\mu_{p^n}(Y); \mathbf{Z}/p^n) \simeq \pi_1 B\mu_{p^n}(Y) \simeq \mu_{p^n}(Y),$$

then the Bott element $\beta_Y(\zeta_n)$ is the image of β_0 under the natural map induced by $B\mu_{p^n}(Y) \subset BGL(Y)^+$.

Now choose $\zeta = (\zeta_n),\ \zeta_n \in \overline{\mathbf{Q}}_p,\ \zeta_n^{p^n} = 1,\ \zeta_{n+1}^p = \zeta_n$, take t to be the associated image in $A_{\mathrm{crys},n}$, let K_1 be the finite extension of $K \subset \bar{K}$ containing ζ_n and V_1 be its ring of integers. Now denote $\beta_n \in K_2(K_1; \mathbf{Z}/p^n)$ and $\tilde{\beta}_n \in K_2(V_1; \mathbf{Z}/p^n)$ by $\beta_{K_1}(\zeta_n)$ and $\beta_{V_1}(\zeta_n)$. We have:

Lemma 3.1

$$\bar{c}^{\acute{e}t}_{i,2i}(\beta_n^i) = (-1)^{i-1}(i-1)!\zeta_n^{\otimes i} \in H^0(K_1, \mathbf{Z}/p^n(i)), \quad \bar{c}^{\acute{e}t}_{l,2i}(\beta_n^i) = 0, l \neq i.$$
$$\bar{c}^{syn}_{i,2i}(\tilde{\beta}_n^i) = (-1)^{i-1}(i-1)!\zeta_n^{\otimes i} \in H^0(V_1, \mathcal{S}_n(i)_X), \quad \bar{c}^{syn}_{l,2i}(\tilde{\beta}_n^i) = 0, l \neq i.$$

Proof : See [24] Lemma 3.1 and [22] Lemma 4.1.

And we recall the main proposition (Proposition 3.2) in [24]:

Proposition 3.2 *Let Y be a smooth scheme of dimension d over $\bar{K}$, and let $p^n \geq 5$. Let $b \geq max\{2d, 2\}, b \geq 3$, for $d = 0$ and $p = 2$, and let $2i - l \geq 0$. There exists an integer $T(d, i, l)$ depending only on d, i, and l such that the kernel and cokernel of the Chern classes*

$$\bar{c}^{\acute{e}t}_{i,l} : gr^i_\gamma K_l(Y; \mathbf{Z}/p^n) \to H^{2i-l}(Y, \mathbf{Z}/p^n(i))$$

are annihilated by $T(d, i, l)$. An odd prime p divides $T(d, i, l)$ if and only if $p \leq d + i + 1$.

Recall (see, e.g., [24] p. 158) for $k, m, n \in \mathbf{N} \cup 0$, the integer $M(k, i, l)$ is defined as follows: Let $q \in \mathbf{N}$ and let w_l be the greatest common divisor of the set of integers $k^P(k^q - 1)$, as k runs over the positive integers and P is large enough with respect to q. Let $M(k)$ be the product of the w_q for $2q < k$. Set $M(k, i, l) = \prod_{2i \leq 2q \leq l+2k+1} M(2l)$. Then $T(d, i, l)$ in the above proposition is defined as

$$T(d, i, l) = (i - 1)! M(d, i, l) M(d, i + 1, l) M(d, i + 1, 2l) M(d, i, 2l) M(2d)^{2d}$$

Also we recall the key lemma (Lemma 3.5) in [24] to construct the comparison morphism:

Lemma 3.3 *Let V_1 be a discrete valuation ring with fraction field K_1 and X be a regular flat scheme over V_1 , and let $j : X_{K_1} \hookrightarrow X$ be the open immersion. Then the restriction*

$$j^* : K_l(X; \mathbf{Z}/p^n) \to K_l(X_{K_1}; \mathbf{Z}/p^n), l > d + 1,$$

is an isomorphism, and the induced map

$$j^* : gr^i_\gamma K_l(X; \mathbf{Z}/p^n) \to gr^i_\gamma K_l(X_{K_1}; \mathbf{Z}/p^n), l > d + 1,$$

has kernel and cokernel annihilated by $M(d, i + 1, 2l)$ and $M(d, i, 2l)$, respectively.

Though we will not need the following result, in fact we can go further: let $\widetilde{X_{V_1}}$ be the log blow up of X_{V_1} such that it is regular, then we have the following commutative diagram:

$$\begin{array}{ccccccc} \longrightarrow & K_l(\widetilde{X_{v_1}}; \mathbf{Z}/p^n) & \longrightarrow & K_l(\widetilde{X_{V_1}}; \mathbf{Z}/p^n) & \longrightarrow & K_l(X_{K_1}; \mathbf{Z}/p^n) & \longrightarrow \\ & \uparrow & & \uparrow & & \uparrow & \\ \longrightarrow & K_l(X_v; \mathbf{Z}/p^n) & \longrightarrow & K_l(X; \mathbf{Z}/p^n) & \longrightarrow & K_l(X_K; \mathbf{Z}/p^n) & \longrightarrow \end{array}$$

Here $\widetilde{X_{v_1}}$, X_v are special fibers of $\widetilde{X_{V_1}}$, X_V, respectively. Now we claim that the map $K_l(X_v; \mathbf{Z}/p^n) \to K_l(\widetilde{X_{v_1}}; \mathbf{Z}/p^n)$ is $e = [V_1 : V]$ times the map induced by the morphism $\widetilde{X_{v_1}} \to X_v$. Since K-theory is the inductive limit of finite dimensional Lie group cohomology, it suffices to show the assertion for K_0; in this case, it follows from $\mathcal{O}_{\widetilde{X_{V_1}}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_v}$ is e times successive extensions of $\mathcal{O}_{\widetilde{X_{v_1}}}$, as $\mathcal{O}_{\widetilde{X_{V_1}}}$ module, and hence $[\mathcal{O}_{\widetilde{X_{V_1}}} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_v}] = e[\mathcal{O}_{\widetilde{X_{v_1}}}]$, hence the claim. The above result shows that, if by abuse of notation, we denote $K_l(X_{\bar{V}}; \mathbf{Z}/p^n)$ the inductive limit of $K_l(\widetilde{X_{V_1}}; \mathbf{Z}/p^n)$, we have $K_l(X_{\bar{V}}; \mathbf{Z}/p^n) \simeq K_l(X_{\bar{K}}; \mathbf{Z}/p^n)$.

3.1 The comparison morphism

Recall we assume that for $0 \leq i \leq r \leq p - 2$ and $n \in \mathbf{N}$, $X^\times$ a proper, vertical log smooth over V of relative dimension d with reduction of Cartier type, the construction of such map is very similar to the rational case in [24]: under the assumption of Proposition 3.2, first we identify étale cohomology as the graded pieces of algebraic K-theory of $X_{\bar{K}}$ under γ-filtration, so it comes from the algebraic K-theory of X_{K_1}, where K_1 is a finite extension of K; since it is log regular, after some log blow up we can find a regular model; then use Lemma 3.3 to map it to the K-theory of this regular model; finally use syntomic Chern class the map to crystalline cohomology. To check this construction is an isomorphism, we follow the approach of [10] Theorem 4.2 and [22] Lemma 4.2; on the other hand, the Gysin map and Grothendieck Riemann-Roch can not be found in the literature yet; we will mimic the crystalline case and establish them in the following sections.

Now define a Galois equivariant transformation

$$\alpha^n_{ab} : H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n(b)) \to \mathrm{Fil}^b(H^a(X^\times_n/E_n, , \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{\mathrm{crys},n}),$$

where on the right-hand side, $E_n \to A_{\mathrm{crys},n}$ is given by $u \to [\underline{\pi}]$, as follows.

The filtration of the right-hand side is induced by the filtration of log crystalline cohomology (i.e., the image of $H^a(X^\times_n/E_n, \mathcal{J}^{[i]}_{X_n/E_n})$ in $H^a(X^\times_n/E_n, \mathcal{O}_{X_n/E_n}))$ and filtration of $\widehat{A_{\mathrm{st},\pi}}$. Let $i = b, l = 2b - a$ satisfying the assumptions of Proposition 3.2, namely, $p^n \geq 5$, $2b - a \geq \max(2d, 2)$, $2b - a \geq 3$, for $d = 0$ and $p = 2$, and now the condition is empty. For $x \in H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n(b))$, by Proposition 3.2, $T(d, b, 2b-a)x \in \mathrm{gr}^b_\gamma K_{2b-a}(X_{\bar{K}}; \mathbf{Z}/p^n)$, take any preimage $x_1 \in F^b_\gamma K_{2b-a}(X_{\bar{K}}; \mathbf{Z}/p^n)$, then $x_1 \in F^b_\gamma K_{2b-a}(X_{K_1}; \mathbf{Z}/p^n)$ for some K_1 finite over K, denote V_1 its corresponding ring of integers. Since X_{V_1} is log regular (and finite and saturated log scheme), by the main result of [23], after some log blow-up, we get a log scheme $Y^\times$ whose underlying scheme is regular and $Y_{K_1} = X_{K_1}$; by Lemma 3.3 we can find $x'_1 \in F^b_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n)$ such that it is the preimage of $M(d, b, 2(2b - a))x_1$, then we have the following composition maps :

$$F^b_\gamma K_{2b-a}(Y; \mathbf{Z}/p^n) \xrightarrow{\varepsilon} H^a(Y, \mathcal{S}_n(b)_{Y^\times}) \underset{\pi^*}{\overset{\simeq}{\longleftarrow}} H^a(X_{V_1}, \mathcal{S}_n(b)_{X^\times_{V_1}}) \xrightarrow{f}$$
$$\mathrm{Fil}^b(H^a(X^\times_n/E_n, , \mathcal{O}_{X_n/E_n}) \otimes_{E_n} \widehat{A_{\mathrm{st},n}}).$$

Here ε is the composition of syntomic Chern class and the natural map

$$H^a(Y, \mathcal{S}_n(b)_Y) \to H^a(Y, \mathcal{S}_n(b)_{Y^\times}).$$

The middle isomorphism follows from the invariance of crystalline cohomology after log blow-up; the last arrow f is the composition map

$$H^a(X_{V_1}, \mathcal{S}_n(b)_{X^\times_{V_1}}) \to H^a((X_n \times_V V_1)^\times/W_n, , \mathcal{J}^{[b]}_{X_n \times_V V_1/W_n}) \to H^a(\bar{X}_n{}^\times/W_n, , \mathcal{J}^{[b]}_{\bar{X}_n/W_n})$$

$$\to (H^a(\bar{X_n}^\times/E_n,,\mathcal{J}^{[b]}_{\bar{X}_n/E_n}))^{N=0} \to (\mathrm{Fil}^b(H^a(X_n^\times/E_n,,\mathcal{O}_{X_n/E_n})\otimes_{E_n}\widehat{A_{\mathrm{st},n}}))^{N=0},$$

the reason the above map factors through the monodromy trivial part follows from [26] Lemma 4.3.8 and the last arrow follows from the natural isomorphism ([26] Proposition 4.5.4)

$$H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n}\widehat{A_{\mathrm{st},n}} \simeq H^a(\bar{X_n}^\times/E_n,\mathcal{O}_{\bar{X}_n/E_n}).$$

Now we claim we have a natural filtered isomorphism

$$(H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n}\widehat{A_{\mathrm{st},n}})^{N=0} \simeq H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n} A_{\mathrm{crys},n},$$

it is done in [25] Proposition 2.12, we repeat the argument here.

Denote $H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n,\pi}\widehat{A_{\mathrm{st},n}}$ by the tensor product of $H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})$ and $\widehat{A_{\mathrm{st},n}}$ over E_n where the map of E_n to $\widehat{A_{\mathrm{st},n}}$ is given by $u\to[\underline{\pi}]$, denote N_1 to be the monodromy operator, which is the usual monodromy operator on $\widehat{A_{\mathrm{st},n}}$, but acts trivially on $H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})$. We can check we still have

$$N(\mathrm{Fil}^l(H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n,\pi}\widehat{A_{\mathrm{st},n}}))\subset \mathrm{Fil}^{l-1}(H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n,\pi}\widehat{A_{\mathrm{st},n}}),$$

and $N_1\phi = p\phi N_1$. Now apply [26] Proposition 1.6.15, we have the following horizontal (compatible with monodromy operations on both sides) filtered $\widehat{A_{\mathrm{st},n}}$ linear isomorphism (Notice that in order to make it a Galois equivariant map, now the Galois action on the right-hand side is given as $\sigma\to\exp(\epsilon(\sigma)N)\otimes\sigma$, where ϵ is the one cocyle which is used to define the Galois action of $\hat{A}_{\mathrm{st},\pi}$ in Section 2.1)

$$H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n}\widehat{A_{\mathrm{st},n}} \simeq H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n,\pi}\widehat{A_{\mathrm{st},n}}$$

$$x\otimes 1\to \Sigma_{i\geq 0}(\prod_{0\leq l\leq i-1}(N-l)(x))\otimes(\frac{X_\pi+1}{[\underline{\pi}]}-1)^{[i]},$$

where $\frac{X_\pi+1}{[\underline{\pi}]}$ exists in $\widehat{A_{\mathrm{st},n}}$ by [26], Lemma 1.6.5 (where X_π refers to $v_\beta-1$ in the statement of [26]), then taking monodromy trivial part on both sides, we see the right-hand side is $H^a(X_n^\times/E_n,\mathcal{O}_{X_n/E_n})\otimes_{E_n} A_{\mathrm{crys},n}$, hence the claim.

Now in addition, we assume the prime p does not divide $T(d,b,2b-a)$ and $M(d,b,2(2b-a))$; by Proposition 3.2, it is equivalent to say $p>d+2b-a+1$. So we can define

$$\alpha^n_{ab}(x) := f(\pi^*)^{-1}\varepsilon(j^*)^{-1}\bar{c}^{\acute{e}t,-1}_{b,2b-a}(x).$$

Now we need to show that it is well-defined; this follows from the argument in [24]. First, it is independent of the regular model Y we choose. If there is another model, say Y_1, then we can blow up X_{V_1} with the center containing the previous two schemes, then by the main result of [23], blow up further and we may get a regular scheme $\widetilde{Y}$ such that $\widetilde{Y} \times_{V_1} K_1 = X_{K_1}$, and by the construction, we have natural maps from $\widetilde{Y}$ to Y and Y_1, hence the result.

Under the identification of $H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n(b))$ and $H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes \beta^b$ through the logarithm map, after tensoring A_{crys} on the étale side, we get a Galois equivariant filtered comparison map. Now we can state the main theorem:

Theorem 3.4 *For any proper log scheme $X^\times$ vertical log smooth over $(V, \mathbf{N})$ of relative dimension d with reduction of Cartier type, if $p^n \geq 5$, $2b - a \geq max\{2d, 2\}$, $2b - a \geq 3$, for $d = 0$ and $p = 2$, and $p > d + 2b - a + 1$. For $0 \leq a \leq b \leq p - 2$, we have a Galois equivalent almost isomorphism*

$$\alpha^n_{ab} : H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes Fil^b(A_{crys,n}) \to Fil^b(H^a(X^\times_n/E_n, , \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{crys,n})$$

By the result of [23], after some log blow-up, for any $X^\times$ with assumption in the above theorem, we can find a vertical semistable scheme. Since our construction of comparison map is functorial, it suffices to prove the theorem when X is vertical semistable (and then we can work on the Zariski site instead of the étale site).

We will talk about Gysin sequence and diagonal class in the next proposition. In the smooth case, the closed immersion between two smooth schemes must be regular immersion; similarly the exact closed immersion between two vertical semistable schemes is regular; to see this, locally as in the beginning of the proof of Lemma 2.4., we can lift both schemes and morphisms to $V_n[u]$, hence now both schems are smooth over V_n, hence we reduce to smooth case, hence the result.

