A KAZHDAN–LUSZTIG ALGORITHM FOR WHITTAKER MODULES

by

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ABSTRACT

This dissertation develops the structure theory of the category Whittaker modules for a complex semisimple Lie algebra. We establish a character theory that distinguishes isomorphism classes of Whittaker modules in the Grothendieck group of the category, then use the localization functor of Beilinson and Bernstein to realize Whittaker modules geometrically as certain twisted $\mathcal{D}$-modules on the associated flag variety (so called "twisted Harish-Chandra sheaves"). The main result of this document is an algorithm for computing the multiplicities of irreducible Whittaker modules in the composition series of standard Whittaker modules, which are generalizations of Verma modules. This algorithm establishes that the multiplicities are determined by a collection of polynomials we refer to as Whittaker Kazhdan–Lusztig polynomials.

For Val.

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CHAPTER 1

INTRODUCTION

A fundamental goal in representation theory is to understand all representations of semisimple Lie groups. The algebraization of this problem leads to the study of modules over complex semisimple Lie algebras. For a complex semisimple Lie algebra $\mathfrak{g}$, understanding all $\mathfrak{g}$-modules is a daunting task, and a full classification has only been obtained for the simplest example: the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ [Blo81]. One way to make this task more manageable is to study subcategories of modules subject to certain restrictions, then relax the restrictions to expand the categories and observe which aspects of the structure carry over to the larger category. Block's classification of irreducible $\mathfrak{sl}(2, \mathbb{C})$-modules suggests two natural categories to consider: highest weight modules and nondegenerate Whittaker modules. Highest weight modules have been studied extensively in the past fifty years, and the category has been shown to have a rich underlying combinatorial structure. A celebrated example of this structure was Beilinson–Bernstein's [BB81] and Brylinski–Kashiwara's [BK81] proofs of the Kazhdan–Lusztig conjecture [KL79] which established that multiplicities of irreducible highest weight modules in Verma modules are determined by the Kazhdan–Lusztig polynomials. (See, for example, [Hum08] for a survey of this and other results.) Nondegenerate Whittaker modules were introduced in [Kos78] as an algebraic tool for determining which representations of a semisimple Lie group admit a Whittaker model, and Kostant showed that the category of nondegenerate Whittaker modules has a very simple structure. This dissertation is concerned with a category of $\mathfrak{g}$-modules which contains both the category of highest weight modules and the category of nondegenerate Whittaker modules as full subcategories. This is the category of Whittaker modules. The main result of this project is the development of an algorithm (Theorem 6.1) for computing the multiplicities of irreducible Whittaker modules in the composition series of standard Whittaker modules (Definition 3.9). These multiplicities are determined by a collection of polynomials which we refer to as Whittaker Kazhdan–Lusztig poynomials.

More specifically, let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{b}$ be a fixed Borel subalgebra of $\mathfrak{g}$ with nilpotent radical $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. The category $\mathcal{N}$ of Whittaker modules contains all $\mathcal{U}(\mathfrak{g})$-modules which are finitely generated, $\mathcal{Z}(\mathfrak{g})$-finite, and $\mathcal{U}(\mathfrak{n})$-finite. Let $\mathfrak{h}$ be the abstract Cartan subalgebra of $\mathfrak{g}$ [Mil93, §2]. For a choice of $\lambda \in \mathfrak{h}^*$ and a Lie algebra morphism $\eta \in \mathfrak{n}^*$, McDowell constructed a standard Whittaker module $M(\lambda, \eta)$ (Definition 3.9), and showed that all irreducible Whittaker modules $L(\lambda, \eta)$ appear uniquely as quotients of $M(\lambda, \eta)$ [McD85]. When $\eta = 0$, the $M(\lambda, 0)$ are Verma modules, and when η acts nontrivially on all root subspaces of $\mathfrak{g}$ corresponding to simple roots (we say such η are *nondegenerate*), the $M(\lambda, \eta)$ are the irreducible modules studied by Kostant in [Kos78]. McDowell also showed that Whittaker modules have finite length composition series [McD85, §2 Thm. 2.8][1], so a natural problem is to determine the multiplicities of the irreducible constituents of a standard Whittaker module. These multiplicities were determined for integral λ in [MS97] and for arbitrary λ in [Bac97] by relating subcategories of Whittaker modules to certain blocks of BGG category $\mathcal{O}$. These papers established multiplicity results for Whittaker modules, but they did not develop the combinatorial structure of Kazhdan–Lusztig polynomials that was established in [KL79, BB81, BK81] for the category of highest weight modules. In this dissertation, we develop a Kazhdan–Lusztig theory for the category of Whittaker modules by using Beilinson–Bernstein localization to realize Whittaker modules geometrically as a certain category of twisted sheaves of $\mathcal{D}$-modules on the flag variety of $\mathfrak{g}$, following [MS14].

The first step in using localization to study Whittaker modules is to realize $\mathcal{N}$ as a category of twisted Harish-Chandra modules. Given a connected algebraic group K with Lie algebra $\mathfrak{k}$ and a morphism $\phi : K \to \mathrm{Int}(\mathfrak{g})$ inducing an injection of $\mathfrak{k}$ into $\mathfrak{g}$, the pair $(\mathfrak{g}, K)$ is called a *Harish-Chandra pair* if $\mathfrak{g}$ acts on the flag variety X of $\mathfrak{g}$ with finitely many orbits. For a Harish-Chandra pair $(\mathfrak{g}, K)$ and a Lie algebra morphism $\eta \in \mathfrak{k}^*$, one can define an abelian category $\mathcal{M}_{fg}(\mathfrak{g}, K, \eta)$ consisting of finitely generated $\mathcal{U}(\mathfrak{g})$-modules that

[1]This fact follows immediately from the geometric description of Whittaker modules introduced by Miličić and Soergel in [MS14].

admit an algebraic action of K such that the differential of the K action differs from the restricted $\mathfrak{g}$-action by η (Definition 3.7). We refer to objects in this category as *η-twisted Harish-Chandra modules*. For a maximal ideal J_θ in $\mathcal{Z}(\mathfrak{g})$ corresponding to a Weyl group orbit $\theta \subset \mathfrak{h}^*$, denote by $\mathcal{U}_\theta$ the quotient of $\mathcal{U}(\mathfrak{g})$ by the two-sided ideal generated by J_θ. We denote the subcategory of $\mathcal{M}_{fg}(\mathfrak{g}, K, \eta)$ consisting of modules with central character determined by θ by $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \eta)$. Let N be the unipotent subgroup of $\mathrm{Int}(\mathfrak{g})$ whose Lie algebra is $\mathfrak{n}$. Then $(\mathfrak{g}, N)$ is a Harish-Chandra pair, and we can realize $\mathcal{N}$ in terms of twisted Harish-Chandra modules for this pair. Indeed, any object in $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ is a Whittaker module, and each standard and irreducible Whittaker module is in $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ for some orbit θ and morphism η (Lemma 3.8).

This description allows us to use the localization theory of Beilinson–Bernstein to study Whittaker modules. For each $\lambda \in \mathfrak{h}^*$, Beilinson and Bernstein constructed a sheaf of twisted differential operators $\mathcal{D}_\lambda$ on the flag variety X of $\mathfrak{g}$ [BB81] whose global sections $\Gamma(X, \mathcal{D}_\lambda)$ are equal to $\mathcal{U}_\theta$, where θ is the Weyl-group orbit of λ in $\mathfrak{h}^*$. Then the global sections functor Γ maps quasicoherent $\mathcal{D}_\lambda$-modules into $\mathcal{U}(\mathfrak{g})$-modules with central character determined by θ; that is, $\mathcal{U}_\theta$-modules. Beilinson and Bernstein also defined a localization functor $\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ by $\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$ and showed that if λ is regular and antidominant, Γ and Δ_λ are inverse functors, which establishes an equivalence of the category $\mathcal{M}(\mathcal{U}_\theta)$ with $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$.

Applying the localization functor Δ_λ to the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$, we obtain a geometric category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ of *η-twisted Harish-Chandra sheaves* (Section 4.2), which are N-equivariant $\mathcal{D}_\lambda$-modules satisfying a compatibility condition determined by η. This category consists of holonomic $\mathcal{D}_\lambda$-modules, so its objects have finite length composition series and there is a well-defined duality in the category [BGK+87]. The morphism η determines a subgroup W_Θ of the Weyl group W of $\mathfrak{g}$, and from the parameters $\eta \in \mathfrak{n}^*$, $C \in W_\Theta \backslash W$, and $\lambda \in \mathfrak{h}^*$, we construct a standard sheaf $\mathcal{I}(w^C, \lambda, \eta)$, costandard sheaf $\mathcal{M}(w^C, \lambda, \eta)$, and irreducible sheaf $\mathcal{L}(w^C, \lambda, \eta)$ (Section 4.3). Here w^C is the longest element in the coset C (Section 5.2). The precise relationship between the algebraic category $\mathcal{N}$ and the geometric category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ is given by the following theorem, which appears in Chapter 5.

Theorem 1.1. *(i) Let $\lambda \in \mathfrak{h}^*$ be antidominant, $\eta \in \mathfrak{n}^*$, and $C \in W_\Theta \backslash W$. Then*

$$\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = M(w^C\lambda, \eta).$$

(ii) If $\lambda \in \mathfrak{h}^*$ is also regular, then for any $C \in W_\Theta \backslash W$, we have

$$\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta).$$

These theorems let us formulate a geometric algorithm for computing the multiplicities of composition factors of a standard η-twisted Harish-Chandra sheaf and then translate that algorithm to the algebraic setting to determine multiplicities of irreducible Whittaker modules in standard Whittaker modules.

The statement of the algorithm is completely combinatorial. Let W be the Weyl group of a reduced root system Σ, $\Pi \subset \Sigma$ the collection of simple roots, and $S \subset W$ the corresponding set of simple reflections. Let $\Theta \subset \Pi$ be a subset of simple roots, and let $W_\Theta \subset W$ be the sub-Weyl group generated by reflections through Θ. Let $\mathcal{H}_\Theta$ be the free $\mathbb{Z}[q, q^{-1}]$-module with basis δ_C, $C \in W_\Theta \backslash W$. For any $\alpha \in \Pi$, we define a $\mathbb{Z}[q, q^{-1}]$-module endomorphism by

$$T_\alpha(\delta_C) = \begin{cases} 0 & \text{if } Cs_\alpha = C; \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C. \end{cases}$$

Here the order relation on cosets is the Bruhat order on longest coset representatives (Section 5.2). The main result of this dissertation is a geometric proof of the following theorem, which appears in Chapter 6.

Theorem 1.2. *There exists a unique function $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying the following properties.*

(i) For $C \in W_\Theta \backslash W$,

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) For $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$ such that $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D).$$

The function $\varphi : W_\Theta \backslash W \longrightarrow \mathcal{H}_\Theta$ determines a unique family $\{P_{CD} | C, D \in W_\Theta \backslash W, D \leq C\}$ of polynomials in $\mathbb{Z}[q]$ such that $\varphi(C) = \sum_{D \leq C} P_{CD} \delta_D$ for $C \in W_\Theta \backslash W$. These are the *Whittaker Kazhdan–Lusztig polynomials*. To prove the theorem, we define φ geometrically[2] by pulling back irreducible η-twisted Harish-Chandra sheaves to Bruhat cells and computing the rank of the resulting $\mathcal{D}$-module (Equation 6.1). Defining φ in this way relates this combinatorial statement of the theorem to multiplicities of irreducible sheaves in the category $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ in the composition series of standard sheaves in the same category, which in turn allows us to use the results in Chapter 5 to deduce multiplicity results about Whittaker modules. Specifically, this theorem establishes an algorithm for computing the multiplicities of irreducible Whittaker modules in standard Whittaker modules in the following way. Using Theorem 1.2, one computes the matrix $(P_{CD}(-1))_{C,D \in W_\Theta \backslash W}$, which is lower triangular and has 1's on the diagonal. Let $(\mu_{CD})_{C,D \in W_\Theta \backslash W}$ be the inverse matrix. The following corollary[3] accomplishes the goal of this dissertation.

Corollary 1.3. *The multiplicity of the irreducible Whittaker module $L(-w^D \rho, \eta)$ in the standard Whittaker module $M(-w^C \rho, \eta)$ is μ_{CD}.*

By twisting by a homogeneous invertible $\mathcal{O}_X$-module, we immediately obtain an analogue of Corollary 1.3 for standard Whittaker modules $M(\mu, \eta)$ corresponding to regular weights $\mu \in P(\Sigma)$.

This document is organized in the following way. Chapter 2 establishes preliminaries and notation. It is split into algebraic (Lie theory) and geometric (algebraic $\mathcal{D}$-modules) preliminaries, with the emphasis on geometry. In Chapter 3, we introduce the category of Whittaker modules and prove some fundamental structural results. In Section 3.2, we develop a character theory for Whittaker modules which plays a critical role in establishing their connection to twisted Harish-Chandra sheaves. Chapter 4 develops the structure of the category of η-twisted Harish-Chandra sheaves. More background about $\mathcal{D}$-modules on homogeneous spaces is found in Section 4.1. In Chapter 5, we establish the connection

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[2]This theorem also has a straightforward proof using purely combinatorial methods, see [Soe97, Thm. 3.1]. However, the geometric proof in this document relates Theorem 1.2 to Whittaker modules, creating a link between the combinatorics described in [Soe97, §2 §3] and the category $\mathcal{N}$.

[3]Here ρ is the half sum of positive roots.

between Whittaker modules and twisted Harish-Chandra sheaves. Chapter 6 contains the proof of the main theorem and its relationship to multiplicities of Whittaker modules. Chapter 7 reformulates Theorem 1.2 in the language of Hecke algebras to explicitly compare the Whittaker Kazhdan–Lusztig polynomials P_{CD} to other types of Kazhdan–Lusztig polynomials arising in the combinatorics literature. More specifically, Chapter 7 establishes the relationship between Whittaker Kazhdan–Lusztig and Kazhdan–Lusztig polynomials of [KL79], as well as the relationship between Theorem 6.1 and the Kazhdan–Lusztig algorithm for generalized Verma modules established in [Milb, Ch. 6 §3 Thm. 3.5]. Chapter 8 summarizes the new results in this manuscript and describes future research directions. For brevity, we omit proofs of results that can be found in [Milb] and [Mila], and refer the curious reader to these very thorough resources on algebraic $\mathcal{D}$-modules.

CHAPTER 2

PRELIMINARIES AND NOTATION

We begin this document by establishing some algebraic and geometric background. The familiar reader can skip this section and use it as a reference.

2.1 Algebraic Preliminaries

We start with some basic properties of Lie algebras. We list only properties that will be explicitly used in future arguments, and refer readers to [Bou05] for a detailed treatment of the subject.

- For a Lie algebra L, we denote by L^* the set of homomorphisms from L to $\mathbb{C}$. For a L-module V and $\lambda \in L^*$, define

  $$V_\lambda = \{v \in V | X \cdot v = \lambda(X)v \text{ for all } X \in L\}, \text{and}$$

  $$V^\lambda = \{v \in V | (X - \lambda(X))^k \cdot v = 0 \text{ for all } X \in L \text{ and some } k \in \mathbb{Z}_{\geq 0}\}.$$

  If $V_\lambda \neq 0$, we call V_λ the *L-weight space* of V corresponding to λ. If $V^\lambda \neq 0$, we call V^λ the *generalized L-weight space* of V corresponding to λ, and say λ is a *L-weight* of V. Clearly $V_\lambda \subseteq V^\lambda$.

- For a Lie algebra L, denote by $\mathcal{U}(L)$ the universal enveloping algebra of L. If $S \subset \mathcal{U}(L)$ is a subset, we say that an L-module V is *S-finite* if for any $v \in V$, the orbit Sv is finite dimensional.

- Let L be a nilpotent Lie algebra and let V be a $\mathcal{U}(L)$-finite L-module. If the action of $x \in L$ on any finite dimensional L-invariant subspace of V is triangularizable, then V^λ is a L-submodule of V and

  $$V = \bigoplus_{\lambda \in L^*} V^\lambda.$$

- Let L be a nilpotent Lie algebra and let U, V, and W be $\mathcal{U}(L)$-finite L-modules such that the action of $x \in L$ is triangularizable on every L-invariant finite dimensional subspace of U, V, and W. If

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

is a short exact sequence of L-modules, then there is a short exact sequence

$$0 \longrightarrow U^\lambda \longrightarrow V^\lambda \longrightarrow W^\lambda \longrightarrow 0$$

of generalized L-weight spaces for any $\lambda \in L^*$ obtained by restricting the maps in the original sequence to the generalized L-weight spaces.

- Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra, and $\mathfrak{h}^*$ its dual. Let Σ be the root system of $(\mathfrak{g}, \mathfrak{h})$ in $\mathfrak{h}^*$, Σ^+ a system of positive roots in Σ, and Π a basis of simple roots in Σ^+. The Lie algebra $\mathfrak{g}$ decomposes into root subspaces $\mathfrak{g} = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} | [H, X] = \alpha(H)X, H \in \mathfrak{h}\}$. If $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilpotent radical of $\mathfrak{b}$, then $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.

- Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, $\mathcal{Z}(\mathfrak{g})$ be the center of $\mathcal{U}(\mathfrak{g})$, and $\mathcal{U}(\mathfrak{g})_0 = \{X \in \mathcal{U}(\mathfrak{g}) | (\operatorname{ad} H)(X) = 0, H \in \mathfrak{h}\}$ be the commutant of $\mathfrak{h}$ in $\mathcal{U}(\mathfrak{g})$. Using the Poincare–Birkhott–Witt (PBW) theorem, one can show [Hum08, Ch. 1 §1.7] that $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{g})_0 \subset \mathcal{U}(\mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}$. Therefore, there is a well-defined homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ given by projection to the $\mathcal{U}(\mathfrak{h})$-coordinate. This is the (untwisted) *Harish-Chandra homomorphism*.

- Let $P(\Sigma) = \{\lambda \in \mathfrak{h}^* | \alpha^\vee(\lambda) \in \mathbb{Z}\}$ be the weight lattice of $\mathfrak{g}$. We say that a weight $\lambda \in \mathfrak{h}^*$ is *integral* if $\lambda \in P(\Sigma)$. We say that a weight $\lambda \in \mathfrak{h}^*$ is *regular* if $\alpha^\vee(\lambda) \neq 0$ for all $\alpha \in R$. We say that a weight $\lambda \in \mathfrak{h}^*$ is *antidominant* if $\alpha^\vee(\lambda)$ is not a strictly positive integer for any $\alpha \in \Sigma^+$.

2.2 Geometric Preliminaries

Here we record some basic properties of twisted sheaves of differential operators and modules over twisted sheaves of differential operators. For a detailed treatment of this subject, see [HMSW87, Mil93, Milb].

- Let X be a smooth complex algebraic variety and $n = \dim X$. Denote by $\mathcal{O}_X$ the structure sheaf of X, $\mathcal{D}_X$ the sheaf of differential operators on X, $\mathcal{T}_x$ the tangent sheaf on X, Ω_X the cotangent sheaf on X, and ω_X the invertible $\mathcal{O}_X$-module of differential n-forms on X. Denote by $i_X : \mathcal{O}_X \to \mathcal{D}_X$ the natural inclusion.

- Assume that X admits a transitive action of a complex linear algebraic group G. We denote the category of G-homogeneous quasicoherent $\mathcal{O}_X$-modules on X by $\mathcal{M}_{qc}(\mathcal{O}_X, G)$. This category can be characterized by the following theorem [MP].

  Theorem 2.1. *For $x \in X$, let $B_x \subset G$ be the stabilizer of x. Then the functor T_x which assigns to an object $\mathcal{F}$ in $\mathcal{M}_{qc}(\mathcal{O}_X, G)$ the geometric fiber $T_x(\mathcal{F})$ of $\mathcal{F}$ at x is an equivalence of the category of G-homogeneous quasicoherent $\mathcal{O}_X$-modules with the category of algebraic representations of B_x.*

- A *twisted sheaf of differential operators on* X is a pair $(\mathcal{D}, i)$ of a sheaf $\mathcal{D}$ of associative $\mathbb{C}$-algebras with identity on X and a homomorphism $i : \mathcal{O}_X \to \mathcal{D}$ of $\mathbb{C}$-algebras with identity that is locally isomorphic to the pair $(\mathcal{D}_X, i_X)$; that is, if X has a cover by open sets U, then for each U, there is a $\mathbb{C}$-algebra isomorphism $\varphi_U : \mathcal{D}|_U \to \mathcal{D}_U$ such that $\varphi_U \circ i = i_X$.

- For $f : Y \to X$ a morphism of smooth algebraic varieties and $\mathcal{D}$ a twisted sheaf of differential operators on X, we define

  $$\mathcal{D}_{Y\to X} = f^*(\mathcal{D}) = \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{D}.$$

  Then $\mathcal{D}_{Y\to X}$ is a left $\mathcal{O}_Y$-module for left multiplication and a right $f^{-1}\mathcal{D}$-module for right multiplication on the second factor. Denote by $\mathcal{D}^f$ the sheaf of differential $\mathcal{O}_Y$-module endomorphisms of $\mathcal{D}_{Y\to X}$ which are also $f^{-1}\mathcal{D}$-module endomorphisms. There is a natural morphism of sheaves of algebras $i_f : \mathcal{O}_Y \to \mathcal{D}^f$, and the pair $(\mathcal{D}^f, i_f)$ is a twisted sheaf of differential operators on Y.

- Let $\mathcal{D}$ be a twisted sheaf of differential operators on X and $\mathcal{L}$ an invertible $\mathcal{O}_X$-module. The *twist* of $\mathcal{D}$ by $\mathcal{L}$ is the sheaf $\mathcal{D}^{\mathcal{L}}$ of differential $\mathcal{O}_X$-module endomorphisms of $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ that commute with the right $\mathcal{D}$-action. Because $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}$ is an $\mathcal{O}_X$-module for left multiplication, there is a natural homomorphism $i_{\mathcal{L}} : \mathcal{O}_X \to \mathcal{D}^{\mathcal{L}}$, and $(\mathcal{D}^{\mathcal{L}}, i_{\mathcal{L}})$ is a twisted sheaf of differential operators on X. If $f : Y \to X$ is a morphism of smooth algebraic varieties as above, $(\mathcal{D}^{\mathcal{L}})^f = (\mathcal{D}^f)^{f^*(\mathcal{L})}$.

- If X is a homogeneous space for a group G with Lie algebra $\mathfrak{g}$, then a *homogeneous twisted sheaf of differential operators* on X is a triple $(\mathcal{D}, \gamma, \alpha)$, where $\mathcal{D}$ is a twisted sheaf of differential operators on X, γ is the algebraic action of G on X, and $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ is a morphism of algebras such that the following three conditions are satisfied:

  (i) the multiplication in $\mathcal{D}$ is G-equivariant;

  (ii) the differential of the G-action on $\mathcal{D}$ agrees with the action $T \mapsto [\alpha(\xi), T]$ for $\xi \in \mathfrak{g}$ and $T \in \mathcal{D}$; and

  (iii) the map $\alpha : \mathcal{U}(\mathfrak{g}) \to \Gamma(X, \mathcal{D})$ is a morphism of G-modules.

  For $x \in X$, denote by B_x the stabilizer of x in G and $\mathfrak{b}_x$ its Lie algebra. For each B_x-invariant linear form $\lambda \in \mathfrak{b}_x^*$, one can construct a homogeneous twisted sheaf of differential operators $\mathcal{D}_{X,\lambda}$ [HMSW87, App. A §1] and all homogeneous twisted sheaves of differential operators on X occur in this way.

- If $\mathcal{A}$ is a sheaf of $\mathbb{C}$-algebras on X, we denote by $\mathcal{A}^\circ$ the opposite sheaf of $\mathbb{C}$-algebras on X. Then if $(\mathcal{D}, i)$ is a twisted sheaf of differential operators on a smooth algebraic variety X, $(\mathcal{D}^\circ, i)$ is also a twisted sheaf of differential operators on X. In particular, the pair $(\mathcal{D}_X^\circ, i_X)$ is a twisted sheaf of differential operators, and it is naturally isomorphic to $(\mathcal{D}_X^{\omega_X}, i_{\omega_X})$. If X is a homogeneous space and δ is the B_x-invariant linear form which is the differential of the representation of B_x on the top exterior power of the cotangent space at x, then $(\mathcal{D}_{X,\lambda})^\circ$ is naturally isomorphic to $\mathcal{D}_{X,-\lambda+\delta}$.

- Let $\mathcal{D}$ be a twisted sheaf of differential operators on X. We can view left $\mathcal{D}$-modules as right $\mathcal{D}^\circ$-modules and vice-versa. In other words, the category $\mathcal{M}_{qc}^L(\mathcal{D})$ of quasicoherent left $\mathcal{D}$-modules on X is isomorphic to the category $\mathcal{M}_{qc}^R(\mathcal{D}^\circ)$ of quasicoherent right $\mathcal{D}^\circ$-modules on X. This relationship allows us to freely use right or left modules depending on the particular situation.

- For a category $\mathcal{M}_{qc}(\mathcal{D})$ of quasicoherent $\mathcal{D}$-modules, we denote by $\mathcal{M}_{coh}(\mathcal{D})$ the corresponding subcategory of coherent $\mathcal{D}$-modules. For a coherent $\mathcal{D}$-module $\mathcal{V}$, we can define the *characteristic variety* $\mathrm{Ch}\mathcal{V}$ of $\mathcal{V}$ in the same way as the non-twisted case [Mila, Ch. III §3]. Because this construction is local, the results in the non-twisted case carry over to our setting. In particular, we have the following structure:

  (i) $\mathrm{Ch}\mathcal{V}$ is a conical subvariety of the cotangent bungle $T^*(X)$.

  (ii) $\dim(\mathrm{Ch}\mathcal{V}) \geq \dim(X)$.

  If $\dim(\mathrm{Ch}\mathcal{V}) = \dim(X)$, we say that $\mathcal{V}$ is a *holonomic* $\mathcal{D}$-module. Holonomic $\mathcal{D}$-modules form a thick subcategory $\mathcal{M}_{hol}(\mathcal{D})$ of $\mathcal{M}_{coh}(\mathcal{D})$. If $\mathcal{V}$ in $\mathcal{M}_{coh}(\mathcal{D})$ is coherent as an $\mathcal{O}_X$-module, we call $\mathcal{V}$ a *connection*. Connections are locally free as $\mathcal{O}_X$-modules and their characteristic variety is the zero section of $T^*(X)$, so they are holonomic.

- For an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *twist functor* from $\mathcal{M}^L_{qc}(\mathcal{D})$ into $\mathcal{M}^L_{qc}(\mathcal{D}^{\mathcal{L}})$ by

$$\mathcal{V} \mapsto (\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}) \otimes_{\mathcal{D}} \mathcal{V}$$

  for $\mathcal{V} \in \mathcal{M}^L_{qc}(\mathcal{D})$. The twist functor is an equivalence of categories. If X is a homogeneous space for G and $\mathcal{L}$ is the invertible G-homogeneous quasicoherent $\mathcal{O}_X$-module determined by the character of B_x with differential $\mu \in \mathfrak{b}_x^*$, then $(\mathcal{D}_{X,\lambda})^{\mathcal{L}} = \mathcal{D}_{X,\lambda+\mu}$.

- For a morphism $f : Y \to X$ of smooth algebraic varieties and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *inverse image functor* $f^+ : \mathcal{M}^L_{qc}(\mathcal{D}) \to \mathcal{M}^L_{qc}(\mathcal{D}^f)$ by

$$f^+(\mathcal{V}) = \mathcal{D}_{Y\to X} \otimes_{f^{-1}\mathcal{D}} f^{-1}\mathcal{V}$$

  for $\mathcal{V} \in \mathcal{M}^L_{qc}(\mathcal{D})$. In general, f^+ is right exact with left derived functor Lf^+. If f is an open immersion, then f^+ is exact and $f^+(\mathcal{V}) = \mathcal{V}|_Y$. If f is a submersion, then f^+ is exact.

- For a morphism $f : Y \to X$ of smooth algebraic varieties and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *extraordinary inverse image functor* $f^! : D^b(\mathcal{M}^L_{qc}(\mathcal{D})) \to D^b(\mathcal{M}^L_{qc}(\mathcal{D}^f))$ by

$$f^! = Lf^+ \circ [\dim Y - \dim X].$$

If f is an immersion, then $f^!$ is the right derived functor of the left exact functor $L^{\dim Y - \dim X} f^+ : \mathcal{M}_{qc}^L(\mathcal{D}) \to \mathcal{M}_{qc}^L(\mathcal{D}^f)$. In this setting, we refer to the functor $L^{\dim Y - \dim X} f^+$ as $f^!$, and for $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D})$, we refer to the i^{th}-cohomology modules $H^k f^!(D(\mathcal{V}))$ as $R^k f^!(\mathcal{V})$.

- For a morphism $f : Y \to X$ of smooth algebraic varieties and a twisted sheaf $\mathcal{D}$ of differential operators on X, we define the *direct image functor* $f_+ : D^b(\mathcal{M}_{qc}^R(\mathcal{D}^f)) \to D^b(\mathcal{M}_{qc}^R(\mathcal{D}))$ by
$$f_+(\mathcal{W}^\cdot) = Rf_\bullet(\mathcal{W}^\cdot \otimes_{\mathcal{D}^f}^L \mathcal{D}_{Y \to X}),$$
for $\mathcal{W}^\cdot \in D^b(\mathcal{M}^R(\mathcal{D}^f))$. Here $Rf_\bullet$ is the right derived functor of the sheaf-theoretic direct image functor $f_\bullet$. If f is an immersion, f_+ is the right derived functor of the left exact functor $H^0 \circ f_+ \circ D : \mathcal{M}_{qc}^R(\mathcal{D}^f) \to \mathcal{M}_{qc}^R(\mathcal{D})$. In this setting, we refer to $H^0 \circ f_+ \circ D$ by f_+. If f is an open immersion, then $f_+ = Rf_\bullet$ is the sheaf-theoretic direct image. If f is affine, then f_+ is exact.

- For a module $\mathcal{V} \in \mathcal{M}_{qc}^R(\mathcal{D})$, denote by $\Gamma_Y(\mathcal{V})$ the $\mathcal{D}$-module of local sections of $\mathcal{V}$ supported in Y. The functor $\Gamma_Y : \mathcal{M}_{qc}^R(\mathcal{D}) \to \mathcal{M}_{qc}^R(\mathcal{D})$ is a left exact functor, and we denote by $R\Gamma_Y : D^b(\mathcal{M}_{qc}^R(\mathcal{D})) \to D^b(\mathcal{M}_{qc}^R(\mathcal{D}))$ its right derived functor. The following equivalence of categories is very useful in computations.

Theorem 2.2. *(Kashiwara) If Y is a closed smooth subvariety of a smooth algebraic variety X, $i : Y \to X$ the natural immersion, and $\mathcal{D}$ a twisted sheaf of differential operators on X, then the functor*
$$i_+ : \mathcal{M}_{qc}^R(\mathcal{D}^i) \to \mathcal{M}_{qc}^R(\mathcal{D})$$
establishes an equivalence of categories between $\mathcal{M}_{qc}^R(\mathcal{D}^i)$ and the full subcategory $\mathcal{M}_{qc,Y}^R(\mathcal{D})$ of $\mathcal{M}_{qc}(\mathcal{D})$ consisting of modules supported in Y. The quasiinverse of i_+ is $i^!$. In particular, if $\mathcal{V}$ is a quasicoherent $\mathcal{D}^i$-module, then $i^!(i_+(\mathcal{V})) = \mathcal{V}$, and if $\mathcal{U}$ is a quasicoherent $\mathcal{D}$-module, then $i_+(i^!(\mathcal{U})) = \Gamma_Y(\mathcal{U})$.

The following corollary to Kashiwara's theorem will be frequently used in future computations.

