ANALYSIS OF SPATIAL PARRONDO GAMES WITH SPATIALLY DEPENDENT GAME *A*

by

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STATEMENT OF DISSERTATION APPROVAL

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ABSTRACT

Parrondo games with spatial dependence have been studied by Ethier and Lee. More precisely, they studied Toral's Parrondo games with N players arranged in a circle. The players play either game A or game B. In game A, a randomly chosen player wins or loses one unit according to the toss of a fair coin. In game B, which depends on parameters $p_0, p_1, p_2, p_3 \in [0,1]$, a randomly chosen player wins or loses one unit according to the toss of a p_m-coin, where $m \in \{0,1,2,3\}$ depends on the winning or losing status of the player's two nearest neighbors. In this dissertation, we study a spatially dependent game A, which we call game A', introduced by Xie and others and considered by Ethier and Lee. Noting that game A' is fair, we say that the Parrondo effect occurs if game B is losing or fair and the random mixture $C' := \gamma A' + (1-\gamma)B$ [respectively, the nonrandom periodic pattern $C' := (A')^r B^s$] is winning. With $p_1 = p_2$ and the parameter space being the unit cube, we investigate numerically the region in which the Parrondo effect appears. We give sufficient conditions for the ergodicity of an interacting particle system in $\{0,1\}^{\mathbf{Z}}$ corresponding to the random mixture $C' := \gamma A' + (1-\gamma)B$ by applying a theorem of Liggett, and also by means of "annihilating duality". We also show that $\lim_{N\to\infty} \mu^N_{(\gamma,1-\gamma)'}$ and $\lim_{N\to\infty} \mu^N_{[r,s]'}$ exist under certain conditions, where $\mu^N_{(\gamma,1-\gamma)'}$ denotes the mean profit per turn at equilibrium to the N players playing the random mixture $C' := \gamma A' + (1-\gamma)B$, and $\mu^N_{[r,s]'}$ denotes the mean profit per turn at equilibrium to the N players playing the nonrandom periodic pattern $C' := (A')^r B^s$.

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I must express my gratitude to my family Myoung Bun, Youlmin, and Youlhee, for their continued love and understanding. I am indebted to my mother, Seunghee Chang and sisters, Sungeun and Sungsil for supporting me and being patient. Also, this dissertation is dedicated to the memory of my beloved father, Deokjoon Choi.

CHAPTER 1

INTRODUCTION

The Parrondo effect, in which there is a reversal in direction in some system parameter when two similar dynamics are combined, is the result of an underlying nonlinearity. It was first described by Spanish physicist J. M. R. Parrondo in 1996 in the context of games of chance: He showed that it is possible to combine two losing games to produce a winning one. His motivation was to provide a simplified model of the so-called flashing Brownian ratchet of Ajdari and Prost [1]. Other versions of Parrondo's games followed, including Toral's [19] spatially dependent games. These games were modified by Xie et al. [21], and it is the goal of this dissertation to explore the latter games in greater depth than was done by Ethier and Lee [10].

1.1 Parrondo's original capital-dependent games (1996)

The original capital-dependent games of Parrondo were motivated by the flashing Brownian ratchet of Ajdari and Prost [1]. This object is well explained in a figure (and caption) from Parrondo and Dinis [2]; see Figure 1.1. (An earlier version of this figure is in Faucheux and others [12].)

Parrondo's idea was to discretize space and time in the flashing Brownian ratchet, replacing continuous-time Markov processes by discrete-time Markov chains, which could be interpreted as cumulative profit in a sequence of games of chance. His games can be described as follows.

First, we define a p-coin to be a coin whose probability of heads is p. Let $p_0 := \frac{1}{10}$ and $p_1 := \frac{3}{4}$. In game A, the player tosses a $\frac{1}{2}$-coin. The rules of game B are more complicated. In game B, if the player's current cumulative capital is a multiple of 3, a p_0-coin is tossed, otherwise a p_1-coin is tossed. So game B is capital-dependent. In both games, the player wins one unit with heads and loses one unit with tails. Figure 1.2 explains these rules via a diagram.

The player's cumulative profit from game A behaves as a simple symmetric random walk in $\mathbf{Z}$. The player's cumulative profit from game B behaves as an asymmetric random walk in $\mathbf{Z}$ with state-dependent probabilities. These probabilities were chosen to make the game asymptotically fair. Nevertheless, the random mixture $C := \frac{1}{2}A + \frac{1}{2}B$ (toss a fair coin to decide which game to play, A or B) is a winning game. (See Figure 1.3.) Moreover, repeated periodic patterns such as ABB, AAB, and $AABB$ are winning as well (and these patterns are analogous to the flashing Brownian ratchet). The only exception is the pattern AB, which is fair. (See Figure 1.4.)

Other forms of Parrondo's games have been introduced, such as history-dependent games (Parrondo, Harmer, and Abbott [18]) and multiplayer games (e.g., Dinis and Parrondo [17]; Toral, [19], [20]), but it is Toral's [19] spatially dependent games that we want to focus on.

1.2 Toral's (2001) spatially dependent games

Toral [19] introduced what he called *cooperative* Parrondo games with spatial dependence. (We prefer the term *spatially dependent* Parrondo games so as to avoid conflict with the field of cooperative game theory.) The games depend on an integer parameter $N \geq 3$, the number of players, and four probability parameters, p_0, p_1, p_2, p_3. The players are arranged in a circle and labeled from 1 to N (so that players 1 and N are nearest neighbors). At each turn, a player is chosen at random to play. Suppose player x is chosen. In game A, he tosses a fair coin. In game B, he tosses a p_m-coin (i.e., a coin whose probability of heads is p_m), where $m \in \{0, 1, 2, 3\}$ depends on the winning or losing status of his two nearest neighbors. A player's status as winner (1) or loser (0) is decided by the result of his most recent game. Specifically,

$$m = \begin{cases} 0 & \text{if } x-1 \text{ and } x+1 \text{ are both losers,} \\ 1 & \text{if } x-1 \text{ is a loser and } x+1 \text{ is a winner,} \\ 2 & \text{if } x-1 \text{ is a winner and } x+1 \text{ is a loser,} \\ 3 & \text{if } x-1 \text{ and } x+1 \text{ are both winners,} \end{cases}$$

where $N+1 := 1$ and $0 := N$ because of the circular arrangement of players. Player x wins one unit with heads and loses one unit with tails. See Figure 1.5 for clarification.

These games have been studied in detail in a series of papers by Ethier and Lee ([6], [7], [8], [9]). For example, with Toral's [19] choice of parameters, namely $(p_0, p_1, p_2, p_3) =$

$(1, 0.16, 0.16, 0.7)$, we can compute the asymptotic profit per turn to the set of N players, for $3 \leq N \leq 19$. See Table 1.1. In most cases the Parrondo effect (two fair or losing games combine to win) is present. In the cited papers, a strong law of large numbers and a central limit theorem are obtained. In particular, the asymptotic cumulative profits per turn exist and are the means in the SLLN (see Table 1.1). Further, it seems clear that these means converges as $N \to \infty$. This has been proved under certain conditions (see Ethier and Lee [8]).

1.3 The spatially dependent games of Xie and Others (2011)

Notice that Toral's [19] game A is not spatially dependent (the coin tossed does not depend on the status of the nearest neighbors). Xie and others. [21] proposed a modification of game A that *is* spatially dependent as well as being a fair game. To distinguish, we call that game A'. The games depend on an integer parameter $N \geq 3$, the number of players, and four probability parameters, p_0, p_1, p_2, p_3. The players are arranged in a circle and labeled from 1 to N (so that players 1 and N are nearest neighbors). At each turn, a player is chosen at random to play. Suppose player x is chosen. In game A', he chooses one of his two nearest neighbors at random and competes with that neighbor by tossing a fair coin. The results is a transfer of one unit from one of the players to the other, hence the wealth of the set of N players is unchanged. In game B, he tosses a p_m-coin (i.e., a coin whose probability of heads is p_m), where m depends on the status of his nearest neighbors. A player's status as winner (1) or loser (0) is decided by the result of his most recent game. Specifically,

$$m = \begin{cases} 0 & \text{if } x-1 \text{ and } x+1 \text{ are both losers,} \\ 1 & \text{if } x-1 \text{ is a loser and } x+1 \text{ is a winner,} \\ 2 & \text{if } x-1 \text{ is a winner and } x+1 \text{ is a loser,} \\ 3 & \text{if } x-1 \text{ and } x+1 \text{ are both winners,} \end{cases}$$

where $N+1 := 1$ and $0 := N$ because of the circular arrangement of players. Player x wins one unit with heads and loses one unit with tails. See Figure 1.6 for clarification.

These games were studied by Xie and others [21] and Ethier and Lee [10]. Only the random mixture case was treated, and convergence of the means has not yet been addressed. Our aim in this thesis is to fill in these gaps in the literature. Further, we want to understand this model as well as Toral's model is understood.

We begin by establishing a strong law of large numbers and a central limit theorem, especially in the periodic pattern case, in Chapter 2. In Chapter 3, we compute various means numerically and use computer graphics to visualize the Parrondo region. Then we address the issue of convergence of means, which involves certain interacting particle systems. We establish ergodicity of the interacting particle systems in Chapter 4 under certain conditions, and we extend this in Chapter 5 using "annihilating duality." Chapter 6 then proves the convergence, both in the random mixture setting and in the periodic pattern setting.

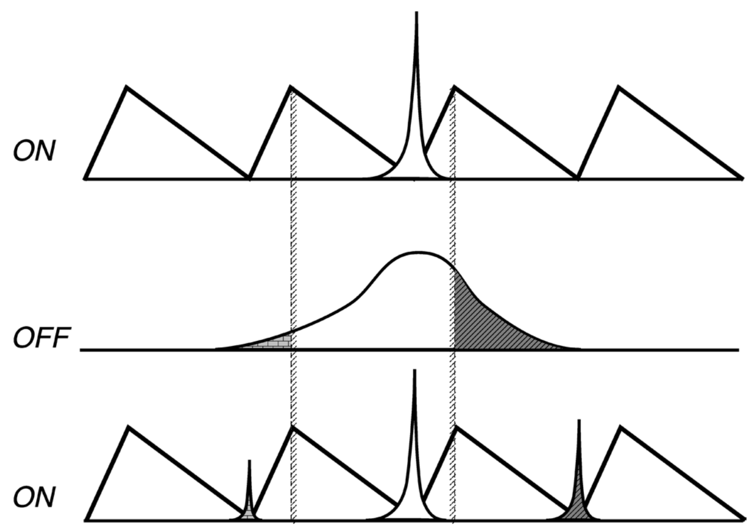

Figure 1.1. The flashing ratchet at work. The figure represents three snapshots of the potential and the density of particles. Initially (upper figure), the potential is on and all the particles are located around one of the minima of the potential. Then the potential is switched off and the particles diffuse freely, as shown in the centred figure, which is a snapshot of the system immediately before the potential is switched on again. Once the potential is connected again, the particles in the darker region move to the right-hand minimum whereas those within the small grey region move to the left. Due to the asymmetry of the potential, the ensemble of particles move, on average, to the right. (Figure and caption used by permission from Parrondo and Dinis, 2004 [17].)

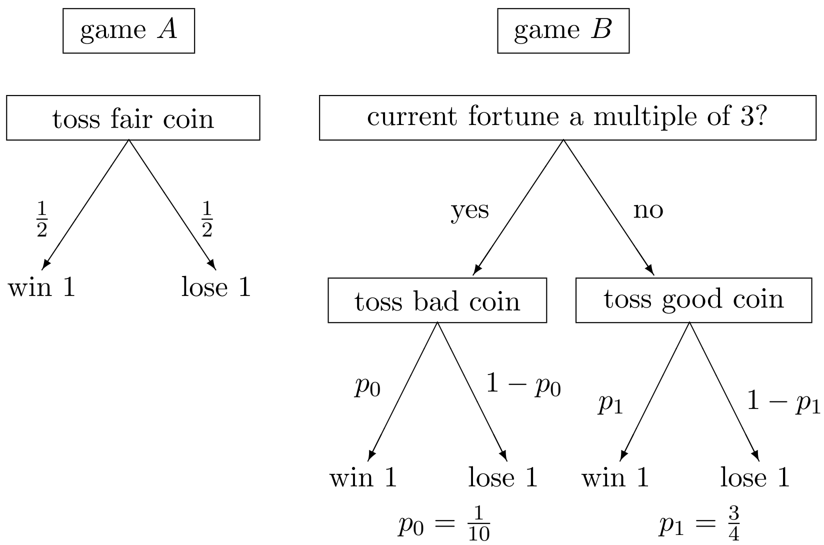

Figure 1.2. Parrondo's capital-dependent games without a bias parameter.

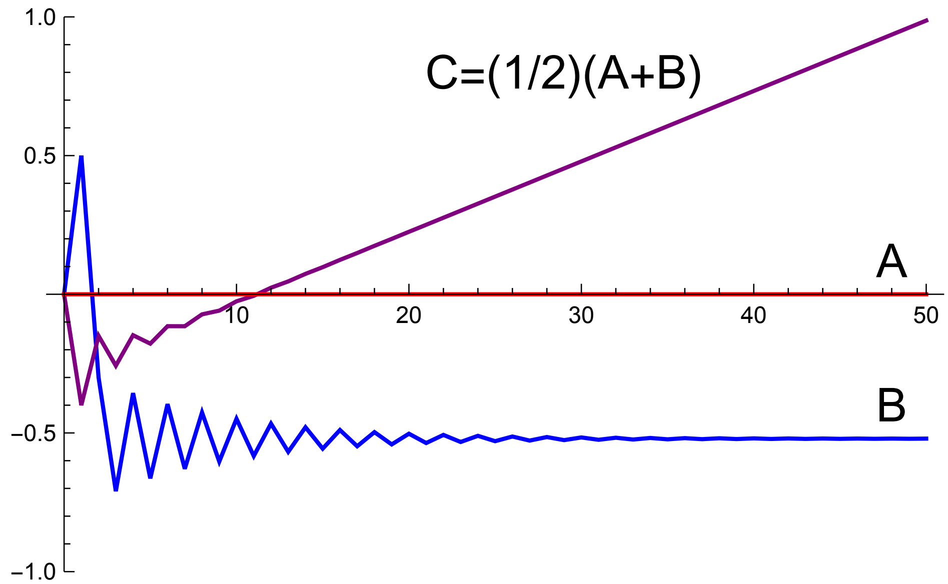

Figure 1.3. Cumulative expected profit from Parrondo's capital-dependent games.

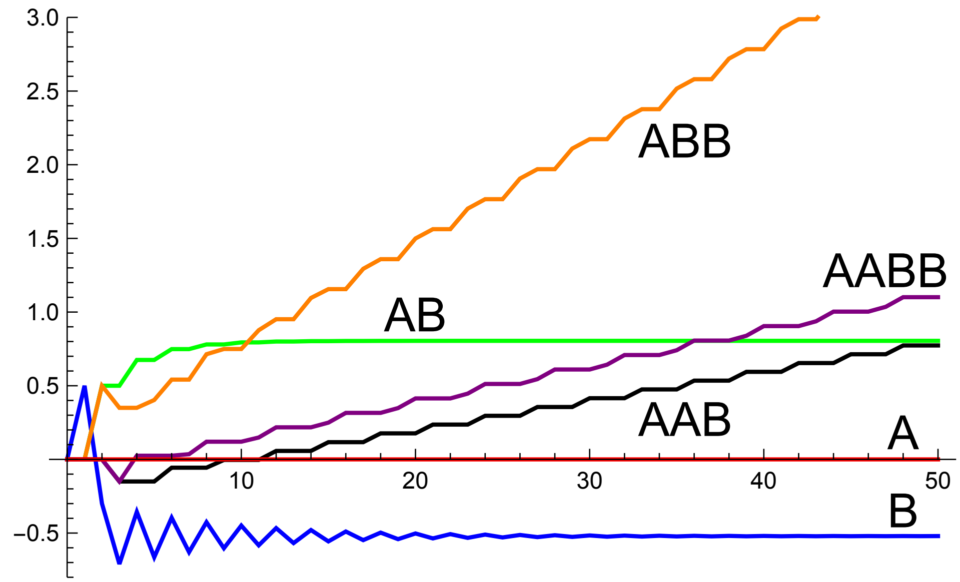

Figure 1.4. Cumulative expected profit from Parrondo's capital-dependent games, cont.

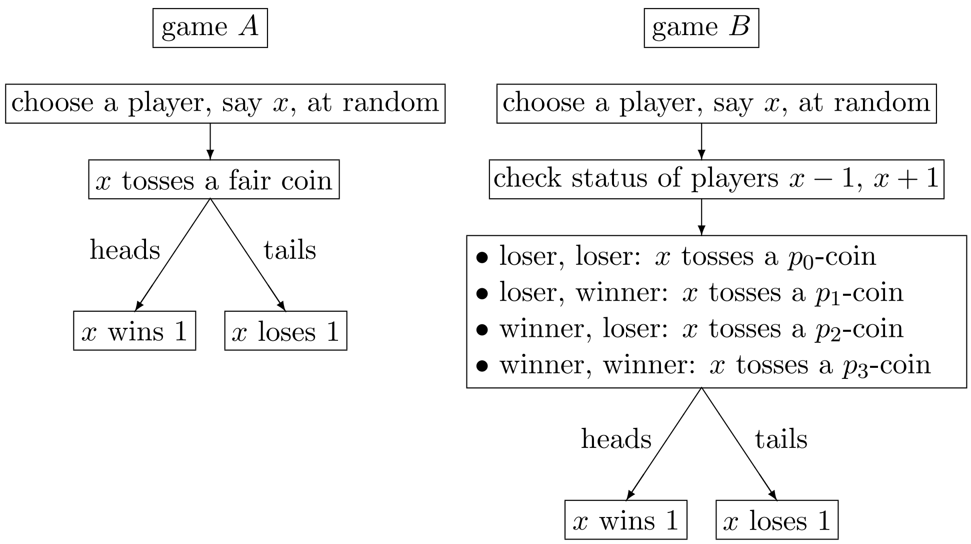

Figure 1.5. Toral's spatially dependent, or cooperative, Parrondo games, with parameters $N \geq 3$ and $p_0, p_1, p_2, p_3 \in [0,1]$. (A player's status as winner or loser depends on the result of his most recent game. Players are labeled from 1 to N; player 0 is player N and player $N+1$ is player 1. A p-coin is one for which the probability of heads is p.)

Table 1.1. Means for Toral's (2001) games, assuming $(p_0, p_1, p_2, p_3) = (1, 0.16, 0.16, 0.7)$. Entries give μ_B and μ_C with $C := \frac{1}{2}A + \frac{1}{2}B$, $C := AB$, $C := ABB$, $C := ABBB$, $C := AAB$, $C := AABB$, and $C := AAAB$ to six significant digits (in most cases) for $3 \le N \le 18$. Notice that $\mu_B < 0$ for $3 \le N \le 19$ except for $N = 4, 7, 8$, so the Parrondo effect is present except in 26 of the 113 $(= 16 \times 7 + 1)$ cases. The seven blank entries were not computed. The $N = \infty$ row gives the limits as $N \to \infty$. (Used by permission from Ethier and Lee [6], [7].)

N	μ_B	$\mu_{(A+B)/2}$	μ_{AB}	μ_{ABB}	μ_{ABBB}	μ_{AAB}	μ_{AABB}	μ_{AAAB}
3	−0.0909091	−0.0183774	−0.00695879	−0.0274821	−0.0402157	0.00067249	−0.0148718	0.00179203
4	0.0799608	0.0171357	0.00877041	0.0234583	0.0356946	0.00352220	0.0101194	0.00244238
5	−0.00219465	0.00405176	0.00466232	0.00501198	0.00434917	0.00320648	0.00465517	0.00240873
6	−0.0189247	0.00463310	0.00497503	0.00590528	0.00513509	0.00325099	0.00498178	0.00241857
7	0.00350598	0.00482261	0.00496767	0.00621483	0.00637676	0.00326314	0.00497331	0.00242540
8	0.000698188	0.00479021	0.00494802	0.00604194	0.00599064	0.00327193	0.00495138	0.00243115
9	−0.00189233	0.00479036	0.00493507	0.00598135	0.00588386	0.00327802	0.00493728	0.00243582
10	−0.000332809	0.00479099	0.00492347	0.00593756	0.00584200	0.00328237	0.00492494	0.00243961
11	−0.000466527	0.00479089	0.00491339	0.00589846	0.00578690	0.00328558	0.00491438	0.00244272
12	−0.000676916	0.00479089	0.00490464	0.00586697	0.00574489	0.00328800	0.00490531	0.00244529
13	−0.000562901	0.00479089	0.00489699	0.00584063	0.00571065	0.00328986	0.00489744	0.00244745
14	−0.000569340	0.00479089	0.00489026	0.00581820	0.00568131	0.00329133	0.00489056	0.00244927
15	−0.000586184	0.00479089	0.00488431	0.00579891	0.00565623	0.00329249	0.00488449	0.00245083
16	−0.000578161	0.00479089	0.00487900	0.00578213	0.00563452	0.00329343	0.00487912	0.00245217
17	−0.000578345	0.00479089	0.00487426	0.00576740	0.00561552	0.00329420	0.00487432	0.00245334
18	−0.000579652	0.00479089	0.00486999	0.00575438	0.00559876	0.00329483	0.00487001	0.00245437
19	−0.000579095	0.00479089						
∞		0.00479089	0.00479089	0.00554084	0.00532972	0.00329853	0.00479089	0.00246903

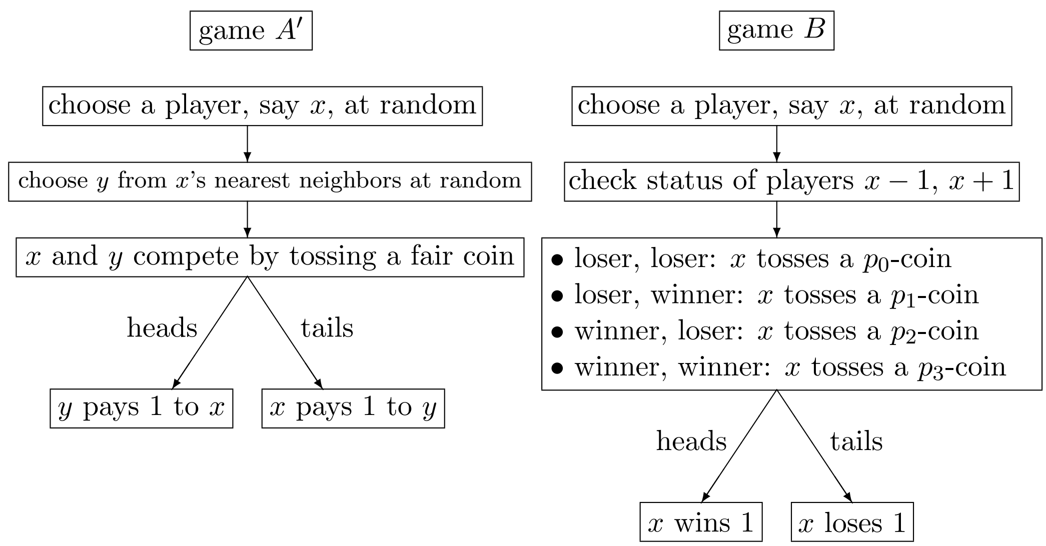

Figure 1.6. Satially dependent, or cooperative, Parrondo games of Xie and others [21], with parameters $N \geq 3$ and $p_0, p_1, p_2, p_3 \in [0,1]$. (A player's status as winner or loser depends on the result of his most recent game. Players are labeled from 1 to N; player 0 is player N and player $N+1$ is player 1. A p-coin is one for which the probability of heads is p.)

CHAPTER 2

SLLN/CLT FOR THE GAMES OF XIE AND OTHERS

In this chapter, we restate the strong law of large numbers (SLLN) and the central limit theorem (CLT) of Ethier and Lee [5], and we apply them to the Parrondo games of Xie and others. [21].

2.1 SLLN and CLT of Ethier and Lee

Ethier and Lee [5] proved an SLLN and a CLT for the Parrondo player's sequence of profits, motivated by game B and the random mixture $C := \gamma A + (1-\gamma)B$. A subsequent version, stated later, treats the case of periodic patterns.

Consider an irreducible aperiodic Markov chain $\{X_n\}_{n\geq 0}$ with finite state space Σ_0. It evolves according to the one-step transition matrix $\boldsymbol{P} = (P_{ij})_{i,j\in\Sigma_0}$. Let us denote its unique stationary distribution by the row vector $\boldsymbol{\pi} = (\pi_i)_{i\in\Sigma_0}$. Let $w : \Sigma_0 \times \Sigma_0 \mapsto \mathbf{R}$ be an arbitrary function, which we write as a matrix $\boldsymbol{W} = (w(i,j))_{i,j\in\Sigma_0}$ and refer to as the *payoff matrix*. Define the sequences $\{\xi_n\}_{n\geq 1}$ and $\{S_n\}_{n\geq 1}$ by

$$\xi_n := w(X_{n-1}, X_n), \qquad n \geq 1, \tag{2.1}$$

and

$$S_n := \xi_1 + \cdots + \xi_n, \qquad n \geq 1. \tag{2.2}$$

Let $\boldsymbol{\Pi}$ denote the square matrix each of whose rows is $\boldsymbol{\pi}$, and let $\boldsymbol{Z} := (\boldsymbol{I} - (\boldsymbol{P} - \boldsymbol{\Pi}))^{-1}$ denote the *fundamental matrix*. Denote by $\dot{\boldsymbol{P}}$ and $\ddot{\boldsymbol{P}}$ the Hadamard (entrywise) products $\boldsymbol{P} \circ \boldsymbol{W}$ and $\boldsymbol{P} \circ \boldsymbol{W} \circ \boldsymbol{W}$ (so $\dot{P}_{ij} := P_{ij} w(i,j)$ and $\ddot{P}_{ij} := P_{ij} w(i,j)^2$). Let $\mathbf{1} := (1, 1, \ldots, 1)^T$ and define

$$\mu := \boldsymbol{\pi}\dot{\boldsymbol{P}}\mathbf{1} \quad \text{and} \quad \sigma^2 := \boldsymbol{\pi}\ddot{\boldsymbol{P}}\mathbf{1} - (\boldsymbol{\pi}\dot{\boldsymbol{P}}\mathbf{1})^2 + 2\boldsymbol{\pi}\dot{\boldsymbol{P}}(\boldsymbol{Z} - \boldsymbol{\Pi})\dot{\boldsymbol{P}}\mathbf{1}. \tag{2.3}$$

Theorem 2.1 (Ethier and Lee [5])**.** *Under the above assumptions, and with the distribution of X_0 arbitrary,*

$$\frac{S_n}{n} \to \mu \text{ a.s.}$$

and, if $\sigma^2 > 0$,

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \to_d N(0,1).$$

If $\mu = 0$ and $\sigma^2 > 0$, then $-\infty = \liminf_{n\to\infty} S_n < \limsup_{n\to\infty} S_n = \infty$ a.s.

To illustrate this theorem, let us consider the capital-dependent Parrondo games of Section 1.1. The underlying Markov chain $\{X_n\}_{n\geq 0}$ has state space $\Sigma_0 := \{0, 1, 2\}$ and one-step transition matrix

$$\boldsymbol{P}_B := \begin{pmatrix} 0 & 1/10 & 9/10 \\ 1/4 & 0 & 3/4 \\ 3/4 & 1/4 & 0 \end{pmatrix}.$$

Its unique stationary distribution is $\boldsymbol{\pi}_B = (1/13)(5, 2, 6)$. The payoff matrix has the form

$$\boldsymbol{W} := \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

We find that

$$\mu_B = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \boldsymbol{1} = 0.$$

We can apply the same argument to

$$\boldsymbol{P}_A := \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

with unique stationary distribution $\boldsymbol{\pi}_A = (1/3)(1, 1, 1)$ to get

$$\mu_A = \boldsymbol{\pi}_A \dot{\boldsymbol{P}}_A \boldsymbol{1} = 0,$$

a result that is obvious without calculation. Finally, the $(\frac{1}{2}, \frac{1}{2})$ random mixture has one-step transition matrix

$$\boldsymbol{P}_C := \frac{1}{2}(\boldsymbol{P}_A + \boldsymbol{P}_B) = \begin{pmatrix} 0 & 3/10 & 7/10 \\ 3/8 & 0 & 5/8 \\ 5/8 & 3/8 & 0 \end{pmatrix}$$

with unique stationary distribution $\boldsymbol{\pi}_C = (1/709)(245, 180, 284)$. We get

$$\mu_C = \boldsymbol{\pi}_C \dot{\boldsymbol{P}}_C \boldsymbol{1} = \frac{18}{709} \approx 0.0253879.$$

This is perhaps the best known example of Parrondo's paradox, and the SLLN justifies the conclusion: Two fair games combine to win.

We can also derive a CLT, which requires the fundamental matrix

$$\boldsymbol{Z}_B := (\boldsymbol{I} - (\boldsymbol{P}_B - \boldsymbol{\Pi}_B))^{-1} = \frac{1}{2197}\begin{pmatrix} 1725 & -38 & 510 \\ -95 & 1938 & 354 \\ 425 & 118 & 1654 \end{pmatrix}.$$

We find that

$$\sigma_B^2 = \boldsymbol{\pi}_B \ddot{\boldsymbol{P}}_B \boldsymbol{1} - (\boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \boldsymbol{1})^2 + 2\boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B (\boldsymbol{Z}_B - \boldsymbol{\Pi}_B) \dot{\boldsymbol{P}}_B \boldsymbol{1} = \left(\frac{9}{13}\right)^2 \approx 0.479290.$$

Similarly,

$$\boldsymbol{Z}_A := (\boldsymbol{I} - (\boldsymbol{P}_A - \boldsymbol{\Pi}_A))^{-1} = \frac{1}{9}\begin{pmatrix} 7 & 1 & 1 \\ 1 & 7 & 1 \\ 1 & 1 & 7 \end{pmatrix},$$

hence,

$$\sigma_A^2 = \boldsymbol{\pi}_A \ddot{\boldsymbol{P}}_A \boldsymbol{1} - (\boldsymbol{\pi}_A \dot{\boldsymbol{P}}_A \boldsymbol{1})^2 + 2\boldsymbol{\pi}_A \dot{\boldsymbol{P}}_A (\boldsymbol{Z}_A - \boldsymbol{\Pi}_A) \dot{\boldsymbol{P}}_A \boldsymbol{1} = 1,$$

as is obvious without the formula. Finally,

$$\boldsymbol{Z}_C := (\boldsymbol{I} - (\boldsymbol{P}_C - \boldsymbol{\Pi}_C))^{-1} = \frac{1}{502681}\begin{pmatrix} 392265 & 22884 & 87532 \\ 23585 & 408580 & 70516 \\ 80305 & 39900 & 382476 \end{pmatrix},$$

and we conclude that

$$\sigma_C^2 = \boldsymbol{\pi}_C \ddot{\boldsymbol{P}}_C \boldsymbol{1} - (\boldsymbol{\pi}_C \dot{\boldsymbol{P}}_C \boldsymbol{1})^2 + 2\boldsymbol{\pi}_C \dot{\boldsymbol{P}}_C (\boldsymbol{Z}_C - \boldsymbol{\Pi}_C) \dot{\boldsymbol{P}}_C \boldsymbol{1} = \frac{311313105}{356400829} \approx 0.873492.$$

In each case we have a CLT.

Next we turn to another SLLN and CLT of Ethier and Lee [5] , this one motivated by the case of periodic patterns.

Let $\boldsymbol{P}_A$ and $\boldsymbol{P}_B$ be one-step transition matrices for Markov chains in a finite state space Σ_0. Fix integers $r, s \geq 1$. Assume that $\boldsymbol{P} := \boldsymbol{P}_A^r \boldsymbol{P}_B^s$, as well as all cyclic permutations of $\boldsymbol{P}_A^r \boldsymbol{P}_B^s$, are ergodic, and let the row vector $\boldsymbol{\pi}$ be the unique stationary distribution of $\boldsymbol{P}$. Let $\boldsymbol{\Pi}$ be the square matrix each of whose rows is equal to $\boldsymbol{\pi}$, and let $\boldsymbol{Z} := (\boldsymbol{I} - (\boldsymbol{P} - \boldsymbol{\Pi}))^{-1}$ be the fundamental matrix of $\boldsymbol{P}$. Given a real-valued function w on $\Sigma_0 \times \Sigma_0$, define the payoff matrix $\boldsymbol{W} := (w(i,j))_{i,j\in\Sigma_0}$. Define $\dot{\boldsymbol{P}}_A := \boldsymbol{P}_A \circ \boldsymbol{W}$, $\dot{\boldsymbol{P}}_B := \boldsymbol{P}_B \circ \boldsymbol{W}$, $\ddot{\boldsymbol{P}}_A := \boldsymbol{P}_A \circ \boldsymbol{W} \circ \boldsymbol{W}$, $\ddot{\boldsymbol{P}}_B := \boldsymbol{P}_B \circ \boldsymbol{W} \circ \boldsymbol{W}$, where $\circ$ denotes the Hadamard (entrywise) product. Let

$$\mu_{[r,s]} := \frac{1}{r+s}\left[\sum_{u=0}^{r-1} \boldsymbol{\pi} \boldsymbol{P}_A^u \dot{\boldsymbol{P}}_A \boldsymbol{1} + \sum_{v=0}^{s-1} \boldsymbol{\pi} \boldsymbol{P}_A^r \boldsymbol{P}_B^v \dot{\boldsymbol{P}}_B \boldsymbol{1}\right],$$

and

$$\sigma_{[r,s]}^2 = \frac{1}{r+s}\left[\sum_{u=0}^{r-1} [\boldsymbol{\pi} \boldsymbol{P}_A^u \ddot{\boldsymbol{P}}_A \boldsymbol{1} - (\boldsymbol{\pi} \boldsymbol{P}_A^u \dot{\boldsymbol{P}}_A \boldsymbol{1})^2]\right.$$

$$
\begin{aligned}
&+\sum_{v=0}^{s-1}[\boldsymbol{\pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^v\ddot{\boldsymbol{P}}_B\mathbf{1}-(\boldsymbol{\pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^v\dot{\boldsymbol{P}}_B\mathbf{1})^2]\\
&+2\sum_{0\le u<v\le r-1}\boldsymbol{\pi}\boldsymbol{P}_A^u\dot{\boldsymbol{P}}_A(\boldsymbol{P}_A^{v-u-1}-\boldsymbol{\Pi}\boldsymbol{P}_A^v)\dot{\boldsymbol{P}}_A\mathbf{1}\\
&+2\sum_{u=0}^{r-1}\sum_{v=0}^{s-1}\boldsymbol{\pi}\boldsymbol{P}_A^u\dot{\boldsymbol{P}}_A(\boldsymbol{P}_A^{r-u-1}-\boldsymbol{\Pi}\boldsymbol{P}_A^r)\boldsymbol{P}_B^v\dot{\boldsymbol{P}}_B\mathbf{1}\\
&+2\sum_{0\le u<v\le s-1}\boldsymbol{\pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^u\dot{\boldsymbol{P}}_B(\boldsymbol{P}_B^{v-u-1}-\boldsymbol{\Pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^v)\dot{\boldsymbol{P}}_B\mathbf{1}\\
&+2\bigg(\sum_{u=0}^{r-1}\sum_{v=0}^{r-1}\boldsymbol{\pi}\boldsymbol{P}_A^u\dot{\boldsymbol{P}}_A\boldsymbol{P}_A^{r-u-1}\boldsymbol{P}_B^s(\boldsymbol{Z}-\boldsymbol{\Pi})\boldsymbol{P}_A^v\dot{\boldsymbol{P}}_A\mathbf{1}\\
&\qquad+\sum_{u=0}^{r-1}\sum_{v=0}^{s-1}\boldsymbol{\pi}\boldsymbol{P}_A^u\dot{\boldsymbol{P}}_A\boldsymbol{P}_A^{r-u-1}\boldsymbol{P}_B^s(\boldsymbol{Z}-\boldsymbol{\Pi})\boldsymbol{P}_A^r\boldsymbol{P}_B^v\dot{\boldsymbol{P}}_B\mathbf{1}\\
&\qquad+\sum_{u=0}^{s-1}\sum_{v=0}^{r-1}\boldsymbol{\pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^u\dot{\boldsymbol{P}}_B\boldsymbol{P}_B^{s-u-1}(\boldsymbol{Z}-\boldsymbol{\Pi})\boldsymbol{P}_A^v\dot{\boldsymbol{P}}_A\mathbf{1}\\
&\qquad+\sum_{u=0}^{s-1}\sum_{v=0}^{s-1}\boldsymbol{\pi}\boldsymbol{P}_A^r\boldsymbol{P}_B^u\dot{\boldsymbol{P}}_B\boldsymbol{P}_B^{s-u-1}(\boldsymbol{Z}-\boldsymbol{\Pi})\boldsymbol{P}_A^r\boldsymbol{P}_B^v\dot{\boldsymbol{P}}_B\mathbf{1}\bigg)\bigg],
\end{aligned}
$$

where $\mathbf{1}$ denotes a column vector of 1s with entries indexed by Σ_0. Let $\{X_n\}_{n\ge0}$ be a nonhomogeneous Markov chain in Σ_0 with one-step transition matrices $\boldsymbol{P}_A,\ldots,\boldsymbol{P}_A$ (r times), $\boldsymbol{P}_B,\ldots,\boldsymbol{P}_B$ (s times), $\boldsymbol{P}_A,\ldots,\boldsymbol{P}_A$ (r times), $\boldsymbol{P}_B,\ldots,\boldsymbol{P}_B$ (s times), and so on. For each $n\ge1$, define $\xi_n:=w(X_{n-1},X_n)$ and $S_n:=\xi_1+\cdots+\xi_n$.