Since the diagonal embedding $X \hookrightarrow X \times X$ is not a regular embedding, by [19] Proposition 4.10, we can factor diagonal embedding through an exact closed immersion composed with an log étale map, but for vertical semistable schemes, we can do it explicitly. Now, for simplicity assume the special fiber of X consists of s Cartier divisors $\{D_1, \cdots, D_s\}$ with simple normal crossing; now we blow up $D_i \times D_i, i = 1, \cdots, s$, we get a scheme $\widetilde{X \times X}$, since the pullback of the center of blow-up is (π), by universal property, we get a map $X \hookrightarrow \widetilde{X \times X}$; certainly it is an closed immersion; in order to say it is an exact closed immersion, it suffices to show that :

Lemma 3.5 *$\widetilde{X \times X}$ is regular.*

Proof : The assertion is étale local, so we may assume locally $X \times X$ is isomorphic to $\text{Spec}V[x_1, x_2, \ldots, x_d, y_1, y_2, y_d]/(x_1 \cdot x_2 \cdots x_s - \pi, y_1 \cdot y_2 \cdots y_s - \pi)$. Blow up it with center on

$$\mathbb{V}((x_1, y_1)(x_2, y_2)\dots(x_s, y_s)),$$

(vanishing set with defining ideal $(x_1, y_1)(x_2, y_2)\dots(x_s, y_s)$), localize $x_1 \cdot x_2 \cdot x_3 \cdots x_s$; for example, by direct computation the affine coordinate ring is

$$V[x_1, x_2, \dots, x_d, y_1, y_2, \cdots y_d, y_1/x_1, y_2/x_2, \dots, y_s/x_s],$$

quotient $(x_1 \cdot x_2 \cdot x_3 \cdots x_n - \pi, y_1 \cdot y_2 \cdot y_3 \cdots y_s - \pi)$ then directly check this ring is isomorphic to

$$V[x_1, \dots, x_d, y_1, \cdots, y_d, y_1/x_1, \cdots, y_s/x_s]/$$

$$(x_1 \cdots x_s - \pi, y_1/x_1 \ \cdots y_s/x_s - 1),$$

which is regular.

To check the above assertion, the general case is very similar but a bit complicated; for simplicity, let us assume $d = s = 4$, and we localize $x_1 \cdot x_2 \cdot y_3 \cdot y_4$; after quotient, the coordinate ring will be

$$V[x_1, x_2, y_3, y_4, y_1/x_1, y_2/x_2, x_3/y_3, x_4/y_4]/$$

$$(x_1 \cdot x_2 \cdot y_3 \cdot y_4 \cdot x_3/y_3, \cdot x_4/y_4 - \pi,\ y_1/x_1 \cdot y_2/x_2 - x_3/y_3 \cdot x_4/y_4)$$

which is regular. (Quotient a regular sequence (x_1, x_2, y_3, y_4), we get

$$k[y_1/x_1,\ y_2/x_2,\ x_3/y_3,\ x_4/y_4]/(y_1/x_1 \cdot y_2/x_2 - x_3/y_3 \cdot x_4/y_4),$$

which is regular.)

Here is our main proposition, which is an analogue of [22] Lemma 4.2:

Proposition 3.6 *(1) α^n_{ab} commutes with products.*

(2) For an exact closed immersion $i : Y \to X$ of relative dimension j with X, Y vertical semistable, we have

$$\alpha^n_{2j,b}(cl^{\acute{e}t}(Y_K)\zeta_n^{b-j}) = i_*(1_{Y_n})t^{b-j}$$

where the Gysin map

$$i_* : H^{\cdot}(Y_n^{\times}/E_n, \mathcal{O}_{Y_n/E_n}) \to H^{\cdot+2j}(X_n^{\times}/E_n, \mathcal{J}^{[j]}_{X_n/E_n})$$

is defined as the adjoint of i^* with respect to Poincaré duality. (So $i_*(1_{Y_n})$ is the constant appeared in the projection formula).

(3) In particular, for irreducible $X_{\bar{K}}$, the following diagram commutes:

$$\begin{array}{ccc}
H^{2d}_{ét}(X_{\bar{K}}, \mathbf{Z}/p^n(b)) & \xrightarrow{tr^{ét}} & \mathbf{Z}/p^n(b-d) \\
\downarrow{\scriptstyle \alpha^{n'}_{2d,b}} & & \downarrow{\scriptstyle t^{b-d}} \\
Fil^b(H^{2d}(X_n^{\times}/E_n), \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{crys,n} & \xrightarrow{tr^{cry}} & A_{crys,n}(d-b)
\end{array}$$

Proof : As in [22], Lemma 4.2, the first assertion follows from Lemma 2.5. For (3), we observe that commutativity of cycle class implies commutativity of trace maps: by étale base change, we may assume the residue field of V is algebraically closed; in this case, the trace maps on both sides can be characterized by sending each cycle class of closed point to 1, hence the claim.

So it suffices to prove (2). First, we recall the argument in [22], Lemma 4.2, in smooth case; first, we prove for $0 \leq i < j$, $\bar{c}^{\mathrm{syn}}_{i0}([i_*(\mathcal{O}_Y)]) = 0$, then we can apply Whitney sum formula

$$\bar{c}^{\text{ét}}_{j,2(b-j)}([O_{Y_K}]\beta_n^{b-j}) = (-1)^{b-1}(b-1)!\mathrm{cl}^{\text{ét}}(Y_K)\zeta_n^{b-j},$$
$$\bar{c}^{\mathrm{syn}}_{j,2(b-j)}([O_Y]\beta_n^{b-j}) = (-1)^{b-1}(b-1)!\bar{c}^{\mathrm{syn}}_{j,0}([O_Y])\zeta_n^{b-j}.$$

On the other hand, by the construction of the map, $[O_{Y_K}]\beta_n^{b-j}$ will map to $[O_Y]\beta_n^{b-j}$, hence $\mathrm{cl}^{\text{ét}}(Y_K)$ will map to $\bar{c}^{\mathrm{syn}}_{j,0}([O_Y])$, hence the claim.

For a vertical semistable scheme Y (we omit the subscript for simplicity) of codimension j in X with closed immersion $i : Y \hookrightarrow X$, the same argument as the above, first we show that for $0 \leq i < j$, $\bar{c}^{\mathrm{syn}}_{i0}([i_*(\mathcal{O}_Y)]) = 0$. This result relies on the Grothendieck Riemann-Roch type theorem (without denominators), i.e., denoted $c_X(-) = 1 + \bar{c}^{\mathrm{syn}}_{10}([-]) + \bar{c}^{\mathrm{syn}}_{20}([-]) + \cdots + \bar{c}^{\mathrm{syn}}_{d0}([-])$, the total Chern class on X, we want to prove for each l, there exists an universal power series P_l of degree $l - j$ (which is zero if $l < j$), such that

$$\bar{c}^{\mathrm{syn}}_{l0}([i_*(\mathcal{O}_Y)]) = i_*(P_l(c(\mathcal{O}_Y), c(N))),$$

where N is the normal bundle of Y in X and i is the Gysin map generalizing the Gysin map in crystalline cohomology (and we will show that it coincides with the definition in above proposition), we will define the Gysin map and prove this formula in the following sections.

In order to prove Grothendieck-Riemann-Roch theorem, one crucial ingredient is the functoriality of cycle class map under a Cartesian Tor independent diagram (Proposition 4.1.5), for this reason, the author is forced to give a down-to-earth construction of a cycle class map; on the other hand, the proof strongly relies on the Poincaré duality, so we will show show that this construction is compatible with the adjoint of pull back map under Poincaré duality. Now assume we already show the existence of Gysin map, and show it is the adjoint of the pullback under Poincaré duality, and Grothendieck Riemann-Roch formula. Now we are ready to give a proof. Since α_{ab} commutes with cup product by Proposition 3.6, it has a left inverse α_{ab}^{-1}. Since α_{ab} is functorial, it also commutes with Künneth products (with respect to the diagonal immersion $\delta : X_n \hookrightarrow \widetilde{X_n \times X_n}$, since cup product is the composition of Künneth product and the diagonal pullback and we have Künneth formula by Proposition 2.1).

To prove α_{ab}^{-1} is a right inverse as well, it suffices to show that α_{ab}^{-1} commutes with cup products as well. Since α_{ab} commutes with Künneth isomorphism, take its associated Poincaré dual, α_{ab}^{-1} also commutes with Künneth isomorphism. So now it suffices to show that it commutes with Δ^*, which is equivalent to say α_{ab} commutes with Δ_*; since by Proposition 3.6. α_{ab} commutes with cycle classes, by projection formula, it suffices to show that Δ^* is surjective, which follows from it admits a retraction

$$X \xrightarrow{\Delta} \widetilde{X \times X} \longrightarrow X \times X \xrightarrow{\mathrm{pr}_1} X,$$

where pr_1 means projection on the first factor, hence the proof.

Remark 3.7 *The reason we can use Poincaré duality in the above argument is, on the étale side, we identify $H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n(b))$ with $H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes \beta^b$; on the crystalline side, $\mathcal{J}_{X_n/E_n}$ is generated by $t - p$, which is also in A_{crys}.*

CHAPTER 4

GYSIN SEQUENCE OF VERTICAL SEMISTABLE SCHEMES IN LOG CRYSTALLINE COHOMOLOGY

In this chapter, we will develop a cycle class map for exact regular closed immersions between log schemes smooth over E_n, following [3]. We honestly follow the approach of Berthelot, and at many places, weaken the smooth condition by log-smooth or flat. To simplify notation, since log crystalline cohomology is blow-up invariant, it suffices to show for the vertical semistable case; then we can work on the Zariski site, instead of the étale site.

4.1 Hyperextension functor

For a vertical semistable scheme $X^\times$ over $(V, \mathbf{N})$, denoted $\widetilde{X \times X}$ blow-up of $X \times X$ with center on the diagonal, so the canonical closed immersion $X \hookrightarrow \widetilde{X \times X}$ is an exact closed immersion, denoted $\mathcal{I}$ by its defining ideal, so we have $\omega^{\cdot}_{X_n/V_n} \simeq \mathcal{I}/\mathcal{I}^2$.

Following [18], now we can define the log version of jet sheaves as follows, define $\mathcal{J}_m = \mathcal{O}_{\widetilde{X_n \times X_n}}/\mathcal{I}^{m+1}$. Equipped the $\mathcal{O}_X$ algebra structure by the (induced) first projection π_1, and we have canonical differential operator $D_n : \mathcal{O}_{X_n} \to \mathcal{J}_m$ given by π_2^*, the projection on the second component.

Definition 4.1 *Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves of $\mathcal{O}_{X_n}$ modules. Let $\mathcal{F} \to \mathcal{J}_m \otimes_{\pi_2^*(\mathcal{O}_{X_n})} \mathcal{F}$ be the canonical map induced by D_m, let $D : \mathcal{F} \to \mathcal{G}$ be a homomorphism of $\mathcal{O}_{X_n}$ modules. We say D is a differential operator of order $\leq m$ if there exists a homomorphism of $\mathcal{O}_{X_n}$ modules $u : \mathcal{J}_m \otimes_{\pi_2^*(\mathcal{O}_{X_n})} \mathcal{F} \to \mathcal{G}$ (where the former one is defined as left $\mathcal{O}_{X_n}$ module using the $\mathcal{O}_{X_n}$ module structure on $\mathcal{J}_m$) such that $D = u \circ (D_m \otimes_{\pi_2^*(\mathcal{O}_{X_n})} 1)$.*

So now denote $\mathcal{C}(X_n)$ be the category of sheaves of differential operators of order ≤ 1, i.e., a complexes of sheaves of $\mathcal{O}_{X_n}$ modules whose differentials are differential operators of order ≤ 1, with morphisms are morphisms as complexes.

Define $\mathcal{J}^0 = \mathcal{O}_{X_n}, \mathcal{J}^i = \mathcal{J}_1 \otimes_{\pi_2(\mathcal{O}_{X_n})} \mathcal{J}^{i-1}$, where π_2 is the morphism induced by the second projection of $X \times X \to X$, we view it as a left $\pi_1^*(\mathcal{O}_{X_n})$ and right $\pi_2^*(\mathcal{O}_{X_n})$ module. So we have the graded sheaf $\mathcal{J}^{\cdot} = \oplus \mathcal{J}^i$, where the multiplication $\mathcal{J}^i \otimes \mathcal{J}^j \to \mathcal{J}^{i+j}$ is given by

$$(\alpha_1 \otimes \cdots \otimes \alpha_i \otimes f) \otimes (\beta_1 \otimes \cdots \otimes \beta_j \otimes g) \to (\alpha_1 \otimes \cdots \otimes \alpha_i \otimes f\beta_1 \otimes \cdots \otimes \beta_j \otimes g).$$

Denote D by the element $1 \in \mathcal{J}^1$, so the $\mathcal{J}^{\cdot}$ module which are complexes are exactly $\mathcal{C}^{\cdot} := \mathcal{J}^{\cdot}/D \otimes D$ modules; using the splitting $\mathcal{C}^1 = \mathcal{J}_1 = \mathcal{O}_{X_n} \oplus \omega_{X_n/V_n}$, we have the induced map $\omega_{X_n/V_n} \otimes_{\mathcal{O}_{X_n}} \mathcal{C}^j \to \mathcal{C}^{j+1}$; from this and by iteration, we get a ω_{X_n/V_n} action on $\mathcal{C}^{\cdot}$; by the same argument as in [18] p. 103, $\mathcal{C}^{\cdot} = \omega^{\cdot}_{X_n/V_n}/\mathcal{I}$, where $\mathcal{I}$ is the ideal generated by $D^2 = 0,\ D\varphi + (-1)^{i+1}\varphi D = 0,\ \phi \in \omega^i_{X/V_n}$. In sum, we have:

1. ([18] (2.1)) The category of complexes with differential operators of order ≤ 1 has enough injectives.
2. ([18] (2.4)) Every complex with differential operators of order ≤ 1 has a natural structure of graded module over $\omega^{\cdot}_{X_n/V_n}$. Injective complexes give rise to injective graded $\omega^{\cdot}_{X_n/V_n}$ modules.

For $\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot} \in \mathcal{C}(X_n)$, denote $\mathcal{H}om^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ as all the morphisms of $\omega^{\cdot}_{X_n/V_n}$ modules $\mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}[k]$ (i.e., a family of maps $\mathcal{F}^i \to \mathcal{G}^{i+k}$ compatible with differential structures), let $\mathcal{H}om^{\cdot}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}) = \oplus \mathcal{H}om^k(\mathcal{G}^{\cdot}, \mathcal{F}^{\cdot}) \in \mathcal{C}(X)$, whose differential structure $D_k : \mathcal{H}om^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}) \to \mathcal{H}om^{k+1}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ is given by formula

$$D_k(\varphi) = D_G \circ \varphi + (-1)^{k+1} \varphi \circ D_F,$$

where $\varphi \in \mathcal{H}om^k(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}), D_G$ (resp. D_F) are corresponding differential operators of $\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}$. Let $\underline{\mathrm{Ext}}^{\cdot,q}_{C(X_n)}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ (resp. $\mathrm{Ext}^{\cdot,q}_{C(X_n)}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$) be the q-th derived functor of $\mathcal{H}om^{\cdot}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$ (resp. $\Gamma(\mathcal{H}om^{\cdot}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot}))$).

Proposition 4.2 *([3] II Proposition 5.3.2)*

If $I^{\cdot}$ is an injective object in the category of graded $\omega^{\cdot}_{X_n/V_n}$ module, then I^q is an injective $\mathcal{O}_{X_n}$ module.

Proof : It follows from the canonical isomorphism

$$\mathrm{Hom}_A(M, I^q) \simeq \mathrm{Hom}^q(M \otimes_A \omega^{\cdot}_{X_n/V_n}, I^{\cdot}),$$

since $\omega^{\cdot}_{X_n/V_n}$ is a flat $\mathcal{O}_{X_n}$ module.

Definition 4.3 *With the notation above, define a (sheaf of) double complex* $\underline{K}^{\cdot\cdot}$ *by* $\underline{K}^{pq} = \underline{Hom}^p(E^{\cdot}, I^{\cdot q})$*, then the hyperext of* $E^{\cdot}, F^{\cdot}$*, denoted by* $\underline{Ext}^i_{\omega^{\cdot}_{X_n/V_n}}(E^{\cdot}, F^{\cdot})$*, is the sheaf cohomology associated to the double complex* $\underline{K}^{\cdot\cdot}$*.*

Note that by definition, we have a spectral sequence

$$E_1^{p,q} = \underline{\mathrm{Ext}}^{p,q}_{C(X_n)}(E^{\cdot}, F^{\cdot}) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{\omega^{\cdot}_{X_n/V_n}}(E^{\cdot}, F^{\cdot}).$$

Proposition 4.4 *([3] II Proposition 5.4.2)*

There is a canonical isomorphism

$$\underline{Ext}^{pq}_{C(X_n)}(E^{\cdot}, \omega^{\cdot}_{X_n/V_n}) \simeq \underline{Ext}^q_{\mathcal{O}_{X_n/V_n}}(E^{d-p}, \omega^d_{X_n/V_n}).$$

Proof : Let $I^{\cdot}$ be a injective resolution of ω_{X_n/V_n}, since ω_{X_n/V_n} is locally free $\mathcal{O}_{X_n/V_n}$ module, $(\omega^{d-\cdot}_{X_n/V_n})^{\vee} \otimes I^{\cdot}$ is a injective resolution of $\omega^{\cdot}_{X_n/V_n}$. Thus we have

$$\underline{\mathrm{Ext}}^{pq}_{C(X_n)}(E^{\cdot}, \omega^{\cdot}_{X/V_n}) = \underline{H}^q(E^{\cdot}, (\omega^{d-\cdot}_{X_n/V_n})^{\vee} \otimes I^{\cdot}) = \underline{H}^q(\mathrm{Hom}_{\mathcal{O}_{X_n/V_n}}(E^{d-p}, I^{\cdot})).$$

Hence the claim.