Corollary 2.3. *Let Y be a smooth subvariety of a smooth algebraic variety X and $j : Y \to X$ the natural immersion. Then for any $\mathcal{D}^j$-module $\mathcal{W} \in \mathcal{M}_{qc}^R(\mathcal{D}^j)$, $j^!(j_+(D(\mathcal{W})) = D(\mathcal{W})$.*

Proof. Let $X' = Y - \partial Y$. By construction, X' is an open dense subset of X, and Y is a closed subset of X'. This allows us to write j as the composition of a closed immersion i and an open immersion k:

$$\begin{tikzcd} Y \arrow[rr, "j"] \arrow[dr, "i"'] & & X \\ & X' \arrow[ur, "k"'] & \end{tikzcd}$$

Because $i : Y \to X'$ is a closed immersion, Theorem 2.2 implies that $i^! \circ i_+ \simeq id$. Because $k : X' \to X$ is an open immersion, $k_+ = Rk_\bullet$, and k^+ is restriction to Y. Therefore, $k^+ \circ k_+$ is isomorphic to the identity functor. Furthermore, because $\dim X' = \dim X$, $k^!(D(\mathcal{V})) = Lk^+[0](D(\mathcal{V})) = k^+(\mathcal{V})$ for $\mathcal{V} \in \mathcal{M}^R_{qc}(\mathcal{D})$. Using these facts, we conclude that for any $\mathcal{W}$ in $\mathcal{M}^R_{qc}(\mathcal{D}^j)$,

$$\begin{aligned} j^!(j_+(D(\mathcal{W})) &= i^!(k^!(k_+(i_+(D(\mathcal{W}))))) \\ &= i^!(i_+(D(\mathcal{W}))) \\ &= D(\mathcal{W}). \end{aligned}$$

This proves our result. $\square$

- Let $i : Y \to X$ be the immersion of a closed subvariety. If $\mathcal{J}_Y$ is the ideal of $\mathcal{O}_X$ consisting of germs vanishing on Y, we can define an increasing filtration of $\mathcal{D}_{Y\to X}$ by (left $\mathcal{D}^i$, right $i^{-1}\mathcal{O}_X$)-modules by

$$F_p\mathcal{D}_{Y\to X} = \{T \in \mathcal{D}_{Y\to X} | T\varphi = 0 \text{ for } \varphi \in (\mathcal{J}_Y)^{p+1}\},$$

for $p \in \mathbb{Z}_+$. We call this filtration the filtration by *normal degree*. By Kashiwara's theorem, it induces a natural $\mathcal{O}_X$-module filtration on $\mathcal{D}$-modules supported on Y. Namely, if $\mathcal{W} \in \mathcal{M}^R_{qc}(\mathcal{D}^i)$,

$$F_p i_+(\mathcal{W}) = i_\bullet(\mathcal{W} \otimes_{\mathcal{D}^i} F_p\mathcal{D}_{Y\to X}).$$

The associated graded module has the form

$$\mathrm{Gr} i_+(\mathcal{W}) = i_\bullet(\mathcal{W} \otimes_{\mathcal{O}_Y} S(\mathcal{N}_{X|Y})), \tag{2.1}$$

where $\mathcal{N}_{X|Y} = i^*(\mathcal{T}_X)/\mathcal{T}_Y$ denotes the normal sheaf of Y, and $S(\mathcal{N}_{X|Y})$ is the corresponding sheaf of symmetric algebras [HMSW87, App. A §3.3].

- The relationship between the twist functor and the direct image functor is the following.

  Proposition 2.4. *(Projection Formula) Let $f : Y \to X$ be a morphism of smooth complex algebraic varieties, and let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the following diagram commutes.*

$$\begin{tikzcd} D(\mathcal{D}^f) \arrow[r, "f_+"] \arrow[d, "f^*(\mathcal{L})\otimes_{\mathcal{O}_Y} -"'] & D(\mathcal{D}) \arrow[d, "\mathcal{L}\otimes_{\mathcal{O}_X} -"] \\ D((\mathcal{D}^{\mathcal{L}})^f) \arrow[r, "f_+"] & D(\mathcal{D}^{\mathcal{L}}) \end{tikzcd}$$

- The interaction between $\mathcal{D}$-module functors and fiber products is captured by base change.

  Theorem 2.5. *(Base Change Formula) Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of smooth complex algebraic varieties such that the fiber product $X \times_Z Y$ is a smooth algebraic variety, and let $\mathcal{D}$ be a twisted sheaf of differential operators on Z. Then the commutative diagram*

$$\begin{tikzcd} X \times_Z Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

  determines an isomorphism

$$g^! \circ f_+ = q_+ \circ p^!$$

  of functors from $D^b(\mathcal{M}(\mathcal{D}^f))$ to $D^b(\mathcal{M}(\mathcal{D}^g))$.

CHAPTER 3

A CATEGORY OF $\mathfrak{n}$-FINITE MODULES

In this chapter, we describe our category of interest. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra, and $\mathcal{Z}(\mathfrak{g})$ the center of $\mathcal{U}(\mathfrak{g})$. Let $\mathfrak{h}$ be the (abstract) Cartan subalgebra of $\mathfrak{g}$ [Mil93, §2]. Let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$, and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$ its nilpotent radical. Let $\Pi \subset \Sigma^+ \subset \Sigma \subset \mathfrak{h}^*$ be the corresponding set of simple roots and positive roots (respectively) inside the root system of $\mathfrak{g}$. Let W be the Weyl group of $\mathfrak{g}$, and denote by $\rho \in \mathfrak{h}^*$ the half-sum of positive roots. We are interested in the following category of $\mathfrak{g}$-modules.

Definition 3.1. Let $\mathcal{N}$ be the category of $\mathfrak{g}$-modules which are

(i) finitely generated as $\mathcal{U}(\mathfrak{g})$-modules,

(ii) $\mathcal{Z}(\mathfrak{g})$-finite, and

(iii) $\mathcal{U}(\mathfrak{n})$-finite.

We refer to objects in this category as *Whittaker modules*.

This category is a natural generalization of Bernstein–Gelfand–Gelfand's (BGG) category $\mathcal{O}$. Indeed, if condition (ii) is replaced by a $\mathfrak{h}$-semisimplicity condition, the resulting category is exactly BGG category $\mathcal{O}$ [Hum08]. A key difference between $\mathcal{N}$ and $\mathcal{O}$ is that when the $\mathfrak{h}$-semisimplicity condition is relaxed to $\mathcal{Z}(\mathfrak{g})$-finiteness, the existence of weight space decompositions is lost. However, the finiteness conditions (ii) and (iii) provide us with other useful decompositions of $\mathcal{N}$ that lead to structural results reminiscent of those in BGG category $\mathcal{O}$. The goal of this section is to describe these decompositions.

Condition (ii) in Definition 3.1 leads to our first categorical decomposition of $\mathcal{N}$. We start by recalling some standard terminology. A *central character* is algebra morphism $\chi : \mathcal{Z}(\mathfrak{g}) \to \mathbb{C}$. If a $\mathfrak{g}$-module V has the property that $z \cdot v = \chi(z)v$ for $z \in \mathcal{Z}(\mathfrak{g}), v \in V$, then we say V *has central character* χ. If for any $z \in \mathcal{Z}(\mathfrak{g})$, $z - \chi(z)$ acts locally nilpotently on a $\mathfrak{g}$-module V, we say V has *generalized central character*. The finite-generation and $\mathcal{Z}(\mathfrak{g})$-finiteness conditions in Definition 3.1 imply that the annihilator $I_V \subset \mathcal{Z}(\mathfrak{g})$ of a Whittaker module V is an ideal of finite codimension in $\mathcal{Z}(\mathfrak{g})$. Therefore, there exists a finite dimensional commuting family of operators $\mathcal{Z}(\mathfrak{g})/I_V$ acting on V which leads to a decomposition of V into infinitesimal blocks

$$V = \bigoplus_{\chi \in S} V^{\chi}.$$

Here S is a finite indexing set, $V^{\chi} = \{v \in V | (z - \chi(z))^k \cdot v = 0, z \in \mathcal{Z}(\mathfrak{g}), \text{ for some } k \in \mathcal{N}\}$ and $\chi : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ is a central character. Note that because the action of $\mathcal{Z}(\mathfrak{g})$ commutes with the action of $\mathfrak{g}$, each V^{χ} is a $\mathfrak{g}$-submodule of V.

We can rephrase this decomposition in terms of Weyl group orbits in $\mathfrak{h}^*$. Fix $\lambda \in \mathfrak{h}^*$, and let $\theta = W \cdot \lambda$ be the Weyl group orbit of λ in $\mathfrak{h}^*$. We can uniquely associate a maximal ideal $J_\theta \subset \mathcal{Z}(\mathfrak{g})$ to θ in the following way. The Harish-Chandra homomorphism $\gamma : \mathcal{Z}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ (Section 2.1) leads to an isomorphism of the center of $\mathcal{U}(\mathfrak{g})$ with Weyl group invariant polynomials on $\mathfrak{h}^*$,

$$\mathcal{Z}(\mathfrak{g}) \simeq P(\mathfrak{h}^*)^W.$$

(See, for example, [Hum08] for a detailed description of how this isomorphism is constructed from γ.) This induces a bijection between maximal ideals of $\mathcal{Z}(\mathfrak{g})$ and maximal ideals of $P(\mathfrak{h}^*)^W$. By Hilbert's Nullstellensatz [Har77, Ch. I Thm. 1.3A], maximal ideals in $P(\mathfrak{h}^*)$ correspond to elements of $\mathfrak{h}^*$, and maximal ideals in $P(\mathfrak{h}^*)^W$ correspond to Weyl group orbits of these elements. So to a Weyl group orbit θ, we can associate a unique maximal ideal J_θ of $\mathcal{Z}(\mathfrak{g})$. In particular, $J_\theta = \ker \chi_\theta$ for the central character $\chi_\theta : \mathcal{Z}(\mathfrak{g}) \longrightarrow \mathbb{C}$ defined by $z \mapsto (\lambda - \rho)(\gamma(z))$.

Let $\mathcal{N}_\theta$ be the full subcategory of $\mathcal{N}$ consisting of modules annihilated by J_θ; that is, modules with central character χ_θ. Let $\mathcal{N}_{\hat{\theta}}$ be the full subcategory of $\mathcal{N}$ consisting of modules annihilated by some power of J_θ; that is, modules with generalized central character χ_θ. Our decomposition of V into V^{χ} above is a decomposition of a Whittaker module into the direct sum of finitely many submodules which are objects in $\mathcal{N}_{\hat{\theta}}$ for distinct θ. This implies the following lemma.

Theorem 3.2. *There is a categorical decomposition*

$$\mathcal{N} = \bigoplus_{\theta \in \mathfrak{h}^*/W} \mathcal{N}_{\hat{\theta}}.$$

In particular, every object in $\mathcal{N}$ can be decomposed into the direct sum of finitely many objects in different $\mathcal{N}_{\hat{\theta}}$.

There is another categorical decomposition of $\mathcal{N}$ that follows from condition (iii) in Definition 3.1. For $V \in \mathcal{N}$ and $\eta \in \mathfrak{n}^*$, let V^η be the generalized η-weight space (Section 2.1). Such a V^η is non-zero only if $\eta|_{[\mathfrak{n},\mathfrak{n}]} = 0$; that is, if η is a Lie algebra morphism [Bou05, Ch. VII, §1, no. 3, Prop. 9.(iii)]. We refer to $\eta \in \mathfrak{n}^*$ with this property as $\mathfrak{n}$-*characters*. Because $\mathfrak{n}$ is a nilpotent Lie algebra and any module $V \in \mathcal{N}$ is $\mathcal{U}(\mathfrak{n})$-finite, we have the following lemma by [Bou05, Ch. VII, §1, no. 3, Prop. 8.(i)].

Lemma 3.3. *Let V be an object in the category $\mathcal{N}$. Then*

$$V = \bigoplus_{\eta \in \mathfrak{n}^*} V^\eta.$$

Conditions (i) and (ii) in Definition 3.1 imply that the generalized $\mathfrak{n}$-weight spaces V^η are invariant under the action by $\mathfrak{n}$ [Bou05, Ch. VII, §1, no. 3, Prop. 9.(i)]. In this setting, they are also invariant under the action of $\mathfrak{g}$.

Lemma 3.4. *Let V be an object in $\mathcal{N}$. For any $\eta \in \mathfrak{n}^*$, V^η is a $\mathfrak{g}$-module.*

Proof. Consider the action map

$$\begin{aligned} a\colon \mathfrak{g} \otimes_{\mathbb{C}} V &\longrightarrow V \\ X \otimes v &\longmapsto X \cdot v \end{aligned}$$

The tensor product of the adjoint action on $\mathfrak{g}$ with the $\mathfrak{g}$-action on V gives $\mathfrak{g} \otimes_{\mathbb{C}} V$ the structure of a $\mathfrak{g}$-module. The following calculation shows that a is a $\mathfrak{g}$-module morphism. For $X, Y \in \mathfrak{g}$ and $v \in V$,

$$\begin{aligned}a(X \cdot (Y \otimes v)) &= a(X \cdot Y \otimes v + Y \otimes X \cdot v)\\&= a([X,Y] \otimes v + Y \otimes X \cdot v)\\&= [X,Y] \cdot v + Y \cdot X \cdot v\\&= X \cdot Y \cdot v - Y \cdot X \cdot v + Y \cdot X \cdot v\\&= X \cdot Y \cdot v\\&= X \cdot a(Y \otimes v).\end{aligned}$$

Thus by restriction, a is also a $\mathfrak{n}$-module morphism, and for any $\mathfrak{n}$-weight $\eta \in \mathfrak{n}^*$, the action map descends to a morphism of generalized $\mathfrak{n}$-weight spaces

$$a\colon (\mathfrak{g} \otimes_{\mathbb{C}} V)^{\eta} \longrightarrow V^{\eta}.$$

Because $\mathfrak{n}$ is a nilpotent Lie algebra, every element of $\mathfrak{n}$ is ad-nilpotent by Engel's theorem [Hum72, Ch. I §3 Thm. 3.1]. This implies that the generalized $\mathfrak{n}$-weight space of $\mathfrak{g}$ corresponding to $\eta = 0$ is all of $\mathfrak{g}$. That is,

$$\mathfrak{g}^0 = \{X \in \mathfrak{g} |\, \mathrm{ad}(Y)^k X = 0 \text{ for some } k \in \mathbb{Z}_{\geq 0} \text{ for all } Y \in \mathfrak{n}\} = \mathfrak{g}.$$

For any two $\mathcal{U}(\mathfrak{n})$-finite modules V and W and $\eta, \eta' \in \mathfrak{n}^*$, a standard calculation shows that $V^{\eta} \otimes W^{\eta'} \subset (V \otimes W)^{\eta+\eta'}$. Therefore,

$$\mathfrak{g} \otimes_{\mathbb{C}} V^{\eta} = \mathfrak{g}^0 \otimes_{\mathbb{C}} V^{\eta} \subseteq (\mathfrak{g} \otimes_{\mathbb{C}} V)^{0+\eta} = (\mathfrak{g} \otimes_{\mathbb{C}} V)^{\eta}.$$

Hence, a maps $\mathfrak{g} \otimes_{\mathbb{C}} V^{\eta}$ into V^{η} so V^{η} is $\mathfrak{g}$-stable. □

Our next result is that the sum in Lemma 3.3 is finite.

Lemma 3.5. *Let V be an object in $\mathcal{N}$. Then there is some finite set $S \subset \mathfrak{n}^*$ so that*

$$V = \bigoplus_{\eta \in S} V^{\eta}$$

Proof. Conditions (i) and (iii) of Definition 3.1 imply that V is generated by an $\mathfrak{n}$-invariant finite dimensional subspace U. Consider the $\mathfrak{n}$-module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U$, where $\mathfrak{n}$ acts by the tensor product of the adjoint action on $\mathcal{U}(\mathfrak{g})$ and restriction of the $\mathfrak{g}$ action on U to $\mathfrak{n}$. The $\mathfrak{n}$-weights of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U$ are sums of $\mathfrak{n}$-weights of $\mathcal{U}(\mathfrak{g})$ and $\mathfrak{n}$-weights of U. Because U is finite dimensional, it has finitely many $\mathfrak{n}$-weights. Because $\mathfrak{n}$ is a nilpotent Lie algebra, Engel's theorem [Hum72, Ch. I §3 Thm. 3.1] implies that $\operatorname{ad}\mathfrak{n}$ acts nilpotently on $\mathcal{U}(\mathfrak{n})$, so the only $\mathfrak{n}$-weight of $\mathcal{U}(\mathfrak{g})$ is $\eta = 0$. Thus, the $\mathfrak{n}$-weights of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U$ are exactly the finitely many $\mathfrak{n}$-weights of U.

There is a surjective action map of $\mathfrak{g}$-modules given by

$$\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} U \longrightarrow \mathcal{U}(\mathfrak{g})U = V$$
$$X \otimes u \longmapsto X \cdot u.$$

This is a $\mathfrak{n}$-module morphism by the calculation in the proof of Lemma 3.4, and it descends to a morphism of generalized $\mathfrak{n}$-weight spaces which is also surjective. Thus, any $\mathfrak{n}$-weight of V is a $\mathfrak{n}$-weight of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} V$, and there are only finitely many such $\mathfrak{n}$-weights. □

Denote by $\mathcal{N}_\eta$ the full subcategory of Whittaker modules V with the property that $V = V^\eta$. Lemma 3.3, Lemma 3.4, and Lemma 3.5 imply the following theorem.

Theorem 3.6. *There is a categorical decomposition*

$$\mathcal{N} = \bigoplus_{\eta \in \mathfrak{n}^*} \mathcal{N}_\eta.$$

In particular, every object in $\mathcal{N}$ can be decomposed into the direct sum of finitely many objects in different $\mathcal{N}_\eta$.

Let $\mathcal{N}_{\theta,\eta} = \mathcal{N}_\theta \cap \mathcal{N}_\eta$. By Schur's lemma [Hum72, Ch. II §6 Lem. 6.1], any irreducible $\mathfrak{g}$-module V has a central character, so any irreducible object in $\mathcal{N}$ lies in some $\mathcal{N}_\theta$. Additionally, because the V_η are $\mathfrak{g}$-submodules of V, any irreducible object of $\mathcal{N}$ also lies in some V_η. Therefore, any irreducible Whittaker modules lies in $\mathcal{N}_{\theta,\eta}$ for some Weyl group orbit θ and some $\eta \in \mathfrak{n}^*$.

Next we explore a different perspective of the category $\mathcal{N}_{\theta,\eta}$. Let K be a connected algebraic group with Lie algebra $\mathfrak{k}$, and ϕ a morphism of K onto the group of inner automorphisms $\operatorname{Int}(\mathfrak{g})$ such that the differential of ϕ induces an injection of $\mathfrak{k}$ into $\mathfrak{g}$. In this way, we can view $\mathfrak{k}$ as a subalgebra of $\mathfrak{g}$. We say that $(\mathfrak{g}, K)$ is a *Harish-Chandra pair* if K acts on the flag variety X of $\mathfrak{g}$ with finitely many orbits.

Fix a Harish-Chandra pair $(\mathfrak{g}, K)$, and a $\mathfrak{n}$-character $\eta \in \mathfrak{n}^*$.

Definition 3.7. An *η-twisted Harish-Chandra module* is a triple (π, ν, V) such that

(i) (π, V) is a finitely generated $\mathcal{U}(\mathfrak{g})$-module,

(ii) (ν, V) is an algebraic representation of K, and

(iii) the differential of the K-action on V induces a $\mathcal{U}(\mathfrak{k})$-module structure on V such that for any $\xi \in \mathfrak{k}$,

$$\pi(\xi) = \nu(\xi) + \eta(\xi).$$

Denote by $\mathcal{M}_{fg}(\mathfrak{g}, K, \eta)$ the category of η-twisted Harish-Chandra modules.

Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$ be the Weyl group orbit of λ. Let J_θ be the corresponding maximal ideal of $\mathcal{Z}(\mathfrak{g})$ described previously. Define $\mathcal{U}_\theta = \mathcal{U}(\mathfrak{g})/J_\theta\mathcal{U}(\mathfrak{g})$. Then denote by $\mathcal{M}_{fg}(\mathcal{U}_\theta, K, \eta)$ the full subcategory of $\mathcal{M}_{fg}(\mathfrak{g}, K, \eta)$ consisting of modules which are actually $\mathcal{U}_\theta$-modules; that is, modules annihilated by J_θ. These are precisely the objects of $\mathcal{M}_{fg}(\mathfrak{g}, K, \eta)$ with central character χ_θ.

Let G be a Lie group such that $G =$Int$(\mathfrak{g})$, and $N \subset G$ a subgroup such that Lie$(N) = \mathfrak{n}$. We will show that the category $\mathcal{M}_{fg}(\mathcal{U}_\theta, N, \eta)$ is equivalent to $\mathcal{N}_{\theta,\eta}$. Let V be an object in $\mathcal{N}_{\theta,\eta}$, and $\mathbb{C}_{-\eta}$ the one-dimensional $\mathfrak{n}$-module where $\mathfrak{n}$ acts by $-\eta$. Consider the induced $\mathfrak{g}$-module $V \otimes \mathbb{C}_{-\eta}$. This module is $\mathcal{U}(\mathfrak{n})$-finite and $V \otimes \mathbb{C}_{-\eta} = (V \otimes \mathbb{C}_{-\eta})_0$; that is, for any $v \in V \otimes \mathbb{C}_{-\eta}$, $\mathfrak{n}^k \cdot v = 0$ for sufficiently large $k \in \mathbb{N}$. This implies that we can exponentiate the $\mathfrak{n}$-action to get an algebraic N-action on $V \otimes \mathbb{C}_{-\eta}$ whose differential is the $\mathfrak{n}$-action. There is a natural isomorphism $V \longrightarrow V \otimes \mathbb{C}_{-\eta}$ given by sending $v \in V$ to $v \otimes 1 \in V \otimes \mathbb{C}_{-\eta}$. This isomorphism gives us an algebraic action of N on V whose differential differs from the original action of $\mathfrak{n}$ by η. Thus, $V \in \mathcal{M}(\mathcal{U}_\theta, N, \eta)$. This proves the following lemma.

Lemma 3.8. *We have an equivalence of categories.*

$$\mathcal{N}_{\theta,\eta} = \mathcal{M}(\mathcal{U}_\theta, N, \eta).$$

This association lets us use the localization functor of Beilinson and Bernstein to study the category of Whittaker modules geometrically. In particular, by localizing objects in $\mathcal{M}(\mathcal{U}_\Theta, N, \eta)$, one obtains a category of η-twisted holonomic $\mathcal{D}$-modules which are equivariant for the action of N. We will discuss the details of this construction in Section 4.2, but we remark here that this correspondence immediately implies that objects in $\mathcal{N}$ have finite length composition series. This fact was also proven algebraically by McDowell in [McD85, §2 Thm. 2.8].

3.1 Standard and Simple Modules

In [McD85], McDowell introduced a class of induced modules in $\mathcal{N}$ that generalize the Verma modules of BGG category $\mathcal{O}$, and showed that all irreducible objects in $\mathcal{N}$ arise as quotients of these "standard modules." In this section, we review this construction following [Luk04]. We show that standard modules decompose into $\mathfrak{h}^\Theta$-weight spaces for the action of a certain subalgebra $\mathfrak{h}^\Theta \subset \mathfrak{h}$ determined by a character $\eta \in \mathfrak{n}^*$, and that these $\mathfrak{h}^\Theta$-weight spaces have finite length composition series. Then we show that all modules in $\mathcal{N}_\eta$ admit generalized $\mathfrak{h}^\Theta$-weight space decompositions, which is the key piece of structure needed in Section 3.2 to establish a character theory for $\mathcal{N}$.

For the remainder of this subsection, fix a character $\eta \in \mathfrak{n}^*$. For $\alpha \in \Sigma$, let $\mathfrak{g}_\alpha$ be the root space corresponding to α (Section 2.1). The character η determines a subset $\Theta \subset \Pi$ of the simple roots in the following way:

$$\Theta = \{\alpha \in \Pi : \eta|_{\mathfrak{g}_\alpha} \neq 0\}.$$

Because $\eta|_{[\mathfrak{n},\mathfrak{n}]} = 0$, η only acts nontrivially on weight spaces corresponding to simple roots, so Θ is indeed a subset of simple roots. If $\Theta = \Pi$, we say that η is *nondegenerate*. We call a Whittaker module $V \in \mathcal{N}_\eta$ for η nondegenerate a *nondegenerate Whittaker module*. The cyclically generated "Whittaker modules" studied by Kostant in [Kos78] are nondegenerate Whittaker modules in our terminology.

Let $\Sigma_\Theta \subset \Sigma$ be the sub-root system generated by Θ, and $\Sigma_\Theta^+ = \Sigma^+ \cap \Sigma_\Theta$ the corresponding set of positive roots. Let W_Θ be the Weyl group of Σ_Θ, and $\rho_\Theta = \frac{1}{2}\sum_{\alpha\in\Sigma_\Theta^+}\alpha$. Let

$$\mathfrak{n}_\Theta = \bigoplus_{\alpha\in\Sigma_\Theta^+}\mathfrak{g}_\alpha, \mathfrak{u}_\Theta = \bigoplus_{\alpha\in\Sigma^+\setminus\Sigma_\Theta^+}\mathfrak{g}_\alpha, \bar{\mathfrak{n}}_\Theta = \bigoplus_{\alpha\in-\Sigma_\Theta^+}\mathfrak{g}_\alpha, \text{ and } \bar{\mathfrak{u}}_\Theta = \bigoplus_{\alpha\in-\Sigma^+\setminus-\Sigma_\Theta^+}\mathfrak{g}_\alpha.$$

In this way, the character η determines a reductive subalgebra $\ell_\Theta = \bar{\mathfrak{n}}_\Theta \oplus \mathfrak{h} \oplus \mathfrak{n}_\Theta$ of $\mathfrak{g}$ and a parabolic subalgebra $\mathfrak{p}_\Theta = \ell_\Theta \oplus \mathfrak{u}_\Theta$. The reductive subalgebra decomposes into the direct sum of a semisimple subalgebra $\mathfrak{s}_\Theta$ and its center $\mathfrak{z}_\Theta$. The semisimple subalgebra $\mathfrak{s}_\Theta$ in this decomposition is the derived subalgebra $[\ell_\Theta, \ell_\Theta]$ [Hum72, Ch. V §19 Prop. 1.(a)], and it is easy to check that the center $\mathfrak{z}_\Theta$ is the subalgebra $\mathfrak{h}^\Theta = \{H \in \mathfrak{h} \mid \alpha(H) = 0, \alpha \in \Theta\} \subset \mathfrak{h}$.

Let $\gamma_\Theta : \mathcal{Z}(\ell_\Theta) \to \mathcal{U}(\mathfrak{h})$ be the untwisted Harish-Chandra homomorphism of $\mathcal{Z}(\ell_\Theta)$ [Hum08, Ch. 1 §7]. Fix $\lambda \in \mathfrak{h}^*$, and define $\varphi_{\Theta,\lambda} : \mathcal{U}(\mathfrak{h}) \longrightarrow \mathbb{C}$ to be the homomorphism sending $H \in \mathfrak{h}$ to $(\lambda - \rho_\Theta)(H) \in \mathbb{C}$. The homomorphism

$$\Omega_{\Theta,\lambda} = \varphi_{\Theta,\lambda} \circ \gamma_\Theta : \mathcal{Z}(\ell_\Theta) \longrightarrow \mathbb{C} \tag{3.1}$$

is a central character of $\mathcal{Z}(\ell_\Theta)$. As explained in the preceding section, there is a corresponding maximal ideal in $\mathcal{Z}(\ell_\Theta)$. This gives us a map associating elements of $\mathfrak{h}^*$ to maximal ideals in $\mathcal{Z}(\ell_\Theta)$:

$$\xi_\Theta : \mathfrak{h}^* \longrightarrow \mathrm{Max}\mathcal{Z}(\ell_\Theta)$$

$$\lambda \mapsto \ker(\Omega_{\Theta,\lambda})$$

From the data $(\lambda, \eta) \in \mathfrak{h}^* \times \mathfrak{n}^*$, we construct a ℓ_Θ-module

$$Y(\lambda, \eta) = \mathcal{U}(\ell_\Theta)/\xi_\Theta(\lambda)\mathcal{U}(\ell_\Theta) \otimes_{\mathcal{U}(\mathfrak{n}_\Theta)} \mathbb{C}_\eta.$$

Here $\mathbb{C}_\eta$ is the one-dimensional $\mathcal{U}(\mathfrak{n}_\Theta)$-module where $\mathfrak{n}_\Theta$ acts by η. This induced module $Y(\lambda, \eta)$ is irreducible. Indeed, if we restrict η to $\mathfrak{n}_\Theta$, it is nondegenerate, so as an $\mathfrak{s}_\Theta$-module, $Y(\lambda, \eta)$ is isomorphic to the nondegenerate Whittaker module $Y_{\xi,\eta}$ defined in [Kos78, 3.6.1] for $\xi = \Omega_{\Theta,\lambda}$. One of Kostant's primary results in [Kos78] is that all nondegenerate Whittaker modules constructed in this way are irreducible. McDowell shows in [McD85, Prop. 2.3] that $Y(\lambda, \eta)$ is also irreducible as an ℓ_Θ-module. We use $Y(\lambda, \eta)$ to construct standard modules in $\mathcal{N}$.

Definition 3.9. The *standard Whittaker module* in $\mathcal{N}$ associated to $\lambda \in \mathfrak{h}^*$ and the character $\eta \in \mathfrak{n}^*$ is the $\mathfrak{g}$-module

$$M(\lambda, \eta) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}_\Theta)} Y(\lambda - \rho + \rho_\Theta, \eta).$$

Here $Y(\lambda - \rho + \rho_\Theta, \eta)$ is a $\mathcal{U}(\mathfrak{p}_\Theta)$-module by letting $\mathfrak{u}_\Theta$ act trivially and $M(\lambda, \eta)$ is a $\mathfrak{g}$-module by left multiplication on the first coordinate.

To get a sense for this construction, it is useful to examine extreme values of η. If $\eta = 0$, then Θ is empty, and $M(\lambda, 0) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} Y(\lambda - \rho, 0)$ is a Verma module of highest weight $\lambda - \rho$. If η is nondegenerate, then $M(\lambda, \eta) = Y(\lambda, \eta)$ is an irreducible Whittaker module with central character χ_θ, as in [Kos78]. Here $\theta = w \cdot \lambda$.

Next we analyze the action of the center $\mathfrak{h}^\Theta$ of ℓ_Θ on $M(\lambda,\eta)$. We will show that it acts semisimply. For a module V in $\mathcal{N}_\eta$ and linear functional $\mu \in \mathfrak{h}^{\Theta *}$, let V_μ be the corresponding $\mathfrak{h}^\Theta$-weight space, and V^μ the corresponding generalized $\mathfrak{h}^\Theta$-weight space (Section 2.1). As in Section 2.1, if $V^\mu \neq 0$, we say μ is a $\mathfrak{h}^\Theta$-*weight* of V. As ℓ_Θ-modules, $M(\lambda,\eta) \simeq \mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda-\rho+\rho_\Theta,\eta)$, where $\mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda-\rho+\rho_\Theta,\eta)$ is a ℓ_Θ-module by the tensor product of the adjoint action of ℓ_Θ on $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ with the action of ℓ_Θ on $Y(\lambda-\rho+\rho_\Theta,\eta)$. Indeed, the map

$$\begin{aligned} p : \mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda-\rho+\rho_\Theta,\eta) &\longrightarrow M(\lambda,\eta) \\ Y \otimes y &\longmapsto Y \otimes y \end{aligned}$$

gives an isomorphism between the two vector spaces, and one can check that p is an ℓ_Θ-module morphism, and thus a $\mathfrak{h}^\Theta$-module morphism. This implies that p induces an isomorphism of $\mathfrak{h}^\Theta$-weight spaces, and all $\mathfrak{h}^\Theta$-weights of $M(\lambda,\eta)$ are $\mathfrak{h}^\Theta$-weights of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda-\rho+\rho_\Theta,\eta)$. This observation allows us to describe the $\mathfrak{h}^\Theta$-weights and $\mathfrak{h}^\Theta$-weight spaces of standard Whittaker modules very explicitly.

