THE GEOMETRY OF Out(F_N) THROUGH COMPLETELY SPLIT TRAIN TRACKS

by

Derrick Wigglesworth

A dissertation submitted to the faculty of
The University of Utah
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

The University of Utah

May 2018

The University of Utah Graduate School

STATEMENT OF DISSERTATION APPROVAL

The dissertation of **Derrick Wigglesworth**
has been approved by the following supervisory committee members:

Mladen Bestvina ,	Chair(s)	**21 Feb 2018** Date Approved
Kenneth Bromberg ,	Member	**21 Feb 2018** Date Approved
Jonathan M. Chaika ,	Member	**21 Feb 2018** Date Approved
Mark Feighn ,	Member	**20 Feb 2018** Date Approved
Kevin Wortman ,	Member	**21 Feb 2018** Date Approved

by **Davar Khoshnevisan** , Chair/Dean of
the Department/College/School of **Mathematics**
and by **David B. Kieda** , Dean of The Graduate School.

ABSTRACT

We prove that abelian subgroups of the outer automorphism group of a free group are quasi-isometrically embedded. Our proof uses recent developments in the theory of train track maps by Feighn-Handel. As an application, we prove the rank conjecture for $\mathrm{Out}(F_n)$.

Then, in joint work with Radhika Gupta, we show that an outer automorphism acts loxodromically on the cyclic splitting complex if and only if it has a filling lamination and no generic leaf of the lamination is carried by a vertex group of a cyclic splitting. This is a direct analog for the cyclic splitting complex of Handel and Mosher's theorem on loxodromics for the free splitting complex.

As a step towards proving that all of the loxodromics for this complex are WPD elements, we show that such outer automorphisms have virtually cyclic centralizers.

For my father, Doc

CONTENTS

LIST OF FIGURES

NOTATION AND SYMBOLS

$\mathbb{P}\mathcal{O}_n$	Culler-Vogtmann's projectivized outer space
$\mathcal{O}_n$	unprojectivized outer space
$\overline{\mathbb{P}\mathcal{O}_n}$	The closure of outer space
$\partial\mathbb{P}\mathcal{O}_n$	The boundary of outer space
$\mathcal{R}_n$	The rose with n petals
$\partial^2 F_n$	The double boundary $((\partial F_n \times \partial F_n - \Delta)/\mathbb{Z}_2)$
$\mathcal{B}$	The space of lines
$\mathcal{B}(G)$	The space of line in G
$\mathcal{L}(\phi)$	The set of attracting laminations for ϕ
Λ	An element of $\mathcal{L}(\phi)$
PF_Λ	The expansion factor homomorphism associated to a lamination Λ
$\tau(\phi)$	the infimal translation distance for ϕ acting on $\mathbb{P}\mathcal{O}_n$
PF_H	the product of PF_Λ for all attracting laminations for the subgroup H
ω	A comparison homomorphism for ϕ
Ω	The product of PF_H and the comparison homomorphisms for H
$\mathcal{FS}_n$	The free splitting complex
$\mathcal{FZ}_n$	The cyclic splitting complex
$\mathcal{FF}_n$	The free factor complex

ACKNOWLEDGEMENTS

I would like to thank my advisor Mladen Bestvina for many hours of his time and for his patience. I would also like to thank Mark Feighn for his encouragement and support. I would like to express my gratitude to Radhika Gupta for countless hours of stimulating conversations about math. Finally, I would like to thank MSRI for its hospitality and partial support. While writing this work, the author was partially supported by the U.S. National Science Foundation grant of Mladen Bestvina (DMS-1607236) and also acknowledges support from U.S. National Science Foundation grants DMS-1246989 and DMS-0932078.

Thank you also to Mr. Ish for always believing in me, to my mother for always supporting me, and to my wife for being awesome.

CHAPTER 1

DISTORTION FOR ABELIAN SUBGROUPS OF OUT(F_N)

In this chapter, we prove that abelian subgroups of $\mathrm{Out}(F_n)$ are undistorted. We begin with an introduction to provide some context to this result and an outline of what follows.

1.1 Introduction

Given a finitely generated group G, a finitely generated subgroup H is *undistorted* if the inclusion $H \hookrightarrow G$ is a quasi-isometric embedding with respect to the word metrics on G and H for some (any) finite generating sets. A standard technique for showing that a subgroup is undistorted involves finding a space on which G acts nicely and constructing a height function on this space satisfying certain properties: elements which are large in the word metric on H should change the height function by a lot, elements of a fixed generating set for G should change the function by a uniformly bounded amount. In this chapter, we use a couple of variations of this method.

Let R_n be the wedge of n circles and let F_n be its fundamental group, the free group of rank $n \geq 2$. The outer automorphism group of the free group, $\mathrm{Out}(F_n)$, is defined as the quotient of $\mathrm{Aut}(F_n)$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(F_n)$ draws parallels with the study of mapping class groups. Furthermore, many theorems concerning $\mathrm{Out}(F_n)$ and their proofs are inspired by analogous theorems and proofs in the context of mapping class groups. Both groups satisfy the Tits alternative (see [McC85] and [BFH00]), both have finite virtual cohomological dimension (see [Har86] and [CV86]), and both have Serre's property FA to name a few. Importantly, this approach to the study of $\mathrm{Out}(F_n)$ has yielded a classification of its elements in analogy with the Nielsen-Thurston classification of elements of the mapping class group [BH92], along with constructive ways for finding good representatives of these elements [FH14].

In [FLM01], the authors proved that infinite cyclic subgroups of the mapping class group are undistorted. Their proof also implies that higher rank abelian subgroups are undistorted. In [Ali02], Alibegović proved that infinite cyclic subgroups of $\mathrm{Out}(F_n)$ are undistorted. In contrast with the mapping class group setting, Alibegović's proof does not directly apply to higher rank subgroups: the question of whether all abelian subgroups of $\mathrm{Out}(F_n)$ are undistorted has been left open. In this chapter, we answer this in the affirmative.

Theorem 1.25. *Abelian subgroups of $Out(F_n)$ are undistorted.*

We remark that this theorem was proved for some specific instances of rank 2 abelian subgroups in [BDtM09, Lemma 9.1]. This theorem has implications for various open problems in the study of $\mathrm{Out}(F_n)$. In [BM08], Behrstock and Minsky prove that the geometric rank of the mapping class group is equal to the maximal rank of an abelian subgroup of the mapping class group. As an application of Theorem 1.25, we prove the analogous result in the $\mathrm{Out}(F_n)$ setting.

Corollary 1.27. *The geometric rank of $Out(F_n)$ is $2n-3$, which is the maximal rank of an abelian subgroup of $Out(F_n)$.*

We remark that in principle, this could have been done earlier by using the techniques in [Ali02] to show that a specific maximal rank abelian subgroup is undistorted.

In the course of proving Theorem 1.25, we show that, up to finite index, only finitely many marked graphs are needed to get good representatives of every element of an abelian subgroup of $\mathrm{Out}(F_n)$. In the setting of mapping class groups, the analogous statement is that for a surface S and an abelian subgroup H of $\mathrm{MCG}(S)$ there is a Thurston decomposition of S into disjoint subsurfaces which is respected by every element of H. This can also be viewed as a version of the Kolchin Theorem of [BFH05] for abelian subgroups. We prove:

Proposition 1.12. *For any abelian subgroup H of $Out(F_n)$, there exists a finite index subgroup H' such that every $\phi \in H'$ can be realized as a CT on one of finitely many marked graphs.*

The chapter is outlined as follows. In section 1.3 we prove that the translation distance of an arbitrary element ϕ of $\mathrm{Out}(F_n)$ acting on outer space is the maximum of the logarithm of the expansion factors associated to the exponentially growing strata in a relative train track map for ϕ. This result was obtained previously and independently by Richard Wade in his thesis [Wad12]. This is the analog for $\mathrm{Out}(F_n)$ of Bers' result [Ber78] that the translation distance of a mapping class f acting on Teichmüller space endowed with the Teichmüller metric is the maximum of the logarithms of the dilatation constants for the pseudo-Anosov components in the Thurston decomposition of f.

In section 1.4 we then use our result on translation distance to prove the main theorem in the special case where the abelian subgroup H has "enough" exponential data. More precisely, we will prove the result under the assumption that the collection of expansion factor homomorphisms determines an injective map $H \to \mathbb{Z}^N$.

In section 1.5 we prove Proposition 1.12 and then use this in section 1.6 to prove the main result in the case that H has "enough" polynomial data. This is the most technical part of the paper because we need to obtain significantly more control over the types of subpaths that can occur in nice circuits in a marked graph than was previously available. The bulk of the work goes towards proving Proposition 1.13. This result provides a connection between the comparison homomorphisms introduced in [FH09] (which are only defined on subgroups of $\mathrm{Out}(F_n)$) and Alibegović's twisting function. We then use this connection to complete the proof of our main result in the polynomial case.

This chapter concludes with section 1.7 where we consolidate results from previous sections to prove Theorem 1.25. The methods used in sections 1.4 and 1.6 can be carried out with minimal modification in the general setting.

1.2 Preliminaries

This section is meant to review background material that we will rely upon in the sequel. The reader may feel comfortable skimming or skipping this section.

1.2.1 Outer space

Culler and Vogtmann's *outer space*, $\mathbb{P}\mathcal{O}_n$, is defined in [CV86] as the space of simplicial, free, and minimal isometric actions of F_n on simplicial metric trees up to F_n-equivariant homothety. We denote by $\mathcal{O}_n$ the *unprojectivized outer space*, in which the trees are considered up to isometry, rather than homothety. Each of these spaces is equipped with a natural (right) action of $\mathrm{Out}(F_n)$.

Outer space has a (nonsymmetric) metric defined in analogy with the Teichmüller metric on Teichmüller space. The distance from T to T' is defined as the logarithm of the infimal Lipschitz constant among all F_n-equivariant maps $f\colon T \to T'$.

1.2.2 Marked graphs

We recall some basic definitions from [BH92]. Identify F_n with $\pi_1(\mathcal{R}_n, *)$ where $\mathcal{R}_n$ is a rose with n petals. A *marked graph* G is a graph of rank n, all of whose vertices have valence at least two, equipped with a homotopy equivalence $m\colon \mathcal{R} \to G$ called a *marking*. The marking determines an identification of F_n with $\pi_1(G, m(*))$. A homotopy equivalence $f\colon G \to G$ induces an outer automorphism of $\pi_1(G)$ and hence an element ϕ of $\mathrm{Out}(F_n)$. If f sends vertices to vertices and the restriction of f to edges is an immersion then we say that f is a *topological representative* of ϕ. All homotopy equivalences will be assumed to map vertices to vertices and the restriction to any edge will be assumed to be an immersion.

1.2.3 Paths, circuits, and tightening

Let Γ be either a marked graph or an F_n-tree. A *path* in Γ is either an isometric immersion of a (possibly infinite) closed interval $\sigma\colon I \to \Gamma$ or a constant map $\sigma\colon I \to \Gamma$. If σ is a constant map, the path will be called *trivial*. If I is finite, then any map $\sigma\colon I \to \Gamma$ is homotopic rel endpoints to a unique path $[\sigma]$. We say that $[\sigma]$ is obtained by *tightening* σ. If $f\colon \Gamma \to \Gamma$ is continuous and σ is a path in Γ, we define $f_\#(\sigma)$ as $[f(\sigma)]$. If the domain of σ is finite and Γ is either a graph or a simplicial tree, then the image has a natural decomposition into edges $E_1E_2\cdots E_k$ called the *edge path associated to* σ. If Γ is a tree, we may use $[x, x']$ to denote the unique geodesic path connecting x and x'.

A *circuit* is an immersion $\sigma\colon S^1 \to \Gamma$. For any path or circuit, let $\overline{\sigma}$ be σ with its orientation reversed. A decomposition of a path or circuit into subpaths is a *splitting* for $f\colon \Gamma \to \Gamma$ and is denoted $\sigma = \ldots \sigma_1 \cdot \sigma_2 \ldots$ if $f^k_\#(\sigma) = \ldots f^k_\#(\sigma_1) f^k_\#(\sigma_2) \ldots$ for all $k \geq 1$.

1.2.4 Turns, directions, and train track structures

Let Γ be a F_n-tree. A direction d based at $p \in \Gamma$ is a component of $\Gamma - \{p\}$. A *turn* is an unordered pair of directions based at the same point. If Γ is a graph, then a direction based at $p \in \Gamma$ is an F_n-orbit of directions in its universal cover based at lifts of p. In the case that Γ is a graph or a simplicial tree, and p is a vertex, we may identify directions at p with edges emanating from p and we will do this frequently in the sequel. A *train track structure* on Γ is an equivalence relation on the set of directions at each point $p \in \Gamma$. The classes of this relation are called *gates*. A turn (d, d') is *legal* if d and d' do not belong to the same gate. A path is legal if it only crosses legal turns.

A train track structure on a graph is defined by passing to its universal cover. If G is a graph and $f\colon G \to G$ is a homotopy equivalence, then f induces a train track structure on G as follows. The map f determines a map Df on the directions in G by definining $Df(E)$ to be the first edge in the edge path $f(E)$. We then declare $E_1 \sim E_2$ if $D(f^k)(E_1) = D(f^k)(E_2)$ for some $k \geq 1$.

1.2.5 Relative train track maps and CTs

A *filtration* for a topological representative $f\colon G \to G$ of an outer automorphism ϕ, where G is a marked graph, is an increasing sequence of f-invariant subgraphs $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$. We let $H_i = \overline{G_i \setminus G_{i-1}}$ and call H_i the *i-th stratum*. A turn with one edge in H_i and the other in G_{i-1} is called *mixed* while a turn with both edges in H_i is called a *turn in* H_i. If $\sigma \subset G_i$ does not contain any illegal turns in H_i, then we say σ is *i-legal*.

We denote by M_i the submatrix of the transition matrix for f obtained by deleting all rows and columns except those labeled by edges in H_i. For the topological representatives that will be of interest to us, the transition matrices M_i will come in three flavors: M_i may be a zero matrix, it may be the 1×1 identity matrix, or it may be an irreducible matrix with Perron-Frobenius eigenvalue $\lambda_i > 1$. We will call H_i a *zero* (Z), *nonexponentially growing* (NEG), or *exponentially growing* (EG) stratum, respectively. Any stratum which is not a zero stratum is called an *irreducible stratum*.

Definition 1.1 ([BH92])**.** *We say that $f\colon G \to G$ is a* relative train track map *representing $\phi \in Out(F_n)$ if for every exponentially growing stratum H_r, the following hold:*

(RTT-i) *Df maps the set of oriented edges in H_r to itself; in particular all mixed turns are legal.*

(RTT-ii) *If $\sigma \subset G_{r-1}$ is a nontrivial path with endpoints in $H_r \cap G_{r-1}$, then so is $f_\#(\sigma)$.*

(RTT-iii) *If $\sigma \subset G_r$ is r-legal, then $f_\#(\sigma)$ is r-legal.*

Suppose that $u < r$, that H_u is irreducible, H_r is EG and each component of G_r is noncontractible, and that for each $u < i < r$, H_i is a zero stratum which is a component of G_{r-1} and each vertex of H_i has valence at least two in G_r. Then we say that H_i is *enveloped by* H_r and we define $H_r^z = \bigcup_{k=u+1}^{r} H_k$.

A path or circuit σ in a representative $f\colon G \to G$ is called a *periodic Nielsen path* if $f_\#^k(\sigma) = \sigma$ for some $k \geq 1$. If $k = 1$, then σ is a *Nielsen path*. A Nielsen path is *indivisible* if it cannot be written as a concatenation of nontrivial Nielsen paths. If w is a closed root-free Nielsen path and E_i is an edge such that $f(E_i) = E_i w^{d_i}$, then we say E *is a linear edge* and we call w the *axis* of E. If E_i, E_j are distinct linear edges with the same axis such that $d_i \neq d_j$ and $d_i, d_j > 0$, then we call a path of the form $E_i w^* \overline{E}_j$ an *exceptional path*. In the same scenario, if d_i and d_j have different signs, we call such a path a *quasi-exceptional path*. We say that x and y are *Nielsen equivalent* if there is a Nielsen path σ in G whose endpoints are x and y. We say that a periodic point $x \in G$ is *principal* if neither of the following conditions hold:

- x is not an endpoint of a nontrivial periodic Nielsen path and there are exactly two periodic directions at x, both of which are contained in the same EG stratum.
- x is contained in a component C of periodic points that is topologically a circle and each point in C has exactly two periodic directions.

A relative train track map f is called *rotationless* if each principal periodic vertex is fixed and if each periodic direction based at a principal vertex is fixed. We remark that there is a closely related notion, whose definition we will omit, of an outer automorphism ϕ being rotationless. We will simply rely on the following fact from [FH09], which (combined with the definition of a CT) provides a connection between these two notions:

Theorem 1.2 ([FH09, Corollary 3.5])**.** *There exists $k > 0$ depending only on n, so that ϕ^k is rotationless for every $\phi \in Out(F_n)$.*

Theorem 1.3 ([FH09, Corollary 3.14])**.** *For each abelian subgroup A of $Out(F_n)$, the set of rotationless elements in A is a subgroup of finite index in A.*

For an EG stratum, H_r, we call a nontrivial path $\sigma \subset G_{r-1}$ with endpoints in $H_r \cap G_{r-1}$ a *connecting path for* H_r. Let E be an edge in an irreducible stratum, H_r and let σ be a maximal subpath of $f_\#^k(E)$ in a zero stratum for some $k \geq 1$. Then we say that σ is *taken*. A nontrivial path or circuit σ is called *completely split* if it has a splitting $\sigma = \tau_1 \cdot \tau_2 \cdots \tau_k$ where each of the τ_i's is a single edge in an irreducible stratum, an indivisible Nielsen path, an exceptional path, or a connecting path in a zero stratum which is both maximal and taken. We say that a relative train track map is *completely split* if $f(E)$ is completely split for every edge E in an irreducible stratum *and* if for every taken connecting path σ in a zero stratum, $f_\#(\sigma)$ is completely split.

Definition 1.4 ([FH11]). *A relative train track map $f\colon G \to G$ and filtration $\mathcal{F}$ given by $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$ is said to be a CT if it satisfies the following properties.*

(Rotationless) *$f\colon G \to G$ is rotationless.*

(Completely split) *$f\colon G \to G$ is completely split.*

(Filtration) *$\mathcal{F}$ is reduced. The core of each filtration element is a filtration element.*

(Vertices) *The endpoints of all indivisible periodic (necessarily fixed) Nielsen paths are (necessarily principal) vertices. The terminal endpoint of each nonfixed NEG edge is principal (and hence fixed).*

(Periodic edges) *Each periodic edge is fixed and each endpoint of a fixed edge is principal. If the unique edge E_r in a fixed stratum H_r is not a loop then G_{r-1} is a core graph and both ends of E_r are contained in G_{r-1}.*

(Zero strata) *If H_i is a zero stratum, then H_i is enveloped by an EG stratum H_r, each edge in H_i is r-taken and each vertex in H_i is contained in H_r and has link contained in $H_i \cup H_r$.*

(Linear edges) *For each linear E_i there is a closed root-free Nielsen path w_i such that $f(E_i) = E_i w_i^{d_i}$ for some $d_i \neq 0$. If E_i and E_j are distinct linear edges with the same axes then $w_i = w_j$ and $d_i \neq d_j$.*

(NEG Nielsen paths) *If the highest edges in an indivisible Nielsen path σ belong to an NEG stratum then there is a linear edge E_i with w_i as in (Linear Edges) and there exists $k \neq 0$ such that $\sigma = E_i w_i^k \bar{E}_i$.*

(EG Nielsen paths) *If H_r is EG and ρ is an indivisible Nielsen path of height r, then $f|G_r = \theta \circ f_{r-1} \circ f_r$ where :*

1. *$f_r : G_r \to G^1$ is a composition of proper extended folds defined by iteratively folding ρ.*

2. *$f_{r-1} : G^1 \to G^2$ is a composition of folds involving edges in G_{r-1}.*

3. *$\theta : G^2 \to G_r$ is a homeomorphism.*

We remark that several of the properties in Definition 1.4 use terms that have not been defined. We will not use these properties in the sequel. The main result for CTs is the following existence theorem:

Theorem 1.5 ([FH11, Theorem 4.28])**.** *Every rotationless $\phi \in Out(F_n)$ is represented by a CT $f\colon G \to G$.*

For completely split paths and circuits, all cancellation under iteration of $f_\#$ is confined to the individual terms of the splitting. Moreover, $f_\#(\sigma)$ has a complete splitting which refines that of σ. Finally, just as with improved relative train track maps introduced in [BFH00], every circuit or path with endpoints at vertices eventually is completely split.

1.2.6 Axes for conjugacy classes

Let Γ be the universal cover of the marked graph G. Each nontrivial $c \in F_n$ acts by a *covering translation* $T_c\colon \Gamma \to \Gamma$ which is a hyperbolic isometry, and therefore has an *axis* which we denote by A_c. The projection of A_c to G is the circuit corresponding to the conjugacy class c. If E is a linear edge in a CT so that $f(E) = Ew^d$ as in (Linear edges), then we say w is the axis of E.

1.2.7 Lines and laminations

We briefly recall some definitions, but the reader is directed to [BFH00] for details. The *space of abstract lines*, $\widetilde{\mathcal{B}} = (\partial F_n \times \partial F_n - \Delta)/\mathbb{Z}_2$ is the set of unordered distinct pairs of points in the boundary of F_n; $\widetilde{\mathcal{B}}$ is equipped with the topology of cylinder sets. The action of F_n on ∂F_n induces an action on $\widetilde{\mathcal{B}}$. The quotient of $\widetilde{\mathcal{B}}$ by this action is *the space of lines in* $\mathcal{R}$ and is called $\mathcal{B}$. It is given the quotient topology, which satisfies none of the separation axioms.

A marking on a graph, G, defines an F_n-equivariant homeomorphism between $\partial^2 F_n$ and $\tilde{\mathcal{B}}(\Gamma)$. The quotient of $\tilde{\mathcal{B}}(\Gamma)$ by the F_n action is the *space of lines* in G and is denoted $\mathcal{B}(G)$.

A closed subset Λ of $\mathcal{B}$ is an *attracting lamination for ϕ* if it is the closure of a single point β that is *bireccurrent* (every finite subpath σ of β occurs infinitely many times as an unoriented subpath of each end of β), has an *attracting neigborhood* (there is some open $U \ni \beta$ so that $\phi^k(\gamma) \to \beta$ for all $\gamma \in U$), and which is not carried by a rank one ϕ-periodic free factor. The collection of lines in Λ satisfying the above properties are called the *generic leaves* of Λ.

Associated to each $\phi \in \mathrm{Out}(F_n)$ is a finite ϕ-invariant set of *attracting laminations*, denoted by $\mathcal{L}(\phi)$. In the coordinates given by a relative train track map $f\colon G \to G$ representing ϕ, the attracting laminations for ϕ are in bijection with the EG strata of G. See [BFH00] for details.