Theorem 2.2 (Ethier and Lee [5]). *Under the above assumptions, and with the distribution of X_0 arbitrary,*

$$\frac{S_n}{n}\to\mu_{[r,s]}\ \text{ a.s.}$$

and, if $\sigma^2_{[r,s]}>0$, then

$$\frac{S_n-n\mu_{[r,s]}}{\sqrt{n\sigma^2_{[r,s]}}}\to_d N(0,1)\ \textit{ as } n\to\infty.$$

To illustrate this result, we consider the capital-dependent Parrondo games of Section 1.1, and we take $r=s=2$. Then

$$\boldsymbol{P}=\boldsymbol{P}_A^2\boldsymbol{P}_B^2=\frac{1}{320}\begin{pmatrix}162&59&99\\151&58&111\\111&47&162\end{pmatrix}.$$

Its unique stationary distribution is $\boldsymbol{\pi}=(1/6357)(2783,1075,2499)$, and the fundamental matrix is

$$\boldsymbol{Z} = \frac{1}{525348837}\begin{pmatrix} 569627023 & 10027235 & -54305421 \\ 22416463 & 532826915 & -29894541 \\ -58953137 & -14383645 & 598685619 \end{pmatrix}.$$

In this example, $\dot{\boldsymbol{P}}_A\boldsymbol{1} = \boldsymbol{0}$, $\ddot{\boldsymbol{P}}_A = \boldsymbol{P}_A$, and $\ddot{\boldsymbol{P}}_B = \boldsymbol{P}_B$, and this simplfies the mean and variance formulas considerably. Specifically, we have

$$\mu_{[2,2]} = \frac{1}{4}\boldsymbol{\pi}\boldsymbol{P}_A^2(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1}$$

and

$$\begin{aligned}\sigma^2_{[2,2]} = \frac{1}{4}\big[&2 + 2 - (\boldsymbol{\pi}\boldsymbol{P}_A^2\dot{\boldsymbol{P}}_B\boldsymbol{1})^2 - (\boldsymbol{\pi}\boldsymbol{P}_A^2\boldsymbol{P}_B\dot{\boldsymbol{P}}_B\boldsymbol{1})^2 \\ &+ 2\boldsymbol{\pi}\dot{\boldsymbol{P}}_A(\boldsymbol{P}_A - \boldsymbol{\Pi}\boldsymbol{P}_A^2)(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} + 2\boldsymbol{\pi}\boldsymbol{P}_A\dot{\boldsymbol{P}}_A(\boldsymbol{I} - \boldsymbol{\Pi}\boldsymbol{P}_A^2)(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} \\ &+ 2\boldsymbol{\pi}\boldsymbol{P}_A^2\dot{\boldsymbol{P}}_B(\boldsymbol{I} - \boldsymbol{\Pi}\boldsymbol{P}_A^2\boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} \\ &+ 2\boldsymbol{\pi}\dot{\boldsymbol{P}}_A\boldsymbol{P}_A\boldsymbol{P}_B^2(\boldsymbol{Z} - \boldsymbol{\Pi})\boldsymbol{P}_A^2(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} + 2\boldsymbol{\pi}\boldsymbol{P}_A\dot{\boldsymbol{P}}_A\boldsymbol{P}_B^2(\boldsymbol{Z} - \boldsymbol{\Pi})\boldsymbol{P}_A^2(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} \\ &+ 2\boldsymbol{\pi}\boldsymbol{P}_A^2\dot{\boldsymbol{P}}_B\boldsymbol{P}_B(\boldsymbol{Z} - \boldsymbol{\Pi})\boldsymbol{P}_A^2(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1} + 2\boldsymbol{\pi}\boldsymbol{P}_A^2\boldsymbol{P}_B\dot{\boldsymbol{P}}_B(\boldsymbol{Z} - \boldsymbol{\Pi})\boldsymbol{P}_A^2(\boldsymbol{I} + \boldsymbol{P}_B)\dot{\boldsymbol{P}}_B\boldsymbol{1}\big].\end{aligned}$$

We conclude that

$$\mu_{[2,2]} = \frac{4}{163} \approx 0.0245399 \quad \text{and} \quad \sigma^2_{[2,2]} = \frac{1923037543}{2195688729} \approx 0.875824.$$

These numbers are consistent with Ethier and Lee [5].

2.2 Application to game B

The Markov chain formalized by Mihailović and Rajković [16] keeps track of the status (loser or winner, 0 or 1) of each of the $N \geq 3$ players of game B, which was described in Chapter 1. Its state space is the product space

$$\Sigma := \{\eta = (\eta(1), \eta(2), \ldots, \eta(N)) : \eta(x) \in \{0,1\} \text{ for } x = 1, \ldots, N\} = \{0,1\}^N$$

with 2^N states. Let $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0,1,2,3\}$. Of course $\eta(0) := \eta(N)$ and $\eta(N{+}1) := \eta(1)$ because of the circular arrangement of players. Also, let η_x be the element of Σ equal to η except at the xth coordinate. For example, $\eta_1 := (1{-}\eta(1), \eta(2), \eta(3), \ldots, \eta(N))$.

The one-step transition matrix $\boldsymbol{P}_B$ for this Markov chain depends not only on N but on four parameters, $p_0, p_1, p_2, p_3 \in [0,1]$. It has the form

$$P_B(\eta, \eta_x) := \begin{cases} N^{-1} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ N^{-1} q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \qquad x = 1, \ldots, N, \ \eta \in \Sigma, \tag{2.4}$$

and

$$P_B(\eta, \eta) := N^{-1} \bigg(\sum_{x:\eta(x)=0} q_{m_x(\eta)} + \sum_{x:\eta(x)=1} p_{m_x(\eta)} \bigg), \qquad \eta \in \Sigma, \tag{2.5}$$

where $q_m := 1 - p_m$ for $m = 0, 1, 2, 3$, and empty sums are 0. The Markov chain is irreducible and aperiodic if $0 < p_m < 1$ for $m = 0, 1, 2, 3$. Under slightly weaker assumptions (see Ethier and Lee [8]), the Markov chain is ergodic, which suffices. For example, if p_0 is arbitrary and $0 < p_m < 1$ for $m = 1, 2, 3$, or if $0 < p_m < 1$ for $m = 0, 1, 2$ and p_3 is arbitrary, then ergodicity holds.

It appears at first glance that the theorem does not apply in the context of game B because the payoffs are not completely specified by the one-step transitions of the Markov chain. Specifically, a transition from a state η to itself results whenever a loser loses or a winner wins, so the transition does not determine the payoff.

Our original Markov chain has state space $\Sigma := \{0,1\}^N$ and its one-step transition matrix $\boldsymbol{P}_B$ is given by (2.4) and (2.5). Assuming it is ergodic, let $\boldsymbol{\pi}$ denote its unique stationary distribution. The approach in Ethier and Lee [6] augments the state space, letting $\Sigma^* := \Sigma \times \{1, 2, \ldots, N\}$ and keeping track not only of the status of each player as described by $\eta \in \Sigma$ but also of the label of the next player to play, say x. The new one-step transition matrix $\boldsymbol{P}_B^*$ has the form

$$P_B^*((\eta, x), (\eta_x, y)) := \begin{cases} N^{-1} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ N^{-1} q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \qquad (\eta, x) \in \Sigma^*, \ y = 1, 2, \ldots, N,$$

and

$$P_B^*((\eta, x), (\eta, y)) := \begin{cases} N^{-1} q_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ N^{-1} p_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \qquad (\eta, x) \in \Sigma^*, \ y = 1, 2, \ldots, N,$$

where $q_m := 1 - p_m$ for $m = 0, 1, 2, 3$ and $m_x(\eta) := 2\eta(x-1) + \eta(x+1)$. This remains an ergodic Markov chain, and its unique stationary distribution $\boldsymbol{\pi}^*$ is given by $\pi^*(\eta, x) = N^{-1}\pi(\eta)$. Further, the payoff matrix now has each nonzero entry equal to ± 1, so the theorem applies.

However, there is a drawback to this approach, namely that it is not clear that the variance parameter $(\sigma^*)^2$ is the same as the original one, σ^2. (It is easy to verify that $\mu^* = \mu$.) Therefore, we take a different approach, namely the one used by Ethier and Lee [11] in their study of two-dimensional spatial models.

Here a different augmentation of Σ is more effective. We let $\Sigma^\circ := \Sigma \times \{-1, 1\}$ and keep track not only of $\eta \in \Sigma$ but also of the profit from the last game played, say $s \in \{-1, 1\}$. The new one-step transition matrix $\boldsymbol{P}_B^\circ$ has the form, for every $(\eta, s) \in \Sigma^\circ$,

$$P_B^\circ((\eta, s), (\eta_x, 1)) := \begin{cases} N^{-1} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ 0 & \text{if } \eta(x) = 1, \end{cases} \tag{2.6}$$

$$P_B^\circ((\eta, s), (\eta_x, -1)) := \begin{cases} 0 & \text{if } \eta(x) = 0, \\ N^{-1} q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \tag{2.7}$$

for $x = 1, \ldots, N$, and

$$P_B^\circ((\eta, s), (\eta, 1)) := N^{-1} \sum_{x:\eta(x)=1} p_{m_x(\eta)}, \tag{2.8}$$

$$P_B^\circ((\eta, s), (\eta, -1)) := N^{-1} \sum_{x:\eta(x)=0} q_{m_x(\eta)}, \tag{2.9}$$

where $q_m := 1 - p_m$ for $m = 0, 1, 2, 3, 4$, and $m_x(\eta) = 2\eta(x-1) + \eta(x+1)$. There are two inaccessible states, $(\mathbf{0}, 1)$ and $(\mathbf{1}, -1)$, but the Markov chain remains ergodic. Let $\boldsymbol{\pi}^\circ$ denote the unique stationary distribution, which has entry 0 at each of the two inaccessible states. The payoff function w° can now be defined by

$$w^\circ((\eta, s), (\eta_x, t)) = t \text{ if } \eta(x) = (1-t)/2, \qquad w^\circ((\eta, s), (\eta, t)) = t$$

for all $(\eta, s) \in \Sigma^\circ$, $x = 1, 2, \ldots, N$, and $t \in \{-1, 1\}$, and $w^\circ = 0$ otherwise. This allows us to define the matrix $\boldsymbol{W}^\circ$ and then $\dot{\boldsymbol{P}}_B^\circ := \boldsymbol{P}_B^\circ \circ \boldsymbol{W}^\circ$ and $\ddot{\boldsymbol{P}}_B^\circ := \boldsymbol{P}_B^\circ \circ \boldsymbol{W}^\circ \circ \boldsymbol{W}^\circ$, the Hadamard (or entrywise) products. Theorem 2.1 yields the following.

Theorem 2.3. *Let $0 < p_m < 1$ for $m = 0, 1, 2$ or for $m = 1, 2, 3$, so that the Markov chain with one-step transition matrix $\boldsymbol{P}_B^\circ$ is ergodic, and let the row vector $\boldsymbol{\pi}_B^\circ$ be its unique stationary distribution. Define*

$$\mu_B^\circ = \boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ \mathbf{1}, \qquad (\sigma_B^\circ)^2 = \boldsymbol{\pi}_B^\circ \ddot{\boldsymbol{P}}_B^\circ \mathbf{1} - (\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ \mathbf{1})^2 + 2\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ (\boldsymbol{Z}_B^\circ - \mathbf{1}\boldsymbol{\pi}_B^\circ) \dot{\boldsymbol{P}}_B^\circ \mathbf{1}.$$

where $\mathbf{1}$ denotes a column vector of 1s with entries indexed by Σ_B° and $\boldsymbol{Z}_B^\circ := (\boldsymbol{I} - (\boldsymbol{P}_B^\circ - \mathbf{1}\boldsymbol{\pi}_B^\circ))^{-1}$ is the fundamental matrix. (Notice that $\mathbf{1}\boldsymbol{\pi}_B^\circ$ is the square matrix each of whose *rows is equal to* $\boldsymbol{\pi}_B^\circ$*.) Let* $\{X_n^\circ\}_{n\geq 0}$ *be a time-homogeneous Markov chain in* Σ° *with one-step transition matrix* $\boldsymbol{P}_B^\circ$*, and let the initial distribution be arbitrary. For each* $n \geq 1$*, define* $\xi_n := w^\circ(X_{n-1}^\circ, X_n^\circ)$ *and* $S_n := \xi_1 + \cdots + \xi_n$*. Then* $\lim_{n\to\infty} n^{-1} S_n = \mu_B^\circ$ a.s. *and, if* $(\sigma_B^\circ)^2 > 0$*, then* $(S_n - n\mu_B^\circ)/\sqrt{n(\sigma_B^\circ)^2} \to_d N(0,1)$ *as* $n \to \infty$*.*

We next show that there is a simpler expression for this mean and variance. Let us define

$$\mu_B := \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \mathbf{1}, \qquad \sigma_B^2 := \boldsymbol{\pi}_B \ddot{\boldsymbol{P}}_B \mathbf{1} - (\boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \mathbf{1})^2 + 2\boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B (\boldsymbol{Z}_B - \mathbf{1}\boldsymbol{\pi}_B)\dot{\boldsymbol{P}}_B \mathbf{1}, \tag{2.10}$$

where $\mathbf{1}$ is the column vector of 1s of the appropriate dimension, $\dot{\boldsymbol{P}}_B$ is $\boldsymbol{P}_B$ with each q_m replaced by $-q_m$, and $\ddot{\boldsymbol{P}}_B = \boldsymbol{P}_B$. This "rule of thumb" for $\dot{\boldsymbol{P}}_B$ requires some caution: It must be applied before any simplifications to $\boldsymbol{P}_B$ are made using $q_m = 1 - p_m$. Of course, $\boldsymbol{\pi}_B$ is the unique stationary distribution, and $\boldsymbol{Z}_B$ is the fundamental matrix, of $\boldsymbol{P}_B$.

Theorem 2.4.

$$\mu_B^\circ = \mu_B \tag{2.11}$$

and

$$(\sigma_B^\circ)^2 = \sigma_B^2. \tag{2.12}$$

Remark. Before proving this, let us explain its significance. μ_B° and $(\sigma_B^\circ)^2$ are the mean and variance that appear in the SLLN and the CLT. They are defined in terms of $\boldsymbol{P}_B^\circ$, the augmented one-step transition matrix. μ_B and σ_B^2 are defined analogously in terms of $\boldsymbol{P}_B$, the original one-step transition matrix, using the rule of thumb.

Proof. To emphasize the fact that $P_B^\circ((\eta, s), (\zeta, t))$ does not depend on s, we write it temporarily as $P_B^\circ((\eta, \cdot), (\zeta, t))$. This leads to

$$\mu_B^\circ = \boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ \mathbf{1} = \sum_{\eta,s,\zeta,t} \pi_B^\circ(\eta, s)\dot{P}_B^\circ((\eta,\cdot),(\zeta,t)) = \sum_{\eta,\zeta} \pi_B(\eta)\dot{P}_B(\eta,\zeta) = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \mathbf{1} = \mu_B. \tag{2.13}$$

To show that $(\sigma_B^\circ)^2 = \sigma_B^2$, we need to show that

$$\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ (\boldsymbol{Z}_B^\circ - \mathbf{1}\boldsymbol{\pi}_B^\circ)\dot{\boldsymbol{P}}_B^\circ \mathbf{1} = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B (\boldsymbol{Z}_B - \mathbf{1}\boldsymbol{\pi}_B)\dot{\boldsymbol{P}}_B \mathbf{1}.$$

Now, by Kemeny and Snell [14], $\boldsymbol{Z}_B - \mathbf{1}\boldsymbol{\pi}_B = \sum_{m=1}^\infty (\boldsymbol{P}_B^{m-1} - \mathbf{1}\boldsymbol{\pi}_B)$, so it is enough to show that

$$\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ ((\boldsymbol{P}_B^\circ)^{m-1} - \mathbf{1}\boldsymbol{\pi}_B^\circ)\dot{\boldsymbol{P}}_B^\circ \mathbf{1} = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B (\boldsymbol{P}_B^{m-1} - \mathbf{1}\boldsymbol{\pi}_B)\dot{\boldsymbol{P}}_B \mathbf{1}, \qquad m \geq 1,$$

or that

$$\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ (\boldsymbol{P}_B^\circ)^{m-1} \dot{\boldsymbol{P}}_B^\circ \mathbf{1} = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \boldsymbol{P}_B^{m-1} \dot{\boldsymbol{P}}_B \mathbf{1}, \qquad m \geq 1.$$

Given $m \geq 1$, we have

$$\begin{aligned}
&\boldsymbol{\pi}_B^\circ \dot{\boldsymbol{P}}_B^\circ (\boldsymbol{P}_B^\circ)^{m-1} \dot{\boldsymbol{P}}_B^\circ \mathbf{1} \\
&\quad = \sum_{\eta,s,\zeta,t,\xi,u,\phi,v} \pi_B^\circ(\eta,s) \dot{P}_B^\circ((\eta,\cdot),(\zeta,t))(P_B^\circ)^{m-1}((\zeta,\cdot),(\xi,u)) \dot{P}_B^\circ((\xi,\cdot),(\phi,v)) \\
&\quad = \sum_{\eta,\zeta,t,\xi,\phi,v} \pi_B(\eta) \dot{P}_B^\circ((\eta,\cdot),(\zeta,t)) P_B^{m-1}(\zeta,\xi) \dot{P}_B^\circ((\xi,\cdot),(\phi,v)) \\
&\quad = \sum_{\eta,\zeta,\xi,\phi} \pi_B(\eta) \dot{P}_B(\eta,\zeta) P_B^{m-1}(\zeta,\xi) \dot{P}_B(\xi,\phi) \\
&\quad = \boldsymbol{\pi}_B \dot{\boldsymbol{P}}_B \boldsymbol{P}_B^{m-1} \dot{\boldsymbol{P}}_B \mathbf{1},
\end{aligned}$$

which completes the proof.

2.3 Application to game $C' := \gamma A' + (1-\gamma)B$

This case is not much different from the previous one. Notice that, if game A' is played, the profit to the set of N players is 0, since game A' simply redistributes capital among the players. So we can use the same augmentation of the state space as before, except that 0 is now a possible value of the profit from the last game played. In other words, $\Sigma^\circ := \Sigma \times \{-1, 0, 1\}$. The transition probabilities require some new notation. Let $\eta^{x,x\pm1,\pm1}$ be the element of Σ representing the players' status after player x plays player $x \pm 1$ and wins (1) or loses (-1). Of course player 0 in player N and player $N+1$ is player 1. For example, $\eta^{1,2,-1} = (0, 1, \eta(3), \ldots, \eta(N))$ (player 1 competes against player 2 and loses, leaving player 1 a loser and player 2 a winner, regardless of their previous status). Then

$$P_{C'}^\circ((\eta,s),(\eta_x,1)) = \begin{cases} (1-\gamma)N^{-1}p_{m_x(\eta)} & \text{if } \eta(x)=0, \\ 0 & \text{if } \eta(x)=1, \end{cases} \tag{2.14}$$

$$P_{C'}^\circ((\eta,s),(\eta_x,-1)) = \begin{cases} 0 & \text{if } \eta(x)=0, \\ (1-\gamma)N^{-1}q_{m_x(\eta)} & \text{if } \eta(x)=1, \end{cases} \tag{2.15}$$

$$P_{C'}^\circ((\eta,s),(\eta^{x,x-1,-1},0)) = \gamma(4N)^{-1}, \tag{2.16}$$

$$P_{C'}^\circ((\eta,s),(\eta^{x,x-1,1},0)) = \gamma(4N)^{-1}, \tag{2.17}$$

$$P_{C'}^\circ((\eta,s),(\eta^{x,x+1,-1},0)) = \gamma(4N)^{-1}, \tag{2.18}$$

$$P_{C'}^\circ((\eta,s),(\eta^{x,x+1,1},0)) = \gamma(4N)^{-1}, \tag{2.19}$$

for $x = 1, 2, \ldots, N$, and

$$P^\circ_{C'}((\eta, s), (\eta, 1)) = (1-\gamma)N^{-1} \sum_{x:\eta(x)=1} p_{m_x(\eta)}, \tag{2.20}$$

$$P^\circ_{C'}((\eta, s), (\eta, -1)) = (1-\gamma)N^{-1} \sum_{x:\eta(x)=0} q_{m_x(\eta)}. \tag{2.21}$$

Of course, we could also define $\boldsymbol{P}_{C'} = \gamma \boldsymbol{P}_{A'} + (1-\gamma)\boldsymbol{P}_B$. We notice that Theorems 2.5 and 2.6 hold in this framework without change.

Theorem 2.5. *Let $0 < p_m < 1$ for $m = 0, 1, 2$ or for $m = 1, 2, 3$, so that the Markov chain with one-step transition matrix $\boldsymbol{P}^\circ_{C'} := \gamma \boldsymbol{P}^\circ_{A'} + (1-\gamma)\boldsymbol{P}^\circ_B$ is ergodic, and let the row vector $\boldsymbol{\pi}^\circ_{C'}$ be its unique stationary distribution. Define*

$$\mu^\circ_{(\gamma,1-\gamma)'} = \boldsymbol{\pi}^\circ_{C'} \dot{\boldsymbol{P}}^\circ_{C'} \boldsymbol{1}, \quad (\sigma^\circ_{(\gamma,1-\gamma)'})^2 = \boldsymbol{\pi}^\circ_{C'} \ddot{\boldsymbol{P}}^\circ_{C'} \boldsymbol{1} - (\boldsymbol{\pi}^\circ_{C'} \dot{\boldsymbol{P}}^\circ_{C'} \boldsymbol{1})^2 + 2\boldsymbol{\pi}^\circ_{C'} \dot{\boldsymbol{P}}^\circ_{C'} (\boldsymbol{Z}^\circ_{C'} - \boldsymbol{1}\boldsymbol{\pi}^\circ_{C'}) \dot{\boldsymbol{P}}^\circ_{C'} \boldsymbol{1}.$$

where $\boldsymbol{1}$ denotes a column vector of 1s with entries indexed by Σ° and $\boldsymbol{Z}^\circ_{C'} := (\boldsymbol{I} - (\boldsymbol{P}^\circ_{C'} - \boldsymbol{1}\boldsymbol{\pi}^\circ_{C'}))^{-1}$ is the fundamental matrix. (Notice that $\boldsymbol{1}\boldsymbol{\pi}^\circ_{C'}$ is the square matrix each of whose rows is equal to $\boldsymbol{\pi}^\circ_{C'}$.) Let $\{X^\circ_n\}_{n\geq 0}$ be a time-homogeneous Markov chain in Σ° with one-step transition matrix $\boldsymbol{P}^\circ_{C'}$, and let the initial distribution be arbitrary. For each $n \geq 1$, define $\xi_n := w^\circ(X^\circ_{n-1}, X^\circ_n)$ and $S_n := \xi_1 + \cdots + \xi_n$. Then $\lim_{n\to\infty} n^{-1} S_n = \mu^\circ_{(\gamma,1-\gamma)'}$ a.s. and, if $(\sigma^\circ_{(\gamma,1-\gamma)'})^2 > 0$, then $(S_n - n\mu^\circ_{(\gamma,1-\gamma)'})/\sqrt{n(\sigma^\circ_{(\gamma,1-\gamma)'})^2} \to_d N(0,1)$ as $n \to \infty$.

Let us define

$$\mu_{(\gamma,1-\gamma)'} := \boldsymbol{\pi}_{C'} \dot{\boldsymbol{P}}_{C'} \boldsymbol{1}, \quad \sigma^2_{(\gamma,1-\gamma)'} := \boldsymbol{\pi}_{C'} \ddot{\boldsymbol{P}}_{C'} \boldsymbol{1} - (\boldsymbol{\pi}_{C'} \dot{\boldsymbol{P}}_{C'} \boldsymbol{1})^2 + 2\boldsymbol{\pi}_{C'} \dot{\boldsymbol{P}}_{C'} (\boldsymbol{Z}_{C'} - \boldsymbol{1}\boldsymbol{\pi}_{C'}) \dot{\boldsymbol{P}}_{C'} \boldsymbol{1}, \tag{2.22}$$

where $\boldsymbol{1}$ is the column vector of 1s of the appropriate dimension, $\dot{\boldsymbol{P}}_{C'}$ is $(1-\gamma)\dot{\boldsymbol{P}}_B$ with each q_m replaced by $-q_m$, and $\ddot{\boldsymbol{P}}_{C'} = (1-\gamma)\boldsymbol{P}_B$. This "rule of thumb" for $\dot{\boldsymbol{P}}_{C'}$ requires some caution: It must be applied before any simplifications to $\boldsymbol{P}_{C'}$ are made using $q_m = 1 - p_m$. Of course, $\boldsymbol{\pi}_{C'}$ is the unique stationary distribution, and $\boldsymbol{Z}_{C'}$ is the fundamental matrix, of $\boldsymbol{P}_{C'}$. Notice that $\dot{\boldsymbol{P}}^\circ_{A'} = \boldsymbol{0}$, so $\dot{\boldsymbol{P}}^\circ_{C'} = (1-\gamma)\dot{\boldsymbol{P}}^\circ_B$ and $\ddot{\boldsymbol{P}}^\circ_{C'} = (1-\gamma)\ddot{\boldsymbol{P}}^\circ_B$.

Theorem 2.6.

$$\mu^\circ_{(\gamma,1-\gamma)'} = \mu_{(\gamma,1-\gamma)'} \tag{2.23}$$

and

$$(\sigma^\circ_{(\gamma,1-\gamma)'})^2 = \sigma^2_{(\gamma,1-\gamma)'}. \tag{2.24}$$

2.4 Application to game $C' := (A')^r B^s$

Next we need versions of the SLLN and the CLT suited to game $C' := (A')^r B^s$. The key result is Theorem 2.2.

For the same reason as before, the theorem does not apply directly to $\boldsymbol{P}_{A'}$ and $\boldsymbol{P}_B$. Therefore, we again consider the Markov chains in the augmented state space $\Sigma^\circ := \Sigma \times \{-1, 0, 1\}$ with one-step transition matrix $\boldsymbol{P}_{A'}^\circ$ and $\boldsymbol{P}_B^\circ$. The definitions are as in (2.14)–(2.21) with $\gamma = 1$ or $\gamma = 0$. With $\boldsymbol{W}^\circ$ as before, the theorem applies.

Fix $r, s \geq 1$. Assume that $\boldsymbol{P}^\circ := (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^s$, as well as all cyclic permutations of $(\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^s$, are ergodic, and let the row vector $\boldsymbol{\pi}^\circ$ be the unique stationary distribution of $\boldsymbol{P}^\circ$. Let

$$\mu_{[r,s]'}^\circ := \frac{1}{r+s} \sum_{v=0}^{s-1} \boldsymbol{\pi}^\circ (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^v \dot{\boldsymbol{P}}_B^\circ \boldsymbol{1}$$

and

$$\begin{aligned}(\sigma_{[r,s]'}^\circ)^2 = 1 - \frac{1}{r+s} \sum_{v=0}^{s-1} (\boldsymbol{\pi}^\circ (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^v \dot{\boldsymbol{P}}_B^\circ \boldsymbol{1})^2 \\ + \frac{2}{r+s} \bigg[\sum_{0 \leq u < v \leq s-1} \boldsymbol{\pi}^\circ (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^u \dot{\boldsymbol{P}}_B^\circ ((\boldsymbol{P}_B^\circ)^{v-u-1} - \boldsymbol{1}\boldsymbol{\pi}^\circ (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^v) \dot{\boldsymbol{P}}_B^\circ \boldsymbol{1} \\ + \sum_{u=0}^{s-1} \sum_{v=0}^{s-1} \boldsymbol{\pi}^\circ (\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^u \dot{\boldsymbol{P}}_B^\circ (\boldsymbol{P}_B^\circ)^{s-u-1} (\boldsymbol{Z}^\circ - \boldsymbol{1}\boldsymbol{\pi}^\circ)(\boldsymbol{P}_{A'}^\circ)^r (\boldsymbol{P}_B^\circ)^v \dot{\boldsymbol{P}}_B^\circ \boldsymbol{1} \bigg].\end{aligned}$$

Let $\{X_n^\circ\}_{n \geq 0}$ be a temporally nonhomogeneous Markov chain in Σ° with one-step transition matrices $\boldsymbol{P}_{A'}^\circ, \ldots, \boldsymbol{P}_{A'}^\circ$ (r times), $\boldsymbol{P}_B^\circ, \ldots, \boldsymbol{P}_B^\circ$ (s times), $\boldsymbol{P}_{A'}^\circ, \ldots, \boldsymbol{P}_{A'}^\circ$ (r times), $\boldsymbol{P}_B^\circ, \ldots, \boldsymbol{P}_B^\circ$ (s times), and so on. For each $n \geq 1$, define $\xi_n := w^\circ(X_{n-1}^\circ, X_n^\circ)$ and $S_n := \xi_1 + \cdots + \xi_n$.

Theorem 2.7. *Under the above assumptions, and with the distribution of X_0 arbitrary,*

$$\frac{S_n}{n} \to \mu_{[r,s]'}^\circ \quad \text{a.s.}$$

and, if $(\sigma_{[r,s]'}^\circ)^2 > 0$, then

$$\frac{S_n - n\mu_{[r,s]'}^\circ}{\sqrt{n(\sigma_{[r,s]'}^\circ)^2}} \to_d N(0,1) \ \textit{as}\ n \to \infty.$$

Again there are simpler expressions for this mean and variance. We define $\mu_{[r,s]'}$ in terms of $\boldsymbol{\pi}$, $\boldsymbol{P}_{A'}$, $\boldsymbol{P}_B$, and $\dot{\boldsymbol{P}}_B$ in the same way that $\mu_{[r,s]'}^\circ$ was defined in terms of $\boldsymbol{\pi}^\circ$, $\boldsymbol{P}_{A'}^\circ$, $\boldsymbol{P}_B^\circ$, and $\dot{\boldsymbol{P}}_B^\circ$. ($\dot{\boldsymbol{P}}_B$ is defined by the rule of thumb.) Finally, $\sigma_{[r,s]'}^2$ is defined analogously to $(\sigma_{[r,s]'}^\circ)^2$.

Theorem 2.8.

$$\mu^{\circ}_{[r,s]'} = \mu_{[r,s]'} \tag{2.25}$$

and

$$(\sigma^{\circ}_{[r,s]'})^2 = \sigma^2_{[r,s]'}. \tag{2.26}$$

Proof. Eq. (2.25) follows exactly as in (2.13). Eq. (2.26) is proved in the same way as (2.24).

CHAPTER 3

NUMERICAL COMPUTATIONS

In this chapter, we compute vairious means numerically by using the reduced state space and use computer graphics to visualize the *Parrondo region* of the Parrondo games of Xie at al.

3.1 State-space reduction

Let us begin by explaining what we mean by state-space reduction, which is an important method for simplifying our computations.

In general, consider an *equivalence relation* $\sim$ on a finite set E. By definition, $\sim$ is *reflexive* ($x \sim x$), *symmetric* ($x \sim y$ implies $y \sim x$), and *transitive* ($x \sim y$ and $y \sim z$ imply $x \sim z$). It is well known that an equivalence relation partitions the set E into *equivalence classes.* The set of all equivalence classes, called the *quotient set*, will be denoted by $\bar{E}$. Let us write $[x] := \{y \in E : y \sim x\}$ for the equivalence class containing x. Then $\bar{E} = \{[x] : x \in E\}$.

Now suppose $X_0, X_1, X_2, \ldots$ is a (time-homogeneous) Markov chain in E with transition matrix $\boldsymbol{P}$. In particular, $P(x, y) = \mathrm{P}(X_{t+1} = y \mid X_t = x)$ for all $x, y \in E$ and $t = 0, 1, 2, \ldots$. Under what conditions on $\boldsymbol{P}$ is $[X_0], [X_1], [X_2], \ldots$ a Markov chain in the "reduced" state space $\bar{E}$? A sufficient condition, apparently due to Kemeny and Snell ([14], p. 124), is that $\boldsymbol{P}$ be *lumpable* with respect to $\sim$. By definition, this means that, for all $x, x', y \in E$,

$$x \sim x' \quad \text{implies} \quad \sum_{y' \in [y]} P(x, y') = \sum_{y' \in [y]} P(x', y'). \tag{3.1}$$

Moreover, if (3.1) holds, then the Markov chain $[X_0], [X_1], [X_2], \ldots$ in $\bar{E}$ has transition matrix $\bar{\boldsymbol{P}}$ given by

$$\bar{P}([x], [y]) := \sum_{y' \in [y]} P(x, y'). \tag{3.2}$$

Notice that (3.1) ensures that (3.2) is well defined.