The irreducible ℓ_Θ-module $Y(\lambda-\rho+\rho_\Theta,\eta)$ has central character $\Omega_{\Theta,\lambda-\rho+\rho_\Theta}$ (equation 3.1). Because $\mathfrak{h}^\Theta$ is isomorphic to the center of ℓ_Θ, it is a subset of $\mathcal{Z}(\ell_\Theta)$. In particular, we can apply the Harish-Chandra homomorphism γ_Θ to elements of $\mathfrak{h}^\Theta$, and because γ_Θ is a projection onto the $\mathcal{U}(\mathfrak{h})$-component, elements of $\mathfrak{h}^\Theta$ are fixed by γ_Θ. So for any $H \in \mathfrak{h}^\Theta$, and $Y \in Y(\lambda-\rho+\rho_\Theta,\eta)$,

$$H \cdot Y = \Omega_{\Theta,\lambda-\rho+\rho_\Theta}(H)Y = ((\lambda-\rho)(H))Y.$$

For any $\nu \in \mathfrak{h}^*$, we use bold to denote the restriction of ν to $\mathfrak{h}^{\Theta *}$; that is, $\boldsymbol{\nu} = \nu|_{\mathfrak{h}^\Theta} \in \mathfrak{h}^{\Theta *}$. By the discussion above, we see that $\mathfrak{h}^\Theta$ acts on the irreducible ℓ_Θ-module $Y(\lambda-\rho+\rho_\Theta,\eta)$ by $\boldsymbol{\lambda}-\boldsymbol{\rho} \in \mathfrak{h}^{\Theta *}$; i.e.

$$Y(\lambda-\rho+\rho_\Theta,\eta) = Y(\lambda-\rho+\rho_\Theta,\eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}}.$$

Next we define an order relation on $\mathfrak{h}^{\Theta *}$. Let $\Pi - \Theta = \{\alpha_1, \alpha_2, \cdots, \alpha_p\}$. Then $\{\boldsymbol{\alpha_1}, \cdots, \boldsymbol{\alpha_p}\}$ is a basis for $\mathfrak{h}^{\Theta *}$. For $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathfrak{h}^{\Theta *}$, say that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if

$$\boldsymbol{\beta} - \boldsymbol{\alpha} = c_1\boldsymbol{\alpha_1} + c_2\boldsymbol{\alpha_2} + \cdots + c_p\boldsymbol{\alpha_p}$$

for $c_i \in \mathbb{Z}_{\geq 0}$. This defines a partial order on $\mathfrak{h}^{\Theta *}$ [McD85, §1 Prop. 1.8(a)]. Now, we are ready to show that $M(\lambda, \eta)$ decomposes into $\mathfrak{h}^{\Theta}$-weight spaces. First we can analyze the action of $\mathfrak{h}^{\Theta}$ on $\mathcal{U}(\bar{\mathfrak{u}}_{\Theta})$.

Lemma 3.10. *$\mathfrak{h}^{\Theta}$ acts semisimply on $\mathcal{U}(\bar{\mathfrak{u}}_{\Theta})$.*

Proof. Let $Y_i \in \mathfrak{g}_{-\alpha_i}$ for $\alpha_i \in \Pi \backslash \Theta$. By the PBW theorem, $Y_1^{k_1} \cdots Y_p^{k_p}$ form a basis of $\mathcal{U}(\bar{\mathfrak{u}}_{\Theta})$, where $k_i \in \mathbb{Z}_{\geq 0}$. For $H \in \mathfrak{h}^{\Theta}$,

$$\begin{aligned} H \cdot Y_1^{k_1} \cdots Y_p^{k_p} &= [H, Y_1^{k_1}] Y_2^{k_2} \ldots Y_p^{k_p} + Y_1^{k_1} [H, Y_2^{k_2}] \ldots Y_p^{k_p} + \ldots + Y_1^{k_1} \ldots Y_{p-1}^{k_{p-1}} [H, Y_p^{k_p}] \\ &= (-k_1 \boldsymbol{\alpha_1} - k_2 \boldsymbol{\alpha_2} - \ldots - k_p \boldsymbol{\alpha_p})(H) Y_1^{k_1} \cdots Y_p^{k_p}. \end{aligned}$$

So $\mathfrak{h}^{\Theta}$ acts on any element of $\mathcal{U}(\bar{\mathfrak{u}}_{\Theta})$ by a scalar, and the possible $\mathfrak{h}^{\Theta}$-weights are

$$\boldsymbol{\mu} = -k_1 \boldsymbol{\alpha_1} - \ldots - k_p \boldsymbol{\alpha_p}$$

for $k_i \in \mathbb{Z}_{\geq 0}$. This implies that in the ordering described above, $\boldsymbol{\mu} \leq 0$, with equality if and only if $k_i = 0$ for all i. Therefore,

$$\mathcal{U}(\bar{\mathfrak{u}}_{\Theta}) = \bigoplus_{\boldsymbol{\mu} \leq 0} \mathcal{U}(\bar{\mathfrak{u}}_{\Theta})_{\boldsymbol{\mu}}$$

for $\boldsymbol{\mu} = -\sum_{i=1}^{p} k_i \boldsymbol{\alpha_i}$. $\square$

The fact that $\mathfrak{h}^{\Theta}$ acts semisimply on both $\mathcal{U}(\bar{\mathfrak{u}}_{\Theta})$ and $Y(\lambda - \rho + \rho_{\Theta}, \eta)$ implies that $M(\lambda, \eta)$ decomposes into $\mathfrak{h}^{\Theta}$-weight spaces. In particular,

$$M(\lambda, \eta) = \bigoplus_{\boldsymbol{\nu} \leq \boldsymbol{\lambda} - \boldsymbol{\rho}} M(\lambda, \eta)_{\boldsymbol{\nu}}$$

where $M(\lambda, \eta)_{\boldsymbol{\lambda} - \boldsymbol{\rho}} \simeq Y(\lambda - \rho + \rho_{\Theta})$, and $M(\lambda, \eta)_{\boldsymbol{\nu}} \simeq \mathcal{U}(\bar{\mathfrak{u}}_{\Theta})_{\boldsymbol{\mu}} \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_{\Theta}, \eta)$ for $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$. The following proposition lists the basic properties of standard Whittaker modules.

Proposition 3.11. *(i) $M(\lambda, \eta) = M(\mu, \eta)$ if and only if $W_{\Theta} \cdot \lambda = W_{\Theta} \cdot \mu$.*

(ii) $M(\lambda, \eta)$ has a unique irreducible quotient $L(\lambda, \eta)$.

(iii) $L(\lambda, \eta) = L(\mu, \eta)$ if and only if $W_{\Theta} \cdot \lambda = W_{\Theta} \cdot \mu$.

Proof. By [Kos78, Thm. 3.6.1], $Y(\lambda - \rho + \rho_{\Theta}, \eta)$ is completely determined by its central character $\Omega_{\Theta, \lambda - \rho + \rho_{\Theta}}$, and two weights determine the same central character if and only if they lie in the same Weyl group orbit [Hum08, Ch. 1 §10 Thm. 1.10]. This establishes (i). Any submodule $N \subset M(\lambda, \eta)$ is the sum of $\mathfrak{h}^\Theta$-weight spaces,

$$N = \bigoplus N_\mu.$$

Each $N_\mu \subset M(\lambda,\eta)_\mu$ is a ℓ_Θ-submodule. The highest $\mathfrak{h}^\Theta$-weight space of $M(\lambda,\eta)$, $M(\lambda,\eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}}$, is irreducible over ℓ_Θ, and generates $M(\lambda,\eta)$ over $\mathfrak{g}$. This implies that any submodule N must be contained in the sum of $\mathfrak{h}^\Theta$-weight spaces corresponding to $\boldsymbol{\mu} < \boldsymbol{\lambda} - \boldsymbol{\rho}$. We can consider the sum of all proper submodules, which is itself a proper submodule, and is necessarily the unique maximal submodule. This implies that $M(\lambda,\eta)$ has a unique irreducible quotient. To prove (iii), it is enough to observe that $L(\lambda,\eta)$ is uniquely determined by $M(\lambda,\eta)$, which is uniquely determined by $Y(\lambda-\rho+\rho_\Theta,\eta)$. By our remarks earlier, $Y(\lambda-\rho+\rho_\Theta,\eta) = Y(\mu-\rho+\rho_\Theta,\eta)$ if and only if $\lambda \in W_\Theta \cdot \mu$. □

McDowell showed that every irreducible object in $\mathcal{N}$ is obtained in this way [McD85, §2 Thm. 2.9].

Proposition 3.12. *Every simple object in $\mathcal{N}$ is isomorphic to $L(\lambda,\eta)$ for some $\lambda \in \mathfrak{h}^*$ and $\eta \in \mathfrak{n}^*$.*

The $\mathfrak{h}^\Theta$-weight spaces of $M(\lambda,\eta)$ have a richer structure than just that of $\mathfrak{h}^\Theta$-modules. We will explore this structure in the following proposition, but first we must introduce some notation. For any ℓ_Θ-module V, we can restrict the action of ℓ_Θ on V to an action of the semisimple Lie algebra $\mathfrak{s}_\Theta \subset \ell_\Theta$, and we denote the corresponding $\mathfrak{s}_\Theta$-module by $\overline{V}$. Note that as vector spaces, $\overline{V} = V$. The bar is used to indicate that we are considering the space as an $\mathfrak{s}_\Theta$-module and results about modules over semisimple Lie algebras can be applied. Let $\mathcal{N}(\mathfrak{s}_\Theta)$ be the category of finitely generated, $\mathcal{Z}(\mathfrak{s}_\Theta)$-finite, $\mathcal{U}(\mathfrak{n}_\Theta)$-finite $\mathfrak{s}_\Theta$-modules. In other words, this is the category $\mathcal{N}$ for the semisimple Lie algebra $\mathfrak{s}_\Theta$.

Proposition 3.13. *Let $M(\lambda,\eta) = \bigoplus_{\boldsymbol{\nu} \leq \boldsymbol{\lambda}-\boldsymbol{\rho}} M(\lambda,\eta)_{\boldsymbol{\nu}}$ be the decomposition of a standard Whittaker module in $\mathcal{N}_\eta$ into $\mathfrak{h}^\Theta$-weight spaces. For each $\boldsymbol{\nu} \in \mathfrak{h}^{\Theta *}$,*

(i) $M(\lambda,\eta)_{\boldsymbol{\nu}}$ is a finite length ℓ_Θ-module, and

(ii) $\overline{M(\lambda,\eta)_{\boldsymbol{\nu}}}$ is an object in $\mathcal{N}(\mathfrak{s}_\Theta)$.

Proof. If $\eta = 0$, then $\mathfrak{h}^\Theta = \emptyset$ and $\mathfrak{s}_\Theta = \mathfrak{g}$. In this setting, the assertion is trivially true, so we assume $\eta \neq 0$. The action of ℓ_Θ commutes with the action of $\mathfrak{h}^\Theta$, so the $\mathfrak{n}$-weight spaces of $M(\lambda,\eta)$ are ℓ_Θ-stable. This proves that $M(\lambda,\eta)_\nu$ are ℓ_Θ-modules. The vector space $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_\mu$ is finite dimensional because there are only finitely many ways that we can express a given $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$ as a negative sum of roots in $\Pi \backslash \Theta$. This implies that $M(\lambda,\eta)_\nu$ is the tensor product of a finite dimensional ℓ_Θ-module with an irreducible Whittaker module. Such modules are of finite length and have composition factors which are irreducible Whittaker modules (for $\eta|_{\mathfrak{n}_\Theta}$) by [Kos78, §4 Thm. 4.6], which proves (i). Because categories of Whittaker modules are closed under extensions [MS97, §1], this in turn implies that $\overline{M(\lambda,\eta)_\nu}$ is an object in $\mathcal{N}(\mathfrak{s}_\Theta)$. □

The $\mathfrak{h}^\Theta$-weight space structure of $M(\lambda,\eta)$ described in proposition 3.13 is also inherited by its unique irreducible quotient $L(\lambda,\eta)$. Additionally, because the unique maximal submodule $N \subset M(\lambda,\eta)$ described in the proof of Proposition 3.11 has $\mathfrak{h}^\Theta$-weights which are strictly less than $\boldsymbol{\lambda}-\boldsymbol{\rho}$, $L(\lambda,\eta)$ has a unique maximal $\mathfrak{h}^\Theta$-weight, $\boldsymbol{\lambda}-\boldsymbol{\rho}$, with respect to the partial order on $\mathfrak{h}^{\Theta *}$, and all other weights of $L(\lambda,\eta)$ lie in a cone below this "highest" weight. The highest $\mathfrak{h}^\Theta$-weight space of a standard module in $\mathcal{N}$ and the highest $\mathfrak{h}^\Theta$-weight space of its unique irreducible quotient are both isomorphic to an irreducible Whittaker module: $M(\lambda,\eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}} = L(\lambda,\eta)_{\boldsymbol{\lambda}-\boldsymbol{\rho}} = Y(\lambda-\rho+\rho_\Theta,\eta)$.

We finish this section by showing that all modules in $\mathcal{N}_\eta$ decompose into generalized $\mathfrak{h}^\Theta$-weight spaces, and these weight spaces are modules in $\mathcal{N}(\mathfrak{s}_\Theta)$. As above, we use an overline to indicate that we are considering an ℓ_Θ-module to be an $\mathfrak{s}_\Theta$-module, and before stating the theorem, we will describe this relationship more explicitly for generalized $\mathfrak{h}^\Theta$-weight spaces. Let V in $\mathcal{N}_\eta$ and $\boldsymbol{\mu} \in \mathfrak{h}^{\Theta *}$ be a $\mathfrak{h}^\Theta$-weight of V. Let $\mathbb{C}_\mu$ be the one dimensional irreducible $\mathfrak{h}^\Theta$-module where $\mathfrak{h}^\Theta$ acts by $\boldsymbol{\mu}$. We have an isomorphism of ℓ_Θ-modules $V^\mu \simeq \overline{V^\mu} \otimes_{\mathbb{C}} \mathbb{C}_\mu$, where the tensor module is defined by the action

$$(Y+H)\cdot v \otimes z = g \cdot v \otimes z + v \otimes h \cdot z$$

for $Y \in \mathfrak{s}_\Theta$, $H \in \mathfrak{h}^\Theta$, $v \in \overline{V^\mu}$ and $z \in \mathbb{C}_\mu$. Because $\mathfrak{h}^\Theta$ acts by scalars on generalized $\mathfrak{h}^\Theta$-weight spaces, it is clear that V^μ is irreducible if and only if $\overline{V^\mu}$ is irreducible. Additionally, for irreducible $\mathfrak{s}_\Theta$-modules $\overline{V}$ and $\overline{W}$, $\overline{V} \simeq \overline{W}$ if and only if $V \simeq W$ as ℓ_Θ-modules. Now we are ready to state the main theorem of this section.

Lemma 3.14. *Any object V in $\mathcal{N}_\eta$ admits a decomposition*

$$V = \bigoplus_{\mu \in \mathfrak{h}^{\Theta *}} V^\mu$$

where the generalized $\mathfrak{h}^\Theta$-weight spaces V^μ are finite length ℓ_Θ-modules. Moreover, if we restrict the ℓ_Θ-action to the semisimple part $\mathfrak{s}_\Theta \subset \ell_\Theta$ and denote the resulting $\mathfrak{s}_\Theta$-module by $\overline{V^\mu}$, the generalized $\mathfrak{h}^\Theta$-weight spaces $\overline{V^\mu}$ of V are objects in $\mathcal{N}(\mathfrak{s}_\Theta)$.

Proof. By Theorem 3.2, it is enough to consider $V \in \mathcal{N}_{\theta,\eta}$. By [MS97, §1], these categories are stable under subquotients and extensions. The $\mathfrak{h}^\Theta$-semisimplicity of irreducible modules in $\mathcal{N}_{\theta,\eta}$ implies that all modules in $\mathcal{N}_{\theta,\eta}$ are $\mathcal{U}(\mathfrak{h}^\Theta)$-finite. Because objects in $\mathcal{N}$ are finite length and exact sequences of $\mathfrak{g}$-modules in $\mathcal{N}_{\theta,\eta}$ descend to exact sequences of $\mathfrak{h}^\Theta$-weight spaces, the assertion follows from induction in the length of V. □

3.2 Character Theory

In this section, we use the decomposition of a module in $\mathcal{N}_\eta$ into generalized $\mathfrak{h}^\Theta$-weight spaces to define a character theory in the category of Whittaker modules. Our main result is that the character of a module V in $\mathcal{N}_\eta$ completely determines its class in the Grothendieck group $K\mathcal{N}_\eta$.

We begin by recalling the Grothendieck group of an abelian category. Let $\mathcal{C}$ be an abelian category. Let $F\mathcal{C}$ be the free abelian group on the set of isomorphism classes of objects in $\mathcal{C}$. For an object $C \in \mathcal{C}$, let $[C]$ be the corresponding element in $F\mathcal{C}$. Let $\mathcal{E} \subset F\mathcal{C}$ be the subgroup generated by $\{[B] - [A] - [C]\}$ for all short exact sequences

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of objects in $\mathcal{C}$. The *Grothendieck group* $K\mathcal{C}$ of the category $\mathcal{C}$ is the quotient group $F\mathcal{C}/\mathcal{E}$. Let $\{A_i : i \in I\}$ be a set of nonisomorphic representatives of simple objects in $\mathcal{C}$. Assume in addition that objects in $\mathcal{C}$ have finite length. Then

$$K\mathcal{C} \simeq \bigoplus_{i \in I} \mathbb{Z}[A_i]$$

as abelian groups. For two objects $B, C \in \mathcal{C}$, [B]=[C] in $K\mathcal{C}$ if and only if B and C have the same composition factors.

Fix a character $\eta \neq 0$, and let $K\mathcal{N}(\mathfrak{s}_\Theta)$ be the Grothendieck group of the category $\mathcal{N}(\mathfrak{s}_\Theta)$.

Definition 3.15. Let V be an object in $\mathcal{N}_\eta$. The *character* of V is

$$\operatorname{ch} V = \sum_{\mu \in \mathfrak{h}^{\Theta *}} [\overline{V^\mu}] e^\mu$$

where $[\overline{V^\mu}]$ is the element $1 \otimes [\overline{V^\mu}] \in \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_\Theta)$ and e^μ is a formal variable parameterized by $\mu \in \mathfrak{h}^{\Theta *}$.

If $\eta = 0$ and $V \in \mathcal{N}_0$, then we define

$$\operatorname{ch} V = [V] \in K\mathcal{N}.$$

A standard Whittaker module is completely determined by its character.

Proposition 3.16. *The following are equivalent.*

(i) $\operatorname{ch} M(\lambda, \eta) = \operatorname{ch} M(\nu, \eta)$.

(ii) $M(\lambda, \eta) = M(\nu, \eta)$.

Proof. It is clear that (ii) implies (i). Assume that $\operatorname{ch} M(\lambda, \eta) = \operatorname{ch} M(\nu, \eta)$. Then $M(\lambda, \eta)$ and $M(\nu, \eta)$ have the same $\mathfrak{h}^\Theta$-weights, and $[\overline{M(\lambda, \eta)_\mu}] = [\overline{M(\nu, \eta)_\mu}]$ for any such $\mathfrak{h}^\Theta$-weight $\boldsymbol{\mu}$. This implies that $\boldsymbol{\lambda} - \boldsymbol{\rho}$ is an $\mathfrak{h}^\Theta$-weight of $M(\nu, \eta)$, so $\boldsymbol{\lambda} - \boldsymbol{\rho} \leq \boldsymbol{\nu} - \boldsymbol{\rho}$. But also, $\boldsymbol{\nu} - \boldsymbol{\rho}$ is an $\mathfrak{h}^\Theta$-weight of $M(\lambda, \eta)$, so $\boldsymbol{\nu} - \boldsymbol{\rho} \leq \boldsymbol{\lambda} - \boldsymbol{\rho}$ and thus $\boldsymbol{\lambda} - \boldsymbol{\rho} = \boldsymbol{\nu} - \boldsymbol{\rho}$. Because $M(\lambda, \eta)_{\boldsymbol{\lambda} - \boldsymbol{\rho}} = Y(\lambda - \rho + \rho_\Theta, \eta)$ and $M(\nu, \eta)_{\boldsymbol{\nu} - \boldsymbol{\rho}} = Y(\nu - \rho + \rho_\Theta, \eta)$, we have

$$[\overline{Y(\lambda - \rho + \rho_\Theta, \eta)}] = [\overline{Y(\nu - \rho + \rho_\Theta, \eta)}] \in \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_\Theta).$$

Because $\overline{Y(\lambda - \rho + \rho_\Theta)}$ and $\overline{Y(\nu - \rho + \rho_\Theta)}$ are irreducible $\mathfrak{s}_\Theta$-modules, $[\overline{Y(\lambda - \rho + \rho_\Theta, \eta)}] = [\overline{Y(\nu - \rho + \rho_\Theta, \eta)}]$ implies that $\overline{Y(\lambda - \rho + \rho_\Theta, \eta)} \simeq \overline{Y(\nu - \rho + \rho_\Theta, \eta)}$ as $\mathfrak{s}_\Theta$-modules. Irreducible Whittaker modules for a fixed η are completely determined by their central character [Kos78, §3 Thm. 3.6.1], so both modules have central character $\Omega_{\Theta, \lambda - \rho + \rho_\Theta}$. This is only possible if $W_\Theta \cdot \lambda = W_\Theta \cdot \nu$, which implies by Proposition 3.11 (i) that $M(\lambda, \eta) = M(\nu, \eta)$. □

Because any module V in $\mathcal{N}_{\theta, \eta}$ has central character χ_θ, there are only finitely many irreducible modules in the category $\mathcal{N}_{\theta, \eta}$. Let $\{L(\lambda_1, \eta), \ldots, L(\lambda_m, \eta)\}$ be the distinct irreducible modules in $\mathcal{N}_{\theta, \eta}$. Any module V in $\mathcal{N}_{\theta, \eta}$ must have composition factors on this list, so by Lemma 3.14, the $\mathfrak{h}^{\Theta}$-weights $\boldsymbol{\mu}$ of V that show up in the character must be of the form $\boldsymbol{\mu} = \boldsymbol{\lambda_i} - \boldsymbol{\rho} - \sum_{j=1}^{p} m_j \boldsymbol{\alpha_j}$ for $1 \leq i \leq m$ and $m_j \in \mathbb{Z}_{\geq 0}$. Let $S_0 = \{\boldsymbol{\lambda_1} - \boldsymbol{\rho}, \ldots, \boldsymbol{\lambda_m} - \boldsymbol{\rho}\} \subset \mathfrak{h}^{\Theta *}$ be the collection of highest weights of irreducible objects in $\mathcal{N}_{\theta,\eta}$.

If V and W are isomorphic objects in $\mathcal{N}_{\theta,\eta}$, then $\operatorname{ch} V = \operatorname{ch} W$. Hence we have a well-defined map

$$\operatorname{ch} : F\mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\boldsymbol{\mu} \leq S_0} \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_{\Theta}) e^{\boldsymbol{\mu}}$$

given by $\operatorname{ch}[V] = \operatorname{ch} V$. Here, $\boldsymbol{\mu} \leq S_0$ means that $\boldsymbol{\mu} \leq \boldsymbol{\lambda_i} - \boldsymbol{\rho}$ for some $\boldsymbol{\lambda_i} - \boldsymbol{\rho} \in S_0$.

Because the coefficients of $e^{\boldsymbol{\mu}}$ in the character of a module V are tensor products of complex numbers with isomorphism classes in the Grothendieck group $K\mathcal{N}(\mathfrak{s}_{\Theta})$, for any short exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

of objects in $\mathcal{N}_{\theta,\eta}$, $\operatorname{ch} V = \operatorname{ch} U + \operatorname{ch} W$. Therefore, ch descends to a homomorphism

$$\operatorname{ch} : K\mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\boldsymbol{\mu} \leq S_0} \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_{\Theta}) e^{\boldsymbol{\mu}}$$

which we call by the same name. Our main result of this section is the following.

Theorem 3.17. $\operatorname{ch} : K\mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\boldsymbol{\mu} \leq S_0} \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_{\Theta}) e^{\boldsymbol{\mu}}$ *is an injective homomorphism.*

Proof. Because modules in $\mathcal{N}$ have finite length, an isomorphism class in the Grothendieck group of an object V in $\mathcal{N}_{\theta,\eta}$ is the sum of the isomorphism classes of its composition factors. If $\{L(\nu_1, \eta), \ldots, L(\nu_k, \eta)\}$ is a set of non-isomorphic composition factors of a module $V \in \mathcal{N}_{\theta,\eta}$, then because ch is a homomorphism,

$$\operatorname{ch}[V] = \sum_{i=1}^{k} a_i \operatorname{ch}[L(\nu_i, \eta)],$$

where a_i is the multiplicity of $L(\nu_i, \eta)$ in V and $[V] \in K\mathcal{N}_{\theta,\eta}$. An element $[V] \in K\mathcal{N}_{\theta,\eta}$ is in the kernal of ch when

$$\operatorname{ch}[V] = \sum_{i=1}^{k} a_i \operatorname{ch}[L(\nu_i, \eta)] = 0.$$

Therefore, to show that ch is injective, it is enough to show that the set $\{\operatorname{ch}[L(\lambda_1, \eta)], \ldots, \operatorname{ch}[L(\lambda_m, \eta)]\}$ is linearly independent.

Consider a nontrivial linear combination

$$b_1 \operatorname{ch}[L(\lambda_1, \eta)] + \cdots + b_m \operatorname{ch}[L(\lambda_m, \eta)] = 0.$$

As before, let $S_0 = \{\boldsymbol{\lambda_1} - \boldsymbol{\rho}, \ldots, \boldsymbol{\lambda_m} - \boldsymbol{\rho}\} \subset \mathfrak{h}^{\Theta *}$ be the collection of the highest $\mathfrak{h}^{\Theta}$-weights of the irreducible objects in $\mathcal{N}_{\theta,\eta}$. Note that the elements $\{\lambda_i\}_{i=1}^m \subset \mathfrak{h}^*$ are distinct, but it is possible that when restricted to $\mathfrak{h}^{\Theta}$, $\boldsymbol{\lambda_i} = \boldsymbol{\lambda_j}$ for some $i \neq j$, so S_0 might have repeated elements. Choose a maximal element of this set, $\boldsymbol{\lambda_j} - \boldsymbol{\rho}$. Because $\boldsymbol{\lambda_j} - \boldsymbol{\rho}$ is a maximal element of S_0, it can only appear as a highest weight of modules in $\{L(\lambda_1, \eta), \ldots, L(\lambda_m, \eta)\}$.

Because the linear combination of irreducible characters vanishes, the coefficient of $e^{\boldsymbol{\lambda_j} - \boldsymbol{\rho}}$ must vanish as well. That coefficient is

$$b_{i_1}[\overline{L(\lambda_{i_1}, \eta)}_{\boldsymbol{\lambda_j} - \boldsymbol{\rho}}] + \cdots + b_{i_n}[\overline{L(\lambda_{i_n}, \eta)}_{\boldsymbol{\lambda_j} - \boldsymbol{\rho}}] = 0$$

where $\{\lambda_{i_1}, \ldots, \lambda_{i_n}\} \subset \{\lambda_1, \ldots, \lambda_m\}$ are the elements of $\mathfrak{h}^*$ so that $\boldsymbol{\lambda_{i_1}} - \boldsymbol{\rho} = \cdots = \boldsymbol{\lambda_{i_n}} - \boldsymbol{\rho} = \boldsymbol{\lambda_j} - \boldsymbol{\rho}$. The highest $\mathfrak{h}^{\Theta}$-weight space of an irreducible module in $\mathcal{N}$ is an irreducible Whittaker module for $\mathfrak{s}_{\Theta}$, namely

$$L(\lambda_i, \eta)_{\boldsymbol{\lambda_i} - \boldsymbol{\rho}} = Y(\lambda_i - \rho + \rho_{\Theta}, \eta).$$

Therefore, we have a vanishing linear combination of isomorphism classes of irreducible objects in $\mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}(\mathfrak{s}_{\Theta})$:

$$b_{i_i}[\overline{Y(\lambda_{i_1} - \rho + \rho_{\Theta}, \eta)}] + \cdots + b_{i_n}[\overline{Y(\lambda_{i_n} - \rho + \rho_{\Theta}, \eta)}] = 0$$

Each of the classes in the above sum must be distinct because the corresponding irreducible modules are non-isomorphic. Distinct isomorphism classes of irreducible objects in a Grothendieck group must be linearly independent, so we conclude that $b_{i_1} = \cdots = b_{i_n} = 0$. This contradicts the assumption that $b_i \neq 0$, so a nontrivial linear combination of irreducible characters cannot exist, and ch must be injective. □

This immediately implies the following corollary.

Corollary 3.18. *Let V and W be objects in $\mathcal{N}_{\theta,\eta}$. Then the following are equivalent:*

(i) $\operatorname{ch} V = \operatorname{ch} W$.

(ii) V and W have the same composition factors.

We complete this section with an explicit calculation of the character of a standard Whittaker module. Let $M(\lambda,\eta)$ be the standard Whittaker module determined by $\lambda \in \mathfrak{h}^*$ and $\eta \in \mathfrak{n}^*$. Recall that as ℓ_Θ-modules, $M(\lambda,\eta) = \mathcal{U}(\bar{\mathfrak{u}}_\Theta) \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\Theta, \eta)$. The Cartan subalgebra $\mathfrak{h}$ acts semisimply on $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$, and the collection of $\mathfrak{h}$-weights of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ are

$$Q = \left\{ - \sum_{\alpha \in \Sigma^+ \backslash \Sigma^+_\Theta} m_\alpha \alpha \,\middle|\, m_\alpha \in \mathbb{Z}_{\geq 0} \right\}.$$

As described in Section 3.1, $M(\lambda,\eta)$ decomposes into $\mathfrak{h}^\Theta$-weight spaces of the form

$$M(\lambda,\eta)_\nu = \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} \otimes_{\mathbb{C}} Y(\lambda - \rho + \rho_\Theta, \eta)$$

for $\boldsymbol{\mu} \leq 0$ in $\mathfrak{h}^{\Theta *}$. The $\mathfrak{h}^\Theta$-weight space of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ corresponding to a $\mathfrak{h}^\Theta$-weight $\boldsymbol{\mu} \leq 0$ is the sum of the $\mathfrak{h}$-weight spaces of $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ corresponding to $\mathfrak{h}$-weights that restrict to $\boldsymbol{\mu}$ on $\mathfrak{h}^\Theta$; i.e. for $\mu \in \mathfrak{h}^\Theta$,

$$\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_\kappa.$$

We define a function $p : Q \to \mathbb{N}$ by $p(\kappa) = \dim \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_\kappa$.[1] By [McD85, §2 Lem. 2.2(b)], each $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}}$ is a finite-dimensional ℓ_Θ-module, so the $\mathfrak{s}_\Theta$-module $\overline{M(\lambda,\eta)_\nu}$ is the direct sum of a finite-dimensional $\mathfrak{s}_\Theta$-module and an irreducible $\mathfrak{s}_\Theta$-module. This allows us to apply [Kos78, §4 Thm. 4.6] and conclude that $\mathfrak{n}_\Theta$ acts on $\overline{M(\lambda,\eta)_\nu}$ by the nondegenerate character $\eta|_{\mathfrak{n}_\Theta}$ and that $\overline{M(\lambda,\eta)_\nu}$ has composition series length equal to $\dim \mathcal{U}(\bar{\mathfrak{u}}_\Theta)_{\boldsymbol{\mu}} = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} p(\kappa)$. Furthermore, [Kos78, §4 Thm. 4.6] implies that the composition factors of $\overline{M(\lambda,\eta)_\nu}$ are

$$\{Y(\lambda - \rho + \rho_\Theta + \kappa, \eta) \mid \kappa \in Q \text{ and } \boldsymbol{\kappa} = \boldsymbol{\mu}\}.$$

This implies that in the Grothendieck group $K\mathcal{N}(\mathfrak{s}_\Theta)$,

$$[\overline{M(\lambda,\eta)_\nu}] = \sum_{\kappa \in Q, \kappa|_{\mathfrak{h}^\Theta} = \boldsymbol{\mu}} p(\kappa)[\overline{Y(\lambda - \rho + \rho_\Theta + \kappa, \eta)}].$$

Therefore,

$$\operatorname{ch} M(\lambda,\eta) = \sum_{\nu \in \mathfrak{h}^{\Theta *}} [\overline{M(\lambda,\eta)_\nu}] e^{\nu} = \sum_{\kappa \in Q} p(\kappa)[\overline{Y(\lambda - \rho + \rho_\Theta + \nu, \eta)}] e^{\boldsymbol{\lambda} - \boldsymbol{\rho} + \boldsymbol{\kappa}}. \tag{3.2}$$

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[1] This function can be interpreted combinatorially as counting the number of distinct ways that $\nu \in \mathfrak{h}^*$ can be expressed as a sum of roots in $\Sigma^+ \backslash \Sigma^+_\Theta$. This is a slight modification of Kostant's partition function.