1.2.8 Expansion factors

For each attracting lamination $\Lambda^+ \in \mathcal{L}(\phi)$, there is an associated *expansion factor homomorphism*, $PF_{\Lambda^+}\colon \mathrm{Stab}_{\mathrm{Out}(F_n)}(\Lambda^+) \to \mathbb{Z}$ which was studied in [BFH97, BFH00]. We briefly describe the essential features of PF_{Λ^+} here, but the reader is directed to the original sources for more details on lines, laminations, and expansion factor homomorphisms. For each $\psi \in \mathrm{Stab}(\Lambda^+)$, at most one of $\mathcal{L}(\psi)$ and $\mathcal{L}(\psi^{-1})$ can contain Λ^+. If neither $\mathcal{L}(\psi)$ nor $\mathcal{L}(\psi^{-1})$ contains Λ^+, then $PF_{\Lambda^+}(\psi) = 0$. Let $f\colon G \to G$ be a relative train track map representing ψ. If $\Lambda^+ \in \mathcal{L}(\psi)$ and H_r is the EG stratum of G associated to Λ^+ with corresponding PF eigenvalue λ_r, then $PF_{\Lambda^+}(\psi) = \log \lambda_r$. Conversely, if $\Lambda^+ \in \mathcal{L}(\psi^{-1})$, then $PF_{\Lambda^+}(\psi) = -\log \lambda_r$, where λ_r is the PF eigenvalue for the EG stratum of a RTT representative of ψ^{-1} which is associated to Λ^+. The image of PF_{Λ^+} is a discrete subset of $\mathbb{R}$ which we will frequently identify with $\mathbb{Z}$.

For $\phi \in \mathrm{Out}(F_n)$, each element $\Lambda^+ \in \mathcal{L}(\phi)$ has a *paired lamination* in $\mathcal{L}(\phi^{-1})$ which is denoted by Λ^-. The paired lamination is characterized by the fact that it has the same free factor support as Λ^+. That is, the minimal free factor carrying Λ^+ is the same as that which carries Λ^-. We denote the pair $\{\Lambda^+, \Lambda^-\}$ by $\Lambda^\pm$.

1.3 Translation lengths in $\mathbb{P}\mathcal{O}_n$

In this section, we will compute the translation distance for an arbitrary element of $\mathrm{Out}(F_n)$ acting on outer space. As is standard, for $\phi \in \mathrm{Out}(F_n)$ we define the translation distance of ϕ on outer space as $\tau(\phi) = \lim_{n\to\infty} \frac{d(x, x\cdot\phi^n)}{n}$. It is straightforward to check that this is independent of $x \in \mathbb{P}\mathcal{O}_n$. For the remainder of this section $\phi \in \mathrm{Out}(F_n)$ will be fixed, and $f\colon G \to G$ will be a relative train track map representing ϕ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_m = G$.

Lemma 1.6. *If H_r is an exponentially growing stratum of G, then there exists a metric ℓ on G such that $\ell(f_\#(E)) \geq \lambda_r \ell(E)$ for every edge $E \in H_r$, where λ_r is the Perron-Frobenius eigenvalue associated to H_r.*

Proof. Let M_r be the transition matrix for the exponentially growing stratum, H_r and let $\mathbf{v}$ be a left eigenvector for the PF eigenvalue λ_r with components $(\mathbf{v})_i$. Normalize $\mathbf{v}$ so that $\sum(\mathbf{v})_i = 1$. For $E_i \in H_r$ define $\ell(E_i) = (\mathbf{v})_i$. If $E \notin H_r$ define $\ell(E) = 1$. We now check the condition on the growth of edges in the EG stratum H_r.

If E is an edge in H_r, (RTT-iii) implies that $f(E)$ is r-legal. Now write $f_\#(E) = f(E)$ as an edge path, $f_\#(E) = E_1E_2\ldots E_j$, and we have

$$\ell(f(E)) = \ell(f_\#(E)) = \sum_{i=1}^{j} \ell(E_i) \geq \sum_{i=1}^{j} \ell(E_i \cap H_r) = \lambda_r \ell(E)$$

completing the proof of the lemma. □

We define the r-length ℓ_r of a path or circuit in G by ignoring the edges in other strata. Explicitly, $\ell_r(\sigma) = \ell(\sigma \cap H_r)$, where $\sigma \cap H_r$ is considered as a disjoint union of sub-paths of σ. Note that the definition of ℓ and the proof of the previous lemma show that $\ell_r(f_\#(E_i)) = \lambda_r \ell(E_i)$.

Lemma 1.7. *If σ is an r-legal reduced edge path in G and ℓ is the metric defined in Lemma 1.6, then $\ell_r(f_\#\sigma) = \lambda_r \ell_r(\sigma)$.*

Proof. We write $\sigma = a_1b_1a_2\cdots b_j$ as a decomposition into maximal subpaths where $a_j \subset H_r$ and $b_j \subset G_{r-1}$ as in Lemma 5.8 of [BH92]. Applying the lemma, we conclude that $f_\#(\sigma) = f(a_1)\cdot f_\#(b_1)\cdot f(a_2)\cdot \ldots \cdot f_\#(b_j)$. Thus,

$$\ell_r(f_\#\sigma) = \sum_i \ell_r(f(a_i)) + \sum_i \ell_r(f_\#(b_i)) = \sum_i \ell_r(f(a_i)) = \sum_i \lambda_r \ell_r(a_i) = \lambda_r \ell_r(\sigma)$$

This completes the proof of the lemma. □

Theorem 1.8 ([Wad12]). *Let $\phi \in Out(F_n)$ with $f\colon G \to G$ a RTT representative. For each EG stratum H_r of f, let λ_r be the associated PF eigenvalue. Then $\tau(\phi) = \max\{0, \log \lambda_r \mid H_r \text{ is an EG stratum}\}$.*

Proof. We first show that $\tau(\phi) \geq \log \lambda_r$ for every EG stratum H_r. Let $x = (G, \ell, \mathrm{id})$ where ℓ is the length function provided by Lemma 1.6. Recall [FM11] that the logarithm of the factor by which a candidate loop is stretched gives a lower bound on the distance between two points in $\mathbb{P}\mathcal{O}_n$. Let σ be an r-legal circuit contained in G_r of height r and let $C = \ell_r(\sigma)/\ell(\sigma)$. (RTT-iii) implies that $f^n_\#(\sigma)$ is r-legal for all n, so repeatedly applying Lemma 1.7, we have

$$\frac{\ell(f^n_\#\sigma)}{\ell(\sigma)} \geq \frac{\ell_r(f^n_\#\sigma)}{\ell(\sigma)} = \frac{\ell_r(f^n_\#\sigma)}{\ell_r(f^{n-1}_\#\sigma)} \frac{\ell_r(f^{n-1}_\#\sigma)}{\ell_r(f^{n-2}_\#\sigma)} \cdots \frac{\ell_r(f_\#\sigma)}{\ell_r(\sigma)} \frac{\ell_r(\sigma)}{\ell(\sigma)} \geq \lambda_r^n C$$

Rearranging the inequality, taking logarithms and using the result of [FM11] yields

$$\frac{d(x, x\cdot\phi^n)}{n} \geq \frac{\log(\lambda_r^n C)}{n} = \log \lambda_r + \frac{\log C}{n}$$

Taking the limit as $n \to \infty$, we have a lower bound on the translation distance of ϕ.

For the reverse inequality, fix $\epsilon > 0$. We must find a point in outer space which is moved by no more than $\epsilon + \max\{0, \log \lambda_r\}$. The idea is to choose a point in the simplex of $\mathbb{P}\mathcal{O}_n$ corresponding to a relative train track map for ϕ in which each stratum is much larger than the previous one. This way, the metric will see the growth in every EG stratum. Let $f\colon G \to G$ be a relative train track map as before, but assume that each NEG stratum consists of a single edge. This is justified, for example by choosing f to be a CT [FH11]. Let K be the maximum edge length of the image of any edge of G. Define a length function on G as follows:

$$\ell(E) = \begin{cases} (K/\epsilon)^r & \text{if } E \text{ is the unique edge in the NEG stratum } H_r \\ (K/\epsilon)^r & \text{if } E \text{ is an edge in the zero stratum } H_r \\ (K/\epsilon)^r \cdot v_i & \text{if } E_i \in H_r \text{ and } H_r \text{ is an EG stratum with } \vec{v} \text{ as above} \end{cases}$$

The logarithm of the maximum amount that any edge is stretched in a difference of markings map gives an upper bound on the Lipschitz distance between any two points. So we just check the factor by which every edge is stretched. Clearly the stretch factor for edges in fixed strata is 1. If E is the single edge in an NEG stratum, H_i, then

$$\frac{\ell(f(E))}{\ell(E)} \leq \frac{\ell(E) + K\max\{\ell(E') \mid E' \in G_{i-1}\}}{\ell(E)} = \frac{(K/\epsilon)^i + K(K/\epsilon)^{i-1}}{(K/\epsilon)^i} = 1 + \epsilon$$

Similarly, if E is an edge in the zero stratum, H_i, then

$$\frac{\ell(f(E))}{\ell(E)} \leq \frac{K(K/\epsilon)^{i-1}}{(K/\epsilon)^i} = \epsilon$$

We will use the notation $\ell_r^{\downarrow}(\sigma)$ to denote the length of the intersection of σ with G_{r-1}. So for any path σ contained in G_r, we have $\ell(\sigma) = \ell_r(\sigma) + \ell_r^{\downarrow}(\sigma)$. Now, if E_i is an edge in the EG stratum, H_r, with normalized PF eigenvector $\mathbf{v}$ then

$$\frac{\ell(f(E_i))}{\ell(E_i)} = \frac{\ell_r(f(E_i)) + \ell_r^{\downarrow}(f(E_i))}{\ell(E_i)} = \lambda_r + \frac{\ell_r^{\downarrow}(f(E_i))}{\ell(E_i)} \leq \lambda_r + \frac{K(K/\epsilon)^{r-1}}{(K/\epsilon)^r(\mathbf{v})_i} = \lambda_r + \frac{\epsilon}{(\mathbf{v})_i}$$

Since the vector $\mathbf{v}$ is determined by f, after decreasing ϵ appropriately, we have that

$$\frac{\ell(f(E))}{\ell(E)} \leq \max\{\lambda_r, 1\} + \epsilon$$

for every edge of G. This is equivalent to the statement that the distance (G, ℓ, ρ) is moved by ϕ is less than $\max\{\log(\lambda_r), 0\} + \epsilon$, which completes the proof. □

Now that we have computed the translation distance of an arbitrary ϕ acting on outer space, we'll use this result to establish our main result in a special case.

1.4 The exponential case

In this section, we'll analyze the case that the abelian subgroup $H = \langle \phi_1, \ldots, \phi_k \rangle$ has enough exponential data so that the entire group is seen by the so called lambda map. More precisely, given an attracting lamination Λ^+ for an outer automorphism ϕ, let $PF_{\Lambda^+} \colon \mathrm{Stab}(\Lambda^+) \to \mathbb{Z}$ be the expansion factor homomorphism defined by Corollary 3.3.1 of [BFH00]. In [FH09, Corollary 3.14], the authors prove that every abelian subgroup of $\mathrm{Out}(F_n)$ has a finite index subgroup which is rotationless (meaning that every element of the subgroup is rotationless). Distortion is unaffected by passing to a finite index subgroup, so there is no loss in assuming that H is rotationless. Now let $\mathcal{L}(H) = \bigcup_{\phi \in H} \mathcal{L}(\phi)$ be the set of attracting laminations for elements of H. By [FH09, Lemma 4.4], $\mathcal{L}(H)$ is a finite set of H-invariant laminations. Define $PF_H \colon H \to \mathbb{Z}^{\#\mathcal{L}(H)}$ by taking the collection of expansion factor homomorphisms for attracting laminations of the subgroup H. In what follows, we will need to interchange PF_{Λ^+} for PF_{Λ^-} and for that we will need the following lemma.

Lemma 1.9. *If $\Lambda^+ \in \mathcal{L}(\phi)$ and $\Lambda^- \in \mathcal{L}(\phi^{-1})$ are paired laminations then $\frac{PF_{\Lambda^+}}{PF_{\Lambda^-}}$ is a constant map. That is, PF_{Λ^+} and PF_{Λ^-} differ by a multiplicative constant, and so determine the same homomorphism.*

Proof. First, Corollary 1.3(2) of [HM14] gives that $\mathrm{Stab}(\Lambda^+) = \mathrm{Stab}(\Lambda^-)$ (which we will henceforth refer to as $\mathrm{Stab}(\Lambda^\pm)$), so the ratio in the statement is always well defined. Now PF_{Λ^+} and PF_{Λ^-} each determine a homomorphism from $\mathrm{Stab}(\Lambda^\pm)$ to $\mathbb{R}$ and it suffices to show that these homomorphisms have the same kernel. Suppose $\psi \notin \ker PF_{\Lambda^+}$ so that by [BFH00, Corollary 3.3.1] either $\Lambda^+ \in \mathcal{L}(\psi)$ or $\Lambda^+ \in \mathcal{L}(\psi^{-1})$. After replacing ψ by ψ^{-1} if necessary, we may assume $\Lambda^+ \in \mathcal{L}(\psi)$. Now ψ has a paired lamination $\Lambda^-_\psi \in \mathcal{L}(\psi^{-1})$ which a priori could be different from Λ^-. But Corollary 1.3(1) of [HM14] says that in fact $\Lambda^-_\psi = \Lambda^-$ and therefore that $\Lambda^- \in \mathcal{L}(\psi^{-1})$. A final application of [BFH00, Corollary 3.3.1] gives that $\psi \notin \ker PF_{\Lambda^-}$. This concludes the proof. □

Theorem 1.10. *If PF_H is injective, then H is undistorted in $Out(F_n)$.*

Proof. Let k be the rank of H and start by choosing laminations $\Lambda_1, \dots, \Lambda_k \in \mathcal{L}(H)$ so the restriction of the function PF_H to the coordinates determined by $\Lambda_1, \dots, \Lambda_k$ is still injective. First note that $\{\Lambda_1, \dots, \Lambda_k\}$ cannot contain an attracting-repelling lamination pair by Lemma 1.9.

Next, pass to a finite index subgroup of H and choose generators ϕ_i so that after reordering the Λ_i's if necessary, each generator satisfies $PF_H(\phi_i) = (0, \dots, 0, PF_{\Lambda_i}(\phi_i), 0, \dots, 0)$. Let $* \in \mathbb{P}\mathcal{O}_n$ be arbitrary and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k} \in H$. We complete the proof one orthant at a time by replacing some of the ϕ_i's by their inverses so that all the p_i's are nonnegative. Next, after replacing some of the Λ_i's by their paired laminations (again using Lemma 1.9), we may assume that $PF_H(\psi)$ has all coordinates nonnegative.

By Theorem 1.8, the translation distance of ψ is the maximum of the Perron-Frobenius eigenvalues associated to the EG strata of a relative train track representative f of ψ. Some, but not necessarily all, of $\Lambda_1, \dots, \Lambda_k$ are attracting laminations for ψ. Those Λ_i's which are in $\mathcal{L}(\psi)$ are associated to EG strata of f. For such a stratum, the logarithm of the PF eigenvalue is $PF_{\Lambda_i}(\psi)$ and the fact that PF_{Λ_i} is a homomorphism implies

$$PF_{\Lambda_i}(\psi) = PF_{\Lambda_i}(\phi_1^{p_1} \cdots \phi_k^{p_k}) = p_1 PF_{\Lambda_j}(\phi_1) + \ldots + p_k PF_{\Lambda_j}(\phi_k) = p_i PF_{\Lambda_i}(\phi_i)$$

Thus, the translation distance of ψ acting on outer space is

$$\begin{aligned}\tau(\psi) &= \max\{\log \lambda \mid \lambda \text{ is PF eigenvalue associated to an EG stratum of } \psi\}\\ &\geq \max\{PF_{\Lambda_i}(\psi) \mid \Lambda_i \text{ is in } \mathcal{L}(\psi) \text{ and } 1 \leq i \leq k\}\\ &= \max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\}\end{aligned}$$

In the last equality, the maximum is taken over a larger set, but the only values added to the set were 0.

Let S be a symmetric (i.e., $S^{-1} = S$) generating set for $\mathrm{Out}(F_n)$ and let $D_1 = \max_{s \in S} d(*, * \cdot s)$. If we write ψ in terms of the generators $\psi = s_1 s_2 \cdots s_l$, then

$$\begin{aligned}d(*, * \cdot \psi) &\leq d(*, * \cdot s_l) + d(* \cdot s_l, * \cdot s_{l-1} s_l) + \ldots + d(* \cdot (s_2 \ldots s_l), * \cdot (s_1 \ldots s_l))\\ &= d(*, * \cdot s_l) + d(*, * \cdot s_{l-1} + \ldots + d(*, * \cdot s_1) \leq D_1 |\psi|_{\mathrm{Out}(F_n)}\end{aligned}$$

Let $K_1 = \min\{PF_{\Lambda_i^{\pm}}(\phi_j^{\pm}) \mid 1 \leq i, j \leq k\}$. Rearranging this and combining these inequalities, we have

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_1} d(*, * \cdot \psi) \geq \frac{1}{D_1} \tau(\psi) \geq \frac{1}{D_1} \max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\} \geq \frac{K_1}{D_1} \max\{p_i\}$$

We have thus proved that the image of H under the injective homomorphism PF_H is undistorted in $\mathbb{Z}^k$. To conclude the proof, recall that any injective homomorphism between abelian groups is a quasi-isometric embedding. □

Now that we have established our result in the exponential setting, we move on to the polynomial case. First we prove a general result about CTs representing elements of abelian subgroups.

1.5 Abelian subgroups are virtually finitely filtered

In this section, we prove an analog of [BFH05, Theorem 1.1] for abelian subgroups. In that paper, the authors prove that any unipotent subgroup of $\mathrm{Out}(F_n)$ is contained in the subgroup $\mathcal{Q}$ of homotopy equivalences respecting a fixed filtration on a fixed graph

G. They call such a subgroup "filtered." While generic abelian subgroups of $\mathrm{Out}(F_n)$ are *not* unipotent, we prove that they are virtually filtered. Namely, that such a subgroup is virtually contained in the union of finitely many $\mathcal{Q}$'s. First, we review the comparison homomorphisms introduced in [FH09].

1.5.1 Comparison homomorphisms

Feighn and Handel defined certain homomorphisms to $\mathbb{Z}$ which measure the growth of linear edges and quasi-exceptional families in a CT representative. Though they can be given a canonical description in terms of principal lifts, we will only need their properties in coordinates given by a CT. Presently, we will define these homomorphisms and recall some basic facts about them. Complete details on comparison homomorphisms can be found in [FH09].

Comparison homomorphisms are defined in terms of principal sets for the subgroup H. The exact definition of a principal set is not important for us. We only need to know that a *principal set* $\mathcal{X}$ for an abelian subgroup H is a subset of ∂F_n which defines a lift $s\colon H \to \mathrm{Aut}(F_n)$ of H to the automorphism group. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two principal sets for H that define distinct lifts s_1 and s_2 to $\mathrm{Aut}(F_n)$. Suppose further that $\mathcal{X}_1 \cap \mathcal{X}_2$ contains the endpoints of an axis A_c. Since H is abelian, $s_1 \cdot s_2^{-1}\colon H \to \mathrm{Aut}(F_n)$ defined by $s_1 \cdot s_2^{-1}(\phi) = s_1(\phi) \cdot s_2(\phi)^{-1}$ is a homomorphism. It follows from [FH11, Lemma 4.14] that for any $\phi \in H$, $s_1(\phi) = s_2(\phi) i_c^k$ for some k, where $i_c\colon \mathrm{Aut}(F_n) \to \mathrm{Aut}(F_n)$ denotes conjugation by c. Therefore $s_1 \cdot s_2^{-1}$ defines homomorphism into $\langle i_c \rangle$, which we call the *comparison homomorphism* determined by $\mathcal{X}_1$ and $\mathcal{X}_2$. Generally, we will use the letter ω for comparison homomorphisms.

For a rotationless abelian subgroup H, there are only finitely many comparison homomorphisms [FH09, Lemma 4.3]. Let K be the number of distinct comparison homomorphisms and (as before) let N be the number of attracting laminations for H. The map $\Omega\colon H \to \mathbb{Z}^{N+K}$ defined as the product of the comparison homomorphisms and expansion factor homomorphisms is injective [FH09, Lemma 4.6]. An element $\phi \in H$ is called *generic* if every coordinate of $\Omega(\phi) \in \mathbb{Z}^{N+K}$ is nonzero. If ϕ is generic and $f\colon G \to G$ is a CT representing ϕ, then there is a correspondence between the comparison homomorphisms for H and the linear edges and quasi-exceptional families in G described in the introduction to §7 of [FH09] which we briefly describe now. There is a comparison homomorphism ω_{E_i} for each linear edge E_i in G. If $f(E_i) = E_i \cdot u^{d_i}$, then $\omega_{E_i}(\phi) = d_i$. There is also a comparison homomorphism for each quasi-exceptional family, $E_i u^* \overline{E}_j$ which is denoted by $\omega_{E_i u^* \overline{E}_j}$. If E_i is as before and $f(E_j) = E_j u^{d_j}$, then $\omega_{E_i u^* \overline{E}_j}$ and $\omega(\phi) = d_i - d_j$. We illustrate this correspondence with an example.

Example 1.11. *Let $G = R_3$ be the rose with three petals labeled a, b, and c. For $i, j \in \mathbb{Z}$, define $g_{i,j} \colon G \to G$ as follows:*

$$g_{i,j} \colon \begin{array}{l} a \mapsto a \\ b \mapsto ba^i \\ c \mapsto ca^j \end{array}$$

Each $g_{i,j}$ determines an outer automorphism of F_3 which we denote by $\phi_{i,j}$. The automorphisms $\phi_{i,j}$ all lie in the rank two abelian subgroup $H = \langle \phi_{0,1}, \phi_{1,0} \rangle$. The subgroup H has three comparison homomorphisms which are easily understood in the coordinates of a CT for a generic element of H. The element $\phi_{2,1}$ is generic in H, and $g_{2,1}$ is a CT representing it. Two of the comparison homomorphisms manifest as ω_b and ω_c where $\omega_b(\phi_{i,j}) = i$ and $\omega_c(\phi_{i,j}) = j$. The third homomorphism is denoted by $\omega_{ba^*\overline{c}}$ and it measures how a path of the form $ba^*\overline{c}$ changes when $g_{i,j}$ is applied. Since $g_{i,j}(ba^*\overline{c}) = ba^{*+i-j}\overline{c}$, we have $\omega_{ba^*\overline{c}}(\phi_{i,j}) = i - j$.

In the sequel, we will rely heavily on this correspondence between the comparison homomorphisms of H and the linear edges and quasi-exceptional families in a CT for a generic element of H. We now prove the main result of this section.

Proposition 1.12. *For any abelian subgroup H of $Out(F_n)$, there exists a finite index subgroup H' such that every $\phi \in H'$ can be realized as a CT on one of finitely many marked graphs.*

Most of the proof consists of restating and combining results of Feighn and Handel from [FH09]. We refer the reader to §6 of their paper for the relevant notation and most of the relevant results.