For Parrondo games with one-dimensional spatial dependence, the state space, assuming $N \geq 3$ players, is

$$\Sigma := \{\eta = (\eta(1), \eta(2), \ldots, \eta(N)) : \eta(x) \in \{0,1\} \text{ for } x = 1, 2, \ldots, N\} = \{0,1\}^N,$$

which has 2^N states. A state $\eta \in \Sigma$ describes the status of each of the N players, 0 for losers and 1 for winners. We can also think of Σ as the set of N-bit binary representations of the integers $0, 1, \ldots, 2^N - 1$, thereby giving a natural ordering to the vectors in Σ.

Ethier and Lee [6] used the following equivalence relation on Σ: $\eta \sim \zeta$ if and only if $\zeta = \eta_\sigma := (\eta(\sigma(1)), \ldots, \eta(\sigma(N)))$ for a permutation σ of $(1, 2, \ldots, N)$ belonging to the cyclic group G of order N of the rotations of the players. If, in addition, $p_1 = p_2$, the permutation σ can belong to the dihedral group G of order $2N$ of the rotations and reflections of the players. They verified the lumpability condition, with the result that the size of the state space was reduced by a factor of nearly N (or $2N$ if $p_1 = p_2$) for large N. It should be noted that a sufficient condition for the lumpability condition in this setting is that, for every $\eta, \zeta \in \Sigma$,

$$P(\eta_\sigma, \zeta_\sigma) = P(\eta, \zeta) \quad \text{for all } \sigma \in G \tag{3.3}$$

or for all σ in a subset of G that generates G.

To fully justify this, the following lemma is useful.

Lemma 3.1 (Ethier and Lee [7]). *Fix $N \geq 3$, let G be a subgroup of the symmetric group S_N. Let $\boldsymbol{P}$ be the one-step transition matrix for a Markov chain in Σ with a unique stationary distribution $\boldsymbol{\pi}$. Assume that*

$$P(\eta_\sigma, \zeta_\sigma) = P(\eta, \zeta), \qquad \sigma \in G,\ \eta, \zeta \in \Sigma. \tag{3.4}$$

Then $\pi(\eta_\sigma) = \pi(\eta)$ for all $\sigma \in G$ and $\eta \in \Sigma$.

Let us say that $\eta \in \Sigma$ is equivalent to $\zeta \in \Sigma$ (written $\eta \sim \zeta$) if there exists $\sigma \in G$ such that $\zeta = \eta_\sigma$, and let us denote the equivalence class containing η by $[\eta]$. Then, in addition, $\boldsymbol{P}$ induces a one-step transition matrix $\bar{\boldsymbol{P}}$ for a Markov chain in the quotient set (i.e., the set of equivalence classes) $\bar{\Sigma}$ defined by the formula

$$\bar{P}([\eta], [\zeta]) := \sum_{\zeta' \in [\zeta]} P(\eta, \zeta'), \tag{3.5}$$

Furthermore, if $\bar{\boldsymbol{P}}$ has a unique stationary distribution $\bar{\boldsymbol{\pi}}$, then the unique stationary distribution $\boldsymbol{\pi}$ is given by $\pi(\eta) = \bar{\pi}([\eta])/|[\eta]|$, where $|[\eta]|$ denotes the cardinality of the equivalence class $[\eta]$.

The lemma will apply to $\boldsymbol{P}_{A'}$ and $\boldsymbol{P}_B$ (hence $\boldsymbol{P}_{C'}$) if we can verify (3.4) for G being the cyclic group of rotations or, if $p_1 = p_2$, the dihedral group of rotations and reflections.

The practical effect of this is that we can reduce the size of the state space (namely, 2^N) to what we will call its *effective size*, which is simply the number of equivalence classes. For example, if $N = 3$, there are eight states and four equivalence classes, namely

$$0 = \{000\}, \quad 1 = \{001, 010, 100\}, \quad 2 = \{011, 101, 110\}, \quad 3 = \{111\}.$$

Notice that we label equivalence classes by the number of 1s each element has. If $N = 4$, there are 16 states and six equivalence classes, namely

$$\begin{aligned} 0 &= \{0000\}, \\ 1 &= \{0001, 0010, 0100, 1000\}, \\ 2 &= \{0011, 0110, 1001, 1100\}, \\ 2' &= \{0101, 1010\}, \\ 3 &= \{0111, 1011, 1101, 1110\}, \\ 4 &= \{1111\}. \end{aligned}$$

In these two cases, it does not matter which of the two equivalence relations we use; the result is the same.

The number of equivalence classes with G being the group of cyclic permutations follows the sequence A000031 in the *The On-Line Encyclopedia of Integer Sequences* (Sloan, 2016), described as the number of necklaces with N beads of two colors when turning over is not allowed. There is an explicit formula in terms of Euler's phi-function. If $p_1 = p_2$, we can reverse the order of the players, and the number of equivalence classes with G being the dihedral group follows the sequence A000029 in the *OEIS*, described as the number of necklaces with N beads of two colors when turning over is allowed. Again, there is an explicit formula. See Table 3.1.

To illustrate our approach in a tractable case, we focus on the case $N = 4$. Here Σ has 16 states, ordered as the 4-bit binary representations of the number 0–15. First, $\boldsymbol{P}_B$ has the form

$$\boldsymbol{P}_B := \frac{1}{4}\begin{pmatrix}
d_0 & p_0 & p_0 & 0 & p_0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_0 & d_1 & 0 & p_1 & 0 & p_0 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_0 & 0 & d_2 & p_2 & 0 & 0 & p_1 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_1 & q_2 & d_3 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 & p_2 & 0 & 0 & 0 & 0 \\
q_0 & 0 & 0 & 0 & d_4 & p_0 & p_2 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\
0 & q_0 & 0 & 0 & q_0 & d_5 & 0 & p_3 & 0 & 0 & 0 & 0 & 0 & p_3 & 0 & 0 \\
0 & 0 & q_1 & 0 & q_2 & 0 & d_6 & p_2 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 & 0 \\
0 & 0 & 0 & q_1 & 0 & q_3 & q_2 & d_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\
q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_8 & p_1 & p_0 & 0 & p_2 & 0 & 0 & 0 \\
0 & q_2 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & d_9 & 0 & p_1 & 0 & p_2 & 0 & 0 \\
0 & 0 & q_0 & 0 & 0 & 0 & 0 & 0 & q_0 & 0 & d_{10} & p_3 & 0 & 0 & p_3 & 0 \\
0 & 0 & 0 & q_2 & 0 & 0 & 0 & 0 & 0 & q_1 & q_3 & d_{11} & 0 & 0 & 0 & p_3 \\
0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_2 & 0 & 0 & 0 & d_{12} & p_1 & p_2 & 0 \\
0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 & q_2 & 0 & 0 & q_1 & d_{13} & 0 & p_3 \\
0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & q_3 & 0 & q_2 & 0 & d_{14} & p_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_3 & 0 & 0 & 0 & q_3 & 0 & q_3 & q_3 & d_{15}
\end{pmatrix},$$

where the diagonal entries are chosen to make the row sums equal to 1:

$$\begin{aligned}
d_0 &:= 4q_0, \\
d_1 = d_2 = d_4 = d_8 &:= p_0 + q_0 + q_1 + q_2, \\
d_3 = d_6 = d_9 = d_{12} &:= p_1 + p_2 + q_1 + q_2, \\
d_5 = d_{10} &:= 2(p_0 + q_3), \\
d_7 = d_{11} = d_{13} = d_{14} &:= p_1 + p_2 + p_3 + q_3, \\
d_{15} &:= 4p_3,
\end{aligned}$$

and $q_m := 1 - p_m$ for $m = 0, 1, 2, 3$. This is consistent with Eq. (12) of Xie and others [21].

For the equivalence relation mentioned above, there are six equivalence classes, namely, $\{0000\}$, $\{0001, 0010, 0100, 1000\}$, $\{0011, 0110, 1001, 1100\}$, $\{0101, 1010\}$, $\{0111, 1011, 1101, 1110\}$, and $\{1111\}$. Denoting the states by their decimal representations (0–15), the equivalence classes are $\{0\}$, $\{1, 2, 4, 8\}$, $\{3, 6, 9, 12\}$, $\{5, 10\}$, $\{7, 11, 13, 14\}$, and $\{15\}$. It will be convenient to reorder the states temporarily. Within each equivalence class, we order elements so that each is a fixed rotation of the preceding one, that is, $\{0000\}$, $\{1000, 0100, 0010, 0001\}$, $\{1100, 0110, 0011, 1001\}$, $\{1010, 0101\}$, $\{1110, 0111, 1011, 1101\}$, and $\{1111\}$, or $\{0\}$, $\{8, 4, 2, 1\}$, $\{12, 6, 3, 9\}$, $\{10, 5\}$, $\{14, 7, 11, 13\}$, and $\{15\}$. We now order states in this order: 0, 8, 4, 2, 1, 12, 6, 3, 9, 10, 5, 14, 7, 11, 13, 15, which leads to an alternative form for the transition matrix, namely

$$\boldsymbol{P}_B := \frac{1}{4}\left(\begin{array}{c|cccc|cccc|cc|cccc|c}
d_0 & p_0 & p_0 & p_0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
q_0 & d_8 & 0 & 0 & 0 & p_2 & 0 & 0 & p_1 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_0 & 0 & d_4 & 0 & 0 & p_1 & p_2 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 \\
q_0 & 0 & 0 & d_2 & 0 & 0 & p_1 & p_2 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q_0 & 0 & 0 & 0 & d_1 & 0 & 0 & p_1 & p_2 & 0 & p_0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & q_2 & q_1 & 0 & 0 & d_{12} & 0 & 0 & 0 & 0 & 0 & p_2 & 0 & 0 & p_1 & 0 \\
0 & 0 & q_2 & q_1 & 0 & 0 & d_6 & 0 & 0 & 0 & 0 & p_1 & p_2 & 0 & 0 & 0 \\
0 & 0 & 0 & q_2 & q_1 & 0 & 0 & d_3 & 0 & 0 & 0 & 0 & p_1 & p_2 & 0 & 0 \\
0 & q_1 & 0 & 0 & q_2 & 0 & 0 & 0 & d_9 & 0 & 0 & 0 & 0 & p_1 & p_2 & 0 \\
\hline
0 & q_0 & 0 & q_0 & 0 & 0 & 0 & 0 & 0 & d_{10} & 0 & p_3 & 0 & p_3 & 0 & 0 \\
0 & 0 & q_0 & 0 & q_0 & 0 & 0 & 0 & 0 & 0 & d_5 & 0 & p_3 & 0 & p_3 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & q_2 & q_1 & 0 & 0 & q_3 & 0 & d_{14} & 0 & 0 & 0 & p_3 \\
0 & 0 & 0 & 0 & 0 & 0 & q_2 & q_1 & 0 & 0 & q_3 & 0 & d_7 & 0 & 0 & p_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q_2 & q_1 & q_3 & 0 & 0 & 0 & d_{11} & 0 & p_3 \\
0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & q_2 & 0 & q_3 & 0 & 0 & 0 & d_{13} & p_3 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_3 & q_3 & q_3 & q_3 & d_{15}
\end{array}\right).$$

The lumpability condition requires that, within each block, row sums be equal. That this condition is met can be seen at a glance. Moreover, we can also see that the sufficient condition (3.3) holds as well. Because of how we ordered the states, this condition requires that each block be constant along each diagonal parallel to the main diagonal (assuming periodic boundary conditions).

We conclude that

$$\bar{\boldsymbol{P}}_B = \frac{1}{4}\begin{pmatrix}
4q_0 & 4p_0 & 0 & 0 & 0 & 0 \\
q_0 & p_0+q_0+q_1+q_2 & p_1+p_2 & p_0 & 0 & 0 \\
0 & q_1+q_2 & p_1+p_2+q_1+q_2 & 0 & p_1+p_2 & 0 \\
0 & 2q_0 & 0 & 2(p_0+q_3) & 2p_3 & 0 \\
0 & 0 & q_1+q_2 & q_3 & p_1+p_2+p_3+q_3 & p_3 \\
0 & 0 & 0 & 0 & 4q_3 & 4p_3
\end{pmatrix}. \tag{3.6}$$

We turn next to game A'. Again there are 16 states (namely, the 4-bit binary representations of the integers 0–15) and the transition matrix has the form

$$\boldsymbol{P}_{A'} := \frac{1}{8}\begin{pmatrix}
0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 & 2 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 2 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 2 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 2 & 2 & 0
\end{pmatrix}.$$

To verify the lumpability condition we reorder the states and rewrite the matrix in block form as we did for $\boldsymbol{P}_B$:

$$\boldsymbol{P}_{A'} := \frac{1}{8}\left(\begin{array}{c|cccc|cccc|cc|cccc|c}
0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 1 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 2 & 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 2 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 & 0
\end{array}\right).$$

Again the condition is clearly met, and we have

$$\bar{\boldsymbol{P}}_{A'} = \frac{1}{4}\begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 \\
0 & 2 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 1 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 4 & 0
\end{pmatrix}. \tag{3.7}$$

We next check (3.4) in the general case when G is the cyclic subgroup of rotations of $(1, 2, \ldots, N)$, that is, the group generated by

$$(\sigma(1), \sigma(2), \ldots, \sigma(N)) := (2, 3, \ldots, N, 1). \tag{3.8}$$

Indeed, for such σ,

$$\begin{aligned} P_B(\eta_\sigma, (\eta_x)_\sigma) &= P_B(\eta_\sigma, (\eta_\sigma)_{\sigma^{-1}(x)}) \\ &= \begin{cases} N^{-1} p_{m_{\sigma^{-1}(x)}(\eta_\sigma)} & \text{if } \eta_\sigma(\sigma^{-1}(x)) = 0 \\ N^{-1} q_{m_{\sigma^{-1}(x)}(\eta_\sigma)} & \text{if } \eta_\sigma(\sigma^{-1}(x)) = 1 \end{cases} \\ &= \begin{cases} N^{-1} p_{m_x(\eta)} & \text{if } \eta(x) = 0 \\ N^{-1} q_{m_x(\eta)} & \text{if } \eta(x) = 1 \end{cases} \\ &= P_B(\eta, \eta_x) \end{aligned} \tag{3.9}$$

for $x = 1, \ldots, N$ and all $\eta \in \Sigma$, where the third equality uses

$$m_x(\eta_\sigma) = m_{\sigma(x)}(\eta). \tag{3.10}$$

If $p_1 = p_2$, then (3.9) also applies to the order-reversing permutation (or reflection) of $(1, 2, \ldots, N)$,

$$(\sigma(1), \sigma(2), \ldots, \sigma(N)) := (N, N-1, \ldots, 2, 1). \tag{3.11}$$

For $\boldsymbol{P}_{A'}$, we can verify the lumpability condition (3.3) by observing that, if $(\sigma(1), \ldots, \sigma(N)) = (2, 3, \ldots, N, 1)$, then

$$\begin{aligned} P_{A'}(\eta_\sigma, \zeta_\sigma) &= (4N)^{-1} \sum_{x=1}^{N} [\delta((\eta_\sigma)^{x,x-1,-1}, \zeta_\sigma) + \delta((\eta_\sigma)^{x,x-1,1}, \zeta_\sigma) \\ &\qquad + \delta((\eta_\sigma)^{x,x+1,-1}, \zeta_\sigma) + \delta((\eta_\sigma)^{x,x+1,1}, \zeta_\sigma)] \\ &= (4N)^{-1} \sum_{x=1}^{N} [\delta((\eta_\sigma)^{\sigma^{-1}(x),\sigma^{-1}(x-1),-1}, \zeta_\sigma) + \delta((\eta_\sigma)^{\sigma^{-1}(x),\sigma^{-1}(x-1),1}, \zeta_\sigma) \\ &\qquad + \delta((\eta_\sigma)^{\sigma^{-1}(x),\sigma^{-1}(x+1),-1}, \zeta_\sigma) + \delta((\eta_\sigma)^{\sigma^{-1}(x),\sigma^{-1}(x+1),1}, \zeta_\sigma)] \\ &= (4N)^{-1} \sum_{x=1}^{N} [\delta((\eta^{x,x-1,-1})_\sigma, \zeta_\sigma) + \delta((\eta^{x,x-1,1})_\sigma, \zeta_\sigma) \\ &\qquad + \delta((\eta^{x,x+1,-1})_\sigma, \zeta_\sigma) + \delta((\eta^{x,x+1,1})_\sigma, \zeta_\sigma)] \\ &= (4N)^{-1} \sum_{x=1}^{N} [\delta(\eta^{x,x-1,-1}, \zeta) + \delta(\eta^{x,x-1,1}, \zeta) + \delta(\eta^{x,x+1,-1}, \zeta) + \delta(\eta^{x,x+1,1}, \zeta)] \\ &= P_{A'}(\eta, \zeta) \end{aligned}$$

since $(\eta_\sigma)^{\sigma^{-1}(x),\sigma^{-1}(x\pm1),s} = (\eta^{x,x\pm1,s})_\sigma$. If $(\sigma(1),\ldots,\sigma(N)) = (N, N-1,\ldots,2,1)$, then the same sequence of identities holds.

A fairly explicit formula for $\bar{\boldsymbol{P}}_B$ is given in Ethier and Lee (2012a). First, define the function $s : \bar{\Sigma} \mapsto \{0,1,\ldots,N\}$ by $s([\eta]) := \eta(1)+\eta(2)+\cdots+\eta(N)$; it counts the number of 1s in each element of an equivalence class. Then

$$\bar{P}_B([\eta],[\zeta]) = \begin{cases} N^{-1}\big(\sum_{x:\eta(x)=0} q_{m_x(\eta)} + \sum_{x:\eta(x)=1} p_{m_x(\eta)}\big) & \text{if } [\zeta]=[\eta] \\ N^{-1}\sum_{x:\eta(x)=1,\eta_x\sim\zeta} q_{m_x(\eta)} & \text{if } s([\zeta]) = s([\eta])-1 \\ N^{-1}\sum_{x:\eta(x)=0,\eta_x\sim\zeta} p_{m_x(\eta)} & \text{if } s([\zeta]) = s([\eta])+1 \\ 0 & \text{otherwise} \end{cases}$$

for all $[\eta],[\zeta]\in\bar{\Sigma}$. A formula for $\bar{\boldsymbol{P}}_{A'}$ is

$$\bar{P}_{A'}([\eta],[\zeta]) = \begin{cases} (4N)^{-1}\sum_{x=1}^N \big(1_{[\zeta]}(\eta^{x,x-1,-1}) + 1_{[\zeta]}(\eta^{x,x-1,1}) \\ \qquad + 1_{[\zeta]}(\eta^{x,x+1,-1}) + 1_{[\zeta]}(\eta^{x,x+1,1})\big) & \text{if } |s([\zeta])-s([\eta])|\le 1 \\ 0 & \text{otherwise} \end{cases}$$

for all $[\eta],[\zeta]\in\bar{\Sigma}$. This generalizes (3.7).

3.2 Means and variances

We saw in Theorems 2.6 and 2.8 that the means and variances that appear in the SLLNs and CLTs of Sections 2.2–2.4 (namely, μ_B°, $\mu^\circ_{(\gamma,1-\gamma)'}$, $\mu^\circ_{[r,s]'}$, $(\sigma_B^\circ)^2$, $(\sigma^\circ_{(\gamma,1-\gamma)'})^2$, and $(\sigma^\circ_{[r,s]'})^2$) are equal to the corresponding quantities defined in terms of the original transition matrices (namely, μ_B, $\mu_{(\gamma,1-\gamma)'}$, $\mu_{[r,s]'}$, σ_B^2, $\sigma^2_{(\gamma,1-\gamma)'}$, and $\sigma^2_{[r,s]'}$). We claim that the corresponding quantities defined in terms of the reduced transition matrices (namely, $\bar{\mu}_B$, $\bar{\mu}_{(\gamma,1-\gamma)'}$, $\bar{\mu}_{[r,s]'}$, $\bar{\sigma}_B^2$, $\bar{\sigma}^2_{(\gamma,1-\gamma)'}$, and $\bar{\sigma}^2_{[r,s]'}$) are also equal. First, we define

$$\bar{\mu}_B := \bar{\boldsymbol{\pi}}_B \dot{\bar{\boldsymbol{P}}}_B \mathbf{1}, \tag{3.12}$$

$$\bar{\mu}_{(\gamma,1-\gamma)'} := (1-\gamma)\bar{\boldsymbol{\pi}}_{C'}\dot{\bar{\boldsymbol{P}}}_B\mathbf{1}, \tag{3.13}$$

$$\bar{\mu}_{[r,s]'} := \frac{1}{r+s}\sum_{v=0}^{s-1}\bar{\boldsymbol{\pi}}\bar{\boldsymbol{P}}_{A'}^r\bar{\boldsymbol{P}}_B^v\dot{\bar{\boldsymbol{P}}}_B\mathbf{1}, \tag{3.14}$$

$$\bar{\sigma}_B^2 := \bar{\boldsymbol{\pi}}_B\ddot{\bar{\boldsymbol{P}}}_B\mathbf{1} - (\bar{\boldsymbol{\pi}}_B\dot{\bar{\boldsymbol{P}}}_B\mathbf{1})^2 + 2\bar{\boldsymbol{\pi}}_B\dot{\bar{\boldsymbol{P}}}_B(\bar{\boldsymbol{Z}}_B - \mathbf{1}\bar{\boldsymbol{\pi}}_B)\dot{\bar{\boldsymbol{P}}}_B\mathbf{1}, \tag{3.15}$$

$$\bar{\sigma}^2_{(\gamma,1-\gamma)'} := \bar{\boldsymbol{\pi}}_{C'}\ddot{\bar{\boldsymbol{P}}}_{C'}\mathbf{1} - (\bar{\boldsymbol{\pi}}_{C'}\dot{\bar{\boldsymbol{P}}}_{C'}\mathbf{1})^2 + 2\bar{\boldsymbol{\pi}}_{C'}\dot{\bar{\boldsymbol{P}}}_{C'}(\bar{\boldsymbol{Z}}_{C'} - \mathbf{1}\bar{\boldsymbol{\pi}}_{C'})\dot{\bar{\boldsymbol{P}}}_{C'}\mathbf{1}, \tag{3.16}$$

$$\bar{\sigma}^2_{[r,s]'} := 1 - \frac{1}{r+s}\sum_{v=0}^{s-1}(\bar{\boldsymbol{\pi}}\bar{\boldsymbol{P}}_{A'}^r\bar{\boldsymbol{P}}_B^v\dot{\bar{\boldsymbol{P}}}_B\mathbf{1})^2$$
$$+ \frac{2}{r+s}\bigg[\sum_{0\le u<v\le s-1}\bar{\boldsymbol{\pi}}\bar{\boldsymbol{P}}_{A'}^r\bar{\boldsymbol{P}}_B^u\dot{\bar{\boldsymbol{P}}}_B(\bar{\boldsymbol{P}}_B^{v-u-1} - \mathbf{1}\bar{\boldsymbol{\pi}}\bar{\boldsymbol{P}}_{A'}^r\bar{\boldsymbol{P}}_B^v)\dot{\bar{\boldsymbol{P}}}_B\mathbf{1}$$

$$+\sum_{u=0}^{s-1}\sum_{v=0}^{s-1}\bar{\boldsymbol{\pi}}\bar{\boldsymbol{P}}_{A'}^{r}\bar{\boldsymbol{P}}_{B}^{u}\dot{\bar{\boldsymbol{P}}}_{B}\bar{\boldsymbol{P}}_{B}^{s-u-1}(\bar{\boldsymbol{Z}}-\mathbf{1}\bar{\boldsymbol{\pi}})\bar{\boldsymbol{P}}_{A'}^{r}\bar{\boldsymbol{P}}_{B}^{v}\dot{\bar{\boldsymbol{P}}}_{B}\mathbf{1}\Bigg]. \tag{3.17}$$

Theorem 3.2.

$$\mu_B = \bar{\mu}_B, \qquad \mu_{(\gamma,1-\gamma)'} = \bar{\mu}_{(\gamma,1-\gamma)'}, \qquad \mu_{[r,s]'} = \bar{\mu}_{[r,s]'}$$

and

$$\sigma_B^2 = \bar{\sigma}_B^2, \qquad \sigma_{(\gamma,1-\gamma)'}^2 = \bar{\sigma}_{(\gamma,1-\gamma)'}^2, \qquad \sigma_{[r,s]'}^2 = \bar{\sigma}_{[r,s]'}^2.$$

Proof. A result of Ethier and Lee [7] implies that, if $\boldsymbol{Q}$ is a G-invariant square (not necessarily stochastic) matrix (i.e., $Q(\eta_\sigma, \zeta_\sigma) = Q(\eta, \zeta)$ for all $\eta, \zeta \in \Sigma$ and all $\sigma \in G$), then

$$\boldsymbol{\pi}\boldsymbol{Q}\mathbf{1} = \bar{\boldsymbol{\pi}}\bar{\boldsymbol{Q}}\mathbf{1}.$$

Repeated application of this identity gives the desired conclusions.

The formulas for the means and variances with bars are computable for $3 \le N \le 18$, at least. We give results for the three choices of the parameter vector (p_0, p_1, p_2, p_3) treated by Ethier and Lee [6] in Table 3.2, 3.3, and 3.4 and three other choices in Table 3.5, 3.6, and 3.7.

3.3 Computer graphics

Ethier and Lee [10] sketched, for games A', B, and $C' := \frac{1}{2}A' + \frac{1}{2}B$, the Parrondo and anti-Parrondo regions when $3 \le N \le 9$. They assumed that $p_1 = p_2$ and relabeled p_3 as p_2. In other words, their parameter vector was of the form (p_0, p_1, p_1, p_2). (The reason for this simplification is that a three-dimensional figure is easier to visualize than a four-dimensional figure.) See Figure 3.1, which includes only the cases $3 \le N \le 8$. The figures for games A', B, and C' are distinctively different from those for games A, B, and $C := \frac{1}{2}A + \frac{1}{2}B$. In both cases, the general shape of the Parrondo and anti-Parrondo regions does not change much, once $N \ge 5$.

Here we do the same for games A', B, and $(A')^r B^s$ for $[r, s]' = [1, 1]', [2, 1]', [1, 2]', [2, 2]'$ and $3 \le N \le 6$ in Figure 3.2, 3.3, 3.4, and 3.5 respectively. Larger N could be considered, but it would be very time-consuming.

Table 3.1. The size and effective size of the state space when there are N players. (Used by permission from Ethier and Lee [6].)

number of players N	size of state space 2^N	effective size not assuming $p_1 = p_2$	effective size assuming $p_1 = p_2$
3	8	4	4
4	16	6	6
5	32	8	8
6	64	14	13
7	128	20	18
8	256	36	30
9	512	60	46
10	1024	108	78
11	2048	188	126
12	4096	352	224
13	8192	632	380
14	16384	1182	687
15	32768	2192	1224
16	65536	4116	2250
17	131072	7712	4112
18	262144	14602	7685
19	524288	27596	14310
20	1048576	52488	27012

Table 3.2. Mean profit per turn at equilibrium in the games of Xie and others (2011), assuming $(p_0, p_1, p_2, p_3) = (1, 0.16, 0.16, 0.7)$. Results are given to six significant digits. Notice that $\mu_B < 0$ for $3 \le N \le 19$ except for $N = 4, 7, 8$, so the Parrondo effect is present except in 28 of the 113 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3). The blank entries were not computed. The behavior in the $\mu_{[1,1]'}$ column cannot easily be explained.

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	−0.0909091	−0.0766158	−0.105479	−0.102038	−0.0993971	−0.0724638	−0.0773252	−0.0547919
4	0.00799608	0.0156538	0.00471698	0.0148270	0.0239996	0.00325815	0.0125698	0.00247440
5	−0.00219465	0.00565126	0.00593697	0.00950811	0.00863794	0.00345975	0.00774689	0.00248827
6	−0.0189247	0.00671656	0.00640351	0.00955597	0.00994894	0.00363075	0.00745377	0.00255559
7	0.00350598	0.00680337	0.00660065	0.00923799	0.00960154	0.00380363	0.00724937	0.00263585
8	0.000698188	0.00678290	0.00670338	0.00901760	0.00922885	0.00393034	0.00713139	0.00270983
9	−0.00189233	0.00678314	0.00676079	0.00887095	0.00900601	0.00402382	0.00705972	0.00277296
10	−0.000332809	0.00678338	0.00679458	0.00876090	0.00884263	0.00409432	0.00701287	0.00282563
11	−0.000466527	0.00678336	0.00681519	0.00867466	0.00871398	0.00414879	0.00698037	0.00286945
12	−0.000676916	0.00678336	0.00682799	0.00860524	0.00861168	0.00419181	0.00695667	0.00290610
13	−0.000562901	0.00678336	0.00683598	0.00854800	0.00852823	0.00422647	0.00693865	0.00293700
14	−0.000569340	0.00678336	0.00684090	0.00849991	0.00845874	0.00425488	0.00692447	0.00296329
15	−0.000586184	0.00678336	0.00684381	0.00845891	0.00839996	0.00427852	0.00691300	0.00298586
16	−0.000578161	0.00678336	0.00684537	0.00842351	0.00834957	0.00429845	0.00690350	0.00300539
17	−0.000578345	0.00678336	0.00684603	0.00839260	0.00830588	0.00431545	0.00689547	0.00302245
18	−0.000579652	0.00678336	0.00684607	0.00836539	0.00826762	0.00433011	0.00688859	0.00303744
19	−0.000579095	0.00678336						
∞		0.00678336	0.00678336	0.00792947	0.00768253	0.00451510	0.00678336	0.00325825

Table 3.3. Mean profit per turn at equilibrium in the games of Xie and others [21], assuming $(p_0, p_1, p_2, p_3) = (0.7, 0.68, 0.68, 0)$. Results are given to six significant digits. Notice that $\mu_B < 0$ for $3 \le N \le 19$ except $N = 3, 5$, so the Parrondo effect is present except in 15 of the 107 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3). The blank entries were not computed.

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	0.0710383	0.0525560	0.0636364	0.0684422	0.0697086	0.0453956	0.0547457	0.0346926
4	−0.0425713	0.00095265	0.0131579	0.00461772	−0.00332027	0.00948905	0.00383213	0.00732323
5	0.00257895	0.00765099	0.0114243	0.00947342	0.00771190	0.00914217	0.00893883	0.00720557
6	−0.0102930	0.00684126	0.0103803	0.00812693	0.00550684	0.00876089	0.00887906	0.00705384
7	−0.00722622	0.00691714	0.00975869	0.00772430	0.00542861	0.00845243	0.00877155	0.00690363
8	−0.00808338	0.00691038	0.00932190	0.00733991	0.00509927	0.00821008	0.00862704	0.00677033
9	−0.00784318	0.00691100	0.00900193	0.00706166	0.00488742	0.00801817	0.00848795	0.00665614
10	−0.00790952	0.00691094	0.00875695	0.00684162	0.00470929	0.00786365	0.00836210	0.00655914
11	−0.00789119	0.00691095	0.00856340	0.00666505	0.00456481	0.00773709	0.00825093	0.00647657
12	−0.00789624	0.00691095	0.00840660	0.00652010	0.00444435	0.00763177	0.00815337	0.00640584
13	−0.00789485	0.00691095	0.00827699	0.00639908	0.00434270	0.00754288	0.00806773	0.00634480
14	−0.00789523	0.00691095	0.00816807	0.00629653	0.00425580	0.00746692	0.00799229	0.00629171
15	−0.00789513	0.00691095	0.00807523	0.00620855	0.00418071	0.00740130	0.00792555	0.00624519
16	−0.00789516	0.00691095	0.00799517	0.00613226	0.00411520	0.00734408	0.00786619	0.00620414
17	−0.00789515	0.00691095	0.00792541	0.00606547	0.00405756	0.00729375	0.00781314	0.00616766
18	−0.00789515	0.00691095						
19	−0.00789515	0.00691095						
∞	−0.00789515	0.00691095	0.00691095	0.00506459	0.00316518	0.00651648	0.00691095	0.00556811

Table 3.4. Mean profit per turn at equilibrium in the games of Xie and others [21], assuming $(p_0, p_1, p_2, p_3) = (0.1, 0.6, 0.6, 0.75)$. Results are given to six significant digits. Notice that $\mu_B < 0$ for $3 \leq N \leq 19$, so the Parrondo effect is present in all 107 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3). The blank entries were not computed.

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	−0.190476	0.0250737	0.0459318	0.0459668	0.0355311	0.0285193	0.0313076	0.0210062
4	−0.0858189	0.0175362	0.0143678	0.0270909	0.0320082	0.00877193	0.0193958	0.00639330
5	−0.0389980	0.0169208	0.0153125	0.0243523	0.0288300	0.00901530	0.0166844	0.00648456
6	−0.0183165	0.0168327	0.0157452	0.0240900	0.0280593	0.00924263	0.0165391	0.00658522
7	−0.00924232	0.0168224	0.0160005	0.0239721	0.0276476	0.00941580	0.0165074	0.00667774
8	−0.00528548	0.0168213	0.0161641	0.0238960	0.0273667	0.00954592	0.0165118	0.00675629
9	−0.00356984	0.0168212	0.0162764	0.0238393	0.0271612	0.00964554	0.0165279	0.00682153
10	−0.00282963	0.0168212	0.0163577	0.0237943	0.0270043	0.00972360	0.0165471	0.00687564
11	−0.00251155	0.0168211	0.0164188	0.0237576	0.0268804	0.00978610	0.0165662	0.00692084
12	−0.00237531	0.0168211	0.0164664	0.0237269	0.0267800	0.00983717	0.0165840	0.00695896
13	−0.00231709	0.0168211	0.0165043	0.0237008	0.0266969	0.00987957	0.0166002	0.00699142
14	−0.00229226	0.0168211	0.0165351	0.0236782	0.0266269	0.00991530	0.0166148	0.00701933
15	−0.00228169	0.0168211	0.0165607	0.0236585	0.0265671	0.00994578	0.0166279	0.00704355
16	−0.00227719	0.0168211	0.0165822	0.0236412	0.0265154	0.00997208	0.0166396	0.00706473
17	−0.00227528	0.0168211	0.0166006	0.0236258	0.0264703	0.00999500	0.0166501	0.00708341
18	−0.00227446	0.0168211						
19	−0.00227412	0.0168211						
∞		0.0168211	0.0168211	0.0233648	0.0257907	0.0103217	0.0168211	0.00737111

Table 3.5. Mean profit per turn at equilibrium in the games of Xie and others [21], assuming $(p_0, p_1, p_2, p_3) = (0, 0.8, 0.8, 0.5)$. Results are given to six significant digits. Notice that $\mu_B = -1$ for $3 \le N \le 17$, so the Parrondo effect is present in all 105 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3).