CHAPTER 4

A CATEGORY OF TWISTED SHEAVES

In this chapter, we introduce the geometric objects that correspond to Whittaker modules. Throughout this chapter, $\mathfrak{g}$ is a complex reductive Lie algebra, $\mathfrak{h}$ is the abstract Cartan subalgebra of $\mathfrak{g}$ [Mil93, §2], and X is the flag variety of $\mathfrak{g}$. Denote by $\Sigma^+ \subset \Sigma \subset \mathfrak{h}^*$ the corresponding set of positive roots in the root system of $\mathfrak{g}$ and by W the Weyl group of Σ.

4.1 $\mathcal{D}$-modules on Flag Varieties

We start by recalling a few essential facts about $\mathcal{D}$-modules on flag varieties. This section includes background that will be used in the main arguments of Chapters 5 and 6.

4.1.1 Beilinson–Bernstein Localization

A key ingredient in this story is the localization theory of Beilinson and Bernstein, which we briefly review here. Full details can be found in [BB81, Milb]. In [BB81], Beilinson and Bernstein construct a twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X for each $\lambda \in \mathfrak{h}^*$. (In the notation of Section 2, $\mathcal{D}_\lambda = \mathcal{D}_{X,\lambda+\rho}$.) They show that for any $\mu \in \theta = W \cdot \lambda$, the global sections $\Gamma(X, \mathcal{D}_\mu)$ of $\mathcal{D}_\mu$ are equal to $\mathcal{U}_\theta$. This implies that the global sections functor Γ maps quasicoherent $\mathcal{D}_\lambda$-modules into $\mathcal{U}(\mathfrak{g})$-modules with central character χ_θ; that is, there is a left exact functor

$$\Gamma : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{U}_\theta).$$

Beilinson and Bernstein define a *localization functor*

$$\Delta_\lambda : \mathcal{M}(\mathcal{U}_\theta) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$$

by $\Delta_\lambda(V) = \mathcal{D}_\lambda \otimes_{\mathcal{U}_\theta} V$ for $V \in \mathcal{M}(\mathcal{U}_\theta)$. This functor is right exact and is a left adjoint to Γ. In [BB81], it is shown that for antidominant regular $\lambda \in \mathfrak{h}^*$, Δ_λ is an equivalence of categories, and its inverse is Γ.

4.1.2 Translation Functors

Fix $\lambda \in \mathfrak{h}^*$, and let $\mathcal{D}_\lambda$ be the corresponding homogeneous twisted sheaf of differential operators. Any μ in the weight lattice $P(\Sigma)$ (Section 2.1) naturally determines a G-homogeneous invertible $\mathcal{O}_X$-module $\mathcal{O}(\mu)$ on X. Twisting by such $\mathcal{O}_X$-modules defines a functor

$$-(\mu) : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_{\lambda+\mu})$$

by $\mathcal{V}(\mu) = \mathcal{O}(\mu) \otimes_{\mathcal{O}_X} \mathcal{V}$ for $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$. We call this functor the *geometric translation functor*. It is evidently an equivalence of categories, and it also induces an equivalence of categories on $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ (resp. $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$) with $\mathcal{M}_{qc}(\mathcal{D}_{\lambda+\mu})$ (resp. $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\mu})$).

4.1.3 Intertwining Functors

Let θ be a Weyl group orbit in $\mathfrak{h}^*$ consisting of regular elements. Then by Section 4.1.1, the bounded derived category $D^b(\mathcal{M}(\mathcal{U}_\theta))$ of $\mathcal{U}_\theta$-modules is equivalent to the bounded derived category $D^b(\mathcal{M}_{qc}(\mathcal{D}_\lambda))$ of quasicoherent $\mathcal{D}_\lambda$-modules for any $\lambda \in \theta$. In particular, for any $\lambda, \mu \in \mathfrak{h}^*$, $D^b(\mathcal{M}_{qc}(\mathcal{D}_\lambda))$ and $D^b(\mathcal{M}_{qc}(\mathcal{D}_\mu))$ are equivalent, and this equivalence is given by the functor $L\Delta_\mu \circ R\Gamma$ from $D^b(\mathcal{M}_{qc}(\mathcal{D}_\mu))$ into $D^b(\mathcal{M}_{qc}(\mathcal{D}_\lambda))$. In this section, we give a geometric construction of a functor isomorphic to this functor. We follow the construction in [Milb, Ch. 3 §3] and for brevity, we omit proofs that can be found in that document.

Define an action of $G = \mathrm{Int}(\mathfrak{g})$ on $X \times X$ by

$$g(x, y) = (gx, gy)$$

for $g \in G$ and $x, y, \in X$. The G-orbits are smooth subvarieties of $X \times X$, and can be parameterized in the following way. Given x, y in X and corresponding Borel subalgebras $\mathfrak{b}_x, \mathfrak{b}_y$, we can choose a Cartan subalgebra $\mathfrak{c}$ contained in $\mathfrak{b}_x \cap \mathfrak{b}_y$. Let $\mathfrak{n}_x = [\mathfrak{b}_x, \mathfrak{b}_x]$ and $\mathfrak{n}_y = [\mathfrak{b}_y, \mathfrak{b}_y]$. Then $\mathfrak{b}_x$ and $\mathfrak{b}_y$ determine a specialization [Mil93, §2] of $(\mathfrak{h}^*, \Sigma, \Sigma^+)$ into $(\mathfrak{c}^*, R, R_x^+)$, and $(\mathfrak{c}^*, R, R_y^+)$, respectively, where R is the root system of $(\mathfrak{g}, \mathfrak{c})$, $R_x^+ \subset R$ is the collection of positive roots determined by $\mathfrak{n}_x$, and $R_y^+ \subset R$ is the collection of positive roots determined by $\mathfrak{n}_y$. The positive root systems R_x^+ and R_y^+ are related by $w(R_x^+) = R_y^+$ for some Weyl group element $w \in W$, and this w does not depend on choice of Cartan subalgebra in $\mathfrak{b}_x \cap \mathfrak{b}_y$. We say that $\mathfrak{b}_y$ is in *relative position* w with respect to $\mathfrak{b}_x$. Let

$s : \mathfrak{h}^* \to \mathfrak{c}^*$ be the specialization determined by $\mathfrak{b}_x$, and $s' : \mathfrak{h}^* \to \mathfrak{c}^*$ be the specialization determined by $\mathfrak{b}_y$. Then $s' = s \circ w$, so $\mathfrak{b}_x$ is in relative position w^{-1} to $\mathfrak{b}_y$. For $w \in W$, let

$$Z_w = \{(x, y) \in X \times X | \mathfrak{b}_y \text{ is in relative position } w \text{ with respect to } \mathfrak{b}_x\}. \tag{4.1}$$

This gives us a parameterization of G-orbits in $X \times X$ [Milb, Ch. 3 §3 Lem. 3.1].

Lemma 4.1. *(i) Sets Z_w for $w \in W$ are smooth subvarieties of $X \times X$.*

(ii) The map $w \mapsto Z_w$ is a bijection of W onto the set of G-orbits in $X \times X$.

Denote by p_1 and p_2 the projections of Z_w onto the first and second factors of $X \times X$, respectively. Then p_i for $i = 1, 2$ are locally trivial fibrations with fibers isomorphic to affine spaces of dimension $\ell(w)$. Additionally, they are affine morphisms [Milb, Ch. 3 §3 Lem. 3.2]. Let $\omega_{Z_w|X}$ be the invertible $\mathcal{O}_{Z_w}$-module of top degree relative differential forms for the projection $p_1 : Z_w \to X$ and let $\mathcal{T}_w$ be its inverse sheaf. Then $\mathcal{T}_w = p_1^*(\mathcal{O}(\rho - w\rho))$, and there is a natural isomorphism [Milb, Ch. 3 §3 Lem. 3.3]

$$(\mathcal{D}_{w\lambda})^{p_1} = (\mathcal{D}_\lambda^{p_2})^{\mathcal{T}_w}.$$

The morphism $p_2 : Z_w \to X$ is a surjective submersion, so the inverse image functor

$$p_2^+ : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_\lambda^{p_2})$$

is exact. Because twisting by an invertible sheaf is also an exact functor, we can define a functor

$$LI_w : D^b(\mathcal{M}(\mathcal{D}_\lambda)) \to D^b(\mathcal{M}(\mathcal{D}_{w\lambda}))$$

by the formula

$$LI_w(\mathcal{V}^\cdot) = p_{1+}(\mathcal{T}_w \otimes_{Z_w} p_2^+(\mathcal{V}^\cdot))$$

for $\mathcal{V}^\cdot \in D^b(\mathcal{M}(\mathcal{D}_\lambda))$. This is the left derived functor of the functor

$$I_w : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_{w\lambda}),$$

where for $\mathcal{V} \in \mathcal{M}(\mathcal{D}_\lambda)$,

$$I_w(\mathcal{V}) = H^0 p_{1+}(\mathcal{T}_w \otimes_{\mathcal{O}_{Z_w}} p_2^+(\mathcal{V})).$$

We call the right exact functor I_w the *intertwining functor* attached to $w \in W$. This functor is the geometric analogue to the functor described at the beginning of this section. It establishes our desired equivalence of derived categories [Milb, Ch. 3 §3 Thm 3.20].

Proposition 4.2. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then LI_w is an equivalence of the category $D^b(\mathcal{M}(\mathcal{D}_\lambda))$ with $D^b(\mathcal{M}(\mathcal{D}_{w\lambda}))$.*

We complete this section by describing a useful "product formula," [Milb, Ch. 3 §3 Cor. 3.8, Lem. 3.15] and an estimate on cohomological dimension of intertwining functors.

Proposition 4.3. *Let $w, w' \in W$ be such that $\ell(w'w) = \ell(w') + \ell(w)$. Then*

$$LI_{w'w} = LI_{w'} \circ LI_w$$

and

$$I_{w'w} = I_{w'} \circ I_w.$$

Given $w \in W$, put

$$\Sigma_w^+ = \{\alpha \in \Sigma^+ | w\alpha \in -\Sigma^+\},$$

and given $\lambda \in \mathfrak{h}^*$, put

$$\Sigma_\lambda = \{\alpha \in \Sigma | \alpha^\vee(\lambda) \in \mathbb{Z}\}.$$

This gives us a useful estimate on the left cohomological dimension of intertwining functors [Milb, Ch. 3 §3 Thm. 3.21].

Proposition 4.4. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$. Then the left cohomological dimension of I_w is less than or equal to $|\Sigma_w^+ \cap \Sigma_\lambda|$.*

A corollary to this is the following [Milb, Ch. 3 §3 Cor. 3.22].

Corollary 4.5. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be such that $\Sigma_w^+ \cap \Sigma_\lambda = \emptyset$. Then*

$$I_w : \mathcal{M}(\mathcal{D}_\lambda) \to \mathcal{M}(\mathcal{D}_{w\lambda})$$

is an equivalence of categories.

Recall that for any $S \subset \Sigma^+$, we say that $\lambda \in \mathfrak{h}^*$ is *S-antidominant* if it is α-antidominant for all $\alpha \in S$. The following theorem [Milb, Ch. 3 §3 Thm. 3.23] will play a role in future arguments.

Theorem 4.6. *Let $w \in W$ and $\lambda \in \mathfrak{h}^*$ be Σ_w^+-antidominant. Then the functors $R\Gamma \circ LI_w$ and $R\Gamma$ from $D^b(\mathcal{M}(\mathcal{D}_\lambda))$ into $D^b(\mathcal{M}(\mathcal{U}_\Theta))$ are isomorphic.*

4.1.4 Intertwining Functors for Simple Reflections and U-functors

In this section, we examine intertwining functors I_{s_α} attached to simple reflections $\alpha \in \Pi$, and define related U-functors, following [Milb, Ch. 3 §8]. These functors will be critical to the arguments in Chapter 6. By Corollary 4.5, if $\alpha^\vee(\lambda)$ is not an integer, I_{s_α} is an equivalence of the categories $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ and $\mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$. A more interesting case is when $\alpha^\vee(\lambda)$ is an integer. This is what we will examine now.

Let $\alpha \in \Pi$ be a simple root, and denote by X_α the variety of parabolic subalgebras of type α. Let p_α be the natural projection of X onto X_α, and let $Y_\alpha = X \times_{X_\alpha} X$ be the fiber product of X with X relative to the morphism p_α. Denote by q_1 and q_2 the projections of Y_α onto the first and second factors, respectively. Then we have the following commutative diagram.

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

There is a natural embedding of Y_α into $X \times X$ that identifies Y_α with the closed subvariety $Z_1 \cup Z_{s_\alpha}$ of $X \times X$. Under this identification, Z_1 is a closed subvariety of Y_α, and Z_{s_α} is an open, dense, affinely imbedded subvariety of Y_α [Milb, Ch. 3 §8 Lem. 8.1].

Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha^\vee(\lambda)$ is an integer. Let $\mathcal{L}$ be the invertible O_{Y_α}-module on Y_α given by

$$\mathcal{L} = q_1^*(\mathcal{O}((-p+1)s_\alpha\rho + \alpha) \otimes_{\mathcal{O}_{Y_\alpha}} q_2^*(\mathcal{O}((-p+1)\rho))^{-1}.$$

This allows us to define functors

$$U^j : \mathcal{M}_{qc}(\mathcal{D}_\lambda) \to \mathcal{M}_{qc}(\mathcal{D}_{s_\alpha\lambda})$$

by the formula

$$U^j(\mathcal{V}) = H^j q_{1+}(q_2^+(\mathcal{V}) \otimes_{\mathcal{O}_{Y_\alpha}} \mathcal{L})$$

for $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ [Milb, Ch. 3 §8, Lem. 8.2]. These functors first appeared in [Milb] as geometric analogues to the U_α functors in [Vog79], and they play a critical role in the algorithm of Chapter 6 for their semisimplicity properties. Because the fibers of q_1 are one-dimensional, $U^j = 0$ for $j \neq -1, 0, 1$. If $\mathcal{V}$ is irreducible, the relationship between $U^j(\mathcal{V})$ and $I_{s_\alpha}(\mathcal{V})$ is captured in the following theorem [Milb, Ch. 3 §8 Thm. 8.4].

Theorem 4.7. *Let $\lambda \in \mathfrak{h}^*$ be such that $p = -\alpha^\vee(\lambda)$ is an integer, and $\mathcal{V} \in \mathcal{M}_{qc}(\mathcal{D}_\lambda)$ an irreducible $\mathcal{D}_\lambda$-module. Then either*

(i) $U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = \mathcal{V}(p\alpha)$ *and* $U^0(\mathcal{V}) = 0$, *and in this case* $I_{s_\alpha}(\mathcal{V}) = 0$ *and* $L^{-1}I_{s_\alpha}(\mathcal{V}) = \mathcal{V}(p\alpha)$; *or*

(ii) $U^{-1}(\mathcal{V}) = U^1(\mathcal{V}) = 0$, *and in this case* $L^{-1}I_{s_\alpha}(\mathcal{V}) = 0$ *and the sequence*

$$0 \to U^0(\mathcal{V}) \to I_{s_\alpha}(\mathcal{V}) \to \mathcal{V}(p\alpha) \to 0$$

is exact. The module $U^0(\mathcal{V})$ *is the largest quasicoherent* $\mathcal{D}_{s_\alpha\lambda}$*-submodule of* $I_{s_\alpha}(\mathcal{V})$ *different from* $I_{s_\alpha}(\mathcal{V})$.

4.1.5 Holonomic Duality

In this section, we list some results on duality of coherent $\mathcal{D}$-modules. Let $\lambda \in \mathfrak{h}^*$ and $\theta = W \cdot \lambda$. Let $D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$ be the derived category of bounded complexes of coherent $\mathcal{D}_\lambda$-modules. For any complex $\mathcal{V}^\cdot$, we have a duality functor

$$\mathbb{D} : D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda)) \to D^b_{coh}(\mathcal{M}(\mathcal{D}_{-\lambda}))$$

given by the formula

$$\mathbb{D}(\mathcal{V}^\cdot) = RHom_{\mathcal{D}_\lambda}(\mathcal{V}^\cdot, D(\mathcal{D}_\lambda))[\dim X].$$

This operation commutes with translation functors.

Lemma 4.8. *For any weight $\nu \in P(\Sigma)$, the following diagram of functors is commutative*

$$\begin{tikzcd} D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda)) \arrow[r, "\mathbb{D}"] \arrow[d, "-(\nu)"'] & D^b_{coh}(\mathcal{M}(\mathcal{D}_{-\lambda})) \arrow[d, "-(-\nu)"] \\ D^b_{coh}(\mathcal{M}(\mathcal{D}_{\lambda+\nu})) \arrow[r, "\mathbb{D}"] & D^b_{coh}(\mathcal{M}(\mathcal{D}_{-\lambda-\nu})) \end{tikzcd}.$$

Proof. Let $\mathcal{V}^\cdot$ be a complex in $D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$. Then $\mathcal{V}(\nu)$ is a complex in $D^b_{coh}(\mathcal{M}(\mathcal{D}_{\lambda+\nu}))$ and

$$\begin{aligned}\mathbb{D}(\mathcal{V}^{\cdot}(\nu)) &= R\operatorname{Hom}_{\mathcal{D}_{\lambda+\nu}}(\mathcal{V}^{\cdot}(\nu), D(\mathcal{D}_{\lambda+\nu}))[\dim X] \\ &= R\operatorname{Hom}_{\mathcal{D}_{\lambda+\nu}}(\mathcal{O}(\nu)\otimes_{\mathcal{O}_X}\mathcal{V}^{\cdot}, \mathcal{O}(\nu)\otimes_{\mathcal{O}_X} D(\mathcal{D}_\lambda)\otimes_{\mathcal{O}_X}\mathcal{O}(-\nu))[\dim X] \\ &= R\operatorname{Hom}_{\mathcal{D}_{\lambda+\nu}}(\mathcal{O}(\nu)\otimes_{\mathcal{O}_X}\mathcal{V}^{\cdot}, \mathcal{O}(\nu)\otimes_{\mathcal{O}_X} D(\mathcal{D}_\lambda))[\dim X]\otimes_{\mathcal{O}_X}\mathcal{O}(-\nu) \\ &= R\operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{V}^{\cdot}, D(\mathcal{D}_\lambda))[\dim X]\otimes_{\mathcal{O}_X}\mathcal{O}(-\nu) \\ &= \mathbb{D}(\mathcal{V}^{\cdot})(-\nu).\end{aligned}$$

This completes the proof. □

In the case of holonomic $\mathcal{D}_\lambda$-modules, we can use this duality on derived categories to define a notion of duality on modules. Let $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$ be the thick subcategory of $\mathcal{M}_{coh}(\mathcal{D}_\lambda)$ consisting of holonomic $\mathcal{D}_\lambda$-modules. If $\mathcal{V}$ is an object in $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$, then $\mathbb{D}(D(\mathcal{V}))$ is a complex in $D^b_{coh}(\mathcal{D}_{-\lambda})$ with holonomic cohomology, and $H^p(\mathbb{D}(D(\mathcal{V}))) = 0$ for $p \neq 0$. Therefore, we can define a functor

$$^* : \mathcal{M}_{hol}(\mathcal{D}_\lambda) \to \mathcal{M}_{hol}(\mathcal{D}_{-\lambda})$$

by

$$\mathcal{V}^* = H^0(\mathbb{D}(D(\mathcal{V}))).$$

This is the *holonomic duality functor*. We have the following result.

Theorem 4.9. *(i) The functor $\mathcal{V} \mapsto \mathcal{V}^*$ from $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$ to $\mathcal{M}_{hol}(\mathcal{D}_{-\lambda})$ is an antiequivalence of categories.*

(ii) The functor $\mathcal{V} \mapsto (\mathcal{V}^*)^*$ is isomorphic to the identity functor on $\mathcal{M}_{hol}(\mathcal{D}_\lambda)$.

4.1.6 Inverses of Intertwining Functors

In this section, we use the duality functors introduced in Section 4.1.5 to describe inverses of the intertwining functors of Section 4.1.3, following [Milb, Ch. 3 §4]. As in earlier sections, we omit proofs that can be found in that document. Our first result is the following [Milb, Ch. 3 §4 Lem. 4.2].

Lemma 4.10. *Let $\lambda \in \mathfrak{h}^*$ and $\theta = W\cdot\lambda$. For any $\mathcal{V}^{\cdot} \in D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$,*

$$R\Gamma(\mathbb{D}(\mathcal{V}^{\cdot})) = R\operatorname{Hom}_{\mathcal{D}_\lambda}(\mathcal{V}^{\cdot}, D(\mathcal{D}_\lambda))[\dim X].$$

In other words, the functors $R\Gamma \circ \mathbb{D}$ and $R\operatorname{Hom}_{\mathcal{D}_\lambda}(-, D(\mathcal{D}_\lambda))[\dim X]$ from $D^b_{coh}(\mathcal{M}(\mathcal{D}_\lambda))$ into $D^b(\mathcal{M}(\mathcal{U}_\theta))$ are isomorphic.

Let θ be a regular orbit. For such orbits θ, the homological dimension of the ring $\mathcal{U}_\theta$ is finite [Milb, Ch. 3 §1 Thm. 1.4], and the principal antiautomorphism of $\mathcal{U}(\mathfrak{g})$ induces an isomorphism of the ring opposite to $\mathcal{U}_\theta$ with $\mathcal{U}_{-\theta}$, where $-\theta = W \cdot -\lambda$ for $\lambda \in \theta$. Let $D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$ be the bounded derived category of finitely generated $\mathcal{U}_\theta$-modules. We define a covariant duality functor from $D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta))$ into $D^b(\mathcal{M}_{fg}(\mathcal{U}_{-\theta}))$ by

$$\mathbb{D}_{alg}(V^\cdot) = R\operatorname{Hom}_{\mathcal{U}_\theta}(V^\cdot, D(\mathcal{U}_\theta)).$$

By construction, we have $\mathbb{D}^2_{alg} \simeq id$. The relationship between this algebraic duality and the functor $\mathbb{D}$ defined in Section 4.1.5 is given by the following lemma [Milb, Ch. 3 §4 Lem. 4.3].

Lemma 4.11. *Let $\lambda \in \mathfrak{h}^*$ be regular. Then the following diagram of functors commutes.*

$$\begin{tikzcd} D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda)) \arrow[r, "\mathbb{D}"] \arrow[d, "R\Gamma"'] & D^b(\mathcal{M}_{coh}(\mathcal{D}_{-\lambda})) \arrow[d, "R\Gamma"] \\ D^b(\mathcal{M}_{fg}(\mathcal{U}_\theta)) \arrow[r, "\mathbb{D}_{alg}[\dim X]"] & D^b(\mathcal{M}_{fg}(\mathcal{U}_{-\theta})) \end{tikzcd} .$$

We will use this relationship to compute an inverse for intertwining functors. Let α be a simple root and $\mathcal{V}^\cdot$ a complex in $D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda))$. If λ is α-antidominant, then by Theorem 4.6

$$R\Gamma(\mathcal{V}^\cdot) = R\Gamma(LI_{s_\alpha}(\mathcal{V}^\cdot)).$$

This implies that

$$R\Gamma(\mathbb{D}(\mathcal{V}^\cdot)) = \mathbb{D}_{alg}(R\Gamma(\mathcal{V}^\cdot))[\dim X] = \mathbb{D}_{alg}(R\Gamma(LI_{s_\alpha}(\mathcal{V}^\cdot)))[\dim X] = R\Gamma(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))).$$

Here $\mathbb{D}(\mathcal{V}^\cdot)$ is in $D^b(\mathcal{M}_{coh}(\mathcal{D}_{-\lambda}))$ and $\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))$ is in $D^b(\mathcal{M}_{coh}(\mathcal{D}_{-s_\alpha\lambda}))$. Therefore, $-s_\alpha\lambda$ is α-antidominant, and so

$$R\Gamma(\mathbb{D}(\mathcal{V}^\cdot)) = R\Gamma(LI_{s_\alpha}(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot)))).$$

Because $\mathbb{D}(\mathcal{V}^\cdot)$ and $LI_{s_\alpha}(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot)))$ are in $D^b(\mathcal{M}_{coh}(\mathcal{D}_{-\lambda}))$ and $R\Gamma$ is an equivalence of categories, we have

$$\mathbb{D}(\mathcal{V}^\cdot) = LI_{s_\alpha}(\mathbb{D}(LI_{s_\alpha}(\mathcal{V}^\cdot))).$$

Therefore,

$$LI_{s_\alpha} \circ (\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D}) \simeq 1$$

on $D^b(\mathcal{M}_{coh}(\mathcal{D}_{-\lambda}))$. All of these functors commute with twists, so this relationship holds in general; i.e. for $w \in W$,

$$LI_w \circ (\mathbb{D} \circ LI_{w^{-1}} \circ \mathbb{D}) \simeq 1.$$

This proves the main result of this section.

Theorem 4.12. *The quasiinverse of the intertwining functor $LI_w : D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda)) \to D^b(\mathcal{M}_{coh}(\mathcal{D}_{w\lambda}))$ is equal to*

$$\mathbb{D} \circ LI_{w^{-1}} \circ \mathbb{D} : D^b(\mathcal{M}_{coh}(\mathcal{D}_{w\lambda})) \to D^b(\mathcal{M}_{coh}(\mathcal{D}_\lambda)).$$

4.2 Twisted Harish-Chandra Sheaves

The geometric category that emerges as an analogue to the category of Whittaker modules is a certain subcategory of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$. Fix a Harish-Chandra pair $(\mathfrak{g}, K)$ and linear form $\lambda \in \mathfrak{h}^*$. Let $\eta : \mathfrak{k} \to \mathbb{C}$ be a Lie algebra morphism; that is, a linear form on $\mathfrak{k}$ which vanishes on $[\mathfrak{k}, \mathfrak{k}]$. We say that $\mathcal{V}$ is a $(\mathcal{D}_\lambda, K, \eta)$-module if

(i) $\mathcal{V}$ is a coherent $\mathcal{D}_\lambda$-module,

(ii) $\mathcal{V}$ is a K-homogeneous $\mathcal{O}_X$-module, and

(iii) $\pi(\xi) = \mu(\xi) + \eta(\xi)$ for all $\xi \in \mathfrak{k}$, and the morphism

$$\mathcal{D}_\lambda \otimes \mathcal{V} \to \mathcal{V}$$

is K-equivariant. Here π is induced by the $\mathcal{D}_\lambda$-action and μ is the differential of the K-action.

We denote by $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ the category of $(\mathcal{D}_\lambda, K, \eta)$-modules, and we refer to the objects in this category as *η-twisted Harish-Chandra sheaves*. Clearly the cohomology modules of η-twisted Harish-Chandra sheave are η-twisted Harish-Chandra modules. Moreover, the localization functor Δ_λ maps η-twisted Harish-Chandra modules to η-twisted Harish-Chandra sheaves. This category of twisted Harish-Chandra sheaves carries much of the same structure as the non-twisted category described in [Milb, Ch. 4]. The next two results are proven exactly as in the non-twisted case [MS14, §1 Lem. 1.1, Cor. 1.2].

Lemma 4.13. *Any η-twisted Harish-Chandra sheaf is holonomic.*

In particular, this immediately implies that any η-twisted Harish-Chandra sheaf has finite length.

Corollary 4.14. *Any η-twisted Harish-Chandra sheaf is of finite length.*

This in turn implies that any η-twisted Harish-Chandra module is of finite length.

Corollary 4.15. *Any η-twisted Harish-Chandra module is of finite length.*

The first example of twisted Harish-Chandra modules arose in the localization theory of Harish-Chandra modules for semisimple Lie groups with infinite center. An analysis of this example can be found in Appendix B of [HMSW87].

4.3 Standard and Simple Sheaves

In this section, we describe the classification of irreducible η-twisted Harish-Chandra sheaves for a Harish-Chandra pair $(\mathfrak{g}, K)$. To do this, we define standard η-twisted Harish-Chandra sheaves and show that all irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are subsheaves of such standard sheaves. This classification mirrors the untwisted case [Milb, Ch. 4 §5]. We begin with a preliminary result.

Lemma 4.16. *Let $\mathcal{V}$ be an irreducible object in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$. Then the support of $\mathcal{V}$ is the closure of a unique K-orbit Q in X.*

Proof. Because K is connected, the η-twisted Harish-Chandra sheaf $\mathcal{V}$ is irreducible if and only if it is irreducible as a $\mathcal{D}_\lambda$-module. Therefore, the support of $\mathcal{V}$ is an irreducible closed subvariety of X. The support of $\mathcal{V}$ must also be K-invariant, so it is a union of K-orbits. Because K acts on X with finitely many orbits, there must be a unique orbit Q in $\mathrm{supp}(\mathcal{V})$ such that $\dim Q = \dim(\mathrm{supp}(\mathcal{V}))$. Because $\mathrm{supp}(\mathcal{V})$ is closed, we conclude that $\overline{Q} = \mathrm{supp}(\mathcal{V})$. $\square$

Let $\mathcal{V}$ be an irreducible η-twisted Harish-Chandra sheaf and Q the K-orbit such that $\overline{Q} = \mathrm{supp}(\mathcal{V})$. Let $i : Q \to X$ be the natural inclusion. Then $(\mathcal{D}_\lambda)^i$ is a K-homogeneous twisted sheaf of differential operators on Q. Fix $x \in Q$, and let $\mathfrak{b}_x$ be the corresponding

Borel subalgebra of G. Let S_x denote the stabilizer in K of x. Then the Lie algebra of S_x is $\mathfrak{k} \cap \mathfrak{b}_x$. Let $\mathfrak{c}$ be a Cartan subalgebra in $\mathfrak{g}$ contained in $\mathfrak{b}_x$, and $s : \mathfrak{h}^* \to \mathfrak{c}^*$ the specialization at x. Let μ denote the restriction of the specialization of $\lambda + \rho$ to $\mathfrak{k} \cap \mathfrak{b}_x$. Then $(\mathcal{D}_\lambda)^i = \mathcal{D}_{Q,\mu}$ [HMSW87, App. A].

Lemma 4.17. *$i^!(\mathcal{V})$ is an irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-module.*

Proof. By Kashiwara's equivalence of categories (Theorem 2.2), the inverse image $i^!(\mathcal{V})$ is an irreducible $\mathcal{D}_{Q,\mu}$-module. By the compatibility condition in the definition of η-twisted Harish-Chandra sheaves, $i^!(\mathcal{V})$ is also a K-homogeneous $\mathcal{O}_Q$-module such that the differential of the K action differs from the action of $\mathfrak{k}$ through $\mathcal{D}_{Q,\mu}$ by η. Therefore, $i^!(\mathcal{V})$ is an irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_{Q,\mu}, K, \eta)$. $\square$

Because $\mathcal{V}$ is holonomic by Lemma 4.13, and $i^!$ preserves holonomicity [Mila, Ch. V §6], $i^!(\mathcal{V})$ is a holonomic $\mathcal{D}_{Q,\mu}$-module with support equal to Q. This implies that there is some open dense subset $U \subset Q$ so that $i^!(\mathcal{V})|_U$ is a connection, and thus a coherent $\mathcal{O}_U$-module. Because $i^!(\mathcal{V})$ is also K-invariant, $i^!(\mathcal{V})$ must be coherent as an $\mathcal{O}_Q$-module on all of Q, hence a connection on Q. (Generally, we can see by this argument that any irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-module is a K-homogeneous $\mathcal{D}_{Q,\mu}$-connection on Q.)