Proof. First replace H by a finite index rotationless subgroup [FH09, Corollary 3.14]. The proof is by induction on the rank of H. The base case follows directly from [FH09, Lemma 6.18]. Let $H = \langle \phi \rangle$ and let $f^\pm \colon G^\pm \to G^\pm$ be CT's for ϕ and ϕ^{-1} which are both generic in H. The definitions then guarantee that $\mathbf{i} = (i, i, \ldots, i)$ for $i > 0$ is both generic and admissible. Lemma 6.18 then says that $f^{\pm}_{\mathbf{i}} \colon G^{\pm} \to G^{\pm}$ is a CT representing $\phi^{\pm}_{\mathbf{i}} = \phi^{\pm i}$, so we are done.

Assume now that the claim holds for all abelian subgroups of rank less than k, and let $H = \langle \phi_1, \ldots, \phi_k \rangle$. The set of generic elements of H is the complement of a finite [FH09, Lemma 4.3] collection of hyperplanes. Every nongeneric element, ϕ, lies in a rank $(k-1)$ abelian subgroup of H: the kernel of the corresponding comparison homomorphism. By induction and the fact that there are only finitely many hyperplanes, every nongeneric element has a CT representative on one of finitely many marked graphs. We now add a single marked graph for each sector defined by the complement of the hyperplanes.

Let ϕ be generic and let $f \colon G \to G$ be a CT representative. Let $\mathcal{D}(\phi)$ be the disintegration of ϕ as defined in [FH09] and recall that $\mathcal{D}(\phi) \cap H$ is finite index in H [FH09, Theorem 7.2]. Let Γ be the semigroup of generic elements of $\mathcal{D}(\phi) \cap H$ that lie in the same sector of H as ϕ (i.e., for every $\gamma \in \Gamma$ and every coordinate ω of Ω, the signs of $\omega(\gamma)$ and $\omega(\phi)$ agree). The claim is that every element of Γ can be realized as a CT on the marked graph G and we will show this by explicitly reconstructing the generic tuple $\mathbf{a}$ such that $\gamma = [f_{\mathbf{a}}]$. Fix $\gamma \in \Gamma$ and let $\phi_{\mathbf{a}_1}, \ldots, \phi_{\mathbf{a}_k}$ be a generating set for H with $\mathbf{a}_i$ generic [FH09, Corollary 6.20]. Write γ as a word in the generators, $\gamma = \phi^{j_1}_{\mathbf{a}_1} \cdots \phi^{j_k}_{\mathbf{a}_k}$ and define $\mathbf{a} = j_1\mathbf{a}_1 + \ldots + j_k\mathbf{a}_k$. Since the admissibility condition is a set of homogeneous linear equations which must be preserved under taking linear combinations, as long as every coordinate of $\mathbf{a}$ is nonnegative, $\mathbf{a}$ must be admissible. To see that every coordinate of $\mathbf{a}$ is in fact positive, let ω be a coordinate of Ω^{ϕ}. Using the fact that ω is a homomorphism to $\mathbb{Z}$ and repeatedly applying [FH09, Lemma 7.5] to the $\phi_{\mathbf{a}_i}$'s, we have

$$\begin{aligned}
\omega(\gamma) &= j_1\omega(\phi_{\mathbf{a}_1}) + j_2\omega(\phi_{\mathbf{a}_2}) + \ldots + j_k\omega(\phi_{\mathbf{a}_k}) \\
&= j_1(\mathbf{a}_1)_s\omega(\phi) + j_2(\mathbf{a}_2)_s\omega(\phi) + \ldots + j_k(\mathbf{a}_k)_s\omega(\phi) \\
&= (j_1\mathbf{a}_1 + j_2\mathbf{a}_2 + \ldots + j_k\mathbf{a}_k)_s\omega(\phi) \\
&= (\mathbf{a})_s\omega(\phi)
\end{aligned}$$

where $(\mathbf{a})_s$ denotes the s-th coordinate of the vector $\mathbf{a}$. Since γ and ϕ were assumed to be generic and to lie in the same sector, we conclude that every coordinate of $\mathbf{a}$ is positive. The injectivity Ω^{ϕ} [FH09, Lemma 7.4] then implies that $\gamma = [f_{\mathbf{a}}]$. That $\mathbf{a}$ is in fact generic follows from the fact, which is directly implied by the definitions, that if $\mathbf{a}$ is a generic tuple, then $\phi_{\mathbf{a}}$ is a generic element of H. Finally, we apply [FH09, Lemma 6.18] to conclude that $f_{\mathbf{a}}\colon G \to G$ is a CT. Thus, every element of Γ has a CT representative on the marked, filtered graph G. Repeating this argument in each of the finitely many sectors and passing to the intersection of all the finite index subgroups obtained this way yields a finite index subgroup H' and finitely many marked graphs, so that every generic element of H' can be realized as a CT on one of the marked graphs. The nongeneric elements were already dealt with using the inductive hypothesis, so the proof is complete. □

1.6 The polynomial case

In [Ali02], the author introduced a function that measures the twisting of conjugacy classes about an axis in F_n and used this function to prove that cyclic subgroups of UPG are undistorted. In order to use the comparison homomorphisms in conjunction with this twisting function, we need to establish a result about the possible terms occuring in completely split circuits. After establishing this connection, we use it to prove (Theorem 1.24) the main result under the assumption that H has "enough" polynomial data.

In the last section, we saw the correspondence between comparison homomorphisms and certain types of paths in a CT. In order to use the twisting function from [Ali02], our goal is to find circuits in G with single linear edges or quasi-exceptional families as subpaths, and moreover to do so in such a way that we can control cancellation at the ends of these subpaths under iteration of f. This is the most technical section of the paper, and the one that most heavily relies on the use of CTs. The main result is Proposition 1.13.

1.6.1 Completely split circuits

One of the main features of train track maps is that they allow one to understand how cancellation occurs when tightening $f^k(\sigma)$ to $f^k_\#(\sigma)$. In previous incarnations of train track maps, this cancellation was understood inductively based on the height of the path σ. One of the main advantages of completely split train track maps is that the way cancellation can occur is now understood directly, rather than inductively.

Given a CT $f\colon G \to G$ representing ϕ, the set of allowed terms in completely split paths would be finite were it not for the following two situations: a linear edge $E \mapsto Eu$ gives rise to an infinite family of INPs of the form $Eu^*\overline{E}$, and two linear edges with the same axis $E_1 \mapsto E_1 u^{d_1}$, $E_2 \mapsto E_2 u^{d_2}$ (with d_1 and d_2 having the same sign) give rise to an infinite family of exceptional paths of the form $E_1 u^* \overline{E}_2$. To see that these are the only two subtleties, one only needs to know that there is at most one INP of height r for each EG stratum H_r. This is precisely [FH09, Corollary 4.19].

To connect Feighn-Handel's comparison homomorphisms to Alibegović's twisting function, we would like to show that every linear edge and exceptional family occurs as a term in the complete splitting of some completely split circuit. We will in fact show something stronger:

Proposition 1.13. *There is a completely split circuit σ containing every allowable term in its complete splitting. That is, the complete splitting of σ contains at least one instance of every*

- *edge in an irreducible stratum (fixed, NEG, or EG),*
- *maximal, taken connecting subpath in a zero stratum,*
- *infinite family of INPs $Eu^*\overline{E}$,*
- *infinite family of exceptional paths $E_1 u^* \overline{E}_2$.*

The proof of this proposition will require a careful study of completely split paths. With that aim, we define a directed graph that encodes the complete splittings of such paths. Given a CT $f\colon G \to G$ representing ϕ define a di-graph $\mathcal{CSP}(f)$ (or just $\mathcal{CSP}$ when f is clear) whose vertices are oriented allowed terms in completely split paths. More precisely, there are two vertices for each edge in an irreducible stratum: one labeled by E and one labeled by $\overline{E}$ (which we will refer to at τ_E and $\tau_{\overline{E}}$). There are two vertices for each maximal taken connecting path in a zero stratum: one for σ and one for $\overline{\sigma}$ (which will be referred to as τ_σ and $\tau_{\overline{\sigma}}$). Similarly, there are two vertices for each family of exceptional paths, two vertices for each INP of EG height, and *one* vertex for each infinite family of NEG Nielsen paths. There is only one vertex for each family of indivisible Nielsen path σ whose height is NEG because σ and $\overline{\sigma}$ determine the same initial direction. There is an edge connecting two vertices τ_σ and $\tau_{\sigma'}$ in $\mathcal{CSP}(f)$ if the path $\sigma\sigma'$ is completely split with splitting given by $\sigma \cdot \sigma'$. This is equivalent to the turn $(\overline{\sigma}, \sigma')$ being legal by the uniqueness of complete splittings [FH11, Lemma 4.11].

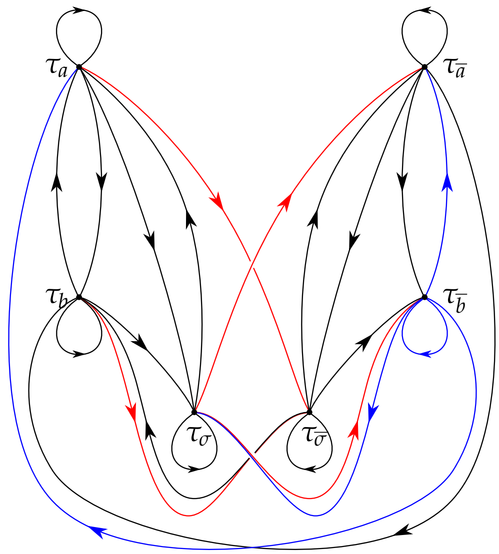

Figure 1.1. The graph of $\mathcal{CSP}(f)$ for Example 1.14

Any completely split path (resp. circuit) σ with endpoints at vertices in G defines a directed edge path (resp. directed loop) in $\mathcal{CSP}(f)$ given by reading off the terms in the complete splitting of σ. Conversely, a directed path or loop in $\mathcal{CSP}(f)$ yields a not quite well defined path or circuit σ in G which is necessarily completely split. The only ambiguity lies in how to define σ when the path in $\mathcal{CSP}(f)$ passes through a vertex labeled by a Nielsen path of NEG height or a quasi-exceptional family.

Example 1.14. *Consider the rose R_2 consisting of two edges a and b with the identity marking. Let $f\colon R_2 \to R_2$ be defined by $a \mapsto ab$, $b \mapsto bab$. This is a CT representing a fully irreducible outer automorphism. There is one indivisible Nielsen path $\sigma = ab\overline{a}\overline{b}$. The graph $\mathcal{CSP}(f)$ is shown in Figure 1.1. The blue edges represent the fact that each of the paths $\overline{b} \cdot \overline{b}$, $\overline{b} \cdot \overline{a}$, $\overline{b} \cdot \sigma$, and $\overline{b} \cdot a$ is completely split.*

Remark 1.15. *A basic observation about the graph $\mathcal{CSP}$ is that every vertex τ_σ has at least one incoming and at least one outgoing edge. While this is really just a consequence of the fact that every vertex in a CT has at least two gates, a bit of care is needed to justify this formally. Indeed, let v be the initial endpoint of σ. If there is some legal turn (E, σ) at v where E is an edge in an irreducible stratum, then $\overline{E} \cdot \sigma$ is completely split so there is an edge in $\mathcal{CSP}$ from $\tau_{\overline{E}}$ to τ_σ. The other possibility is that the only legal turns $(\_\_, \sigma)$ at v consist of an edge in a zero stratum H_i. In* *this case, (Zero Strata) guarantees that v is contained in the EG stratum H_r which envelops H_i and that the link of v is contained in $H_i \cup H_r$. In particular, there are a limited number of possibilities for σ; σ may be a taken connecting subpath in H_i, an edge in H_r, or an EG INP of height r. In the first two cases, σ is a term in the complete splitting of $f_\#^k(E)$ for some edge E. By increasing k if necessary, we can guarantee that σ is not the first or last term in this splitting. Therefore, there is a directed edge in $\mathcal{CSP}$ with terminal endpoint τ_σ. In the case that σ is an INP, σ has a first edge E_0 which is necessarily of EG height. We have already established that there is a directed edge in $\mathcal{CSP}$ pointed to τ_{E_0}, so we just observe that any vertex in $\mathcal{CSP}$ with a directed edge ending at E_0 will also have a directed edge terminating at τ_σ. The same argument shows that there is an edge in $\mathcal{CSP}$ emanating from τ_σ.*

The statement of Proposition 1.13 can now be rephrased as a statement about the graph $\mathcal{CSP}$. Namely, that there is a directed loop in $\mathcal{CSP}$ which passes through every vertex.

We will need some basic terminology from the study of directed graphs. We say a di-graph Γ is *strongly connected* if every vertex can be connected to every other vertex in Γ by a directed edge path. In any di-graph, we may define an equivalence relation on the vertices by declaring $v \sim w$ if there is a directed edge path from v to w and vice versa (we are required to allow the trivial edge path so that $v \sim v$). of Γ. The equivalence classes of this relation partition the vertices of Γ into *strongly connected components*.

We will prove that $\mathcal{CSP}(f)$ is connected and has one strongly connected component. From this, Proposition 1.13 follows directly. The proof proceeds by induction on the core filtration of G, which is the filtration obtained from the given one by considering only the filtration elements which are their own cores. Because the base case is in fact more difficult than the inductive step, we state it as a lemma.

Lemma 1.16. *If $f\colon G \to G$ is a CT representing a fully irreducible automorphism, then $\mathcal{CSP}(f)$ is connected and strongly connected.*

Proof. Under these assumptions, there are two types of vertices in $\mathcal{CSP}(f)$: those labeled by edges, and those labeled by INPs. We denote by $\mathcal{CSP}_e$ the subgraph consisting of only the vertices which are labeled by edges. Recall that τ_E denotes the vertex in $\mathcal{CSP}$ corresponding to the edge E. If the leaves of the attracting lamination are nonorientable, then we can produce a path in $\mathcal{CSP}_e$ starting at τ_E, then passing through every other vertex in $\mathcal{CSP}_e$, and finally returning to τ_E by looking at a long segment of a leaf of the attracting lamination. More precisely, (Completely Split) says that $f^k(E)$ is a completely split path for all $k \geq 0$ and the fact that f is a train track map says that this complete splitting contains no INPs. Moreover, irreducibility of the transition matrix and nonorientability of the lamination implies that for sufficiently large k this path not only contains every edge in G (with both orientations), but contains the edge E followed by every other edge in G with both of its orientations, and then the edge E again. Such a path in G exactly shows that $\mathcal{CSP}_e$ is connected and strongly connected.

We isolate the following remark for future reference.

Remark 1.17. *If there is an indivisible Nielsen path σ in G, write its edge path $\sigma = E_1E_2\dots E_k$ (recall that all INPs in a CT have endpoints at vertices). If $\tau_{\sigma'}$ is any vertex in $\mathcal{CSP}$ with a directed edge pointing to τ_{E_1}, then $\sigma' \cdot \sigma$ is completely split since the turn $(\overline{\sigma}', \sigma)$ must be legal. Hence there is also a directed edge in $\mathcal{CSP}$ from $\tau_{\sigma'}$ to τ_σ. The same argument shows that there is an edge in $\mathcal{CSP}$ from τ_σ to some vertex $\tau' \neq \tau_\sigma$.*

Since $\mathcal{CSP}_e$ is strongly connected, and the remark implies that each vertex τ_σ (for σ an INP in G) has directed edges coming from and going back into $\mathcal{CSP}_e$, we conclude that $\mathcal{CSP}$ is strongly connected in the case that leaves of the attracting lamination are nonorientable.

Now choose an orientation on the attracting lamination Λ. If we imagine an ant following the path in G determined by a leaf of Λ, then at each vertex v we see the ant arrive along certain edges and leave along others. Let E be an edge with initial vertex v so that E determines a gate $[E]$ at v. We say that $[E]$ is a *departure gate* at v if E occurs in some (any) oriented leaf λ. Similarly, we say the gate $[E]$ is an *arrival gate* at v if the edge $\overline{E}$ occurs in λ. Some gates may be both arrival and departure gates.

Suppose now that there is some vertex v in G that has at least two arrival gates and some vertex w that has at least two departure gates. As before, we will produce a path in $\mathcal{CSP}_e$ that shows this subgraph has one strongly connected component. Start at any edge in G and follow a leaf λ of the lamination until you have crossed every edge with its forward orientation. Continue following the leaf until you arrive at v, say through the gate $[\overline{E}]$. Since v has two arrival gates, there is some edge E' which occurs in λ with the given orientation and whose terminal vertex is v ($[\overline{E'}]$ is a second arrival gate). Now turn onto $\overline{E'}$. Since $[\overline{E}]$ and $[\overline{E'}]$ are distinct gates, this turn is legal. Follow $\overline{\lambda}$ going backwards until you have crossed every edge of G (now in the opposite direction). Finally, continue following $\overline{\lambda}$ until you arrive at w, where there are now two arrival gates because you are going backwards. Use the second arrival gate to turn around a second time, and follow λ (now in the forwards direction again) until you cross the edge you started with. By construction, this path in G is completely split and every term in its complete splitting is a single edge. The associated path in $\mathcal{CSP}_e$ passes through every vertex and then returns to the starting vertex, so $\mathcal{CSP}_e$ is strongly connected. In the presence of an INP, Remark 1.17 completes the proof of the lemma under the current assumptions.

We have now reduced to the case where the lamination is orientable *and* either every vertex has only one departure gate or every vertex has only one arrival gate. The critical case is the latter of the two, and we would like to conclude in this situation that there is an INP. Example 1.14 illustrates this scenario. Some edges are colored red to illustrate the fact that in order to turn around and get from the vertices labeled by a and b to those labeled by $\overline{a}$ and $\overline{b}$, one must use an INP. The existence of an INP in this situation is provided by the following lemma.

Lemma 1.18. *Assume $f\colon G \to G$ is a CT representing a fully irreducible rotationless automorphism. Suppose that the attracting lamination is orientable and that every vertex has exactly one arrival gate. Then G has an INP, σ, and the initial edges of σ and $\overline{\sigma}$ are oriented consistently with the orientation of the lamination.*

We postpone the proof of this lemma and explain how to conclude our argument. If every vertex has one arrival gate, then we apply the lemma to conclude that there must be an INP. Since INPs have exactly one illegal turn, using the previous argument, we can turn around once. Now if we are again in a situation where there is only one arrival gate, then we can apply the lemma a second time (this time with the orientation of Λ reversed) to obtain the existence of a second INP, allowing us to turn around a second time. $\square$

We remark that since there is at most one INP in each EG stratum of a CT, Lemma 1.18 implies that if the lamination is orientable, then some vertex of G must have at least 3 gates.

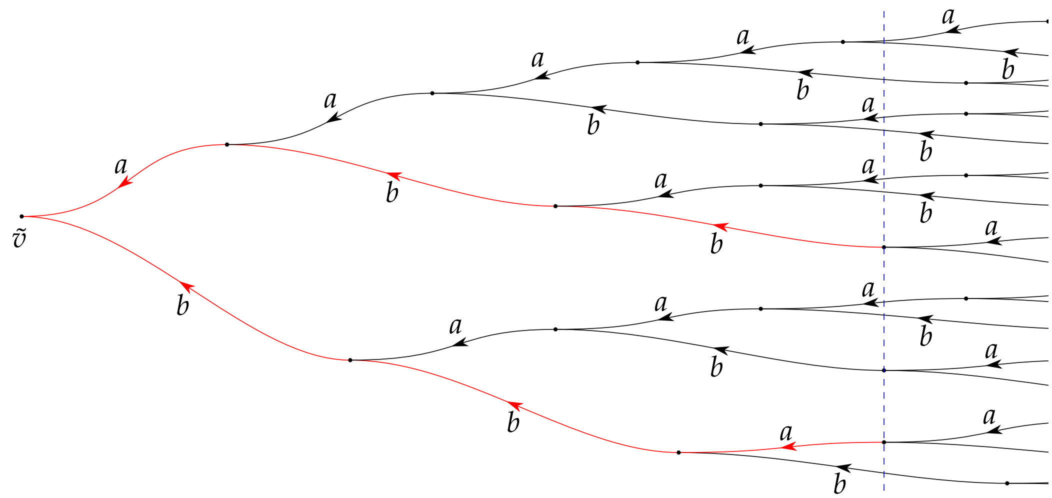

Figure 1.2. The tree T for Example 1.14. The red path connects two vertices of the same height.

Proof of Lemma 1.18. There is a vertex of G that is fixed by f since [FH11, Lemma 3.19] guarantees that every EG stratum contains at least one principal vertex and principal vertices are fixed by (Rotationless). Choose such a vertex v and let $\tilde{v} \in \Gamma$ be a lift of v to the universal cover Γ of G. Let g be the unique arrival gate at $\tilde{v}$. Lift f to a map $\tilde{f}\colon \Gamma \to \Gamma$ fixing $\tilde{v}$. Let T be the infinite subtree of Γ consisting of all embedded rays $\gamma\colon [0, \infty) \to \Gamma$ starting at $\tilde{v}$ and leaving every vertex through its unique arrival gate. That is $\gamma(0) = \tilde{v}$ and whenever $\gamma(t)$ is a vertex, $D\gamma(t)$ should be the unique arrival gate at $\gamma(t)$. Refer to Figure 1.2 for the tree T for Example 1.14.

First, we claim that $\tilde{f}(T) \subset T$. To see this, notice that since f is a topological representative, it suffices to show that $\tilde{f}(p) \in T$ for every vertex p of T. Notice that vertices p of T are characterized by two things: first $[\tilde{v}, p]$ is legal, and second, for every edge E in the edge path of $[\tilde{v}, p]$, the gate $[E]$ is the unique arrival gate at the initial endpoint of E. Now $[\tilde{v}, \tilde{f}(p)] = \tilde{f}([\tilde{v}, p])$ is legal because f is a train track map. Moreover, every edge E in the edge path of $[\tilde{v}, p]$ occurs (with orientation) in a leaf $\overline{\lambda}$ of the lamination. Since $\tilde{f}$ takes leaves to leaves preserving orientation, the same is true for $\tilde{f}([\tilde{v}, p])$. The gate determined by every edge in the edge path of $\overline{\lambda}$ is the unique arrival gate at that vertex. Thus, for every edge E in the edge path of $\tilde{f}([\tilde{v}, p])$, $[E]$ is the unique arrival gate at that vertex, which means that $\tilde{f}(p) \in T$.

Endow G with a metric using the left PF eigenvector of the transition matrix so that for every edge of G, we have $\ell(f(E)) = \nu\,\ell(E)$ where ν is the PF eigenvalue of the transition matrix. Lift the metric on G to a metric on Γ and define a height function on the tree T by measuring the distance to $\tilde{v}$: $h(p) = d(p, \tilde{v})$. Since legal paths are stretched by exactly ν, we have that for any $p \in T$, $h(\tilde{f}(p)) = \nu h(p)$.

Now let w and w' be two distinct lifts of v with the same height, $h(w) = h(w')$. To see that this is possible, just take α and β to be two distinct ($\langle\alpha, \beta\rangle \simeq F_2$) circuits in G based at v which are obtained by following a leaf of the lamination. The initial vertices of the lifts of $\alpha\beta$ and $\beta\alpha$ which end at $\tilde{v}$ are distinct lifts of v which are contained in T, and have the same height.