Proposition 4.5 *([3] II Proposition 5.4.4)*

$$\underline{Ext}^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{X_n/V_n}, F^{\cdot}) \simeq \underline{H}^i(F^{\cdot}), \quad Ext^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{X_n/V_n}, F^{\cdot}) \simeq H^i(F^{\cdot}).$$

Proof : Let $I^{\cdot\cdot}$ be the injective resolution of $F^{\cdot}$, then the first one follows from the isomorphism

$$\underline{\mathrm{Hom}}^{\cdot}(\omega^{\cdot}_{X_n/V_n}, I^{\cdot q}) \simeq I^{\cdot q}.$$

That means the E_1 term spectral sequence of both sides are isomorphic, and the spectral sequence is regular, thus the target is also isomorphic; the same argument works for the second one by taking the global section.

Proposition 4.6 *([3] VI Proposition 3.1.6) Let* $i : Y \to X$ *is an closed immersion with* Y *vertical semistable over* V *of codimension* j*. Then there is an canonical isomorphism*

$$\underline{Ext}^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}) \simeq \underline{H}^{i-2j}(\omega^{\cdot}_{Y_n/V_n}),$$
$$Ext^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}) \simeq H^{i-2j}(\omega^{\cdot}_{Y_n/V_n}).$$

In particular, that means for $i < 2j$*, the above objects are all zero.*

Proof : Like Proposition 4.5, we consider the E_1 term spectral sequence on both sides.

Recall after Definition 4.3. we have a spectral sequence

$$E_1^{p,q} = \underline{\mathrm{Ext}}^{p,q}_{C(X_n)}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}) \Rightarrow \underline{\mathrm{Ext}}^{p+q}_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n})$$

and by Proposition 4.4. we have

$$\underline{\mathrm{Ext}}^{p,q}_{C(X_n)}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}) \simeq \underline{\mathrm{Ext}}^{q}_{\mathcal{O}_{X_n/V_n}}(\omega^{d-p}_{Y_n/V_n}, \omega^{d}_{X_n/V_n})$$

(recall d is the rank of $\omega^1_{X_n/V_n}$ as an $\mathcal{O}_{X_n/V_n}$ module). Since Y is of locally of complete intersection in X, by [16] III Proposition 7.2, we have

$$\underline{\mathrm{Ext}}^{q}_{\mathcal{O}_{X_n}}(\omega^{d-p}_{Y_n/V_n}, \omega^{d}_{X_n/V_n}) = 0, \text{for } q \neq j,$$

$$\underline{\mathrm{Ext}}^{j}_{\mathcal{O}_{X_n}}(\omega^{d-p}_{Y_n/V_n}, \omega^{d}_{X_n/V_n}) \simeq (\omega^{d-p}_{Y_n/V_n})^{\vee} \otimes \omega^{d}_{X_n/V_n} \otimes \omega_{Y_n/X_n} \simeq \omega^{p-j}_{Y_n/V_n},$$

where $\omega_{Y_n/X_n} = \wedge^d(\mathcal{I}_{Y_n}/\mathcal{I}^2_{Y_n})^{\vee}$ and $\mathcal{I}_{Y_n}$ is an ideal sheaf of X_n defining Y_n. The last isomorphism comes from the isomorphism

$$\omega^{d}_{X_n/V_n} \otimes \omega_{Y_n/X_n} \simeq \omega^{d-j}_{Y_n/V_n}.$$

Now it suffices to show that the differential

$$d_1^{p,q} : E_1^{p,q} \to E_1^{p,q+1}$$

is the usual differential

$$\omega^{p-j}_{Y_n/V_n} \to \omega^{p+1-j}_{Y_n/V_n}$$

through the isomorphisms. Since the question is local, we may assume Y_n is defined by a regular sequence $t_1, \ldots, t_j \in \Gamma(X_n, \mathcal{O}_{X_n})$. Denote $\underline{C}^{\cdot}_{t_1,\ldots,t_j}(\omega^{\cdot}_{X_n/V_n})$ the Čech resolution of $\omega^{\cdot}_{X_n/V_n}$, so we have an exact triangle :

$$0 \to \omega^{\cdot}_{X_n/V_n} \to \underline{C}^{\cdot}_{t_1,\ldots,t_j}(\omega^{\cdot}_{X_n/V_n}) \to \underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) \to 0,$$

here $\underline{H}^{\cdot l}_{Y_n}(\omega^{\cdot}_{X_n/V_n})$ is defined by $\underline{H}^{k,l}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) = \underline{H}^{l}_{Y_n}(\omega^{k}_{X_n/V_n})$. Notice that $\underline{C}^{\cdot}_{t_1,\ldots,t_j}(\omega^{\cdot}_{X_n/V_n})$ and $\underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n})$ are both differential operators of order ≤ 1, so we get a resolution of $\omega^{\cdot}_{X_n/V_n}$ in the category of $C(X_n)$. On the other hand, since we invert the defining ideal of Y_n in X_n, for all $l \geq 0$

$$\underline{\mathrm{Ext}}^{\cdot,l}(\omega^{\cdot}_{Y_n/V_n}, \underline{C}^{\cdot}_{t_1,\ldots,t_j}(\omega^{\cdot}_{X_n/V_n})) = 0.$$

So that means

$$\underline{\mathrm{Ext}}^{\cdot,l}(\omega^{\cdot}_{Y_n/V_n}, \underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n})) \simeq \underline{\mathrm{Ext}}^{\cdot,j+l}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}),$$

and use Koszul resolution on $\mathcal{O}_{Y_n}$ means the above is 0 unless $l = 0$; in sum, we have $\underline{\mathrm{Ext}}^{\cdot,l}(\omega^{\cdot}_{Y_n/V_n}, \underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n})) = 0$ while $l > 0$.

To calculate $\underline{\mathrm{Hom}}^p(\omega^{\cdot}_{Y_n/V_n}, \underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n}))$, first notice that since for any k, $\omega^k_{X_n/V_n}$ is locally free, so for $I^{\cdot}$ an injective resolution of $\mathcal{O}_{X_n}$, $I^{\cdot} \otimes \omega^k_{X_n/V_n}$ is an injective resolution of $\omega^k_{X_n/V_n}$, that means

$$\underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) \simeq \underline{H}^j_{Y_n}(\mathcal{O}_{X_n}) \otimes \omega^{\cdot}_{X_n/V_n}.$$

Since

$$\underline{H}^j_{Y_n}(\mathcal{O}_{X_n}) \simeq \lim_{\rightarrow} \mathcal{O}_{X_n}/\mathcal{I}^{(n)}_{Y_n},$$

where $\mathcal{I}_{Y_n}$ is the sheaf of ideals generated by $t_1, \ldots, t_j$.

So $\underline{\mathrm{Hom}}^p(\omega^{\cdot}_{Y_n/V_n}, \underline{H}^{\cdot j}_{Y_n}(\omega^{\cdot}_{X_n/V_n}))$ is equivalent to a section of $\underline{H}^j_{Y_n}(\mathcal{O}_{X_n/V_n}) \otimes \omega^p_{X_n/V_n}$ which is annihilated by $t_1, \ldots, t_j, dt_1, \ldots, dt_j$. So that means the isomorphism

$$\omega^{p-j}_{Y_n/V_n} \simeq \underline{\mathrm{Hom}}^p(\omega^{\cdot}_{Y_n/V_n}, \underline{H}^j_{Y_n}(\mathcal{O}_{X_n}) \otimes \omega^{\cdot}_{X_n/V_n})$$

is given by

$$\omega \to \omega \wedge \frac{dt_1 \wedge \cdots \wedge dt_j}{t_1 \ldots t_j}$$

and the claim follows.

For $\mathrm{Ext}^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n})$, by [3] VI Proposition 2.2.5, there is a E_2 term spectral sequence

$$E_2^{p,q} = H^p(X, \underline{\mathrm{Ext}}^q_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n})) \Rightarrow \mathrm{Ext}^{p+q}_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n}),$$

hence the result follows from the result of $\underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X_n/V_n}}(\omega^{\cdot}_{Y_n/V_n}, \omega^{\cdot}_{X_n/V_n})$.

Now we want to define crystalline version of differential complex of order ≤ 1.

Assume $X_n^{\times}$ admits a smooth lifting $X_n'^{\times}$ on E_n (so the defining ideal is also a PD-ideal). Recall that by [21] p.66, we have a log linearization functor L, which sends the category of $\mathcal{O}_{X_n'}$ module with log HPD differential operators to the category of crystals of $\mathcal{O}_{X_n/E_n}$ module. In particular, $L(\omega^{\cdot}_{X_n'/E_n})$ is a complex of $\mathcal{O}_{X_n/E_n}$ module. For any $\mathcal{O}_{X_n'}$ module E, by [21], p. 65, define $L(E)_{(U,T,\delta)} = \mathcal{O}_{T \times_{X_n} X_n'} \otimes_{\mathcal{O}_{X_n'}} E$, so we can define differential operators of order ≤ 1 with respect to $L(\omega^{\cdot}_{X_n'/E_n})$ as a complex of sheaves of $\mathcal{O}_{X_n'/E_n}$ modules; on the other hand, for another $\mathcal{O}_{X_n'}$ module F, $L(E \otimes_{\mathcal{O}_{X_n'}} F) = L(E) \otimes_{\mathcal{O}_{X_n'}} F$, that means the log linearization functor sends differential operator of order ≤ 1 to differential operators of order ≤ 1 with respect to $L(\omega^{\cdot}_{X_n'/E_n})$. Now observe that we have the following:

Proposition 4.7 *([3] VI Proposition 2.3.1)*

Let the assumption be as above. For $\mathcal{O}_{X'_n}$ (on the Zariski site) module E and $L(\mathcal{O}_{X'_n})$ module F, we have the following adjunction formula

$$\underline{Hom}_{L(\mathcal{O}_{X'_n})}(L(E),F) \simeq \underline{Hom}_{\mathcal{O}_{X'_n}}(E, u_{X_n/E_n*}(F)),$$

where $u_{X_n/E_n} : (X_n/E_n)_{crys} \to (X_n)_{Zar}$ is defined as in [3] 5.18, which is given by: For $F \in (X_n/E_n)_{crys}$ and $j : U \hookrightarrow X_n$ an open immersion, $u_{X_n/E_n*}(F)(U) = \Gamma((U/E_n)_{crys}, j^*_{crys}(F))$; on the other hand, for $E \in (X_n)_{Zar}$, we have

$$(u^*_{X_n/E_n}(E))(U,T,M_T,\delta) = E(U).$$

Proof : First, we mention that there exists such morphism since

$$u_{X_n/E_n*}(L(\mathcal{O}_{X'_n})) \simeq \mathcal{O}_{X'_n}$$

Since we assume X_n admits a log-smooth lifting, $\Gamma(X_n, \underline{\mathrm{Hom}}_{L(\mathcal{O}_{X'_n})}(L(E),F))$ is isomorphic to the equalizer of the following diagram:

$$\mathrm{Ker}[\Gamma(X'_n, \underline{\mathrm{Hom}}(L(E),F)_{(X_n,X'_n)} \rightrightarrows \Gamma(D_{X_n}(X'^2_n), \underline{\mathrm{Hom}}(L(E),F)_{(X_n,D_{X_n}(X'^2_n))})]$$

(By abuse of notation here, we skip some data on the log crystalline site). On the other hand, we have

$$\underline{\mathrm{Hom}}(L(E),F)_{(X_n,X'_n)} \simeq \underline{\mathrm{Hom}}_{L(\mathcal{O}_{X'_n})_{(X_n,X'_n)}}(L(E)_{(X_n,X'_n)}, F_{(X_n,X'_n)})$$
$$\simeq \underline{\mathrm{Hom}}_{D_{X'_n/E_n}(1)}(D_{X'_n/E_n}(1) \otimes_{\mathcal{O}_{X'_n}} E, F_{(X_n,X'_n)}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_{X'_n}}(E, F_{(X_n,X_n)}).$$

Similarly we have,

$$\underline{\mathrm{Hom}}(L(E),F)_{(X_n,D_{X_n}(X'^2_n))} \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_{X'_n}}(E, F_{(X_n,D_{X_n}(X'^2_n))})$$

So that means

$$\underline{\mathrm{Hom}}_{L(O_{X'_n})}(L(E),F) \simeq \mathrm{Ker}[\mathrm{Hom}_{\mathcal{O}_{X'_n}}(E, F_{(X_n,X'_n)}) \rightrightarrows \mathrm{Hom}_{\mathcal{O}_{X'_n}}(E, F_{(X,D_{X_n}(X'^2_n))})]$$
$$\simeq \underline{\mathrm{Hom}}_{O_{X'_n}}(E, \mathrm{Ker}(F_{(X_n,X'_n)} \rightrightarrows F_{(X_n,D_{X_n}(X'^2_n))})).$$

By the same reason, we know the following diagram

$$u_{X_n/E_n*}(E)|_{X_n} \to E_{(X_n,X'_n)} \rightrightarrows E_{(X_n,D_{X_n}(X'^2_n))}$$

is exact. Put it into the above equality, hence the claim.

Using exactly the same method, we have:

Lemma 4.8 *([3] VI Lemma 2.3.3)*

Let F be an $L(\mathcal{O}_{X'_n})$ module. Then there exists an canonical isomorphism

$$u_{X_n/E_n*}(L(D_{X'_n/E_n}(1))\otimes_{L(\mathcal{O}_{X'_n})} F)\simeq D_{X'_n/E_n}(1)\otimes_{\mathcal{O}_{X'_n}} u_{X_n/E_n*}(F).$$

Combining the above proposition and lemma, we can get, let E, F be as above, then there is natural 1-1 correspondence between the differential operators of order ≤ 1 with respect to $L(\omega_{X'_n/E_n})$ from $L(E)$ to F, and the differential operators of order ≤ 1 from E to $u_{X_n/E_n*}(F)$. In particular, let $E^{\cdot}$ be a complex of differential operator of order ≤ 1, and $K^{\cdot}$ be a complex of differential operator of order ≤ 1 with respect to $L(\omega^{\cdot}_{X'_n/E_n})$, then there is a natural 1-1 correspondence between the homomorphism from $L(E^{\cdot})$ to $K^{\cdot}$ and homomorphism between $E^{\cdot}$ and $u_{X_n/E_n*}(K^{\cdot})$. So that means for $I^{\cdot}$, an injective object in the category of complex of differential operators of order ≤ 1 with respect to $L(\omega^{\cdot}_{X'_n/E_n})$, $u_{X_n/E_n*}(I^{\cdot})$ is an injective object in the category of complexes of differential operator of order ≤ 1.