Therefore, to each irreducible object $\mathcal{V}$ in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$, we can attach a pair (Q, τ) consisting of a K-orbit Q and an irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-module τ such that

(i) $\operatorname{supp}\mathcal{V} = \overline{Q}$, and

(ii) $i^!(\mathcal{V}) = \tau$.

We call the pair (Q, τ) the *standard data* attached to $\mathcal{V}$.

Definition 4.18. Let Q be a K-orbit in X, $i : Q \to X$ be the natural inclusion, and τ an irreducible $\mathcal{M}(\mathcal{D}_{Q,\mu}, K, \eta)$-module. Then $\mathcal{I}(Q, \tau) = i_+(\tau)$ is a coherent holonomic $(\mathcal{D}_\lambda, K, \eta)$-module. We call $\mathcal{I}(Q, \tau)$ the *standard η-twisted Harish-Chandra sheaf* attached to (Q, τ).

As in the untwisted case [Milb, Ch. 4 §5 Thm. 5.3], standard η-twisted Harish-Chandra sheaves have unique irreducible subsheaves.

Lemma 4.19. *Let Q be a K-orbit in X, $i : Q \to X$ be the natural inclusion, and τ an irreducible $\mathcal{M}(\mathcal{D}_{Q,\mu}, K, \eta)$-module. Then the standard η-twisted Harish-Chandra sheaf $\mathcal{I}(Q,\tau)$ has a unique irreducible subsheaf $\mathcal{L}(Q,\tau)$.*

Moreover, the quotient $\mathcal{I}(Q,\tau)/\mathcal{L}(Q,\tau)$ is an η-twisted Harish-Chandra sheaf supported in the boundary of $\overline{Q}$. The classification of irreducible η-twisted Harish-Chandra sheaves is given in the following result [MS14, §3].

Theorem 4.20. *(i) Any irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ with standard data (Q,τ) is isomorphic to $\mathcal{L}(Q,\tau)$.*

(ii) Let Q and Q' be K-orbits in X, and $i : Q \to X$ and $i' : Q' \to X$ the natural inclusions. Let μ' be the restriction of the specialization $\lambda + \rho$ to $\mathfrak{k} \cap \mathfrak{b}_{x'}$ for a fixed $x' \in Q'$, and let τ and τ' be irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$ and $(\mathcal{D}_{Q',\mu'}, K, \eta)$-modules, respectively. Then $\mathcal{L}(Q,\tau) = \mathcal{L}(Q',\tau')$ if and only if $Q = Q'$ and $\tau = \tau'$.

By this classification, we see that understanding irreducible η-twisted Harish-Chandra sheaves reduces to understanding irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-modules on every K-orbit Q. By the argument below the proof of Lemma 4.17, any irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-module is an irreducible $\mathcal{D}_{Q,\mu}$-connection on Q. We can describe all η-twisted irreducible $\mathcal{D}_{Q,\mu}$-connections in the following way. Let $x \in Q$. Let B_x be the Borel subgroup of $\mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathfrak{b}_x$. Any K-homogeneous $\mathcal{O}_Q$-module is completely determined by the action of the stabilizer $S_x = \phi^{-1}(\phi(K) \cap B_x)$ in the geometric fiber at x. If τ is an irreducible $(\mathcal{D}_{Q,\mu}, K, \eta)$-module, it must also be irreducible as a K-homogeneous $\mathcal{O}_Q$-module by the compatibility condition, so the representation of S_x in the geometric fiber of τ is irreducible. Moreover, its differential is a direct sum of a number of copies of the linear form $\mu - \eta|_{\mathfrak{k}\cap\mathfrak{b}_x}$ on $\mathfrak{k} \cap \mathfrak{b}_x$.

Therefore, our problem reduces to finding irreducible K-homogeneous $\mathcal{D}_{Q,\mu}$-connections on K-orbits Q where the η-compatibility condition on the actions of K and $\mathcal{D}_{Q,\mu}$ is satisfied. By the argument above, the following condition describes such modules.

If τ is a K-homogeneous $\mathcal{O}_Q$-module, then we say that τ is *compatible* with $(\lambda,\eta) \in \mathfrak{h}^* \times \eta^*$ if

$$\mu(\xi) = \nu(\xi) + \eta(\xi)$$

for any $\xi \in \mathfrak{k} \cap \mathfrak{b}_x$, where ν is the differential of the S-action on the geometric fiber $T_x(\tau)$. We have proven the following proposition.

Proposition 4.21. *The following statements are equivalent.*

(i) τ *is an irreducible* $(\mathcal{D}_{Q,\mu}, K, \eta)$*-module.*

(ii) τ *is an irreducible* K*-homogeneous* $\mathcal{O}_Q$*-module compatible with* $(\lambda, \eta) \in \mathfrak{h}^* \times \mathfrak{k}^*$*.*

4.4 Costandard Sheaves

In this section, we construct costandard objects in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, K, \eta)$ using the holonomic duality functor defined in Section 4.1.5. Let Q be a K-orbit in X and τ an irreducible K-homogeneous connection on Q compatible with $(\lambda + \rho, \eta) \in \mathfrak{h}^* \times \mathfrak{k}^*$. Let $\mathcal{L}(Q, \tau)$ be the corresponding irreducible η-twisted Harish-Chandra sheaf, and $\mathcal{I}(Q, \tau)$ the corresponding standard η-twisted Harish-Chandra sheaf. Then $\mathcal{L}(Q, \tau)$ is an irreducible holonomic $\mathcal{D}_\lambda$-module supported on the closure of the orbit Q by Lemma 4.13 and Lemma 4.19. Therefore, by Theorem 4.9, $\mathcal{L}(Q, \tau)^*$ is an irreducible holonomic $\mathcal{D}_{-\lambda}$-module whose support is contained in the closure of Q.

Let $X' = X - \partial Q$. Then $j : Q \to X'$ is a closed immersion, and $k : X' \to X$ is an open immersion. We have an exact sequence of η-twisted Harish-Chandra sheaves

$$0 \to \mathcal{L}(Q, \tau) \to \mathcal{I}(Q, \tau) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q} = \mathcal{I}(Q, \tau)/\mathcal{L}(Q, \tau)$ is supported on ∂Q. Because k is an open immersion, k^+ is exact, and for any $\mathcal{D}_\lambda$-module $\mathcal{V}$, $k^+(\mathcal{V}) = \mathcal{V}|_{X'}$. Therefore, by restricting to X', we see that $\mathcal{L}(Q, \tau)|_{X'} = \mathcal{I}(Q, \tau)|_{X'}$. Because duality is local, we have

$$\mathcal{L}(Q, \tau)^*|_{X'} = (\mathcal{L}(Q, \tau)|_{X'})^* = (\mathcal{I}(Q, \tau)|_{X'})^* = j_+(\tau)^*.$$

Moreover, by Kashiwara's equivalence of categories, j_+ commutes with duality, so we have

$$\mathcal{L}(Q, \tau)^*|_{X'} = j_+(\tau^*).$$

On the other hand, τ^* is an irreducible η-twisted K-homogeneous connection on Q compatible with $(-\lambda + \rho, \eta)$. Hence,

$$\mathcal{L}(Q, \tau)^*|_{X'} = j_+(\tau^*) = \mathcal{L}(Q, \tau^*))|_{X'},$$

and we have the following result.

Lemma 4.22.

$$\mathcal{L}(Q,\tau)^* = \mathcal{L}(Q,\tau^*).$$

Dualizing, we get

$$\mathcal{L}(Q,\tau^*)^* = \mathcal{L}(Q,\tau).$$

Denote by $\mathcal{M}(Q,\tau)$ the η-twisted Harish-Chandra sheaf $\mathcal{I}(Q,\tau^*)^*$. We call this the *co-standard η-twisted Harish-Chandra sheaf* attached to the geometric data (Q,τ). There is a natural inclusion $\mathcal{L}(Q,\tau^*) \to \mathcal{I}(Q,\tau^*)$. By dualizing, we get a natural epimorphism $\mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau)$, so $\mathcal{L}(Q,\tau)$ is a quotient of $\mathcal{M}(Q,\tau)$. The main properties of costandard η-twisted Harish-Chandra sheaves are the following.

Proposition 4.23. *(i) The length of $\mathcal{M}(Q,\tau)$ is equal to the length of $\mathcal{I}(Q,\tau)$.*

(ii) The irreducible η-twisted Harish-Chandra sheaf $\mathcal{L}(Q,\tau)$ is the unique irreducible quotient of $\mathcal{M}(Q,\tau)$. The kernal of this projection is supported on the boundary ∂Q of Q.

Proof. Because duality preserves irreducibility, the composition series of $\mathcal{M}(Q,\tau)$ is obtained by dualizing the composition series of $\mathcal{I}(Q,\tau^*)$. Because $\mathcal{L}(Q',\tau'^*)^* = \mathcal{L}(Q',\tau')$ for any irreducible η-twisted Harish-Chandra sheaf $\mathcal{L}(Q',\tau')$, the composition factors of $\mathcal{M}(Q,\tau)$ must be equal to those of $\mathcal{I}(Q,\tau)$. This proves (i).

We have a short exact sequence of $\mathcal{D}_{-\lambda}$-modules

$$0 \to \mathcal{L}(Q,\tau^*) \to \mathcal{I}(Q,\tau^*) \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$ is a holonomic $\mathcal{D}_{-\lambda}$-module supported in ∂Q. Applying holonomic duality to this, we get a short exact sequence of $\mathcal{D}_\lambda$-modules

$$0 \to \mathcal{Q}^* \to \mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau) \to 0.$$

Because duality preserves support, this implies that the kernel $\mathcal{Q}^*$ of the projection map $\mathcal{M}(Q,\tau) \to \mathcal{L}(Q,\tau)$ is supported in ∂Q.

If $\mathcal{U} \subset \mathcal{M}(Q,\tau)$ is a maximal $\mathcal{D}_\lambda$-submodule different from $\mathcal{Q}^*$, then $\mathcal{L}(Q',\tau') = \mathcal{M}(Q,\tau)/\mathcal{U}$ is an irreducible $\mathcal{D}_\lambda$-module that is not isomorphic to $\mathcal{L}(Q,\tau)$. By dualizing, this implies that $\mathcal{L}(Q',\tau'^*)$ is an irreducible $\mathcal{D}_\lambda$-submodule of $\mathcal{I}(Q,\tau^*)$ which is not isomorphic to $\mathcal{L}(Q,\tau^*)$, but $\mathcal{L}(Q,\tau^*)$ is the unique irreducible $\mathcal{D}_\lambda$-submodule of $\mathcal{I}(Q,\tau^*)$, so this is impossible. This implies (ii). $\square$

4.5 Standard and Simple Sheaves for the Pair $(\mathfrak{g}, N)$

Let $K = N$. Let $\mathfrak{b}$ be the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{n} = \mathrm{Lie} N$. The pair $(\mathfrak{g}, N)$ is a Harish-Chandra pair. This section is dedicated to describing the standard η-twisted Harish-Chandra sheaves in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ using the classification described in Section 4.3. The category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ is the geometric analogue of the category $\mathcal{N}_{\theta,\eta}$. (We will make this statement precise in the following section.) By the discussion in Section 4.3, standard objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are parameterized by pairs (Q, τ), where Q is an N-orbit and τ is an irreducible N-homogeneous connection in $\mathcal{M}_{coh}(\mathcal{D}_{Q,\mu}, N, \eta)$. (Recall that μ is the restriction of the specialization of $\lambda + \rho$ at a point $x \in Q$ to $\mathfrak{n} \cap \mathfrak{b}_x$.) We can describe these pairs more explicitly.

The N-orbits on X are Bruhat cells $C(w)$, $w \in W$. The Bruhat cell $C(w)$ contains all Borel subalgebras in relative position w to $\mathfrak{b}$. Now we will describe the previous compatibility condition in this special case. Assume that a Bruhat cell $C(w)$ admits an irreducible N-homogeneous connection τ. Let $\mathfrak{b}_w$ be a fixed Borel subalgebra in $C(w)$, and $\mathfrak{n}_w = [\mathfrak{b}_w, \mathfrak{b}_w]$. Fix a Cartan subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ contained in $\mathfrak{b} \cap \mathfrak{b}_w$. Let R be the root system of $(\mathfrak{g}, \mathfrak{c})$, and R^+ the set of positive roots determined by $\mathfrak{n}$. Denote by $s : \mathfrak{h}^* \to \mathfrak{c}^*$ the specialization determined by $\mathfrak{b}$. Then $\mathfrak{n}_w$ is spanned by the root subspaces corresponding to roots in $s(w(\Sigma^+))$. We make two key observations:

- Because $\mathfrak{n} \cap \mathfrak{b}_w \subset \mathfrak{n}_w$, $\mu = 0$.

- Because the stabilizer S_w of $\mathfrak{b}_w$ in N is unipotent, the only irreducible algebraic representation of S_w is the trivial representation. This implies that the only possible action of S_w on the geometric fiber of τ is the trivial action, so the only irreducible N-homogeneous $\mathcal{O}_{C(w)}$-module on $C(w)$ is $\mathcal{O}_{C(w)}$.

Therefore, a connection with the properties described in Proposition 4.21 exists on $C(w)$ if and only if $\eta|_{\mathfrak{n} \cap \mathfrak{n}_w} = 0$. Moreover, if such a connection τ exists, it is isomorphic to $\mathcal{O}_{C(w)}$.

For each $\alpha \in \Sigma$, we denote by $\mathfrak{g}_\alpha$ the root subspace in $\mathfrak{g}$ corresponding to the root $s(\alpha) \in R$.[1] Then the subalgebra $\mathfrak{n} \cap \mathfrak{n}_w$ is spanned by the root subspaces $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma^+ \cap w(\Sigma^+)$.

[1] Note that this differs from the usual convention.

Hence, $\eta|_{\mathfrak{n}\cap\mathfrak{n}_w} = 0$ if and only if $\eta|_{\mathfrak{g}_\alpha} = 0$ for all $\alpha \in \Sigma^+ \cap w(\Sigma^+)$. Let Π be the set of simple roots in Σ corresponding to Σ^+, and let

$$\Theta = \{\alpha \in \Pi | \eta|_{\mathfrak{g}_\alpha} \neq 0\}.$$

This leads us to the following result.

Lemma 4.24. *The following statements are equivalent.*

(i) $\eta|_{\mathfrak{n}\cap\mathfrak{n}_w} = 0$.

(ii) $\Theta \cap w(\Sigma^+) = \emptyset$.

Proof. From the discussion above, we have that $\eta|_{\mathfrak{n}\cap\mathfrak{n}_w} = 0$ if and only if $\eta|_{\mathfrak{g}_\alpha} = 0$ for all $\alpha \in \Sigma^+ \cap w(\Sigma^+)$. Then by the definition of Θ, $\eta|_{\mathfrak{g}_\alpha} = 0$ for all $\alpha \in \Sigma^+ \cap w(\Sigma^+)$ if and only if $\Theta \cap (\Sigma^+ \cap w(\Sigma^+)) = \Theta \cap w(\Sigma^+) = \emptyset$. $\square$

Let P_Θ be the standard parabolic subgroup of $\operatorname{Int}\mathfrak{g}$ corresponding to Θ. The following result relates conditions (i) and (ii) of Lemma 4.24 to P_Θ-orbits in X and appears in [MS14, §4 Lem. 4.1].

Lemma 4.25. *The following conditions are equivalent.*

(i) $\Theta \cap w(\Sigma^+) = \emptyset$.

(ii) $C(w)$ *is the Bruhat cell open in one of the* P_Θ*-orbits in* X.

(iii) w *is the longest element in one of the right* W_Θ*-cosets of* W.

This allows us to classify irreducible and standard objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. For a coset $C \in W_\Theta \backslash W$, let w^C be the longest element in C. By [Milb, Ch. 6 §1 Thm. 1.4], this longest element is unique. Then by Lemma 4.24 and Lemma 4.25, there exists a compatible irreducible connection $\mathcal{O}_{C(w^C)}$ on the Bruhat cell $C(w^C)$, and we denote by $\mathcal{I}(w^C, \lambda, \eta)$ the corresponding standard η-twisted Harish-Chandra sheaf in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$, and by $\mathcal{L}(w^C, \lambda, \eta)$ the unique irreducible subsheaf of $\mathcal{L}(w^C, \lambda, \eta)$. We've established the following theorem.

Theorem 4.26. *The irreducible objects in the category* $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ *are the modules* $\mathcal{L}(w^C, \lambda, \eta)$*, where* w^C *is the longest element in a coset* $C \in W_\Theta \backslash W$.

We complete this section by computing inverse images of standard and irreducible η-twisted Harish-Chandra sheaves to obtain a result that will be of use to us in future sections. Fix parameters $\lambda \in \mathfrak{h}^*$, $\eta \in \mathfrak{n}^*$, and $C \in W_\Theta \backslash W$. Let $C(w^C)$ be the corresponding Bruhat cell that admits an irreducible compatible connection $\mathcal{O}_{C(w^C)}$. Let $X' = X - \partial C(w^C)$, $i_{w^C} : C(w^C) \to X$ the canonical immersion, and $j_{w^C} : C(w^C) \to X'$. The following diagram is commutative.

$$\begin{tikzcd} C(w^C) \arrow[rr, "i_{w^C}"] \arrow[dr, "j_{w^C}"'] & & X \\ & X' \arrow[ur, "k_{w^C}"'] & \end{tikzcd}$$

Here j_{w^C} is a closed immersion and k_w^C is an open immersion. For a coherent $\mathcal{D}_\lambda$-module $\mathcal{V}$ on X, $k^+_{w^C}(\mathcal{V}) = \mathcal{V}|_{X'}$, and $i^+_{w^C}(\mathcal{V}) = j^+_{w^C}(\mathcal{V}|_{X'})$. Denote by $\mathcal{I}$ the standard η-twisted Harish-Chandra sheaf $\mathcal{I}(w^C, \lambda, \eta)$ and by $\mathcal{L}$ its irreducible subsheaf $\mathcal{L}(w^C, \lambda, \eta)$. Then we have a short exact sequence of η-twisted Harish-Chandra sheaves

$$0 \to \mathcal{L} \to \mathcal{I} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q} = \mathcal{I}/\mathcal{L}$, and $\operatorname{supp}\mathcal{Q} = \partial C(w^C)$. Because k_{w^C} is an open immersion, $k^+_{w^C}$ is exact, so by applying $k^+_{w^C}$, we get another short exact sequence

$$0 \to \mathcal{L}|_{X'} \to \mathcal{I}|_{X'} \to \mathcal{Q}|_{X'} \to 0.$$

Because $\operatorname{supp}\mathcal{Q} = \partial C(w^C)$, $\mathcal{Q}|_{X'} = 0$, and we conclude that

$$\mathcal{L}|_{X'} \simeq \mathcal{I}|_{X'}.$$

Furthermore, because $\dim C(w^C) = \ell(w^C)$, for any $k \in \mathbb{Z}$, we have the relationship

$$R^{n-\ell(w^C)-k} i^!_{w^C} = L^{-k} i^+_{w^C},$$

where $n = \dim X$. So for a $\mathcal{D}_\lambda$-module $\mathcal{V}$ on X and $m \in \mathbb{Z}$, $R^m i^!_{w^C}(\mathcal{V}) = R^m j^!_{w^C}(\mathcal{V}|_{X'})$. Therefore, we conclude that

$$R^m i^!_{w^C}(\mathcal{I}) = R^m i^!_{w^C}(\mathcal{I}|_{X'}) = R^m j^!_{w^C}(\mathcal{L}|_{X'}) = R^m i^!_{w^C}(\mathcal{L}).$$

We have proven the following lemma.

Lemma 4.27. *Let $\lambda \in \mathfrak{h}^*$, $\eta \in \mathfrak{n}^*$, and $C \in W_\Theta \backslash W$. Let $C(w^C)$ be the corresponding Bruhat cell and $i_{w^C} : C(w^C) \to X$ the canonical immersion. Then for any $m \in \mathbb{Z}$,*

$$R^m i^!_{w^C}(\mathcal{I}(w^C, \lambda, \eta)) = R^m i^!_{w^C}(\mathcal{L}(w^C, \lambda, \eta)).$$

4.6 Intertwining Functors on Standard and Costandard Sheaves

In this section, we examine the action of intertwining functors on standard and costandard η-twisted Harish-Chandra sheaves in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. These results will be critical in establishing the relationship between $\mathcal{N}_{\theta,\eta}$ and $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. Let $\alpha \in \Pi$, $w \in W$, and p_i for $i = 1, 2$ the projections of Z_{s_α} (equation 4.1) onto the first and second coordinates, respectively. As in Section 4.5, let $\mathfrak{b}$ be the unique Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{n} = \text{Lie}N$. We start with a useful lemma.

Lemma 4.28. *The projection $p_1 : Z_{s_\alpha} \to X$ induces an immersion of $p_2^{-1}(C(w))$ into X, and its image is equal to $C(ws_\alpha)$.*

Proof. Using the definition of Z_{s_α}, and the set-up below Proposition 4.21, we see that

$$\begin{aligned} p_2^{-1}(C(w)) &= \{(x,y) \in X \times X | y \in C(w) \text{ and } \mathfrak{b}_y \text{ is in relative position } s_\alpha \text{ to } \mathfrak{b}_x\} \\ &= \{(x,y) \in X \times X | \mathfrak{b}_y \text{ is in relative position } w \text{ to } \mathfrak{b} \\ &\qquad \text{and } \mathfrak{b}_x \text{ is in relative position } s_\alpha \text{ to } \mathfrak{b}_y\} \\ &= \{(x,y) \in X \times X | \mathfrak{b}_y \text{ is in relative position } w \text{ to } \mathfrak{b} \\ &\qquad \text{and } \mathfrak{b}_x \text{ is in relative position } ws_\alpha \text{ to } \mathfrak{b}\} \\ &= C(ws_\alpha) \times C(w). \end{aligned}$$

Therefore, p_1 induces an immersion of $p_2^{-1}(C(w))$ into X and its image is equal to $C(ws_\alpha)$. □

Our first result is the following proposition.

Proposition 4.29. *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha > C$ and let $\lambda \in \mathfrak{h}^*$ be arbitrary. Then*

$$LI_{s_\alpha}(D(\mathcal{I}(w^C, \lambda, \eta))) = D(\mathcal{I}(w^C s_\alpha, s_\alpha\lambda, \eta)).$$

Proof. By Theorem 2.5, we have the following commutative diagram.

$$\begin{tikzcd} p_2^{-1}(C(w^C)) \arrow[r, "j"] \arrow[d, "pr_2"'] & Z_{s_\alpha} \arrow[d, "p_2"] \\ C(w^C) \arrow[r, "i_{w^C}"] & X \end{tikzcd}$$

Here p_2 and $pr_2 = p_2|_{p_2^{-1}(C(w^C))}$ are surjective submersions and j and i_{w^C} are affine immersions, so p_2^+, pr_2^+, i_{w^C+}, and j_+ are all exact. Then, using the definition of $\mathcal{I}(w^C, \lambda, \eta)$, the fact that $\dim Z_{s_\alpha} - \dim X = \dim p_2^{-1}(C(w^C)) - \dim C(w^C)$, and Theorem 2.5, we see that

$$\begin{aligned} p_2^+(\mathcal{I}(w^C, \lambda, \eta)) &= p_2^+(i_{w^C+}(\mathcal{O}_{C(w^C)})) \\ &= j_+(pr_2^+(\mathcal{O}_{C(w^C)})) \\ &= j_+(\mathcal{O}_{p_2^{-1}(C(w^C))}). \end{aligned}$$

Applying the projection formula of Proposition 2.4 to the morphism p_1, the line bundle $\mathcal{L} = \mathcal{O}(\rho - s_\alpha\rho)$, and the twisted sheaf of differential operators $\mathcal{D}_\lambda$ on X, we have the following commutative diagram.

$$\begin{tikzcd} D(\mathcal{D}_\lambda^{p_1}) \arrow[r, "p_{1+}"] \arrow[d, "p_1^*(\mathcal{L})\otimes_{\mathcal{O}_{Z_\alpha}} -"'] & D(\mathcal{D}_\lambda) \arrow[d, "\mathcal{L}\otimes_{\mathcal{O}_X} -"] \\ D((\mathcal{D}_\lambda^{\mathcal{L}})^{p_1}) \arrow[r, "p_{1+}"] & D(\mathcal{D}_\lambda^{\mathcal{L}}) \end{tikzcd}$$

Using this commutative diagram, the definition of intertwining functors, and the fact that $\mathcal{T}_{s_\alpha} = p_1^*(\mathcal{O}(\rho - s_\alpha\rho))$, we compute

$$\begin{aligned} LI_{s_\alpha}(D(\mathcal{I}(w^C, \lambda, \eta)) &= p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} p_2^+(\mathcal{I}(w^C, \lambda, \eta))) \\ &= p_{1+}(\mathcal{T}_{s_\alpha} \otimes_{\mathcal{O}_{Z_{s_\alpha}}} j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})) \\ &= p_{1+}(p_1^*(\mathcal{O}(\rho - s_\alpha\rho)) \otimes_{\mathcal{O}_{Z_{s_\alpha}}} j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})) \\ &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} p_{1+}(j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})). \end{aligned}$$

By Lemma 4.28, the diagram

$$\begin{tikzcd} p_2^{-1}(C(w^C)) \arrow[r, "j"] \arrow[d, "pr_1"'] & Z_{s_\alpha} \arrow[d, "p_1"] \\ C(w^C s_\alpha) \arrow[r, "i_{w^C s_\alpha}"] & X \end{tikzcd}$$

commutes. Picking up our previous computation, this lets us further conclude that

$$\begin{aligned}\mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} p_{1+}(j_+(\mathcal{O}_{p_2^{-1}(C(w^C))})) &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} i_{w^C s_\alpha +}(pr_{1+}(\mathcal{O}_{p_2^{-1}(C(w^C))})) \\ &= \mathcal{O}(\rho - s_\alpha\rho) \otimes_{\mathcal{O}_X} i_{w^C s_\alpha +}(\mathcal{O}_{C(w^C s_\alpha)}) \\ &= D(\mathcal{I}(w^C s_\alpha, s_\alpha\lambda, \eta)).\end{aligned}$$

This completes the proof. □

For $C \in W_\Theta\backslash W$, let $\mathcal{M}(w^C, \lambda, \eta)$ be the corresponding costandard η-twisted Harish-Chandra sheaf in the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. Our second result is the following.

Proposition 4.30. *Let $C \in W_\Theta\backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$, and $\lambda \in \mathfrak{h}^*$ be arbitrary. Then*

$$I_{s_\alpha}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta),$$

and

$$L^p I_{s_\alpha}(\mathcal{M}(w^C, \lambda, \eta)) = 0 \textit{ for } p \neq 0.$$

Proof. By Proposition 4.29 applied to the coset Cs_α and linear form $-\lambda \in \mathfrak{h}^*$, we have

$$D(\mathcal{I}(w^C, -\lambda, \eta)) = LI_{s_\alpha}(D(\mathcal{I}(w^C s_\alpha, -s_\alpha\lambda, \eta))).$$

Applying holonomic duality, we get

$$\begin{aligned}D(\mathcal{M}(w^C, \lambda, \eta)) &= \mathbb{D}(LI_{s_\alpha}(D(\mathcal{I}(w^C s_\alpha, -s_\alpha\lambda, \eta)))) \\ &= (\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D})(D(\mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta)))\end{aligned}$$

By Theorem 4.12, $\mathbb{D} \circ LI_{s_\alpha} \circ \mathbb{D}$ is the inverse of the intertwining functor LI_{s_α}, so applying LI_{s_α} to both sides of the above equation proves the proposition. □

Combined with Theorem 4.6, this implies the following result.

Theorem 4.31. *If $\lambda \in \mathfrak{h}^*$ is α-antidominant, and $C \in W_\Theta\backslash W$ is such that $Cs_\alpha < C$, we have*

$$H^p(X, \mathcal{M}(w^C, \lambda, \eta)) = H^p(X, \mathcal{M}(w^C s_\alpha, s_\alpha\lambda, \eta))$$

for any $p \in \mathbb{Z}_+$.

CHAPTER 5

GEOMETRIC DESCRIPTION OF WHITTAKER MODULES

In this chapter, we establish the connection between the category of Whittaker modules and the category of twisted Harish-Chandra sheaves. We begin by reviewing the existing results in the nondegenerate setting.

5.1 The Nondegenerate Case

Let $\eta \in \mathfrak{n}^*$ be a nondegenerate character. Then $\Theta = \Pi$, $P_\Theta = G$, and $\ell_\Theta = \mathfrak{g}$. In this section, we review the existing results on nondegenerate Whittaker modules and nondegenerate twisted Harish-Chandra sheaves. This setting was examined algebraically by Kostant in [Kos78], and geometrically by Miličić and Soergel in [MS14].

By Theorem 4.26, there exists only one irreducible object $\mathcal{L}(w_0, \lambda, \eta)$ in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ corresponding to the compatible irreducible connection on the open Bruhat cell $C(w_0)$. Let $\mathcal{I}(w_0, \lambda, \eta)$ be the corresponding standard twisted Harish-Chandra sheaf. Because the quotient $\mathcal{I}(w_0, \lambda, \eta)/\mathcal{L}(w_0, \lambda, \eta)$ is supported in the complement of $C(w_0)$, it must be zero. Therefore, $\mathcal{I}(w_0, \lambda, \eta) = \mathcal{L}(w_0, \lambda, \eta)$ is irreducible. Furthermore, $\Gamma(X, \mathcal{I}(w_0, \lambda, \eta))$ is equal to the space $R(C(w_0))$ of regular functions on the affine variety $C(w_0) \simeq \mathbb{C}^{\ell(w_0)}$. Therefore, if $\lambda \in \mathfrak{h}^*$ is antidominant, $\Gamma(X, \mathcal{I}(w_0, \lambda, \eta))$ is an irreducible Whittaker module. This implies that there exists a unique irreducible object in the category $\mathcal{N}_{\theta,\eta}$. The unique irreducible objects in the categories $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ and $\mathcal{N}_{\theta,\eta}$ were described explicitly in [MS14, §5 Thm. 5.1, Thm. 5.2]

Theorem 5.1. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate and $\lambda \in \mathfrak{h}^*$. Then the only irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ is $\mathcal{D}_\lambda \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$.*

The corresponding theorem in the algebraic category was originally proven by Kostant in [Kos78, §3 Thm. 3.6.1], but falls immediately from the geometric result.

Theorem 5.2. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate. Then the only irreducible module in $\mathcal{N}_{\theta,\eta}$ is $\mathcal{U}_\theta \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$.*

These theorems demonstrate that for nondegenerate η, the categories $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ and $\mathcal{N}_{\theta,\eta}$ are extremely simple. Indeed, if $\mathcal{V}$ is an arbitrary object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$, then its restriction to the open cell $C(w_0)$ is an N-homogeneous connection. For a generic point $x \in C(w_0)$, the stabilizer in N of x is trivial, so this connection is equal to a sum of copies of the irreducible connection on $C(w_0)$. Moreover, there is a natural morphism φ of $\mathcal{V}$ into a sum of copies of $\mathcal{I}(w_0, \lambda, \eta) = \mathcal{L}(w_0, \lambda, \eta)$ because restriction is a left adjoint to direct image. The kernal and cokernal of φ are supported in the complement of $C(w_0)$, so they must be zero, which leads to the following semisimplicity results [MS14, §5 Thm. 5.5, Thm 5.6].