Let τ be the unique embedded segment connecting w to w' in T. By [FH11, Lemma 4.25], $\tilde{f}^k_\#(\tau)$ is completely split for all sufficiently large k. Moreover, the endpoints of $\tilde{f}^k_\#(\tau)$ are distinct since the restriction of $\tilde{f}$ to the lifts of v is injective. This is simply because $\tilde{f}\colon (\Gamma, \tilde{v}) \to (\Gamma, \tilde{v})$ represents an automorphism of F_n and lifts of v correspond to elements of F_n. Now observe that the endpoints $\tilde{f}^k_\#(\tau)$ have the same height and for any pair of distinct vertices with the same height, the unique embedded segment connecting them must contain an illegal turn. This follows from the definition of T and the assumption that every vertex has a unique arrival gate. Therefore, the completely split path $\tilde{f}^k_\#(\tau)$ contains an illegal turn. In particular, it must have an INP in its complete splitting. That the initial edges of σ and $\overline{\sigma}$ are oriented consistently with the orientation on λ is evident from the construction. □

The key to the inductive step is provided by the "moving up through the filtration" lemma from [FH09] which explicitly describes how the graph G can change when moving from one element of the core filtration to the next. Recall the *core filtration* of G is the filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$ obtained by restricting to those filtration elements which are their own cores. For each G_{l_i}, the i-th stratum of the core filtration is defined to be $H^c_{l_i} = \bigcup_{j=l_{i-1}+1}^{l_i} H_j$. Finally, we let $\Delta\chi^-_i = \chi(G_{l_{i-1}}) - \chi(G_{l_i})$ denote the negative of the change in Euler characteristic.

Lemma 1.19 ([FH09, Lemma 8.3]). *1. If $H^c_{l_i}$ does not contain any EG strata then one of the following holds.*

(a) $l_i = l_{i-1} + 1$ and the unique edge in $H^c_{l_i}$ is a fixed loop that is disjoint from $G_{l_{i-1}}$.

(b) $l_i = l_{i-1} + 1$ and both endpoints of the unique edge in $H^c_{l_i}$ are contained in $G_{l_{i-1}}$.

(c) $l_i = l_{i-1} + 2$ and the two edges in $H^c_{l_i}$ are nonfixed and have a common initial endpoint that is not in $H_{l_{i-1}}$ and terminal endpoints in $G_{l_{i-1}}$.

In case 1a, $\Delta_i\chi^- = 0$; in cases 1b and 1c, $\Delta_i\chi^- = 1$.

2. *If $H^c_{l_i}$ contains an EG stratum, then H_{l_i} is the unique EG stratum in $H^c_{l_i}$ and there exists $l_{i-1} \leq u_i < l_i$ such that both of the following hold.*

   (a) *For $l_{i_1} < j \leq u_i$, H_j is a single nonfixed edge E_j whose terminal vertex is in $G_{l_{i-1}}$ and whose initial vertex has valence one in G_{u_i}. In particular, G_{u_i} deformation retracts to $G_{l_{i-1}}$ and $\chi(G_{u_i}) = \chi(G_{l_{i-1}})$.*

   (b) *For $u_i < j < l_i$, H_j is a zero stratum. In other words, the closure of $G_{l_i} \setminus G_{u_i}$ is the extended EG stratum $H^z_{l_i}$.*

   If some component of $H^c_{l_i}$ is disjoint from G_{u_i} then $H^c_{l_i} = H_{l_i}$ is a component of G_{l_i} and $\Delta_i\chi^- \geq 1$; otherwise $\Delta_i\chi^- \geq 2$.

As we move up through the core filtration, we imagine adding new vertices to $\mathcal{CSP}$ and adding new edges connecting these vertices to each other and to the vertices already present. Thus, we define $\mathcal{CSP}_{l_i}$ to be the subgraph of $\mathcal{CSP}$ consisting of vertices labeled by allowable terms in G_{l_i}. Here we use the fact that the restriction of f to each connected component of an element of the core filtration is a CT.

The problem with proving that $\mathcal{CSP}$ is strongly connected by induction on the core filtration is that $\mathcal{CSP}_{l_i}$ may have multiple connected components. This only happens, however, if G_{l_i} has more than one connected component in which case $\mathcal{CSP}_{l_i}$ will have multiple connected components. If any component of G_{l_i} is a topological circle (necessarily consisting of a single fixed edge E), then $\mathcal{CSP}_{l_i}$ will have two connected components for this circle.

Lemma 1.20. *For every $1 \leq i \leq k$, the number of strongly connected components of $\mathcal{CSP}_{l_i}(f)$ is equal to*

$$2 \cdot \#\left\{\textit{components of } G_{l_i} \textit{that are circles}\right\} + \#\left\{\textit{components of } G_{l_i} \textit{that are not circles}\right\}$$

Proof. Lemma 1.16 establishes the base case when H_1^c is exponentially growing. If H_1^c is a circle, then $\mathcal{CSP}_1$ has exactly two vertices, each with a self loop, so the lemma clearly holds. We now proceed to the inductive step, which is case-by-case analysis based on Lemma 1.19. We set some notation to be used throughout: E will be an edge with initial vertex v and terminal vertex w (it's possible that $v = w$). We denote by $G_{l_i}^v$ the component of G_{l_i} containing v and similarly for w. Let $\mathcal{CSP}_{l_i}^v$ be the component(s) of $\mathcal{CSP}_{l_i}$ containing paths which pass through v. In the case that $G_{l_i}^v$ is a topological circle, there will be two such components.

In case 1a of Lemma 1.19, $\mathcal{CSP}_{l_i}$ is obtained from $\mathcal{CSP}_{l_{i-1}}$ by adding two new vertices: τ_E and $\tau_{\overline{E}}$. Each new vertex has a self loop, and no other new edges are added. So the number of connected components of $\mathcal{CSP}$ increases by two. Each component is strongly connected.

In case 1b, there are several subcases according to the various possibilities for the edge E, and the topological types of $G_{l_{i-1}}^v$ and $G_{l_{i-1}}^w$. First, suppose that E is a fixed edge. Then $\mathcal{CSP}_{l_i}$ is obtained from $\mathcal{CSP}_{l_{i-1}}$ by adding two new vertices. There are no new INPs since the restriction of f to each component of G_{l_i} is a CT and any INP is of the form provided by (NEG Nielsen Paths) or (EG Nielsen Paths). As in Remark 1.15, the vertex τ_E has an incoming edge with initial endpoint τ and an outgoing edge with terminal endpoint τ'. Moreover, $\tau \in \mathcal{CSP}_{l_{i-1}}^v$ and $\tau' \in \mathcal{CSP}_{l_{i-1}}^w$. We then have a directed edge from $\sigma \in \mathcal{CSP}_{l_{i-1}}^w$ to $\tau_{\overline{E}}$ and a directed edge from $\tau_{\overline{E}}$ to $\sigma' \in \mathcal{CSP}_{l_{i-1}}^v$. Hence, there are directed paths in $\mathcal{CSP}_{l_i}$ connecting the two strongly connected subgraphs $\mathcal{CSP}_{l_{i-1}}^v$ and $\mathcal{CSP}_{l_{i-1}}^w$ to each other, and passing through all new vertices. Therefore, there is one strongly connected component of $\mathcal{CSP}_{l_i}$ corresponding to the component of G_{l_i} containing v (and w). This component cannot be a circle, since it contains at least two edges. In the case that $G_{l_{i-1}}^v$ (resp. $G_{l_{i-1}}^w$) is a topological circle, we remark that there are incoming (resp. outgoing) edges in $\mathcal{CSP}_{l_i}^v$ (resp. $\mathcal{CSP}_{l_i}^w$) to τ_E from each of the components of $\mathcal{CSP}_{l_{i-1}}^v$ (resp. $\mathcal{CSP}_{l_{i-1}}^w$). See Figure 1.3.

Suppose now that E is a nonfixed NEG edge. There are two new vertices in $\mathcal{CSP}_{l_i}$ labeled τ_E and $\tau_{\overline{E}}$. The argument given in the previous paragraph goes through once we notice that if $v \neq w$, then $G_{l_{i-1}}^w$ cannot be a circle since this would imply that w is not a principal vertex in G_{l_i} (see first bullet point in the definition) contradicting the fact that $f|_{G_{l_i}}$ is a CT ((Vertices) is not satisfied).

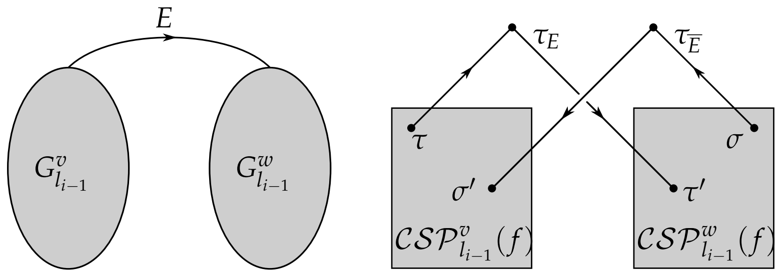

Figure 1.3. A possibility for G_{l_i} and the graph $\mathcal{CSP}_{l_i}$ when $H^c_{l_i}$ is a single NEG edge

If E is a nonlinear edge, then we are done. If E is linear, then there will be other new vertices in $\mathcal{CSP}_{l_i}$. There will be a new vertex for the family of NEG Nielsen paths $Eu^*\overline{E}$. The fact that we have concluded the inductive step for the vertex τ_E along with remark 1.17 shows that this new vertex is in the same strongly connected component as τ_E. There will also be two vertices for each family of exceptional paths $Eu^*\overline{E'}$. For the exact same reasons, these vertices are also in this strongly connected component. This concludes the proof in case 1b of Lemma 1.19.

The arguments given thus far apply directly to case 1c of Lemma 1.19. We remark that in this case, neither of the components of $G_{l_{i-1}}$ containing the terminal endpoints of the new edges can be circles for the same reason as before.

The most complicated way that G (and hence $\mathcal{CSP}$) can change is when $H^c_{l_i}$ contains an EG stratum. In case 2 of Lemma 1.19, if some component of $H^c_{l_i}$ is disjoint from G_{u_i}, then $H^c_{l_i}$ is a component of G_{l_i} and the restriction of f to this component is a fully irreducible. In particular, $\mathcal{CSP}_{l_i}$ has one more strongly connected component than $\mathcal{CSP}_{l_{i-1}}$ by Lemma 1.16.

Though case 2 of Lemma 1.19 describes G_{l_i} as being built from $G_{l_{i-1}}$ in three stages from bottom to top, somehow it is easier to prove $\mathcal{CSP}_{l_i}$ has the correct number of connected components by going from top to bottom.

By looking at a long segment of a leaf of the attracting lamination for H_{l_i}, we can see as in Lemma 1.16 that the vertices in $\mathcal{CSP}_{l_i}$ labeled by edges in the EG stratum H_{l_i} are in at most two different strongly connected components. In fact, we can show that these vertices are all in the same strongly connected component. Since we are working under the assumption that no component of $H^c_{l_i}$ is disjoint from G_{u_i}, we can use one of the components of G_{u_i} to turn around on a leaf of the lamination. Indeed, choose some component G^1 of G_{u_i} which intersects H_{l_i}. Let E be an EG edge in H_{l_i} with terminal vertex $w \in G^1$. Note that if G^1 deformation retracts onto a circle with vertex v, then some EG edge in H_{l_i} must be incident to v, since otherwise $f|_{G_{l_i}}$ would not be a CT. Thus, by replacing E if necessary, we may assume in this situation that w is on the circle. Using the inductive hypothesis and the fact that mixed turns are legal, we can connect the vertex τ_E to the vertex $\tau_{\overline{E}}$ in $\mathcal{CSP}_{l_i}$. Then we can follow a leaf of the lamination going backwards until we return to w, say along E'. If $E = E'$, then the leaves of the lamination were nonorientable in the first place, and all the vertices labeled by edges in H_{l_i} are in the same strongly connected component of $\mathcal{CSP}_{l_i}$. Otherwise, apply the inductive hypothesis again and use the fact that mixed turns are legal to get a path from $\tau_{E'}$ to $\tau_{\overline{E}'}$. This shows all vertices labeled by edges in H_{l_i} are in the same strongly connected component of $\mathcal{CSP}_{l_i}$. We will henceforth denote the strongly connected component of $\mathcal{CSP}_{l_i}$ which contains all these vertices by $\mathcal{CSP}^{EG}_{l_i}$.

If there is an INP σ of height H_{l_i}, its first and last edges are necessarily in H_{l_i}. Remark 1.17 then implies that τ_σ and $\tau_{\overline{\sigma}}$ are in $\mathcal{CSP}^{EG}_{l_i}$. Recall that the only allowable terms in complete splittings which intersect zero strata are connecting paths which are both maximal and *taken*. In particular, each vertex in $\mathcal{CSP}_{l_i}$ corresponding to such a connecting path is in the aforementioned strongly connected component, $\mathcal{CSP}^{EG}_{l_i}$.

Now let E be an NEG edge in $H^c_{l_i}$ with terminal vertex w. There is necessarily an outgoing edge from τ_E into $\mathcal{CSP}^w_{l_{i-1}}$ and an incoming edge to τ_E from $\mathcal{CSP}^{EG}_{l_i}$. If the graph $G^w_{l_{i-1}}$ is not a topological circle, then the corresponding component $\mathcal{CSP}^w_{l_{i-1}}$ is already strongly connected and there is a directed edge from this graph back to $\tau_{\overline{E}}$ and from there back into $\mathcal{CSP}^{EG}_{l_i}$. Thus, this subgraph is contained in the strongly connected component $\mathcal{CSP}^{EG}_{l_i}$. On the other hand, if $G^w_{l_{i-1}}$ is a topological circle, then there is a directed edge from τ_E back into $\mathcal{CSP}^{EG}_{l_i}$ because mixed turns are legal, and as before, some edge in H_{l_i} must be incident to w. Thus all the vertices in $\mathcal{CSP}_{l_i}$ labeled by NEG edges are in the strongly connected component $\mathcal{CSP}^{EG}_{l_i}$, as are all vertices in $\mathcal{CSP}^w_{l_{i-1}}$ for w as above.

The same argument and the inductive hypothesis shows that for any component of $G_{l_{i-1}}$ which intersects H_{l_i}, the corresponding strongly connected component(s) of $\mathcal{CSP}_{l_{i-1}}$ are also in $\mathcal{CSP}^{EG}_{l_i}$. The only thing remaining is to deal with NEG Nielsen paths and families of exceptional paths. Both of these are handled by Remark 1.17 and the fact that we have already established that $\mathcal{CSP}_{l_i}^{EG}$ contains all vertices of the form τ_E or $\tau_{\overline{E}}$ for NEG edges in $H_{l_i}^c$. We have shown that every vertex of a strongly connected component of $\mathcal{CSP}_{l_{i-1}}$ coming from a component of $G_{l_{i-1}}$ which intersects $H_{l_i}^c$ is in the strongly connected component $\mathcal{CSP}_{l_i}^{EG}$. In particular, there is only one strongly connected component of $\mathcal{CSP}_{l_i}$ for the component of G_{l_i} which contains edges in $H_{l_i}^c$. This completes the proof of the proposition. □

In the proof of Theorem 1.24, we will need to consider a weakening of the complete splitting of paths and circuits. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element. Given a CT $f\colon G \to G$, we define the graph $\mathcal{CSP}^{QE}(f)$ by adding two vertices to $\mathcal{CSP}(f)$ for each QE-family (one for $E_i u^* \overline{E}_j$ and one for $E_j u^* \overline{E}_i$). For every vertex τ_σ with a directed edge terminating at τ_{E_i} add an edge from τ_σ to $\tau_{E_i u^* \overline{E}_j}$ and similarly for every edge emanating from $\tau_{\overline{E}_j}$, add an edge to the same vertex beginning at $\tau_{E_i u^* \overline{E}_j}$. Do the same for the vertex $\tau_{E_j u^* \overline{E}_i}$. As before, every completely split path σ gives rise to a directed edge path in $\mathcal{CSP}^{QE}$ corresponding to its QE-splitting. It follows immediately from the definition and Proposition 1.13 that

Corollary 1.21. *There is a completely split circuit σ containing every allowable term in its QE-splitting.*

We are now ready to prove our main result in the polynomial case.

1.6.2 Polynomial subgroups are undistorted

In this subsection, we will complete the proof of our main result in the polynomial case. We first recall the height function defined by Alibegović in [Ali02]. Given two conjugacy classes $[u], [w]$ of elements of F_n, define the *twisting of* $[w]$ *about* $[u]$ as

$$\mathrm{tw}_u(w) = \max\{k \mid w = au^k b \text{ where } u, w \text{ are a cyclically reduced conjugates of } [u], [w]\}$$

Then define the twisting of $[w]$ by $\mathrm{tw}(w) = \max\{\mathrm{tw}_u(w) \mid u \in F_n\}$. Alibegović proved the following lemma using bounded cancellation, which we restate for convenience. A critical point is that D_2 is independent of w.

Lemma 1.22 ([Ali02, Lemma 2.4]). *There is a constant D_2 such that $tw(s(w)) \leq tw(w) + D_2$ for all conjugacy classes w and all $s \in S$, our symmetric finite generating set of $Out(F_n)$.*

Since we typically work with train tracks, we have a similar notion of twisting adapted to that setting. Let τ be a path or circuit in a graph G and let σ be a circuit in G. Define the *twisting of τ about σ* as

$$\mathrm{tw}_\sigma(\tau) = \max\{k \mid \tau = \alpha\sigma^k\beta \text{ where the path } \alpha\sigma^k\beta \text{ is immersed}\}$$

Then define $\mathrm{tw}(\tau) = \max\{\mathrm{tw}_\sigma(\tau) \mid \sigma \text{ is a circuit}\}$. The bounded cancellation lemma of [Coo87] directly implies

Lemma 1.23. *If $\rho \colon R_n \to G$ and $[w]$ is a conjugacy class in $F_n = \pi_1(R_n)$, then $tw(\rho(w)) \geq tw(w) - 2C_\rho$.*

We are now ready to prove nondistortion for polynomial abelian subgroups. Recall the map $\Omega \colon H \to \mathbb{Z}^{N+K}$ was defined by taking the product of comparison and expansion factor homomorphisms. In the following theorem, we will denote the restriction of this map to the last K coordinates (those corresponding to comparison homomorphisms) by Ω_{comp}.

Theorem 1.24. *Let H be a rotationless abelian subgroup of $Out(F_n)$ and assume that the map from H into the collection of comparison factor homomorphisms $\Omega_{comp} \colon H \to \mathbb{Z}^K$ is injective. Then H is undistorted.*

Proof. The first step is to note that it suffices to prove the generic elements of H are uniformly undistorted. This is just because the set of nongeneric elements of H is a finite collection of hyperplanes, so there is a uniform bound on the distance from a point in one of these hyperplanes to a generic point.

We set up some constants now for later use. This is just to emphasize that they depend only on the subgroup we are given and the data we have been handed thus far. Let $\mathcal{G}$ be the finite set of marked graphs provided by Proposition 1.12 and define K_2 as the maximum of $BCC(\rho_G)$ and $BCC(\rho_G^{-1})$ as G varies over the finitely many marked graphs in $\mathcal{G}$. Lemma 1.23 then implies that $\mathrm{tw}(\rho(w)) \geq \mathrm{tw}(w) - K_2$ for any conjugacy class w and any of the finitely many marked graphs in $\mathcal{G}$. Let D_2 be the constant from Lemma 1.22.

Fix a minimal generating set $\phi_1, \ldots, \phi_k$ for H and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k}$ be generic in H. Let $f\colon G \to G$ be a CT representing ψ with G chosen from $\mathcal{G}$ and let ω be the comparison homomorphism for which $\omega(\psi)$ is the largest. The key point is that given ψ, Corollary 1.21 will provide a split circuit σ for which the twisting will grow by $|\omega(\psi)|$ under application of the map f.

Indeed, let σ be the circuit provided by Corollary 1.21. As we discussed in section 1.5.1, there is a correspondence between the comparison homomorphisms for H and the set of linear edges and quasi-exceptional families in G. Assume first that ω corresponds to the linear edge E with axis u, so that by definition $f(E) = E \cdot u^{\omega(\psi)}$. Since the splitting of $f_\#(\sigma)$ refines that of σ and E is a term in the complete splitting of σ, $f_\#(\sigma)$ not only contains the path $E \cdot u^{\omega(\psi)}$, but in fact splits at the ends of this subpath. Under iteration, we see that $f_\#^t(\sigma)$ contains the path $E \cdot u^{t\omega(\psi)}$, and therefore $\mathrm{tw}(f_\#^t(\sigma)) \geq t|\omega(\psi)|$. This isn't quite good enough for our purposes, so we will argue further to conclude that for some t_0,

$$\mathrm{tw}(f_\#^{t_0}(\sigma)) - \mathrm{tw}(f_\#^{t_0-1}(\sigma)) \geq |\omega(\psi)| \tag{1.1}$$

Suppose for a contradiction that no such t exists. Then for every t, we have $\mathrm{tw}(f_\#^t(\sigma)) - \mathrm{tw}(f_\#^{t-1}(\sigma)) \leq |\omega(\psi)| - 1$. Using a telescoping sum and repeatedly applying this assumption, we obtain $\mathrm{tw}(f_\#^t(\sigma)) - \mathrm{tw}(\sigma) \leq t|\omega(\psi)| - t$. Combining and rearranging inequalities, this implies

$$\mathrm{tw}(\sigma) \geq \mathrm{tw}(f_\#^t(\sigma)) + t - t|\omega(\psi)| \geq t|\omega(\psi)| + t - t|\omega(\psi)| = t$$

for all t, a contradiction. This establishes the existence of t_0 satisfying equation (1.1).

The above argument works without modification in the case that ω corresponds to a family of quasi-exceptional paths. We now address the minor adjustment needed in the case that ω corresponds to a family of exceptional paths, $E_i u^* \overline{E}_j$. Let $f(E_i) = E_i u^{d_i}$ and $f(E_j) = E_j u^{d_j}$. Since σ contains both $E_i u^* \overline{E}_j$ and $E_j u^* \overline{E}_i$ in its complete splitting, we may assume without loss that $d_i > d_j$. The only problem is that the exponent of u in the term $E_i u^* \overline{E}_j$ occuring in the complete splitting of σ may be negative, so that $\mathrm{tw}(f_\#^t(\sigma))$ may be less than $t|\omega(\psi)|$. In this case, just replace σ by a sufficiently high iterate so that the exponent is positive.