Theorem 4.9 *([3] VI Theorem 2.4.1)*

Let E, F be two $\mathcal{O}_{X_n/E_n}$ module (on the restricted log crystalline site, we are forced to work on it, while X'_n is log smooth over E_n it does not affect the cohomology). Denote $\underline{Ext}^i_{\mathcal{O}_{X_n/E_n}}(E,F)$ (resp. $Ext^i_{\mathcal{O}_{X_n/E_n}}(E,F)$) the i-th derived functor of $\underline{Hom}_{\mathcal{O}_{X_n/E_n}}(E,F)$ (resp. $\Gamma(X_n,(\underline{Hom}_{\mathcal{O}_{X_n/E_n}}(E,F)))$. For all i, there is canonical isomorphisms :

$$\underline{Ext}^i_{\mathcal{O}_{X_n/E_n}}(E,F)\to \underline{Ext}^i_{L(\omega^{\cdot}_{X'_n/E_n})}(E\otimes L(\omega^{\cdot}_{X'_n/E_n}), F\otimes L(\omega^{\cdot}_{X'_n/E_n})),$$

$$Ext^i_{O_{X_n/E_n}}(E,F)\to Ext^i_{L(\omega^{\cdot}_{X'_n/E_n})}(E\otimes L(\omega^{\cdot}_{X'_n/E_n}), F\otimes L(\omega^{\cdot}_{X'_n/E_n})).$$

Proof : Here we closely follow the proof of [3] VI Theorem 2.4.1. Let (U,T,M_T,i,δ) be a restricted log crystalline site, namely it is a crystalline site equipped with an E_n morphism $g: T\to X'_n$, so we have $L(\mathcal{O}_{X'_n})_{(U,T)}\simeq g^*(D_{X'_n/E_n}(1))$, which is flat over $\mathcal{O}_{X_n/E_n}$ since $D_{X'_n/E_n}(1))$ is flat over $\mathcal{O}_{X'_n}$ (after log blow-up, étale locally it is $D_{X'_n}(X'_n\times \mathbf{A}^d)$).

For $K^{\cdot}$, an object of differential operator of order ≤ 1, we have adjunction formula

$$\mathrm{Hom}^m(E\otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^{\cdot}_{X'_n/E_n}), K^{\cdot})\simeq \mathrm{Hom}_{\mathcal{O}_{X_n/E_n}}(E, K^m).$$

This means for $K^{\cdot}$ injective, K^m is an injective $\mathcal{O}_{X_n/E_n}$ module. So that means for $q\geq 1$, $\underline{\mathrm{Ext}}^{\cdot q}(E\otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^{\cdot}_{X'_n/E_n}), K^{\cdot})$ is acyclic. So the above result implies one can use $I^{\cdot\cdot}$, injective resolution of $F \otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^{\cdot}_{X'_n/E_n})$, to compute $\underline{\mathrm{Ext}}^i_{L(\omega^{\cdot}_{X'_n/E_n})}(E \otimes L(\omega^{\cdot}_{X'_n/E_n}), F \otimes L(\omega^{\cdot}_{X'_n/E_n}))$. Again by adjunction formula,

$$\mathrm{Hom}^p(E \otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^{\cdot}_{X'_n/E_n}), I^{\cdot q}) \simeq \mathrm{Hom}_{\mathcal{O}_{X_n/W_n}}(E, I^{p,q}).$$

But $I^{p,\cdot}$ is an injective resolution of $F \otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^p_{X'_n/E_n})$, so the double complex $I^{\cdot\cdot}$ is quasi-isomorphic to $F \otimes_{\mathcal{O}_{X_n/E_n}} L(\omega^{\cdot}_{X'_n/E_n})$, hence by Poincaré lemma an injective resolution of F, hence we prove the first isomorphism. The other isomorphism follows from a similar method.

Theorem 4.10 *([3] VI Theorem 2.4.2)*

There exists canonical isomorphisms

$$\underline{Ext}^i_{L(\omega_{X'_n/E_n})}(L(K^{\cdot}), L(M^{\cdot})) \simeq \underline{Ext}^i_{\omega_{X'_n/E_n}}(K^{\cdot}, M^{\cdot}).$$

Proof : We sketch the approach in [3]. Let $I^{\cdot\cdot}$ be a resolution of $L(M^{\cdot})$ in the category of differential operator of order ≤ 1 with respect to $L(\omega^{\cdot}_{X'_n/E_n})$, so by the discussion before Theorem 4.9, $u_{X_n/E_n*}(I^{\cdot q})$ is an injective object in $C(X'_n)$. Also $I^{p\cdot}$ is an injective resolution of $L(M^p)$. So $u_{X_n/E_n*}(I^{\cdot})$ is an injective resolution of $u_{X_n/E_n*}(L(M^p))$, which is equal to M^p. Finally, by adjunction formula, we have a canonical isomorphism

$$\underline{\mathrm{Hom}}^p_{X_n}(L(K^{\cdot}), I^{\cdot q}) \simeq \underline{\mathrm{Hom}}^p_{X_n}(K^{\cdot}, u_{X_n/E_n*}(I^{\cdot q})),$$

hence the result .

Since the hyperext does not always factors through derived category, in order to get a Gysin map, we need to do something more:

Theorem 4.11 *([3] VI Theorem 3.2.1)*

Let $X'^{\times}_n$, $Y'^{\times}_n$ be two integral ([19] Definition 4.3) log smooth scheme over E_n, and let $Y'^{\times}_n \hookrightarrow X'^{\times}_n$ be exact closed immersion. Then the functorial homomorphism

$$\underline{Ext}^i_{\omega^{\cdot}_{X'_n/E_n}}(\omega^{\cdot}_{Y'_n/E_n}, \omega^{\cdot}_{X'_n/E_n}) \to \underline{Ext}^i_{\omega^{\cdot}_{X'_n/E_n}}(D_{Y'_n}(X'_n) \otimes_{\mathcal{O}_{X'_n}} \omega^{\cdot}_{X'_n/E_n}, \omega^{\cdot}_{X'_n/E_n})$$

induced from the morphism of complexes

$$D_{Y'_n}(X'_n) \otimes_{\mathcal{O}_{X'_n}} \omega^{\cdot}_{X'_n/E_n} \to O_{Y'_n} \otimes_{\mathcal{O}_{X'_n}} \omega^{\cdot}_{X'_n/E_n} \simeq \omega^{\cdot}_{Y'_n/E_n},$$

is an isomorphism.

Proof : Here we closely follow the argument in [3] and indicate where we modify the proof.

Step1. Let $\mathcal{J}^{\cdot} = \mathrm{Ker}(D_{Y'_n}(X'_n) \otimes_{\mathcal{O}_{X'_n}} \omega^{\cdot}_{X'_n/E_n} \to \omega^{\cdot}_{Y'_n/E_n})$. So it suffices to prove that

$$\underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X'_n/E_n}}(\mathcal{J}^{\cdot}, \omega^{\cdot}_{X'_n/E_n}) = 0.$$

Since p is nilpotent in E_n, the above statement is equivalent to

$$\underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X'_n/E_n}}(\mathcal{J}^{\cdot}, p^k \cdot \omega^{\cdot}_{X'_n/E_n}/p^{k+1} \cdot \omega^{\cdot}_{X'_n/E_n}) = 0.$$

Assume this theorem is proved when the base scheme is of characteristic p, (i.e., the base scheme is E_1), since $X'^{\times}_n$ is integral and log smooth over E_n, by [19] Corollary 4.5, the morphism between underlying schemes is flat, hence $\omega^1_{X'_n/E_n}$ is a flat $\mathcal{O}_{E_n}$ module. Let f be the canonical map $X'^{\times}_n \to E_n$, then we have $f^*(p^k \cdot \mathcal{O}_{E_n}/p^{k+1} \cdot \mathcal{O}_{E_n}) \otimes_{\mathcal{O}_{X'_n}} \omega^{\cdot}_{X'_n/E_n} \simeq p^k \cdot \omega^{\cdot}_{X'_n/E_n}/p^{k+1}\omega^{\cdot}_{X'_n/E_n}$. Now let $S^{\times}$ be the closed subscheme of E_n whose defining ideal sheaf is the kernel of the mutiplication by p^k homomorphism $\mathcal{O}_{E_n} \to p^k\mathcal{O}_{E_n}/p^{k+1}\mathcal{O}_{E_n}$ and we define X'' and Y'' the associated reduction of X_n and Y_n by this ideal. Then we have $p^k\omega^{\cdot}_{X'_n/E_n}/p^{k+1}\omega^{\cdot}_{X'_n/E_n} \simeq \omega^{\cdot}_{X''/S}$. Let

$$\mathcal{J}'^{\cdot} := \mathrm{Ker}(D_{Y''}(X'') \otimes_{\mathcal{O}_{X''}} \omega^{\cdot}_{X''/S} \to \omega^{\cdot}_{Y''/S}).$$

Since we have a canonical ismorphism $D_{Y'_n}(X'_n) \otimes_{\mathcal{O}_{E_n}} \mathcal{O}_S \simeq D_{Y''}(X'')$, we have $\mathcal{J} \otimes_{\mathcal{O}_{X'_n}} \mathcal{O}_{X''} \simeq \mathcal{J}'$. Since by hypothesis $\underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X''/S}}(\mathcal{J}'^{\cdot}, \omega^{\cdot}_{X''/S}) = 0$, it suffices to show that the canonical homomorphism

$$\underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X''/S}}(\mathcal{J}'^{\cdot}, \omega^{\cdot}_{X''/S}) \to \underline{\mathrm{Ext}}^i_{\omega^{\cdot}_{X''/S}}(\mathcal{J}^{\cdot}, \omega^{\cdot}_{X''/S})$$

is an isomorphism, which we prove it in Step 2.

Step 2. Since both sides are the target of the spectral sequences with

$$E_1^{p,q} = \underline{\mathrm{Ext}}^{p,q}(\mathcal{J}'^{\cdot}, \omega^{\cdot}_{X''/S}) \ (E_1^{p,q} = \underline{\mathrm{Ext}}^{p,q}(\mathcal{J}^{\cdot}, \omega^{\cdot}_{X'_n/E_n}))$$

and such spectral sequence are regular, so it suffices to show that the natural homomorphism on E_1 term is an isomorphism. Since the question is local, we may assume every scheme here is affine, and $E' = \Gamma(S, O_S)$, $B = \Gamma(X'_n, O_{X'_n})$, $B' = \Gamma(X'', O_{X''})$, $J^{\cdot} = \Gamma(X'_n, \mathcal{J}^{\cdot})$

(resp. $J'^{\cdot} = \Gamma(X'', \mathcal{J}'^{\cdot})$) , since $\mathcal{J}^{\cdot}$ (resp. $\mathcal{J}'^{\cdot}$) are sheaves of quasicoherent $\mathcal{O}_{X'_n}$ (resp. $\mathcal{O}_{X''}$), we have

$$\underline{\mathrm{Ext}}^{\cdot q}(\mathcal{J}'^{\cdot}, \omega^{\cdot}_{X''/E'}) \simeq \mathrm{Ext}^{\cdot q}(J'^{\cdot}, \omega^{\cdot}_{B'/E'}), \quad \underline{\mathrm{Ext}}^{\cdot q}(\mathcal{J}^{\cdot}, \omega^{\cdot}_{X/E_n}) \simeq \mathrm{Ext}^{\cdot q}(J^{\cdot}, \omega^{\cdot}_{B/E_n}).$$

Since $\omega^{\cdot}_{B'/E'} \simeq \omega^{\cdot}_{B/E_n} \otimes_{E_n} E'$, $J'^{\cdot} \simeq J^{\cdot} \otimes_{E_n} E' \simeq J^{\cdot} \otimes_{\omega^{\cdot}_{B/E_n}} \omega^{\cdot}_{B'/E'}$, so we have the following spectral sequences of graded modules

$$\mathrm{Ext}^{\cdot q}_{\omega^{\cdot}_{B'/E'}}(\mathrm{Tor}^{\omega^{\cdot}_{B/E_n}}_{\cdot p}(\omega^{\cdot}_{B'/E'}, J^{\cdot}), \omega^{\cdot}_{B'/E'}) \Rightarrow \mathrm{Ext}^{\cdot l}_{\omega^{\cdot}_{B/E_n}}(J^{\cdot}, \omega^{\cdot}_{B'/E'}),$$

$$\mathrm{Tor}^{\omega^{\cdot}_{B/E_n}}_{\cdot q}(\mathrm{Tor}^{E_n}_{\cdot p}(E', \omega^{\cdot}_{B/E_n}), J^{\cdot}) \Rightarrow \mathrm{Tor}^{E_n}_{\cdot l}(E', J^{\cdot}),$$

here $\mathrm{Tor}^{E_n}_{ml}(E', J^{\cdot})$ means the l-th derived functor of the tensor product of two complexes $E', J^{\cdot}$ with shifted m degree, i.e., degree i of the first tensor degree $m+i$ of the second. The second one degenerates since ω_{B/E_n} is flat over E_n; on the other hand, $\omega^{\cdot}_{X/E_n}$, $\omega^{\cdot}_{Y/E_n}$ and $D_{Y'}(X')$ is flat over E_n (by [19] Corollary 4.4) by assumption, so the first one also degenerates, hence we get the desired isomorphism.

Step 3. Now we need to prove the claim for E_1; to simplify the notation, we write X', Y', instead of X'_1, Y'_1. We have the following commutative diagram

$$\begin{array}{ccc}
\underline{\mathrm{Ext}}^{m,q}(\omega^{\cdot}_{Y'/E_1}, \omega^{\cdot}_{X'/E_1}) & \Longrightarrow & \underline{\mathrm{Ext}}^{m+q}_{\omega^{\cdot}_{X'/E_1}}(\omega^{\cdot}_{Y'/E_1}, \omega^{\cdot}_{X'/E_1}) \\
\downarrow & & \downarrow \\
\underline{\mathrm{Ext}}^{m,q}(D_{Y'}(X') \otimes_{\mathcal{O}_{X'}} \omega^{\cdot}_{X'/E_1}, \omega^{\cdot}_{X'/E_1}) & \Longrightarrow & \underline{\mathrm{Ext}}^{m+q}_{\omega^{\cdot}_{X'/E_1}}(D_{Y'}(X') \otimes_{\mathcal{O}_{X'}} \omega^{\cdot}_{X'/E_1}, \omega^{\cdot}_{X'/E_1})
\end{array}$$

The horizontal morphisms are $E_1^{m,q}$ term of two spectral sequences. Again, since the two spectral sequences are regular, we claim that the corresponding homomorphism is isomorphism at E_2 term. For the top horizontal row, as in the construction of de Rham cycle class, we see for $q \neq j$, $E_1^{m,q} = 0$, for $q = d$, $E_1^{\cdot,j} \simeq \omega^{\cdot}_{Y'/E_1}[-j]$. For the bottom $E_1^{p,q}$, take $\mathcal{I}^{\cdot\cdot}$ the injective resolution of $\omega^{\cdot}_{X'/E_1}$, we have

$$\underline{\mathrm{Hom}}^m(D_{Y'}(X') \otimes_{\mathcal{O}'_X} \omega^{\cdot}_{X'/E_1}, \mathcal{I}^{\cdot q}) \simeq \underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(D_{Y'}(X'), \mathcal{I}^{m,q}).$$

so to show that $\underline{\mathrm{Ext}}^{m,q}(D_{Y'}(X') \otimes_{\mathcal{O}_{X'}} \omega^{\cdot}_{X'/E_1}, \omega^{\cdot}_{X'/E_1}) = 0$ for $q \neq j$, it suffices to show that $\underline{\mathrm{Ext}}^q_{\mathcal{O}_{X'}}(D_{Y'}(X'), \omega^m_{X'/E}) = 0$, for $q \neq j$. Now since the question is local, use the same notation as in Step 2, now assume the definning ideal $\mathcal{J}$ of Y' in X' is generated by the regular sequence $t_1, \ldots, t_j$, since there is a retraction $C^{(p)} = B/J^p \to C = B/J$, by [3] I 4.5.1 we have $D_B(J) \simeq C < t_1, \ldots, t_j >$. Then by [3] VI Lemma 3.2.5, $C < t_1, \ldots, t_j >$ is a free module over $C[t_1,\ldots,t_j]/(t_1^p,\ldots,t_j^p)$ with basis consisting of $t_1^{[pq_1]}\ldots t_j^{[pq_j]}$, with $q_i\geq 0$ (pq_i-th divided power of t). So $\mathrm{Ext}_B^q(D_B(J),\omega^m_{B/E_1})\simeq\prod\mathrm{Ext}_B^q(C[t_1,\ldots,t_j]/t_1^p,\ldots,t_j^p,\omega^m_{B/E_1})$, since $t_1^p,\ldots,t_j^p$ is regular and ω^m_{B/E_1} is flat over B, by [16] III Proposition 7.2 we have

$$\mathrm{Ext}_B^q(C[t_1,\ldots,t_d]/t_1^p,\ldots,t_d^p,\omega^m_{B/E_1})=0$$

for $q\neq j$, hence we get the desired vanishing result.