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	−1	0.0700935	0.125641	0.120719	0.0949431	0.0793651	0.0832734	0.0587049
4	−1	0.0312943	0.0277778	0.0518280	0.0569121	0.0173333	0.0391038	0.0127193
5	−1	0.0334457	0.0297026	0.0466364	0.0552000	0.0177825	0.0328559	0.0128710
6	−1	0.0329776	0.0306359	0.0458271	0.0523336	0.0182363	0.0324445	0.0130605
7	−1	0.0329365	0.0311909	0.0455525	0.0512224	0.0185913	0.0323592	0.0132431
8	−1	0.0329318	0.0315485	0.0453778	0.0505181	0.0188624	0.0323574	0.0134018
9	−1	0.0329314	0.0317936	0.0452410	0.0500111	0.0190722	0.0323834	0.0135356
10	−1	0.0329313	0.0319703	0.0451282	0.0496216	0.0192379	0.0324177	0.0136478
11	−1	0.0329313	0.0321027	0.0450332	0.0493112	0.0193714	0.0324528	0.0137424
12	−1	0.0329313	0.0322052	0.0449521	0.0490575	0.0194808	0.0324861	0.0138225
13	−1	0.0329313	0.0322865	0.0448820	0.0488459	0.0195720	0.0325166	0.0138912
14	−1	0.0329313	0.0323523	0.0448208	0.0486666	0.0196490	0.0325441	0.0139505
15	−1	0.0329313	0.0324067	0.0447670	0.0485126	0.0197149	0.0325688	0.0140021
16	−1	0.0329313	0.0324522	0.0447192	0.0483788	0.0197718	0.0325688	0.0140474
17	−1	0.0329313	0.0324907	0.0446700	0.0482615	0.0198214	0.0326108	0.0140874
∞	−1	0.0329313	0.0329313	0.0439298	0.0464462	0.0205333	0.0329313	0.0147142

Table 3.6. Mean profit per turn at equilibrium in the games of Xie and others [21], assuming $(p_0, p_1, p_2, p_3) = (0.78, 0.65, 0.65, 0)$. Results are given to six significant digits. Notice that $\mu_B < 0$ for $3 \leq N \leq 17$ except for $N = 3, 5$, so the Parrondo effect is present except in 14 of the 105 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3). The blank entries were not computed.

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	0.0649566	0.0478973	0.0569837	0.0623296	0.0637036	0.0409447	0.0502669	0.0313571
4	−0.0388363	0.00577713	0.0173010	0.0105370	0.0034869	0.0125786	0.00889721	0.00973808
5	0.00672837	0.0117080	0.0156329	0.0146990	0.0131555	0.0121857	0.0133209	0.00958235
6	−0.0119357	0.0109211	0.0146005	0.0132671	0.0108107	0.0117909	0.0131938	0.00941062
7	−0.00448927	0.0110144	0.0139764	0.0128216	0.0107077	0.0114764	0.0130462	0.00924977
8	−0.00756841	0.0110033	0.0135325	0.0123876	0.0102846	0.0112301	0.0128741	0.00911021
9	−0.00631011	0.0110047	0.0132049	0.0120781	0.0100324	0.0110351	0.0127156	0.00899195
10	−0.00682669	0.0110045	0.0129524	0.0118314	0.00981392	0.0108779	0.0125752	0.00889207
11	−0.00661495	0.0110045	0.0127520	0.0116336	0.00963960	0.0107491	0.0124525	0.00880731
12	−0.00670177	0.0110045	0.0125890	0.0114710	0.00949408	0.0106417	0.0123457	0.00873485
13	−0.00666618	0.0110045	0.0124539	0.0113350	0.00937165	0.0105509	0.0122523	0.00867238
14	−0.00668077	0.0110045	0.0123400	0.0112197	0.00926711	0.0104733	0.0121703	0.00861808
15	−0.00667479	0.0110045	0.0122427	0.0111207	0.00917686	0.0104061	0.0120980	0.00857051
16	−0.00667724	0.0110045	0.0121586	0.0110348	0.00909819	0.0103475	0.0120337	0.00852853
17	−0.00667623	0.0110045	0.0120853	0.0109595	0.00902901	0.0102959	0.0119764	0.00849125
∞		0.0110045	0.0110045	0.00982296	0.00795942	0.00949246	0.0110045	0.00787643

Table 3.7. Mean profit per turn at equilibrium in the games of Xie and others [21], assuming $(p_0, p_1, p_2, p_3) = (0.9, 0.54, 0.54, 0.05)$. Results are given to six significant digits. Notice that $\mu_B < 0$ for $3 \le N \le 17$ except for $N = 3, 5$, so the Parrondo effect is present except in 14 of the 105 cases. The entries corresponding to $N = \infty$ are limits as $N \to \infty$ (see Theorem 6.3). The blank entries were not computed.

N	μ_B	$\mu_{(1/2,1/2)'}$	$\mu_{[1,1]'}$	$\mu_{[1,2]'}$	$\mu_{[1,3]'}$	$\mu_{[2,1]'}$	$\mu_{[2,2]'}$	$\mu_{[3,1]'}$
3	0.0186100	0.0138568	0.0163482	0.0180393	0.0184102	0.0118203	0.0146580	0.00906909
4	−0.0116982	0.00305201	0.00641026	0.00478337	0.00286234	0.00469314	0.00405576	0.00364322
5	0.00304411	0.00465818	0.00591223	0.00587933	0.00545615	0.00455730	0.00524184	0.00358348
6	−0.00560270	0.00442670	0.00559742	0.00542162	0.00470623	0.00443058	0.00518684	0.00352406
7	−0.00099606	0.00446002	0.00540436	0.00527306	0.00466731	0.00433109	0.00513152	0.00347081
8	−0.00358987	0.00445522	0.00526572	0.00512618	0.00450849	0.00425351	0.00507162	0.00342552
9	−0.00217231	0.00445591	0.00516276	0.00502269	0.00442020	0.00419216	0.00501795	0.00338753
10	−0.00296007	0.00445581	0.00508303	0.00493948	0.00434044	0.00414271	0.00497098	0.00335562
11	−0.00252629	0.00445583	0.00501949	0.00487281	0.00427808	0.00410213	0.00493028	0.00332863
12	−0.00276636	0.00445582	0.00496765	0.00481785	0.00422566	0.00406830	0.00489498	0.00330560
13	−0.00263387	0.00445583	0.00492455	0.00477187	0.00418172	0.00403968	0.00486424	0.00328578
14	−0.00270711	0.00445582	0.00488816	0.00473281	0.00414416	0.00401518	0.00483730	0.00326856
15	−0.00266666	0.00445582	0.00485701	0.00469924	0.00411176	0.00399397	0.00481355	0.00325349
16	−0.00268901	0.00445582	0.00483005	0.00467007	0.00408351	0.00397545	0.00479249	0.00324019
17	−0.00267666	0.00445582	0.00480648	0.00464450	0.00405867	0.00395913	0.00477371	0.00322838
∞		0.00445582	0.00445582	0.00425571		0.00370360	0.00445582	0.00303347

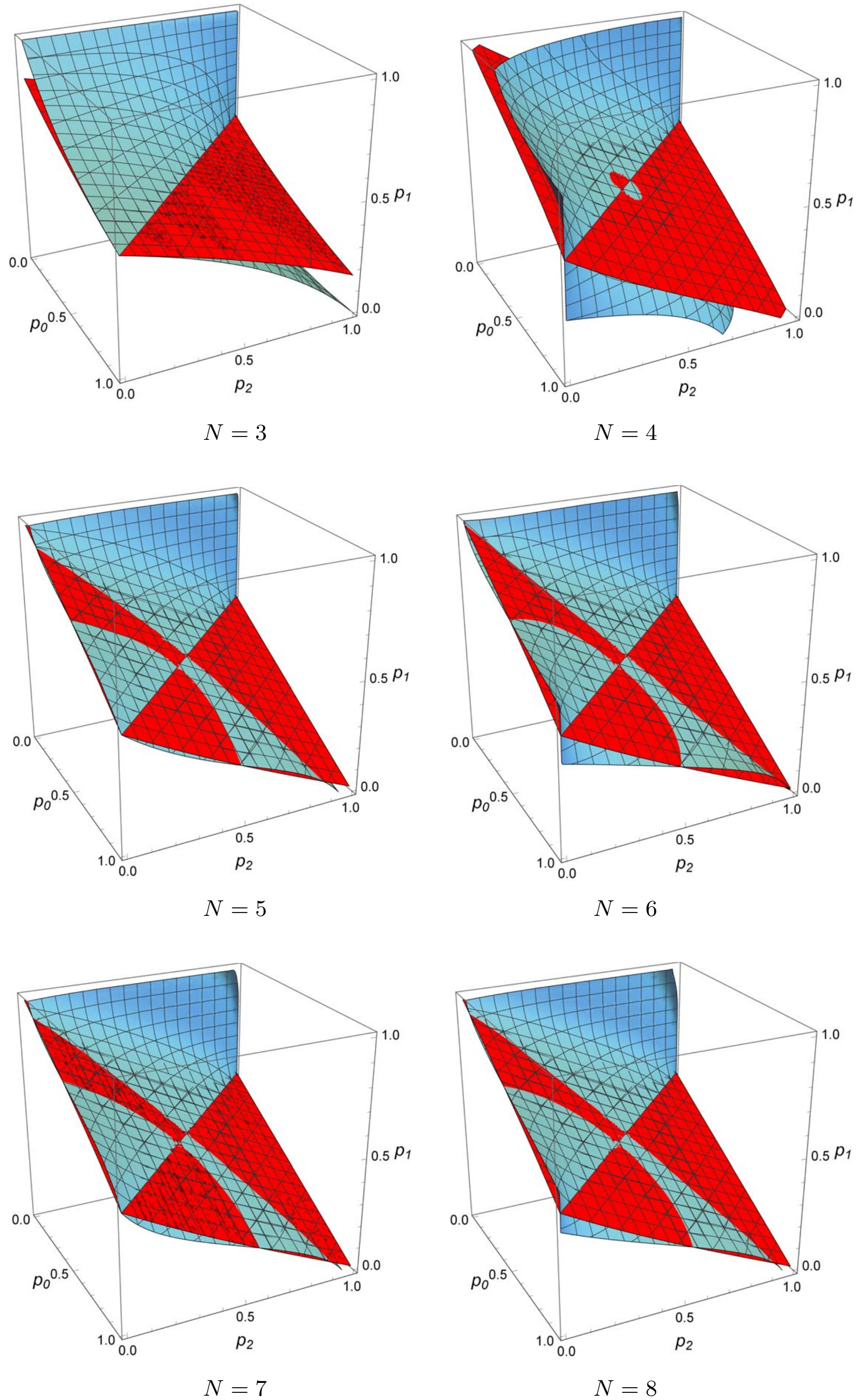

Figure 3.1. For $3 \leq N \leq 8$ and $\gamma = 1/2$, the blue surface is the surface $\mu_B = 0$, and the red surface is the surface $\mu_{(1/2,1/2)'} = 0$, in the (p_0, p_2, p_1) unit cube. The Parrondo region is the region on or below the blue surface and above the red surface, while the anti-Parrondo region is the region on or above the blue surface and below the red surface. Here (p_0, p_1, p_1, p_3) is relabeled as (p_0, p_1, p_1, p_2). (Used by permission from Ethier and Lee [10].)

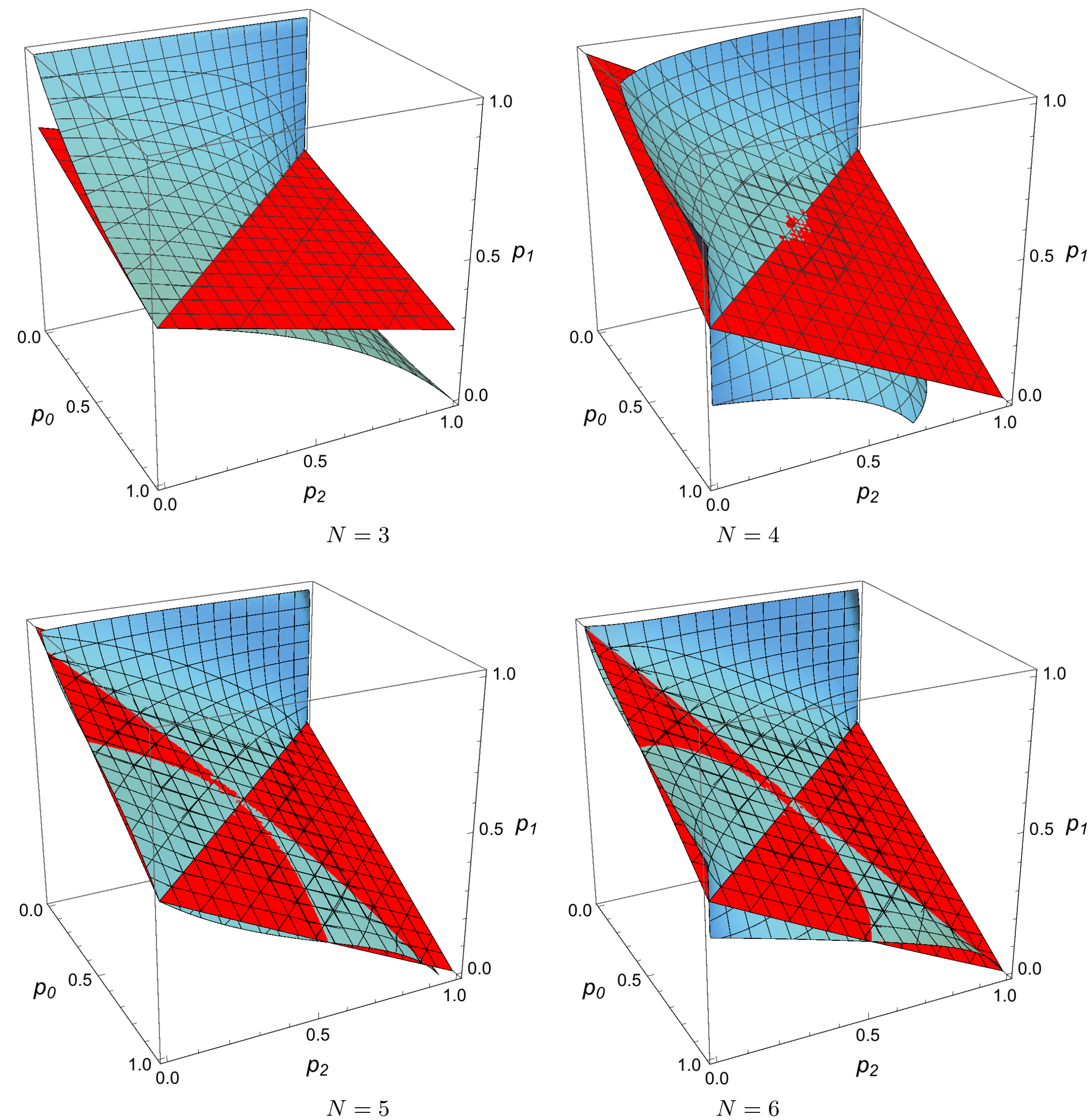

Figure 3.2. For $3 \leq N \leq 6$, the blue surface is the surface $\mu_B = 0$, and the red surface is the surface $\mu_{[1,1]'} = 0$, in the (p_0, p_2, p_1) unit cube. The Parrondo region is the region on or below the blue surface and above the red surface, while the anti-Parrondo region is the region on or above the blue surface and below the red surface. Here (p_0, p_1, p_1, p_3) is relabeled as (p_0, p_1, p_1, p_2).

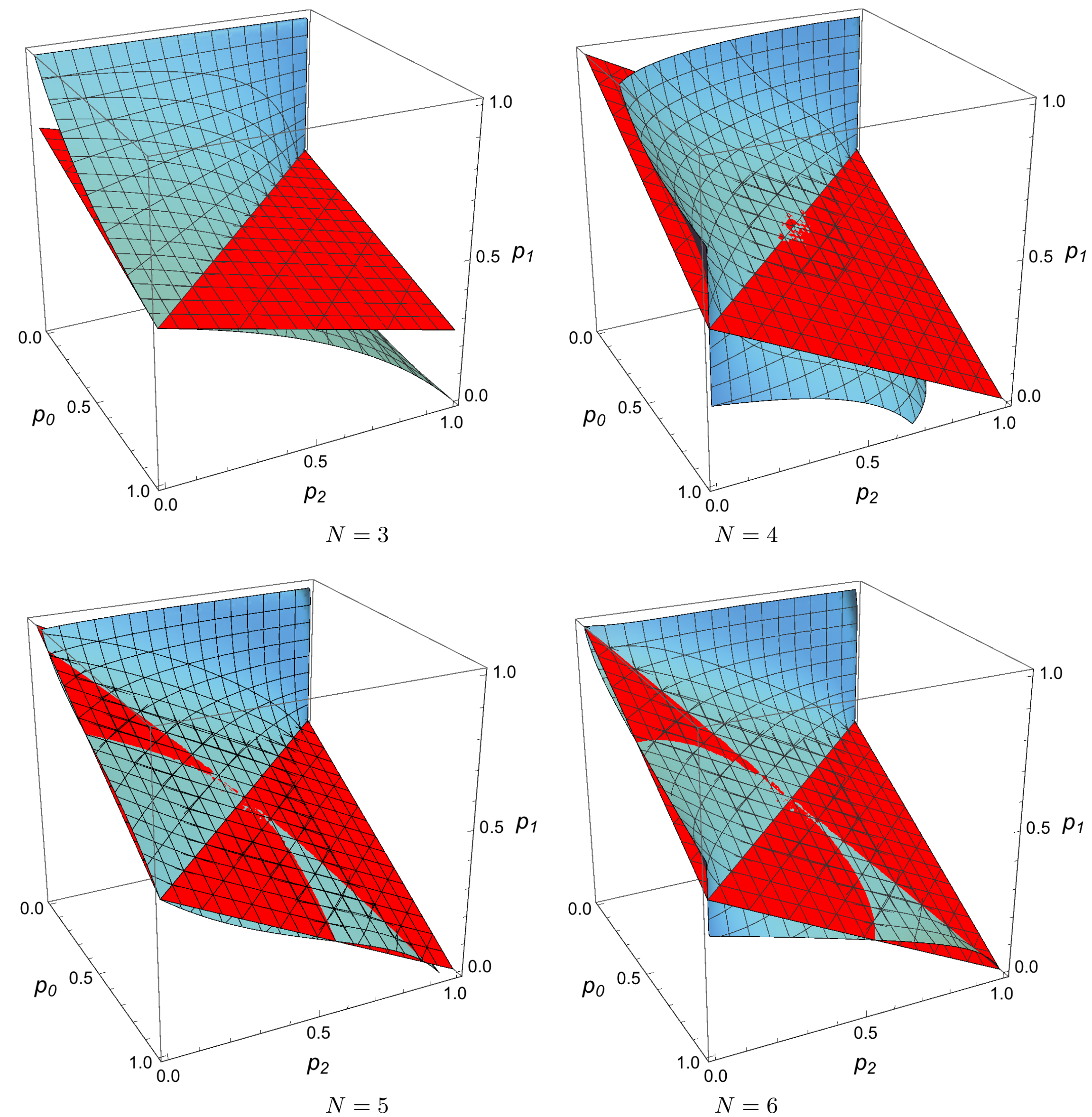

Figure 3.3. For $3 \leq N \leq 6$, the blue surface is the surface $\mu_B = 0$, and the red surface is the surface $\mu_{[2,1]'} = 0$, in the (p_0, p_2, p_1) unit cube. The Parrondo region is the region on or below the blue surface and above the red surface, while the anti-Parrondo region is the region on or above the blue surface and below the red surface. Here (p_0, p_1, p_1, p_3) is relabeled as (p_0, p_1, p_1, p_2).

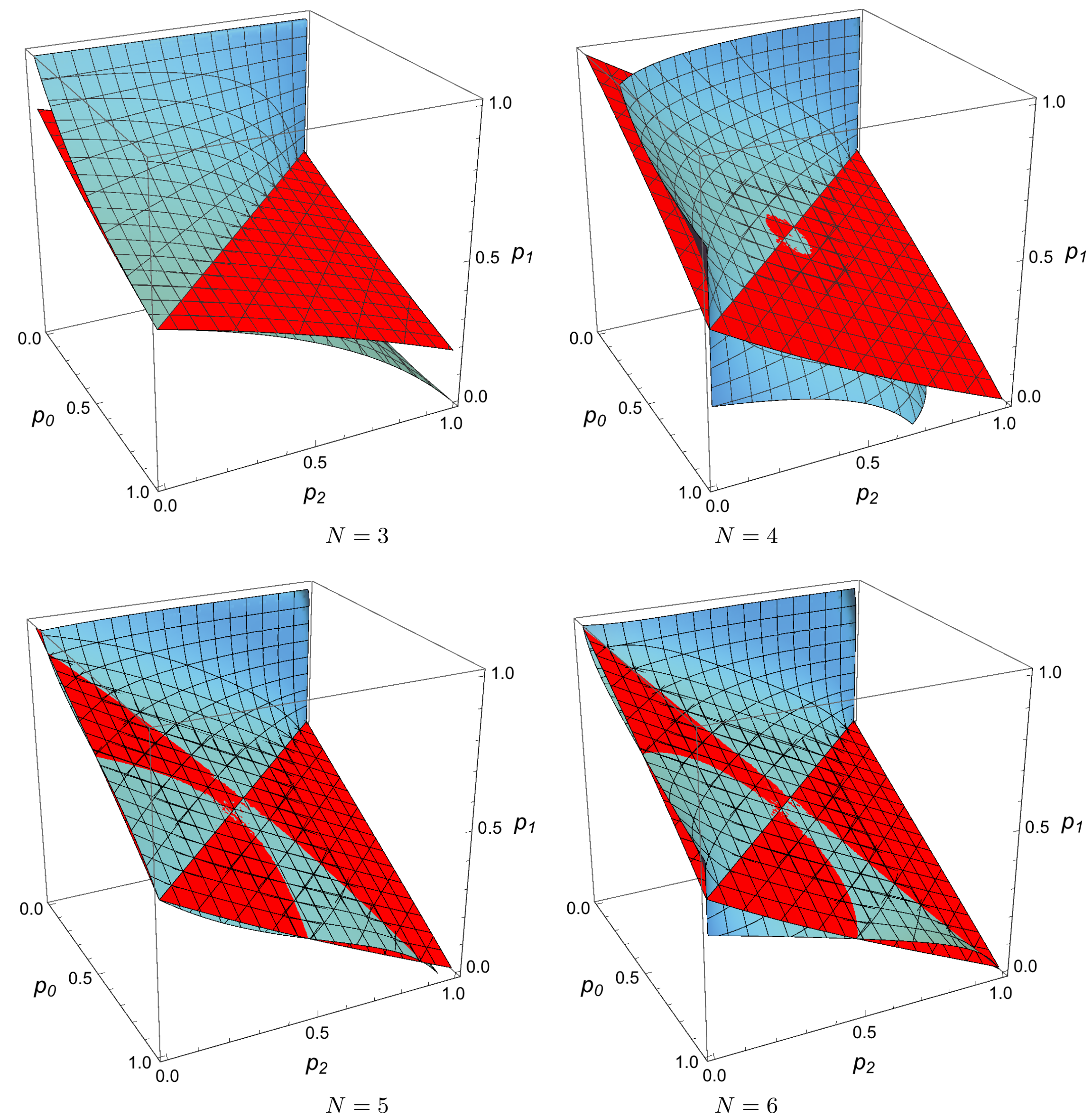

Figure 3.4. For $3 \le N \le 6$, the blue surface is the surface $\mu_B = 0$, and the red surface is the surface $\mu_{[1,2]'} = 0$, in the (p_0, p_2, p_1) unit cube. The Parrondo region is the region on or below the blue surface and above the red surface, while the anti-Parrondo region is the region on or above the blue surface and below the red surface. Here (p_0, p_1, p_1, p_3) is relabeled as (p_0, p_1, p_1, p_2).

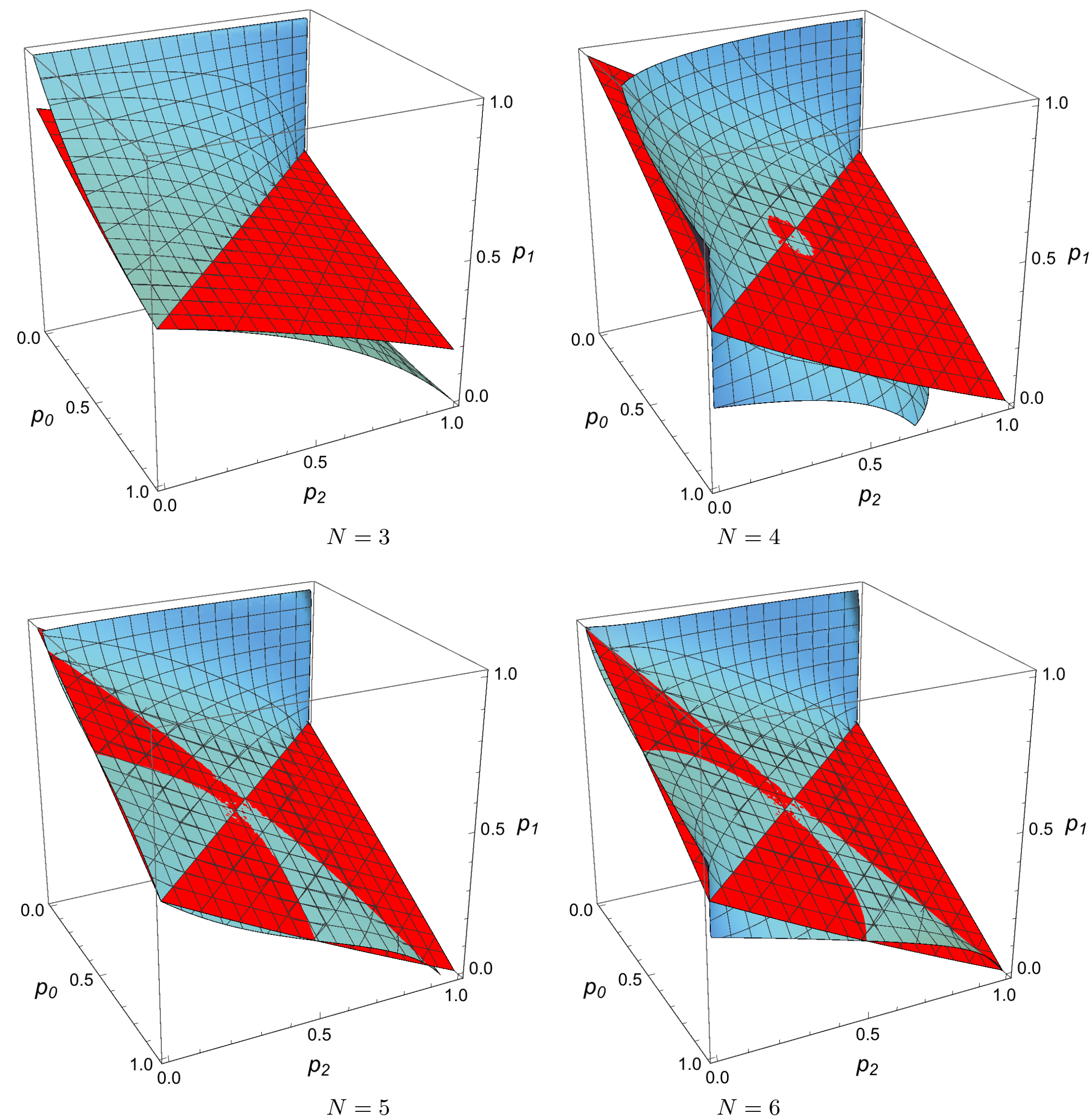

Figure 3.5. For $3 \leq N \leq 6$, the blue surface is the surface $\mu_B = 0$, and the red surface is the surface $\mu_{[2,2]'} = 0$, in the (p_0, p_2, p_1) unit cube. The Parrondo region is the region on or below the blue surface and above the red surface, while the anti-Parrondo region is the region on or above the blue surface and below the red surface. Here (p_0, p_1, p_1, p_3) is relabeled as (p_0, p_1, p_1, p_2).

CHAPTER 4

A BASIC SUFFICIENT CONDITION FOR ERGODICITY

So far, the state space of our basic Markov chain has been $\Sigma_N := \{0,1\}^N$, though we have usually omitted the subscript N for convenience. Now we want to regard the players, originally labeled from 1 to N, as labeled from l_N to r_N, where

$$l_N := \begin{cases} -(N-1)/2 & \text{if } N \text{ is odd,} \\ -N/2 & \text{if } N \text{ is even,} \end{cases} \qquad r_N := \begin{cases} (N-1)/2 & \text{if } N \text{ is odd,} \\ N/2-1 & \text{if } N \text{ is even.} \end{cases}$$

Then we can speed up time, playing N games per unit of time, and our process is described in the limit as $N \to \infty$ by an interacting particle system in the state space $\Sigma := \{0,1\}^{\mathbf{Z}}$. The details of this limit operation are postponed to Chapter 6. Initially, our concern is with the ergodicity of the limiting interacting particle system.

In this chapter, we apply Liggett's [15] sufficient condition for ergodicity of an interacting particle system, first to $\Omega_{A'}$, the interacting particle system corresponding to game A', then to Ω_B, the spin system corresponding to game B (this part has already been done), and finally to $\Omega_{C'}$, the interacting particle system corresponding to the random mixture $C' := \gamma A' + (1-\gamma)B$.

4.1 Analysis of $\Omega_{A'}$

The generator $\Omega_{A'}$ of the interacting particle system corresponding to game A' can be described as follows. For $\eta \in \Sigma := \{0,1\}^{\mathbf{Z}}$ and $x \in \mathbf{Z}$, define η_x and ${}_x\eta_{x+1}$ in Σ by

$$\eta_x(y) := \begin{cases} 1-\eta(x) & \text{if } y = x, \\ \eta(y) & \text{otherwise,} \end{cases} \qquad \text{and} \qquad {}_x\eta_{x+1}(y) := \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

Then

$$(\Omega_{A'} f)(\eta) := \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)] \tag{4.1}$$

for $f \in C(\Sigma)$ depending on only finitely many coordinates, where

$$c'(x,\eta) := \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}].$$

To see where (4.1) comes from, recall from Section 2.3 the notation $\eta^{x,x\pm1,\pm1}$. We define

$$\eta^{x,x-1,-1}(y) := \begin{cases} 1 & \text{if } y = x-1, \\ 0 & \text{if } y = x, \\ \eta(y) & \text{otherwise,} \end{cases} \qquad \eta^{x,x-1,1}(y) := \begin{cases} 0 & \text{if } y = x-1, \\ 1 & \text{if } y = x, \\ \eta(y) & \text{otherwise,} \end{cases}$$

$$\eta^{x,x+1,-1}(y) := \begin{cases} 0 & \text{if } y = x, \\ 1 & \text{if } y = x+1, \\ \eta(y) & \text{otherwise,} \end{cases} \qquad \eta^{x,x+1,1}(y) := \begin{cases} 1 & \text{if } y = x, \\ 0 & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned}
&\sum_x \left(\frac{1}{4} f(\eta^{x,x-1,-1}) + \frac{1}{4} f(\eta^{x,x-1,1}) + \frac{1}{4} f(\eta^{x,x+1,-1}) + \frac{1}{4} f(\eta^{x,x+1,1}) - f(\eta) \right) \\
&= \frac{1}{4}\sum_x [f(\eta^{x,x-1,-1}) - f(\eta)] + \frac{1}{4}\sum_x [f(\eta^{x,x-1,1}) - f(\eta)] \\
&\qquad + \frac{1}{4}\sum_x [f(\eta^{x,x+1,-1}) - f(\eta)] + \frac{1}{4}\sum_x [f(\eta^{x,x+1,1}) - f(\eta)] \\
&= \frac{1}{2}\sum_x [f(\eta^{x-1,x,1}) - f(\eta)] + \frac{1}{2}\sum_x [f(\eta^{x,x+1,-1}) - f(\eta)] \\
&= \frac{1}{2}\sum_{x:(\eta(x-1),\eta(x))=(0,0)} [f(\eta_{x-1}) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x-1),\eta(x))=(0,1)} [f({}_{x-1}\eta_x) - f(\eta)] \\
&\quad + \frac{1}{2}\sum_{x:(\eta(x-1),\eta(x))=(1,1)} [f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(0,0)} [f(\eta_{x+1}) - f(\eta)] \\
&\quad + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(1,0)} [f({}_x\eta_{x+1}) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(1,1)} [f(\eta_x) - f(\eta)] \\
&= \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(0,0)} [f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(0,1)} [f({}_x\eta_{x+1}) - f(\eta)] \\
&\quad + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(1,1)} [f(\eta_{x+1}) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(0,0)} [f(\eta_{x+1}) - f(\eta)] \\
&\quad + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(1,0)} [f({}_x\eta_{x+1}) - f(\eta)] + \frac{1}{2}\sum_{x:(\eta(x),\eta(x+1))=(1,1)} [f(\eta_x) - f(\eta)] \\
&= \frac{1}{2}\sum_{x:\eta(x)=\eta(x+1)} [f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_{x:\eta(x)=\eta(x+1)} [f(\eta_{x+1}) - f(\eta)] + \frac{1}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)] \\
&= \frac{1}{2}\sum_{x:\eta(x)=\eta(x+1)} [f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_{x:\eta(x-1)=\eta(x)} [f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)] \\
&= \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)].
\end{aligned}$$

Thus, the generator $\Omega_{A'}$ is the sum of a spin system generator and an exclusion process generator.

Liggett's [15] Theorem I.4.1 (page 31) gives a sufficient condition for ergodicity of an interacting particle system. The requirement is that $M < \varepsilon$, where M is given by (I.3.8) (page 26) defined in terms of $c_T(u)$ (page 23) which in turn is defined in terms of $c_T(\eta, d\zeta)$ (page 22), and ε is also defined in terms of $c_T(\eta, d\zeta)$ (page 24). The various formulas are given below.