Now write ψ in terms of the generators $\psi = s_1 s_2 \cdots s_p$ so that for any conjugacy class w, by repeatedly applying Lemma 1.22 we obtain

$$\mathrm{tw}(s_1(s_2 \cdots s_p(w))) \leq \mathrm{tw}(s_2(s_3 \cdots s_p(w))) + D_2 \leq \ldots \leq \mathrm{tw}(w) + pD_2$$

so that, $D_2|\psi|_{\mathrm{Out}(F_n)} \geq \mathrm{tw}(\psi(w)) - \mathrm{tw}(w)$. Applying this inequality to the circuit $f_\#^{t_0-1}(\sigma)$ just constructed, and letting w be the conjugacy class $\rho^{-1}(f_\#^{t_0-1}(\sigma))$, we have

$$\begin{aligned}
|\psi|_{\mathrm{Out}(F_n)} &\geq \frac{1}{D_2}\left[\mathrm{tw}(\psi(w)) - \mathrm{tw}(w)\right] \\
&\geq \frac{1}{D_2}\left[\mathrm{tw}(f_\#^{t_0}(\sigma)) - \mathrm{tw}(f_\#^{t_0-1}(\sigma))\right] - \frac{2K_2}{D_2} \\
&\geq \frac{1}{D_2}|\omega(\psi)| - \frac{2K_2}{D_2}
\end{aligned}$$

The second inequality is justified by Lemma 1.23 and the third uses the property of σ established in (1.1) above. Since ω was chosen to be largest coordinate of $\Omega_{comp}(\psi)$ and Ω_{comp} is injective, the proof is complete. □

1.7 The mixed case

There are no additional difficulties with the mixed case since both the distance function on $\mathbb{P}\mathcal{O}_n$ and Alibegović's twisting function are well suited for dealing with outer automorphisms whose growth is neither purely exponential nor purely polynomial. Consequently, for an element ψ of an abelian subgroup H, if the image of ψ is large under PF_H then we can use $\mathbb{P}\mathcal{O}_n$ to show that $|\psi|_{\mathrm{Out}(F_n)}$ is large, and if the image is large under Ω_{comp} then we can use the methods from §1.6 to show $|\psi|_{\mathrm{Out}(F_n)}$ is large. The injectivity of Ω [FH09, Lemma 4.6] exactly says that if $|\psi|_H$ is large, then at least one of the aforementioned quantities must be large as well.

Theorem 1.25. *Abelian subgroups of $Out(F_n)$ are undistorted.*

Proof. Assume, by passing to a finite index subgroup, that H is rotationless. By [FH09, Lemma 4.6], the map $\Omega\colon H \to \mathbb{Z}^{N+K}$ is injective. Choose a minimal generating set for H and write $H = \langle \phi_1, \ldots, \phi_k \rangle$. The restriction of Ω to the first N coordinates is precisely the map PF_H from section 1.4. Choose k coordinates of Ω so that the restriction Ω_π to those coordinates is injective. Let $PF_{\Lambda_1}, \ldots, PF_{\Lambda_l}$ be the subset of the chosen coordinates corresponding to expansion factor homomorphisms. Pass to a finite index subgroup of H

and choose generators so that $\Omega_\pi(\phi_i) = (0, \ldots, PF_{\Lambda_i}(\phi_i), \ldots, 0)$ for $1 \leq i \leq l$. Now we proceed as in the proofs of Theorems 1.10 and 1.24.

Fix a basepoint $* \in \mathbb{P}\mathcal{O}_n$ and let $\psi = \phi_1^{p_1} \cdots \phi_k^{p_k}$ in H. We may assume without loss that ψ is generic in H (again, it suffices to prove that generic elements are uniformly undistorted). Replace the ϕ_i's by their inverses if necessary to ensure that all p_i's are nonnegative. Then, for each of the first l coordinates of Ω_π, replace Λ_i by its paired lamination if necessary (Lemma 1.9) to ensure that $PF_{\Lambda_i}(\psi) > 0$. Look at the coordinates of $\Omega_\pi(\psi)$ and pick out the one with the largest absolute value. We first consider the case where the largest coordinate corresponds to an expansion factor homomorphism PF_{Λ_j}. We have already arranged that $PF_{\Lambda_j}(\psi) > 0$.

By Theorem 1.8, the translation distance of ψ is the maximum of the Perron-Frobenius eigenvalues associated to the EG strata of a relative train track representative f of ψ. Since ψ is generic and the first l coordinates of Ω_π are nonnegative, $\{\Lambda_1, \ldots, \Lambda_l\} \subset \mathcal{L}(\psi)$. Each Λ_i is associated to an EG stratum of f. For such a stratum, the logarithm of the PF eigenvalue is $PF_{\Lambda_i}(\psi)$. Just as in the proof of Theorem 1.10, for each $1 \leq i \leq l$, we have that $PF_{\Lambda_i}(\psi) = p_i PF_{\Lambda_i}(\phi_i)$. So the translation distance of ψ acting on Outer Space is

$$\tau(\psi) \geq \max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq l\}$$

The inequality is because there may be other laminations in $\mathcal{L}(\psi)$. Just as in Theorem 1.10, we have

$$d(*, * \cdot \psi) \leq D_1 |\psi|_{\mathrm{Out}(F_n)}$$

where $D_1 = \max_{s \in S} d(*, * \cdot s)$. Let $K_1 = \min\{PF_{\Lambda_i^{\pm}}(\phi_j^{\pm}) \mid 1 \leq i \leq l, 1 \leq j \leq k\}$. Then we have

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_1} d(*, * \cdot \psi) \geq \frac{1}{D_1}\tau(\psi) \geq \frac{1}{D_1}\max\{p_i PF_{\Lambda_i}(\phi_i) \mid 1 \leq i \leq k\} \geq \frac{K_1}{D_1}\max\{p_i\}$$

We now handle the case where the largest coordinate of $\Omega_\pi(\psi)$ corresponds to a comparison homomorphism ω. Let $\mathcal{G}$ be the finite set of marked graphs provided by Proposition 1.12 and let $f \colon G \to G$ be a CT for ψ where $G \in \mathcal{G}$. Define K_2 exactly as in the proof of Theorem 1.24 so that $\mathrm{tw}(\rho(w)) \geq \mathrm{tw}(w) - K_2$ for all conjugacy classes w and any marking or inverse marking of the finitely many marked graphs in $\mathcal{G}$. The construction of the completely split circuit σ satisfying equation (1.1) given in the polynomial case works without modification in our current setting, where the comparison homomorphism ω in equation (1.1) is the coordinate of Ω_π which is largest in absolute value.

Using this circuit and defining $w = \rho^{-1}(f_\#^{t_0-1}\sigma)$, the inequalities and their justifications in the proof of Theorem 1.24 now apply verbatim to the present setting to conclude

$$|\psi|_{\mathrm{Out}(F_n)} \geq \frac{1}{D_2}\max\{|\omega(\psi)| \mid \omega \in \Omega_\pi\}| - \frac{2K_2}{D_2}$$

We have thus shown that the image of H under Ω_π undistorted. Since Ω_π is injective, it is a quasi-isometric embedding of H into $\mathbb{Z}^k$, so the theorem is proved. □

We conclude by proving the rank conjecture for $\mathrm{Out}(F_n)$. The maximal rank of an abelian subgroup of $\mathrm{Out}(F_n)$ is $2n-3$, so Theorem 1.25 gives a lower bound for the geometric rank of $\mathrm{Out}(F_n)$: $\mathrm{rankOut}(F_n) \leq 2n-3$. The other inequality follows directly from the following result, whose proof we sketch below.

Theorem 1.26. *If G has virtual cohomological dimension $k \geq 3$, then $\mathrm{rank}(G) \leq k$.*

The virtual cohomological dimension of $\mathrm{Out}(F_n)$ is $2n-3$ [CV86]. Thus, for $n \geq 3$, we have:

Corollary 1.27. *The geometric rank of $Out(F_n)$ is $2n-3$, which is the maximal rank of an abelian subgroup of $Out(F_n)$.*

Proof of 1.26. Let $G' \leq G$ be a finite index subgroup whose cohomological dimension is k. Since G is quasi-isometric to its finite index subgroups, we have $\mathrm{rank}(G') = \mathrm{rank}(G)$. A well known theorem of Eilenberg-Ganea [EG57] provides the existence of a $k-$dimensional CW complex X which is a $K(G', 1)$. By Švarc-Milnor, it suffices to show that there can be no quasi-isometric embedding of $\mathbb{R}^{k+1}$ into the universal cover $\tilde{X}$. Suppose for a contradiction that $f \colon \mathbb{R}^{k+1} \to \tilde{X}$ is such a map. The first step is to replace f by a continuous quasi-isometry f' which is a bounded distance from f. This is done using the "connect-the-dots argument" whose proof is sketched in [SW02]. The key point is that $\tilde{X}$ is *uniformly contractible*. That is, for every r, there is an $s = s(r)$, such that any continuous map of a finite simplicial complex into X whose image is contained in an r-ball is contractible in an $s(r)$-ball.

It is a standard fact [Hat02, Theorem 2C.5] that X may be replaced with a simplicial complex of the same dimension so that $\tilde{X}$ may be assumed to be simplicial. We now construct a cover $\mathcal{U}$ of the simplicial complex $\tilde{X}$ whose nerve is equal to the barycentric subdivision of $\tilde{X}$. The cover $\mathcal{U}$ has one element for each cell of $\tilde{X}$. For each vertex v, the set $U_v \in \mathcal{U}$ is a small neighborhood of v. For each i-cell, σ, Define U_σ by taking a sufficiently small neighborhood of $\sigma \setminus \bigcup_{\sigma' \in \tilde{X}^{(i-1)}} U_{\sigma'}$ to ensure that $U_\sigma \cap \tilde{X}^{(i-1)} = \emptyset$. The key property of $\mathcal{U}$ is that all $(k+2)-$fold intersections are necessarily empty because the dimension of the barycentric subdivision of $\tilde{X}$ is equal to $\dim(\tilde{X})$.

Since we have arranged f to be continuous, we can pull back the cover just constructed to obtain a cover $\mathcal{V} = \{f^{-1}(U)\}_{U \in \mathcal{U}}$ of $\mathbb{R}^{k+1}$. Since the elements of $\mathcal{U}$ are bounded, and f is a quasi-isometric embedding, the elements of $\mathcal{V}$ are bounded as well. The intersection pattern of the elements of $\mathcal{V}$ is exactly the same as the intersection pattern of elements of $\mathcal{U}$. But the cover $\mathcal{U}$ was constructed so that any intersection of $(k+2)$ elements is necessarily empty. Thus, we have constructed a cover of $\mathbb{R}^{k+1}$ by bounded sets with no $(k+2)-$fold intersections. We will contradict the fact that the Lebesgue covering dimension of any compact subset of $\mathbb{R}^{k+1}$ is $k+1$. Let K be compact in $\mathbb{R}^{k+1}$ and let $\mathcal{V}'$ be an arbitrary cover of K. Let δ be the constant provided by the Lebesgue covering Lemma applied to $\mathcal{V}'$. Since the elements of $\mathcal{V}$ are uniformly bounded, we can scale them by a single constant to obtain a cover of K whose sets have diameter $< \delta/3$. Such a cover is necessarily a refinement of $\mathcal{V}'$, but has multiplicity $k+1$. This contradicts the fact that K has covering dimension $k+1$ so the theorem is proved. $\square$

CHAPTER 2

LOXODROMIC ELEMENTS FOR THE CYCLIC SPLITTING COMPLEX

In this chapter, we identify those outer automorphisms that act loxodromically on the cyclic splitting complex. We begin with an introduction to provide some context for this result and an outline of what follows.

2.1 Introduction

The study of the mapping class group of a closed orientable surface S has benefited greatly from its action on the curve complex, $\mathcal{C}(S)$, which was shown to be hyperbolic in [MM99]. Curve complexes have been used for bounded cohomology of subgroups of mapping class groups, rigidity results, and myriad other applications.

The outer automorphism group of a finite rank free group F_n, denoted by $\mathrm{Out}(F_n)$, is defined as the quotient of $\mathrm{Aut}(F_n)$ by the inner automorphisms, those which arise from conjugation by a fixed element. Much of the study of $\mathrm{Out}(F_n)$ draws parallels with the study of mapping class groups. This analogy, however, is far from perfect; there are several $\mathrm{Out}(F_n)$-complexes that act as analogs for the curve complex. Among them are the free splitting complex $\mathcal{FS}_n$, the cyclic splitting complex $\mathcal{FZ}_n$, and the free factor complex $\mathcal{FF}_n$, all of which have been shown to be hyperbolic [HM13b, Man14, BF14]. Just as curve complexes have yielded useful information about mapping class groups, so too have these complexes furthered our understanding of $\mathrm{Out}(F_n)$.

The three hyperbolic $\mathrm{Out}(F_n)$-complexes mentioned above are related via Lipschitz maps, $\mathcal{FS}_n \to \mathcal{FZ}_n \to \mathcal{FF}_n$. The loxodromics for $\mathcal{FF}_n$ have been identified with the set of fully irreducible outer automorphisms [BF14]. In [HM14], the authors proved that an outer automorphism, ϕ, acts loxodromically on $\mathcal{FS}_n$ precisely when ϕ has a *filling lamination*, that is, some element of the finite set of laminations associated to ϕ (see [BFH00]) is not carried by a vertex group of any free splitting. In this paper, we focus our attention on the isometry type of outer automorphisms, considered as elements of $\mathrm{Isom}(\mathcal{FZ}_n)$.

The cyclic splitting complex $\mathcal{FZ}_n$, introduced in [Man14], is defined as follows: vertices are one-edge splittings of F_n with edge stabilizer either trivial or $\mathbb{Z}$ and k-simplicies correspond to a collections of $k+1$ vertices, each of which is compatible with a k-edge $\mathcal{Z}$-splitting. In this chapter, we determine precisely which outer automorphisms act loxodromically on $\mathcal{FZ}_n$.

In [BFH00], the authors associate to each $\phi \in \mathrm{Out}(F_n)$ a finite set of attracting laminations, denoted by $\mathcal{L}(\phi)$. We say that a lamination $\Lambda \in \mathcal{L}(\phi)$ is *$\mathcal{Z}$-filling* if no generic leaf of Λ is carried by a vertex group of a one-edge $\mathcal{Z}$-splitting; we say that ϕ has a $\mathcal{Z}$-filling lamination if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}$-filling. We prove

Theorem 2.1. *An outer automorphism, ϕ, acts loxodromically on the cyclic splitting complex if and only if it has a $\mathcal{Z}$-filling lamination. Furthermore, if ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then a power of ϕ fixes a point in $\mathcal{FZ}$.*

In [HW15], Horbez and Wade showed that every isometry of $\mathcal{FZ}_n$ is induced by an outer automorphism. Combining their result with [HM14, Theorem 1.1] and Theorem 2.1, this amounts to a classification of the isometries of $\mathcal{FZ}_n$.

Corollary 2.2 (Classification of isometries of $\mathcal{FZ}_n$)**.** *The following hold for all $\phi \in Isom(\mathcal{FZ}_n)$.*

1. *The action of ϕ on $\mathcal{FZ}_n$ is loxodromic if and only if some element of $\mathcal{L}(\phi)$ is $\mathcal{Z}$-filling.*

2. *If the action of ϕ on $\mathcal{FZ}_n$ is not loxodromic, then it has bounded orbits (there are no parabolic isometries).*

The proof of Theorem 2.1 relies on the description of the boundary of $\mathcal{FZ}_n$ due to Horbez [Hor14]; points in the boundary of $\mathcal{FZ}_n$ are equivalence classes of $\mathcal{Z}$-averse trees. The proof is carried out as follows. In Section 2.3, we extend the theory of folding paths to the boundary of Culler & Vogtmann's outer space, $\mathbb{P}\mathcal{O}_n$, defining a folding path guided by ϕ which is entirely contained in $\partial\mathbb{P}\mathcal{O}_n$. In Section 2.4, we show that the limit of the folding path thus constructed is $\mathcal{Z}$-averse. In Section 2.5, we show that an outer automorphism with a filling but not $\mathcal{Z}$-filling lamination fixes (up to taking a power) a point in $\mathcal{FZ}_n$ and conclude with a proof of Theorem 2.1.

2.2 More preliminaries

In this section, we review the background material that is necessary for what follows, but which was not introduced in Chapter 1.

2.2.1 Isometries of metric spaces

Let X be a Gromov hyperbolic metric space. We say that an infinite order isometry g of X is *loxodromic* if it acts with positive translation length on X: $\lim_{N\to\infty} \frac{d(x,g^N(x))}{N} > 0$ for some (any) $x \in X$. Every loxodromic element has exactly two limit points in the Gromov boundary of X.

2.2.2 The compactification of outer space

An F_n-tree is an $\mathbb{R}$-tree with an isometric action of F_n. An F_n-tree is called *very small* if the action is minimal, arc stabilizers are either trivial or maximal cyclic, and tripod stabilizers are trivial. Outer space can be mapped into $\mathbb{R}^{F_n}$ by the map $T \mapsto (\|g\|_T)_{g\in F_n}$, where $\|g\|_T$ denotes the translation length of g in T. This was shown in [CM87] to be a continuous injection. The closure of $\mathbb{P}\mathcal{O}_n$ under the embedding into $\mathbb{P}\mathbb{R}^{F_n}$ is compact and was identified in [BF94] and [CL95] with the space of all very small F_n-trees. We denote by $\overline{\mathbb{P}\mathcal{O}_n}$ the closure of outer space and by $\partial\mathbb{P}\mathcal{O}_n$ its boundary.

2.2.3 Free factor systems

A free factor system of F_n is a finite collection of conjugacy classes of proper free factors of F_n of the form $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$, where $k \geq 0$ and $[\cdot]$ denotes the conjugacy class of a subgroup, such that there exists a free factorization $F_n = A_1 * \cdots * A_k * F_N$. We refer to the free factor F_N as the *cofactor* of $\mathcal{A}$ keeping in mind that it is not unique, even up to conjugacy.

The main geometric example of a free factor system is as follows: suppose G is a marked graph and K is a subgraph whose noncontractible connected components are denoted $C_1, \ldots, C_k$. Let $[A_i]$ be the conjugacy class of a free factor of F_n determined by $\pi_1(C_i)$. Then $\mathcal{A} = \{[A_1], \ldots, [A_k]\}$ is a free factor system. We say $\mathcal{A}$ is *realized by* K and we denote it by $\mathcal{F}(K)$.

2.2.4 Optimal morphisms and train track maps

Given two F_n-trees Γ and Γ', an F_n-equivariant map $f\colon \Gamma \to \Gamma'$ is called a *morphism* if every segment of Γ can be subdivided into finitely many subintervals such that f is an isometry when restricted to each subinterval. Just as with graphs, a morphism between F_n-trees induces a train track structure on the domain, Γ. A morphism is called *optimal* if there are at least two gates at each point of Γ.

A morphism is called a *train track map* if f is an embedding on each edge and legal turns are sent to legal turns. When Γ is a graph, train track maps are defined by passing to the universal cover. For more details on train track maps, the reader is referred to [BF10, BH92]. The reader should not that this definition does not conflict with the definition of a train track map given in Chapter 1.

2.2.5 A little more on CTs

We now restate the existence theorem for CTs more precisely, as we will need this stronger version in what follows.

Theorem 2.3 ([FH11, Theorem 4.28]). *Given a rotationless $\phi \in Out(F_n)$ and a nested sequence of ϕ-invariant free factor systems, there is a CT representing ϕ such that each of the free factor systems is realized by some filtration element.*

2.2.6 Subspaces of lines

A finitely generated subgroup A of F_n determines a subset of the boundary of F_n called $\partial A \subset \partial F_n$. We say that A *carries* the lamination Λ if there is some lift $\widetilde{\beta}$ of a generic leaf of Λ whose endpoints are in ∂A.

2.2.7 Bounded backtracking

Let $f\colon T \to T'$ be a continuous map between two $\mathbb{R}$-trees T and T'. We say that f has bounded backtracking if the f image of any path $[p, q]$ is contained in a C-neighborhood of $[f(p), f(q)]$. The smallest such C is called the *bounded backtracking constant* of f, and is denoted $\mathrm{BBT}(f)$.

2.2.8 Folding paths

Let T and T' be two simplicial F_n-trees in $\overline{\mathcal{O}_n}$ such that the set of point stabilizers of T and T' are the same. In [GL07b, Section 3], Guirardel and Levitt construct a *canonical optimal folding path* $(T_t)_{t\in\mathbb{R}^+}$ guided by an optimal morphism $f\colon T \to T'$. The tree T_t is constructed as follows. Given $a, b \in T$ with $f(a) = f(b)$, the *identification time* of a and b is defined as $\tau(a,b) = \sup_{x\in[a,b]} d_{T'}(f(x), f(a))$. Define $L := \frac{1}{2}\mathrm{BBT}(f)$. For each $t \in [0, L]$, one defines an equivalence relation $\sim_t$ by $a \sim_t b$ if $f(a) = f(b)$ and $\tau(a,b) < t$. The tree T_t is then a quotient of T by the equivalence relation $\sim_t$. The authors prove that for each $t \in [0, L]$, T_t is an $\mathbb{R}$-tree. The collection of trees $(T_t)_{t\in[0,L]}$ comes equipped with F_n-equivariant morphisms $f_{s,t}\colon T_t \to T_s$ for all $t < s$ and these maps satisfy the semi-flow property: for all $r < s < t$, we have $f_{t,s} \circ f_{s,r} = f_{t,r}$. Moreover $T_L = T'$ and $f_{L,0} = f$. The set of data $(T_t)_{t\in[0,L]}$, $(f_{s,t}\colon T_t \to T_s)_{t<s\in[0,L]}$ is called the *connection data*.

2.2.9 Transverse families and transverse coverings

A subtree Y of a tree T is called *closed* [Gui04, Definition 2.4] if $Y \cap \sigma$ is either empty or a path in T for all paths $\sigma \subset T$; recall that paths are defined on closed intervals. A *transverse family* [Gui04, Definition 4.6] of an $\mathbb{R}$-tree T is a family $\mathcal{Y}$ of nondegenerate closed subtrees of T such that any two distinct subtrees in $\mathcal{Y}$ intersect in at most one point. If every path in T intersects only finitely many subtrees in $\mathcal{Y}$, then the transverse family is called a *transverse covering*.

2.2.10 Mixing and indecomposable trees

A tree $T \in \overline{\mathbb{P}\mathcal{O}_n}$ is *mixing* if for all finite subarcs $I, J \subset T$, there exist $g_1, \dots, g_k \in F_n$ such that $J \subseteq g_1 I \cup g_2 I \cup \cdots \cup g_k I$ and $g_i I \cap g_{i+1} I \neq \emptyset$ for all $i \in \{1, \dots, k-1\}$. A tree $T \in \overline{\mathbb{P}\mathcal{O}_n}$ is called *indecomposable* [Gui08] if it is mixing and $g_i \cap g_{i+1} I$ is a nondegenerate arc for each $i \in \{1, \dots, k-1\}$. An F_n-tree is indecomposable if and only if it has no transverse family containing a proper subtree.

2.2.11 Cyclic splitting complex and $\mathcal{Z}$-averse trees

Let $\mathcal{Z}$ be the set of cyclic subgroups of F_n. A *$\mathcal{Z}$-splitting* is a minimal, simplicial F_n-tree whose edge stabilizers belong to the set $\mathcal{Z}$; it is a *one-edge splitting* if there is one F_n orbit of edges. A *cyclic splitting* is a one-edge $\mathcal{Z}$-splitting whose edge stabilizer is infinite cyclic.

Two $\mathcal{Z}$-splittings are *equivalent* if they are F_n-equivariantly homeomorphic. Given two $\mathcal{Z}$-splittings T and T', T is a *refinement* of T' if there is a collapse map from T to T', that is, T' is obtained from T by equivariantly collapsing a set of edges of T. Two $\mathcal{Z}$-splittings are *compatible* if there is a common refinement. A tree T is *$\mathcal{Z}$-incompatible* if the set of $\mathcal{Z}$-splittings that are compatible with T is empty. The cyclic splitting complex $\mathcal{FZ}_n$ is the simplicial complex whose vertices are equivalence classes of one-edge $\mathcal{Z}$-splittings and whose k-simplicies are collections of $k+1$ pairwise compatible one-edge $\mathcal{Z}$-splittings. In [Man14], Mann showed that $\mathcal{FZ}_n$ is a δ-hyperbolic space.