Consider the complex

$$C^{\cdot}_{t_1\ldots t_j}(\omega_{X'/E_1})\to\mathcal{H}^{\cdot j}_{Y'}(\omega_{X'/E_1})\to 0,$$

which is a resolution of ω_{X'/E_1}. In order to use this resolution to compute these two spectral sequences, it suffices to show that for $q\geq 1$, each term of this resolution is annihilated by the functor $\underline{\mathrm{Ext}}^{\cdot q}(\omega_{Y'/E_1},\cdot)$; this is done in the proof of Proposition 4.6. Combined with the above results, that means both spectral sequences will degenerate at E_2 term. To prove the isomorphism at E_2 term, it suffices to prove the following map of the associated complexes

$$E_1^{\cdot,j}:\underline{\mathrm{Hom}}^{\cdot}(\omega^{\cdot}_{Y'/E_1},\mathcal{H}^{\cdot j}_{Y'}(\omega^{\cdot}_{X'/E_1}))\to\underline{\mathrm{Hom}}^{\cdot}(D_{Y'}(X')\otimes_{\mathcal{O}_{X'}}\omega^{\cdot}_{X'/E_1},\mathcal{H}^{\cdot j}_{Y'}(\omega^{\cdot}_{X'/E_1}))$$

is a quasi-isomorphism. It suffices to do it locally, using notations as above, that means it suffices to show that

$$\mathrm{Hom}^{\cdot}(\omega^{\cdot}_{C/E_1},H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1}))\to\mathrm{Hom}^{\cdot}(D_B(J)\otimes_B\omega^{\cdot}_{B/E_1},H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1}))$$

is a quasi-isomorphism.

The rest of the argument honestly follows from the argument in [3], so we skip the computations and sketch the main point.

First, for all k, we have

$$\mathrm{Hom}^k(D_B(J)\otimes_B\omega^{\cdot}_{B/E_1},H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1}))$$

$$\simeq\mathrm{Hom}_B(D_B(J),H^j_{Y'}(\omega^{\cdot}_{B/E_1}))\simeq\prod_{\underline{q}}\mathrm{Hom}_B(C^{(p)}\cdot t^{[p\underline{q}]},H^j_y(\omega^k_{B/E_1})),$$

where $\underline{q}=(q_1,\cdots,q_j)$ and $t^{[p\underline{q}]}=t_1^{[pq_1]}\cdots t_d^{[pq_j]}$. Since we have

$$H^j_{Y'}(\omega^k_{B/E_1})\simeq\varinjlim_{l}\omega^k_{B/E_1}/J^{(l)}\cdot\omega^k_{B/E_1},$$

where J is the defining ideal of Y' in X'. Notice that we have a canonical inclusion

$$C^{(p)}\otimes_B\omega^k_{B/E_1}=\omega^k_{B/E_1}/J^{(p)}\cdot\omega^k_{B/E_1}\hookrightarrow\varinjlim_{n}\omega^k_{B/A}/J^{(n)}\cdot\omega^k_{B/E_1},$$

we can choose retraction $C \to C^{(p)}$, so we can view $\mathrm{Hom}_B(D_B(J), H^d_Y(\omega^k_{B/E_1}))$ as a C module, then since

$$\mathrm{Hom}_B(D_B(J), H^j_{Y'}(\omega^k_{B/E_1})) \simeq \mathrm{Hom}_B(D_B(J), H^j_{Y'}(B)) \otimes_B \omega^k_{B/E_1},$$

for every $\varphi \in \mathrm{Hom}_B(D_B(J), H^j_{Y'}(\omega^k_{B/E_1}))$,

$$\varphi = \Sigma_{\underline{m},\omega} a_{\underline{m},\omega} \cdot \psi_{\underline{m}} \otimes \omega$$

where $a_{\underline{m},\omega} \in C, \omega \in \omega^k_{B/E_1}$,$\psi_{\underline{m}} \in \mathrm{Hom}_B(D_B(J), H^j_{Y'}(B))$, for $\underline{m} = p \cdot \underline{q}' + \underline{r}$,

$$\psi_{\underline{m}}(t^{[p\underline{q}]}) = \begin{cases} 0, & \text{if } \underline{q} \neq \underline{q}' \\ (-1)^{|\underline{m}|+|\underline{r}|} \cdot \psi_{\underline{r}}(1) = (-1)^{|\underline{m}|+|\underline{r}|} \cdot \underline{t}^{[p-1-\underline{r}]}, & \text{if } \underline{q} = \underline{q}'. \end{cases}$$

From [3] II 5.2.4, there exists a canonical isomorphism

$$\mathrm{Hom}^{\cdot}(D_B(J) \otimes_B \omega^{\cdot}_{B/E_1}, H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1})) \simeq \mathrm{Hom}^{\cdot}(D_B(J) \otimes_B \omega^{\cdot}_{B/E_1}, H^j_{Y'}(B) \otimes_B \omega^{\cdot}_{B/E_1})$$
$$\simeq \mathrm{Hom}_B(D_B(J), H^j_{Y'}(B)) \otimes_B \omega^{\cdot}_{B/E_1}.$$

Under this isomorphism, the differential is taken in the following way

$$d(\Sigma_{\underline{m},\omega} a_{\underline{m},\omega} \cdot \psi_{\underline{m}} \otimes \omega) = \Sigma_{\underline{m},\omega} a_{\underline{m},\omega} \cdot \psi_{\underline{m}} \otimes d(\omega) + \nabla(\Sigma_{\underline{m},\omega} a_{\underline{m},\omega} \cdot \psi_{\underline{m}}) \wedge \omega,$$

where ∇ is the homomorphism $\mathrm{Hom}_B(D_B(J), H^d_Y(B)) \to \mathrm{Hom}_B(D_B(J), H^d_Y(B)) \otimes_B \omega^1_{B/E_1}$ induced from the connections on $D_B(J)$ and $H^d_Y(B)$.

Choose $y_l \in C, l = 1, \ldots, n-s$ such that $dt_i, i = 1, \ldots, s, dy_l/y_l$ form a basis of ω^1_{B/E_1}, by assumption we have $\frac{d}{dt_i}(a_{\underline{m}w}) = 0$. On the other hand, by the definition of connection on $\mathrm{Hom}_B(D_B(J), H^j_{Y'}(B))$, we have

$$\nabla(\frac{d}{dy_j})(\psi_{\underline{m}})(t^{[p\underline{q}]}) = \frac{d}{dy_j}(\psi_{\underline{m}})(t^{[p\underline{q}]})) - \psi_{\underline{m}}(\frac{d}{dy_j}(t^{[p\underline{q}]})).$$

Both terms on the right-hand sides are zero by the definition of $\psi_{\underline{m}}$, so we have $\frac{d}{dy_j})(\psi_{\underline{m}}) = 0$. For $\varphi = \sum_{\underline{m}} a_{\underline{m}} \cdot \psi_{\underline{m}}$, by definition we have

$$\nabla(\varphi) = \sum_i \nabla(\frac{d}{dt_i})(\varphi) \otimes dt_i + \sum_j \nabla(\frac{d}{dy_j})(\varphi) \otimes dy_j.$$

Combined with these results above, it is equal to

$$\sum_i (\sum_{\underline{m}} a_{\underline{m}} \cdot \psi_{\underline{m+1}_i}) \otimes dt_i + \sum_j (\sum_{\underline{m}} \frac{d}{dy_j}(a_{\underline{m}}) \cdot \psi_{\underline{m}}) \otimes dy_j.$$

Now write ω'^1_{B/E_1} the submodule of ω^1_{B/E_1} generated by dy_l/y_l and Ω''^1_{B/E_1} the submodule of ω^1_{B/E_1} generated by dt_i. Consider the complex

$$\mathrm{Hom}_B(D_B(J), H^d_{Y'}(B)) \otimes_B \omega'^i_{B/E_1} \otimes_B \Omega''^{\cdot}_{B/E_1},$$

by [3] VI Lemma 3.2.10 (notice that Ω''^1_{B/E_1} is the ordinary, not log differential), fix i, the associated complex is acyclic on degree $\neq d$, its cohomology on degree d is isomorphic to ω^i_{C/E_1}. Because of such acyclicity result, the spectral sequence degenerates and is quasi-isomorphic to $\mathrm{Hom}_B(D_B(J), H^j_{Y'}(B)) \otimes_B \omega^{\cdot}_{B/E_1}$.

Finally, in the construction of de Rham cycle class (see the beginning of the next section), we have a quasi-isomorphism

$$\omega^{\cdot}_{C/E_1}[-j] \to \mathrm{Hom}^{\cdot}(\omega^{\cdot}_{C/E_1}, H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1}))$$

which is defined by $\omega\ (\in \omega^k_{C/E_1}) \to \omega \wedge dt_1 \wedge \cdots \wedge dt_j/t_1 \cdots t_j$. Since the kernel of $D_B(J) \to C$ consists of $t^{[p\underline{q}]}$, for $\underline{q} \neq 0$, consider the composition map

$$\omega^{\cdot}_{C/E_1}[-j] \to \mathrm{Hom}^{\cdot}(\omega^{\cdot}_{C/E_1}, H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1})) \to \mathrm{Hom}^{\cdot}(D_B(J) \otimes_B \omega^{\cdot}_{B/E_1}, H^{\cdot j}_{Y'}(\omega^{\cdot}_{B/E_1})),$$

$$\omega \to (-1)^j \cdot \psi_0 \otimes \omega \wedge dt_1/t_1 \wedge \cdots \wedge dt_l/t_l,$$

which is injective by definition. Combine this map and the map above (map back to $\mathrm{Hom}_B(D_B(J), H^d_{Y'}(B)) \otimes_B \omega'^i_{B/E_1}$, then to $\omega^{\cdot}_{C/E_1}$), the image of ω in the target is $(-1)^j\omega$, hence the surjectivity and the claim.

Recall that we have a spectral sequence

$$E_2^{m,q} = H^m(X', \underline{\mathrm{Ext}}^q_{\omega^{\cdot}_{X'/E_n}}(\omega^{\cdot}_{Y'/E_n}, \omega^{\cdot}_{X'/E_n})) \Rightarrow \mathrm{Ext}^{m+q}_{\omega^{\cdot}_{X'/E_n}}(\omega^{\cdot}_{Y'/E_n}, \omega^{\cdot}_{X'/E_n})),$$

$$E_2^{m,q} = H^m(X', \underline{\mathrm{Ext}}^q_{\omega^{\cdot}_{X'/E_n}}(\mathcal{D}_{Y'}(X') \otimes_{O_{X'}} \omega^{\cdot}_{X'/E_n}, \omega^{\cdot}_{X'/E_n})) \Rightarrow$$
$$\mathrm{Ext}^{m+q}_{\omega^{\cdot}_{X'/E_n}}(\mathcal{D}_Y(X) \otimes_{O_X} \omega^{\cdot}_{X/E_n}, \omega^{\cdot}_{X/E_n}).$$

So Theorem 4.2. is also true for $\mathrm{Ext}^{\cdot}_{\omega^{\cdot}_{X'/S}}$.

4.2 Crystalline cycle class

We follow the approach of [3] VI 3.1. Fix a base log DP-scheme $(V_n, \mathbf{N})$, $X_n^\times$, $Y_n^\times$ are vertical semistable schemes over $(V_n, \mathbf{N})$ with exact closed immersion $i : Y_n^\times \hookrightarrow X_n^\times$. So every closed point of X_n over V_n is defined by a regular sequence. So for a differential complex of order less or equal than 1 which is quasicoherent and flat over $\mathcal{O}_{X_n}$ (e.g., $\omega^{\cdot}_{X_n/V_n}$), [3] VI (1.5.6) still holds, that means, as in Section 2, denoted X_n^l by the closed points of relative dimension l, we have

$$\forall i \neq l, \mathcal{H}^{\cdot,i}_{X_n^l/X_n^{l+1}}(\omega^{\cdot}_{X_n/V_n}) = 0.$$

In particular, we can form a cousin complex of $\omega^{\cdot}_{X_n/V_n}$, whose (p,q) term is

$$\mathcal{H}^{q,p}_{X_{p,n}/X_{p+1,n}}(\omega^{\cdot}_{X_n/V_n}) = \mathcal{H}^p_{X_{p,n}/X_{p+1,n}}(\omega^q_{X_n/V_n}),$$

which is a resolution of $\omega^{\cdot}_{X_n/V_n}$.

Again, since $\omega^{\cdot}_{X_n/V_n}$ is flat over $\mathcal{O}_{X_n}$, we have the isomorphism

$$\mathcal{H}^{\cdot,i}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) \simeq \mathcal{H}^{i}_{Y_n}(\mathcal{O}_{X_n}) \otimes \omega^{\cdot}_{X_n/V_n}.$$

So now let $\mathcal{I}_{Y_n}$ be the defining ideal sheaf of Y_n in X_n, since $\mathcal{I}_{Y_n}/\mathcal{I}^2_{Y_n}$ is locally free of rank j we have the short exact sequence

$$0 \to \mathcal{I}_{Y_n}/\mathcal{I}^2_{Y_n} \to \omega_{X_n/V_n} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_{Y_n} \to \omega_{Y_n/V_n} \to 0,$$

which induces a map

$$\omega^{i}_{Y_n/V_n} \to \omega^{i+j}_{X_n/V_n} \otimes_{\mathcal{O}_{X_n}} (\wedge^j(\mathcal{I}_{Y_n}/\mathcal{I}^2_{Y_n}))^{\vee} \simeq \underline{Ext}^j_{\mathcal{O}_{X_n}}(\mathcal{O}_{Y_n}, \omega^{i+j}_{X_n/V_n}) \to \mathcal{H}^j_{Y_n}(\omega^{i+j}_{X_n/V_n}).$$

The middle isomorphism comes from [16] III, 7.2 since $Y_n \to X_n$ is a regular closed immersion, and the last one comes from the canonical map of functors

$$\mathcal{H}om_{\mathcal{O}_{X_n}}(\mathcal{O}_{Y_n}, \cdot) \to \Gamma_{Y_n}.$$

So locally if we assume Y_n is defined by a regular sequence $t_1, \ldots, t_j$, taking Koszul resolution of $\mathcal{O}_{X_n}$, by [3] VI 3.1.3, the above map is given by $\omega \to \omega \wedge \frac{dt_1 \wedge \cdots \wedge dt_j}{t_1 \ldots t_j}$, where $\omega \in \omega^i_{Y_n/V_n}$. Combined with the natural map

$$\mathcal{H}^{\cdot,j}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) \simeq R\Gamma_{Y_n}(\omega^{\cdot}_{X_n/V_n}) \to \omega^{\cdot}_{X_n/V_n},$$

where the first isomorphism comes from the fact $\mathcal{H}^{\cdot,i}_{Y_n}(\omega^{\cdot}_{X_n/V_n}) = 0, \forall i \neq j$, we get the de Rham cycle class.

Using the generalization developed in the last section, now we can handle the cycle class map in crystalline cohomology.

Proposition 4.12 *([3] VI Proposition 3.3.1)*

Let $i : Y_n^{\times} \to X_n^{\times}$ are two vertical semistable schemes over V_n such that i is an exact closed immersion, assume these two schemes have both log smooth lifting $Y_n'^{\times}$, $X_n'^{\times}$ over E_n with exact closed immersion $i' : Y_n'^{\times} \to X_n'^{\times}$ extending i, then for all q, there exists canonical isomorphisms

$$\underline{Ext}^q_{\mathcal{O}_{X/E_n}}(i_{crys*}(\mathcal{O}_{Y_n/E_n}), \mathcal{O}_{X_n/E_n}) \simeq \underline{Ext}^q_{\omega^{\cdot}_{X_n'/E_n}}(\omega^{\cdot}_{Y_n'/E_n}, \omega^{\cdot}_{X_n'/E_n})$$

Proof : Since $i_{\mathrm{crys}*}(\mathcal{O}_{Y'/E_n})$ is a crystal on $(X_n'/E_n)^{\log}_{\mathrm{crys}}$ (since it is an exact closed immersion, it follows from classical case), it is defined by

$$i_{\mathrm{crys}*}(\mathcal{O}_{Y_n/E_n})_{(X_n, X_n')} = \mathcal{D}_{Y_n,\gamma}(X_n')$$

where γ is the divided power equipped on E_n (so this notation means the DP-structure of $\mathcal{D}_{Y_n,\gamma}(X_n')$ is compatible with the DP-structure of E_n), directly checking we see it is isomorphic to $\mathcal{D}_{Y_n'}(X_n')$, applying Theorem 4.11 we get the result.