Theorem 5.3. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate. Then all objects in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ are finite sums of irreducible objects $\mathcal{D}_\lambda \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$.*

Theorem 5.4. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate. Then all objects in $\mathcal{N}_{\theta,\eta}$ are finite sums of irreducible objects $\mathcal{U}_\theta \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$.*

5.2 Cosets in the Weyl Group

Before analyzing the degenerate case, we list some combinatorial properties of cosets in Weyl groups that will be essential in future arguments. For brevity, we omit the proofs that can be found in [Milb, Ch. 6 §1].

Let Σ be a reduced root system and Σ^+ a set of positive roots. Let Π be the corresponding set of simple roots, and for $w \in W$, let

$$\Sigma_w^+ = \Sigma^+ \cap \{-w^{-1}(\Sigma^+)\} = \{\alpha \in \Sigma^+ | w\alpha \in -\Sigma^+\}.$$

Let $\Theta \subset \Pi$, and let $\Sigma_\Theta \subset \Sigma$ be the root subsystem generated by Θ. Let $S_\Theta = \{s_\alpha | \alpha \in \Theta\}$ be the set of simple reflections corresponding to Θ, and denote the Weyl group generated by these reflections by W_Θ. Then $W_\Theta \subset W$ is a subgroup. The length function ℓ on W restricted to W_Θ gives the length function on W_Θ. We define

$$W^\Theta = \{w \in W | \Sigma_w^+ \cap \Theta = \emptyset\} = \{w \in W | \Theta \subset w^{-1}(\Sigma^+)\}.$$

Theorem 5.5. *Every element $w \in W$ has a unique decomposition in the form $w = w't$, with $w' \in W^{\Theta}$ and $t \in W_{\Theta}$. Additionally, $\ell(w) = \ell(w') + \ell(t)$.*

Let w_{Θ} be the longest element in W_{Θ}.

Theorem 5.6. *(i) Each left W_{Θ}-coset in W has a unique shortest element.*

(ii) If w is the shortest element in a left W_{Θ}-coset C, ww_{Θ} is the unique longest element in this coset.

(iii) Each right W_{Θ}-coset in W has a unique shortest element.

(iv) If w is the shortest element in a right W_{Θ}-coset C, $w_{\Theta}w$ is the unique longest element in this coset.

The antiautomorphism $w \mapsto w^{-1}$ preserves W_{Θ}, w_{Θ}, and the length function $\ell : W \to \mathbb{Z}_{+}$. It also maps left W_{Θ}-cosets to right W_{Θ}-cosets. Therefore, the set W^{Θ} is the section of the left W_{Θ}-cosets in W consisting of the shortest elements of each coset. This implies that the shortest elements of right W_{Θ}-cosets in W are the elements of the set

$$W_R^{\Theta} = \{w \in W | w^{-1} \in W^{\Theta}\} = \{w \in W | \Theta \subset w(\Sigma^{+})\}.$$

This implies the following result.

Lemma 5.7. *The set*

$${}^{\Theta}W = \{w \in W | \Theta \subset -w(\Sigma^{+})\}$$

is the section of the of right W_{Θ}-cosets in W consisting of the longest elements of each coset.

This also gives us an analogue to Theorem 5.5.

Theorem 5.8. *Every element $w \in W$ has a unique decomposition in the form $w = tw'$ for $t \in W_{\Theta}$ and $w' \in W_R^{\Theta}$. In addition, $\ell(w) = \ell(t) + \ell(w')$.*

Proof. Let $w \in W$. Then by 5.5, there is unique decomposition $w^{-1} = v's$ for $v' \in W^{\Theta}$ and $s \in W_{\Theta}$, and $\ell(w^{-1}) = \ell(v') + \ell(s)$. So $w = s^{-1}v'^{-1}$, where $s^{-1} \in W_{\Theta}$ and $v'^{-1} \in W_R^{\Theta}$, and this decomposition is unique. Furthermore,

$$\ell(w) = \ell(w^{-1}) = \ell(v') + \ell(s) = \ell(s^{-1}) + \ell(v'^{-1}).$$

This completes the proof. $\square$

For a right W_Θ-coset C, we denote by w^C the unique longest element in C. We define an order relation on the set $W_\Theta\backslash W$ by transferring the Bruhat order on the collection of longest coset elements. Specifically, we say that $C < D$ for two cosets $C, D \in W_\Theta\backslash W$ if $w^C < w^D$ in the Bruhat order on W. The key result on this order relation is the following.

Proposition 5.9. *Let C be a right W_Θ-coset in W and $\alpha \in \Pi$. Then we have the following three possibilities:*

(i) $Cs_\alpha = C$;

(ii) $Cs_\alpha > C$*, and in this case,* $w^{Cs_\alpha} = w^C s_\alpha$*, and* $\ell(ws_\alpha) = \ell(w) + 1$ *for any* $w \in C$;

(iii) $Cs_\alpha < C$*, and in this case* $w^{Cs_\alpha} = w^C s_\alpha$*, and* $\ell(ws_\alpha) = \ell(w) - 1$ *for any* $w \in C$.

Let C be a right W_Θ-coset and $w \in C$ its shortest element. Then $w^C = w_\Theta w$ by 5.6, and $\ell(w^C) = \ell(w) + \ell(w_\Theta)$ by 5.8. This implies that $w_\Theta < w_\Theta w = w^C$ in the Bruhat order, so $W_\Theta < C$ in the order relation on $W_\Theta\backslash W$. This implies that W_Θ is the smallest element in $W_\Theta\backslash W$. Furthermore, W_Θ is the unique smallest element in $W_\Theta\backslash W$:

Proposition 5.10. *Let $C \in W_\Theta\backslash W$. Assume that for any $\alpha \in \Pi$, we have either $Cs_\alpha = C$, or $Cs_\alpha > C$. Then $C = W_\Theta$.*

In particular, for any coset $C \neq W_\Theta$, there exists a simple root α such that $Cs_\alpha < C$. This implies that the set ${}^\Theta W$ contains Weyl group elements of every length greater $\ell(w_\Theta)$.

Theorem 5.11. *Let $j = \ell(w_\Theta)$, and let m be the length of the longest element of W. Then for any n with $j \leq n \leq m$, the set ${}^\Theta W_{\leq n} = \{w^C \in {}^\Theta W | \ell(w^C) \leq n\}$ contains an element of length k for every $j \leq k \leq n$.*

Proof. We proceed by induction in n. The base case is when $n = j$. In this case, w_Θ has length j and by 5.10, ${}^\Theta W = \{w_\Theta\}$, so the theorem holds. Let n be such that $j < n < m$, and assume that ${}^\Theta W_{\leq n}$ contains an element of length k for every $j \leq k \leq n$. Choose $w^C \in {}^\Theta W$ of length n. Then because $n \neq m$, there exists some $\alpha \in \Pi$ such that $\ell(w^C s_\alpha) = \ell(w^C) + 1 = n + 1$. We claim that $w^C s_\alpha$ is the longest element in a coset. By 5.9, we have three possibilities: either $Cs_\alpha = C$, $Cs_\alpha < C$, or $Cs_\alpha > C$. Because w^C is the longest element of C, and $\ell(w^C s_\alpha) > \ell(w^C)$, $w^C s_\alpha \notin C$, so the first possibility cannot happen. If $Cs_\alpha < C$, then 5.9 implies that $w^{Cs_\alpha} = w^C s_\alpha$, and $\ell(w^C s_\alpha) = \ell(w^C) - 1$. But this is not the case, so the second possibility cannot happen. We conclude that $Cs_\alpha > C$, and $w^{Cs_\alpha} = w^C s_\alpha$; i.e. $w^C s_\alpha \in {}^\Theta W_{\leq n+1}$. We are done by induction. $\square$

In particular, the set ${}^\Theta W$ contains elements of length k for every $\ell(w_\Theta) \leq k \leq m$, so we can apply inductive arguments to the length of elements of ${}^\Theta W$. Finally, we record the following fact.

Lemma 5.12. *If $w \in {}^\Theta W$ and $t \in W_\Theta$, we have*

$$\ell(tw) = \ell(w) - \ell(t).$$

Let B be a Borel subgroup of G, and P_Θ the standard parabolic subgroup of G of type Θ containing B. Then the P_Θ-orbits in the flag variety X are B-invariant, so they must be unions of Bruhat cells. We end this section by describing how the structure of $W_\Theta \backslash W$ determines the structure of P_Θ-orbits in X.

Lemma 5.13. *Let O be a P_Θ-orbit in X and $C(w) \subset O$. Then*

$$O = \bigcup_{t \in W_\Theta} C(tw).$$

This result establishes a bijection between $W_\Theta \backslash W$ and the set of P_Θ-orbits in X. Let $C \in W_\Theta \backslash W$ and let O be the corresponding P_Θ-orbit in X. Then by Lemma 5.12,

$$\dim O = \max_{t \in W_\Theta} \dim C(tw^C) = \max_{t \in W_\Theta} \ell(tw^C) = \ell(w^C).$$

Therefore, $C(w^C)$ is the open Bruhat cell in O. This implies the following result.

Proposition 5.14. *The map attaching to a P_Θ-orbit O in the flag varitey X the unique Bruhat cell $C(w)$ open in O is a bijection between the set of all P_Θ-orbits in X and the set of Bruhat cells $C(w)$ with $w \in {}^\Theta W$.*

We end this section with a geometric interpretation of the order relation on $W_\Theta \backslash W$.

Proposition 5.15. *Let $C \in W_\Theta \backslash W$, and let O be the corresponding P_Θ-orbit in X. Then the closure of O consists of all P_Θ-orbits in X corresponding to $D \leq C$.*

5.3 Global Sections of Twisted Harish-Chandra Sheaves

In this section, we prove that global sections of costandard twisted Harish-Chandra sheaves are standard Whittaker modules. This allows us to use geometric arguments to draw conclusions about our algebraic category of Whittaker modules, which will be essential in the interpretation of the algorithm developed in Chapter 6. Our main tool in this section is the character theory developed in Section 3.2.

We begin by examining the nondegenerate case. Let w_0 be the longest element of the Weyl group W of $\mathfrak{g}$.

Proposition 5.16. *Let $\eta \in \mathfrak{n}^*$ be nondegenerate and $\lambda \in \mathfrak{h}^*$. Then*

$$\Gamma(X, \mathcal{M}(w_0, \lambda, \eta)) = M(w_0\lambda, \eta).$$

Proof. If η is nondegenerate, then $W = W_\Theta$, so by Theorem 5.1, there exists a unique irreducible object $\mathcal{L}(w_0, \lambda, \eta) = \mathcal{I}(w_0, \lambda, \eta) = \mathcal{M}(w_0, \lambda, \eta) = \mathcal{D}_\lambda \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta$ in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$. Assume λ is antidominant, and let $\theta = W \cdot \lambda$. Then by Theorem 5.2,

$$\Gamma(X, \mathcal{M}(w_0, \lambda, \eta)) = \mathcal{U}_\theta \otimes_{\mathcal{U}(\mathfrak{n})} \mathbb{C}_\eta = M(w_0\lambda, \eta).$$

Now, let $w \in W$ be arbitrary. By Theorem 4.6 and the preceeding argument, we have

$$D(M(w_0\lambda, \eta)) = R\Gamma(D(\mathcal{M}(w_0, \lambda, \eta))) = R\Gamma(LI_w(D(\mathcal{M}(w_0, \lambda, \eta)))) = R\Gamma(\mathcal{C}^{\cdot}),$$

where $\mathcal{C}^{\cdot}$ is a complex in $D^b(\mathcal{D}_{w\lambda})$ such that for any $i \in \mathbb{Z}$, $\mathcal{C}^i$ is a finite sum of copies of the unique irreducible object $\mathcal{M}(w_0, w\lambda, \eta)$. (See Theorem 5.1.) Because $D(M(w_0\lambda, \eta))$ is a complex with a single irreducible object in the zero degree and zeros elsewhere and $R\Gamma$ is an equivalence of derived categories, the equality above implies that

$$LI_w(D(\mathcal{M}(w_0, \lambda, \eta))) = D(\mathcal{M}(w_0, w\lambda, \eta)).$$

Therefore,

$$\Gamma(X, \mathcal{M}(w_0, w\lambda, \eta)) = M(w_0\lambda, \eta) = M(w_0 w\lambda, \eta).$$

This completes the proof of the proposition. $\square$

Next, we prove the result for the costandard twisted Harish-Chandra sheaf corresponding to the smallest P_Θ-orbit. This is the bulk of the argument.

Lemma 5.17. *Let $\eta \in \mathfrak{n}^*$ be arbitrary and $\lambda \in \mathfrak{h}^*$. Then*

$$\Gamma(X, \mathcal{M}(w_\Theta, \lambda, \eta)) = M(w_\Theta \lambda, \eta).$$

Here w_Θ is the longest element in the Weyl group W_Θ determined by Θ. We will prove the lemma in a series of steps. Our first step is to realize the standard sheaf corresponding to the smallest P_Θ-orbit as the direct image of a twisted Harish-Chandra sheaf for the flag variety of ℓ_Θ. Let $P(w_\Theta)$ be the P_Θ-orbit with open Bruhat cell $C(w_\Theta) \subset P(w_\Theta)$ (see Section 4.5). Because w_Θ is minimal in set ${}^\Theta W$ of longest coset elements by Proposition 5.10, $P(w_\Theta)$ is a closed subvariety of X. Because $P(w_\Theta)$ is an orbit of an algebraic group action, it is also a smooth subvariety of X. In fact, P_Θ-orbit $P(w_\Theta)$ is isomorphic to the flag variety of ℓ_Θ. (Indeed, by Lemma 5.13, $P(w_\Theta) = \bigcup_{t \in W_\Theta} C(tw_\Theta) = \bigcup_{w \in W_\Theta} C(w)$.) Let

$$i_{w_\Theta} : C(w_\Theta) \to P(w_\Theta), j : P(w_\Theta) \to X, \text{ and } i : C(w_\Theta) \to X$$

be the natural inclusions. This expresses $i = j \circ i_{w_\Theta}$ as the composition of an open immersion and a closed immersion. By definition, $\mathcal{I}(w_\Theta, \lambda, \eta) = j_+(\mathcal{F})$, where $\mathcal{F} = i_{w_\Theta +}(\mathcal{O}_{C(w_\Theta)})$, and $\mathcal{O}_{C(W_\Theta)}$ is the N-homogeneous connection in $\mathcal{M}_{coh}(D^i_\lambda, N, \eta)$ described in Section 4.5.

Lemma 5.18. *$\mathcal{F} = \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta})$ is the standard object in the category $\mathcal{M}_{coh}(\mathcal{D}_{P(w_\Theta), \lambda+\rho}, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ corresponding to the open Bruhat cell $C(w_\Theta) \subset P(w_\Theta)$.*

Proof. As described above, we can view $P(w_\Theta)$ as the flag variety for ℓ_Θ, and the character $\eta|_{\mathfrak{n}_\Theta}$ is nondegenerate on ℓ_Θ. The irreducible N-homogeneous connection $\mathcal{O}_{C(w_\Theta)}$ is compatible with $(\lambda, \eta) \in \mathfrak{h}^* \times \mathfrak{n}^*$ by construction (Section 4.3). We can restrict the N-action to $N_\Theta \subset N$, and consider $\mathcal{O}_{C(w_\Theta)}$ as an irreducible N_Θ-homogeneous connection compatible with $(\lambda, \eta|_{\mathfrak{n}_\Theta}) \in \mathfrak{h}^* \times \mathfrak{n}^*_\Theta$. This allows us to interpret $\mathcal{F} = i_{w_\Theta +}(\mathcal{O}_{C(w_\Theta)})$ as the standard sheaf on the flag variety of ℓ_Θ induced from the irreducible N_Θ-homogeneous connection $\mathcal{O}_{C(w_\Theta)}$ on $C(w_\Theta)$ in $\mathcal{M}_{coh}((\mathcal{D}^j_\lambda)^i, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$. (Note that because $\eta|_{\mathfrak{n}_\Theta}$ is nondegenerate, this is the only standard $\eta|_{\mathfrak{n}_\Theta}$-twisted Harish-Chandra sheaf in the category $\mathcal{M}_{coh}(\mathcal{D}^j_\lambda, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ by the results in Section 5.1.) Because

$$\mathcal{D}^j_\lambda = (\mathcal{D}_{X, \lambda+\rho})^j = \mathcal{D}_{P(w_\Theta), \lambda+\rho} = \mathcal{D}_{\lambda+\rho-\rho_\Theta},$$

we have that

$$\mathcal{F} = \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta}).$$

This completes the proof. $\square$

Our next step is to use the normal degree filtration introduced in Section 2.2 to analyze the global sections of the standard sheaf $\mathcal{I}(w_\Theta, \lambda, \eta)$. We will do so using the character theory established in Section 3.2. By Lemma 5.18, we can express our standard sheaf $\mathcal{I}(w_\Theta, \lambda, \eta) = j_+(\mathcal{F})$, where $\mathcal{F} = \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta, \eta|_{\mathfrak{n}_\Theta})$. Because $j : P(w_\Theta) \to X$ is a closed immersion, this implies that $\mathcal{I}(w_\Theta, \lambda, \eta)$ has a filtration by normal degree, $F_n\mathcal{I}(w_\Theta, \lambda, \eta)$. Let $Gr\mathcal{I}(w_\Theta, \lambda, \eta)$ be the corresponding graded sheaf. Let $ch : \mathcal{N}_{\theta,\eta} \longrightarrow \prod_{\mu \leq S_0} \mathbb{C} \otimes_{\mathbb{Z}} K\mathcal{N}([\ell_\Theta, \ell_\Theta])e^\mu$ be the character function described in Section 3.2.

Lemma 5.19. *$ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) = ch\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$.*

Proof. By construction (see [Har77]), we have

$$\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)) = \varinjlim \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)).$$

For each $n \in \mathbb{Z}_+$, we have an exact sequence

$$0 \to F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta) \to F_n\mathcal{I}(w_\Theta, \lambda, \eta) \to Gr_n\mathcal{I}(w_\Theta, \lambda, \eta) \to 0.$$

We claim that $H^p(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$. To see this, recall that by construction, $Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)$ is the sheaf-theoretic direct image of a sheaf on $P(w_\Theta)$ which has a finite filtration such that the graded pieces are standard $\eta|_{\mathfrak{n}_\Theta}$-twisted Harish-Chandra sheaves on the flag variety $P(w_\Theta)$ of ℓ_Θ. These have vanishing cohomologies by Lemma 5.16, which implies the claim. The short exact sequence above gives rise to a long exact sequence

$$\begin{aligned} 0 &\to \Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \\ &\to H^1(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to H^1(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to H^1(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \cdots \end{aligned}$$

Using induction and the preceding paragraph, we see that $H^p(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$ (and therefore, $H^p(X, \mathcal{I}(w_\Theta, \lambda, \eta)) = 0$ for $p > 0$). This implies that for each $n \in \mathbb{Z}_+$, we have a short exact sequence

$$0 \to \Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \to 0.$$

(If $\lambda \in \mathfrak{h}^*$ is antidominant, the existence of this short exact sequence falls from the exactness of Γ, but notice that the argument above holds for arbitrary $\lambda \in \mathfrak{h}^*$.) This gives us a filtration of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$, with corresponding graded module

$$\begin{aligned}\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) &= \bigoplus \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \\ &= \bigoplus \Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta))/\Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)).\end{aligned}$$

Because character sums over short exact sequences, we have

$$ch\Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) = ch\Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) - Ch\Gamma(X, F_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta)).$$

Now we compute character, using the fact that character distributes through direct sums.

$$\begin{aligned}ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) &= ch \bigoplus_{n \in \mathbb{Z}_+} \Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \\ &= \bigoplus_{n \in \mathbb{Z}_+} ch\Gamma(X, Gr_n\mathcal{I}(w_\Theta, \lambda, \eta)) \\ &= \sum_{n \in \mathbb{Z}_+} (ch\Gamma(X, F_n\mathcal{I}(w_\Theta, \lambda, \eta)) - ch\Gamma(X, f_{n-1}\mathcal{I}(w_\Theta, \lambda, \eta))) \\ &= ch\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)).\end{aligned}$$

This completes the proof. □

This reduces our calculation of the character of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$ to the calculation of the character of $\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta))$. Before completing this calculation, we need a few more supporting lemmas.

The Borel subalgebra $\mathfrak{b}$ acts on $\bar{\mathfrak{u}}_\Theta$ by the adjoint action, and this extends to an action of $\mathfrak{b}$ on the universal enveloping algebra $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$. The $\mathfrak{h}$-weights of this action are

$$Q = \left\{ - \sum_{\alpha \in \Sigma^+ \backslash \Sigma_\Theta^+} m_\alpha \alpha \,\middle|\, m_\alpha \in \mathbb{Z}_{\geq 0} \right\}.$$

Let $\mathcal{N}_{X|P(w_\Theta)} = j^*(\mathcal{T}_X)/\mathcal{T}_{P(w_\Theta)}$ be the normal sheaf of $P(w_\Theta)$ and $S(\mathcal{N}_{X|P(w_\Theta)})$ the corresponding sheaf of symmetric algebras.

Lemma 5.20. *As $\mathcal{O}_{P(w_\Theta)}$-modules,*

$$S(\mathcal{N}_{X|P(w_\Theta)}) = \bigoplus_{\mu \in Q} \mathcal{O}(\mu).$$

Proof. The normal sheaf $\mathcal{N}_{X|P(w_\Theta)}$ is a P_Θ-homogeneous $\mathcal{O}_{P(w_\Theta)}$-module. For any $x \in P(w_\Theta)$, there is an equivalence of categories between $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$ and the category of algebraic representations of $B_x = stab_{P_\Theta}\{x\}$ given by taking the geometric fiber of a sheaf $\mathcal{F}$ in $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$ (Theorem 2.1). Let $x_0 \in X$ be the point corresponding to B. The P_Θ-orbit of x_0 in X is the unique closed P_Θ-orbit [Bor91, Ch. IV], so it must be equal to $P(w_\Theta)$. In particular, $x_0 \in P(w_\Theta)$, so the functor T_{x_0} is an equivalence of the category $\mathcal{M}_{qc}(\mathcal{O}_{P(w_\Theta)}, P_\Theta)$ with the category of algebraic representations of B. The tangent space $T_{x_0}(X)$ is isomorphic to $\mathfrak{g}/\mathfrak{b}$, and the subspace $T_{x_0}(P(w_\Theta))$ is equal to $\mathfrak{p}_\Theta/\mathfrak{b}$ under this isomorphism. Therefore, the geometric fiber of the normal bundle at x_0 is isomorphic to $\mathfrak{g}/\mathfrak{p}_\Theta \simeq \bar{\mathfrak{u}}_\Theta$. We conclude from this discussion that $\mathcal{N}_{X|P(w_\Theta)}$ corresponds to the Adjoint representation of B on $\bar{\mathfrak{u}}_\Theta$, or, equivalently, the adjoint representation of $\mathfrak{b}$ on $\bar{\mathfrak{u}}_\Theta$.

Therefore, to analyze the $\mathcal{O}_{P(w_\Theta)}$-module $S(\mathcal{N}_{X|P(w_\Theta)})$, we can examine the symmetric algebra $S(\overline{\mathfrak{u}_\Theta})$, viewed as a $\mathfrak{b}$-module under the inherited action of the adjoint representation of $\mathfrak{b}$ on $\bar{\mathfrak{u}}_\Theta$. The universal enveloping algebra $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ has a PBW filtration such that the corresponding graded module $Gr\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ is isomorphic to $S(\bar{\mathfrak{u}}_\Theta)$. Under the adjoint action, $\mathcal{U}(\bar{\mathfrak{u}}_\Theta)$ decomposes into $\mathfrak{h}$-weight spaces corresponding to weights in Q. Therefore, the $\mathfrak{b}$-module $S(\bar{\mathfrak{u}}_\Theta)$ decomposes into $\mathfrak{h}$-weight spaces corresponding to the same weights in Q.

For $k \in \mathbb{Z}_{\geq 0}$, consider $V = S^k(\bar{\mathfrak{u}}_\Theta)$. There is a $\mathfrak{b}$-invariant filtration

$$0 = f_0 V \subset F_1 V \subset \cdots \subset F_n V = V$$

such that $F_i V / F_{i-1} V = \mathbb{C}_\mu$, where $\mu \in Q$ is an $\mathfrak{h}$-weight of $S^k(\bar{\mathfrak{u}}_\Theta)$. This induces a filtration of $\mathcal{V} = S^k(\mathcal{N}_{X|P(w_\Theta)})$

$$0 = f_0 \mathcal{V} \subset F_1 \mathcal{V} \subset \cdots \subset F_n \mathcal{V} = \mathcal{V}$$

where each $F_i\mathcal{V}$ is a P_Θ-homogeneous subsheaf and $F_i\mathcal{V}/F_{i+1}\mathcal{V} = \mathcal{O}_{P(w_\Theta)}(\mu)$. This proves the result. □

Lemma 5.21. *For $\lambda, \mu \in \mathfrak{h}^*$,*

$$\mathcal{I}(w_\Theta, \lambda, \eta|_{\mathfrak{n}_\Theta}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(\mu) = \mathcal{I}(w_\Theta, \lambda + \mu, \eta|_{\mathfrak{n}_\Theta}).$$

Proof. The twist functor (Section 2.2)

$$\mathcal{V} \mapsto \mathcal{V} \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(\mu)$$

is an equivalence of the categories $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ and $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\mu}, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$. Each of these categories has a unique irreducible object (because $\eta|_{\mathfrak{n}_\Theta}$ is nondegenerate, Theorem 5.1), so twisting must take the unique irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$ to the unique irreducible object in $\mathcal{M}_{coh}(\mathcal{D}_{\lambda+\mu}, N_\Theta, \eta|_{\mathfrak{n}_\Theta})$. This proves the lemma. □

Lemma 5.22. *As a left $\mathcal{D}_\lambda$-module, the graded sheaf has a decomposition*

$$Gr\mathcal{I}(w_\Theta, \lambda, \eta) = j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)).$$

Proof. Recall that the direct image functor $j_+ : \mathcal{M}^R(\mathcal{D}^j_\lambda) \to \mathcal{M}^R(\mathcal{D}_\lambda)$ is naturally defined on right modules. We can apply equation 2.1 to the left $\mathcal{D}^j_\lambda$-module $\mathcal{F}$ by first twisting by $\omega_{P(w_\Theta)}$. Then the graded module (which is naturally a right $\mathcal{D}_\lambda$-module) can be considered a left $\mathcal{D}_\lambda$-module with a second twist by ω_X; that is,

$$Gr\mathcal{I}(w_\Theta, \lambda, \eta) = j_\bullet(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \omega_{P(w_\Theta)|X}),$$

where $\omega_{P(w_\Theta)|X} = \omega_{P(w_\Theta)} \otimes_{\mathcal{O}_{P(w_\Theta)}} j^*(\omega_X^{-1})$ is the invertible $\mathcal{O}_{P(w_\Theta)}$-module of top degree relative differential forms for the morphism j. The result then follows from the fact that $\omega_{P(w_\Theta)|X} = \mathcal{O}(2\rho_\Theta - 2\rho)$. □

Now we are ready to prove Lemma 5.17.

Proof. Using the preceding lemmas, Kashiwara's theorem, and equation 3.2, we can show that the character of $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$ is equal to the character of $M(w_\Theta\lambda, \eta)$. Here $\lambda \in \mathfrak{h}^*$ and $\eta \in \mathfrak{n}^*$ are arbitrary.

$$
\begin{aligned}
ch\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)) &= ch\Gamma(X, Gr\mathcal{I}(w_\Theta, \lambda, \eta)) \\
&= ch\Gamma(X, j_*(\mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho))) \\
&= ch\Gamma(P(w_\Theta), \mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} S(\mathcal{N}_{X|P(w_\Theta)}) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)) \\
&= ch\Gamma(P(w_\Theta), \mathcal{F} \otimes_{\mathcal{O}_{P(w_\Theta)}} \bigoplus_{\mu \in Q} \mathcal{O}(\mu) \otimes_{\mathcal{O}_{P(w_\Theta)}} \mathcal{O}(2\rho_\Theta - 2\rho)) \\
&= ch\Gamma(P(w_\Theta), \bigoplus_{\mu \in Q} \mathcal{I}(w_\Theta, \lambda + \rho - \rho_\Theta + \mu + 2\rho_\Theta - 2\rho, \eta|_{\mathfrak{n}_\Theta})) \\
&= ch \bigoplus_{\mu \in Q} Y(\lambda - \rho + \rho_\Theta + \mu, \eta|_{\mathfrak{n}_\Theta}) \\
&= \sum_{\mu \in Q} [\overline{Y(\lambda - \rho + \rho_\Theta + \mu, \eta)}] e^{\overline{\lambda - \rho + \mu}} \\
&= chM(\lambda, \eta) \\
&= chM(w_\Theta \lambda, \eta).
\end{aligned}
$$

Because $\mathcal{I}(w_\Theta, \lambda, \eta)$ is irreducible, $\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta))$ is irreducible or zero. Because it has nonzero character, it cannot be zero, and must be irreducible. Also, $M(w_\Theta \lambda, \eta)$ is irreducible. Irreducible Whittaker modules are completely determined by their character (Corollary 3.18), so we conclude that

$$\Gamma(X, \mathcal{I}(w_\Theta, \lambda, \eta)) = M(w_\Theta \lambda, \eta).$$

This completes the proof of the lemma. □

Finally, we are ready to prove our desired result.

Theorem 5.23. *Let $\lambda \in \mathfrak{h}^*$ be antidominant, $C \in W_\Theta \backslash W$, and $\eta \in \mathfrak{n}^*$ be arbitrary. Then*

$$\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = M(w^C \lambda, \eta).$$

Proof. By Proposition 4.30, if $\alpha \in \Pi$ is such that $Cs_\alpha < C$,

$$I_{s_\alpha}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w^C s_\alpha, s_\alpha \lambda, \eta),$$

and

$$L^p I_{s_\alpha}(\mathcal{M}(w^C, \lambda, \eta)) = 0 \text{ for } p \neq 0.$$

Therefore, if $\alpha \in \Pi$ is such that $Cs_\alpha > C$,

$$I_{s_\alpha}(\mathcal{M}(w^C s_\alpha, \lambda, \eta)) = \mathcal{M}(w^C, s_\alpha \lambda, \eta).$$

If $\alpha \in \Pi - \Theta$, then $\ell(w_\Theta s_\alpha) = \ell(w_\Theta) + 1$ and $W_\Theta s_\alpha > W_\Theta$, so

$$I_{s_\alpha}(\mathcal{M}(w_\Theta s_\alpha, \lambda, \eta)) = \mathcal{M}(w_\Theta, s_\alpha \lambda, \eta).$$

We will now expand this result to a larger class of $w \in W$. For a coset $C \in W_\Theta \backslash W$, let $w^C \in C$ be the longest coset element, and let $w_C \in C$ be the unique shortest coset element. (See Theorem 5.6.) Then by Theorem 5.6 and Theorem 5.5, we have $w_\Theta w_C = w^C$, and $\ell(w_\Theta w_C) = \ell(w_\Theta) + \ell(w_C) = \ell(w^C)$.