In [Hor14], Horbez, characterized the boundary of the cyclic splitting complex as the set of $\mathcal{Z}$*-averse* trees. A tree in $\overline{\mathbb{P}\mathcal{O}_n}$ is called $\mathcal{Z}$-averse if it is not compatible with any F_n-tree in $\mathbb{P}\mathcal{O}_n$ that is itself compatible with a $\mathcal{Z}$-splitting. Let $\mathcal{X}(F_n)$ denote the set of $\mathcal{Z}$-averse trees.

2.3 Folding in the boundary of outer space

Throughout this section, ϕ will be an outer automorphism with a $\mathcal{Z}$-filling lamination Λ^+_ϕ. Our first goal is to extract from ϕ a folding path converging to a tree in $\partial\mathbb{P}\mathcal{O}_n$ which "witnesses" the lamination Λ^+_ϕ. As the assumption on ϕ implies that it is fully irreducible relative to some free factor system $\mathcal{A}$, we let $f\colon T \to T$ be the universal cover of a relative train track representative of ϕ realizing the invariant free factor system $\mathcal{A}$. Let $G = T/F_n$ be the quotient graph, which comes with a filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_r = G$ such that $\mathcal{F}(G_{r-1}) = \mathcal{A}$ and H_r is an EG stratum with Perron-Frobenius eigenvalue λ_ϕ. Let T_r (resp. T_{r-1}) denote the full preimage of H_r (resp. G_{r-1}) under the quotient map $T \to G$. We will henceforth consider T as a point in unprojectivized outer space $\mathcal{O}_n$, whereby f may be thought of as an F_n-equivariant map $T \to T\cdot\phi$.

Let T_0' be the tree obtained from T by equivariantly collapsing the $\mathcal{A}$-minimal subtree. Our present aim is to construct a folding path ending at $T^+_\phi := \lim_{n\to\infty} T_0'\phi^n/\lambda^n_\phi$. To accomplish this, we will construct simplicial trees T_0, T_1 and define an optimal morphism $f_0\colon T_0 \to T_1$. From this we will obtain a periodic canonical optimal folding path $(f_t)_{t\in[0,L]}$ which will end at T^+_ϕ. It is worth noting that the natural map $f_0'\colon T_0' \to T_0'\phi$ induced by f is neither optimal nor a morphism as there may be nondegenerate intervals which are mapped to points.

2.3.1 Constructing T_0

The following is based on the construction in the proof of [BH92, Lemma 5.10]. Define a measure μ on T with support contained in the set $\{x \in T_r : f^k(x) \in T_r \text{ for all } k \geq 0\}$ as follows: choose a Perron Frobenius (PF) eigenvector $\vec{v}$ corresponding to the PF eigenvalue λ_ϕ. For an edge e in T_r, let $\mu(e) = v_e$ where v_e is the component of $\vec{v}$ corresponding to e. Define $\mu(e) = 0$ for all edges $e \in T_{r-1}$. Let V be the set of vertices of T and let $V_m := \{x \in T : f^m(x) \in V\}$. Subdividing T at V_m divides each edge into segments that map to edge paths under f^m. If a is such a segment then define $\mu(a) = \mu(f^m(a))/\lambda_\phi^m$. The definition of μ together with the fact that relative train track maps take r-legal paths to r-legal paths implies:

Lemma 2.4. *If $[x,y]$ is an r-legal path in T, then $\mu(f_\#([x,y])) = \lambda_\phi \mu([x,y])$. If $[x,y]$ contains an initial or terminal segment of some edge in T_r, then $\mu([x,y]) > 0$.*

The measure μ defines a pseudometric d_μ on T. Collapsing the sets of μ-measure zero to make d_μ into a metric, we obtain a tree T_0.

Lemma 2.5. *T_0 is simplicial.*

Proof. We will show that the F_n-orbit of any point in T_0 must be discrete. Let $x \in T_0$ and choose a point $\tilde{x} \in p^{-1}(x)$. The F_n-orbit of $\tilde{x}$ in T is discrete, and to understand the orbit of x, we need only understand $\mu([\tilde{x}, g\tilde{x}])$ for $g \in F_n$. If $[\tilde{x}, g\tilde{x}]$ contains no edges in T_r, then $\mu([\tilde{x}, g\tilde{x}]) = 0$, in which case $g \in \mathrm{Stab}(x)$. Otherwise, the segment contains an edge in T_r, and hence has positive μ-measure. Since there are only finitely many F_n-orbits of edges in T_r, there is a lower bound on the μ-measure of $[\tilde{x}, g\tilde{x}]$. Hence, there is a lower bound on $d_{T_0}(x, gx)$. This concludes the proof. $\square$

2.3.2 Defining $f_0 \colon T_0 \to T_1$

Let T_1 be the tree $\lambda_\phi^{-1} T_0 \cdot \phi$: the leading coefficient indicates that the metric has been scaled by λ_ϕ^{-1}. The relative train track map $f\colon T \to T \cdot \phi$ naturally induces a map $f_0 \colon T_0 \to T_1$. For each $x \in T_0$, its preimage $p^{-1}(x)$ is a connected subtree of T with μ-measure zero. The definition of μ guarantees that the f-image of this set is also connected and has μ-measure zero. Therefore $p \circ f \circ p^{-1}(x)$ is a single point in $T_0 \cdot \phi$, which is identified with

T_1 and we define $f_0 := p \circ f \circ p^{-1}$.

Lemma 2.6. *f_0 is an optimal morphism.*

Proof. We first show that f_0 is a morphism, which will follow from the definition of μ and properties of relative train track maps. Given a nondegenerate segment $[x, x']$ in T_0, choose $\tilde{x} \in p^{-1}(x)$ and $\tilde{x}' \in p^{-1}(x')$. The intersection of $[\tilde{x}, \tilde{x}']$ with the vertices of T is a finite set $\{\tilde{x}_1, \ldots, \tilde{x}_{k-1}\}$. Let $\tilde{x}_0 := \tilde{x}$ and $\tilde{x}_k := \tilde{x}'$. Taking the p-image of $\tilde{x}_i$ for $i \in \{0, \ldots, k\}$ yields a subdivision of $[x, x']$ into finitely many subsegments $[x_i, x_{i+1}]$, some of which may be degenerate. We will ignore the degenerate subdivisions: they occur as the projections of edges in T_{r-1} (all of which have μ-measure zero).

We claim that f_0 is an isometry in restriction to each of these subsegments. Indeed, let $e = [\tilde{x}_i, \tilde{x}_{i+1}]$ be an edge in T. Assume without loss of generality that $x_i \neq x_{i+1}$ so that $\mu(e) \neq 0$ and e is therefore an edge in T_r. It is an immediate consequence of Lemma 2.4 that for each $y \in e$, $\mu([f(\tilde{x}_i), f(y)]) = \lambda_\phi \mu([\tilde{x}_i, y])$ and hence that f_0 is an isometry in restriction to $[x_i, x_{i+1}]$.

We now address the optimality of f_0. There are three types of points to consider: points in the interior of an edge, vertices with trivial stabilizer, and vertices with nontrivial stabilizer. We have already established that f_0 is an isometry in restriction to edges, so there are two gates at each $x \in T_0$ contained in the interior of an edge. If $x \in T_0$ is a vertex with trivial stabilizer, then $p^{-1}(x)$ is a vertex (Lemma 2.4) contained in $T_r \setminus T_{r-1}$. As f is a relative train track map, there are at least two gates at $p^{-1}(x)$ and each is necessarily contained in T_r. A short path in T containing $p^{-1}(x)$ entering through the first gate and leaving through the second will be legal. Lemma 2.4 again gives that f_0 is an isometry in restriction to such a path, so there are at least two gates at x.

Now let $x \in T_0$ be a vertex with nontrivial stabilizer. Then $p^{-1}(x)$ is a subtree which is the inverse image of a component of G_{r-1} under the quotient map $T \to G$. Let $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ be distinct vertices in $T_r \cap T_{r-1}$ and let d (resp. d') be a direction based at $\tilde{x}$ (resp. $\tilde{x}'$) corresponding to an edge e (resp. e') in T_r. Lemma 2.4 provides that d and d' determine distinct directions at x. As mixed turns are legal, the path $\bar{e} \cup [\tilde{x}, \tilde{x}'] \cup e'$ in T is r-legal. A final application of Lemma 2.4 gives that the restriction of f_0 to the p-image of this path is an isometry, and hence that there are at least two directions at x. $\square$

The reader will note that we have actually proved

Lemma 2.7. *f_0 is a train track map.*

Next, we use this map to construct a folding path starting at T_0. This folding path will converge in $\partial\mathbb{P}\mathcal{O}_n$ to a tree T_L. We then prove that T_L is in fact the tree T_ϕ^+ defined above.

2.3.3 Folding T_0

Applying the canonical folding path construction, we obtain a folding path $(T_t)_{t\in[0,L_0]}$ guided by $f_0\colon T_0 \to T_1$ which begins at T_0 and ends at T_1, where $L_0 = \frac{1}{2}\mathrm{BBT}(f_0)$. Adapting a construction of Handel-Mosher [HM11, Section 7.1], we now extend this to a *periodic fold path guided by* f_0. For each $i \in \mathbb{N}$, let $T_i = \lambda_\phi^{-i} T_0 \cdot \phi^i$, whence we have optimal morphisms $f_i\colon T_i \to T_{i+1}$ satisfying $\mathrm{BBT}(f_i) = \lambda_\phi^{-i}\mathrm{BBT}(f_0)$. For each i, inductively define $L_i := L_{i-1} + \frac{1}{2}\mathrm{BBT}(f_i)$ and extend the folding path (which has so far been defined on $[0, L_{i-1}]$) using f_i to a folding path $(T_t)_{t\in[0,L_i]}$. Define $L := \lim_{i\to\infty} L_i$, which is finite as $\mathrm{BBT}(f_i)$ is a geometric sequence. We have thus defined the trees $(T_t)_{t\in[0,L)}$.

We now describe the maps $f_{t,s}$ for $s, t \in [0, L)$ with $s < t$. Indeed, given s, t, there is a natural choice of a map $f_{t,s}\colon T_s \to T_t$. Suppose $s \in [L_i, L_{i+1})$ and $t \in [L_j, L_{j+1})$. Then

$$f_{t,s} := f_{t,L_j} \circ f_{j-1} \circ f_{j-2} \circ \ldots \circ f_{i+1} \circ f_{L_{i+1},s}$$

The semi-flow property for the connection data follows from the definitions. Though our setting differs slightly from that of [BF14], Proposition 2.2 (5) can still be applied to give that each tree T_t has a well defined train track structure.

Along with the connection data, the fold path $(T_t)_{t\in[0,L)}$ forms a directed system in the category of F_n-equivariant metric spaces and distance nonincreasing maps. As direct limits exist in this category, let $T_L := \varinjlim T_t$ and let $f_{L,t}$ be the direct limit maps. The proof of the following proposition is contained in Section 7.3 of [HM11], though it is not stated in this way. While Handel-Mosher deal with trees in $\mathcal{O}_n$ rather than $\partial\mathbb{P}\mathcal{O}_n$, the reader will easily verify that their proof goes through directly in our setting.

Proposition 2.8 ([HM11]). *T_L is a nontrivial, minimal, $\mathbb{R}$-tree. Moreover T_t converges to T_L in the length function topology.*

We have now described two trees in the boundary of outer space: $T_\phi^+ = \lim_{n\to\infty} T_0'\phi^n$

and T_L. We observe that both T_0 and T_0' are points in the relative outer space $\mathcal{O}(F_n, \mathcal{A})$, which inherits the subspace topology from $\overline{\mathcal{O}_n}$. Moreover, ϕ is fully irreducible relative to $\mathcal{A}$, and as such, it acts with north-south dynamics on $\mathcal{O}(F_n, \mathcal{A})$ [Gup16]. Recall that for each $i \in \mathbb{N}$, $T_{L_i} = T_0 \cdot \phi^i / \lambda_\phi^i$, and that $L_i \to L$. As T_L is the limit of the fold path $(T_t)_{t \in [0,L)}$, we conclude

Lemma 2.9. $T_L = T_\phi^+$.

2.4 Stable tree is $\mathcal{Z}$-averse

Our present aim is to understand T_ϕ^+; we would like to show that it is $\mathcal{Z}$-averse. In this section, we will use the leaves of the topmost lamination Λ_ϕ^+ to construct a transverse covering of T_ϕ^+, then use the transverse covering to achieve our goal.

Definition 2.10 (Transverse family)**.** *Let $I = [x, y]$ be a nondegenerate arc in T_ϕ^+ which is a segment of a leaf of Λ_ϕ^+. Define Y_I as the union of all arcs J such that there exists $g_1, \ldots, g_m \in F_n$ with $J \subseteq g_1 I \cup \cdots \cup g_m I$ and such that $g_i I \cap g_{i+1} I$ is nondegenerate for each $i \in \{1, \ldots, m-1\}$. The collection $\mathcal{Y} = \{gY_I\}_{g \in F_n}$ is a transverse family in T_ϕ^+ since, by definition, distinct F_n-translates of Y_I intersect in a point or not at all. This construction is due to Guirardel-Levitt.*

Lemma 2.11. *With notation as above, Y_I is an indecomposable tree. Moreover, $\mathcal{Y} = \{gY_I\}_{g \in F_n}$ is a transverse covering of T_ϕ^+.*

Proof. We first show that Y_I is indecomposable. It is enough to show that every arc $J \subseteq Y_I$ can be covered by finitely many translates with nondegenerate overlap of the fixed arc I, and conversely that I can be covered by finitely many translates of J with nondegenerate overlap. Indeed, let $J = [x', y']$ be a nondegenerate arc in T_ϕ^+ and recall that $I = [x, y]$. The definition of Y_I guarantees that J can be covered by finitely many translates of I, so we are left to show the converse.

The construction in Section 2.3 provides an optimal folding path $(T_t)_{t \in [0,L]}$, and optimal morphisms $f_{s,t} \colon T_t \to T_s$ for all $s, t \in [0, L]$ with $s > t$ which satisfy the semi-flow property. Since (T_t) is a folding path, for any z in $T_L = T_\phi^+$, the set $f_{L,0}^{-1}(z)$ is a discrete set of points in T_0. Let $x_0 \in f_{L,0}^{-1}(x)$ and $y_0 \in f_{L,0}^{-1}(y)$ be points in T_0 chosen so that (x_0, y_0) contains no points in $f_{L,0}^{-1}(x) \cup f_{L,0}^{-1}(y)$ and define $I_0 = [x_0, y_0]$. Define J_0 by choosing $x_0' \in f_{L,0}^{-1}(x')$

and $y'_0 \in f_{L,0}^{-1}(y')$ similarly. Define the arc I_t (resp. J_t) in T_t by $I_t := [f_{t,0}(I_0)]$ (resp. $J_t := [f_{t,0}(J_0)]$). The definitions of I_0 and J_0 guarantee that $[f_{L,0}(I_0)] = I$ and similarly for J_0. The semiflow property of the maps $f_{s,t}$ gives that for all $s, t \in [0, L]$ with $s > t$, we have $[f_{s,t}(I_t)] = I_s$.

Now choose t large enough so that I_t is a crosses every turn taken by a leaf of Λ_ϕ^+; this is possible because I is itself a leaf segment. By enlarging t if necessary, we may arrange that J_t also crosses every turn taken by a leaf. While J_t may have illegal turns, I_t can nonetheless be covered by finitely many translates of the interval J_t with nondegenerate overlaps. This is because every turn in I_t can be covered by a legal turn in J_t. Thus finitely many translates of $[f_{L,t}(J_t)] = J$ cover $[f_{L,t}(I_t)] = I$ with nondegenerate overlaps.

We now show that $\mathcal{Y}$ is in fact a transverse covering. As before, by choosing t sufficiently large, we may assume that I_t crosses every turn taken by a leaf. Let J be an arc in T_ϕ^+ with preimage J_t in T_t, which is necessarily a concatenation of finitely many leaf segments. As I_t crosses every turn taken by a leaf, each of these leaf segments can be covered by finitely many translates of I_t and we have a covering of J_t by finitely many translates of I_t (with degenerate overlaps at illegal turns). Using $f_{L,t}$, we conclude that finitely many translates of $I = [f_{L,t}(I_t)]$ cover $J = [f_{L,t}(I_t)]$. □

Lemma 2.12. *T_ϕ^+ is mixing.*

Proof. Since T_ϕ^+ has a transverse covering by translates of an indecomposable tree Y_I, T_ϕ^+ is mixing. □

Lemma 2.13. *If $T \in \overline{\mathcal{O}_n}$ is mixing, then either T is indecomposable, or T splits as a graph of actions with one orbit of subtrees.*

Proof. Follows from [Rey12, Lemma 5.5]. Alternatively, the proof is straightforward from the definitions. □

For convenience of the reader, we recall the following essential fact:

Proposition 2.14 ([Hor14, Proposition 4.3]). *If $T \in \overline{\mathcal{O}_n}$ is mixing, then T is $\mathcal{Z}$-averse if and only if T is $\mathcal{Z}$-incompatible.*

Proposition 2.15. *T_ϕ^+ is $\mathcal{Z}$-averse.*

Proof. If the transverse covering $\mathcal{Y}$ is trivial, so that $T_\phi^+ = Y_I$, then T_ϕ^+ is indecomposable and hence $\mathcal{Z}$-averse. So suppose that $\mathcal{Y}$ is nontrivial and assume, for a contradiction, that T_ϕ^+ is not $\mathcal{Z}$-averse. As T_ϕ^+ is mixing, Proposition 2.14 implies that it is compatible with a $\mathcal{Z}$-splitting S. Lemma 1.18 of [Gui08] states that for a subgroup H of F_n, if the H-minimal subtree T_H of T_ϕ^+ is indecomposable, then H is elliptic in S. Since Y_I is indecomposable (Lemma 2.11), letting $H = \mathrm{Stab}(Y_I)$, we conclude that $\mathrm{Stab}(Y_I)$ is contained in a vertex group of the cyclic splitting S. Hence, Λ_ϕ^+ is carried by a vertex group S, contradicting the assumption that it is $\mathcal{Z}$-filling. □

2.5 Filling but not $\mathcal{Z}$-filling laminations

In this section, we endeavor to study filling laminations which are not $\mathcal{Z}$-filling. We then use this understanding to establish the following proposition, which is a restatement of the second claim in Theorem 2.1. This section concludes with a proof of the first statement in Theorem 2.1.

Proposition 2.16. *Let ϕ be an automorphism with a filling lamination Λ_ϕ^+ that is not $\mathcal{Z}$-filling, so that Λ_ϕ^+ is carried by a vertex group of a cyclic splitting S. Then there is a cyclic splitting S' that is fixed by a power of ϕ.*

The splitting S' is canonical in the sense that the vertex group which carries Λ_ϕ^+ is as small as possible. The proof of Proposition 2.16 will require an excursion into the theory of JSJ-decompositions; the reader is referred to [FP06] for details about JSJ theory.

We say a lamination is *elliptic* in an F_n-tree T if it is is carried by a vertex stabilizer of T. Let $\mathfrak{S}$ be the set of all one-edge $\mathcal{Z}$-splittings in which the lamination Λ_ϕ^+ is elliptic. Since Λ_ϕ^+ is filling, the set $\mathfrak{S}$ does not contain any free splittings.

Definition 2.17 (Types of pairs of splittings [RS97])**.** *Let $S = A *_C B$ (or $A*_C$) and $S' = A' *_{C'} B'$ (or $A'*_{C'}$) be one-edge cyclic splittings with corresponding Bass-Serre trees T and T'. We say S is* hyperbolic *with respect to S' if there is an element $c \in C$ that acts hyperbolically on T'. We say S is* elliptic *with respect to S' if C is fixes a point of T'. We say this pair is* hyperbolic-hyperbolic *if each splitting is hyperbolic with respect to the other. We define elliptic-*

elliptic, hyperbolic-elliptic and elliptic-hyperbolic splittings similarly.

Lemma 2.18. *With notation as above, suppose that $S, S' \in \mathfrak{S}$, and assume without loss that Λ^+_ϕ is carried by the vertex groups A and A'. Then Λ^+_ϕ is elliptic in the minimal subtree of A in T', denoted T'_A and in the minimal subtree of A' in T, denoted $T_{A'}$.*

Proof. Since A and A' both carry Λ^+_ϕ, their intersection $A \cap A'$ also carries Λ^+_ϕ. The vertex stabilizers of $T_{A'}$ are precisely the intersection of vertex stabilizers of T with A', namely the conjugates of $A \cap A'$. Thus Λ^+_ϕ is carried by a vertex group of $T_{A'}$. □

Lemma 2.19. *With notation as above, suppose that S, S' are one-edge cyclic splittings in $\mathfrak{S}$. Then S and S' are either hyperbolic-hyperbolic or elliptic-elliptic.*

Proof. The following is based on the proof of [FP06, Proposition 2.2]. We will address the case that both the splittings are free products with amalgamations; when one or both are HNN extensions, the proof is similar. Toward a contradiction, suppose C is hyperbolic in T' and C' is elliptic in T. Without loss of generality, we may assume that C' fixes the vertex stabilized by A in T. Suppose first that both A' and B' fix vertices in T. The two subgroups cannot fix the same vertex because they generate F_n. On the other hand, if the vertices are distinct, then C' fixes an edge in T. Hence C' must be a finite index subgroup of C, in contradiction to the assumption that C is hyperbolic in T'. Thus, one of A' or B' does not fix a vertex in T.

Assume without loss that A' does not fix a vertex of T. The minimal subtree of A' in T, $T_{A'}$, gives a minimal splitting of A' over an infinite index subgroup of C (i.e., a free splitting). As C' is elliptic in T, it is also elliptic in $T_{A'}$. Blowing up the vertex stabilized by A' in T' to the free splitting of A' just obtained, we get a free splitting of F_n. Lemma 2.18 implies that Λ^+_ϕ is elliptic in this free splitting, which is a contradiction. □

In [FP06], the existence of a JSJ decompositions for splittings with slender edge groups ([FP06, Theorem 5.13]) is established via an iterative process: one starts with a pair of splittings, and produces a new splitting which is a common refinement (in the case of an elliptic-elliptic pair) [FP06, Proposition 5.10], or an enclosing subgroup (in the case of a hyperbolic-hyperbolic pair) [FP06, Proposition 5.8]. One then repeats this process for all the splittings under consideration, and uses an accessibility result due to Bestvina-Feighn

[BF91] to conclude that the process stops after finitely many iterations. In order to use Fujiwara-Papasoglu's techniques, we need only ensure that if two splittings to belong to the set $\mathfrak{S}$, then the splittings created in this process also belong to $\mathfrak{S}$. By examining the construction of an enclosing subgroup for a pair of hyperbolic-hyperbolic splittings (Proposition 4.7) and using Lemma 2.18, we see that the enclosing graph decomposition of F_n for this pair of splittings indeed belongs to $\mathfrak{S}$. Similarly, by the construction of refinement for two elliptic-elliptic splittings and Lemma 2.18, we see that the refined splitting is also contained in $\mathfrak{S}$. This discussion implies that a JSJ decomposition exists for cyclic splittings of F_n in which Λ^+_ϕ is elliptic.