Our first goal is to calculate M and ε in the case of the interacting particle system with generator $\Omega_{A'}$. For each $\eta \in \Sigma := \{0,1\}^{\mathbf{Z}}$ and finite $T \subset \mathbf{Z}$, $c_T(\eta, d\zeta)$ is assumed to be a finite positive measure on $\{0,1\}^T$. Define η^ζ by

$$\eta^\zeta(x) = \begin{cases} \zeta(x) & \text{if } x \in T \\ \eta(x) & \text{if } x \notin T \end{cases}$$

for $\zeta \in \{0,1\}^T$. What is $c_T(\eta, d\zeta)$? We answer this using

$$\begin{aligned}
(\Omega_{A'} f)(\eta) &= \sum_T \int_{\{0,1\}^T} c_T(\eta, d\zeta)[f(\eta^\zeta) - f(\eta)] \\
&= \sum_x \int_{\{0,1\}} c_{\{x\}}(\eta, d\zeta)[f(\eta^\zeta) - f(\eta)] \\
&\quad + \sum_x \int_{\{0,1\}\times\{0,1\}} c_{\{x,x+1\}}(\eta,\ d\zeta)[f(\eta^\zeta) - f(\eta)] \\
&= \sum_x \sum_{\zeta\in\{0,1\}} c_{\{x\}}(\eta, \{\zeta\})[f(\eta^\zeta) - f(\eta)] \\
&\quad + \sum_x \sum_{\zeta\in\{(0,0),(0,1),(1,0),(1,1)\}} c_{\{x,x+1\}}(\eta, \{\zeta\})[f(\eta^\zeta) - f(\eta)] \\
&= \sum_x c_{\{x\}}(\eta, \{1-\eta(x)\})[f(\eta^{1-\eta(x)}) - f(\eta)] \\
&\quad + \sum_{x:(\eta(x),\eta(x+1))=(1,0)} c_{\{x,x+1\}}(\eta, \{(0,1)\})[f({}_x\eta_{x+1}) - f(\eta)] \\
&\quad + \sum_{x:(\eta(x),\eta(x+1))=(0,1)} c_{\{x,x+1\}}(\eta, \{(1,0)\})[f({}_x\eta_{x+1}) - f(\eta)] \\
&= \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] \\
&\quad + \frac{1}{2} \sum_{x:(\eta(x),\eta(x+1))=(1,0)} [f({}_x\eta_{x+1}) - f(\eta)] + \frac{1}{2} \sum_{x:(\eta(x),\eta(x+1))=(0,1)} [f({}_x\eta_{x+1}) - f(\eta)] \\
&= \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] + \frac{1}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)]. \qquad (4.2)
\end{aligned}$$

We conclude that

$c_{\{x\}}(\eta, G) = \delta_{1-\eta(x)}(G)c'(x,\eta)$ and $c_{\{x,x+1\}}(\eta, H) = \delta_{(1-\eta(x),1-\eta(x+1))}(H)\frac{1}{2}\mathbf{1}_{\{\eta(x)\neq\eta(x+1)\}}$, where δ_u is the unit mass concentrated at u. Here $G \subset \{0,1\}$ and $H \subset \{(0,0),(0,1),(1,0),(1,1)\}$.

We can now evaluate M. For $u \in \mathbf{Z}$ and finite $T \subset \mathbf{Z}$, let $c_T(u) := \sup\{\|c_T(\eta, d\zeta) - c_T(\eta', d\zeta)\|_{\text{TV}} : \eta(y) = \eta'(y)\ \forall\ y \neq u\}$, where $\|\cdot\|_{\text{TV}}$ denotes the total variation norm of a measure on $\{0,1\}^T$.[1] Then

$$
\begin{aligned}
M &= \sup_{x\in\mathbf{Z}} \sum_{T\ni x} \sum_{u:u\neq x} c_T(u) \\
&= \sup_{x\in\mathbf{Z}} \sum_{T\ni x} \sum_{u:u\neq x} \sup_{\eta(x)=\eta'(x)\,\forall\, x\neq u} \|c_T(\eta, d\zeta) - c_T(\eta', d\zeta)\|_{\text{TV}} \\
&= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} \|c_{\{x\}}(\eta, d\zeta) - c_{\{x\}}(\eta_u, d\zeta)\|_{\text{TV}} \\
&\qquad + \sum_{v:v\neq x} \sup_{\eta\in\Sigma} \|c_{\{x,x+1\}}(\eta, d\zeta) - c_{\{x,x+1\}}(\eta_v, d\zeta)\|_{\text{TV}} \\
&\qquad + \sum_{w:w\neq x} \sup_{\eta\in\Sigma} \|c_{\{x-1,x\}}(\eta, d\zeta) - c_{\{x-1,x\}}(\eta_w, d\zeta)\|_{\text{TV}} \Bigg] \\
&= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} \sup_{A\subset\{0,1\}} |c_{\{x\}}(\eta, A) - c_{\{x\}}(\eta_u, A)| \\
&\qquad + \sum_{v:v\neq x} \sup_{\eta\in\Sigma} \sup_{B\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x,x+1\}}(\eta, B) - c_{\{x,x+1\}}(\eta_v, B)| \\
&\qquad + \sum_{w:w\neq x} \sup_{\eta\in\Sigma} \sup_{C\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x-1,x\}}(\eta, C) - c_{\{x-1,x\}}(\eta_w, C)| \Bigg] \\
&= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} |c'(x,\eta) - c'(x,\eta_u)| + 2\cdot\frac{1}{2} \Bigg] \\
&= \sup_{x\in\mathbf{Z}} \sum_{u:u\neq x} \sup_{\eta\in\Sigma} |c'(x,\eta) - c'(x,\eta_u)| + 1 \\
&= \sup_{x\in\mathbf{Z}} \Big[\sup_{\eta\in\Sigma} |c'(x,\eta) - c'(x,\eta_{x+1})| + \sup_{\eta\in\Sigma} |c'(x,\eta) - c'(x,\eta_{x-1})| \Big] + 1 \\
&= 2,
\end{aligned}
\tag{4.3}
$$

where the last step uses Table 4.1; by definition,

$$
\begin{aligned}
c'(x,\eta) &= \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}] \\
c'(x,\eta_{x+1}) &= \frac{1}{2}[\mathbf{1}_{\{\eta_{x+1}(x)=\eta_{x+1}(x+1)\}} + \mathbf{1}_{\{\eta_{x+1}(x)=\eta_{x+1}(x-1)\}}]
\end{aligned}
\tag{4.4}
$$

[1] $\|\mu_1 - \mu_2\|_{\text{TV}} = \sup_A |\mu_1(A) - \mu_2(A)|$. This refers to Durrett [3].

$$= \frac{1}{2}[\mathbf{1}_{\{\eta(x)=1-\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}] \tag{4.5}$$

$$c'(x,\eta_{x-1}) = \frac{1}{2}[\mathbf{1}_{\{\eta_{x-1}(x)=\eta_{x-1}(x+1)\}} + \mathbf{1}_{\{\eta_{x-1}(x)=\eta_{x-1}(x-1)\}}]$$

$$= \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=1-\eta(x-1)\}}] \tag{4.6}$$

Next, we evaluate ε.

$$\begin{aligned}
\varepsilon = \inf_{u\in\mathbf{Z}} \inf_{\eta=\eta' \text{ off } u,\, \eta(u)\neq\eta'(u)} \sum_{T\ni u} [c_T(\eta, \{\zeta\in\{0,1\}^T : \zeta(u)=\eta'(u)\}) \\
+ c_T(\eta', \{\zeta\in\{0,1\}^T : \zeta(u)=\eta(u)\})] \\
= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \big[c_{\{u\}}(\eta, \{\eta_u(u)\}) + c_{\{u\}}(\eta_u, \{\eta(u)\}) \\
+ c_{\{u,u+1\}}(\eta, \{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=\eta_u(u)\}) \\
+ c_{\{u,u+1\}}(\eta_u, \{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=\eta(u)\}) \\
+ c_{\{u-1,u\}}(\eta, \{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=\eta_u(u)\}) \\
+ c_{\{u-1,u\}}(\eta_u, \{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=\eta(u)\})\big] \\
= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \big[c_{\{u\}}(\eta, \{1-\eta(u)\}) + c_{\{u\}}(\eta_u, \{1-\eta_u(u)\}) \\
+ c_{\{u,u+1\}}(\eta, \{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=1-\eta(u)\}) \\
+ c_{\{u,u+1\}}(\eta_u, \{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=\eta(u)\}) \\
+ c_{\{u-1,u\}}(\eta, \{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=1-\eta(u)\}) \\
+ c_{\{u-1,u\}}(\eta_u, \{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=\eta(u)\})\big] \\
= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \left[c'(u,\eta) + c'(u,\eta_u) + 2\cdot\frac{1}{2}\right] \\
= \inf_{x\in\mathbf{Z},\, \eta\in\Sigma} [c'(x,\eta) + c'(x,\eta_x)] + 1 \\
= 2,
\end{aligned} \tag{4.7}$$

where the last step uses Table 4.2; by definition,

$$c'(x,\eta) = \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}], \tag{4.8}$$

$$c'(x,\eta_x) = \frac{1}{2}[\mathbf{1}_{\{\eta_x(x)=\eta_x(x+1)\}} + \mathbf{1}_{\{\eta_x(x)=\eta_x(x-1)\}}]$$

$$= \frac{1}{2}[\mathbf{1}_{\{1-\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{1-\eta(x)=\eta(x-1)\}}]. \tag{4.9}$$

We conclude that $M = \varepsilon = 2$. Since $M < \varepsilon$ is a sufficient condition for ergodicity, the condition is inconclusive for ergodicity in this case. However, these calculations will prove useful in analyzing $\Omega_{C'}$.

4.2 Analysis of Ω_B

Our second goal is to calculate M and ε in the case of the spin system with generator Ω_B. For each $\eta \in \Sigma := \{0,1\}^{\mathbf{Z}}$ and finite $T \subset \mathbf{Z}$, $c_T(\eta, d\zeta)$ is assumed to be a finite positive measure on $\{0,1\}^T$. For each $x \in \mathbf{Z}$, define η_x by

$$\eta_x(y) := \begin{cases} 1-\eta(x) & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x. \end{cases}$$

Then, given parameters $p_0, p_1, p_2, p_3 \in [0,1]$,

$$(\Omega_B f)(\eta) := \sum_x c(x,\eta)[f(\eta_x) - f(\eta)]$$

for $f \in C(\Sigma)$ depending on only finitely many coordinates, where

$$c(x,\eta) = \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \tag{4.10}$$

$q_m := 1 - p_m$ for $m = 0,1,2,3$, and $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0,1,2,3\}$.

What is $c_T(\eta,\, d\zeta)$ here? Recall that, for $\zeta \in \{0,1\}^T$,

$$\eta^\zeta(x) := \begin{cases} \zeta(x) & \text{if } x \in T, \\ \eta(x) & \text{if } x \notin T. \end{cases}$$

We answer this using

$$\begin{aligned}(\Omega_B f)(\eta) &= \sum_T \int_{\{0,1\}^T} c_T(\eta,\, d\zeta)[f(\eta^\zeta) - f(\eta)] \\ &= \sum_x \int_{\{0,1\}} c_{\{x\}}(\eta,\, d\zeta)[f(\eta^\zeta) - f(\eta)] \\ &= \sum_x \sum_{\zeta \in \{0,1\}} c_{\{x\}}(\eta,\, \{\zeta\})[f(\eta^\zeta) - f(\eta)] \\ &= \sum_x c_{\{x\}}(\eta,\, \{1-\eta(x)\})[f(\eta_x) - f(\eta)] \\ &= \sum_x c(x,\eta)[f(\eta_x) - f(\eta)].\end{aligned}$$

We conclude that $c_{\{x\}}(\eta, G) = \delta_{1-\eta(x)}(G)c(x,\eta)$ for all $G \subset \{0,1\}$, where δ_u is the unit mass at u.

We can now evaluate M. For $u \in \mathbf{Z}$ and finite $T \subset \mathbf{Z}$, let $c_T(u) := \sup\{\|c_T(\eta, d\zeta) - c_T(\eta', d\zeta)\|_{\text{TV}} : \eta(y) = \eta'(y)\,\forall\, y \neq u\}$, where $\|\cdot\|_{\text{TV}}$ denotes the total variation norm of a measure on $\{0,1\}^T$. Then

$$M = \sup_{x \in \mathbf{Z}} \sum_{T \ni x} \sum_{u : u \neq x} c_T(u)$$

$$= \sup_{x\in\mathbf{Z}} \sum_{T\ni x} \sum_{u:u\neq x} \sup_{\eta(x)=\eta'(x)\,\forall\, x\neq u} \|c_T(\eta,d\zeta)-c_T(\eta',d\zeta)\|_{\mathrm{TV}}$$

$$= \sup_{x\in\mathbf{Z}} \sum_{u:u\neq x} \sup_{\eta\in\Sigma} \|c_{\{x\}}(\eta,d\zeta)-c_{\{x\}}(\eta_u,d\zeta)\|_{\mathrm{TV}}$$

$$= \sup_{x\in\mathbf{Z}} \sum_{u:u\neq x} \sup_{\eta\in\Sigma} \sup_{A\subset\{0,1\}} |c_{\{x\}}(\eta,A)-c_{\{x\}}(\eta_u,A)|$$

$$= \sup_{x\in\mathbf{Z}} \sum_{u:u\neq x} \sup_{\eta\in\Sigma} |c(x,\eta)-c(x,\eta_u)|.$$

Next,

$$\varepsilon = \inf_{u\in\mathbf{Z}} \inf_{\eta=\eta' \text{ off } u,\ \eta(u)\neq\eta'(u)} \sum_{T\ni u} [c_T(\eta,\{\zeta:\zeta(u)=\eta'(u)\})+c_T(\eta',\{\zeta:\zeta(u)=\eta(u)\})]$$

$$= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} [c_{\{u\}}(\eta,\{\eta_u(u)\})+c_{\{u\}}(\eta_u,\{\eta(u)\})]$$

$$= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} [c_{\{u\}}(\eta,\{1-\eta(u)\})+c_{\{u\}}(\eta_u,\{1-\eta_u(u)\})]$$

$$= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} [c(u,\eta)+c(u,\eta_u)]$$

$$= \inf_{x\in\mathbf{Z},\ \eta\in\Sigma} [c(x,\eta)+c(x,\eta_x)].$$

Therefore the sufficient condition for ergodicity in this case is that

$$\sup_{x\in\mathbf{Z}} \sum_{u:u\neq x} \sup_{\eta\in\Sigma} |c(x,\eta)-c(x,\eta_u)| < \inf_{x\in\mathbf{Z},\ \eta\in\Sigma} [c(x,\eta)+c(x,\eta_x)], \tag{4.11}$$

which is the same result as (III.0.6) of Liggett (1985).

Now we use (4.10), in which case the right side of (4.11) is 1. On the left side of (4.11), the sum has only two terms, corresponding to $u = x \pm 1$. We can evaluate the left side with the help of Table 4.3.

The result is that (4.11) is equivalent to

$$\max(|p_0-p_1|,|p_2-p_3|)+\max(|p_0-p_2|,|p_1-p_3|) < 1.$$

The volume of the subset of the parameter space $[0,1]^4$ for which this inequality holds is, by *Mathematica*, 7/12.

Ethier and Lee [8] used other methods to show that ergodicity holds on a subset of volume estimated to be 0.789.

```
In[3]:= Integrate[Boole[Max[Abs[p0 - p1], Abs[p2 - p3]] + Max[Abs[p0 - p2], Abs[p1 - p3]] < 1],
          {p0, 0, 1}, {p1, 0, 1}, {p2, 0, 1}, {p3, 0, 1}]
Out[3]= 7/12
```

4.3 Analysis of $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$

Our third goal is to calculate M and ε in the case of the interacting particle system with generator $\Omega_{C'}$. For each $\eta \in \Sigma := \{0,1\}^{\mathbf{Z}}$ and finite $T \subset \mathbf{Z}$, $c_T(\eta, d\zeta)$ is assumed to be a finite positive measure on $\{0,1\}^T$. Define η_x, ${}_x\eta_{x+1}$, and η^ζ by

$$\eta_x(y) := \begin{cases} 1-\eta(x) & \text{if } y = x, \\ \eta(y) & \text{if } y \neq x, \end{cases} \qquad {}_x\eta_{x+1}(y) := \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise,} \end{cases}$$

$$\eta^\zeta(x) := \begin{cases} \zeta(x) & \text{if } x \in T, \\ \eta(x) & \text{if } x \notin T, \end{cases}$$

for $\zeta \in \{0,1\}^T$ and $x \in \mathbf{Z}$. Then

$$\begin{aligned}(\Omega_{C'} f)(\eta) &:= \gamma(\Omega_{A'} f)(\eta) + (1-\gamma)(\Omega_B f)(\eta) \\ &= \gamma \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] + \frac{\gamma}{2}\sum_x [f({}_x\eta_{x+1}) - f(\eta)] \\ &\qquad + (1-\gamma)\sum_x c(x,\eta)[f(\eta_x) - f(\eta)],\end{aligned} \tag{4.12}$$

where

$$c'(x,\eta) := \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}], \qquad c(x,\eta) = \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases}$$

$q_m := 1 - p_m$ for $m = 0,1,2,3$, and $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0,1,2,3\}$.

The generator $\Omega_{C'}$ is the sum of a spin system generator and an exclusion process generator. What is $c_T(\eta, d\zeta)$? It should be a mixture of the measures found in Sections 4.1 and 4.2. We conclude that

$$c_{\{x\}}(\eta, G) = \delta_{1-\eta(x)}(G)[\gamma c'(x,\eta) + (1-\gamma)c(x,\eta)],$$

$$c_{\{x,x+1\}}(\eta, H) = \delta_{(1-\eta(x),1-\eta(x+1))}(H)\frac{\gamma}{2}\mathbf{1}_{\{\eta(x)\neq\eta(x+1)\}}.$$

We can now evaluate M. For $u \in \mathbf{Z}$ and finite $T \subset \mathbf{Z}$, let $c_T(u) := \sup\{\|c_T(\eta, d\zeta) - c_T(\eta', d\zeta)\|_{\mathrm{TV}} : \eta(y) = \eta'(y)\,\forall\, y \neq u\}$, where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation norm of a measure on $\{0,1\}^T$. Then

$$\begin{aligned} M &= \sup_{x\in\mathbf{Z}} \sum_{T\ni x} \sum_{u:u\neq x} c_T(u) \\ &= \sup_{x\in\mathbf{Z}} \sum_{T\ni x} \sum_{u:u\neq x} \sup_{\eta(y)=\eta'(y)\,\forall\, y\neq u} \|c_T(\eta, d\zeta) - c_T(\eta', d\zeta)\|_{\mathrm{TV}} \end{aligned}$$

$$= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} \|c_{\{x\}}(\eta, d\zeta) - c_{\{x\}}(\eta_u, d\zeta)\|_{\mathrm{TV}}$$

$$+ \sum_{v:v\neq x} \sup_{\eta\in\Sigma} \|c_{\{x,x+1\}}(\eta, d\zeta) - c_{\{x,x+1\}}(\eta_v, d\zeta)\|_{\mathrm{TV}}$$

$$+ \sum_{w:w\neq x} \sup_{\eta\in\Sigma} \|c_{\{x-1,x\}}(\eta, d\zeta) - c_{\{x-1,x\}}(\eta_w, d\zeta)\|_{\mathrm{TV}} \Bigg]$$

$$= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} \sup_{G\subset\{0,1\}} |c_{\{x\}}(\eta, G) - c_{\{x\}}(\eta_u, G)|$$

$$+ \sum_{v:v\neq x} \sup_{\eta\in\Sigma} \sup_{H\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x,x+1\}}(\eta, H) - c_{\{x,x+1\}}(\eta_v, H)|$$

$$+ \sum_{w:w\neq x} \sup_{\eta\in\Sigma} \sup_{H\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x-1,x\}}(\eta, H) - c_{\{x-1,x\}}(\eta_w, H)| \Bigg]$$

$$= \sup_{x\in\mathbf{Z}} \Bigg[\sum_{u:u\neq x} \sup_{\eta\in\Sigma} |\gamma c'(x,\eta) + (1-\gamma)c(x,\eta) - [\gamma c'(x,\eta_u) + (1-\gamma)c(x,\eta_u)]|$$

$$+ \sup_{\eta\in\Sigma} \sup_{H\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x,x+1\}}(\eta, H) - c_{\{x,x+1\}}(\eta_{x+1}, H)|$$

$$+ \sup_{\eta\in\Sigma} \sup_{H\subset\{(0,0),(0,1),(1,0),(1,1)\}} |c_{\{x-1,x\}}(\eta, H) - c_{\{x-1,x\}}(\eta_{x-1}, H)| \Bigg]$$

$$= \sup_{x\in\mathbf{Z}} \sum_{u=x\pm1} \sup_{\eta\in\Sigma} |\gamma c'(x,\eta) + (1-\gamma)c(x,\eta) - [\gamma c'(x,\eta_u) + (1-\gamma)c(x,\eta_u)]| + 2\cdot\frac{\gamma}{2}$$

$$= \sup_{x\in\mathbf{Z}} \Bigg[\sup_{\eta\in\Sigma} |\gamma[c'(x,\eta) - c'(x,\eta_{x+1})] + (1-\gamma)[c(x,\eta) - c(x,\eta_{x+1})]|$$

$$+ \sup_{\eta\in\Sigma} |\gamma[c'(x,\eta) - c'(x,\eta_{x-1})] + (1-\gamma)[c(x,\eta) - c(x,\eta_{x-1})]| \Bigg] + \gamma$$

$$= \max\left[\left|\frac{\gamma}{2} + (1-\gamma)(p_0 - p_1)\right|, \left|\frac{\gamma}{2} + (1-\gamma)(p_2 - p_3)\right|\right]$$

$$+ \max\left[\left|\frac{\gamma}{2} + (1-\gamma)(p_0 - p_2)\right|, \left|\frac{\gamma}{2} + (1-\gamma)(p_1 - p_3)\right|\right] + \gamma, \tag{4.13}$$

where the last step requires clarification. Notice first that

$$c'(x,\eta) = \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}],$$

$$c'(x,\eta_{x+1}) = \frac{1}{2}[\mathbf{1}_{\{\eta_{x+1}(x)=\eta_{x+1}(x+1)\}} + \mathbf{1}_{\{\eta_{x+1}(x)=\eta_{x+1}(x-1)\}}]$$
$$= \frac{1}{2}[\mathbf{1}_{\{\eta(x)=1-\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}],$$

$$c'(x,\eta_{x-1}) = \frac{1}{2}[\mathbf{1}_{\{\eta_{x-1}(x)=\eta_{x-1}(x+1)\}} + \mathbf{1}_{\{\eta_{x-1}(x)=\eta_{x-1}(x-1)\}}]$$
$$= \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=1-\eta(x-1)\}}],$$

$$c(x,\eta) = \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x)=0, \\ q_{m_x(\eta)} & \text{if } \eta(x)=1, \end{cases}$$

$$c(x,\eta_{x+1}) = \begin{cases} p_{m_x(\eta_{x+1})} & \text{if } \eta(x)=0, \\ q_{m_x(\eta_{x+1})} & \text{if } \eta(x)=1, \end{cases} \qquad c(x,\eta_{x-1}) = \begin{cases} p_{m_x(\eta_{x-1})} & \text{if } \eta(x)=0, \\ q_{m_x(\eta_{x-1})} & \text{if } \eta(x)=1, \end{cases}$$

where $q_m := 1-p_m$ for $m=0,1,2,3$ and $m_x(\eta) := 2\eta(x-1)+\eta(x+1) \in \{0,1,2,3\}$. The last line of (4.13) is from Tables 4.4–4.6 below.

Next, we evaluate ε.

$$\begin{aligned}
\varepsilon &= \inf_{u\in\mathbf{Z}} \inf_{\eta=\eta' \text{ off } u,\ \eta(u)\neq\eta'(u)} \sum_{T\ni u} [c_T(\eta,\{\zeta\in\{0,1\}^T : \zeta(u)=\eta'(u)\}) \\
&\qquad\qquad + c_T(\eta',\{\zeta\in\{0,1\}^T : \zeta(u)=\eta(u)\})] \\
&= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \big[c_{\{u\}}(\eta,\{\eta_u(u)\}) + c_{\{u\}}(\eta_u,\{\eta(u)\}) \\
&\qquad + c_{\{u,u+1\}}(\eta,\{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=\eta_u(u)\}) \\
&\qquad + c_{\{u,u+1\}}(\eta_u,\{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=\eta(u)\}) \\
&\qquad + c_{\{u-1,u\}}(\eta,\{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=\eta_u(u)\}) \\
&\qquad + c_{\{u-1,u\}}(\eta_u,\{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=\eta(u)\})\big] \\
&= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \big[c_{\{u\}}(\eta,\{1-\eta(u)\}) + c_{\{u\}}(\eta_u,\{1-\eta_u(u)\}) \\
&\qquad + c_{\{u,u+1\}}(\eta,\{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=1-\eta(u)\}) \\
&\qquad + c_{\{u,u+1\}}(\eta_u,\{\zeta\in\{0,1\}^{\{u,u+1\}} : \zeta(u)=1-\eta_u(u)\}) \\
&\qquad + c_{\{u-1,u\}}(\eta,\{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=1-\eta(u)\}) \\
&\qquad + c_{\{u-1,u\}}(\eta_u,\{\zeta\in\{0,1\}^{\{u-1,u\}} : \zeta(u)=1-\eta_u(u)\})\big] \\
&= \inf_{u\in\mathbf{Z}} \inf_{\eta\in\Sigma} \big[\gamma[c'(u,\eta)+c'(u,\eta_u)] + (1-\gamma)[c(u,\eta)+c(u,\eta_u)]\big] + 2\cdot\frac{\gamma}{2} \\
&= 1+\gamma, \qquad (4.14)
\end{aligned}$$

where the last line of (4.14) requires clarification. First, notice that

$$c'(u,\eta) = \frac{1}{2}[\mathbf{1}_{\{\eta(u)=\eta(u+1)\}} + \mathbf{1}_{\{\eta(u)=\eta(u-1)\}}],$$

and

$$c'(u,\eta_u) = \frac{1}{2}[\mathbf{1}_{\{\eta_u(u)=\eta_u(u+1)\}} + \mathbf{1}_{\{\eta_u(u)=\eta_u(u-1)\}}] = \frac{1}{2}[\mathbf{1}_{\{1-\eta(u)=\eta(u+1)\}} + \mathbf{1}_{\{1-\eta(u)=\eta(u-1)\}}].$$

The last step of (4.14) is from column **7** of Table 4.7.

We have proved the following theorem.

Theorem 4.1. *The interacting particle system in* $\Sigma := \{0,1\}^{\mathbf{Z}}$ *with generator* $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$, *where* $0 < \gamma < 1$, *is ergodic if*

$$\max\left[\left|\frac{\gamma}{2} + (1-\gamma)(p_0 - p_1)\right|, \left|\frac{\gamma}{2} + (1-\gamma)(p_2 - p_3)\right|\right] + \max\left[\left|\frac{\gamma}{2} + (1-\gamma)(p_0 - p_2)\right|, \left|\frac{\gamma}{2} + (1-\gamma)(p_1 - p_3)\right|\right] < 1. \tag{4.15}$$

The volume of the subset of the parameter space $[0,1]^4$ for which (4.15) holds with $\gamma = 1/2$ is, by *Mathematica*, 5/6. Of the six examples studied in Section 3.2, namely $(p_0, p_1, p_2, p_3) = (1, 0.16, 0.16, 0.7)$, $(0.7, 0.68, 0.68, 0)$, $(0.1, 0.6, 0.6, 0.75)$, $(0, 0.8, 0.8, 0.5)$, $(0.78, 0.65, 0.65, 0)$, and $(0.9, 0.54, 0.54, 0.05)$, the third, fourth, and sixth satisfy (4.15) with $\gamma = 1/2$.

If we assume that $p_1 = p_2$, then the volume of the subset of the parameter space $[0,1]^3$ for which (4.15) holds is, by *Mathematica*, 3/4. In fact, we plot the three-dimensional volume as a function of γ in Figure 4.1.

Notice that the volume is 3/4 if and only if $\gamma \geq 1/3$.

Table 4.1. Calculations for the last step of (4.3).

$(\ldots,\eta(x-1),\eta(x),\eta(x+1),\ldots)$	(4.4), (4.5), (4.6)	$(*)$	$(**)$
$(\ldots,0,0,0,\ldots)$	$1,\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,0,0,1,\ldots)$	$\frac{1}{2},1,0$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,0,1,0,\ldots)$	$0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,0,1,1,\ldots)$	$\frac{1}{2},0,1$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,1,0,0,\ldots)$	$\frac{1}{2},0,1$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,1,0,1,\ldots)$	$0,\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,1,1,0,\ldots)$	$\frac{1}{2},1,0$	$\frac{1}{2}$	$\frac{1}{2}$
$(\ldots,1,1,1,\ldots)$	$1,\frac{1}{2},\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$(*) := |c'(x,\eta) - c'(x,\eta_{x+1})|$, $(**) := |c'(x,\eta) - c'(x,\eta_{x-1})|$.

Table 4.2. Calculations for the last step of (4.7).

$(\ldots,\eta(x-1),\eta(x),\eta(x+1),\ldots)$	$(4.8),(4.9)$	$c'(x,\eta)+c'(x,\eta_x)$
$(\ldots,0,0,0,\ldots)$	$1,0$	1
$(\ldots,0,0,1,\ldots)$	$\frac{1}{2},\frac{1}{2}$	1
$(\ldots,0,1,0,\ldots)$	$0,1$	1
$(\ldots,0,1,1,\ldots)$	$\frac{1}{2},\frac{1}{2}$	1
$(\ldots,1,0,0,\ldots)$	$\frac{1}{2},\frac{1}{2}$	1
$(\ldots,1,0,1,\ldots)$	$0,1$	1
$(\ldots,1,1,0,\ldots)$	$\frac{1}{2},\frac{1}{2}$	1
$(\ldots,1,1,1,\ldots)$	$1,0$	1

Table 4.3. Evaluating the left side of (4.11).

$\eta(x-1)$	$\eta(x+1)$	$m_x(\eta)$	$m_x(\eta_{x-1})$	$m_x(\eta_{x+1})$
0	0	0	2	1
0	1	1	3	0
1	0	2	0	3
1	1	3	1	2

Table 4.4. Calculations for the last step of (4.13).

	1	**2**	**3**	**4**	**5**	**6**
$\big(\eta(x-1),\eta(x),\eta(x+1)\big)$	$c'(x,\eta)$	$c'(x,\eta_{x+1})$	$c'(x,\eta_{x-1})$	$c(x,\eta)$	$c(x,\eta_{x+1})$	$c(x,\eta_{x-1})$
$(0,0,0)$	1	$\frac{1}{2}$	$\frac{1}{2}$	p_0	p_1	p_2
$(0,0,1)$	$\frac{1}{2}$	1	0	p_1	p_0	p_3
$(0,1,0)$	0	$\frac{1}{2}$	$\frac{1}{2}$	q_0	q_1	q_2
$(0,1,1)$	$\frac{1}{2}$	0	1	q_1	q_0	q_3
$(1,0,0)$	$\frac{1}{2}$	0	1	p_2	p_3	p_0
$(1,0,1)$	0	$\frac{1}{2}$	$\frac{1}{2}$	p_3	p_2	p_1
$(1,1,0)$	$\frac{1}{2}$	1	0	q_2	q_3	q_0
$(1,1,1)$	1	$\frac{1}{2}$	$\frac{1}{2}$	q_3	q_2	q_1

Table 4.5. Calculations for the last step of (4.13), continued.

$\big(\eta(x-1), \eta(x), \eta(x+1)\big)$	**7** $\gamma\big(\mathbf{1}-\mathbf{2}\big)$	**8** $(1-\gamma)\big(\mathbf{4}-\mathbf{5}\big)$	**9** $\gamma\big(\mathbf{1}-\mathbf{3}\big)$	**10** $(1-\gamma)\big(\mathbf{4}-\mathbf{6}\big)$
$(0,0,0)$	$\frac{\gamma}{2}$	$(1-\gamma)(p_0-p_1)$	$\frac{\gamma}{2}$	$(1-\gamma)(p_0-p_2)$
$(0,0,1)$	$-\frac{\gamma}{2}$	$(1-\gamma)(p_1-p_0)$	$\frac{\gamma}{2}$	$(1-\gamma)(p_1-p_3)$
$(0,1,0)$	$-\frac{\gamma}{2}$	$(1-\gamma)(q_0-q_1)$	$-\frac{\gamma}{2}$	$(1-\gamma)(q_0-q_2)$
$(0,1,1)$	$\frac{\gamma}{2}$	$(1-\gamma)(q_1-q_0)$	$-\frac{\gamma}{2}$	$(1-\gamma)(q_1-q_3)$
$(1,0,0)$	$\frac{\gamma}{2}$	$(1-\gamma)(p_2-p_3)$	$-\frac{\gamma}{2}$	$(1-\gamma)(p_2-p_0)$
$(1,0,1)$	$-\frac{\gamma}{2}$	$(1-\gamma)(p_3-p_2)$	$-\frac{\gamma}{2}$	$(1-\gamma)(p_3-p_1)$
$(1,1,0)$	$-\frac{\gamma}{2}$	$(1-\gamma)(q_2-q_3)$	$\frac{\gamma}{2}$	$(1-\gamma)(q_2-q_0)$
$(1,1,1)$	$\frac{\gamma}{2}$	$(1-\gamma)(q_3-q_2)$	$\frac{\gamma}{2}$	$(1-\gamma)(q_3-q_1)$

Table 4.6. Calculations for the last step of (4.13), continued.

	$\mathbf{\|7+8\|}$	$\mathbf{\|9+10\|}$
$(0,0,0)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_1)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_2)\right\|$
$(0,0,1)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_1)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_1-p_3)\right\|$
$(0,1,0)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_1)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_2)\right\|$
$(0,1,1)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_1)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_1-p_3)\right\|$
$(1,0,0)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_2-p_3)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_2)\right\|$
$(1,0,1)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_2-p_3)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_1-p_3)\right\|$
$(1,1,0)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_2-p_3)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_0-p_2)\right\|$
$(1,1,1)$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_2-p_3)\right\|$	$\left\|\frac{\gamma}{2}+(1-\gamma)(p_1-p_3)\right\|$

Table 4.7. Calculations for the last step of (4.14).