Lemma 5.24. *For any $C \in W_\Theta \backslash W$,*

$$I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = \mathcal{M}(w_\Theta, w_C \lambda, \eta),$$

and

$$L^p I_{w_C}(\mathcal{M}(w^C, \lambda, \eta)) = 0 \textit{ for } p < 0.$$

Proof. We proceed by induction in $\ell(w_C)$. If $\ell(w_C) = 0$, then $C = W_\Theta$, and the assertion is trivially true. If $\ell(w_C) = 1$, then w_C is a simple reflection s_α and the assertion is true by the preceding remarks. Let $D \in W_\Theta \backslash W$ and assume that

$$I_{w_D}(\mathcal{M}(w^D, \lambda, \eta)) = \mathcal{M}(w_\Theta, w_D \lambda, \eta) \text{ and } L^p I_{w_D}(\mathcal{M}(w^D, \lambda, \eta)) = 0 \text{ for } p < 0.$$

Let $\alpha \in \Pi$ be such that $D s_\alpha > D$. By Proposition 5.9, the shortest element $w_{D s_\alpha}$ in $D s_\alpha$ is $w_D s_\alpha$. Thus,

$$\begin{aligned} I_{w_D s_\alpha}(\mathcal{M}(w^D s_\alpha, \lambda, \eta)) &= I_{w_D}(I_{s_\alpha}(\mathcal{M}(w^D s_\alpha, \lambda, \eta)) \\ &= I_{w_D}(\mathcal{M}(w^D, s_\alpha \lambda, \eta)) \\ &= \mathcal{M}(w_\Theta, w_d s_\alpha \lambda, \eta). \end{aligned}$$

Here the first equality falls from Proposition 4.3 and the second equality from Proposition 4.30. This completes the proof of the lemma by induction. $\square$

Now we return to the proof of Theorem 5.23. Lemma 5.24 implies that for $C \in W_\Theta \backslash W$,

$$L I_{w_C}(D(\mathcal{M}(w^C, \lambda, \eta)) = D(\mathcal{M}(w_\Theta, w_C \lambda, \eta))$$

and

$$R\Gamma(L I_{w_C}(D(\mathcal{M}(w^C, \lambda, \eta))) = R\Gamma(D(\mathcal{M}(w_\Theta, w_C \lambda, \eta))).$$

If $\lambda \in \mathfrak{h}^*$ is antidominant, then by Theorem 4.6,

$$R\Gamma(D(\mathcal{M}(w^C, \lambda, \eta))) = R\Gamma(D(\mathcal{M}(w_\Theta, w_C\lambda, \eta))),$$

and

$$H^p(X, \mathcal{M}(w^C, \lambda, \eta)) = 0 \text{ for } p > 0.$$

Therefore, by Lemma 5.17,

$$\Gamma(X, \mathcal{M}(w^C, \lambda, \eta)) = \Gamma(X, \mathcal{M}(w_\Theta, w_C\lambda, \eta)) = M(w^C\lambda, \eta),$$

which completes the proof of the Theorem. □

It is now straightforward to calculate the global sections of irreducible modules.

Theorem 5.25. *Let $\lambda \in \mathfrak{h}^*$ be regular antidominant. Then, for any $C \in W_\Theta \backslash W$, we have*

$$\Gamma(X, \mathcal{L}(w^C, \lambda, \eta)) = L(w^C\lambda, \eta).$$

Proof. Because λ is regular antidominant, the global sections functor $\Gamma(X, -)$ is an equivalence of categories. Therefore, by Theorem 5.23, the unique irreducible quotient $\mathcal{L}(w^C, \lambda, \eta)$ of $\mathcal{M}(w^C, \lambda, \eta)$ must be mapped to the unique irreducible quotient $L(w^C\lambda, \eta)$ of $M(w^C\lambda, \eta)$ by $\Gamma(X, -)$. □

These results explicitly establish the connection between the category of Whittaker modules and the category of twisted Harish-Chandra sheaves.

CHAPTER 6

A KAZHDAN–LUSZTIG ALGORITHM

The main result of this document is an algorithm to calculate the multiplicity of an irreducible Whittaker module in a standard Whittaker module. This chapter develops the algorithm. This algorithm is inspired by Beilinson and Bernstein's algorithm for calculating the multiplicity of an irreducible $\mathfrak{g}$-module in a Verma module in [BB81]. We base notation and proof structure off of Miličić's interpretation of the Verma module algorithm in [Milb, Ch. 5 §2].

The statement of the algorithm is completely combinatorial. Let W be the Weyl group of a reduced root system Σ. Let $\Pi \subset \Sigma$ be the collection of simple roots, and let $S \subset W$ be the corresponding set of simple reflections. Let $\Theta \subset \Pi$ be a subset of simple roots, and let $W_\Theta \subset W$ be the sub-Weyl group generated by reflections through Θ. Let $\mathcal{H}_\Theta$ be the free $\mathbb{Z}[q, q^{-1}]$-module with basis δ_C, $C \in W_\Theta \backslash W$. Here $\mathbb{Z}[q, q^{-1}]$ is the ring of finite Laurent series in q. Let $\alpha \in \Pi$. Then for $C \in W_\Theta \backslash W$, there are three possibile relationships between α and C: either $Cs_\alpha = C$, $Cs_\alpha > C$, or $Cs_\alpha < C$ (Section 5.2). For any $\alpha \in \Pi$, we define a $\mathbb{Z}[q, q^{-1}]$-module endomorphism by

$$T_\alpha(\delta_C) = \begin{cases} 0 & \text{if } Cs_\alpha = C; \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C; \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C. \end{cases}$$

The main result is the following theorem.

Theorem 6.1. *There exists a unique function $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying the following properties.*

(i) For $C \in W_\Theta \backslash W$,

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) For $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$ such that $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D).$$

The function $\varphi : W_\Theta \backslash W \longrightarrow \mathcal{H}_\Theta$ determines a unique family $\{P_{CD} | C, D \in W_\Theta \backslash W, D \leq C\}$ of polynomials in $\mathbb{Z}[q]$ such that $\varphi(C) = \sum_{D \leq C} P_{CD} \delta_D$ for $C \in W_\Theta \backslash W$. We refer to these polynomials as *Whittaker Kazhdan–Lusztig polynomials* to emphasize the analogy between the relationship of these polynomials and Whittaker modules and the relationship between the Kahdan–Lusztig polynomials and Verma modules. We will discuss the combinatorial properties of these polynomials in Chapter 7.

We prove uniqueness of the function φ using a straightforward combinatorial argument. However, to prove existence of the function, we appeal to geometry[1]. Defining φ geometrically allows us to use the results in Chapter 5 to deduce multiplicity results about Whittaker modules from Theorem 6.1. This is done explicitly in Section 6.1. We begin with uniqueness, and we prove a slightly stronger form. Denote by $W_\Theta \backslash W_{\leq k}$ the set of cosets $C \in W_\Theta \backslash W$ such that $\ell(w^C) \leq k$.

Lemma 6.2. *Let $k \in \mathbb{N}$. Then there exists at most one function $\varphi : W_\Theta \backslash W_{\leq k} \longrightarrow \mathcal{H}_\Theta$ such that the following properties are satisfied.*

(i) For $C \in W_\Theta \backslash W_{\leq k}$,

$$\varphi(C) = \delta_C + \sum_{D < C} P_{CD} \delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

(ii) For $\alpha \in \Pi$ and $C \in W_\Theta \backslash W_{\leq k}$ such that $Cs_\alpha < C$, there exist $c_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D).$$

Proof. We proceed by induction in k. Our base case is when $k = \ell(w^C)$ for $C \in W_\Theta \backslash W$ which is minimal in the order relation on $W_\Theta \backslash W$. By Proposition 5.10, the unique minimal coset is $C = W_\Theta$, so $k = \ell(w_\Theta)$ is our base case. In this case, $W_\Theta \backslash W_{\leq k} = \{W_\Theta\}$. The only possible function $\varphi : W_\Theta \backslash W \longrightarrow \mathcal{H}_\Theta$ which satisfies (i) is $\varphi(W_\Theta) = \delta_{W_\Theta}$, and (ii) is void.

[1] There is also a straightforward combinatorial argument to prove existence of φ, as demonstrated in [Soe97, Thm. 3.1]. However, our geometric proof provides the critical link between these combinatorial objects and Whittaker modules, which is the main intention of this project.

Assume that for $k > \ell(w_\Theta)$, there exists $\varphi : W_\Theta\backslash W_{\leq k} \longrightarrow \mathcal{H}_\Theta$ which satisfies (i) and (ii). Our induction assumption is that $\varphi|_{W_\Theta\backslash W_{\leq k-1}}$ is unique. Let $C \in W_\Theta\backslash W_{\leq k}$ be such that $\ell(w^C) = k$. (We know that such a C exists for any $k > \ell(w_\Theta)$ by Theorem 5.11.) Then by Proposition 5.10, there exists $\alpha \in \Pi$ such that $Cs_\alpha < C$. By (ii),

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \leq C} c_D \varphi(D).$$

Evaluating at $q = 0$ and using (i), we have

$$T_\alpha(\varphi(Cs_\alpha))(0) = \sum_{D\leq C} c_D \left(\delta_D + \sum_{E<D} P_{DE}(0)\delta_C \right) = \sum_{D\leq C} c_D \delta_D.$$

Because $\ell(w^{Cs_\alpha}) = k-1$, the induction assumption implies that the coefficients c_D in this sum are uniquely determined. On the other hand, using the definition of φ and T_α, we compute

$$\begin{aligned} T_\alpha(\varphi(Cs_\alpha)) &= T_\alpha\left(\delta_{Cs_\alpha} + \sum_{D<Cs_\alpha} P_{Cs_\alpha D}\delta_D \right) \\ &= T_\alpha(\delta_{Cs_\alpha}) + \sum_{D<Cs_\alpha} P_{Cs_\alpha D} T_\alpha(\delta_D) \\ &= q\delta_{Cs_\alpha} + \delta_C + \sum_{D<Cs_\alpha} P_{Cs_\alpha D} T_\alpha(\delta_D). \end{aligned}$$

Because all cosets D appearing in the sum are less than Cs_α in the coset order, $\ell(w^D) < k-1$ for any such D. In particular, δ_C does not show up in this sum. Evaluating at zero and setting this equal to our first computation, we conclude that $c_C = 1$. Therefore,

$$\varphi(C) = T(\varphi(Cs_\alpha)) - \sum_{D<C} c_D \varphi(D).$$

The quantities $\varphi(Cs_\alpha)$ and $\varphi(D)$ are uniquely determined by the induction assumption, and the coefficients c_D for $D < C$ are uniquely determined by our previous argument, so the left-hand side must be uniquely determined as well. This shows that the Lemma holds for $W_\Theta\backslash W_{\leq k}$, and we are done by induction. □

The uniqueness of Theorem 6.1 follows immediately from Lemma 6.2. Before we show the existence of the function φ, we will establish a "parity" condition on solutions of Lemma 6.2. This condition will be critical in the upcoming geomtric computations.

We define additive involutions i on $\mathbb{Z}[q,q^{-1}]$ and ι on $\mathcal{H}_\Theta$ by

$$i(q^m) = (-1)^m q^m, \text{ for } m \in \mathbb{Z}, \text{ and}$$
$$\iota(q^m \delta_C) = (-1)^{m+\ell(w^C)} q^m \delta_C, \text{ for } m \in \mathbb{Z} \text{ and } C \in W_\Theta \backslash W.$$

Then $\iota T_\alpha \iota$ is a $\mathbb{Z}[q,q^{-1}]$-linear endomorphism of $\mathcal{H}_\Theta$, and

$$(\iota T_\alpha \iota)(\delta_C) = (-1)^{\ell(w^C)} \iota(T_\alpha(\delta_C)).$$

If $Cs_\alpha = C$, then

$$(\iota T_\alpha \iota)(\delta_C) = (-1)^{\ell(w^C)} \iota(T_\alpha(\delta_C)) = 0 = -T_\alpha(\delta_C).$$

If $Cs_\alpha > C$, then

$$\begin{aligned}
(\iota T_\alpha \iota)(\delta_C) &= (-1)^{\ell(w^C)} \iota(T_\alpha(\delta_C)) = (-1)^{\ell(w^C)} \iota(q\delta_C + \delta_{Cs_\alpha}) \\
&= (-1)^{\ell(w^C)} (\iota(q\delta_C) + \iota(\delta_{Cs_\alpha})) = (-1)^{\ell(w^c)} ((-1)^{1+\ell(w^C)} q\delta_C + (-1)^{\ell(w^C)+1} \delta_{Cs_\alpha}) \\
&= (-1)^{1+2\ell(w^C)} (q\delta_C + \delta_{Cs_\alpha}) = -(q\delta_C + \delta_{Cs_\alpha}) = -T_\alpha(\delta_C).
\end{aligned}$$

Here, the fact that $\iota(\delta_{Cs_\alpha}) = (-1)^{\ell(w^C)+1} \delta_{Cs_\alpha}$ follows from Proposition 5.9. Finally, if $Cs_\alpha < C$,

$$\begin{aligned}
(\iota T_\alpha \iota)(\delta_C) &= (-1)^{\ell(w^C)} \iota(T_\alpha(\delta_C)) = (-1)^{\ell(w^C)} \iota(q^{-1}\delta_C + \delta_{Cs_\alpha}) \\
&= (-1)^{\ell(w^C)} (\iota(q^{-1}\delta_C) + \iota(\delta_{Cs_\alpha})) = (-1)^{\ell(w^c)} ((-1)^{-1+\ell(w^C)} q^{-1}\delta_C + (-1)^{\ell(w^C)-1} \delta_{Cs_\alpha}) \\
&= (-1)^{2\ell(w^C)-1} (q^{-1}\delta_C + \delta_{Cs_\alpha}) = -(q^{-1}\delta_C + \delta_{Cs_\alpha}) = -T_\alpha(\delta_C).
\end{aligned}$$

We conclude from this calculation that $\iota T_\alpha \iota = -T_\alpha$.

Lemma 6.3. *Let $k \in \mathbb{N}$. Let $\varphi : W_\Theta \backslash W_{\leq k} \longrightarrow \mathcal{H}_\Theta$ be a function satisfying properties (i) and (ii) of Lemma 6.2. Then*

$$P_{CD} = q^{\ell(w^C)-\ell(w^D)} Q_{CD},$$

where $Q_{CD} \in \mathbb{Z}[q^2, q^{-2}]$.

Proof. Define a function $\psi : W_\Theta \backslash W_{\leq k} \to \mathcal{H}_\Theta$ by $\psi(C) = (-1)^{\ell(w^C)}\iota(\varphi(C))$. Then

$$\begin{aligned}\psi(C) &= (-1)^{\ell(w^C)}\iota\left(\delta_C + \sum_{D<C} P_{CD}\delta_D\right)\\ &= (-1)^{2\ell(w^C)}\delta_C + \sum_{D<C}(-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD})\delta_D\\ &= \delta_C + \sum_{D<C}(-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD})\delta_D.\end{aligned}$$

The polynomials $(-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD})$ are in $q\mathbb{Z}[q]$, so ψ satisfies (i). We will show that ψ also satisfies (ii), then use Lemma 6.2 to conclude that $\psi = \varphi$. Let $C \in W_\Theta\backslash W_{\leq k}$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$. Then

$$\begin{aligned}T_\alpha(\psi(Cs_\alpha)) &= T_\alpha\left((-1)^{\ell(w^C)-1}\iota(\varphi(Cs_\alpha))\right)\\ &= (-1)^{\ell(w^C)}\left(-T_\alpha(\iota(\varphi(Cs_\alpha)))\right)\\ &= (-1)^{\ell(w^C)}\iota T_\alpha \iota(\iota(\varphi(Cs_\alpha)))\\ &= (-1)^{\ell(w^C)}\iota T_\alpha(\varphi(Cs_\alpha))\\ &= (-1)^{\ell(w^C)}\iota\left(\sum_{D\leq C} c_D\varphi(D)\right)\\ &= (-1)^{\ell(w^C)}\sum_{D\leq C} c_D \iota(\varphi(D))\\ &= \sum_{D\leq C}(-1)^{\ell(w^C)-\ell(w^D)} c_D\psi(D).\end{aligned}$$

This shows that ψ satisfies (ii), so Lemma 6.2 implies that $\varphi = \psi$; that is, that

$$P_{CD} = (-1)^{\ell(w^C)-\ell(w^D)} i(P_{CD}).$$

This relationship implies the result. Indeed, if $\ell(w^C) - \ell(w^D)$ is even, then we can conclude that P_{CD} has no odd-degree terms, so $P_{CD} \in q^2\mathbb{Z}[q^2]$. Therefore, we can pull out an even power of q and are left with an element of $\mathbb{Z}[q^{-2}, q^2]$; i.e. $P_{CD} = q^{\ell(w^C)-\ell(w^D)}Q_{CD}$, for some $Q_{CD} \in \mathbb{Z}[q^{-2}, q^2]$. If $\ell(w^C) - \ell(w^D)$ is odd, then the relationship above implies that P_{CD} has no even-degree terms, so we can pull out an odd power of P_{CD} and are left with an element of $\mathbb{Z}[q^{-2}, q^2]$; i.e. $P_{CD} = q^{\ell(w^C)-\ell(w^D)}Q_{CD}$ for $Q_{CD} \in \mathbb{Z}[q^{-2}, q^2]$. □

Now we are ready to prove existence of φ. Let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$. For $w \in W$, let $i_w : C(w) \longrightarrow X$ be the canonical immersion of the corresponding Bruhat cell into the flag variety. We note the following facts.

- For any $k \in \mathbb{Z}$, $L^{-k}i_w^+(\mathcal{F})$ is an η-twisted N-equivariant connection on $C(w)$, so it is isomorphic to a sum of copies of $\mathcal{O}_{C(w)}$. (See Section 4.3.) We refer to the number of copies of $\mathcal{O}_{C(w)}$ that appear in this decomposition as the $\mathcal{O}$-dimension, and denote it $\dim_{\mathcal{O}}(L^{-k}i_w^+(\mathcal{F}))$.

- Because the dimension of $C(w)$ is $\ell(w)$, for any $k \in \mathbb{Z}$,

$$R^{n-\ell(w)-k}i_w^!(\mathcal{F}) = L^{-k}i_w^+(\mathcal{F}).$$

  Here $n = \dim X$.

We define a map $\nu : \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta) \longrightarrow \mathcal{H}_\Theta$ by

$$\nu(\mathcal{F}) = \sum_{C \in W_\Theta \backslash W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i_{w^C}^!(\mathcal{F}))q^m \delta_C. \tag{6.1}$$

For $C \in W_\Theta \backslash W$, let $\mathcal{I}_C = \mathcal{I}(w^C, -\rho, \eta)$ be the standard sheaf in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ corresponding to the coset C and $\mathcal{L}_C = \mathcal{L}(w^C, -\rho, \eta)$ its unique irreducible subsheaf.

Proposition 6.4. *Let $\varphi(C) = \nu(\mathcal{L}_C)$. Then φ satisfies conditions (i) and (ii) in Theorem 6.1.*

Checking that φ satisfies 6.1 (i) is straightforward.

Lemma 6.5. *Let $\varphi(C) = \nu(\mathcal{L}_C)$. Then*

$$\varphi(C) = \delta_C + \sum_{D < C} P_{CD}\delta_D,$$

where $P_{CD} \in q\mathbb{Z}[q]$.

Proof. We need to show three things:

(a) If $D \not\leq C$, $\dim_{\mathcal{O}}(R^m i_{w^D}^!(\mathcal{L}_C)) = 0$ for all $m \in \mathbb{Z}$,

(b) $\dim_{\mathcal{O}}(R^m i_{w^C}^!(\mathcal{L}_C)) = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}$, and

(c) if $D < C$, $\dim_{\mathcal{O}}(R^m i_{w^D}^!(\mathcal{L}_C)) = 0$ for all $m \leq 0$.

Part (a) follows immediately from the fact that supp$\mathcal{L}_C = \overline{C(w^C)}$ and $D \leq C$ in the coset order if and only if $C(w^D) \subset \overline{C(w^C)}$ (Proposition 5.15). To see part (b), we first observe that by Lemma 4.27 and Corollary 2.3,

$$R^0 i^!_{w^C}(\mathcal{L}_C) = R^0 i^!_{w^C}(\mathcal{I}_C) = R^0 i^!_{w^C}(i_{w^C +}(\mathcal{O}_{C(w^C)})) = \mathcal{O}_{C(w^C)}.$$

So $\dim_{\mathcal{O}}(R^0 i^!_{w^C}(\mathcal{L}_C)) = 1$. Furthermore, for $m \neq 0$,

$$R^m i^!_{w^C}(\mathcal{L}_C) = R^m i^!_{w^C}(\mathcal{I}_C) = R^m i^!_{w^C}(i_{w^C +}(\mathcal{O}_{C(w^C)})) = 0.$$

This proves (b). We end by showing (c). Let $D \in W_\Theta \backslash W$ be a coset so that $D < C$. Because i_w is an immersion, $i^!_{w^D}$ is a right derived functor, so for any $m < 0$, $R^m i^!_{w^D}(\mathcal{V}) = 0$ for any $\mathcal{D}$-module $\mathcal{V}$ on X, so all that remains is to show that $R^0 i^!_{w^D}(\mathcal{L}_C) = 0$. Let $X' = X - \partial C(w^D)$, and let $j_{w^D} : C(w^D) \to X'$ be the natural closed immersion, and $k_{w^D} : X' \to X$ the natural open immersion. Then we have a commutative diagram.

$$\begin{tikzcd} C(w^D) \arrow[rr, "i_{w^D}"] \arrow[dr, "j_{w^D}"'] & & X \\ & X' \arrow[ur, "k_{w^D}"'] & \end{tikzcd}$$

Then, we can see that

$$\begin{aligned} R^0 j_{w^D +}(R^0 i^!_{w^D}(\mathcal{L}_C)) &= R^0 j_{w^D +}(R^0 j^!_{w^D}(R^0 k^!_{w^D}(\mathcal{L}_C))) \\ &= R^0 j_{w^D +}(R^0 j^!_{w^D}(L^0 k^+_{w^D}(\mathcal{L}_C))) \\ &= R^0 j_{w^D +}(R^0 j^!_{w^D}(\mathcal{L}_C|_{X'})) \\ &= R^0 \Gamma_{C(w^D)}(\mathcal{L}_C|_{X'}). \end{aligned}$$

Here the second equality falls from the fact that $\dim X = \dim X'$, the third equality holds because k_{w^D} is an open immersion, and the final equality is from Kashiwara's Theorem (Theorem 2.2). From this calculation, we see that $R^0 j_{w^D +}(R^0 i^!_{w^D}(\mathcal{L}_C))$ is the submodule of $\mathcal{L}_C|_{X'}$ consisting of sections supported on $C(w^D)$. However, because X' is open, $\mathcal{L}|_{X'}$ is irreducible, so this submodule must be zero. We conclude that $R^0 i^!_{w^D}(\mathcal{L}_D) = 0$, which completes the proof of the lemma. □

Our final step in proving Theorem 6.1 is establishing that φ satisfies Theorem 6.1(ii). This is the bulk of the argument. First, we need to introduce another useful functor.

Fix $\alpha \in \Pi$, and let $p_\alpha : X \longrightarrow X_\alpha$ be projection onto the flag variety of parabolic subalgebras of type α. If $P_\alpha \subset G$ is the standard parabolic of type α, then $P_\alpha = B \cup Bs_\alpha B$. Let $C(v)$ be the Bruhat cell corresponding to $v \in W$. Then we have the following facts:

- $C(v) \simeq \mathbb{C}^{\ell(v)}$, so $i_v : C(v) \longrightarrow X$ is an affine morphism.
- $p_\alpha(C(v))$ is also affine, so it is an affine subvariety of X_α.
- p_α is locally trivial, so $p_\alpha^{-1}(p_\alpha(C(v))$ is a smooth, affinely imbedded subvariety of X.

We conclude that $p_\alpha^{-1}(p_\alpha(C(v))) = C(v) \cup C(vs_\alpha)$. One of these orbits is closed in the variety $p_\alpha^{-1}(p_\alpha(C(v)))$ and the other is open and dense. We have two possible scenarios:

1. $\ell(vs_\alpha) = \ell(v) + 1$. Then $\dim(C(vs_\alpha)) > \dim(C(v))$, and so
   - $C(vs_\alpha)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(v)))$,
   - $C(v)$ is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$, and
   - $p_\alpha : C(v) \longrightarrow p_\alpha(C(v))$ is an isomorphism.
2. $\ell(vs_\alpha) = \ell(v) - 1$. Then $\dim(C(vs_\alpha)) < \dim(C(v))$, and so
   - $C(vs_\alpha)$ is closed in $p_\alpha^{-1}(p_\alpha(C(v)))$,
   - $C(v)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(v)))$, and
   - $p_\alpha : C(v) \longrightarrow p_\alpha(C(v))$ is a fibration with fibers ismorphic to an affine line.

We define a family of functors $U_\alpha^k : \mathcal{M}_{qc}(\mathcal{D}_X) \longrightarrow \mathcal{M}_{qc}(\mathcal{D}_X)$ by

$$U_\alpha^k(\mathcal{F}) = p_\alpha^+(H^k p_{\alpha+}(\mathcal{F})).$$

Because the fibers of the projection map $p_\alpha : X \to X_\alpha$ are one-dimensional, U_α^k can be non-zero only for $k \in \{-1, 0, 1\}$. These functors are closely related to the U-functors discussed in Section 4.1.4. Their main utility in our argument comes from their semisimplicity properties.

Lemma 6.6. *Let $C \in W_\Theta \backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$. Then*

(i) $U_\alpha^k(\mathcal{L}_{Cs_\alpha}) = 0$ *for all* $k \neq 0$, *and*

(ii) $U^0_\alpha(\mathcal{L}_{Cs_\alpha})$ *is a direct sum of* $\mathcal{L}_D$ *for* $D \leq C$.

Proof. By construction, $U^0_\alpha(\mathcal{L}_{Cs_\alpha})$ are holonomic $(\mathcal{D}_X, N, \eta)$-modules supported inside $\overline{C(w^C) \cup C(w^C s_\alpha))} = \overline{C(w^C)}$. This implies that $U^0_\alpha(\mathcal{L}_{Cs_\alpha})$ has finite length, and its composition factors must be in the set $\{\mathcal{L}_D | D \in W_\Theta\backslash W \text{ and } D \leq C\}$. Because p_α is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^1$ (in particular, it is a projective morphism of smooth quasi-projective varieties), and $\mathcal{L}_{Cs_\alpha}$ is a semisimple holonomic $\mathcal{D}$-module, the decomposition theorem [Moc11, §1 Thm. 1.4.1] implies that $H^k p_{\alpha+}(\mathcal{L}_{Cs_\alpha})$ are semisimple. By the local triviality of p_α, this in turn implies that $U^0_\alpha(\mathcal{L}_{Cs_\alpha})$ are semisimple, which completes the proof of (ii).

To prove (i), we establish the connection between U^0_α and the results in Section 4.1.4. Let $Y_\alpha = X \times_{X_\alpha} X$ be the fiber product of X with itself relative to the morphism p_α. Denote by q_1 and q_2 the projections of Y_α onto the first and second factors, respectively. Then the following diagram

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{q_2} & X \\ {\scriptstyle q_1}\big\downarrow & & \big\downarrow{\scriptstyle p_\alpha} \\ X & \xrightarrow{p_\alpha} & X_\alpha \end{array}$$

is commutative. By base change (Theorem 2.5),

$$U^0_\alpha(\mathcal{L}_{Cs_\alpha}) = p_\alpha^+(H^k p_{\alpha+}(\mathcal{L}_{Cs_\alpha})) = H^k q_{1+}(q_2^+(\mathcal{L}_{Cs_\alpha})).$$

Because $\mathcal{D}_X = \mathcal{D}_{-\rho}$, we have $U^q_\alpha(\mathcal{L}_{Cs_\alpha})(\alpha) = U^q(\mathcal{L}_{Cs_\alpha})$. This establishes the connection with the U-functors of Section 4.1.4, and to complete the proof, we need to show that we are in case (ii) of Theorem 4.7; i.e. that $L^{-1}I_{s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$. Because $Cs_\alpha < C$, we can apply Proposition 4.29 to the coset Cs_α and conclude that

$$LI_{s_\alpha}(D(\mathcal{I}(w^C s_\alpha, \lambda, \eta))) = D(\mathcal{I}(w^C, s_\alpha\lambda, \eta)).$$

In particular, this implies that $L^{-1}I_{s_\alpha}(\mathcal{I}(w^c s_\alpha, \lambda, \eta)) = 0$, and because $\mathcal{L}_{Cs_\alpha}$ is a submodule of $\mathcal{I}(w^c s_\alpha, \lambda, \eta)$, $L^{-1}I_{s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$ as well. □

We are working toward showing that $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies (ii). We will do so by relating $T_\alpha(\varphi(Cs_\alpha))$ to $U^0_\alpha(\mathcal{L}_{Cs_\alpha})$ and using Lemma 6.6 to obtain our desired decomposition. Let $C \in W_\Theta\backslash W$ and $\alpha \in \Pi$ be such that $Cs_\alpha < C$. Then if w^C is the longest element in C, Proposition 5.9 implies that $w^C s_\alpha$ is the longest element of Cs_α, and $\ell(w^C s_\alpha) = \ell(w) - 1$. Let $D \leq C$. Then by Proposition 5.9, $\ell(w^D) \leq \ell(w^C)$, so $C(w^D) \subset \overline{C(w^C)}$. By assumption, $C(w^C)$ is open and dense in $p_\alpha^{-1}(p_\alpha(C(w^C))) = C(w^C) \cup C(w^C s_\alpha)$, so $\overline{p_\alpha^{-1}(p_\alpha(C(w^C)))} = \overline{C(w^C)}$. Because $C(w^D) \subset \overline{C(w^C)}$, $p_\alpha(C(w^D)) \subset p_\alpha(\overline{C(w^C)})$, so

$$C(w^D) \cup C(w^D s_\alpha) = p_\alpha^{-1}(p_\alpha(C(w^D))) \subset p_\alpha^{-1}(p_\alpha(\overline{C(w^C)})) \subset \overline{p_\alpha^{-1}(p_\alpha(C(w^C)))} = \overline{C(w^C)}.$$

We conclude that both $w^D s_\alpha \leq w^C$ and $w^D \leq w^C$. Because both elements are less than or equal to w^C in the Bruhat order, we can assume without loss of generality that $w^D s_\alpha \leq w^D$; i.e. $\ell(w^D s_\alpha) = \ell(w^D) - 1$ and $C(w^D)$ is open in $Z_\alpha = p_\alpha^{-1}(p_\alpha(C(w^D))) = C(w^D) \cup C(w^D s_\alpha)$.

Here we pause, and address the two possibilities of the relationship between D and α. Either

(a) $w^D s_\alpha \in D$, or

(b) $w^D s_\alpha \notin D$.

The rest of our argument will address each case separately. The following lemma describes the key result in case (a).

Lemma 6.7. *Let $v \in W$ be a Weyl group element such that $v \neq w^C$ is not a longest coset element for any coset $C \in W_\Theta \backslash W$. Let $\mathcal{F} \in \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ be irreducible. Then*

$$R^k i_v^!(\mathcal{F}) = 0$$

for all $k \in \mathbb{Z}$.

Proof. Let $X' = X - \partial C(v)$, and express the canonical immersion i_v as the composition of a closed immersion and an open immersion in the following way.

$$\begin{tikzcd} C(v) \arrow[r, "j_v"] \arrow[rr, bend right, "i_v"'] & X' \arrow[r, "k_v"] & X \end{tikzcd}$$

Then, if $\mathcal{F}$ is an irreducible (D_X, N, η)-module,

$$\begin{aligned} i_v^!(D(\mathcal{F})) &= j_v^!(k_v^!(D(\mathcal{F}))) \\ &= i_v^!(i_{v+}(j_v^!(k_v^!(D(\mathcal{F}))))) \end{aligned}$$

$$= i_v^!(k_{v+}(j_{v+}(j_v^!(k_v^!(D(\mathcal{F}))))))$$
$$= i_c^!(k_{v+}(R\Gamma_{C(v)}(k_v^!(D(\mathcal{F})))))$$
$$= i_v^!(k_{v+}(R\Gamma_{C(v)}(D(\mathcal{F}|_{X'}))))$$

Here the second equality falls from Corollary 2.3, the fourth equality from Theorem 2.2, and the final equality from the fact that $\dim X = \dim X'$ and k_v is an open immersion. Because X' is open in X and $\mathcal{F}$ is irreducible, $\mathcal{F}|_{X'}$ is irreducible as well. For all $k \in \mathbb{Z}$, $R^k\Gamma_{C(v)}\mathcal{F}|_{X'}$ is a submodule of $\mathcal{F}|_{X'}$, so either

(a) $R^k\Gamma_{C(v)}\mathcal{F}|_{X'} = 0$, or

(b) $R^k\Gamma_{C(v)}\mathcal{F}|_{X'} = \mathcal{F}|_{X'}$.