We conclude our foray into JSJ decompositions by using the theory of deformation spaces [For02, GL07a] to show that the set of JSJ splittings of F_n in which Λ^+_ϕ is elliptic is finite. By passing to a power, we will then obtain a ϕ-invariant splitting in $\mathfrak{S}$.

Definition 2.20 (Slide moves [GL07a, Section 7])**.** *Let $e = vw$ and $f = vu$ be adjacent edges in an F_n-tree T such that the vertex stabilizer of f, denoted G_f, is contained in G_e. Assume that e and f are not in the same orbit as nonoriented edges. Define a new tree T' with the same vertex set as T and replacing f by an edge $f' = wu$ equivariantly. Then we say f* slides *across e. Often, a slide move is described on the quotient of T by F_n.*

Definition 2.21 ([GL07a, For02])**.** *The deformation space $\mathcal{D}$ containing a tree T is the set of all trees T' such that there are equivariant maps from T to T' and from T' to T, up to equivariant isometry. A deformation space $\mathcal{D}$ for F_n is* nonascending *if it is irreducible, and no T in $\mathcal{D}$ is such that T/F_n contains a strict ascending loop.*

Definition 2.22 ([For02])**.** *A tree T is reduced if no inclusion of an edge group into either of its vertex group is an isomorphism.*

Theorem 2.23 ([GL07a, Theorem 7.2])**.** *Let $\mathcal{D}$ be a nonascending deformation space. Any two reduced simplicial trees $T, T' \in \mathcal{D}$ may be connected by a finite sequence of slides.*

Lemma 2.24. *There are only finitely many slide moves that can be performed on a reduced cyclic splitting S.*

Proof. First suppose that the splitting S/F_n does not have any loops or circuits. Then it is clear that only finitely many slide moves can be performed on S. If S has a loop, then we can slide an edge f along the loop e only once. Indeed, we have $G_f \subseteq G_e$ and after sliding we have $G_{f'} \subseteq tG_et^{-1}$, where t is the stable letter corresponding to the loop. Since $G_e \cong \mathbb{Z}$ and $G_e \cap tG_et^{-1} = 1$, $G_{f'} \nsubseteq G_e$ which prevents sliding of f' over e. The proof in the case of a circuit is similar. □

Proof of Proposition 2.16. By assumption, there exists a one-edge cyclic splitting S such that Λ_ϕ^+ is elliptic in S. The existence of JSJ decomposition for splittings in $\mathfrak{S}$ implies that the deformation space $\mathcal{D}$ for cyclic splittings in $\mathfrak{S}$ is nonempty. Theorem 2.23 and Lemma 2.24 together imply that the set of reduced trees in $\mathcal{D}$ is finite. As the set of reduced trees in $\mathcal{D}$ is ϕ-invariant, passing to a power yields a reduced cyclic splitting S' in $\mathcal{D}$ which is fixed by ϕ^k. □

Proof of Theorem 2.1 (Loxodromic). Applying Proposition 2.15 to each of ϕ and ϕ^{-1}, we conclude that T_ϕ^+ and T_ϕ^- are both $\mathcal{Z}$-averse. We now argue that these trees are distinct. We denote the dual lamination of a tree T by $L(T)$ [CHL08]. Since the attracting lamination Λ_ϕ^+ and the repelling lamination Λ_ϕ^- are different, and $\Lambda_\phi^\mp \subseteq L(T_\phi^\pm)$ and $\Lambda_\phi^\pm \nsubseteq L(T_\phi^\pm)$, we have that T_ϕ^+ and T_ϕ^- are distinct points in the Gromov boundary of $\mathcal{FZ}_n$ that are fixed by ϕ.

Thus, ϕ fixes two distinct $\mathcal{Z}$-averse trees in the boundary of $\mathcal{FZ}_n$. Furthermore, we saw in Section 2.3 that the $\mathcal{Z}$-splitting T_0' in $\mathcal{FZ}_n$ converges to T_ϕ^+ (resp. T_ϕ^-) under forward (resp. backward) iterates of ϕ. Thus ϕ acts loxodromically on $\mathcal{FZ}_n$.

We now prove the converse: if ϕ acts loxodromically on $\mathcal{FZ}_n$, then ϕ has a $\mathcal{Z}$-filling lamination. Indeed, if ϕ acts loxodromically on $\mathcal{FZ}_n$, then ϕ necessarily act loxodromically on $\mathcal{FS}_n$, and thus has a filling lamination Λ_ϕ^+. If the lamination is not $\mathcal{Z}$-filling, then Proposition 2.16 implies that ϕ fixes a point in $\mathcal{FZ}_n$, contradicting our assumption on ϕ. Thus, Λ_ϕ^+ is $\mathcal{Z}$-filling. □

2.6 Examples

This section will provide several examples exhibiting the range of behaviors of outer automorphisms acting on $\mathcal{FZ}_n$. We begin with an automorphism that acts loxodromically on $\mathcal{FZ}_n$.

Example 2.25 (Loxodromic element)**.** *Let ϕ be a rotationless automorphism with a CT representative $f\colon G \to G$ satisfying the following properties:*

- *f has exactly two strata, each of which is EG and nongeometric*
- *the lamination corresponding to the top stratum of f is filling*

An explicit example satisfying these properties can be constructed using the sage-train-tracks package written by T. Coulbois. The fact that the top lamination is filling guarantees that ϕ acts loxodromically on $\mathcal{FS}_n$. As both strata are nongeometric, [HM13a, Fact 1.42(1a)] guarantees that ϕ does not fix the conjugacy class of any element of F_n, and therefore cannot possibly fix a cyclic splitting. Theorem 2.2 implies that ϕ acts loxodromically.

Example 2.26 (Bounded orbit without fixed point)**.** *Building on Example 2.25, we can construct an automorphism ψ which acts on $\mathcal{FZ}_n$ with bounded orbits but without a fixed point. The reader is referred to Figure 2.1. Let ψ be a three stratum automorphism obtained from f by creating a duplicate of H_2. Explicitly, ψ has a CT representative $f'\colon G' \to G'$ defined as follows. The graph G' is obtained by taking two copies of G and identifying them along G_1. Each edge E of G' is naturally identified with an edge of G, and $f'(E)$ is defined via this identification. Moreover, the marking of G naturally gives a marking of G' (by a larger free group). That f' is a CT is evident from the fact that f is a CT.*

There are three laminations in $\mathcal{L}(\psi)$, none of which is filling. Since the top lamination in $\mathcal{L}(\phi)$ (where ϕ is as in Example 2.25) is filling, we know that $\mathcal{L}(\psi)$ must fill. Thus, ψ acts on $\mathcal{FS}_n$ with bounded orbits. As before, [HM13a, Fact 1.42 (1a)] implies that ψ is atoroidal: each stratum, H_i, may have an INP, ρ_i, but none of these INPs can be closed loops, and they cannot be concatenated to form a closed loop. Therefore does not fix any cyclic splitting and ψ must act on $\mathcal{FZ}_n$ with bounded orbits, but no fixed point.

Example 2.27 (Loxodromic element)**.** *Consider the outer automorphism $\phi\colon F_4 \to F_4$ given by*

$$\phi(a) = ab, \phi(b) = bcab, \phi(c) = d, \phi(d) = cd.$$

In [Rey12], it is shown that the stable tree for ϕ is indecomposable and hence $\mathcal{Z}$-averse. Therefore ϕ acts loxodromically on $\mathcal{FZ}_n$.

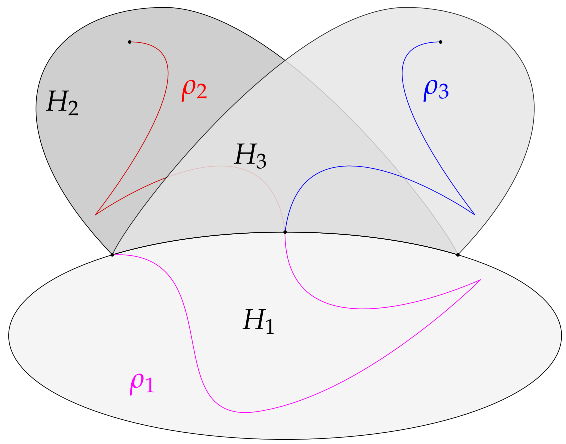

Figure 2.1. A CT representative for the automorphism in Example 2.26, which acts with bounded orbits but no fixed point

Example 2.28 (Fixed point)**.** *Let $\Sigma_{2,1}$ be the surface of genus two with one puncture. There is a unique free homotopy class of separating curve, and it divides $\Sigma_{2,1}$ into two subsurfaces: a once punctured torus and a twice punctured torus. Placing a pseudo-Anosov on each of these subsurfaces and taking the outer automorphism induced by this mapping class yields an element of $Out(F_n)$ that acts loxodromically on $\mathcal{FS}_n$, but fixes a point in $\mathcal{FZ}_n$.*

CHAPTER 3

CENTRALIZERS FOR AUTOMORPHISMS WITH $\mathcal{Z}$-FILLING LAMINATION

This chapter is devoted to a study of the centralizers of automorphisms with filling laminations. We begin with an introduction to provide some context for this endeavor along with an outline of what will follow.

3.1 Introduction

The main result of this chapter is:

Theorem 3.1. *An outer automorphism with a filling lamination has a virtually cyclic centralizer in $Out(F_n)$ if and only if the lamination is $\mathcal{Z}$-filling.*

The key tools used to prove Theorem 3.1 are the completely split train tracks introduced in [FH11] and the disintegration theory for outer automorphisms developed in [FH09]. We first show (Proposition 3.6) that the disintegration of any outer automorphism ϕ, that has a $\mathcal{Z}$-filling lamination, is virtually cyclic. Then we show that Proposition 3.6 implies the centralizer of ϕ is also virtually cyclic. Conversely, in Proposition 3.13, we show that if ϕ has a filling lamination that is not $\mathcal{Z}$-filling, then ϕ commutes with an appropriately chosen partial conjugation.

The method used to prove Theorem 3.1 provides alternate (and simple) proof of the well-known fact due to Bestvina, Feighn and Handel that centralizers of fully irreducible outer automorphisms are virtually cyclic. In [BFH00], the stretch factor homomorphism is used to show that the stabilizer of the lamination of a fully irreducible outer automorphism is virtually cyclic, which implies that the centralizer is also virtually cyclic. In general, not much is known about the centralizers of outer automorphisms. In [RW15], Rodenhausen and Wade describe an algorithm to find the presentation of the centralizer of an outer automorphism that is a Dehn Twist. In [FH09], Feighn and Handel show that the disintegration of an outer automorphism $\mathcal{D}(\phi)$ is contained in the weak center of the centralizer of ϕ. Recently, Algom-Kfir and Pfaff showed [AKP17] that centralizers of fully irreducible outer automorphisms with lone axes are isomorphic to $\mathbb{Z}$. We also mention a result of Kapovich and Lustig [KL11]: automorphisms whose limiting trees are free have virtually cyclic centralizers.

The main motivation for examining the centralizers of loxodromic elements of $\mathcal{FZ}_n$ (and $\mathcal{FS}_n$) is to understand which automorphisms have the potential to be WPD elements for the action of $\mathrm{Out}(F_n)$ on $\mathcal{FS}_n$ or $\mathcal{FZ}_n$.

Corollary 3.2. *Any outer automorphism that is loxodromic for the action of $Out(F_n)$ on $\mathcal{FS}_n$ but elliptic for the action on $\mathcal{FZ}_n$ is not a WPD element for the action on $\mathcal{FS}_n$.*

The result that centralizers of loxodromic elements of $\mathcal{FZ}_n$ are virtually cyclic is a promising sign for the following conjecture:

Conjecture 3.3. *The action of $Out(F_n)$ on $\mathcal{FZ}_n$ is a WPD action. That is, every loxodromic element for the action satisfies WPD.*

3.2 A quick review

We begin with a terse review of disintegration for outer automorphisms.

Given a mapping class f in Thurston normal form, there is a straightforward way of making a subgroup of the mapping class group, called *the disintegration of f*, by "doing one piece at a time." The subgroup is easily seen to be abelian as each pair of generators can be realized as homeomorphisms with disjoint supports. The process of disintegration in $\mathrm{Out}(F_n)$ is analogous, but more difficult.

3.2.1 Disintegration in Out(F_n)

The reader is warned that we will only review those ingredients from [FH09] that will be used directly; the reader is directed there, specifically to §6, for complete details. Given a rotationless outer automorphism ϕ, one can form an abelian subgroup called $\mathcal{D}(\phi)$. The process of disintegrating ϕ begins by creating a finite graph, B, which records the interactions between different strata in a CT representing ϕ. As a first approximation, the components of B correspond to generators of $\mathcal{D}(\phi)$. However, there may be additional relations between strata that are unseen by B, so the number of components of B only gives an upper bound to the rank of $\mathcal{D}(\phi)$.

Let $f\colon G \to G$ be a CT representing the rotationless outer automorphism ϕ. While the construction of $\mathcal{D}(\phi)$ does depend on f, using different representatives will produce subgroups that are commensurable. We will need to consider a weakening of the complete splitting of paths and circuits in f. The *quasi-exceptional splitting* of a completely split path or circuit σ is the coarsening of the complete splitting obtained by considering each quasi-exceptional subpath to be a single element.

Definition 3.4. *Define a finite directed graph B as follows. There is one vertex v_i^B for each nonfixed irreducible stratum H_i. If H_i is NEG, then a v_i^B-path is defined as the unique edge in H_i; if H_i is EG, then a v_i^B-path is either an edge in H_i or a taken connecting path in a zero stratum contained in H_i^z. There is a directed edge from v_i^B to v_j^B if there exists a v_i^B-path κ_i such that some term in the QE-splitting of $f_\#(\kappa_i)$ is an edge in H_j. The components of B are labeled $B_1, \ldots, B_K$. For each B_s, define X_s to be the minimal subgraph of G that contains H_i for each NEG stratum with $v_i^B \in B_s$ and contains H_i^z for each EG stratum with $v_i^B \in B_s$. We say that $X_1, \ldots, X_K$ are the* almost invariant subgraphs *associated to $f\colon G \to G$.*

The reader should note that the number of components of B is left unchanged if an iterate of $f_\#$ is used in the definition, rather than $f_\#$ itself. In the sequel, we will frequently make statements about B using an iterate of $f_\#$.

For each K-tuple $\vec{a} = (a_1, \ldots, a_K)$ of nonnegative integers, define

$$f_{\vec{a}}(E) = \begin{cases} f_\#^{a_i}(E) & \text{if } E \in X_i \\ E & \text{if } E \text{ is fixed by } f \end{cases}$$

It turns out that $f_{\vec{a}}$ is always a homotopy equivalence of G [FH09, Lemma 6.7], but in general $\langle f_{\vec{a}} \mid \vec{a} \text{ is a nonnegative tuple}\rangle$ is not abelian. To obtain an abelian subgroup, one has to pass to a certain subset of tuples which take into account interactions between the almost invariant subgraphs that are unseen by B. The reader is referred to [FH09, Example 6.9] for an example.

Definition 3.5. *A K-tuple $(a_1, \ldots, a_K)$ is called* admissible *if for all axes μ, whenever*

- *X_s contains a linear edge E_i with axis μ and exponent d_i,*

- *X_t contains a linear edge E_j with axis μ and exponent d_j,*
- *there is a vertex v^B of B and a v^B-path $\kappa \subseteq X_r$ such that some element in the quasi-exceptional family $E_i\overline{E}_j$ is a term in the QE-splitting of $f_\#(\kappa)$,*

then $a_r(d_i - d_j) = a_s d_i - a_t d_j$.

The disintegration of ϕ is then defined as $\mathcal{D}(\phi) = \langle f_{\vec{a}} \mid \vec{a} \text{ is admissible}\rangle$

3.2.2 Rotationless abelian subgroups

If an abelian subgroup H is generated by rotationless automorphisms, then all elements of H are rotationless [FH09, Corollary 3.13]. In this case, H is said to be rotationless. Rotationless abelian subgroups of $\mathrm{Out}(F_n)$ come equipped with a finite collection of (nontrivial) homomorphisms to $\mathbb{Z}$. Combining these, one obtains a homomorphism $\Omega\colon H \to \mathbb{Z}^N$ that is injective [FH09, Lemma 4.6]. An element $\psi \in H$ is said to be *generic* if all coordinates of $\Omega(\psi)$ are nonzero.

Every abelian subgroup of $\mathrm{Out}(F_n)$ has finitely many attracting laminations: if H is abelian and $\mathcal{L}(H) := \bigcup_{\phi\in H}\mathcal{L}(\phi)$, then $|\mathcal{L}(H)| < \infty$. For the purposes of this section, we require only two facts concerning Ω. First, there is one coordinate of the homomorphism Ω corresponding to each element of the finite set $\mathcal{L}(H)$ (there are other coordinates, which we will not need, corresponding to the so called "comparison homomorphisms"). Second is the fact that the coordinate of $\Omega(\psi)$ corresponding to $\Lambda \in \mathcal{L}(H)$ is positive if and only if $\Lambda \in \mathcal{L}(\psi)$.

3.3 From disintegrations to centralizers

In this section, we explain how to deduce Theorem 3.1 from the following proposition concerning the disintegration of elements acting loxodromically on $\mathcal{FZ}_n$. The proof of Proposition 3.6 is postponed until the next section.

Proposition 3.6. *If ϕ is rotationless and has a $\mathcal{Z}$-filling lamination, then $\mathcal{D}(\phi)$ is virtually cyclic.*

Proof of Theorem 3.1. Suppose $\psi \in C(\phi)$ has infinite order, and assume that $\langle \phi, \psi\rangle \simeq \mathbb{Z}^2$. If no such element exists, then $C(\phi)$ is virtually cyclic, as there is a bound on the order of a finite subgroup of $\mathrm{Out}(F_n)$ [Cul84]. Now let H_R be the finite index subgroup of $\langle \phi, \psi\rangle$ consisting of rotationless elements [FH09, Corollary 3.14] and let ψ' be a generic element of this subgroup. If the coordinate of $\Omega(\psi)$ corresponding to the $\mathcal{Z}$-filling lamination Λ^+_ϕ is negative, then replace ψ' by $(\psi')^{-1}$, which is also rotationless as Ω is a homomorphism. Since $\Lambda^+_\phi \in \mathcal{L}(\psi')$ is $\mathcal{Z}$-filling, Theorem 2.1 implies that ψ' acts loxodromically on $\mathcal{FZ}_n$. Since ψ' is generic in H_R, [FH09, Theorem 7.2] says that $\mathcal{D}(\psi') \cap \langle \phi, \psi \rangle$ has finite index in $\langle \phi, \psi \rangle$. This contradicts Proposition 3.6, which says that the disintigration of ψ' is virtually cyclic. □

3.4 The proof of Proposition 3.6

The idea of the proof is as follows. We noted above that the number of components in B only gives an upper bound to the rank of $\mathcal{D}(\phi)$; it may happen that there are interactions between the strata of f that are unseen by B (Definition 3.5). We will obtain precise information about the structure of B; it consists of one main component (B_1), and several components consisting of a single point ($B_2, \dots, B_K$). We will then show that the admissibility condition provides sufficiently many constraints so that choosing a_1 determines $a_2, \dots, a_K$. Thus, the set of admissible tuples consists of a line in $\mathbb{Z}^K$.

Let $f\colon G \to G$ be a CT representing ϕ with filtration $\emptyset = G_0 \subset G_1 \subset \cdots \subset G_M = G$. Let $\Lambda^+_\phi \in \mathcal{L}(\phi)$ be filling and let $\ell \in \Lambda^+_\phi$ be a generic leaf. As Λ^+_ϕ is filling, the corresponding EG stratum is necessarily the top stratum, H_M. We will understand the graph B by studying the realization of ℓ in G. The results of [BFH00, §3.1], together with Lemma 4.25 of [FH11] give that the realization of ℓ in G is completely split, and this splitting is unique. Thus, we may consider the QE-splitting of ℓ.

We begin with a lemma that allows the structure of INPs and quasi-exceptional paths to be understood inductively.

Lemma 3.7. *Let H_r be a nonfixed irreducible stratum and let ρ be a path of height $s \geq r$ which is either an INP or a quasi-exceptional path. Assume further that ρ intersects H_r nontrivially. Then one of the following holds:*

- *H_r and H_s are NEG linear strata with the same axis, each consisting of a single edge E_r (resp. E_s), and $\rho = E_s w^* \overline{E}_r$, where w is a closed, root-free Nielsen path of height $< s$.*
- *ρ can be written as a concatenation $\rho = \beta_0\rho_1\beta_1\rho_2\beta_2 \dots \rho_j\beta_j$, where each ρ_i is an INP of height r and each β_i is a path contained in $G - int(H_r)$ (some of the β_i's may be trivial).*

Proof. The proof proceeds by strong induction on the height s of the path ρ. In the base case, $s = r$, and ρ is either an INP of height r or a quasi-exceptional path of the form described. The inductive step breaks into cases according whether H_s is an EG stratum, or an NEG stratum.

If H_s is an EG stratum, then ρ must be an INP, as there are no exceptional paths of EG height. In this case, [FH11, Lemma 4.24 (2)] provides a decomposition of ρ into subpaths of height s and maximal subpaths of height $< s$, and each of the subpaths of height $< s$ is a Nielsen path. The inductive hypothesis then guarantees that each of these Nielsen paths has the desired form. By breaking apart and combining these terms appropriately, we conclude that ρ does as well.

Suppose now that H_s is an NEG stratum and let E_s be the unique edge in H_s. Using (NEG Nielsen Paths), we see that E_s must be a linear edge, and therefore that ρ is either $E_s w^k \overline{E_s}$ or $E_s w^k \overline{E'}$, where E' is another linear edge with the same axis and w is a closed root free Nielsen path of height $< s$. If H_r is NEG linear, and $E' = E_r$, then the first conclusion holds. Otherwise, we may apply the inductive hypothesis to w to obtain a decomposition as desired. This completes the proof. □

We now begin our study of the graph B. We call the component of B containing v_M^B, the vertex corresponding to the topmost stratum of f, the *main component*.

Lemma 3.8. *All nonlinear NEG strata are in the main component of B.*

Proof. Let H_r be a nonlinear NEG stratum, with single edge E_r. It is enough to show that the single edge E_r occurs as a term in the QE-splitting of ℓ, as this implies that there is an edge in B connecting v_M^B to v_r^B. As ℓ is filling, we know that its realization in G must cross E_r. If the corresponding term in the QE-splitting of ℓ is the single edge E, then we are done. The only other possibility is that the corresponding term is an INP or a quasi-exceptional path of some height $s \geq r$. An application of Lemma 3.7 shows that this is impossible, as it implies the existence of an INP of height r or a quasi-exceptional path of the form $E_r w^* \overline{E}'$, contradicting (NEG Nielsen Paths). □

Lemma 3.9. *All EG strata are in the main component of B.*

Proof. Let H_r be an EG stratum. As before, it is enough to show that some (every) edge of

H_r occurs as a term in the QE-splitting of ℓ. There are three types of pieces in a QE-splitting that can cross H_r: a single edge in H_r, an INP of height $\geq r$, or a quasi-exceptional path. In the first case, we are done, so suppose that every time ℓ crosses H_r, the corresponding term in its QE-splitting is an INP or a quasi-exceptional path.