	1	**2**	**3**	**4**	**5**	**6**	**7**
$\big(\eta(u-1), \eta(u), \eta(u+1)\big)$	$c'(u,\eta)$	$c'(u,\eta_u)$	$c(u,\eta)$	$c(u,\eta_u)$	$\gamma\big(\mathbf{1}+\mathbf{2}\big)$	$(1-\gamma)\big(\mathbf{3}+\mathbf{4}\big)$	$\big(\mathbf{5}+\mathbf{6}\big)$
$(0,0,0)$	1	0	p_0	q_0	γ	$1-\gamma$	1
$(0,0,1)$	$\frac{1}{2}$	$\frac{1}{2}$	p_1	q_1	γ	$1-\gamma$	1
$(0,1,0)$	0	1	q_0	p_0	γ	$1-\gamma$	1
$(0,1,1)$	$\frac{1}{2}$	$\frac{1}{2}$	q_1	p_1	γ	$1-\gamma$	1
$(1,0,0)$	$\frac{1}{2}$	$\frac{1}{2}$	p_2	q_2	γ	$1-\gamma$	1
$(1,0,1)$	0	1	p_3	q_3	γ	$1-\gamma$	1
$(1,1,0)$	$\frac{1}{2}$	$\frac{1}{2}$	q_2	p_2	γ	$1-\gamma$	1
$(1,1,1)$	1	0	q_3	p_3	γ	$1-\gamma$	1

```
In[4]:= Integrate[Boole[Max[Abs[1 / 4 + (1 / 2) (p0 - p1)], Abs[1 / 4 + (1 / 2) (p2 - p3)]] +
           Max[Abs[1 / 4 + (1 / 2) (p0 - p2)], Abs[1 / 4 + (1 / 2) (p1 - p3)]] < 1],
        {p0, 0, 1}, {p1, 0, 1}, {p2, 0, 1}, {p3, 0, 1}]
```

Out[4]= $\frac{5}{6}$

```
In[3]:= vol[g_] :=
          Integrate[Boole[Max[Abs[g / 2 + (1 - g) (p0 - p1)], Abs[g / 2 + (1 - g) (p1 - p3)]] < 1 / 2],
            {p0, 0, 1}, {p1, 0, 1}, {p3, 0, 1}];
        Plot[vol[g], {g, 0, 1}, PlotRange → {.5, .8}]
```

Out[4]=

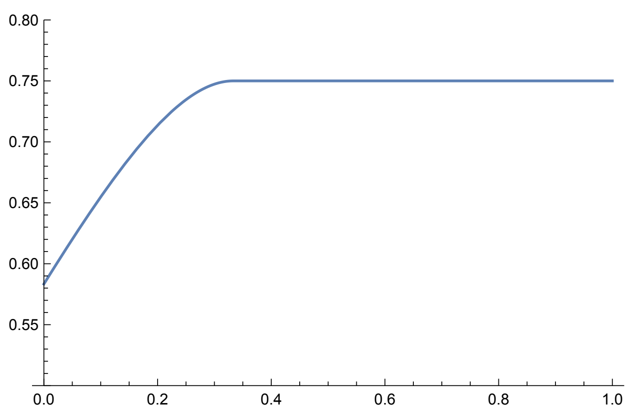

Figure 4.1. Assuming $p_1 = p_2$, the three-dimensional volume of the subset of the parameter space for which (4.15) holds is plotted as a function of γ.

CHAPTER 5

ERGODICITY VIA DUALITY

Duality is a valuable tool for finding conditions under which an interacting particle system is ergodic. We will focus on what is known as annihilating duality because that is the type of duality that works best for Ω_B.

5.1 Duality for Ω_B

We begin with the case of Ω_B, reviewing certain results of Ethier and Lee [8]. Recall that $p_0, p_1, p_2, p_3 \in [0,1]$ are the parameters of the process, $\Sigma := \{0,1\}^{\mathbf{Z}}$ is the state space, and the generator is

$$(\Omega_B f)(\eta) := \sum_{x \in \mathbf{Z}} c(x,\eta)[f(\eta_x) - f(\eta)]$$

for all $f \in C(\Sigma)$ depending on only finitely many coordinates, where η_x is the configuration that differs from η only ar x,

$$c(x,\eta) := \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \tag{5.1}$$

$q_m := 1 - p_m$ for $m = 0,1,2,3$, and $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0,1,2,3\}$.

The state space of the dual process is the countable set $Y := \{A \subset \mathbf{Z} \cup \{\infty\} : A \text{ is finite}\}$. (Here, A is a finite set having nothing to do with game A or game A'.) The duality function to which we will restrict attention is defined on $\Sigma \times Y$ by

$$H_2(\eta, A) = \begin{cases} \prod_{x \in A \cap \mathbf{Z}} [2\eta(x) - 1] = (-1)^{|\{x \in A \cap \mathbf{Z} : \eta(x) = 0\}|} & \text{if } \infty \notin A, \\ -\prod_{x \in A \cap \mathbf{Z}} [2\eta(x) - 1] = -(-1)^{|\{x \in A \cap \mathbf{Z} : \eta(x) = 0\}|} & \text{if } \infty \in A, \end{cases}$$

where $\eta \in X$ and $A \in Y$. The result of Liggett [15] that we will use here is originally due to Holley and Stroock [13].

Lemma 5.1 (Liggett [15], Section III.4). *Consider the spin system generator*

$$(\Omega_2 f)(\eta) := \sum_{x \in \mathbf{Z}} c_2(x,\eta)[f(\eta_x) - f(\eta)],$$

defined for all $f \in C(\Sigma)$ *depending only on finitely many coordinates. If the flip rates* $c_2(x,\eta)$ *have the form*

$$c_2(x,\eta) := \frac{1}{2}c(x)\bigg\{1 - [2\eta(x) - 1]\sum_{A\in Y} p(x,A)H_2(\eta,A)\bigg\}, \tag{5.2}$$

where $c(x) \geq 0$ *for all* $x \in \mathbf{Z}$, $\sup_{x\in\mathbf{Z}} c(x) < \infty$, $p(x,A) \geq 0$ *for all* $x \in \mathbf{Z}$ *and* $A \in Y$, $\sum_{A\in Y} p(x,A) = b(x) \leq 1$ *for all* $x \in \mathbf{Z}$*, and* $\sup_{x\in\mathbf{Z}} c(x)\sum_{A\in Y} p(x,A)|A| < \infty$*, then*

$$\Omega_2 H_2(\eta,A) = \sum_{B\in Y} q_2(A,B)[H_2(\eta,B) - H_2(\eta,A)] - V(A)H_2(\eta,A),$$

where q_2 *is the infinitesimal matrix*

$$q_2(A,B) := \sum_{x\in A\cap\mathbf{Z}} c(x) \sum_{F:F\Delta(A-\{x\})=B} p(x,F) \geq 0, \qquad B \neq A,$$

and $V(A) := \sum_{x\in A\cap\mathbf{Z}} c(x)[1-b(x)] \geq 0$*. This establishes that the spin system with generator* Ω_2 *and the jump process with infinitesimal matrix* q_2 *are in duality with respect to* H_2*. If, in addition,* $\inf_{x\in\mathbf{Z}} c(x)[1-b(x)] > 0$*, then the spin system is ergodic.*

Remark. The nonegativity assumption on $p(x,A)$ is not a serious restriction because the sign of $H_2(\eta,A)$ changes when ∞ is added to the finite set $A \subset \mathbf{Z}$.

Since our flip rates (5.1) are translation invariant [meaning $c(x,\eta) = c(x+1,\eta(\cdot-1))$] and nearest neighbor [meaning $c(x,\eta)$ depends only on $\eta(x-1)$, $\eta(x)$, and $\eta(x+1)$], there are nine parameters necessary to specify (5.2), namely $c(0) = c(x)$ and $p(0,A) = p(x,x+A)$ as A ranges over the eight subsets of $\{-1,0,1\}$ and furthermore each such A may be augmented by including ∞. The basic requirement of our spin system is that $c(x,\eta) + c(x,\eta_x) = 1$ for all $\eta \in \Sigma$ and $x \in \mathbf{Z}$, and this implies that three of the eight probabilities are 0 (namely, the ones corresponding to $A \cap \mathbf{Z} = \{-1,0\}$, $A \cap \mathbf{Z} = \{0,1\}$, and $A \cap \mathbf{Z} = \{-1,0,1\}$) and $c(0)$ is determined. Indeed,

$$\begin{aligned}
1 &= \frac{1}{2}c(0)\bigg\{1 - [2\eta(0) - 1]\sum_{A\cap\mathbf{Z}\subset\{-1,0,1\}} p(0,A)H_2(\eta,A)\bigg\} \\
&\quad + \frac{1}{2}c(0)\bigg\{1 - [2\eta_0(0) - 1]\sum_{A\cap\mathbf{Z}\subset\{-1,0,1\}} p(0,A)H_2(\eta_0,A)\bigg\} \\
&= \frac{1}{2}c(0)\bigg\{1 - [2\eta(0) - 1]\sum_{A\cap\mathbf{Z}=\varnothing,\{-1\},\{1\},\{-1,1\}} p(0,A)H_2(\eta,A)
\end{aligned}$$

$$- [2\eta(0) - 1] \sum_{A\cap\mathbf{Z}=\{0\},\{-1,0\},\{0,1\},\{-1,0,1\}} p(0,A)H_2(\eta,A)\Big\}$$

$$+ \frac{1}{2}c(0)\Big\{1 + [2\eta(0) - 1] \sum_{A\cap\mathbf{Z}=\varnothing,\{-1\},\{1\},\{-1,1\}} p(0,A)H_2(\eta,A)$$

$$+ [2\eta(0) - 1] \sum_{A\cap\mathbf{Z}=\{0\},\{-1,0\},\{0,1\},\{-1,0,1\}} p(0,A)(-H_2(\eta,A))\Big\}$$

$$= c(0)\Big\{1 - \sum_{A\cap\mathbf{Z}=\{0\}} p(0,A)(2\cdot\mathbf{1}_{\{\infty\notin A\}} - 1)$$

$$- [2\eta(0) - 1] \sum_{A\cap\mathbf{Z}=\{-1,0\},\{0,1\},\{-1,0,1\}} p(0,A)H_2(\eta,A)\Big\},$$

and the latter sum must be 0 for the result to be constant in η.

This leaves five remaining parameters, which we will denote by $z_\varnothing$, z_{-1}, z_0, z_1, and $z_{-1,1}$, the interpretation being that $z_A = p(0,A)$ if $z_A \geq 0$ and $z_A = -p(0, A\cup\{\infty\})$ if $z_A < 0$. They must satisfy

$$\begin{aligned}
2^{-1}(1-z_0)^{-1}(1 + z_\varnothing - z_{-1} - z_0 - z_1 + z_{-1,1}) &= p_0,\\
2^{-1}(1-z_0)^{-1}(1 + z_\varnothing - z_{-1} - z_0 + z_1 - z_{-1,1}) &= p_1,\\
2^{-1}(1-z_0)^{-1}(1 + z_\varnothing + z_{-1} - z_0 - z_1 - z_{-1,1}) &= p_2,\\
2^{-1}(1-z_0)^{-1}(1 + z_\varnothing + z_{-1} - z_0 + z_1 + z_{-1,1}) &= p_3,
\end{aligned} \tag{5.3}$$

where $(1-z_0)^{-1}$ is $c(0)$ and the coefficient of each z_A is -1 raised to the number of 0s in A when $(\eta(-1),\eta(0),\eta(1)) = (0,0,0)$, $(0,0,1)$, $(1,0,0)$, and $(1,0,1)$, respectively. The corresponding equations for $\eta(0) = 1$ are

$$\begin{aligned}
2^{-1}(1-z_0)^{-1}(1 - z_\varnothing + z_{-1} - z_0 + z_1 - z_{-1,1}) &= q_0,\\
2^{-1}(1-z_0)^{-1}(1 - z_\varnothing + z_{-1} - z_0 - z_1 + z_{-1,1}) &= q_1,\\
2^{-1}(1-z_0)^{-1}(1 - z_\varnothing - z_{-1} - z_0 + z_1 + z_{-1,1}) &= q_2,\\
2^{-1}(1-z_0)^{-1}(1 - z_\varnothing - z_{-1} - z_0 - z_1 - z_{-1,1}) &= q_3,
\end{aligned}$$

If there is a solution of (5.3) with

$$|z_\varnothing| + |z_{-1}| + |z_0| + |z_1| + |z_{-1,1}| < 1, \tag{5.4}$$

this ensures the ergodicity of the spin system.

The linear system is underdetermined, and a solution to (5.3) is given by

$$\begin{aligned} z_\varnothing &= (p_0 + p_1 + p_2 + p_3 - 2)(1 - z_0)/2, \\ z_{-1} &= -(p_0 + p_1 - p_2 - p_3)(1 - z_0)/2, \\ z_1 &= -(p_0 - p_1 + p_2 - p_3)(1 - z_0)/2, \\ z_{-1,1} &= (p_0 - p_1 - p_2 + p_3)(1 - z_0)/2, \end{aligned} \tag{5.5}$$

in which case (5.4) reduces to, if $0 \le z_0 < 1$,

$$\begin{aligned} &\frac{1}{2}(|p_0 + p_1 + p_2 + p_3 - 2| + |p_0 + p_1 - p_2 - p_3| \\ &\qquad + |p_0 - p_1 + p_2 - p_3| + |p_0 - p_1 - p_2 + p_3|)(1 - z_0) + z_0 < 1, \end{aligned} \tag{5.6}$$

which holds if and only if it holds for $z_0 = 0$. Replacing $|p_0+p_1-p_2-p_3|$ by $(p_0+p_1-p_2-p_3)$ and $-(p_0 + p_1 - p_2 - p_3)$, and similarly for the other three terms, we get 16 inequalities, which are jointly equivalent to

$$p_0, p_1, p_2, p_3 \in (2\overline{p} - 1, 2\overline{p}) \cap (0, 1), \quad \overline{p} := (p_0 + p_1 + p_2 + p_3)/4.$$

The 4-dimensional volume of this region is 2/3. We have just rederived, with a little more detail, a result of Ethier and Lee (2013a).

5.2 Duality for $\Omega_{A'}$

We recall that $\Omega_{A'}$ has the form

$$(\Omega_{A'} f)(\eta) := \sum_{x\in\mathbf{Z}} c'(x, \eta)[f(\eta_x) - f(\eta)] + \frac{1}{2} \sum_{x\in\mathbf{Z}} [f({}_x\eta_{x+1}) - f(\eta)], \tag{5.7}$$

where

$$c'(x, \eta) := \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}].$$

and

$$\eta_x(y) := \begin{cases} 1 - \eta(x) & \text{if } y = x, \\ \eta(y) & \text{if } y \ne x, \end{cases} \qquad {}_x\eta_{x+1}(y) := \begin{cases} \eta(x + 1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x + 1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

Because of the second sum in (5.7), this is not the generator of a spin system (because the spin may change at more than one coordinate at a time). Therefore, the duality method of the preceding section does not apply directly. Nevertheless, there may still be a jump

Markov process in Y that is dual to our interacting particle system with respect to the function H_2. That is what we investigate in this section.

First, we consider the first sum in (5.7), namely

$$(\Omega_1 f)(\eta) := \sum_{x\in\mathbf{Z}} c'(x,\eta)[f(\eta_x) - f(\eta)].$$

Recalling that c_2 involves nine unknown parameters and there are eight equations (for the eight values of $(\eta(x-1),\eta(x),\eta(x+1))$), we can hope for solutions, and indeed there are some. Notice that

$$\begin{aligned} c'(x,\eta) &= \frac{1}{2}\bigg[\frac{1+[2\eta(x)-1][2\eta(x+1)-1]}{2} + \frac{1+[2\eta(x)-1][2\eta(x-1)-1]}{2}\bigg] \\ &= \frac{1}{2}\bigg[1+[2\eta(x)-1]\bigg(\frac{1}{2}H_2(\eta,\{x+1\}) + \frac{1}{2}H_2(\eta,\{x-1\})\bigg)\bigg] \\ &= \frac{1}{2}\bigg[1-[2\eta(x)-1]\bigg(-\frac{1}{2}H_2(\eta,\{x+1\}) - \frac{1}{2}H_2(\eta,\{x-1\})\bigg)\bigg], \end{aligned} \tag{5.8}$$

which, for $0 \le z_0 < 1$, is of the form (5.2) with $c(x) = 1$, $p(x,\{x-1,\infty\}) = \frac{1}{2}$, $p(x,\{x+1,\infty\}) = \frac{1}{2}$, and $p(x,A) = 0$ otherwise. Thus $b(x) = 1$ and the sufficient condition for ergodicity fails, at least for this component of the generator by itself. That is not a concern because we are interested in $\Omega_{A'}$ only for the role it plays in $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$.

Next, let us consider the second sum in (5.7), namely

$$(\Omega_2 f)(\eta) := \frac{1}{2}\sum_{x\in\mathbf{Z}}[f({}_x\eta_{x+1}) - f(\eta)].$$

We notice that

$$H_2({}_x\eta_{x+1}, A) = \begin{cases} H_2(\eta,A) & \text{if } x, x+1 \in A \text{ or if } x, x+1 \notin A, \\ -H_2(\eta,A) & \text{if } x \in A,\ x+1 \notin A \text{ or if } x \notin A,\ x+1 \in A, \end{cases}$$

so

$$\begin{aligned} \Omega_2 H_2(\eta,A) &= \frac{1}{2}\sum_{x\in\mathbf{Z}}[H_2({}_x\eta_{x+1},A) - H_2(\eta,A)] \\ &= \frac{1}{2}\sum_{x\in\mathbf{Z}: x\in A,\, x+1\notin A \text{ or } x\notin A,\, x+1\in A} -2H_2(\eta,A) \\ &= -|\{x\in\mathbf{Z} : x\in A,\ x+1\notin A \text{ or } x\notin A,\ x+1\in A\}|H_2(\eta,A) \\ &= -V_2(A)H_2(\eta,A) \end{aligned}$$

for all $\eta \in \Sigma$ and $A \in Y$. Here $V_2(A) = 0$ if $A = \varnothing$ or $A = \{\infty\}$, but otherwise $V_2(A) \ge 2$. This fits into the duality framework with $q_2(A,B) = 0$ for all $A, B \in Y$.

5.3 Duality for $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$

Now we put our results together. Because the exclusion process part does not contribute to the jump rates of the dual process in Y, we need only consider the spin system parts. These have flip rates in the mixed generator $\Omega_{C'}$ equal to

$$
\begin{aligned}
&\gamma c'(x,\eta) + (1-\gamma)c(x,\eta) \\
&\quad = \frac{\gamma}{2}\Big[1 - [2\eta(x)-1]\Big(-\frac{1}{2}H_2(\eta,\{x+1\}) - \frac{1}{2}H_2(\eta,\{x-1\})\Big)\Big] \\
&\qquad + \frac{1-\gamma}{2}(1-z_0)^{-1}\Big[1 - [2\eta(x)-1]\sum_{A\in Y: A\cap\mathbf{Z}=\varnothing,\{-1\},\{0\},\{1\},\{-1,1\}} z_A H_2(\eta, x+A)\Big] \\
&\quad = \frac{\gamma}{2}(1-z_0)^{-1}\Big[1 - z_0 - [2\eta(x)-1]\Big(-\frac{1-z_0}{2}H_2(\eta,\{x+1\}) - \frac{1-z_0}{2}H_2(\eta,\{x-1\})\Big)\Big] \\
&\qquad + \frac{1-\gamma}{2}(1-z_0)^{-1}\Big[1 - [2\eta(x)-1]\sum_{A\in Y: A\cap\mathbf{Z}=\varnothing,\{-1\},\{0\},\{1\},\{-1,1\}} z_A H_2(\eta, x+A)\Big] \\
&\quad = \frac{\gamma}{2}(1-z_0)^{-1}\Big[1 - [2\eta(x)-1]\Big(-\frac{1-z_0}{2}H_2(\eta,\{x+1\}) + z_0 H_2(\eta,\{x\}) \\
&\qquad\qquad - \frac{1-z_0}{2}H_2(\eta,\{x-1\})\Big)\Big] \\
&\qquad + \frac{1-\gamma}{2}(1-z_0)^{-1}\Big[1 - [2\eta(x)-1]\sum_{A\in Y: A\cap\mathbf{Z}=\varnothing,\{-1\},\{0\},\{1\},\{-1,1\}} z_A H_2(\eta, x+A)\Big],
\end{aligned}
$$

using (5.2) and (5.8) but with the quantities z_A as in (5.5). The condition (5.4) for ergodicity becomes, if $0 \le z_0 < 1$,

$$
\begin{aligned}
1 &> |(1-\gamma)z_\varnothing| + |-\gamma(1-z_0)/2 + (1-\gamma)z_{-1}| + |-\gamma(1-z_0)/2 + (1-\gamma)z_1| \\
&\quad + |(1-\gamma)z_{-1,1}| + |\gamma z_0 + (1-\gamma)z_0| \\
&= \frac{1}{2}\{(1-\gamma)|p_0+p_1+p_2+p_3-2| \\
&\quad + |(1-\gamma)(p_0+p_1-p_2-p_3)+\gamma| \\
&\quad + |(1-\gamma)(p_0-p_1+p_2-p_3)+\gamma| \\
&\quad + (1-\gamma)|p_0-p_1-p_2+p_3|\}(1-z_0) + z_0, \qquad (5.9)
\end{aligned}
$$

As in (5.6), this holds if and only if it holds for $z_0 = 0$. Our sufficient condition for ergodicity is

$$
\begin{aligned}
&(1-\gamma)|p_0+p_1+p_2+p_3-2| + |(1-\gamma)(p_0+p_1-p_2-p_3)+\gamma| \\
&\quad + |(1-\gamma)(p_0-p_1+p_2-p_3)+\gamma| + (1-\gamma)|p_0-p_1-p_2+p_3| < 2. \qquad (5.10)
\end{aligned}
$$

We have proved the following theorem.

Theorem 5.2. *The interacting particle system in* $\Sigma := \{0,1\}^{\mathbf{Z}}$ *with generator* $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$*, where* $0 < \gamma < 1$*, is ergodic if* (5.10) *holds.*

To see that this is an improvement over the result of Chapter 4, take $\gamma = 1/2$. Then the 4-dimensional volume of the region where the inequality is satisfied is equal to 11/12. If $p_1 = p_2$, the 3-dimensional volume of the region where the inequality is satisfied is equal to 5/6.

We continue to assume that $\gamma = 1/2$, so it is actually the union of the two parameter sets (the one from the basic inequality and the one from duality) that is relevant. Its 4-dimensional volume could not be determined, but, assuming $p_1 = p_2$, its 3-dimensional volume is, from *Mathematica*, 7/8:

```
In[1]:= Integrate[Boole[Max[Abs[1 / 4 + (1 / 2) (p0 - p1)], Abs[1 / 4 + (1 / 2) (p1 - p3)]] < 1 / 2 ||
          (1 / 2) Abs[p0 + 2 p1 + p3 - 2] + Abs[(1 / 2) (p0 - p3) + 1 / 2] + Abs[(1 / 2) (p0 - p3) + 1 / 2] +
            (1 / 2) Abs[p0 - 2 p1 + p3] < 2], {p0, 0, 1}, {p1, 0, 1}, {p3, 0, 1}]

Out[1]= 7
        -
        8
```

Of the six examples studied in Chapter 3, only three belong to this union.

CHAPTER 6

CONVERGENCE OF MEANS

We would like to prove that $\lim_{N\to\infty} \mu^N_{(\gamma,1-\gamma)'}$ and $\lim_{N\to\infty} \mu^N_{[r,s]'}$ exist under certain conditions, where $\mu^N_{(\gamma,1-\gamma)'}$ denotes the mean profit per turn at equilibrium to the N players playing the $(\gamma, 1-\gamma)$ random mixture of games A' and B (the Parrondo games of Xie and others [21]), and $\mu^N_{[r,s]'}$ denotes the mean profit per turn at equilibrium to the N players playing games A' and B in the nonrandom periodic pattern $A', A', \ldots, A'$ (r times), $B, B, \ldots, B$ (s times), $A', A', \ldots, A'$ (r times), $B, B, \ldots, B$ (s times), etc. The first result is relatively straightforward, while the second requires more work. The key step for the second result is to prove that the sequence of discrete generators converges to the generator of an interacting particle system. We treat the simple case of $r = s = 1$ first, then $r = s = 2$, and finally the general case.

6.1 Convergence of means in random mixture case

We want to show that our sequence of discrete-time Markov chains, suitably rescaled, converges in distribution to an interacting particle system on $\mathbf{Z}$. The limiting process is characterized in terms of its generator. First, we need to define generators corresponding to game A', game B, and game C'. The state space is

$$\Sigma := \{0,1\}^{\mathbf{Z}} = \{\eta = (\ldots, \eta(-2), \eta(-1), \eta(0), \eta(1), \eta(2), \ldots) : \eta(i) \in \{0,1\} \text{ for all } i \in \mathbf{Z}\}.$$

For $\eta \in \Sigma$ and $x \in \mathbf{Z}$ define $\eta^{x,-1}$ and $\eta^{x,1}$ to be the elements of Σ given by

$$\eta^{x,-1}(y) := \begin{cases} 1 & \text{if } y = x-1, \\ 0 & \text{if } y = x, \\ \eta(y) & \text{otherwise,} \end{cases} \qquad \eta^{x,1}(y) := \begin{cases} 0 & \text{if } y = x, \\ 1 & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases}$$

For example, $\eta^{0,1} := (\ldots, \eta(-2), \eta(-1), 0, 1, \eta(2), \eta(3), \ldots)$. And let η_x be the element of Σ equal to η except at the xth coordinate. Then the generators are

$$(\Omega_{A'}f)(\eta) := \sum_{x\in\mathbf{Z}}\left[\frac{1}{4}f(\eta^{x,x-1,-1}) + \frac{1}{4}f(\eta^{x,x-1,1}) + \frac{1}{4}f(\eta^{x,x+1,-1}) + \frac{1}{4}f(\eta^{x,x+1,1}) - f(\eta)\right]$$
$$= \sum_{x\in\mathbf{Z}}\left[\frac{1}{4}f(\eta^{x,-1}) + \frac{1}{4}f(\eta^{x-1,1}) + \frac{1}{4}f(\eta^{x,1}) + \frac{1}{4}f(\eta^{x+1,-1}) - f(\eta)\right]$$
$$= \sum_{x\in\mathbf{Z}}\left[\frac{1}{2}f(\eta^{x,-1}) + \frac{1}{2}f(\eta^{x,1}) - f(\eta)\right], \tag{6.1}$$

$$(\Omega_B f)(\eta) := \sum_{x\in\mathbf{Z}} c(x,\eta)\big[f(\eta_x) - f(\eta)\big], \tag{6.2}$$

and

$$(\Omega_{C'} f)(\eta) := \big[\gamma\Omega_{A'}f + (1-\gamma)\Omega_B f\big](\eta) \tag{6.3}$$

for functions $f \in C(\Sigma)$ depending on only finitely many coordinates, where

$$c(x,\eta) := \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases} \tag{6.4}$$

$q_m := 1 - p_m$ for $m = 0,1,2,3$, and $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0,1,2,3\}$.

Next, it is necessary to be shown that this interacting particle system is the limit in distribution of the N-player model as $N \to \infty$. Furthermore, we need to adjust the state space by relabeling the players. Specifically, we let

$$\Sigma_N := \{\eta = (\eta(l_N),\ldots,\eta(-1),\eta(0),\eta(1),\ldots,\eta(r_N)) : \eta(x) \in \{0,1\} \text{ for } x = l_N,\ldots,r_N\},$$

where

$$l_N := \begin{cases} -(N-1)/2 & \text{if } N \text{ is odd}, \\ -N/2 & \text{if } N \text{ is even}, \end{cases} \qquad r_N := \begin{cases} (N-1)/2 & \text{if } N \text{ is odd}, \\ N/2 - 1 & \text{if } N \text{ is even}. \end{cases}$$

It should be noted that players l_N and r_N are nearest neighbors. We denote the Markov chain in Σ_N by $\{X_k^N,\ k = 0,1,2,\ldots\}$.

First, let us analyze game A'. The one-step transition matrix $\boldsymbol{P}_{A'}$ of the Markov chain in the state space Σ_N has the form

$$P_{A'}(\eta,\xi) := \frac{1}{4N}\sum_{l_N\le x\le r_N}[\delta(\eta^{x,x-1,-1},\xi) + \delta(\eta^{x,x-1,1},\xi) + \delta(\eta^{x,x+1,-1},\xi) + \delta(\eta^{x,x+1,1},\xi)]$$
$$= \frac{1}{4N}\sum_{l_N\le x\le r_N}[\delta(\eta^{x,-1},\xi) + \delta(\eta^{x-1,1},\xi) + \delta(\eta^{x,1},\xi) + \delta(\eta^{x+1,-1},\xi)]$$
$$= \frac{1}{2N}\sum_{l_N\le x\le r_N}\big[\delta(\eta^{x,-1},\xi) + \delta(\eta^{x,1},\xi)\big],$$

where $\delta(\eta, \xi)$ is the Kronecker delta, which is 1 if $\eta = \xi$ and is 0 otherwise; the sum over x ranges over $\{l_N, \ldots, r_N\}$, and $l_N - 1 := r_N$ and $r_N + 1 := l_N$. Next, we have the one-step transition matrix $\boldsymbol{P}_B$ of the form

$$P_B(\xi, \zeta) := \frac{1}{N} \sum_y [1 - c(y, \xi)] \delta(\xi, \zeta) + \frac{1}{N} \sum_y c(y, \xi) \delta(\xi_y, \zeta),$$

where the sum over y also ranges over $\{l_N, \ldots, r_N\}$; $c(y, \xi)$ is as in (6.4), except that $l_N - 1 := r_N$ and $r_N + 1 := l_N$. In addition, ξ_y is equal to ξ except at the yth coordinate.

We speed up time in the N-player model so that N one-step transitions occur per unit of time. Then the discrete generator corresponding to game A' is

$$\begin{aligned}
(\Omega^N_{A'} f)(\eta) &= N\mathrm{E}\big[f(X^N_1) - f(\eta) \mid X^N_0 = \eta\big] \\
&= N \sum_{\xi \in \Sigma_N} P_{A'}(\eta, \xi) \big[f(\xi) - f(\eta)\big] \\
&= N \sum_{\xi \in \Sigma_N} \frac{1}{2N} \sum_{l_N \le x \le r_N} \big[\delta(\eta^{x,-1}, \xi) + \delta(\eta^{x,1}, \xi)\big] \big[f(\xi) - f(\eta)\big] \\
&= \frac{1}{2} \sum_{l_N \le x \le r_N} \big[f(\eta^{x,-1}) - f(x) + f(\eta^{x,1}) - f(\eta)\big] \\
&= \sum_{l_N \le x \le r_N} \left[\frac{1}{2} f(\eta^{x,-1}) + \frac{1}{2} f(\eta^{x,1}) - f(\eta)\right].
\end{aligned} \tag{6.5}$$

The discrete generator corresponding to game B is

$$\begin{aligned}
(\Omega^N_B f)(\eta) &= N\mathrm{E}\big[f(X^N_1) - f(\eta) \mid X^N_0 = \eta\big] \\
&= N \sum_{\xi \in \Sigma_N} P_B(\eta, \xi) \big[f(\xi) - f(\eta)\big] \\
&= N \bigg[\sum_{l_N \le x \le r_N : \eta(x) = 0} N^{-1} p_{m_x(\eta)} \big[f(\eta_x) - f(\eta)\big] \\
&\qquad + \sum_{l_N \le x \le r_N : \eta(x) = 1} N^{-1} q_{m_x(\eta)} \big[f(\eta_x) - f(\eta)\big] \bigg] \\
&= \sum_{l_N \le x \le r_N : \eta(x) = 0} p_{m_x(\eta)} \big[f(\eta_x) - f(\eta)\big] \\
&\qquad + \sum_{l_N \le x \le r_N : \eta(x) = 1} q_{m_x(\eta)} \big[f(\eta_x) - f(\eta)\big].
\end{aligned} \tag{6.6}$$

Hence the discrete generator corresponding to game C' is

$$(\Omega^N_{C'} f)(\eta) = \big[\gamma \Omega^N_{A'} f + (1 - \gamma) \Omega^N_B f\big](\eta)$$

$$
\begin{aligned}
&= \gamma \sum_{l_N \le x \le r_N} \left[\frac{1}{2} f(\eta^{x,-1}) + \frac{1}{2} f(\eta^{x,1}) - f(\eta) \right] \\
&\quad + (1-\gamma) \Bigg[\sum_{l_N \le x \le r_N : \eta(x)=0} p_{m_x(\eta)} \big[f(\eta_x) - f(\eta) \big] \\
&\qquad\qquad + \sum_{l_N \le x \le r_N : \eta(x)=1} q_{m_x(\eta)} \big[f(\eta_x) - f(\eta) \big] \Bigg]. \qquad (6.7)
\end{aligned}
$$

We define $\psi_N : B(\Sigma) \mapsto B(\Sigma_N)$ by

$$
(\psi_N f)(\eta(l_N), \ldots, \eta(r_N)) := f(\ldots, 1, 1, \eta(l_N), \ldots, \eta(r_N), 1, 1, \ldots). \qquad (6.8)
$$

Lemma 6.1. *If $f \in C(\Sigma)$ depends on η only through the $2K+1$ components $\eta(x)$ for $-K \le x \le K$, then*

$$
(\Omega_{A'}^N \psi_N f)(\eta) = \psi_N(\Omega_{A'} f)(\eta), \qquad (6.9)
$$

$$
(\Omega_B^N \psi_N f)(\eta) = \psi_N(\Omega_B f)(\eta), \qquad (6.10)
$$

and

$$
(\Omega_{C'}^N \psi_N f)(\eta) = \psi_N(\Omega_{C'} f)(\eta) \qquad (6.11)
$$

for all $\eta \in \Sigma_N$ and $N \ge 2K+4$.

Proof. The left side of (6.9) is

$$
\sum_{l_N \le x \le r_N} \left[\frac{1}{2} f(\ldots, 1, 1, \eta^{x,-1}, 1, 1, \ldots) + \frac{1}{2} f(\ldots, 1, 1, \eta^{x,1}, 1, 1, \ldots) - f(\ldots, 1, 1, \eta, 1, 1, \ldots) \right],
$$

where $\eta \in \Sigma_N$, while the right side of (6.9) is

$$
\sum_{x \in \mathbf{Z}} \left[\frac{1}{2} f((\ldots, 1, 1, \eta, 1, 1, \ldots)^{x,-1}) + \frac{1}{2} f((\ldots, 1, 1, \eta, 1, 1, \ldots)^{x,1}) - f(\ldots, 1, 1, \eta, 1, 1, \ldots) \right],
$$

where again $\eta \in \Sigma_N$.

If $l_N+1 \le x \le r_N-1$, then we have $f(\ldots, 1, 1, \eta^{x,-1}, 1, 1 \ldots) = f((\ldots, 1, 1, \eta, 1, 1 \ldots)^{x,-1})$ and $f(\ldots, 1, 1, \eta^{x,1}, 1, 1 \ldots) = f((\ldots, 1, 1, \eta, 1, 1 \ldots)^{x,1})$. Thus a sufficient condition for (6.9) is that $K \le r_N - 1$ and $-K \ge l_N + 1$. Equivalently, it suffices that

$$
K \le \min(r_N - 1, -l_N - 1) = r_N - 1 = \begin{cases} (N-1)/2 - 1 & \text{if } N \text{ is odd,} \\ N/2 - 2 & \text{if } N \text{ is even.} \end{cases}
$$

Therefore it suffices that $N \ge 2K+3$ if N is odd, while $N \ge 2K+4$ if N is even. Hence, $N \ge 2K+4$ is certainly sufficient. Similarly, we can prove (6.10) and (6.11).