Corollary 4.13 *([3] VI Corollary 3.3.3) Let the assumption be as above , then*

$$\forall q < 2j,\ \underline{Ext}^q_{\mathcal{O}_{X_n/E_n}}(i_{crys*}(\mathcal{O}_{Y_n/E_n}),\mathcal{O}_{X_n/E_n}) = 0;$$
$$For\ q = 2j,\ Ext^{2j}_{\mathcal{O}_{X_n/E_n}}(i_{crys*}(\mathcal{O}_{Y_n/E_n}),\mathcal{O}_{X_n/E_n}) \simeq$$
$$\Gamma(X_n, \underline{Ext}^{2j}_{\mathcal{O}_{X_n/E_n}}(i_{crys\,*}(\mathcal{O}_{Y_n/E_n}),\mathcal{O}_{X_n/E_n})).$$

Proof : For the first vanishing result, by Proposition 4.6, we have (since the question is local)

$$\underline{\mathrm{Ext}}^q_{\mathcal{O}_{X_n/E_n}}(i_{\mathrm{crys}*}(\mathcal{O}_{Y_n/E_n}),\mathcal{O}_{X_n/E_n}) \simeq \underline{\mathrm{Ext}}^q_{\omega^{\cdot}_{X'/E_n}}(\omega^{\cdot}_{Y'_n/E_n},\omega^{\cdot}_{X'_n/E_n}),$$

which is isomorphic to $\mathcal{H}^{i-2j}(\omega^{\cdot}_{Y_n/E_n})$.

The second result comes from the spectral sequence, since $E_2^{p,q} = 0$ for $q < 2j$, then such spectral sequence degenerates and is isomorphic to the target at $E_2^{p,2j}$.

By the above corollary, especially the second result, locally for every lifting of Y_n over E_n, we can define the crystalline cycle class as the way we define de Rham cycle class and try to glue them together. We summarize the result in the following theorem.

Theorem 4.14 *([3] VI Theorem 3.3.5) For X_n, Y_n vertical semistable schemes with exact closed immersions $i : Y_n \hookrightarrow X_n$ of relative dimension j, there exists an unique cohomological class*

$$s_{Y_n/X_n} \in Ext^{2j}_{\mathcal{O}_{X_n/E_n}}(i_{crys*}(\mathcal{O}_{Y_n/E_n}),\mathcal{O}_{X_n/E_n})$$

with the following properties: if U is open in X_n such that there exists a smooth lifting U', V' and $i' : V' \hookrightarrow U'$ lifting i, then the restriction of s_{Y_n/X_n} in $Ext^{2d}_{\mathcal{O}_{U_n/E_n}}(i_{crys*}(\mathcal{O}_{Y_n|U/E_n}),\mathcal{O}_{U/E_n})$ is identified by an isomorphism as the de Rham cohomology class of V' in U'.

Proof : It suffices to check the gluing procedure. Let $U''^{\times}$, $V''^{\times}$ be another log smooth scheme over E_n lifting $U^{\times}$, $V^{\times}$ and $i'' : V''^{\times} \hookrightarrow U''^{\times}$ be the exact closed immersion lifting i, since the problem is local, we may assume there exists an E_n-isomorphism $\sigma : U'^{\times} \to U''^{\times}$ inducing automorphism on $V'^{\times} \to V''^{\times}$, and by Proposition 4.4, we can form the commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}^0(\omega^{\cdot}_{V'/E_n}) & \xrightarrow{\simeq} & \underline{\mathrm{Ext}}^{2j}_{\omega^{\cdot}_{U'/E_n}}(\omega^{\cdot}_{V'/E_n},\omega^{\cdot}_{U'/E_n}) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{H}^0(\omega^{\cdot}_{V''/E_n}) & \xrightarrow{\simeq} & \underline{\mathrm{Ext}}^{2j}_{\omega^{\cdot}_{U''/E_n}}(\omega^{\cdot}_{V''/E_n},\omega^{\cdot}_{U''/E_n})
\end{array}$$

hence the existence.

Then as in [3] VI 3.3.6, we can define the cohomology class as the image of s_{Y_n/X_n} (by abuse of notation) in the composition map

$$\mathrm{Ext}^{2j}_{\mathcal{O}_{X_n/E_n}}(i_{\mathrm{crys}*}(\mathcal{O}_{Y_n/E_n}), \mathcal{O}_{X_n/E_n}) \to \mathrm{Ext}^{2j}_{\mathcal{O}_{X_n/E_n}}(\mathcal{J}^{[j]}_{X_n/E_n}, \mathcal{O}_{X_n/E_n})$$

$$\simeq H^{2j}(X_n^{\times}/E_n, \mathcal{J}^{[j]}_{X_n/E_n})$$

and from the above theorem, we can define $i_* : i_{\mathrm{crys}*}(\mathcal{O}_{Y_n/E_n}) \to \mathcal{J}^{[j]}_{X_n/E_n}[2j]$.

Taking global sections and using this isomorphism, one gets

$$i_* : H^k(Y_n^{\times}/E_n, \mathcal{O}_{Y_n/E_n}) \to H^{k+2j}(X_n^{\times}/E_n, \mathcal{J}^{[j]}_{X_n/E_n}).$$

From the definition, we have $i_*(1) = s_{Y_n/X_n}$. On the other hand, by [3] VI Lemma 4.1.1, one has projection formula (the proof works for any topos so we can apply here): for $x \in H^i(X_n^{\times}/E_n, \mathcal{O}_{X_n/E_n}),\ y \in H^l(Y_n^{\times}/E_n, \mathcal{O}_{Y_n/E_n}),\ i_*(y \cdot i^*(x)) = i_*(y) \cdot x$.

We also need intersection formula and the compatibility of cohomological class with respect to Künneth morphism for later use.

Proposition 4.15 *([3] VI Theorem 4.3.12)*

For X, Y, T, Z smooth or vertical semistable scheme with exact closed immersions $i : Y \hookrightarrow X$, $j : T \hookrightarrow Z$ and a Cartesian diagram:

$$\begin{array}{ccc} T & \xrightarrow{g} & Y \\ \Big\downarrow{\scriptstyle j} & & \Big\downarrow{\scriptstyle i} \\ Z & \xrightarrow{f} & X \end{array}$$

where f and i are transverse. Then the canonical filtered homomorphism

$$f^* : H^i(X_n^{\times}/E_n, \mathcal{O}_{X_n/E_n}) \to H^i(Z_n^{\times}/E_n, \mathcal{O}_{Z_n/E_n})$$

sends the cohomology class s_{Y_n/X_n} to s_{T_n/Z_n}.

Proof : It exactly follows from [3] VI Theorem 4.3.12 once one replaces linearization by log linearization and Poincaré lemma by log Poincaré lemma. Let us follow the proof here.

Since both sides are sheaves, it suffices to work locally, namely, assume X_n, Y_n, T_n, Z_n both have log smooth lifting X'_n, Y'_n, T'_n, Z'_n over E_n, and $T'_n = Y'_n \times_{X'_n} Z'_n$, and Y'_n are defined by a regular sequence $t_1, \ldots, t_j$ of $\mathcal{O}_{X'_n}$, so T'_n are defined by a regular sequence $t'_1, \ldots, t'_j$ of $\mathcal{O}_{Z'_n}$, where $t'_i = f'^*(t_i)$.

So now we go back to the de Rham cycle class case. The canonical homomorphism $f'^*(\omega^{\cdot}_{X'_n/E_n}) \to \omega_{Z'_n/E_n}$ induces the corresponding morphism between $\underline{H}^{\cdot d}_{Y'_n}(\omega^{\cdot}_{X'_n/E_n})$ and

$\underline{H}^{\cdot d}_{T'_n}(\omega^{\cdot}_{Z'_n/E_n})$ ($\underline{C}^{\cdot}_{t_1,\dots,t_d}(\omega^{\cdot}_{X'_n/E_n})$ and $\underline{C}^{\cdot}_{t'_1,\dots,t'_j}(\omega^{\cdot}_{Z'_n/E_n})$). Now set $K^{\cdot\cdot} = \underline{C}^{\cdot}_{t_1,\dots,t_j}(\omega^{\cdot}_{X'_n/E_n}) \to \underline{H}^{\cdot d}_{Y'}(\omega^{\cdot}_{X'_n/E_n}) \to 0$ and $K'^{\cdot\cdot} = \underline{C}^{\cdot}_{t'_1,\dots,t'_j}(\omega^{\cdot}_{Z'_n/E_n}) \to \underline{H}^{\cdot d}_{T'_n}(\omega^{\cdot}_{Z'_n/E_n}) \to 0$.

Since $K^{\cdot\cdot}$ and $K'^{\cdot\cdot}$ are resolutions of $\omega^{\cdot}_{X'_n/E_n}$, $\omega^{\cdot}_{Z'_n/E_n}$, respectively, and since the log linearization functor is exact, if we denote $K^{\cdot}$ and $K'^{\cdot}$ the simple complex associated to $K^{\cdot\cdot}$ and $K'^{\cdot\cdot}$, $L_{X'_n}(K^{\cdot}_n)$ and $L_{Z'_n}(K'^{\cdot}_n)$ are resolution of $\mathcal{O}_{X'_n/E_n}$ and $\mathcal{O}_{Z'_n/E_n}$.

From [3] IV Proposition 2.5.3, the natural map $f^*_{\text{crys}}(\mathcal{O}_{X_n/E_n}) \to \mathcal{O}_{Z_n/E_n}$ induces the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_n/E_n} & \longrightarrow & f_{\text{Rcrys}*}(\mathcal{O}_{Z_n/E_n}) \\ \downarrow & & \downarrow \\ L_{X'_n}(K^{\cdot}) & \longrightarrow & f_{\text{Rcrys}*}(L_{Z'_n}(K'^{\cdot})) \end{array}$$

(where Rcrys means the restricted crystalline site) which induces the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where

$$\begin{aligned} A =& u_{X_n/E_n*}(\underline{\text{Hom}}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}}(\mathcal{O}_{Y_n/E_n}), L_{X'_n}(K^{\cdot})[2j])), \\ B =& u_{X_n/E_n*}(\underline{\text{Ext}}^{2d}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}), \mathcal{O}_{X_n/E_n})), \\ C =& u_{X_n/E_n*}(\underline{\text{Hom}}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}}(\mathcal{O}_{Y_n/E_n}), f_{\text{Rcrys}*}(L_{Z'_n}(K^{\cdot})[2j]))), \\ D =& u_{X_n/E_n*}(\underline{\text{Ext}}^{2d}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}), f_{\text{Rcrys}*}(L_{Z'_n}(K^{\cdot})[2j])). \end{aligned}$$

Since we have the canonical isomorphism

$$\begin{aligned} & u_{X_n/E_n*}(\underline{\text{Hom}}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}}(\mathcal{O}_{Y_n/E_n}), f_{\text{Rcrys}*}(L_{Z'_n}(K^{\cdot})[2j]))) \\ & \simeq f_* u_{Z_n/E_n*}((\underline{\text{Hom}}_{\mathcal{O}_{Z_n/E_n}}(i_{\text{crys}}(\mathcal{O}_{T_n/E_n}), f_{\text{Rcrys}*}(L_{Z'_n}(K^{\cdot})[2j])))). \end{aligned}$$

Finally, we get a commutative diagram as above, this time

$$\begin{aligned} A =& u_{X_n/E_n*}(\underline{\text{Hom}}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}), L_{X'_n}(K^{\cdot})[2j])), \\ B =& f_* u_{Z_n/E_n*}(\underline{\text{Hom}}_{\mathcal{O}_{Z_n/E_n}}(j_{\text{crys}*}(\mathcal{O}_{T_n/E_n}), f_{\text{Rcrys}*}(L_{Z'_n}(K^{\cdot})[2j])), \\ C =& u_{X_n/E_n*}(\underline{\text{Ext}}^{2d}_{\mathcal{O}_{X_n/E_n}}(i_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}), \mathcal{O}_{X_n/E_n})), \\ D =& f_* u_{Z_n/E_n*}(\underline{\text{Ext}}^{2d}_{Z_n,\mathcal{O}_{Z_n/E_n}}(j_{\text{crys}*}(\mathcal{O}_{T_n/E_n}), \mathcal{O}_{Z_n/E_n}))). \end{aligned}$$

By definition, the crystalline cycle class, for example, on the upper left corner, is the class such that when evaluating on X_n, it is the de Rham cycle class, so it suffices to check the upper arrow maps cycle class to cycle class. That means it suffices to check the following diagram commutes:

$$\begin{array}{ccc} f^*_{\mathrm{crys}}(i_{\mathrm{crys}*}(\mathcal{O}_{Y_n/E_n})) & \xrightarrow{\sim} & j_{\mathrm{crys}*}(\mathcal{O}_{T_n/E_n}) \\ \downarrow & & \downarrow \\ f^*_{\mathrm{crys}}(L_{X'_n}(K^{\cdot}))[2j] & \longrightarrow & L_{X'_n}(K'^{\cdot})[2j] \end{array}$$

Since each term of the above diagram is a crystal, it suffices to verify the case when evaluation on Z'_n, which is the diagram

$$\begin{array}{ccc} f'^*(D_{Y'_n}(X'_n)) & \xrightarrow{\sim} & D_{T'_n}(Z'_n) \\ \downarrow & & \downarrow \\ f'^*(D_{X'_n/E_n}(1)\otimes K^{\cdot}) & \longrightarrow & D_{Z'_n/E_n}(1)\otimes K'^{\cdot} \end{array}$$

But the vertical arrows are corresponding to cycle class maps; it is given by the map

$$D_{T'_n}(Z'_n) \to D_{Z'_n/E_n}(1)\otimes \mathcal{O}_{T'_n} \to D_{Z'_n/R_n}(1)\otimes \underline{H}^j_{T'_n}(\omega^j_{Z'_n/E_n}),$$

where $\mathcal{O}_{T'_n} \to \underline{H}^j_{T'_n}(\omega^j_{Z'_n/E_n})$ is given by $1 \to dt_1 \wedge \cdots \wedge dt_j/t_1 \cdots t_j$, the same for the other one, hence it commutes.

By the above proposition and the projection formula, we have:

Proposition 4.16 *([3] VI Corollary 4.3.15)*

Let notations be as above. Assume codimension of Y (resp. T) in X (resp. Z) is j (resp. j'), then in $H^{2j+2j'}(X_n^\times/E_n, \mathcal{J}^{[j+j']}_{X_n/E_n})$, we have

$$s_{Y_n/X_n} \cdot s_{Z_n/X_n} = s_{T_n/X_n}.$$

Proof : Now by definition, $s_{Y_n/X_n} = i_*(1)$ and $s_{Z_n/X_n} = j_*(1)$. So by projection formula, we have $i_*(1)\cdot j_*(1) = i_*(1\cdot i^*(j_*(1))) = i_*(1\cdot i^*(s_{Z_n/X_n})) = i_*(j_*(1)) = s_{T_n/X_n}$, as desired.

Proposition 4.17 *([4] VI Corollary 4.3.16)*

Let X Y be two vertical semistable schemes and let X' (resp. Y') be two vertical semistable subschemes of X (resp. Y) of codimension j (resp. j'). Let $Z' = X' \times_V Y'$, if we denote

$$\begin{aligned} k_{X_n,Y_n} : H^{2j}(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \otimes_{O_{E_n}} H^{2j'}(Y_n^\times/E_n, \mathcal{O}_{Y_n/E_n}) \\ \to H^{2j+2j'}(Z_n^\times/E_n, \mathcal{O}_{Z_n/E_n}) \end{aligned}$$

as the filtered map induced by Künneth morphism, then we have

$$k_{X_n,Y_n}(s_{X'_n/X_n} \otimes s_{Y'_n/Y_n}) = s_{Z'_n/Z_n}.$$

Proof : Let us sketch the original proof in [3].

By the compatibility of Künneth product and cup product, $k_{X_n,Y_n}(s_{X'_n/X_n}\otimes s_{Y'_n/Y_n}) = k_{X_n,Y_n}((s_{X'_n/X_n}\otimes 1)(1\otimes s_{Y'_n/Y_n})) = k_{X_n,Y_n}(s_{X'_n/X_n}\otimes 1)\cdot k_{X,_nY_n}(1\otimes s_{Y'_n/Y_n}) = s_{X''_n/Z_n}\cdot s_{Y''_n/Z_n}$, where X'' and Y'' are the inverse image of X' and Y' in Z, so by Proposition 4.7, it is equal to $s_{Z'_n/Z_n}$.