We may therefore write ℓ as a concatenation $\ell = \ldots \gamma_1\sigma_1\gamma_2\sigma_2\ldots$ where each σ_i is a single term in the QE-splitting of ℓ which intersects $int(H_r)$, and each γ_i is a maximal concatenation of terms in the QE-splitting of ℓ which does not intersect $int(H_r)$ (some γ_i's may be trivial). By assumption, each σ_i is an INP or a QEP. Applying Lemma 3.7 to each of the σ_i's, then combining and breaking apart the terms appropriately, we see that ℓ can be written as a concatenation $\ell = \ldots \gamma_1\rho_1\gamma_2\rho_2\ldots$ where each ρ_i is the unique INP of height r or its inverse. Call this INP ρ.

The key to proving Proposition 3.6 is using the information we have about ℓ to find a $\mathcal{Z}$-splitting in which ℓ is carried by a vertex group, thus contradicting our assumption. We now modify G to produce a 2-complex, G'', whose fundamental group is identified with F_n. First assume H_r is nongeometric, so that ρ has distinct endpoints, v_0 and v_1. Let G' be the graph obtained from G by replacing each vertex v_i for $i \in \{0,1\}$ with two vertices, v_i^u and v_i^d (u and d stand for "up" and "down"), which are to be connected by an edge E_i. For each edge E of G that is incident to v_i, connect it in G' to the new vertices as follows: if $E \in H_r$, then E is connected to v_i^d, and if $E \notin H_r$, then E is connected to v_i^u. G' deformation retracts onto G by collapsing the new edges, and this retraction identifies $\pi_1(G')$ with F_n via the marking of G. Let $R = [0,1] \times [0,1]$ be a rectangle and define G'' by gluing $\{i\} \times [0,1]$ homeomorphically onto E_i for $i \in \{0,1\}$, then gluing $[0,1] \times \{0\}$ homeomorphically to the INP ρ. As only three sides of the rectangle have been glued, G'' deformation retracts onto G', and its fundamental group is again identified with F_n. The reader is referred to Figure 3.1.

The construction of G'' differs only slightly if H_r is geometric. In this case, ρ is a closed loop based at v_0 and we blow up v_0 to two vertices, v_0^u and v_0^d, that are connected by an edge E_0. Instead of gluing in a rectangle, we glue in a cylinder $R = S^1 \times [0,1]$; $\{p\} \times [0,1]$ is glued homeomorphically to E_0 where p is a point in S^1, and $S^1 \times \{0\}$ is glued homeomorphically to ρ.

Recall that in G, the leaf ℓ can be written as a concatenation $\ell = \ldots \gamma_1\rho_1\gamma_2\rho_2\ldots$ where

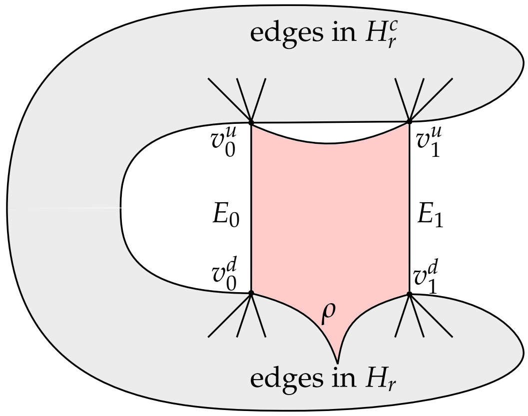

Figure 3.1. G'' when H_r is a nongeometric EG stratum

each ρ_i is either ρ or $\overline{\rho}$. Thus we can realize ℓ in G' as $\ell = \ldots \gamma_1\rho_1'\gamma_2\rho_2' \ldots$ where each ρ_i' is either $E_0\rho\overline{E}_1$ or $E_1\overline{\rho}\overline{E}_0$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, after performing a (proper!) homotopy to the image of ℓ, we can arrange that it never intersects the interior of R, nor the vertical sides of R. Cutting R along its centerline yields a $\mathcal{Z}$-splitting S of F_n, and ℓ is carried by a vertex group of this splitting. If H_r is nongeometric, then S is a free splitting and if H_r is geometric, then S is a cyclic splitting. In either case, if S is nontrivial, then we get a contradiction as our lamination is assumed to be $\mathcal{Z}$-filling. $\square$

Claim 3.10. *The splitting S is nontrivial.*

Proof of Claim 3.10. We first handle the case that H_r is geometric. We have described a one-edge cyclic splitting S which was obtained as follows: cut G' along the edge E_0 to get a free splitting of F_n, then fold $\langle w \rangle$, where w is the conjugacy class of the INP ρ. If $G' - E_0$ is connected, then the free splitting is an HNN extension, and there is no danger of S being trivial as $\text{rank}(F_n) \geq 3$. On the other hand, if $G' - E_0$ is disconnected, then let $G^{d'}$ and $G^{u'}$ be the components of $G' - E_0$ containing v_0^d and v_0^u respectively. The free splitting which is folded to get S is precisely $\pi_1(G^{d'}) * \pi_1(G^{u'})$. In this case, $G^{d'}$ is necessarily a component of G_r and [FH11, Proposition 2.20 (2)] together with (Filtration) imply that this component is a core graph. As H_r is EG, the rank of $\pi_1(G^{d'})$ is at least two and the splitting S is therefore nontrivial. To see that $\text{rank}(\pi_1(G^{u'})) \geq 1$, we need only recall that ℓ is not periodic and is carried by $\pi_1(G^{c'}) * \langle w \rangle$.

In the case that H_r is nongeometric, the splitting obtained above is a free splitting. If $G' - \{E_0, E_1\}$ is connected, then the free splitting is an HNN extension, and as before S is nontrivial. If $G' - \{E_0, E_1\}$ is disconnected, then the component containing v_0^d (and by necessity v_1^d), denoted $G^{d'}$, corresponds to a vertex group of S. By the same reasoning as in the previous case, we get that $\pi_1(G^{d'})$ is nontrivial. As before, the other vertex group of S carries the leaf ℓ and hence S is a nontrivial free splitting. $\square$

Before addressing the NEG linear strata and concluding the proof of Proposition 3.6, we present a final lemma concerning the structure of B.

Lemma 3.11. *Assume H_r is a linear NEG stratum consisting of an edge E_r. If v_r^B is not in the main component of B, then the component of B containing v_r^B is a single point.*

Proof. This follows directly from the definition of B, together with Lemmas 3.8 and 3.9. If H_r is a linear NEG stratum, then the definition of B implies that v_r^B has no outgoing edges. For any edge in B whose terminal vertex is v_r^B, its initial vertex necessarily corresponds to a nonlinear NEG stratum or an EG stratum, and hence is in the main component of B. $\square$

When dealing with an NEG linear stratum, we would like to carry out a similar strategy to the EG case: blow up the terminal vertex, v_0, to an edge and glue in a cylinder, thereby producing a cyclic splitting in which ℓ is carried by a vertex group. The main difficulty in implementing this comes from other linear edges with the same axis; for each such edge, one has to decide whether to glue it in G' to v_0^d or v_0^u.

Let μ be an axis with corresponding unoriented root-free conjugacy class w. Let $\mathcal{E}_\mu$ be the set of linear edges in G with axis μ. Define a relation on $\mathcal{E}_\mu$ by declaring $E \sim_R E'$ if the quasi-exceptional path $Ew^*\overline{E}'$ occurs as a term in the QE-splitting of ℓ or if both E and E' occur as terms in the QE-splitting of ℓ. Then let $\sim$ be the equivalence relation generated by $\sim_R$. Note that all edges in $\mathcal{E}_\mu$ which occur as terms in the QE-splitting of ℓ are in the same equivalence class.

Lemma 3.12. *There is only one equivalence class of $\sim$. Moreover, at least one edge in $\mathcal{E}_\mu$ occurs as a term in the QE-splitting of ℓ.*

Proof. Suppose for a contradiction that there is more than one equivalence class of $\sim$ and

$[E]$ be an equivalence class for which no edge in $[E]$ occurs as a term in the QE-splitting of ℓ. Now build G' as in the proof of Lemma 3.9. Let v_0 be the terminal vertex of the edges in $\mathcal{E}_\mu$ (they all have the same terminal vertex), and define G' by blowing up v_0 into two vertices, v_0^u and v_0^d, which are connected by an edge E_0. The terminal vertex of each edge of $[E]$ is to be glued in G' to v_0^u, while all other edges in G that are incident to v_0 are glued to v_0^d. Define G'' as before, gluing the bottom of a cylinder R along the closed loop w, and gluing the vertical interval above v_0 homeomorphically to the edge E_0.

The definition of $\sim$ guarantees that ℓ is carried by a vertex group of the cyclic splitting determined by cutting along the centerline of R. Indeed, whenever ℓ crosses an edge from $[E]$, the corresponding term in the QE-splitting is either an INP or a quasi-exceptional path $E'w^*\overline{E}''$, where $E', E'' \in [E]$. Repeatedly applying Lemma 3.7 to each of these terms, then rearranging and combining terms appropriately, we see that ℓ can be written in G as a concatenation $\ell = \ldots\gamma_1\rho_1\gamma_2\rho_2\ldots$ where each ρ_i is either $E'w^*\overline{E'}$ or $E'w^*\overline{E''}$ with $E', E'' \in [E]$. Thus we can realize ℓ in G' as $\ell = \ldots\gamma_1\rho_1'\gamma_2\rho_2'\ldots$ where each ρ_i' is $E'E_0w^*\overline{E}_0\overline{E'}$ or $E'E_0w^*\overline{E}_0\overline{E''}$. In G'', each ρ_i' is homotopic rel endpoints to a path that travels along the top of R, rather than down-across-and-up. Thus, we have again produced a cyclic splitting in which ℓ is carried by a vertex group.

We now argue that the splitting is nontrivial. There is a free splitting S which comes from cutting the edge E_0 in G', which cannot be a self loop. The cyclic splitting of interest S' is obtained from S by folding w across the single edge. If $G' - E_0$ is connected, then S' is an HNN extension with edge group $\langle [w] \rangle$. As $\text{rank}(F_n) \geq 3$, the vertex group has rank at least two and we are done. Now suppose E_0 is separating so that $G' - E_0$ consists of two components. Let G'^u be the component containing the vertex v_0^u and let G'^d be the other component. The vertex groups of the splitting S' are $\pi_1(G^d)$ and $\pi_1(G^u) * \langle [w] \rangle$. The fact that v is a principal vertex guarantees that $\pi_1(G^d) \ncong \mathbb{Z}$, and the fact that G is a finite graph without valence one vertices ensures that $\pi_1(G^u)$ is nontrivial.

The proof of the second statement is exactly the same as that of the first. □

Finally, we finish the proof of Proposition 3.6. As before, B_1 is the main component of B, with corresponding almost invariant subgraph X_1. All other components $B_2, \ldots, B_K$ are single points, and each almost invariant subgraph X_i consist of a single linear edge. Let

$(a_1, \ldots, a_K)$ be a K-tuple and suppose that a_1 has been chosen. We claim that imposing the admissibility condition determines all other a_i's.

Suppose first that E_i, E_j are linear edges with the same axis, μ, such that $E_i \in X_1$, $E_j \in X_k$, and $E_i \sim_R E_j$. Let d_i and d_j be the exponents of E_i and E_j respectively. Applying the definition of admissibility with $s = r = 1$, $t = k$, and κ a v^B path such that $f_\#(\kappa)$ contains a quasi-exceptional path of the form $E_i w^* \overline{E}_j$ in its QE-splitting (such a κ must exist as a quasi-exceptional path of this type occurs in the QE-splitting of ℓ), we obtain the relation $a_1(d_i - d_j) = a_1 d_i - a_k d_j$. Thus a_k is determined by a_1.

Now suppose E_i and E_j are as above, but rather than being related by $\sim_R$, we only have that $E_i \sim E_j$. There is a finite chain of $\sim_R$-relations to get from E_i to E_j. At each stage in this chain, the definition of admissibility (applied with $r = 1$ and κ chosen appropriately) will impose a relation that determines the next coordinate from the previous ones. Ultimately, this determines a_k.

We have thus shown that an admissible tuple is completely determined by choosing a_1, and therefore that the set of admissible tuples forms a line in $\mathbb{Z}^K$. Therefore $\mathcal{D}(\phi)$ is virtually cyclic.

3.5 A converse

We conclude this chapter with a converse to Proposition 3.6, which will complete the proof of Theorem 3.1.

Proposition 3.13. *If ϕ has a filling lamination which is not $\mathcal{Z}$-filling, then the centralizer of ϕ in $Out(F_n)$ is not virtually cyclic.*

Proof. Since ϕ has filling lamination which is not $\mathcal{Z}$-filling, it follows by Proposition 2.16 that ϕ fixes a one-edge cyclic splitting S.

Suppose S/F_n is a free product with amalgamation with vertex stabilizers $\langle A, w\rangle$ and B and edge group $\langle w \rangle \subset B$. Consider the Dehn twist D_w given by S as follows: D_w acts as identity on B and conjugation by w on A. The automorphism D_w has infinite order. We claim that D_w and ϕ commute. Indeed, consider a generating set $\{a_1, \ldots, a_k, b_1, \ldots b_m\}$ for F_n such that the a_i's generate A and the b_i's generate B. Without loss of generality, we may assume $\phi(B) = B$ and $\phi(\langle A, w\rangle) = \langle A, w\rangle^b$ for some element $b \in B$. Since D_w is identity on B and $\phi(B) = B$, we have $\phi(D_w(b_i)) = D_w(\phi(b_i))$ for all generators b_i. Since $D_w(a_i) = wa_i\overline{w}$, $\phi(w) = w$ and $\phi(\langle A, w\rangle) = \langle A, w\rangle^b$, we have $D_w(\phi(a_i)) = \phi(D_w(a_i))$ for all generators a_i. Thus D_w and ϕ commute.

We now address the case that S/F_n is an HNN extension. Assume S/F_n has stable letter t, edge group $\langle w\rangle$ and vertex group $\langle A, \bar{t}wt\rangle$. Consider a basis of F_n given by $\{a_1, a_2, \ldots, a_k, t\}$, where the a_i's generate A. Consider the Dehn twist D_w determined by S such that D_w is identity on A and sends t to wt. The automorphism D_w has infinite order. Since $\langle A, \bar{t}wt\rangle$ is ϕ-invariant, for every generator a_i, $\phi(a_i)$ is a word in the a_is and $\bar{t}wt$. Since D_w is identity on A and fixes $\bar{t}wt$, we get $\phi(D_w(a_i)) = D_w(\phi(a_i))$. Again, since $\langle A, \bar{t}wt\rangle$ is ϕ-invariant, $\phi(t)$ is equal to $w^m d\alpha$, where α is some word in $\langle A, \bar{t}wt\rangle$ and $m \in \mathbb{Z}$. On one hand, $\phi(D_w(t)) = \phi(wt) = \phi(w)\phi(t) = ww^m t\alpha$ and on the other hand, $D_w(\phi(t)) = D_w(w^m t\alpha) = w^m D_w(t) D_w(\alpha) = w^m wt\alpha$. Thus D_w and ϕ commute.

Thus when ϕ fixes a cyclic splitting, then an infinite order element other than a power of ϕ exists in the centralizer of ϕ. $\square$

REFERENCES

[AKP17] Yael Algom-Kfir and Catherine Pfaff. Normalizers and centralizers of cyclic subgroups generated by lone axis fully irreducible outer automorphisms. *New York J. Math.*, 23:365–381, 2017.

[Ali02] Emina Alibegović. Translation lengths in $\mathrm{Out}(F_n)$. *Geom. Dedicata*, 92:87–93, 2002.

[BDtM09] Jason Behrstock, Cornelia Dru¸tu, and Lee Mosher. Thick metric spaces, relative hyperbolicity, and quasi-isometric rigidity. *Math. Ann.*, 344(3):543–595, 2009.

[Ber78] Lipman Bers. An extremal problem for quasiconformal mappings and a theorem by Thurston. *Acta Math.*, 141(1-2):73–98, 1978.

[BF91] Mladen Bestvina and Mark Feighn. Bounding the complexity of simplicial group actions on trees. *Invent. Math.*, 103(3):449–469, 1991.

[BF94] Mladen Bestvina and Mark Feighn. Outer limits. *preprint, available at http://andromeda.rutgers.edu/ feighn/papers/outer.pdf*, 1994.

[BF10] Mladen Bestvina and Mark Feighn. A hyperbolic $\mathrm{Out}(F_n)$-complex. *Groups Geom. Dyn.*, 4(1):31–58, 2010.

[BF14] Mladen Bestvina and Mark Feighn. Hyperbolicity of the complex of free factors. *Adv. Math.*, 256:104–155, 2014.

[BFH97] Mladen Bestvina, Mark Feighn, and Michael Handel. Laminations, trees, and irreducible automorphisms of free groups. *Geom. Funct. Anal.*, 7(2):215–244, 1997.

[BFH00] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\mathrm{Out}(F_n)$. I. Dynamics of exponentially-growing automorphisms. *Ann. of Math. (2)*, 151(2):517–623, 2000.

[BFH05] Mladen Bestvina, Mark Feighn, and Michael Handel. The Tits alternative for $\mathrm{Out}(F_n)$. II. A Kolchin type theorem. *Ann. of Math. (2)*, 161(1):1–59, 2005.

[BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math. (2)*, 135(1):1–51, 1992.

[BM08] Jason A. Behrstock and Yair N. Minsky. Dimension and rank for mapping class groups. *Ann. of Math. (2)*, 167(3):1055–1077, 2008.

[CHL08] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. $\mathbb{R}$-trees and laminations for free groups. II. The dual lamination of an $\mathbb{R}$-tree. *J. Lond. Math. Soc. (2)*, 78(3):737–754, 2008.

[CL95] Marshall M. Cohen and Martin Lustig. Very small group actions on **R**-trees and Dehn twist automorphisms. *Topology*, 34(3):575–617, 1995.

[CM87] Marc Culler and John W. Morgan. Group actions on **R**-trees. *Proc. London Math. Soc. (3)*, 55(3):571–604, 1987.

[Coo87] Daryl Cooper. Automorphisms of free groups have finitely generated fixed point sets. *J. Algebra*, 111(2):453–456, 1987.

[Cul84] Marc Culler. Finite groups of outer automorphisms of a free group. In *Contributions to group theory*, volume 33 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 1984.

[CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.

[EG57] Samuel Eilenberg and Tudor Ganea. On the Lusternik-Schnirelmann category of abstract groups. *Ann. of Math. (2)*, 65:517–518, 1957.

[FH09] Mark Feighn and Michael Handel. Abelian subgroups of $\mathrm{Out}(F_n)$. *Geom. Topol.*, 13(3):1657–1727, 2009.

[FH11] Mark Feighn and Michael Handel. The recognition theorem for $\mathrm{Out}(F_n)$. *Groups Geom. Dyn.*, 5(1):39–106, 2011.

[FH14] Mark Feighn and Michael Handel. Algorithmic constructions of relative train track maps and CTs. *ArXiv e-prints*, November 2014.

[FLM01] Benson Farb, Alexander Lubotzky, and Yair Minsky. Rank-1 phenomena for mapping class groups. *Duke Math. J.*, 106(3):581–597, 2001.

[FM11] Stefano Francaviglia and Armando Martino. Metric properties of outer space. *Publ. Mat.*, 55(2):433–473, 2011.

[For02] Max Forester. Deformation and rigidity of simplicial group actions on trees. *Geom. Topol.*, 6:219–267, 2002.

[FP06] K. Fujiwara and P. Papasoglu. JSJ-decompositions of finitely presented groups and complexes of groups. *Geom. Funct. Anal.*, 16(1):70–125, 2006.

[GL07a] Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups Geom. Dyn.*, 1(2):135–181, 2007.

[GL07b] Vincent Guirardel and Gilbert Levitt. The outer space of a free product. *Proc. Lond. Math. Soc.*, 94(3):695–714, 2007.

[Gui04] Vincent Guirardel. Limit groups and groups acting freely on $\mathbb{R}^n$-trees. *Geom. Topol.*, 8:1427–1470 (electronic), 2004.

[Gui08] Vincent Guirardel. Actions of finitely generated groups on $\mathbb{R}$-trees. *Ann. Inst. Fourier (Grenoble)*, 58(1):159–211, 2008.

[Gup16] Radhika Gupta. Loxodromic elements for the relative free factor complex. *arXiv:1612.04261*, December 2016.

[Har86] John L. Harer. The virtual cohomological dimension of the mapping class group of an orientable surface. *Invent. Math.*, 84(1):157–176, 1986.

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[HM11] Michael Handel and Lee Mosher. Axes in outer space. *Mem. Amer. Math. Soc.*, 213(1004):vi+104, 2011.

[HM13a] M. Handel and L. Mosher. Subgroup decomposition in Out(F_n), Part I: Geometric Models. *ArXiv e-prints*, February 2013.

[HM13b] Michael Handel and Lee Mosher. The free splitting complex of a free group, I: hyperbolicity. *Geom. Topol.*, 17(3):1581–1672, 2013.

[HM14] Michael Handel and Lee Mosher. The free splitting complex of a free group II: Loxodromic outer automorphisms. *arXiv:1402.1886*, February 2014.

[Hor14] Camille Horbez. Hyperbolic graphs for free products, and the Gromov boundary of the graph of cyclic splittings. *arXiv:1408.0544*, August 2014.

[HW15] Camille Horbez and Richard D. Wade. Automorphisms of graphs of cyclic splittings of free groups. *Geom. Dedicata*, 178:171–187, 2015.

[KL11] Ilya Kapovich and Martin Lustig. Stabilizers of $\mathbb{R}$-trees with free isometric actions of F_N. *J. Group Theory*, 14(5):673–694, 2011.

[Man14] Brian Mann. Hyperbolicity of the cyclic splitting graph. *Geom. Dedicata*, 173:271–280, 2014.

[McC85] John McCarthy. A "Tits-alternative" for subgroups of surface mapping class groups. *Trans. Amer. Math. Soc.*, 291(2):583–612, 1985.

[MM99] Howard A. Masur and Yair N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999.

[Rey12] Patrick Reynolds. Reducing systems for very small trees. *arXiv:1211.3378*, November 2012.

[RS97] E. Rips and Z. Sela. Cyclic splittings of finitely presented groups and the canonical JSJ decomposition. *Ann. of Math. (2)*, 146(1):53–109, 1997.

[RW15] Moritz Rodenhausen and Richard D. Wade. Centralisers of Dehn twist automorphisms of free groups. *Math. Proc. Cambridge Philos. Soc.*, 159(1):89–114, 2015.

[SW02] Estelle Souche and Bert Wiest. An elementary approach to quasi-isometries of tree $\times$ $\mathbb{R}^n$. In *Proceedings of the Conference on Geometric and Combinatorial Group Theory, Part II (Haifa, 2000)*, volume 95, pages 87–102, 2002.

[Wad12] Richard D Wade. *Symmetries of free and right-angled Artin groups*. PhD thesis, University of Oxford, 2012.