Lemma 6.1 implies that the process $\{X^N_{\lfloor Nt\rfloor}\}$ converges in distribution to the interacting particle system $\{X_t\}$ by Theorem 1.6.5 and 4.2.6 of Ethier and Kurtz [4]. It also implies that, if the interacting particle system has a unique stationary distribution, then the unique stationary distribution of the N-player Markov chain converges to it in the topology of weak convergence, essentially by Proposition I.2.14 of Liggett [15]. Let us assume that the interacting particle system with generator $\Omega_{C'}$ has a unique stationary distribution π, and let us denote the unique stationary distribution of the N-player Markov chain for the $(\gamma, 1-\gamma)$ random mixture of games A' and B by π^N. (We previously denoted the latter by $\boldsymbol{\pi}$ but now it is necessary to make the dependence on N explicit.) We do not use boldface for π^N or π because it is no longer useful or possible, respectively, to think of them as row vectors.) Let us denote their $-1, 1$ two-dimensional marginals by $\pi^N_{-1,1}$ and $\pi_{-1,1}$. Then we have

$$\begin{aligned}
&\mu^N_{(\gamma,1-\gamma)'}\\
&\quad= \pi^N_{-1,1}(0,0)(2p_0-1) + \pi^N_{-1,1}(0,1)(2p_1-1) + \pi^N_{-1,1}(1,0)(2p_2-1) + \pi^N_{-1,1}(1,1)(2p_3-1)\\
&\quad\to \pi_{-1,1}(0,0)(2p_0-1) + \pi_{-1,1}(0,1)(2p_1-1) + \pi_{-1,1}(1,0)(2p_2-1) + \pi_{-1,1}(1,1)(2p_3-1)\\
&\quad=: \mu_{(\gamma,1-\gamma)'}, \qquad (6.12)
\end{aligned}$$

hence $\mu^N_{(\gamma,1-\gamma)'}$, the mean profit per turn at equilibrium to the N players playing the $(\gamma, 1-\gamma)$ random mixture of games A' and B, converges as $N \to \infty$ to a limit that can be expressed in terms of an interacting particle system. We have proved the following.

Theorem 6.2. *Fix $\gamma \in (0,1)$. Assume that the interacting particle system on $\mathbf{Z}$ with generator $\Omega_{C'} := \gamma\Omega_{A'} + (1-\gamma)\Omega_B$ is ergodic with unique stationary distribution π. Then $\lim_{N\to\infty} \mu^N_{(\gamma,1-\gamma)'} = \mu_{(\gamma,1-\gamma)'}$, where $\mu_{(\gamma,1-\gamma)'}$ is as in* (6.12).

6.2 Convergence of generators: Case of $r = s = 1$

We will need $P_{A'}(\eta,\xi)$, $P_B(\xi,\zeta)$, and $(\boldsymbol{P}_{A'}\boldsymbol{P}_B)(\eta,\zeta)$ before we evaluate $(\Omega^N_{[1,1]'}f)(\eta)$, where the discrete generator $\Omega^N_{[1,1]'}$ corresponds to the nonrandom pattern $[1,1]'$, that is, $A'BA'BA'B\cdots$, and with N games played per unit of time.

The state space is

$$\Sigma_N := \{\eta = (\eta(l_N), \ldots, \eta(-1), \eta(0), \eta(1), \ldots, \eta(r_N)) : \eta(x) \in \{0,1\} \text{ for } x = l_N, \ldots, r_N\},$$

where

$$l_N := \begin{cases} -(N-1)/2 & \text{if } N \text{ is odd,} \\ -N/2 & \text{if } N \text{ is even,} \end{cases} \qquad r_N := \begin{cases} (N-1)/2 & \text{if } N \text{ is odd,} \\ N/2-1 & \text{if } N \text{ is even.} \end{cases}$$

For $x = l_N, \ldots, r_N$, let $\eta^{x,1}$ be the element of Σ_N whose yth component is equal to $\eta(y)$ if $y \neq x, x+1$, 0 if $y = x$, 1 if $y = x+1$; let $\eta^{x,-1}$ be the element of Σ_N whose yth component is equal to $\eta(y)$ if $y \neq x, x-1$, 0 if $y = x$, 1 if $y = x-1$. For example, $\eta^{0,1} := (\eta(l_N), \ldots, \eta(-1), 0, 1, \eta(2), \ldots, \eta(r_N))$. Note that $r_N + 1 := l_N$ and $l_N - 1 := r_N$ here, since players l_N and r_N are nearest neighbors.

Then the one-step transition matrix $\boldsymbol{P}_{A'}$ of the Markov chain in the state space Σ_N has the form

$$P_{A'}(\eta, \xi) := \frac{1}{2N} \sum_x \left[\delta(\eta^{x,-1}, \xi) + \delta(\eta^{x,1}, \xi)\right],$$

where $\delta(\eta, \xi)$ is the Kronecker delta, which is 1 if $\eta = \xi$ and is 0 otherwise; the sum over x ranges over $\{l_N, \ldots, r_N\}$. Next, we have the one-step transition matrix $\boldsymbol{P}_B$ of the form

$$P_B(\xi, \zeta) := \frac{1}{N} \sum_y [1 - c(y, \xi)]\delta(\xi, \zeta) + \frac{1}{N} \sum_y c(y, \xi)\delta(\xi_y, \zeta),$$

where the sum over y also ranges over $\{l_N, \ldots, r_N\}$; $c(y, \xi)$ is equal to $p_{m_y(\xi)}$ if $\xi(y) = 0$ and is equal to $q_{m_y(\xi)}$ if $\xi(y) = 1$; here $q_m := 1 - p_m$ for $m = 0, 1, 2, 3$ and $m_y(\xi) := 2\xi(y-1) + \xi(y+1) \in \{0, 1, 2, 3\}$. In addition, ξ_y is equal to ξ except at the yth component.

The discrete generator has the form

$$(\Omega^N_{[1,1]'} f)(\eta) = \frac{N}{2} \sum_\zeta [f(\zeta) - f(\eta)](\boldsymbol{P}_{A'}\boldsymbol{P}_B)(\eta, \zeta).$$

To evaluate this, we begin with

$$\begin{aligned}
&(\boldsymbol{P}_{A'}\boldsymbol{P}_B)(\eta, \zeta) \\
&\quad = \sum_\xi P_{A'}(\eta, \xi) P_B(\xi, \zeta) \\
&\quad = \sum_\xi \left[\frac{1}{2N} \sum_x \left[\delta(\eta^{x,-1}, \xi) + \delta(\eta^{x,1}, \xi)\right]\right] \\
&\qquad\qquad \cdot \left[\frac{1}{N} \sum_y [1 - c(y, \xi)]\delta(\xi, \zeta) + \frac{1}{N} \sum_y c(y, \xi)\delta(\xi_y, \zeta)\right] \\
&\quad = \frac{1}{2N^2} \sum_\xi \sum_{x,y} \left[(1 - c(y, \xi))\delta(\eta^{x,-1}, \xi)\delta(\xi, \zeta) + (1 - c(y, \xi))\delta(\eta^{x,1}, \xi)\delta(\xi, \zeta)\right.
\end{aligned}$$

$$+ c(y,\xi)\delta(\eta^{x,-1},\xi)\delta(\xi_y,\zeta) + c(y,\xi)\delta(\eta^{x,1},\xi)\delta(\xi_y,\zeta)\big]$$

$$= \frac{1}{2N^2}\sum_x\sum_y \big[(1-c(y,\eta^{x,-1}))\delta(\eta^{x,-1},\zeta) + (1-c(y,\eta^{x,1}))\delta(\eta^{x,1},\zeta)\big]$$

$$+ \frac{1}{2N^2}\sum_x\sum_y \big[c(y,\eta^{x,-1})\delta((\eta^{x,-1})_y,\zeta) + c(y,\eta^{x,1})\delta((\eta^{x,1})_y,\zeta)\big]. \tag{6.13}$$

Hence, for $f \in B(\Sigma_N)$,

$$(\Omega^N_{[1,1]'}f)(\eta)$$

$$= \frac{1}{4N}\sum_x\sum_y \big[(1-c(y,\eta^{x,-1}))[f(\eta^{x,-1}) - f(\eta)] + (1-c(y,\eta^{x,1}))[f(\eta^{x,1}) - f(\eta)]\big]$$

$$+ \frac{1}{4N}\sum_x\sum_y \big[c(y,\eta^{x,-1})[f((\eta^{x,-1})_y) - f(\eta)] + c(y,\eta^{x,1})[f((\eta^{x,1})_y) - f(\eta)]\big]. \tag{6.14}$$

Now suppose that $f \in B(\Sigma)$ depends only on $\eta(-(K-2)),\dots,\eta(K-2)$ for some integer $K \geq 2$, and define $f_N := \psi_N f \in B(\Sigma_N)$. Then we can write the discrete generator acting on f_N as

$$(\Omega^N_{[1,1]'}f_N)(\eta)$$

$$= \frac{1}{4N}\Bigg[\sum_{|x|\le K}\sum_{|y|\le K} \big[(1-c(y,\eta^{x,-1}))[f_N(\eta^{x,-1}) - f_N(\eta)] + (1-c(y,\eta^{x,1}))[f_N(\eta^{x,1}) - f_N(\eta)]\big]$$

$$+ \sum_{|x|\le K}\sum_{|y|> K} \big[(1-c(y,\eta^{x,-1}))[f_N(\eta^{x,-1}) - f_N(\eta)] + (1-c(y,\eta^{x,1}))[f_N(\eta^{x,1}) - f_N(\eta)]\big]$$

$$+ \sum_{|x|> K}\sum_{|y|\le K} \big[(1-c(y,\eta^{x,-1}))[f_N(\eta^{x,-1}) - f_N(\eta)] + (1-c(y,\eta^{x,1}))[f_N(\eta^{x,1}) - f_N(\eta)]\big]$$

$$+ \sum_{|x|> K}\sum_{|y|> K} \big[(1-c(y,\eta^{x,-1}))[f_N(\eta^{x,-1}) - f_N(\eta)] + (1-c(y,\eta^{x,1}))[f_N(\eta^{x,1}) - f_N(\eta)]\big]\Bigg]$$

$$+ \frac{1}{4N}\Bigg[\sum_{|x|\le K}\sum_{|y|\le K} \big[c(y,\eta^{x,-1})[f_N((\eta^{x,-1})_y) - f_N(\eta)] + c(y,\eta^{x,1})[f_N((\eta^{x,1})_y) - f_N(\eta)]\big]$$

$$+ \sum_{|x|\le K}\sum_{|y|> K} \big[c(y,\eta^{x,-1})[f_N((\eta^{x,-1})_y) - f_N(\eta)] + c(y,\eta^{x,1})[f_N((\eta^{x,1})_y) - f_N(\eta)]\big]$$

$$+ \sum_{|x|> K}\sum_{|y|\le K} \big[c(y,\eta^{x,-1})[f_N((\eta^{x,-1})_y) - f_N(\eta)] + c(y,\eta^{x,1})[f_N((\eta^{x,1})_y) - f_N(\eta)]\big]$$

$$+ \sum_{|x|> K}\sum_{|y|> K} \big[c(y,\eta^{x,-1})[f_N((\eta^{x,-1})_y) - f_N(\eta)] + c(y,\eta^{x,1})[f_N((\eta^{x,1})_y) - f_N(\eta)]\big]\Bigg]. \tag{6.15}$$

Let us analyze each term in (6.15). The first term becomes $O(N^{-1})$ because

$$\Bigg|\sum_{|x|\le K}\sum_{|y|\le K} \big[(1-c(y,\eta^{x,-1}))[f_N(\eta^{x,-1}) - f_N(\eta)] + (1-c(y,\eta^{x,1}))[f_N(\eta^{x,1}) - f_N(\eta)]\big]\Bigg|$$

$$\le \sum_{|x|\le K}\sum_{|y|\le K}\big[|f_N(\eta^{x,-1})-f_N(\eta)|+|f_N(\eta^{x,1})-f_N(\eta)|\big]$$
$$\le (2K+1)^2\sup_{\eta,x}\Big[\big|f_N(\eta^{x,-1})-f_N(\eta)\big|+\big|f_N(\eta^{x,1})-f_N(\eta)\big|\Big]$$
$$\le 4(2K+1)^2\sup_{\eta}\big|f(\eta)\big|.$$

The fifth term becomes $O(N^{-1})$ in the same way as the first term.

Since $|x|>K$ in the third term, $f_N(\eta^{x,-1})=f_N(\eta^{x,1})=f_N(\eta)$. Hence the third term is zero.

Also $|x|>K$ and $|y|>K$ imply that $f_N(\eta^{x,-1})$, $f_N(\eta^{x,1})$, $f_N((\eta^{x,-1})_y)$, and $f_N((\eta^{x,1})_y)$ are equal to $f_N(\eta)$. This causes the fourth and the eighth terms to be zero.

The second and sixth terms combine because $|y|>K$ implies $f_N((\eta^{x,-1})_y)=f_N(\eta^{x,-1})$ and $f_N((\eta^{x,1})_y)=f_N(\eta^{x,1})$, so we have

$$\frac{1}{4N}\sum_{|x|\le K}\sum_{|y|>K}\big[[f_N(\eta^{x,-1})-f_N(\eta)]+[f_N(\eta^{x,1})-f_N(\eta)]\big]$$
$$=\frac{1}{4}\sum_{|x|\le K}\left(\frac{N-(2K+1)}{N}\right)\big[[f_N(\eta^{x,-1})-f_N(\eta)]+[f_N(\eta^{x,1})-f_N(\eta)]\big]$$
$$=\frac{1}{2}\sum_{|x|\le K}\left[\frac{1}{2}f_N(\eta^{x,-1})+\frac{1}{2}f_N(\eta^{x,1})-f_N(\eta)\right]+O(N^{-1}).$$

Since $|x|>K$ implies $f_N((\eta^{x,-1})_y)=f_N((\eta^{x,1})_y)=f_N(\eta_y)$, the seventh term becomes

$$\frac{1}{4N}\sum_{|x|>K}\sum_{|y|\le K}\big[c(y,\eta^{x,-1})[f_N((\eta^{x,-1})_y)-f_N(\eta)]+c(y,\eta^{x,1})[f_N((\eta^{x,1})_y)-f_N(\eta)]\big]$$
$$=\frac{1}{4}\sum_{|y|\le K}\left(\frac{N-(2K+1)}{N}\right)c(y,\eta)\big[[f_N(\eta_y)-f_N(\eta)]+[f_N(\eta_y)-f_N(\eta)]\big]$$
$$=\frac{1}{2}\sum_{|y|\le K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]+O(N^{-1}),$$

where we are also using $c(y,\eta^{x,-1})=c(y,\eta)$ if $|x|>K$ and $|y|\le K$ with possible exceptions if $|x-y|=1$ or 2. But in such a case, $f_N(\eta_y)-f_N(\eta)=0$ since f depends only on $\eta(-(K-2)),\dots,\eta(K-2)$. Thus we can conclude that

$$(\Omega^N_{[1,1]'}f_N)(\eta)$$
$$=\frac{1}{2}\sum_{|x|\le K}\left[\frac{1}{2}f_N(\eta^{x,-1})+\frac{1}{2}f_N(\eta^{x,1})-f_N(\eta)\right]+\frac{1}{2}\sum_{|y|\le K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]+O(N^{-1})$$
$$=\frac{1}{2}\psi_N\big(\Omega_{A'}f+\Omega_B f\big)(\eta)+O(N^{-1}),$$

where $f \in B(\Sigma)$ depends on η through the $2K-3$ components $\eta(x)$ for $-(K-2) \le x \le K-2$. Here we recall that $\Omega_{A'} f$ and $\Omega_B f$ are as in (6.1) and (6.2).

6.3 Convergence of generators: Case of $r = s = 2$

The discrete generator with $r = s = 2$ has the form

$$(\Omega^N_{[2,2]'} f)(\eta) = \frac{N}{4} \sum_{\zeta} [f(\zeta) - f(\eta)](\boldsymbol{P}^2_{A'} \boldsymbol{P}^2_B)(\eta, \zeta).$$

First, we compute

$$\begin{aligned}
&(\boldsymbol{P}^2_{A'} \boldsymbol{P}^2_B)(\eta, \zeta) \\
&= \sum_{\xi,\tau,\phi} P_{A'}(\eta,\xi) P_{A'}(\xi,\tau) P_B(\tau,\phi) P_B(\phi,\zeta) \\
&= \sum_{\xi,\tau,\phi} \left[\frac{1}{2N} \sum_w [\delta(\eta^{w,-1},\xi) + \delta(\eta^{w,1},\xi)]\right] \left[\frac{1}{2N} \sum_x [\delta(\xi^{x,-1},\tau) + \delta(\xi^{x,1},\tau)]\right] \\
&\qquad \cdot \left[\frac{1}{N} \sum_y (1 - c(y,\tau))\delta(\tau,\phi) + \frac{1}{N} \sum_y c(y,\tau)\delta(\tau_y,\phi)\right] \\
&\qquad \cdot \left[\frac{1}{N} \sum_z (1 - c(z,\phi))\delta(\phi,\zeta) + \frac{1}{N} \sum_z c(z,\phi)\delta(\phi_z,\zeta)\right] \\
&= \frac{1}{4N^4} \sum_{\xi,\tau,\phi} \sum_{w,x,y,z} \Big[(1 - c(y,\tau))(1 - c(z,\phi))\delta(\eta^{w,-1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau,\phi)\delta(\phi,\zeta) \\
&\qquad + (1 - c(y,\tau))c(z,\phi)\delta(\eta^{w,-1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau,\phi)\delta(\phi_z,\zeta) \\
&\qquad + c(y,\tau)(1 - c(z,\phi))\delta(\eta^{w,-1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau_y,\phi)\delta(\phi,\zeta) \\
&\qquad + c(y,\tau)c(z,\phi)\delta(\eta^{w,-1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau_y,\phi)\delta(\phi_z,\zeta) \\
&\qquad + (1 - c(y,\tau))(1 - c(z,\phi))\delta(\eta^{w,-1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau,\phi)\delta(\phi,\zeta) \\
&\qquad + (1 - c(y,\tau))c(z,\phi)\delta(\eta^{w,-1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau,\phi)\delta(\phi_z,\zeta) \\
&\qquad + c(y,\tau)(1 - c(z,\phi))\delta(\eta^{w,-1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau_y,\phi)\delta(\phi,\zeta) \\
&\qquad + c(y,\tau)c(z,\phi)\delta(\eta^{w,-1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau_y,\phi)\delta(\phi_z,\zeta) \\
&\qquad + (1 - c(y,\tau))(1 - c(z,\phi))\delta(\eta^{w,1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau,\phi)\delta(\phi,\zeta) \\
&\qquad + (1 - c(y,\tau))c(z,\phi)\delta(\eta^{w,1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau,\phi)\delta(\phi_z,\zeta) \\
&\qquad + c(y,\tau)(1 - c(z,\phi))\delta(\eta^{w,1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau_y,\phi)\delta(\phi,\zeta) \\
&\qquad + c(y,\tau)c(z,\phi)\delta(\eta^{w,1},\xi)\delta(\xi^{x,-1},\tau)\delta(\tau_y,\phi)\delta(\phi_z,\zeta) \\
&\qquad + (1 - c(y,\tau))(1 - c(z,\phi))\delta(\eta^{w,1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau,\phi)\delta(\phi,\zeta)
\end{aligned}$$

$$
\begin{aligned}
&+ (1-c(y,\tau))c(z,\phi)\delta(\eta^{w,1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau,\phi)\delta(\phi_z,\zeta)\\
&+ c(y,\tau)(1-c(z,\phi))\delta(\eta^{w,1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau_y,\phi)\delta(\phi,\zeta)\\
&+ c(y,\tau)c(z,\phi)\delta(\eta^{w,1},\xi)\delta(\xi^{x,1},\tau)\delta(\tau_y,\phi)\delta(\phi_z,\zeta)\Big]\\
= \frac{1}{4N^4}\sum_w\sum_x\sum_y\sum_z &\Big[(1-c(y,(\eta^{w,-1})^{x,-1}))(1-c(z,(\eta^{w,-1})^{x,-1}))\delta((\eta^{w,-1})^{x,-1},\zeta)\\
&+ (1-c(y,(\eta^{w,-1})^{x,-1}))c(z,(\eta^{w,-1})^{x,-1})\delta(((\eta^{w,-1})^{x,-1})_z,\zeta)\\
&+ c(y,(\eta^{w,-1})^{x,-1})(1-c(z,((\eta^{w,-1})^{x,-1})_y))\delta(((\eta^{w,-1})^{x,-1})_y,\zeta)\\
&+ c(y,(\eta^{w,-1})^{x,-1})c(z,((\eta^{w,-1})^{x,-1})_y)\delta((((\eta^{w,-1})^{x,-1})_y)_z,\zeta)\\
&+ (1-c(y,(\eta^{w,-1})^{x,1}))(1-c(z,(\eta^{w,-1})^{x,1}))\delta((\eta^{w,-1})^{x,1},\zeta)\\
&+ (1-c(y,(\eta^{w,-1})^{x,1}))c(z,(\eta^{w,-1})^{x,1})\delta(((\eta^{w,-1})^{x,1})_z,\zeta)\\
&+ c(y,(\eta^{w,-1})^{x,1})(1-c(z,((\eta^{w,-1})^{x,1})_y))\delta(((\eta^{w,-1})^{x,1})_y,\zeta)\\
&+ c(y,(\eta^{w,-1})^{x,1})c(z,((\eta^{w,-1})^{x,1})_y)\delta((((\eta^{w,-1})^{x,1})_y)_z,\zeta)\\
&+ (1-c(y,(\eta^{w,1})^{x,-1}))(1-c(z,(\eta^{w,1})^{x,-1}))\delta((\eta^{w,1})^{x,-1},\zeta)\\
&+ (1-c(y,(\eta^{w,1})^{x,-1}))c(z,(\eta^{w,1})^{x,-1})\delta(((\eta^{w,1})^{x,-1})_z,\zeta)\\
&+ c(y,(\eta^{w,1})^{x,-1})(1-c(z,((\eta^{w,1})^{x,-1})_y))\delta(((\eta^{w,1})^{x,-1})_y,\zeta)\\
&+ c(y,(\eta^{w,1})^{x,-1})c(z,((\eta^{w,1})^{x,-1})_y)\delta((((\eta^{w,1})^{x,-1})_y)_z,\zeta)\\
&+ (1-c(y,(\eta^{w,1})^{x,1}))(1-c(z,(\eta^{w,1})^{x,1}))\delta((\eta^{w,1})^{x,1},\zeta)\\
&+ (1-c(y,(\eta^{w,1})^{x,1}))c(z,(\eta^{w,1})^{x,1})\delta(((\eta^{w,1})^{x,1})_z,\zeta)\\
&+ c(y,(\eta^{w,1})^{x,1})(1-c(z,((\eta^{w,1})^{x,1})_y))\delta(((\eta^{w,1})^{x,1})_y,\zeta)\\
&+ c(y,(\eta^{w,1})^{x,1})c(z,((\eta^{w,1})^{x,1})_y)\delta((((\eta^{w,1})^{x,1})_y)_z,\zeta). \qquad (6.16)
\end{aligned}
$$

Next, assume that $f \in B(\Sigma)$ depends only on $\eta(-(K-2)),\ldots,\eta(K-2)$ for some integer $K \geq 2$, and put $f_N := \psi_N f \in B(\Sigma_N)$. Then the discrete generator with $r = s = 2$ acting on f_N reduces to

$$
\begin{aligned}
&(\Omega^N_{[2,2]'} f_N)(\eta)\\
&= \frac{1}{16N^3}\sum_w\sum_x\sum_y\sum_z \Big[(1-c(y,(\eta^{w,-1})^{x,-1}))(1-c(z,(\eta^{w,-1})^{x,-1}))\\
&\qquad \cdot \big[f_N((\eta^{w,-1})^{x,-1}) - f_N(\eta)\big]\\
&\qquad + (1-c(y,(\eta^{w,-1})^{x,-1}))c(z,(\eta^{w,-1})^{x,-1})\big[f_N(((\eta^{w,-1})^{x,-1})_z) - f_N(\eta)\big]
\end{aligned}
$$

$$
\begin{aligned}
&+ c(y,(\eta^{w,-1})^{x,-1})(1-c(z,((\eta^{w,-1})^{x,-1})_y))\big[f_N(((\eta^{w,-1})^{x,-1})_y)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,-1})^{x,-1})c(z,((\eta^{w,-1})^{x,-1})_y)\big[f_N((((\eta^{w,-1})^{x,-1})_y)_z)-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,-1})^{x,1}))(1-c(z,(\eta^{w,-1})^{x,1}))\big[f_N((\eta^{w,-1})^{x,1})-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,-1})^{x,1}))c(z,(\eta^{w,-1})^{x,1})\big[f_N(((\eta^{w,-1})^{x,1})_z)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,-1})^{x,1})(1-c(z,((\eta^{w,-1})^{x,1})_y))\big[f_N(((\eta^{w,-1})^{x,1})_y)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,-1})^{x,1})c(z,((\eta^{w,-1})^{x,1})_y)\big[f_N((((\eta^{w,-1})^{x,1})_y)_z)-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,1})^{x,-1}))(1-c(z,(\eta^{w,1})^{x,-1}))\big[f_N((\eta^{w,1})^{x,-1})-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,1})^{x,-1}))c(z,(\eta^{w,1})^{x,-1})\big[f_N(((\eta^{w,1})^{x,-1})_z)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,1})^{x,-1})(1-c(z,((\eta^{w,1})^{x,-1})_y))\big[f_N(((\eta^{w,1})^{x,-1})_y)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,1})^{x,-1})c(z,((\eta^{w,1})^{x,-1})_y)\big[f_N((((\eta^{w,1})^{x,-1})_y)_z)-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,1})^{x,1}))(1-c(z,(\eta^{w,1})^{x,1}))\big[f_N((\eta^{w,1})^{x,1})-f_N(\eta)\big]\\
&+ (1-c(y,(\eta^{w,1})^{x,1}))c(z,(\eta^{w,1})^{x,1})\big[f_N(((\eta^{w,1})^{x,1})_z)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,1})^{x,1})(1-c(z,((\eta^{w,1})^{x,1})_y))\big[f_N(((\eta^{w,1})^{x,1})_y)-f_N(\eta)\big]\\
&+ c(y,(\eta^{w,1})^{x,1})c(z,((\eta^{w,1})^{x,1})_y)\big[f_N((((\eta^{w,1})^{x,1})_y)_z)-f_N(\eta)\big]\Bigg]. \qquad (6.17)
\end{aligned}
$$

We replace $\sum_w \sum_x \sum_y \sum_z$ in (6.17) by

$$\Big(\sum_{|w|\le K}+\sum_{|w|>K}\Big)\Big(\sum_{|x|\le K}+\sum_{|x|>K}\Big)\Big(\sum_{|y|\le K}+\sum_{|y|>K}\Big)\Big(\sum_{|z|\le K}+\sum_{|z|>K}\Big)$$

and conclude the following.

- The term for which all of $|w|$, $|x|$, $|y|$, and $|z|$ are less than or equal to K contribute at most $(16N^3)^{-1}(2K+1)^4\, 2\sup_\eta |f(\eta)|$.

- The terms for which three of $|w|$, $|x|$, $|y|$, and $|z|$ are less than or equal to K contribute at most $4(16N^3)^{-1}(N-(2K+1))(2K+1)^3\, 2\sup_\eta |f(\eta)|$.

- The terms for which two of $|w|$, $|x|$, $|y|$, and $|z|$ are less than or equal to K contribute at most $6(16N^3)^{-1}(N-(2K+1))^2(2K+1)^2\, 2\sup_\eta |f(\eta)|$.

- The term for which none of $|w|$, $|x|$, $|y|$, and $|z|$ is less than or equal to K is equal to 0.

The contributions to (6.17) described in the four cases above will be $O(N^{-1})$. So it is enough to analyze the cases in which one of $|w|$, $|x|$, $|y|$, and $|z|$ is less than or equal to K.

First, we consider the case in which only $|w| \le K$ (or similarly only $|x| \le K$). This case contributes

$$
\begin{aligned}
&\frac{1}{16N^3}\sum_{|w|\le K}\sum_{|x|>K}\sum_{|y|>K}\sum_{|z|>K}\Big[(1-c(y,(\eta^{w,-1})^{x,-1}))(1-c(z,(\eta^{w,-1})^{x,-1}))\\
&\qquad\qquad\qquad\qquad\qquad\cdot\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,-1})^{x,-1}))c(z,(\eta^{w,-1})^{x,-1})\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,-1})^{x,-1})(1-c(z,((\eta^{w,-1})^{x,-1})_y))\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,-1})^{x,-1})c(z,((\eta^{w,-1})^{x,-1})_y)\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,-1})^{x,1}))(1-c(z,(\eta^{w,-1})^{x,1}))\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,-1})^{x,1}))c(z,(\eta^{w,-1})^{x,1})\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,-1})^{x,1})(1-c(z,((\eta^{w,-1})^{x,1})_y))\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,-1})^{x,1})c(z,((\eta^{w,-1})^{x,1})_y)\big[f_N(\eta^{w,-1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,1})^{x,-1}))(1-c(z,(\eta^{w,1})^{x,-1}))\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,1})^{x,-1}))c(z,(\eta^{w,1})^{x,-1})\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,1})^{x,-1})(1-c(z,((\eta^{w,1})^{x,-1})_y))\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,1})^{x,-1})c(z,((\eta^{w,1})^{x,-1})_y)\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,1})^{x,1}))(1-c(z,(\eta^{w,1})^{x,1}))\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + (1-c(y,(\eta^{w,1})^{x,1}))c(z,(\eta^{w,1})^{x,1})\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,1})^{x,1})(1-c(z,((\eta^{w,1})^{x,1})_y))\big[f_N(\eta^{w,1})-f_N(\eta)\big]\\
&\qquad\qquad + c(y,(\eta^{w,1})^{x,1})c(z,((\eta^{w,1})^{x,1})_y)\big[f_N(\eta^{w,1})-f_N(\eta)\big]\Big]\\
&=\frac{1}{16N^3}\sum_{|w|\le K}\sum_{|x|>K}\sum_{|y|>K}\sum_{|z|>K}\big\{2\big[f_N(\eta^{w,-1})-f_N(\eta)\big]+2\big[f_N(\eta^{w,1})-f_N(\eta)\big]\big\}\\
&=\frac{1}{4N^3}\sum_{|w|\le K}\big[N-(2K+1)\big]^3\Big[\frac{1}{2}f_N(\eta^{w,-1})+\frac{1}{2}f_N(\eta^{w,1})-f_N(\eta)\Big]\\
&=\frac{1}{4}\sum_{|w|\le K}\Big[\frac{1}{2}f_N(\eta^{w,-1})+\frac{1}{2}f_N(\eta^{w,1})-f_N(\eta)\Big]+O(N^{-1}). \qquad (6.18)
\end{aligned}
$$

Next, we consider the case in which only $|y| \le K$ (or similarly only $|z| \le K$). This case contributes

$$\frac{1}{16N^3}\sum_{|w|>K}\sum_{|x|>K}\sum_{|y|\le K}\sum_{|z|>K}\Big[(1-c(y,\eta))(1-c(z,\eta))\big[f_N(\eta)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))c(z,\eta))\big[f_N(\eta)-f_N(\eta)\big]$$
$$+c(y,\eta)(1-c(z,\eta_y)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+c(y,\eta)c(z,\eta_y)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))(1-c(z,\eta))\big[f_N(\eta)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))c(z,\eta)\big[f_N(\eta)-f_N(\eta)\big]$$
$$+c(y,\eta)(1-c(z,\eta_y))\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+c(y,\eta)c(z,\eta_y)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))(1-c(z,\eta))\big[f_N(\eta)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))c(z,\eta)\big[f_N(\eta)-f_N(\eta)\big]$$
$$+c(y,\eta)(1-c(z,\eta_y))\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+c(y,\eta)c(z,\eta_y)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))(1-c(z,\eta))\big[f_N(\eta)-f_N(\eta)\big]$$
$$+(1-c(y,\eta))c(z,\eta)\big[f_N(\eta)-f_N(\eta)\big]$$
$$+c(y,\eta)(1-c(z,\eta_y))\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$+c(y,\eta)c(z,\eta_y))\big[f_N(\eta_y)-f_N(\eta)\big]\Big]$$
$$=\frac{1}{4N^3}\sum_{|w|>K}\sum_{|x|>K}\sum_{|y|\le K}\sum_{|z|>K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$=\frac{1}{4N^3}\sum_{|y|\le K}\big[N-(2K+1)\big]^3c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]$$
$$=\frac{1}{4}\sum_{|y|\le K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]+O(N^{-1}), \tag{6.19}$$

where we are also using, for example, $c(y,(\eta^{w,-1})^{x,-1}) = c(y,\eta)$ if $|w| > K$, $|x| > K$, and $|y| \le K$ with possible exceptions if $|w-y| = 1$ or 2 or $|x-y| = 1$ or 2. But in such a case, $f_N(\eta_y) - f_N(\eta) = 0$ since f depends only on $\eta(-(K-2)),\ldots,\eta(K-2)$. Finally, we conclude that

$$(\Omega^N_{[2,2]'}f_N)(\eta)$$
$$=\frac{1}{4}\sum_{|w|\le K}\left[\frac{1}{2}f_N(\eta^{w,-1})+\frac{1}{2}f_N(\eta^{w,1})-f_N(\eta)\right]$$

$$
\begin{aligned}
&+\frac{1}{4}\sum_{|x|\le K}\left[\frac{1}{2}f_N(\eta^{x,-1})+\frac{1}{2}f_N(\eta^{x,1})-f_N(\eta)\right]\\
&+\frac{1}{4}\sum_{|y|\le K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]\\
&+\frac{1}{4}\sum_{|z|\le K}c(z,\eta)\big[f_N(\eta_z)-f_N(\eta)\big]+O(N^{-1})\\
=&\ \frac{1}{2}\sum_{|w|\le K}\left[\frac{1}{2}f_N(\eta^{w,-1})+\frac{1}{2}f_N(\eta^{w,1})-f_N(\eta)\right]+\frac{1}{2}\sum_{|y|\le K}c(y,\eta)\big[f_N(\eta_y)-f_N(\eta)\big]\\
&+O(N^{-1})\\
=&\ \frac{1}{2}\psi_N\big(\Omega_{A'}f+\Omega_B f\big)(\eta)+O(N^{-1}),
\end{aligned}
$$

as desired.