For the convenience of the readers, we sketch the proof of the compatibility of Künneth product and cup product. Now we have the commutative diagram

$$\begin{array}{ccc} Z_n = X_n^\times \times_{E_n} Y_n^\times & \xrightarrow{pr_2} & Y_n^\times \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle g} \\ X_n^\times & \xrightarrow{f} & E_n \end{array}$$

In the complex level, the Künneth morphism is given by the morphism

$$Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Rg_{\text{crys}*}(\mathcal{O}_{Y_n/E_n})$$

$$\to Rh_{\text{crys}*}(Lpr^*_{1\text{crys}}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Lpr^*_{2\text{crys}}(\mathcal{O}_{Y_n/E_n})),$$

where h is the canonical morphism from Z_n to E_n. By adjunction, it suffices to define

$$Lh^*_{\text{crys}}(Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Rg_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}))$$

$$\to Lpr^*_{1\text{crys}}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Lpr^*_{2\text{crys}}(\mathcal{O}_{Y_n/E_n}).$$

So the existence of such morphism comes from the canonical isomorphism

$$Lh^*_{\text{crys}}(Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Rg_{\text{crys}*}(\mathcal{O}_{Y_n/E_n}))$$

$$\simeq Lh^*_{\text{crys}}(Rpr_{1\text{crys}*}(\mathcal{O}_{X_n/E_n}))\otimes^L_{\mathcal{O}_{E_n}} Lh^*_{\text{crys}}(Rpr_{2\text{crys}*}(\mathcal{O}_{Y_n/E_n}))$$

and the canonical morphisms

$$Lh^*_{\text{crys}}(Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n}))\simeq Lp^*_{\text{crys}}(Lf^*_{\text{crys}}(Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})))\to Lp^*_{\text{crys}}(\mathcal{O}_{X_n/E_n}),$$

while the cup product is defined by the following morphism:

$$Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})\otimes^L_{\mathcal{O}_{E_n}} Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n})\to Rf_{\text{crys}*}(\mathcal{O}_{X_n/E_n}\otimes_{\mathcal{O}_{E_n}}\mathcal{O}_{X_n/E_n}),$$

which commutes with the above operations, hence the proof.

4.3 Another characterization of direct image

Now we turn back to the case $H^i(X_n^\times/E_n, \mathcal{O}_{X_n/E_n})$, where $X^\times$ is a log smooth scheme of Cartier type over $(V, \mathbf{N})$. As shown in the beginning, this cohomology has Poincaré duality. For $Y^\times$ another log smooth scheme of Cartier type over $(V, \mathbf{N})$ with exact closed immersion $i : Y^\times \hookrightarrow X^\times$ of codimension j, one can also define the direct image map

$$i_* : H^l(Y_n^\times/E_n, \mathcal{O}_{Y_n/E_n}) \to H^{l+2j}(X_n^\times/E_n, \mathcal{J}^{[j]}_{X_n/E_n})$$

as the adjoint dual of i^* with respect to Poincaré duality. In this section, we want to show the two definitions coincide. It suffices to show that:

Proposition 4.18 *([3] VII Proposition 2.3.1)*

Assme $X^\times$ and $Y^\times$ are vertical semistable schemes. Let m be the dimension of X_1, then we have the following diagram

$$\begin{array}{ccc} H^{2d-2j}(Y_n^\times/E_n, \mathcal{O}_{Y_n/E_n}) & \xrightarrow{G_{Y/X}} & H^{2d}(X_n^\times/E_n, \mathcal{J}^{[j]}_{X_n/E_n}) \\ & {\scriptstyle Tr_g}\searrow \quad \swarrow{\scriptstyle Tr_f} & \\ & E_n[-2d+2j] & \end{array}$$

Proof : Once this proposition is proved, then the claim follows from [3] VII Corollary 2.3.2, i.e., i_* is the adjoint of i^* with respect to Poincaré duality.

First we show that there is a commutative diagram

$$\begin{array}{ccc} H^{d-j}(Y_n^\times/E_n, \mathcal{H}^{d-j}_{Y_n^{d-j}}(\mathcal{O}_{Y_n/E_n})) & \xrightarrow{G'_{Y/X}} & H^d(X_n^\times/E_n, \mathcal{H}^d_{X_n^d}(\mathcal{O}_{X_n/E_n})) \\ \downarrow & & \downarrow \\ H^{2d-2j}(Y_n^\times/E_n, \mathcal{O}_{Y_n/E_n}) & \xrightarrow{G_{Y/X}} & H^{2d}(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \end{array}$$

By [27] Proposition 7.6, $\mathcal{H}^*_{Y_n^{d-j}}(\mathcal{O}_{Y_n/E_n})$ is a crystal of $\mathcal{O}_{Y_n/E_n}$ module. On the other hand, by the definition of $i_{\mathrm{crys}*}$ (resp. $i_{\mathrm{Rcrys}*}$), it commutes with $\Gamma_{Y_n^{d-j}}$. So $R\Gamma_{Y_n^{d-j}}(i_{\mathrm{Rcrys}*}(\mathcal{O}_{Y_n/E_n})) \simeq i_{\mathrm{Rcrys}*}(\mathcal{H}^{d-j}_{Y_n^{d-j}}(\mathcal{O}_{Y_n/E_n}))[-d+j]$. Then $G'_{Y/X}$ is defined as applying the fucntor $R\Gamma_{Y_n^{d-j}}$ on the Gysin map $i_{\mathrm{Rcrys}*}(\mathcal{O}_{Y_n/E_n})[-2j] \to \mathcal{O}_{X_n/E_n}$, and compose with the natural map $\mathcal{H}^m_{Y_n^{d-j}}(\mathcal{O}_{X_n/E_n}) \hookrightarrow \mathcal{H}^d_{X_n^d}(\mathcal{O}_{X_n/E_n})$.

The vertical lines are surjective (recall how we construct Cousin complex of $\mathcal{O}_{X_n/E_n}$ in Proposition 2.2) And since it suffices to show it locally, for $y \in Y$, that means it suffices to show that the following diagram is commutative:

$$\begin{array}{ccc} H^{d-j}(Y_n^\times/E_n, \mathcal{H}^{d-j}_y(\mathcal{O}_{Y_n/E_n})) & \to & H^d(X_n^\times/E_n, \mathcal{H}^{d-j}_y(\mathcal{O}_{X_n/E_n})) \\ & {\scriptstyle \mathrm{Res}_{g,y}}\searrow \quad \swarrow{\scriptstyle \mathrm{Res}_{f,y}} & \\ & E_n & \end{array}$$

Since the problem is local, we assume there exists smooth lifting $X_n'^{\times}, Y_n'^{\times}$ of $X^{\times}, Y^{\times}$ over E_n (with canonical map f', g') and $i' : Y_n'^{\times} \hookrightarrow X_n'^{\times}$ an exact regular closed immersion extending i, defined by a regular sequennce $t_1, \cdots, t_j$, and $y' \in \mathcal{O}_{Y_n'}$ is a point lying over y so the above diagram is equivalent to the following one

$$\begin{array}{ccc} H^{d-j}_{y'}(\omega^{d-j}_{Y_n'/E_n}) & \longrightarrow & H^{d}_{y'}(\omega^{j}_{X_n'/E_n}) \\ & {\scriptstyle \mathrm{Tr}_{g',y'}} \searrow \quad \swarrow {\scriptstyle \mathrm{Tr}_{f',y'}} & \\ & E_n & \end{array}$$

Working on the Zariski site, assume $t_{j+1}, \ldots, t_d$ be a regular sequence generating the maximal ideal of $\mathcal{O}_{Y_n', y'}$, recall

$$H^{d-j}_{y'}(\omega^{d-j}_{Y_n'/E_n}) \simeq \varinjlim \omega^{d-j}_{Y_n'/E_n, y}/t^k_{j+1} \cdots t^k_d,$$

then

$$G'_{Y'/X'}(\omega/t^k_{j+1} \cdots t^k_d) = \frac{dt_1 \wedge \cdots \wedge dt_j \wedge \omega'}{t_1 \cdots t_j \cdot t^k_{j+1} \cdots t^k_d},$$

where ω' is any lifting of $\omega \in \omega^{j}_{X_n'/E_n}$. On the other hand, using the trick as in Lemma 2.2, since we work Zariski locally, we may assume we have a log smooth lifting of X_n' on $(E_n[t], \mathbf{N})$ (where the log structure is given by $1 \to t$ and the map $E_n[t] \to E_n$ is given by $t \to u$), denoted $X''^{\times}$ as the log scheme such that the underlying scheme is smooth over E_n (with trivial log structure) through the naive map $E_n \to E_n[t]$. We may assume there is a lifting $Y''^{\times}$ of $Y'^{\times}$ which is log smooth over $(E_n[t], \mathbf{N})$ such that the underlying scheme is smooth over E_n and we have an exact closed immersion $i'' : Y''^{\times} \hookrightarrow X''^{\times}$ (and let $y'' \in Y_n''$ a point lying over y'). So we go back to classical case ([3] VII Lemma 1.2.5); we have the following commutative diagram:

$$\begin{array}{ccc} H^{d-j+1}_{y''}(\Omega^{d-j+1}_{Y_n''/E_n}) & \longrightarrow & H^{d+1}_{y''}(\Omega^{d+1}_{X_n''/E_n}) \\ & {\scriptstyle \mathrm{Tr}_{g'',y''}} \searrow \quad \swarrow {\scriptstyle \mathrm{Tr}_{f'',y''}} & \\ & E_n & \end{array}$$

But, again in the proof of Poincaré duality, we get a map

$$\mathrm{Tr}_{s,y'} : H^d_{y'}(\omega^d_{X_n'/E_n}) \to H^{d+1}_{y''}(\Omega^{d+1}_{X_n''/E_n}), \quad \frac{\omega}{t_{j+1} \ldots t_d} \to \frac{dt \wedge \omega''}{(t-a) \cdot t_{j+1} \ldots t_d},$$

where we denote $s : X_n' \hookrightarrow X_n''$ the corresponding closed immersion. By the transitivity of trace map, we have $\mathrm{Tr}_{s,y'} \circ \mathrm{Tr}_{f'',y''} = \mathrm{Tr}_{f',y'}$; similar results hold on Y_n side, hence the compatibility.

CHAPTER 5

GROTHENDIECK-RIEMANN-ROCH THEOREM AND UNIQUENESS CRITIERION

Again, since log crystalline cohomology is invariant under log blow-up, we may assume X is vertical semistable. Here we follow [15] II to give a proof about Grothendieck-Riemann-Roch theorem without denominators.

We divide into 2 cases :

(1) $\mathcal{N}$ a locally free sheaf on Y_n, X_n is $\mathbf{P}(\mathcal{N} \oplus \mathcal{O}_{Y_n})$ and i is the inclusion of Y_n as the zero section. In this case, we will show that $c_j(i_*(\mathcal{O}_{Y_n})) = (-1)^{j-1}(j-1)! i_*(1_{Y_n})$. Once this is proved, by [15], Corollary 1.2.8, using projection formula in K-theory, we will have the same formula in case (2) below.

(2) i is an arbitrary closed immersion. In this case, we want to show that, for $\xi \in F_\gamma^q K^0(Y)$,

$$\text{If } \ q > 0, \ \ c_{q+j}(i_*(\xi)) = (-1)^d \frac{(q+j-1)!}{(q-1)!} i_*(c_q \xi),$$
$$\text{If } \ q = 0, \ \ c_j(i_*(\xi)) = (-1)^{j-1}(j-1)! \epsilon(\xi) l_*(1_{Y_n}).$$

For the definition of i, see case 1 and case 2 below, and $\epsilon(\xi)$ means the rank of ξ.

(Case 1) Denote $\pi : X_n \to Y_n$ the projection. We have the standard exact sequence

$$0 \to \mathcal{H} \to \pi^* N \oplus \mathcal{O}_{X_n} \to \mathcal{O}_{X_n}(1) \to 0. \tag{5.1}$$

Over $\mathbb{V}(N)$ the composite map $H \to \pi^* N \oplus \mathcal{O}_{X_n} \to \pi^* N$ is an isomorphism. So the zero section $i(Y_n)$ defined by the ideal which is the image of the composite map $H \to \pi^* N \oplus \mathcal{O}_{X_n} \to \mathcal{O}_{X_n}$. So $i_*(\mathcal{O}_{Y_n})$ has Koszul resolution

$$0 \to \Lambda^d H \to \cdots \to H \to \mathcal{O}_{X_n} \to i_* \mathcal{O}_{Y_n} \to 0. \tag{5.2}$$

From (5.1) we have $c(H)(1+\zeta) = \pi^* c(N)$ (where $\zeta = c_1^{\text{syn}}(\mathcal{O}_{X_n}(1))$) by the additivity of Chern class. So we have

$$c_j(H) = (-1)^j c_0(N)\zeta^j + (-1)^{j-1} c_1(N)\zeta^{j-1} + \cdots + (-1)c_{j-1}(N)\zeta + c_j(N)$$

(Instead of writing $\pi^* c_i(N)$, we just write $c_i(N)$ for simplicity). By the additivity of Chern classes again, from (5.2) we can rewrite the right-hand side as

$$c_j(i_*\mathcal{O}_{Y_n}) = -(j-1)!\Sigma_{s=0}^{j}(-1)^{j-s}c_s(N)\zeta^{j-s}. \tag{5.3}$$

By projective bundle formula of crystalline Chern class, in $H^{2*}(X_n^{\times}/E_n, \mathcal{J}_{X_n/E_n}^{[*]})$ we can write

$$i_*(1_{Y_n}) = \Sigma_{s=0}^{j}\alpha_{j-s}\zeta^s.$$

By (5.3) in order to get desired equality, we need to show

$$\alpha_{j-s} = (-1)^{j-s}c_{j-s}. \tag{5.4}$$

We will prove this by induction on $l = j - s$. Since $1_{Y_n} = \pi_* i_*(1_{Y_n}) = \alpha_0 \pi_*(\zeta^d)$, now we claim $\pi_*(\zeta^d) = 1_Y$, so $\alpha_0\pi_*(\zeta^d)$ equal to $\alpha_0 \cdot 1_{Y_n}$. Observe that we have the following commutative diagram:

$$Y_n \overset{i}{\hookrightarrow} X_n \overset{\pi}{\to} Y_n,$$

where $\pi \circ i = 1_Y$. Since locally i is the map $X_n \hookrightarrow \mathbf{P}^d \times_V X_n$ crystalline cycle class can be defined locally (Theorem 4.14), we can see that $i_*(1_{Y_n}) = \zeta^d$, so $\pi_*(\zeta^d) = \pi_*(i_*(1_{Y_n})) = 1_{Y_n}$, hence the result.

So $\alpha_0 = 1_{Y_n}$. Assume $\alpha_l = (-1)^l c_l(N)$ for all $l < k$, where $k > 0$. Since $c_j(H)\zeta^l = 0$ by dimension reason, we have

$$\zeta^{j+l} - c_1(N)\zeta^{j+k-1} + \cdots + (-1)^{j-1}c_{j-1}(N)\zeta^{k+1} + (-1)^j c_j(N)\zeta^k = 0.$$

So $i_*(1_{Y_n})\zeta^k = i_*(1_{Y_n})\zeta^l - c_d(H)\zeta^k$, which is equal to

$$i_*(1_{Y_n})\zeta^k = \Sigma_{i=0}^{j}(\alpha_l - (-1)^l c_l(N))\zeta^{j+k-l}.$$

Again, applying π_*, by the induction hypothesis and projection formula, we have

$$\pi_*(i_*(1_{Y_n})\zeta^k) = \alpha_k - (-1)^k c_l(N).$$

But the left-hand side is equal to $\pi_* i_*(i^*(\zeta^k)) = i^*(\zeta^k) = [(c_1(i^*\mathcal{O}_{X_n}(1)))^k] = [(c_1(\mathcal{O}_{Y_n}))^k] = 0$, hence the result.

Now we turn to the general case. For $\xi \in \mathrm{Fil}_\gamma^j K^0(Y_n)$, first we assume $q \geq 1$. By the projection formula of crystalline cohomology, we have $i_*(c_q(\xi)) = i_* i^* \pi^*(c_q(\xi)) = \pi^*(c_q(\xi))i_*(1_{Y_n}) = c_q(\pi^*(\xi))i_*(1_{Y_n})$, so we have

$$\begin{aligned}(-1)^j\frac{(q+j-1)!}{(q-1)!}i_*(c_q(\xi)) &= -\frac{(q+j-1)!}{(q-1)!(d-1)!}c_q(\pi^*(\xi)) \cdot (-1)^{j-1}(d-1)!i_*(1_{Y_n}) \\ &= -\frac{(q+j-1)!}{(q-1)!(d-1)!}c_q(\pi^*(\xi)) \cdot c_j(i_*(O)_{Y_n})\end{aligned}$$

by the above results. By the projection formula for K-theory, we have $c_{q+j}(i_*(\xi)) = c_{q+j}(\pi^*(\xi) \cdot i_*(O)_{Y_n})$. Notice that $c_l(\pi^*(\xi)) = 0$ for $1 \leq l \leq q-1$ since $\pi^*\xi \in \mathrm{Fil}_\gamma^j K^0(X_n)$.