6.4 Convergence of generators: General case

The discrete generator for the nonrandom periodic pattern $(A')^r B^s$ has the form, for $f\in B(\Sigma_N)$,

$$
(\Omega^N_{[r,s]'}f)(\eta^0)=\frac{N}{r+s}\sum_{\eta^{r+s}}[f(\eta^{r+s})-f(\eta^0)](\boldsymbol{P}^r_{A'}\boldsymbol{P}^s_B)(\eta^0,\eta^{r+s}).
$$

We begin by evaluating

$$
\begin{aligned}
&(\boldsymbol{P}^r_{A'}\boldsymbol{P}^s_B)(\eta^0,\eta^{r+s})\\
&=\sum_{\eta^1,\eta^2,\ldots,\eta^{r+s-1}}P_{A'}(\eta^0,\eta^1)P_{A'}(\eta^1,\eta^2)\cdots P_{A'}(\eta^{r-1},\eta^r)\\
&\qquad\times P_B(\eta^r,\eta^{r+1})P_B(\eta^{r+1},\eta^{r+2})\cdots P_B(\eta^{r+s-1},\eta^{r+s})\\
&=\sum_{\eta^1,\eta^2,\ldots,\eta^{r+s-1}}\prod_{i=1}^{r}P_{A'}(\eta^{i-1},\eta^i)\prod_{i=r+1}^{r+s}P_B(\eta^{i-1},\eta^i)\\
&=\sum_{\eta^1,\eta^2,\ldots,\eta^{r+s-1}}\prod_{i=1}^{r}\left[\frac{1}{2N}\sum_{x_i}[\delta((\eta^{i-1})^{x_i,-1},\eta^i)+\delta((\eta^{i-1})^{x_i,1},\eta^i)]\right]\\
&\qquad\times\prod_{i=r+1}^{r+s}\left[\frac{1}{N}\sum_{x_i}(1-c(x_i,\eta^{i-1}))\delta(\eta^{i-1},\eta^i)+\frac{1}{N}\sum_{x_i}c(x_i,\eta^{i-1})\delta((\eta^{i-1})_{x_i},\eta^i)\right]\\
&=\frac{1}{2^rN^{r+s}}\sum_{A\subset\{1,\ldots,r\}}\sum_{B\subset\{r+1,\ldots,r+s\}}\sum_{\eta^1,\eta^2,\ldots,\eta^{r+s-1}}\prod_{i\in A^c}\left[\sum_{x_i}\delta((\eta^{i-1})^{x_i,-1},\eta^i)\right]\\
&\qquad\times\prod_{i\in A}\left[\sum_{x_i}\delta((\eta^{i-1})^{x_i,1},\eta^i)\right]\prod_{i\in B^c}\left[\sum_{x_i}(1-c(x_i,\eta^{i-1}))\delta(\eta^{i-1},\eta^i)\right]\\
&\qquad\times\prod_{i\in B}\left[\sum_{x_i}c(x_i,\eta^{i-1})\delta((\eta^{i-1})_{x_i},\eta^i)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2^r N^{r+s}} \sum_{A\subset\{1,\dots,r\}} \sum_{B\subset\{r+1,\dots,r+s\}} \sum_{\eta^1,\eta^2,\dots,\eta^{r+s-1}} \sum_{x_i:i\in A^c} \sum_{x_i:i\in A} \sum_{x_i:i\in B^c} \sum_{x_i:i\in B} \\
&\quad \times \prod_{j\in A^c} \left[\delta((\eta^{j-1})^{x_j,-1}, \eta^j)\right] \prod_{j\in A} \left[\delta((\eta^{j-1})^{x_j,1}, \eta^j)\right] \\
&\quad \times \prod_{j\in B^c} \left[(1 - c(x_j, \eta^{j-1}))\delta(\eta^{j-1}, \eta^j)\right] \prod_{j\in B} \left[c(x_j, \eta^{j-1})\delta((\eta^{j-1})_{x_j}, \eta^j)\right] \\
&= \frac{1}{2^r N^{r+s}} \sum_{A\subset\{1,\dots,r\}} \sum_{B\subset\{r+1,\dots,r+s\}} \sum_{x_i:i\in\{1,2,\dots,r+s\}} \\
&\quad \times \prod_{j\in B^c} \left[1 - c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}})\right] \\
&\quad \times \prod_{j\in B} c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}}) \\
&\quad \times \delta(((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B\}}, \eta^{r+s}),
\end{aligned}
\tag{6.20}
$$

where $A^c := \{1,2,\dots,r\} - A$ and $B^c := \{r+1, r+2, \dots, r+s\} - B$; also $p \in \{1,2,\dots,r\}$ and

$$
a_p = \begin{cases} -1 & \text{if } p \in A^c, \\ 1 & \text{if } p \in A. \end{cases}
$$

Here, for example, $\eta_{\{x_l:l\in B\}}$ denotes η with the spins flipped at each site x_l with $l \in B$. These site labels need not be distinct, so if there are multiple flips at a single site, only the parity of the number of flips is relevant.

Next, assume that $f \in B(\Sigma)$ depends only on $\eta(-(K-2)), \dots, \eta(K-2)$ for some integer $K \geq 2$, and put $f_N := \psi_N f \in B(\Sigma_N)$. Then the discrete generator for the pattern $(A')^r B^s$, acting on f_N, reduces to

$$
\begin{aligned}
(\Omega^N_{[r,s]'} f_N)(\eta^0) &= \frac{N}{r+s} \sum_{\eta^{r+s}} [f_N(\eta^{r+s}) - f_N(\eta^0)](\boldsymbol{P}^r_{A'} \boldsymbol{P}^s_B)(\eta^0, \eta^{r+s}) \\
&= \frac{1}{r+s} \frac{1}{2^r N^{r+s-1}} \sum_{A\subset\{1,\dots,r\}} \sum_{B\subset\{r+1,\dots,r+s\}} \sum_{x_i:i\in\{1,2,\dots,r+s\}} \\
&\quad \times \prod_{j\in B^c} \left[1 - c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}})\right] \\
&\quad \times \prod_{j\in B} c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}}) \\
&\quad \times [f_N(((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B\}}) - f_N(\eta^0)]
\end{aligned}
\tag{6.21}
$$

We replace $\sum_{x_i:i\in\{1,2,\dots,r+s\}}$ in (6.21) by

$$
\left(\sum_{i\in A^c} + \sum_{i\in A}\right) \sum_{|x_i|\leq K} \sum_{|x_m|>K:m\in\{1,2,\dots,r+s\},m\neq i} + \sum_{i\in B} \sum_{|x_i|\leq K} \sum_{|x_m|>K:m\in\{1,2,\dots,r+s\},m\neq i}
\tag{6.22}
$$

since each sum $\sum_{x_i}$ can be written as $\sum_{|x_i|\le K}+\sum_{|x_i|>K}$ resulting in 2^{r+s} multiple sums in which each of those multiple sums with two or more sums of the form $\sum_{|x_i|\le K}$ contributes at most $O(N^{-1})$ and those without the form $\sum_{|x_i|\le K}$, where $i\in\{1,2,\dots,r\}\cup B$ are 0. So it is enough to analyze the cases in which only one of the $|x_i|$'s is less than or equal to K.

We consider first the first term in (6.22). It contributes

$$
\begin{aligned}
&\frac{1}{r+s}\,\frac{1}{2^rN^{r+s-1}}\sum_{A\subset\{1,\dots,r\}}\sum_{B\subset\{r+1,\dots,r+s\}}\Bigg[\sum_{i\in A^c}\sum_{|x_i|\le K}[f_N((\eta^0)^{x_i,-1})-f_N(\eta^0)]\\
&\qquad\qquad+\sum_{i\in A}\sum_{|x_i|\le K}[f_N((\eta^0)^{x_i,1})-f_N(\eta^0)]\Bigg]\\
&\times\sum_{|x_m|>K:m\in\{1,2,\dots,r+s\},m\ne i}\prod_{j\in B^c}\big[1\\
&\qquad-c(x_j,((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}})\big]\\
&\times\prod_{j\in B}c(x_j,((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}})\\
&=\frac{1}{r+s}\,\frac{1}{2^rN^{r+s-1}}\sum_{A\subset\{1,\dots,r\}}\sum_{B\subset\{r+1,\dots,r+s\}}\Bigg[\sum_{i\in A^c}\sum_{|x_i|\le K}[f_N((\eta^0)^{x_i,-1})-f_N(\eta^0)]\\
&\qquad\qquad+\sum_{i\in A}\sum_{|x_i|\le K}[f_N((\eta^0)^{x_i,1})-f_N(\eta^0)]\Bigg]\\
&\times\sum_{|x_m|>K:m\in\{1,2,\dots,r+s\},m\ne i}\prod_{j\in B^c}\big[1-c(x_j,(\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})\big]\\
&\times\prod_{j\in B}c(x_j,(\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})\\
&=\frac{1}{r+s}\,\frac{1}{2^rN^{r+s-1}}\big[N-(2K+1)\big]^{r+s-1}\sum_{A\subset\{1,\dots,r\}}\Bigg[|A^c|\sum_{|x|\le K}[f_N((\eta^0)^{x,-1})-f_N(\eta^0)]\\
&\qquad\qquad+|A|\sum_{|x|\le K}[f_N((\eta^0)^{x,1})-f_N(\eta^0)]\Bigg]\\
&=\frac{1}{r+s}\,\frac{1}{2^rN^{r+s-1}}\big[N-(2K+1)\big]^{r+s-1}\sum_{A\subset\{1,\dots,r\}}\Bigg[|A^c|\sum_{|x|\le K}f_N((\eta^0)^{x,-1})\\
&\qquad\qquad+|A|\sum_{|x|\le K}f_N((\eta^0)^{x,1})-r\sum_{|x|\le K}f_N(\eta^0)\Bigg]\\
&=\frac{1}{r+s}\,\frac{1}{N^{r+s-1}}\big[N-(2K+1)\big]^{r+s-1}\Bigg[\frac{r}{2}\sum_{|x|\le K}f_N((\eta^0)^{x,-1})\\
&\qquad\qquad+\frac{r}{2}\sum_{|x|\le K}f_N((\eta^0)^{x,1})-r\sum_{|x|\le K}f_N(\eta^0)\Bigg]
\end{aligned}
$$

$$= \frac{r}{r+s} \sum_{|x|\le K} \left[\frac{1}{2} f_N((\eta^0)^{x,-1}) + \frac{1}{2} f_N((\eta^0)^{x,1}) - f_N(\eta^0) \right] + O(N^{-1})$$

$$= \frac{r}{r+s} \psi_N(\Omega_{A'} f)(\eta^0) + O(N^{-1}), \tag{6.23}$$

where, in the second equality,

$$\sum_{|x_m|>K: m\in\{1,2,\ldots,r+s\}, m\neq i} \; \sum_{B\subset\{r+1,\ldots,r+s\}} \prod_{j\in B^c} [1 - c(x_j, (\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})] \times \prod_{j\in B} c(x_j, (\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})$$

$$= \sum_{|x_m|>K: m\in\{1,2,\ldots,r+s\}, m\neq i} \; \prod_{j=r+1}^{r+s} [1 - c(x_j, (\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r}) + c(x_j, (\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})]$$

$$= [N - (2K+1)]^{r+s-1},$$

and in the fourth equality,

$$\frac{1}{2^r} \sum_{A\subset\{1,\ldots,r\}} |A^c| = \frac{1}{2^r} \sum_{A\subset\{1,\ldots,r\}} |A| = \frac{1}{2^r} \sum_{i=0}^{r} i \binom{r}{i} = \frac{r}{2}.$$

Next, we consider the second term in (6.22). It contributes

$$\frac{1}{r+s} \frac{1}{2^r N^{r+s-1}} \sum_{A\subset\{1,\ldots,r\}} \sum_{B\subset\{r+1,\ldots,r+s\}} \sum_{i\in B} \sum_{|x_i|\le K} [f_N((\eta^0)_{x_i}) - f_N(\eta^0)] \times \sum_{|x_m|>K: m\in\{1,2,\ldots,r+s\}, m\neq i} \prod_{j\in B^c} [1 - c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l: l\in B, l<j\}})] \times \prod_{j\in B} c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l: l\in B, l<j\}})$$

$$= \frac{1}{r+s} \frac{1}{2^r N^{r+s-1}} \sum_{A\subset\{1,\ldots,r\}} \sum_{i=r+1}^{r+s} \sum_{|x_i|\le K} c(x_i, \eta^0)[f_N((\eta^0)_{x_i}) - f_N(\eta^0)] \times \sum_{|x_m|>K: m\in\{1,2,\ldots,r+s\}, m\neq i} \; \sum_{B\subset\{r+1,\ldots,r+s\}: i\in B} \prod_{j\in B^c} [1 - c(x_j, \eta^0)] \prod_{j\in B-\{i\}} c(x_j, \eta^0) + O(N^{-1})$$

$$= \frac{1}{r+s} \frac{1}{2^r N^{r+s-1}} \big[N - (2K+1)\big]^{r+s-1} \cdot \sum_{A\subset\{1,\ldots,r\}} \sum_{i=r+1}^{r+s} \sum_{|x_i|\le K} c(x_i, \eta^0)[f_N((\eta^0)_{x_i}) - f_N(\eta^0)] + O(N^{-1}) \tag{6.24}$$

$$= \frac{1}{r+s} \frac{1}{2^r N^{r+s-1}} \big[N-(2K+1)\big]^{r+s-1} 2^r \sum_{i=r+1}^{r+s} \sum_{|x_i|\le K} c(x_i,\eta^0)[f_N((\eta^0)_{x_i}) - f_N(\eta^0)]$$
$$+ O(N^{-1})$$
$$= \frac{1}{r+s} \frac{1}{N^{r+s-1}} \big[N-(2K+1)\big]^{r+s-1} s \sum_{|x|\le K} c(x,\eta^0)[f_N((\eta^0)_x) - f_N(\eta^0)] + O(N^{-1})$$
$$= \frac{s}{r+s} \sum_{|x|\le K} c(x,\eta^0)[f_N((\eta^0)_x) - f_N(\eta^0)] + O(N^{-1})$$
$$= \frac{s}{r+s} \psi_N(\Omega_B f)(\eta^0) + O(N^{-1}), \tag{6.25}$$

where the first and second equalities require clarification.

In the first equality we used

$$\sum_{B\subset\{r+1,\ldots,r+s\}} \sum_{i\in B} = \sum_{i=r+1}^{r+s} \sum_{B\subset\{r+1,\ldots,r+s\}:i\in B}$$

and

$$c(x_j,((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B,l<j\}}) = c(x_j,\eta^0)$$

with possible exceptions if

$$\{x_j-1,x_j,x_j+1\} \cap \Bigg[\bigcup_{p\in A^c}\{x_p,x_p-1\} \cup \bigcup_{p\in A}\{x_p,x_p+1\} \cup \bigcup_{p\in B}\{x_p\}\Bigg] \ne \varnothing.$$

That excludes at most $4r+3s$ of the N possible values of x_j, hence involves an error of at most $O(N^{-1})$. In the second equality,

$$\sum_{|x_m|>K:m\in\{1,2,\ldots,r+s\},m\ne i} \sum_{B\subset\{r+1,\ldots,r+s\}:i\in B} \prod_{j\in B^c} [1-c(x_j,\eta^0)] \prod_{j\in B-\{i\}} c(x_j,\eta^0)$$
$$= \sum_{|x_m|>K:m\in\{1,2,\ldots,r+s\},m\ne i} \prod_{j\in\{r+1,\ldots,r+s\}-\{i\}} [1-c(x_j,\eta^0)+c(x_j,\eta^0)]$$
$$= [N-(2K+1)]^{r+s-1}.$$

Therefore, we conclude that

$$(\Omega^N_{[r,s]'}\psi_N f)(\eta^0) = \psi_N\left(\frac{r}{r+s}\Omega_{A'} f + \frac{s}{r+s}\Omega_B f\right)(\eta^0) + O(N^{-1}), \tag{6.26}$$

as desired.

6.5 Convergence of means in periodic pattern

Since (6.26) holds, uniformly over Σ_N, the unique stationary distribution $\boldsymbol{\pi}^N$ of $\boldsymbol{P}_{A'}^r \boldsymbol{P}_B^s$ converges weakly to the unique stationary distribution $\pi^{r/(r+s)}$ of the interacting particle system with generator $\Omega_{C'}$, provided that ergodicity holds for the limiting interacting particle system. Here, $\Omega_{C'} = \gamma\Omega_{A'} + (1-\gamma)\Omega_B$ with $\gamma := r/(r+s)$, specifically

$$(\Omega_{C'} f)(\eta) = \gamma \sum_x c'(x,\eta)[f(\eta_x) - f(\eta)] \\ + \gamma \sum_x [f({}_x\eta_{x+1}) - f(\eta)] + (1-\gamma) \sum_x c(x,\eta)[f(\eta_x) - f(\eta)], \tag{6.27}$$

where η_x is η except at coordinate x, and

$${}_x\eta_{x+1}(y) := \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise,} \end{cases}$$

so ${}_x\eta_{x+1}$ interchanges the spins at adjacent sites x and $x+1$. In addition,

$$c'(x,\eta) := \frac{1}{2}[\mathbf{1}_{\{\eta(x)=\eta(x+1)\}} + \mathbf{1}_{\{\eta(x)=\eta(x-1)\}}] \quad \text{and} \quad c(x,\eta) := \begin{cases} p_{m_x(\eta)} & \text{if } \eta(x) = 0, \\ q_{m_x(\eta)} & \text{if } \eta(x) = 1, \end{cases}$$

where $q_m := 1 - p_m$ for $m = 0, 1, 2, 3$, and $m_x(\eta) := 2\eta(x-1) + \eta(x+1) \in \{0, 1, 2, 3\}$. Our aim is to prove the following theorem.

Theorem 6.3. *Fix integers $r, s \geq 1$ and put $\gamma := r/(r+s)$. Assume that the interacting particle system on $\mathbf{Z}$ with generator $\Omega_{C'}$ as in* (6.27) *is ergodic with unique stationary distribution π^γ. Then $\lim_{N\to\infty} \mu^N_{[r,s]'} = \mu_{(\gamma,1-\gamma)'}$, where $\mu_{(\gamma,1-\gamma)'}$ is as in* (6.12).

Proof. The mean profit per turn to the ensemble of N players playing the nonrandom periodic pattern $(A')^r B^s$ is

$$\mu^N_{[r,s]'} = \frac{1}{r+s} \sum_{v=0}^{s-1} \sum_{\eta\in\Sigma_N} (\boldsymbol{\pi}^N \boldsymbol{P}^r_{A'} \boldsymbol{P}^v_B)(\eta) \frac{1}{N} \sum_z [2p_{m_z(\eta)} - 1]. \tag{6.28}$$

The sum over η in (6.28) can be expressed, using (6.20), as

$$\begin{aligned} &\sum_{\eta^0,\eta} \pi^N(\eta^0) (\boldsymbol{P}^r_{A'} \boldsymbol{P}^v_B)(\eta^0,\eta) \frac{1}{N} \sum_z [2p_{m_z(\eta)} - 1] \\ &= \frac{1}{2^r N^{r+v}} \sum_{\eta^0} \pi^N(\eta^0) \sum_{A\subset\{1,\ldots,r\}} \sum_{B\subset\{r+1,\ldots,r+v\}} \sum_{x_i:i\in\{1,2,\ldots,r+v\}} \\ &\quad \times \prod_{j\in B^c} \big[1 - c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B, l<j\}})\big] \\ &\quad \times \prod_{j\in B} c(x_j, ((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B, l<j\}}) \\ &\quad \times \frac{1}{N} \sum_z [2p_{m_z(((\cdots((\cdots(((\eta^0)^{x_1,a_1})^{x_2,a_2})\cdots)^{x_p,a_p})\cdots)^{x_r,a_r})_{\{x_l:l\in B\}})} - 1] \end{aligned}$$

$$= \frac{1}{2^r N^{r+v}} \sum_{\eta^0} \pi^N(\eta^0) \sum_{A \subset \{1,\ldots,r\}} \sum_{B \subset \{r+1,\ldots,r+v\}} \sum_{x_i : i \in \{1,2,\ldots,r+v\}}$$

$$\times \prod_{j \in B^c} [1 - c(x_j, \eta^0)] \prod_{j \in B} c(x_j, \eta^0) \frac{1}{N} \sum_z [2p_{m_z(\eta^0)} - 1] + O(N^{-1})$$

$$= \frac{1}{2^r N^{r+v}} \sum_{\eta^0} \pi^N(\eta^0) \ (2^r)(1)[N - (2k+1)]^{r+v} \frac{1}{N} \sum_z [2p_{m_z(\eta^0)} - 1] + O(N^{-1})$$

$$= \frac{1}{N} \sum_{z=1}^{N} \sum_{\eta^0} \pi^N(\eta^0)[p_{m_z(\eta^0)} - q_{m_z(\eta^0)}] + O(N^{-1})$$

$$= \sum_{k=0}^{1} \sum_{l=0}^{1} (\pi^N)_{-1,1}(k,l)[2p_{2k+l} - 1] + O(N^{-1})$$

$$= \sum_{k=0}^{1} \sum_{l=0}^{1} (\pi^{r/(r+s)})_{-1,1}(k,l)[2p_{2k+l} - 1] + o(1).$$

So we have, with $\gamma := r/(r+s)$,

$$\mu^N_{[r,s]'} \to (1-\gamma) \sum_{k=0}^{1} \sum_{l=0}^{1} (\pi^\gamma)_{-1,1}(k,l)[2p_{2k+l} - 1] = \mu_{(\gamma,1-\gamma)'},$$

as required.

APPENDIX A

μ_B AND $\mu_{C'}$ FROM *MATHEMATICA*

This code calculates μ_B and $\mu_{C'}$ for $C' := \frac{1}{2}A' + \frac{1}{2}B$ assuming $p_0 = 0, p_1 = p_2 = 0.8$, and $p_3 = 0.5$ with $3 \leq N \leq 10$.

```
DateList[]
ClearAll[``Global `*'']
For[n = 3, n ≤ 10, n++, Print[n]; p0=.; p1=.; p2=.; p3=.; q0=.; q1=.; q2=.; q3=.;
group = PermutationGroup[DihedralGroup[n]//GroupGenerators]; (*dihedral group*)
sigma = IntegerDigits[Range[0, 2^n − 1], 2, n];
Do[permuted[i] = Permute[sigma[[i]], group], {i, 1, 2^n}];
(*orbitofithelementofSigmaunderG, withduplication*)
Do[digit[x_]:=FromDigits[permuted[i][[x]], 2]; list[i] = Table[digit[x], {x, 1, 2n}],
{i, 1, 2^n}]; (*setoforbitelementsindecimalform, withduplication*)
class[1] = {0}; (*the first equivalence class*)
num = 1;
For[j = 2, j ≤ 2^(n − 1), j++,
For[test = 1; k = 1, k ≤ j − 1, k++, If[Sort[list[k]] == Sort[list[j]], test = 0]];
If[test == 1, num = num + 1; class[num] = DeleteDuplicates[Sort[list[j]]]]]];
(*generates list of equivalence classes*)
num = num + 1; (*number of equivalence classes*)
class[num] = {2^n − 1}; (*the last equivalence class*)
For[i = 1, i ≤ num, i++, state[i] = IntegerDigits[class[i], 2, n]];
(*the binary states belonging to equivalence class i*)
For[i = 1, i ≤ num, i++,
ones[i] = (sum = 0; For[k = 1, k ≤ n, k++, If[state[i][[1, k]] == 1, sum+=1]]; sum)];
(*number of ones in each element of equivalence class i*)
```

```
diff[x_, y_]:=(sum = 0; For[k = 1, k ≤ n, k++, sum+=Abs[x[[k]] − y[[k]]]]; sum);
(*the Hamming distance between states x and y*)
p[0] = p0; p[1] = p1; p[2] = p2; p[3] = p3; q[0] = q0; q[1] = q1; q[2] = q2; q[3] = q3;
pBbar = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++,
For[j = 1, j ≤ num, j++,
If[i == j, For[k = 1, k ≤ n, k++,
If[state[i][[1, k]] == 1,
If[k == 1, pBbar[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]],
If[k == 1, pBbar[[i, j]]+=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]],
If[ones[j] == ones[i] + 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] > state[i][[1, k]],
If[k == 1, pBbar[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]],
If[ones[j] == ones[i] − 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] < state[i][[1, k]],
If[k == 1, pBbar[[i, j]]+=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]]]]]]
pBbardot = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++,
For[j = 1, j ≤ num, j++,
If[i == j, For[k = 1, k ≤ n, k++,
If[state[i][[1, k]] == 1,
```

```
If[k == 1, pBbardot[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]],
If[k == 1, pBbardot[[i, j]]-=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]],
If[ones[j] == ones[i] + 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] > state[i][[1, k]],
If[k == 1, pBbardot[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]],
If[ones[j] == ones[i] − 1,
For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] < state[i][[1, k]],
If[k == 1, pBbardot[[i, j]]-=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]]]]]]
one = ConstantArray[1, {num, 1}];
pBbardotone = pBbardot.one;
pAbar = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++,
For[j = 1, j ≤ num, j++, If[i == 1, pAbar[[i, i + 1]] = 1,
If[i == num, pAbar[[i, i − 1]] = 1,
If[ones[j] == ones[i] + 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++,
If[k < n, If[state[i][[1, k]] + state[i][[1, k + 1]] == 0&&
state[j][[m, k]] + state[j][[m, k + 1]] == 1, pAbar[[i, j]]+=1/(2n)],
If[state[i][[1, k]] + state[i][[1, 1]] == 0&&
```

```
state[j][[m,k]] + state[j][[m,1]] == 1, pAbar[[i,j]]+=1/(2n)]]]]]];
If[ones[j] == ones[i] - 1, For[m = 1, m <= Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k <= n, k++,
If[k < n, If[state[i][[1,k]] + state[i][[1,k+1]] == 2&&
state[j][[m,k]] + state[j][[m,k+1]] == 1, pAbar[[i,j]]+=1/(2n)],
If[state[i][[1,k]] + state[i][[1,1]] == 2&&
state[j][[m,k]] + state[j][[m,1]] == 1, pAbar[[i,j]]+=1/(2n)]]]]]];
If[ones[j] == ones[i], For[m = 1, m <= Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 0,
For[k = 1, k <= n, k++,
If[k < n, If[state[i][[1,k]] + state[i][[1,k+1]] == 1&&
state[j][[m,k]] + state[j][[m,k+1]] == 1, pAbar[[i,j]]+=1/(2n)],
If[state[i][[1,k]] + state[i][[1,1]] == 1&&
state[j][[m,k]] + state[j][[m,1]] == 1, pAbar[[i,j]]+=1/(2n)]]]];
If[diff[state[j][[m]], state[i][[1]]] == 2,
For[k = 1, k <= n, k++,
If[k < n,
If[(state[i][[1,k]] == 0&&state[i][[1,k+1]] == 1&&state[j][[m,k]] ==
1&&state[j][[m,k+1]] == 0)||
(state[i][[1,k]] == 1&&state[i][[1,k+1]] == 0&&state[j][[m,k]] ==
0&&state[j][[m,k+1]] == 1), pAbar[[i,j]]+=1/(2n)],
If[(state[i][[1,k]] == 0&&state[i][[1,1]] == 1&&state[j][[m,k]] ==
1&&state[j][[m,1]] == 0)||
(state[i][[1,k]] == 1&&state[i][[1,1]] == 0&&state[j][[m,k]] ==
0&&state[j][[m,1]] == 1), pAbar[[i,j]]+=1/(2n)]]]]]]]]]];
p0 = 0; p1 = 0.8; p2 = p1; p3 = 0.5;
q0 = 1 - p0; q1 = 1 - p1; q2 = 1 - p2; q3 = 1 - p3;
pi = Array[x, {num}];
solB = Solve[{pi == pi.pBbar, pi.one == 1}, pi];
Print[''muB='', N[pi.pBbardotone/.solB]];
```

```
pCbar = (1/2)pAbar + (1/2)pBbar;
pCbardotone = (1/2)pBbardotone;
solC = Solve[{pi == pi.pCbar, pi.one == 1}, pi];
Print[``muC'='', N[pi.pCbardotone/.solC]]]
DateList[]
```

APPENDIX B

GRAPH OF PARRONDO REGION FROM *MATHEMATICA*

This code sketches graph of Parrondo region assuming $p_1 = p_2$ with $N = 3, 4$.

```
ClearAll[‘‘Global `*’’]
For[n = 3, n ≤ 4, n++, Print[n];
Print[DateList[]]; p0=.; p1=.; p2=.; p3=.; q0=.; q1=.; q2=.; q3=.;
group = PermutationGroup[DihedralGroup[n]//GroupGenerators]; (*dihedral group*)
sigma = IntegerDigits[Range[0, 2^n − 1], 2, n];
Do[permuted[i] = Permute[sigma[[i]], group], {i, 1, 2^n}];
(*orbitofithelementofSigmaunderG, withduplication*)
Do[digit[x_]:=FromDigits[permuted[i][[x]], 2];
list[i] = Table[digit[x], {x, 1, 2n}], {i, 1, 2^n}];
(*setoforbitelementsindecimalform, withduplication*)
class[1] = {0}; (*the first equivalence class*)
num = 1;
For[j = 2, j ≤ 2^(n − 1), j++, For[test = 1; k = 1, k ≤ j − 1, k++,
If[Sort[list[k]] == Sort[list[j]], test = 0]];
If[test == 1, num = num + 1; class[num] = DeleteDuplicates[Sort[list[j]]]]]];
(*generates list of equivalence classes*)
num = num + 1; (*number of equivalence classes*)
class[num] = {2^n − 1}; (*the last equivalence class*)
For[i = 1, i ≤ num, i++, state[i] = IntegerDigits[class[i], 2, n]];
(*the binary states belonging to equivalence class i*)
For[i = 1, i ≤ num, i++, ones[i] = (sum = 0;
```

```
For[k = 1, k ≤ n, k++, If[state[i][[1, k]] == 1, sum+=1]]; sum)];
(*number of ones in each element of equivalence class i*)
diff[x_, y_]:=(sum = 0; For[k = 1, k ≤ n, k++, sum+=Abs[x[[k]] − y[[k]]]]; sum);
(*the Hamming distance between states x and y*)
p[0] = p0; p[1] = p1; p[2] = p2; p[3] = p3; q[0] = q0; q[1] = q1; q[2] = q2; q[3] = q3;
pBbar = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++, For[j = 1, j ≤ num, j++, If[i == j,
For[k = 1, k ≤ n, k++, If[state[i][[1, k]] == 1,
If[k == 1, pBbar[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]],
If[k == 1, pBbar[[i, j]]+=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]],
If[ones[j] == ones[i] + 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] > state[i][[1, k]],
If[k == 1, pBbar[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]],
If[ones[j] == ones[i] − 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] < state[i][[1, k]],
If[k == 1, pBbar[[i, j]]+=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbar[[i, j]]+=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]]]]]]
pBbardot = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++, For[j = 1, j ≤ num, j++,
If[i == j, For[k = 1, k ≤ n, k++, If[state[i][[1, k]] == 1,
If[k == 1, pBbardot[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
```

```
pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]],
If[k == 1, pBbardot[[i, j]]-=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]],
If[ones[j] == ones[i] + 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1, For[k = 1, k ≤ n, k++,
If[state[j][[m, k]] > state[i][[1, k]],
If[k == 1, pBbardot[[i, j]]+=p[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]+=p[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]],
If[ones[j] == ones[i] − 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[state[j][[m, k]] < state[i][[1, k]],
If[k == 1, pBbardot[[i, j]]-=q[2state[i][[1, n]] + state[i][[1, k + 1]]]/n,
If[k == n, pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, 1]]]/n,
pBbardot[[i, j]]-=q[2state[i][[1, k − 1]] + state[i][[1, k + 1]]]/n]]]]]]]]]]
one = ConstantArray[1, {num, 1}];
pBbardotone = pBbardot.one;
pAbar = ConstantArray[0, {num, num}];
For[i = 1, i ≤ num, i++, For[j = 1, j ≤ num, j++,
If[i == 1, pAbar[[i, i + 1]] = 1, If[i == num, pAbar[[i, i − 1]] = 1, If[ones[j] == ones[i] + 1,
For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[k < n,
If[state[i][[1, k]] + state[i][[1, k + 1]] == 0
&&state[j][[m, k]] + state[j][[m, k + 1]] == 1, pAbar[[i, j]]+=1/(2n)],
If[state[i][[1, k]] + state[i][[1, 1]] == 0
&&state[j][[m, k]] + state[j][[m, 1]] == 1, pAbar[[i, j]]+=1/(2n)]]]]]];
If[ones[j] == ones[i] − 1, For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 1,
For[k = 1, k ≤ n, k++, If[k < n,
```

```
If[state[i][[1, k]] + state[i][[1, k + 1]] == 2
&&state[j][[m, k]] + state[j][[m, k + 1]] == 1, pAbar[[i, j]]+=1/(2n)],
If[state[i][[1, k]] + state[i][[1, 1]] == 2
&&state[j][[m, k]] + state[j][[m, 1]] == 1, pAbar[[i, j]]+=1/(2n)]]]]]];
If[ones[j] == ones[i],
For[m = 1, m ≤ Length[class[j]], m++,
If[diff[state[j][[m]], state[i][[1]]] == 0,
For[k = 1, k ≤ n, k++,
If[k < n, If[state[i][[1, k]] + state[i][[1, k + 1]] == 1
&&state[j][[m, k]] + state[j][[m, k + 1]] == 1, pAbar[[i, j]]+=1/(2n)],
If[state[i][[1, k]] + state[i][[1, 1]] == 1
&&state[j][[m, k]] + state[j][[m, 1]] == 1, pAbar[[i, j]]+=1/(2n)]]]];
If[diff[state[j][[m]], state[i][[1]]] == 2, For[k = 1, k ≤ n, k++,
If[k < n,
If[(state[i][[1, k]] == 0&&state[i][[1, k + 1]] == 1
&&state[j][[m, k]] == 1&&state[j][[m, k + 1]] == 0)
||(state[i][[1, k]] == 1&&state[i][[1, k + 1]] == 0
&&state[j][[m, k]] == 0&&state[j][[m, k + 1]] == 1), pAbar[[i, j]]+=1/(2n)],
If[(state[i][[1, k]] == 0&&state[i][[1, 1]] == 1
&&state[j][[m, k]] == 1&&state[j][[m, 1]] == 0)
||(state[i][[1, k]] == 1&&state[i][[1, 1]] == 0&&state[j][[m, k]] == 0
&&state[j][[m, 1]] == 1), pAbar[[i, j]]+=1/(2n)]]]]]]]]]];
p2 = p1;
q0 = 1 − p0; q1 = 1 − p1; q2 = 1 − p2; q3 = 1 − p3;
pi = Array[x, {num}];
solB = Solve[{pi == pi.pBbar, pi.one == 1}, pi];
muB:=N[pi.pBbardotone/.solB];
pCbar = pAbar.(pBbar);
pCbardot = pAbar.(pBbardot);
pCbardotone = pCbardot.one;
solC = Solve[{pi == pi.pCbar, pi.one == 1}, pi];
```

```
muC':=N[((0.5) *  pi.pCbardotone)/.solC];
MatrixForm[pBbar];
MatrixForm[pBbardot];
MatrixForm[pBbardotone];
MatrixForm[pAbar];
Print[ContourPlot3D[{muB == 0, muC' == 0}, {p0, 0, 1}, {p1, 0, 1}, {p3, 0, 1},
AxesLabel →   {Style ["p_0  ", Italic, 16] , Style ["p_1", Italic, 16] ,
Style ["  p_3", Italic, 16]} , LabelStyle → 12,
ContourStyle → {RGBColor[135/255, 206/255, 235/255], Red},
ViewPoint → {3.3, −1.6, 1.7}]]
Print[DateList[]]
Print[‘‘************************************************************’’]]
```

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