On the other hand, by the Koszul resolution of $i_*(\mathcal{O}_{Y_n})$, we also know $c_l(i_*(\mathcal{O}_{Y_n})) = 0$ for $1 \leq l \leq j-1$, combined with these two results, since the total Chern class map is a homomorphism of λ-rings, we have

$$\begin{aligned} c_{q+j}(\pi^*(\xi)\cdot i_*(\mathcal{O}_{Y_n})) &= -\frac{(q+j-1)!}{(q-1)!(d-1)!}c_q(\pi^*(\xi))\cdot c_j(i_*(\mathcal{O}_{Y_n})) \\ &= (-1)^j\frac{(q+j-1)!}{(q-1)!}i_*(c_q(\xi)), \end{aligned}$$

hence the result. For $q=0$, write $\xi = \epsilon(\xi)\cdot \mathcal{O}_{Y_n} + \xi'$, where $\xi' \in \mathrm{Fil}^1_\gamma K^0(Y_n)$, the same reason as above, we have $c_j(i_*(\xi)) = c_j(\epsilon(\xi)i_*(\mathcal{O}_{Y_n})) = \epsilon(\xi)c_j(i_*(\mathcal{O}_{Y_n}))$, so it follows from the case $\xi = \mathcal{O}_{Y_n}$.

(Case 2) Blow up $X_n \times \mathbf{P}^1$ along $Y_n \times \infty$ to obtain a scheme W. We have

(1) W is flat over $\mathbf{P}^1$.

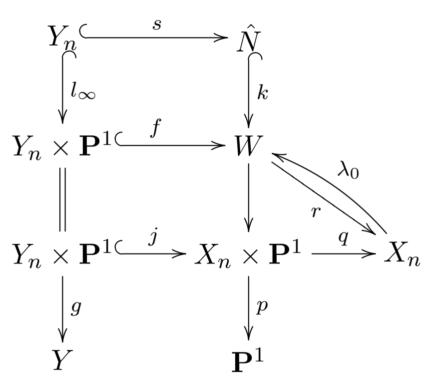
(2) The fiber W_t is, for $t \neq \infty$, isomorphic to X_n and $W_\infty = \hat{N} \cup \tilde{X}_n$ where $\hat{N} = \mathbf{P}(N \oplus \mathcal{O}_{Y_n})$, $N = \mathcal{I}/\mathcal{I}^2$, and $\tilde{X}$ is the blow up of X_n along Y_n. Both scheme theoretically and set theoretically $\hat{N} \cap \tilde{X}_n = \mathbf{P}(N)$, hence $\hat{N}$ and $\tilde{X}_n$ meet transversely.

(3) As divisors on W, $W_\infty = \hat{N} + \tilde{X}_n$. So W_0 is linearly equivalent to $\hat{N} + \tilde{X}_n$.

(4) $W' = W - \tilde{X}_n$ is vertical semistable over $\mathbf{P}^1$ (i.e., Zariski locally the spectrum of W' is isomorphic to $A[t_1,\dots,t_s]/t_1\dots t_u - \pi$, where A is the spectrum of the affine open subset of $\mathbf{P}^1$).

Notice that the above properties hold for smooth case (where W' is smooth over $\mathbf{P}^1$ in that case), since the assertion is local, étale locally the coordinate ring of X is isomorphic to $V_n[x_1,\dots,x_d]/x_1\cdots x_s - \pi$, so we can first check the assertion on $V_n[x_1,\dots,x_d]$, and then base change to $V_n[x_1,\dots,x_d]/x_1\cdots x_s - \pi$; in this case, the blow up procedure commutes with this base change since the center of blow-up are not zero divisors of $V_n[x_1,\dots,x_d]/x_1\cdots x_s - \pi$.

Consider the commutative diagram :



where λ_0 identifies X with W and thus $r \circ \lambda_0 = id_X$. The upper square, in which s is the inclusion of Y as the zero section of $\hat{N}$, is Cartesian.

Denote $\xi \in K^0(Y_n)$. We wish to compute $c_j(i_*(\xi))$ in terms of $i_*(c_j(\xi))$.

$$\textbf{Claim} : i_*(\xi) = \lambda_0^* f_* g^*(\xi). \tag{5.5}$$

Proof : Consider the diagram

$$\begin{array}{ccc} Y_n \times \mathbf{P}^1 & \xhookrightarrow{\ f\ } & W \\ l_0 \big\uparrow & & \big\uparrow \lambda_0 \\ Y_n & \xhookrightarrow{\ i\ } & X_n \end{array}$$

Since f and λ_0 are transverse, by [4] IV 3.1.1, i.e., the corresponding theorem of Proposition 4.15 in K-theory, $\lambda_0^* f_* = i_* l_0^*$, since $g \circ l_0 = id_{Y_n}$, hence the claim.

So we have

$$c_j(i_*(\xi)) = c_j(\lambda_0^* f_* g^*(\xi)) = \lambda_0^* c_j(f_* g^*(\xi)). \tag{5.6}$$

On the other hand, let $t \in H^*(X_n^\times/E_n, \mathcal{O}_{X_n/E_n})$. Then $\lambda_0^*(t) = r_* \lambda_{0*} \lambda_0^*(t) = r_*[t \cdot \lambda_{0*}(1_X)]$ by projection formula (in log crystalline cohomology). Combined with the above result, we have

$$c_j(i_*(\xi)) = r_*(c_j(f_* g^*(\xi)) \cdot \lambda_{0*}(1_X)). \tag{5.7}$$

Consider the Cartesian diagram

$$\begin{array}{ccccc} X_n & \xrightarrow{\ \sim\ } & W_0 & \hookrightarrow & W \\ & & \big\downarrow & & \big\downarrow w \\ & & \{0\} & \xrightarrow{\ \text{cl}\ } & \mathbf{P}^1 \end{array} \qquad (\lambda_0 : X_n \to W) \tag{5.8}$$

By Proposition 4.15. we have $\lambda_{0*}(1_{X_n}) = w^*(cl_*(\{0\}))$. For $\mathbf{P}^1$, the de Rham cycle class and Chern class coincide (the result comes from the local characterization of de Rham cycle class). As $w^*(\mathcal{O}_{\{0\}}) = \mathcal{O}_{W_0}$, we find $\lambda_{0*}(1_{X_n}) = c_1(\mathcal{O}_{W_0})$. Since W_0 is linearly equivalent to $\hat{N} + \hat{Y}_n$, we have

$$c_1(\mathcal{O}_{W_0}) = c_1(\mathcal{O}_{\hat{N}}) + c_1(\mathcal{O}_{\tilde{X}_n}). \tag{5.9}$$

$$\textbf{Claim} : c_j(f_* g^*(\xi)) \cdot c_1(\mathcal{O}_{\tilde{X}_n}) = 0. \tag{5.10}$$

Proof : Since

$$c_j(f_* g^*(\xi)) \in \mathrm{Im}[H^{2j}_{Y_n \times \mathbf{P}^1}(W^\times/E_n, \mathcal{J}^{[j]}_{W/E_n}) \to H^{2j}(W/E_n, \mathcal{J}^{[j]}_{W/E_n})]$$

(For the theory of log-crystalline cohomology with support is exactly parallel with that of crystalline cohomology, see [27] Section 6) and

$$c_1(\mathcal{O}_{\tilde{X}_n}) \in \mathrm{Im}[H^2_{\tilde{X}_n}(W^\times/E_n, \mathcal{J}^{[1]}_{W/E_n}) \to H^2(W^\times/E_n, \mathcal{J}^{[1]}_{W/E_n})].$$

So the product is supported on $Y_n \times \mathbf{P}^1 \cap \tilde{X}_n = \emptyset$, hence the result.

In sum, we have

$$c_j(i_*(\xi)) = r_*[c_j(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}})]. \tag{5.11}$$

$$\textbf{Claim} : c_j(f_*g^*(\xi)) \cdot c_1(\mathcal{O}_{\hat{N}}) = k_*k^*c_j(f_*g^*(\xi)). \tag{5.12}$$

Proof : Take $a \in H^{2j}_{Y_n \times \mathbf{P}^1}(W^\times/E_n, \mathcal{J}^{[j]}_{W/E_n})$ such that its image in $H^{2j}(W^\times/E_n, \mathcal{J}^{[j]}_{W/E_n})$ is $c_j(f_*g^*(\xi))$. Since W' contains $Y_n \times \mathbf{P}^1$, we have

$$H^{2j}_{Y_n \times \mathbf{P}^1}(W^\times/E_n, \mathcal{O}_{W/E_n}) \simeq H^{2j}_{Y_n \times \mathbf{P}^1}(W'^\times/E_n, \mathcal{O}_{W'/E_n}),$$

and $\hat{N} \cap Y_n \times \mathbf{P}^1 = \mathbb{V}(N)$, we can identify $a \cdot c_1(\mathcal{O}_{\hat{N}})$ with $a \cdot c_1(\mathcal{O}_{\mathbb{V}(N)})$ under the above isomorphism. Consider the Cartesian diagram

$$\begin{array}{ccc} \mathbb{V}(N) & \xrightarrow{k'} & W' \\ \downarrow & & \downarrow \\ X_n \times \{\infty\} & \xrightarrow{\mu_\infty} & X_n \times \mathbf{P}^1 \end{array} \tag{5.13}$$

Since the cycle class $\mu_{\infty *}(1_{X_n \times \{\infty\}})$ is induced from $\infty \hookrightarrow \mathbf{P}^1$, in this case, we know it is compatible with de Rham/crystalline Chern class, so we have $\mu_{\infty *}(1_{X_n \times \{\infty\}}) = c_1(\mathcal{O}_{X_n \times \{\infty\}})$. So by transversality of de Rham cycle class, we have $k'_*(1_{\mathbb{V}(N)}) = c_1(\mathcal{O}_{\mathbb{V}(N)})$. But $k_*(1_{\hat{N}})$ restricts to $k'_*(1_{\mathbb{V}(N)})$, so $a \cdot c_1(\mathcal{O}_{\hat{N}}) = a \cdot c_1(\mathcal{O}_{\mathbb{V}(N)})$ in $H^{2j}_{Y \times \mathbf{P}^1}(W^\times/E_n, \mathcal{J}^{[j]}_{W/E_n})$. So $c_j(f_*g^*(\xi)) \cdot c_1(O_{\hat{N}}) = c_j(f_*g^*(\xi)) \cdot k_*(1_{\hat{N}}) = k_*k^*(c_j(f_*g^*(\xi)))$ by the projection formula, hence the result.

Moreover, since f and k are transverse, we have $s_*l^*_\infty = k^*f_*$ (in $K_0(\hat{N})$). Combined with (5.11) and (5.12) we have

$$c_j(i_*(\xi)) = r_*k_*c_j(s_*(\xi)). \tag{5.14}$$

Finally, combined with the fact that $r \circ k \circ s = i$, but for i_* we can use the result from Case 1 and get the desired result. (The argument also shows that $c_l(i_*(\xi)) = 0$ for $l < j$, then replaced $c_j(\xi)$ by $c_l(\xi)$ in the above argument).

Now Following [15] II 1.4, here we explain how to derived a form in the proof of Proposition 3.6 from the information we have.

Let notation be as the above, denoted $p : \hat{N} = \mathbf{P}(N \oplus \mathcal{O}_{Y_n}) \to Y_n$ be the projection map, and then for a vector bundle F on Y we set $P(F, N) = p_*(\prod c(\bigwedge^i H \otimes p^*F)^{(-1)^i})$. First we observe that by Koszul resolution in (Case 1) we have

$$c(i_*F) = i_*(P(F, N)).$$

Notice that we can use splitting principle to check $P(F, N)$ is divisible by $c_j(H)$, by the relation in (Case 1) (5.1), we see $i_*c_j(H) = [\mathcal{O}_{Y_n}]$, hence the degree $l - j$ part on the right-hand side corresponds to $c_l(F)$; for the general case, in (Case 2) (5.14), working on total Chern class, applying the result above, we get the desired result.

5.1 Uniqueness of p-adic period morphism

For the rest of the article, we try to compare the construction here with other people; as mentioned in the beginning there are comparison isomorphisms constructed by Beilinson-Bhatt, Breuil-Tsuji, and Faltings; now we recall the uniqueness criteria of [25], Theorem 3.1:

Theorem 5.1 *Let the assumption be the same as in Theorem 3.4, there is a unique morphism (up to inverting the Bott element)*

$$Fil^b(H^a(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{crys,n}) \to H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes Fil^b(A_{crys,n})$$

making the following diagram commutes :

$$\begin{array}{ccc} F^b_\gamma K_a(X_{\bar{V}}; \mathbf{Z}/p^n) & \xrightarrow{\simeq} & F^b_\gamma K_a(X_{\bar{K}}; \mathbf{Z}/p^n) \\ \downarrow \bar{c}^{syn}_{b,2b-a} & & \downarrow \bar{c}^{\acute{e}t}_{b,2b-a} \\ Fil^b(H^a(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{crys,n}) & \longrightarrow & H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes Fil^b(A_{crys,n}) \end{array}$$

Proof : By the construction of Chapter 3, the bottom horizontal map is exactly the inverse map of the comparison map we constructed in Chapter 3.

So in order to show other comparison morphisms are inverse to ours, it suffices to show that they fit into the above diagram, by splitting principle, it suffices to show that they sends zeroth and first crystalline (syntomic) Chern class to zeroth and first étale Chern class.

For the first one, keep the hypothesis of Theorem 3.4 and now assume V is unramified extension of $\mathbf{Z}_p$, in [8] 3.2.4.4, Breuil showed that the natural map

$$H^a(X_{n,\bar{V}}, \mathcal{S}_n(b)_{X^\times_{\bar{V}}}) \to (H^a(X^\times_{n,\bar{V}}/E_n, , \mathcal{O}_{X_n/E_n}) \otimes_{E_n} \widehat{A_{\mathrm{st},n}})^{N=0,\phi_b=1}$$

is an isomorphism. Then use Tsuji's result [28] Theorem 5.1, the source is isomorphic to $H^a_{\acute{e}t}(X_{\bar{K}}, \mathbf{Z}/p^n(b))$; so that means it suffices to show that Tsuji's construction sends syntomic Chern class to étale Chern classes; for Tsuji's construction, since in [26] Proposition 3.2.4 (3) and [28] Proposition 2.11, Tsuji also showed that his construction maps first syntomic Chern class to first étale Chern class, and by the construction of [26] 2.2, the construction of syntomic Chern class is compatible with crystalline Chern class, and in the line bundle case crystalline Chern class can be defined using Gysin map, so by the Grothendieck-Riemann-Roch formula the construction coincides the K-theoretic definition here, hence the result.

For Faltings and Bhatt, using different ways unconditionally they both constructs

$$H^a(X_n^\times/E_n, \mathcal{O}_{X_n/E_n}) \otimes_{E_n} A_{\mathrm{crys},n} \to H^a_{\text{ét}}(X_{\bar{K}}, \mathbf{Z}/p^n) \otimes A_{\mathrm{crys},n}$$

(again, the Galois action on log crystalline cohomology needed to be twisted as we did in Section 3) which admits an inverse up to β^d. Their construction also maps crystalline Chern class to étale Chern class ([1] p.47, [11] p.252), hence are the same, i.e., the inverse map of the K-theory map.

Notice that after taking inverse limit and tensoring $\mathbf{Q}$, the construction here goes back to Nizioł's construction in [24], now we try to compare it with Beilinson's construction, which is also rational. Finally we notice that in [2] it is already shown it sends de Rham cycle class to crystalline cycle class ([3] p.16), so it suffices to show that Beilinsion's map sends crystalline Chern class to de Rham cycle class, and the result comes from [6] since the inclusion of $\mathcal{A}_{\mathrm{crys}} \hookrightarrow \mathcal{A}_{\mathrm{dR}}$ is the natural inclusion. In sum, we have the following:

Theorem 5.2 *The same hypothesis as the above, the p-adic period isomorphism constructed by Beilinson [1] and Nizioł [24] are inverse to each other.*

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