In case (a), the preceding calculation implies that $R^k i_v^!(\mathcal{F}) = 0$, and we are done. In case (b), we have $\mathrm{supp}\mathcal{F}|_{X'} = \mathrm{supp}R^k\Gamma_{C(v)}\mathcal{F}|_{X'} \subseteq C(v)$. By [Mila, Ch. V §4 Cor. 4.2], $\mathcal{F}$ is the unique irreducible holonomic $\mathcal{D}_X$-module that restricts to $\mathcal{F}|_{X'}$, and $\mathrm{supp}\mathcal{F} = \overline{\mathrm{supp}\mathcal{F}|_{X'}} \subseteq \overline{C(v)}$. There are no irreducible objects in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ with support equal to $\overline{C(v)}$ because v is not a longest coset element, so we must have $\mathrm{supp}\mathcal{F} \subseteq \partial C(v) = \overline{C(v)} - C(v)$. This implies that in case (b),

$$R^k i_v^!(\mathcal{F}) = 0$$

for all $k \in \mathbb{Z}$. $\square$

Now, we return to our previous setting. Let $D \leq C$, and $\alpha \in \Pi$ such that $Cs_\alpha < C$, and assume that $C(w^D)$ is open in $Z_\alpha = p_\alpha^{-1}(p_\alpha(C(w^D)))$. We do not specify at this time whether $Ds_\alpha = D$ or $Ds_\alpha \neq D$. Let $j : Z_\alpha \longrightarrow X$ and $j_D : p_\alpha(C(w^D)) \longrightarrow X_\alpha$ be natural inclusions. Let $q_\alpha : Z_\alpha \longrightarrow p_\alpha(C(w^D))$ be the restriction of p_α to Z_α. Then we have the following fiber product diagram [Mila, Ch. IV §10].

$$\begin{tikzcd} Z_\alpha \arrow[r, "j"] \arrow[d, "q_\alpha"] & X \arrow[d, "p_\alpha"] \\ p_\alpha(C(w^D)) \arrow[r, "j_D"] & X_\alpha \end{tikzcd}$$

Note that because p_α and q_α are surjective submersions, p_α^+ and q_α^+ are exact, so they both lift to functors on the respective derived categories $D^b(\mathcal{M}(\mathcal{D}_X))$ and $D^b(\mathcal{M}(\mathcal{D}_{Z_\alpha}))$. In the calculations below, we denote both the functors on the derived category and the functors on modules by the same name, either p_α^+ or q_α^+. Let d be the codimension of Z_α in X. Note that the codimension of $p_\alpha(C(w^D)) = p_\alpha(Z_\alpha)$ in X_α is also d. Recall that for any immersion $i : Y \to X$ of smooth algebraic varieties, the extraordinary inverse image and the $\mathcal{D}$-module inverse image are related by $i^![\mathrm{codim}(Y)] = Li^+$. By base change (Theorem 2.5), Lemma 6.6, and the relationship described in the previous sentence, we get

$$\begin{aligned}
R^k j^!(U_\alpha^0(\mathcal{L}_{Cs_\alpha})) &= H^k(j^!(p_\alpha^+(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha}))))) \\
&= H^{k+d}(Lj^+(p_\alpha^+(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha}))))) \\
&= H^{k+d}(L(p_\alpha \circ j)^+(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha})))) \\
&= H^{k+d}(L(j_D \circ q_\alpha)^+(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha})))) \\
&= H^{k+d}(q_\alpha^+(Lj_D^+(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha}))))) \\
&= H^k(q_\alpha^+(j_D^!(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha}))))) \\
&= q_\alpha^+(H^k(j_D^!(p_{\alpha+}(D(\mathcal{L}_{Cs_\alpha}))))) \\
&= q_\alpha^+(H^k(q_{\alpha+}(j^!(D(\mathcal{L}_{Cs_\alpha}))))).
\end{aligned}$$

Our next step is to analyze the complex $j^!(D(\mathcal{L}_{Cs_\alpha}))$. Denote by $i : C(w^D) \longrightarrow Z_\alpha$ and $i' : C(w^D s_\alpha) \longrightarrow Z_\alpha$ the canonical affine immersions. Note that i is an open immersion, and i' is a closed immersion. We have the following commutative diagram.

$$\begin{tikzcd}
C(w^D) \arrow[dr, "i"] \arrow[drr, bend left, "i_{w^D}"] & & \\
& Z_\alpha \arrow[r, "j"] & X \\
C(w^D s_\alpha) \arrow[ur, "i'"] \arrow[urr, bend right, "i_{w^D s_\alpha}"'] & &
\end{tikzcd}$$

For any complex $\mathcal{F}^\cdot \in D^b(\mathcal{M}(\mathcal{D}_{Z_\alpha}))$, we have the following distinguished triangle [Mila, Ch. IV §9].

$$i'_+(i'^!(\mathcal{F}^\cdot)) \longrightarrow \mathcal{F}^\cdot \longrightarrow i_+(\mathcal{F}^\cdot|_{C(w^D)}).$$

Applying this to $\mathcal{F}^\cdot = j^!(D(\mathcal{L}_{Cs_\alpha}))$, we get the distinguished triangle

$$i'_+(i'^!(j^!(D(\mathcal{L}_{Cs_\alpha})))) \longrightarrow j^!(D(\mathcal{L}_{Cs_\alpha})) \longrightarrow i_+(j^!(D(\mathcal{L}_{Cs_\alpha}))|_{C(w^D)}).$$

Now, since $j^!(D(\mathcal{L}_{Cs_\alpha}))|_{C(w^D)} = i^+(j^!(D(\mathcal{L}_{Cs_\alpha}))) = i^!(j^!(D(\mathcal{L}_{Cs_\alpha}))) = i^!_{w^D}(D(\mathcal{L}_{Cs_\alpha}))$ because i is an open immersion and $i'^! \circ j^! = i^!_{w^D s_\alpha}$, we simplify this distinguished triangle to

$$i'_{+}(i^{!}_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha}))) \longrightarrow j^{!}(D(\mathcal{L}_{Cs_\alpha})) \longrightarrow i_{+}(i^{!}_{w^D}(D(\mathcal{L}_{Cs_\alpha}))).$$

Applying the functor $q_{\alpha+}$ and using the fact that $(q_\alpha \circ i)_+ = q_{\alpha+} \circ i_+$ and $(q_\alpha \circ i')_+ = q_{\alpha+} \circ i_+$ we get the following distinguished triangle in $D^b(\mathcal{M}(\mathcal{D}_{p_\alpha(C(w^D))}))$:

$$(q_\alpha \circ i')_{+}(i^{!}_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha}))) \longrightarrow q_{\alpha+}(j^{!}(D(\mathcal{L}_{Cs_\alpha}))) \longrightarrow (q_\alpha \circ i)_{+}(i^{!}_{w^D}(D(\mathcal{L}_{Cs_\alpha})))).$$

Because $p_\alpha(C(w^D))$ is an N-orbit in X_α and all $\mathcal{D}$-modules in the arguments above are N-equivariant, the cohomologies of the complexes in this triangle are all sums of copies of $\mathcal{O}_{p_\alpha(C(w^D))}$. Additionally, the map

$$q_\alpha \circ i' : C(w^D s_\alpha) \longrightarrow p_\alpha(C(w^D))$$

is an isomorphism, and the map

$$q_\alpha \circ i : C(w^D) \longrightarrow p_\alpha(C(w^D))$$

is a locally trivial projection with one-dimensional fibers. We conclude that

$$\dim_{\mathcal{O}} H^k((q_\alpha \circ i')_{+}(i^{!}_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})))) = \dim_{\mathcal{O}} R^k i^{!}_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}), \text{ and}$$
$$\dim_{\mathcal{O}} H^k((q_\alpha \circ i)_{+}(i^{!}_{w^D}(D(\mathcal{L}_{Cs_\alpha})))) = \dim R^{k+1} i^{!}_{w^D}(\mathcal{L}_{Cs_\alpha}).$$

From the final distinguished triangle above, we also obtain the long exact sequence in cohomology:

$$\cdots \to H^{k-1}((q_\alpha \circ i)_{+}(i^{!}_{w^D}(D(\mathcal{L}_{Cs_\alpha}))) \to H^k((q_\alpha \circ i')_{+}(i^{!}_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})))) \to$$
$$H^k(q_{\alpha+}(j^{!}(D(\mathcal{L}_{Cs_\alpha}))) \to H^k((q_\alpha \circ i)_{+}(i^{!}_{w^D}(D(\mathcal{L}_{Cs_\alpha}))) \to$$
$$H^{k+1}((q_\alpha \circ i')_{+}(i^{!}_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})))) \to \cdots$$

This is a sequence of $\mathcal{D}_{p_\alpha(C(w^D))}$-modules which are sums of copies of $\mathcal{O}_{p_\alpha(C(w^D))}$.

Now we are ready to prove that $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies 6.1 (ii) by induction in the length of w^C. The base case is when $w^C = w_\Theta$. In this case, for any $\alpha \in \Pi$, either $Cs_\alpha = C$, or $Cs_\alpha > C$, because w_Θ is minimal length in the set of longest coset elements. Therefore, by 5.10, we conclude that $C = W_\Theta$, and 6.1 (ii) is void. Assume that $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies 6.1 (ii) for $C \in W_\Theta \backslash W_{\leq k}$ and some $k \in \mathbb{N}$. By 6.3, $\varphi(C) = \nu(\mathcal{L}_C)$ satisfies the parity condition on $W_\Theta \backslash W_{\leq k}$; that is, for $C \in W_\Theta \backslash W_{\leq k}$ and $D \in W_\Theta \backslash W$, $P_{CD} = q^{\ell(w^C) - \ell(w^D)} Q_{CD}$, for $Q_{CD} \in \mathbb{Z}[q^2, q^{-2}]$. Because

$$P_{CD}(q) = \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(\mathcal{L}_C)) q^m,$$

we conclude that for any $C \in W_\Theta \backslash W_{\leq k}$ and $D \in W_\Theta \backslash W$, if $m \equiv \ell(w^C) - \ell(w^D) - 1 (\text{mod } 2)$, then $R^m i^!_{w^D}(\mathcal{L}_C) = 0$.

Let $C \in W_\Theta \backslash W$ be such that $\ell(w^C) = k+1$. Then $C \neq W_\Theta$, and so we know by 5.10 there exists $\alpha \in \Pi$ such that $Cs_\alpha < C$. By 5.9, the longest element in Cs_α is $w^C s_\alpha$. For any $D \in W_\Theta \backslash W$, we have either

(a) $k \equiv \ell(w^D) - \ell(w^C s_\alpha)$ (mod 2); or

(b) $k \equiv \ell(w^D) - \ell(w^C s_\alpha) - 1$ (mod 2).

In case (a), we have $k+1 \equiv \ell(w^D) - \ell(w^C s_\alpha) - 1$ (mod 2), so the parity condition implies that

$$R^{k+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) = 0.$$

Similarly, in this case, we have $k \equiv \ell(w^D s_\alpha) - \ell(w^C s_\alpha) - 1$ (mod 2). If $D \in W_\Theta \backslash W$ has the property that $Ds_\alpha \neq D$ (i.e. $w^D s_\alpha$ is the longest element in some coset), then we can apply the parity condition again to conclude that

$$R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0.$$

If $D \in W_\Theta \backslash W$ has the property that $Ds_\alpha = D$, then we can use Lemma 6.7 to draw the same conclusion: $R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0$. Either way, we are able to conclude that in case (a),

$$H^k((q_\alpha \circ i)_+(i^!_{w^D}(D(\mathcal{L}_{Cs_\alpha})))) = 0, \text{ and } H^k((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})))) = 0.$$

Then, by the long exact sequence in cohomology, this implies that

$$H^k(q_\alpha(j^!(D(\mathcal{L}_{Cs_\alpha})))) = 0 \text{ for } k \equiv \ell(w^D) - \ell(w^C s_\alpha) \text{ (mod 2)}.$$

In case (b), we have $k \equiv \ell(w^D s_\alpha) - \ell(w^C s_\alpha) - 1$ (mod 2), and $k+1 \equiv \ell(w^D s_\alpha) - \ell(w^C s_\alpha) - 1$ (mod 2). We can conclude immediately from the parity condition that

$$R^k i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) = 0, \text{ and } H^{k-1}((q_\alpha \circ i)_+(i^!_{w^D}(D(\mathcal{L}_{Cs_\alpha})))) = 0.$$

As above, if $Ds_\alpha \neq D$, then $w^D s_\alpha$ is the longest element in some coset, and we can use the parity condition again to conclude that

$$R^{k+1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) = 0, \text{ and } H^{k+1}((q_\alpha \circ i')_+(i^!_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})))) = 0.$$

If $Ds_\alpha = D$, then applying Lemma 6.7 leads us to the same conclusion. We see from these two arguments that the long exact sequence in cohomology has the form

$$\cdots \to 0 \to 0 \to 0 \to * \to * \to * \to 0 \to 0 \to 0 \to * \to * \to * \to 0 \to 0 \to 0 \to \cdots$$

Therefore, if $k \equiv \ell(w^D) - \ell(w^C s_\alpha) - 1 \pmod 2$, then

$$\dim_{\mathcal{O}} H^k(q_{\alpha+}(j^!(D(\mathcal{L}_{Cs_\alpha})))) = \dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{k+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}).$$

Recall that because q_α is a surjective submersion, q_α^+ is exact, and we showed previously that $R^k j^!(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = q_\alpha^+(H^k(q_{\alpha+}(j^!(D(\mathcal{L}_{Cs_\alpha})))))$. We conclude that

(a) If $k \equiv \ell(w^D) - \ell(w^C s_\alpha) \pmod 2$, then $\dim_{\mathcal{O}} R^k j^!(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = 0$; and

(b) If $k \equiv \ell(w^D) - \ell(w^C d_\alpha) - 1 \pmod 2$, then

$$\dim_{\mathcal{O}} R^k j^!(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{k+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}).$$

By restricting further to $C(w^D)$ and $C(w^D s_\alpha)$, we finally obtain our desired parity result. For all $k \in \mathbb{Z}_+$, $D \in W_\Theta \backslash W$, and $\alpha \in \Pi$ such that $Cs_\alpha < C$, we have

$$\dim_{\mathcal{O}} R^k i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^{k+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}), \text{ and}$$

$$\dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \dim_{\mathcal{O}} R^k i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{k-1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}).$$

In addition, if $D \in W_\Theta \backslash W$ has the property that $Ds_\alpha = D$, we can use Lemma 6.7 to further reduce the formulas above. Indeed, by Lemma 6.7, in this case,

$$\dim_{\mathcal{O}} R^{k-1} i^!_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})) = 0, \text{ and}$$
$$\dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(D(\mathcal{L}_{Cs_\alpha})) = 0$$

for all $k \in \mathbb{Z}_+$. By Lemma 6.6, $U^0_\alpha(\mathcal{L}_{Cs_\alpha}) = \bigoplus_{D \leq C} m_{CD} \mathcal{L}_D$ for some $m_{CD} \in \mathbb{Z}_+$, hence Lemma 6.7 also implies that

$$\dim_{\mathcal{O}} R^k i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = 0.$$

Combining this with the relationships above, which hold for all $D \in W_\Theta \backslash W$ regardless of relationship between D and s_α, we conclude that for $D \in W_\Theta \backslash W$ such that $Ds_\alpha = D$,

$$\dim_{\mathcal{O}} R^k i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = 0$$

for all $k \in \mathbb{Z}_+$.

We conclude the proof of the theorem with the following computation.

$$\begin{aligned}
\nu(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) &= \sum_{D \in W_\Theta \backslash W} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha > D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&\quad + \sum_{Ds_\alpha = D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D s_\alpha}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_{Ds_\alpha} \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}}(R^m i^!_{w^D}(U^0_\alpha(\mathcal{L}_{Cs_\alpha})))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^{m-1} i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m \delta_{Ds_\alpha} \\
&\quad + \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m \delta_D \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} (\dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) + \dim_{\mathcal{O}} R^m i^!_{w^D s_\alpha}(\mathcal{L}_{Cs_\alpha}))q^m (\delta_D + q\delta_{Ds_\alpha}) \\
&= \sum_{Ds_\alpha < D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} R^{m+1} i^!_{w^D}(\mathcal{L}_{Cs_\alpha}) q^{m+1} (q^{-1}\delta_D + \delta_{Ds_\alpha}) \\
&\quad + \sum_{Ds_\alpha > D} \sum_{m \in \mathbb{Z}} \dim_{\mathcal{O}} R^m i^!_{w^D}(\mathcal{L}_{Cs_\alpha}))q^m (\delta_{Ds_\alpha} + q\delta_D) \\
&= T_\alpha(\nu(\mathcal{L}_{Cs_\alpha})) = T_\alpha(\varphi(Cs_\alpha)).
\end{aligned}$$

Therefore, for $C \in W_\Theta \backslash W_{\leq k}$ and $\alpha \in \Pi$ such that $Cs_\alpha < C$,

$$T_\alpha(\varphi(Cs_\alpha)) = \nu(U^0_\alpha(\mathcal{L}_{Cs_\alpha})) = \nu(\bigoplus_{D \leq C} c_D \mathcal{L}_D) = \sum_{D \leq C} c_D \nu(\mathcal{L}_D) = \sum_{D \leq C} c_D \varphi(D).$$

This shows that 6.1 (ii) holds on $W_\Theta \backslash W_{\leq k+1}$. By induction, we see that φ satisfies 6.1 (ii), and this completes the proof of Proposition 6.4. It also completes the proof of Theorem 6.1.

6.1 Multiplicities of Irreducible Whittaker Modules in Standard Whittaker Modules

Finally, we will establish the connection between the polynomials P_{CD} and the multiplicities of irreducible Whittaker modules in the composition series of standard Whittaker modules. We start with two preliminary lemmas.

Lemma 6.8. *The evaluation $\nu(-1)$ of the map ν at -1 factors through the Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ of $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$.*

Proof. For an object $\mathcal{F}$ in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$,

$$\nu(\mathcal{F})(-1) = \sum_{C \in W_\Theta \backslash W} \sum_{m \in \mathbb{Z}} (-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}))\delta_C.$$

If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is a short exact sequence in $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$, then for each $C \in W_\Theta \backslash W$, we have a long exact sequence

$$\cdots \xrightarrow{\partial_{m-1}} R^m i^!_{w^C}(\mathcal{F}_1) \xrightarrow{f_m} R^m i^!_{w^C}(\mathcal{F}_2) \xrightarrow{g_m} R^m i^!_{w^C}(\mathcal{F}_3) \xrightarrow{\partial_m} R^{m+1} i^!_{w^C}(\mathcal{F}_1) \to \cdots$$

of N-homogeneous η-twisted connections on $C(w^C)$. For each $m \in \mathbb{Z}$, we have short exact sequences

$$0 \to \ker f_m \to R^m i^!_{w^C}(\mathcal{F}_1) \to \operatorname{im} f_m \to 0,$$
$$0 \to \ker g_m \to R^m i^!_{w^C}(\mathcal{F}_2) \to \operatorname{im} g_m \to 0, \text{ and}$$
$$0 \to \ker \partial_m \to R^m i^!_{w^C}(\mathcal{F}_3) \to \operatorname{im} \partial_m \to 0.$$

The $\mathcal{O}$-dimension sums over short exact sequences, so we have

$$\dim_{\mathcal{O}} R^m i^!_{w^C}(\mathcal{F}_1) = \dim_{\mathcal{O}} \ker f_m + \dim_{\mathcal{O}} \operatorname{im} f_m,$$
$$\dim_{\mathcal{O}} R^m i^!_{w^C}(\mathcal{F}_2) = \dim_{\mathcal{O}} \ker g_m + \dim_{\mathcal{O}} \operatorname{im} g_m, \text{ and}$$
$$\dim_{\mathcal{O}} R^m i^!_{w^C}(\mathcal{F}_3) = \dim_{\mathcal{O}} \ker \partial_m + \dim_{\mathcal{O}} \operatorname{im} \partial_m.$$

Therefore, by multiplying the second equality above by -1, summing over $m \in \mathbb{Z}$, and using the relationships $\ker f_m = \operatorname{im} \partial_{m-1}$, $\ker g_m = \operatorname{im} f_m$, and $\ker \partial_m = \operatorname{im} g_m$, we have

$$\sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_2)) = \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_1)) - \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}} \ker f_m$$
$$+ \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_3)) - \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}} \ker \partial_m$$
$$= \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_1))$$
$$+ \sum_{m\in\mathbb{Z}}(-1)^m \dim_{\mathcal{O}}(R^m i^!_{w^C}(\mathcal{F}_3)).$$

This implies the result. □

Lemma 6.9. $\nu(\mathcal{I}_C) = \delta_C$.

Proof. By definition, $\mathcal{I}_C = i_{w^C+}(\mathcal{O}_{C(w^C)})$. By Kashiwara's theorem (more specifically, Corollary 2.3),

$$R^0 i^!_{w^C}(\mathcal{I}_C) = R^0 i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = \mathcal{O}_{C(w^C)},$$

and for $m \neq 0$,

$$R^m i^!_{w^C}(\mathcal{I}_C) = R^m i^!_{w^C}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = 0.$$

Let $D \neq C$ be another coset in $W_\Theta\backslash W$. Then $i^{-1}_{w^D}(C(w^C)) = 0$, so by base change (Theorem 2.5),

$$R^m i^!_{w^D}(\mathcal{I}_C) = R^m i^!_{w^D}(i_{w^C+}(\mathcal{O}_{C(w^C)})) = 0$$

for all $m \in \mathbb{Z}$. □

Let $\chi : \mathcal{M}_{coh}(\mathcal{D}_X, N, \eta) \to K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ be the natural map of the category $\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta)$ into its Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$.

Theorem 6.10. *Let P_{CD}, $C, D \in W_\Theta\backslash W$ be the polynomials in Theorem 6.1. Then*

$$\chi(\mathcal{L}_C) = \chi(\mathcal{I}_C) + \sum_{D<C} P_{CD}(-1)\chi(\mathcal{I}_D).$$

Proof. By definition, the set $\{\chi(\mathcal{L}_C)\}_{C\in W_\Theta\backslash W}$ forms a basis for the Grothendieck group $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$. Because $\mathcal{I}_C$ contains $\mathcal{L}_C$ as a unique irreducible submodule, and the other composition factors of $\mathcal{I}_C$ are $\mathcal{L}_D$ for $D < C$, we can see that $\chi(\mathcal{I}_C), C \in W_\Theta\backslash W$ form another basis for the Grothendieck group. Therefore, there exist $\lambda_{CD} \in \mathbb{Z}$ such that

$$\chi(\mathcal{L}_C) = \sum_{D\leq C} \lambda_{CD}\chi(\mathcal{I}_D).$$

By Lemma 6.8, $\nu(-1)$ factors through $K(\mathcal{M}_{coh}(\mathcal{D}_X, N, \eta))$ and by Lemma 6.9, $\nu(\mathcal{I}_D) = \delta_D$, so

$$\nu(\mathcal{L}_C)(-1) = \sum_{D \leq C} \lambda_{CD}\nu(\mathcal{I}_D)(-1) = \sum_{D \leq C} \lambda_{CD}\delta_D.$$

By construction, $P_{CC} = 1$ for any $C \in W_\Theta \backslash W$, so $\lambda_{CC} = 1$ and $P_{CD}(-1) = \lambda_{CD}$. This proves the theorem. □

This theorem gives an algorithm for calculating the multiplicities of irreducible Whittaker modules in standard Whittaker modules. One starts by ordering elements of $W_\Theta \backslash W$ by the Bruhat order on longest coset representatives. Then the matrix $(\lambda_{CD})_{C,D \in W_\Theta \backslash W}$ is lower triangular and has 1's on the diagonal. Let $(\mu_{CD})_{C,D \in W_\Theta \backslash W}$ be the inverse matrix. From Theorem 6.10, we have

$$\begin{aligned}
\chi(\mathcal{I}_C) &= \sum_{D \in W_\Theta \backslash W} \sum_{E \in W_\Theta \backslash W} \mu_{CE}\lambda_{ED}\chi(\mathcal{I}_D) \\
&= \sum_{E \in W_\Theta \backslash W} \mu_{CE} \left(\sum_{D \in W_\Theta \backslash W} \lambda_{ED}\chi(\mathcal{I}_D) \right) \\
&= \sum_{E \in W_\Theta \backslash W} \mu_{CD}\chi(\mathcal{L}_E) \\
&= \sum_{E \leq C} \mu_{CE}\chi(\mathcal{L}_C).
\end{aligned}$$

By Theorem 5.23 and Theorem 5.25, we have established the main result of this dissertation.

Corollary 6.11. *The multiplicity of the irreducible Whittaker module $L(-w^D\rho, \eta)$ in the standard Whittaker module $M(-w^C\rho, \eta)$ is μ_{CD}.*

We can get theorems analogous to Theorem 6.10 and Corollary 6.11 for regular weights $\mu \in P(\Sigma)$ by twisting by a homogeneous invertible $\mathcal{O}_X$-module. To establish the same multiplicity results for standard Whittaker modues of arbitrary central character requires further analysis, which we will examine in future work. (See Chapter 8.)

We complete this section with the following observation about the polynomials P_{CD}.

Corollary 6.12. *The coefficients of the polynomials P_{CD} from Theorem 6.1 are non-negative integers.*

Proof. This follows immediately from Proposition 6.4 and the definition of ν. □

CHAPTER 7

WHITTAKER KAZHDAN–LUSZTIG POLYNOMIALS

This chapter relates the Whittaker Kazhdan–Lusztig polynomials P_{CD} of Theorem 6.1 to the combinatorics of Kazhdan–Lusztig polynomials appearing in [Soe97] and [Milb, Ch. 5 §2 §3]. We also establish the relationship between Whittaker Kazhdan–Lusztig polynomials and the polynomials arising in the Kazhdan-Lusztig algorithm for generalized Verma modules in [Milb, Ch. 6 §3 Thm. 3.5], following the philosophy of dual Hecke algebra modules laid out in [Vog82, §12 §13]. To make these associations, we need to introduce the Hecke algebra into our story.

7.1 The Hecke Algebra

Let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$.

Definition 7.1. The *Hecke algebra*[1] $\mathcal{H} = \mathcal{H}(W, S)$ of the Coxeter system (W, S) is the associative algebra over $\mathbb{Z}[q, q^{-1}]$ with generators $\{H_s\}_{s\in S}$ satisfying the relations

(i) (quadratic)

$$(H_s + q)(H_s - q^{-1}) = 0 \text{ for all } s \in S, \text{ and}$$

(ii) (braid) for $s, t \in S$,

$$H_sH_t \cdots H_s = H_tH_s \cdots H_t \text{ if } st \cdots s = ts \cdots t$$

$$H_sH_tH_s \cdots H_t = H_tH_sH_t \cdots H_s \text{ if } sts \cdots t = tst \cdots s.$$

All H_s for $s \in S$ are invertible with $H_s^{-1} = H_s + (q - q^{-1})$. For $w \in W$, we choose a reduced expression $rs \cdots t$ of w and define $H_w \in \mathcal{H}$ by $H_rH_s \cdots H_t$. This element is

[1] As $\mathbb{Z}[q, q^{-1}]$-modules, $\mathcal{H} = \bigoplus_{w\in W} \mathbb{Z}[q, q^{-1}]T_w$ where $T_w := q^{-1}H_w$. The algebra structure defined above is the unique associative algebra structure on the $\mathbb{Z}[q, q^{-1}]$-module $\bigoplus_{w\in W} \mathbb{Z}[q, q^{-1}]T_w$ such that $T_wT_v = T_{wv}$ if $\ell(w) + \ell(v) = \ell(wv)$ and $T_s^2 = q^{-2}T_e + (q^{-2} - 1)T_s$ for $s \in S$ [Bou05, Ch. IV §2 Ex. 23].

independent of choice of reduced expression. If $\ell(w) + \ell(v) = \ell(wv)$, then we have $H_w H_v = H_{wv}$. There is exactly one ring homomorphism

$$\begin{aligned} d : \mathcal{H} &\to \mathcal{H} \\ H &\mapsto \overline{H} \end{aligned}$$

such that $\overline{q} = q^{-1}$ and $\overline{H}_w = (H_{w^{-1}})^{-1}$. This is clearly an involution. We say that $H \in \mathcal{H}$ is *self-dual* if $\overline{H} = H$. For each $s \in S$, the element $C_s := H_s + q$ is self-dual. Indeed, $\overline{C_s} = (H_s)^{-1} + q^{-1} = H_s + (q - q^{-1}) + q^{-1} = C_s$.

7.2 $\mathcal{H}_\Theta$ is a Hecke Algebra Module

Now we return to the setting of Chapter 6. Let W be the Weyl group of a reduced root system Σ with simple roots $\Pi \subset \Sigma$ and corresponding simple reflections $S \subset W$. Then (W, S) is a Coxeter system. Let $\Theta \subset \Pi$ be a fixed subset of simple roots and let $\mathcal{H}_\Theta = \bigoplus_{C \in W_\Theta \backslash W} \mathbb{Z}[q, q^{-1}]\delta_C$ be the $\mathbb{Z}[q, q^{-1}]$-module from Theorem 6.1. Recall that for each $\alpha \in \Pi$, we defined a $\mathbb{Z}[q, q^{-1}]$-linear endomorphism T_α of $\mathcal{H}_\Theta$ by

$$T_\alpha(\delta_C) = \begin{cases} 0 & \text{if } Cs_\alpha = C \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C \end{cases}.$$

Our first observation is that the operators $\{T_\alpha\}_{\alpha \in \Pi}$ generate a Hecke algebra under composition. Indeed, if we define $S_\alpha := T_\alpha - q$, then a computation shows that

(i) $(S_\alpha + q)(S_\alpha - q^{-1}) = 0$, and

(ii) $S_\alpha S_\beta S_\alpha = S_\beta S_\alpha S_\beta$ if $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \in W$.

We conclude that $\langle T_\alpha \rangle_{\alpha \in \Pi} \subset \mathrm{End}_{\mathbb{Z}[q,q^{-1}]}(\mathcal{H}_\Theta)$ is isomorphic to the Hecke algebra $\mathcal{H}$ of (W, S) under the isomorphism $S_\alpha \mapsto H_{s_\alpha}$. Under this isomorphism, $T_\alpha \mapsto C_{s_\alpha}$, where C_{s_α} is the self-dual element described in the preceding section. This gives $\mathcal{H}_\Theta$ the structure of a Hecke algebra module with action given by

$$\begin{aligned} \mathcal{H} \times \mathcal{H}_\Theta &\to \mathcal{H}_\Theta \\ (H_{s_\alpha}, \delta_C) &\mapsto S_\alpha(\delta_C). \end{aligned}$$

This extra structure will allow us to relate Theorem 6.1 to the results in [Soe97, §2 §3]. Our first step is to recognize $\mathcal{H}_\Theta$ as a certain induced module (the antispherical module for the Hecke algebra) in order to extend the duality in $\mathcal{H}$ given by the involution d to a duality in $\mathcal{H}_\Theta$. If $S_\Theta \subset S$ is the subset of simple reflections corresponding to $\Theta \subset \Pi$, then the subalgebra $\mathcal{H}^\Theta$ of $\mathcal{H}$ generated by $\{H_{s_\alpha}\}$ for $\alpha \in \Theta$ is isomorphic to the Hecke algebra of the Coxeter system (W_Θ, S_Θ). The surjection $\mathcal{H}^\Theta \twoheadrightarrow \mathbb{Z}[q,q^{-1}]$ sending $H_{s_\alpha} \mapsto -q$ gives $\mathbb{Z}[q,q^{-1}]$ the structure of a $\mathcal{H}^\Theta$-bimodule, and with this bimodule structure, we can form the induced right $\mathcal{H}$-module

$$\mathcal{N}^\Theta := \mathbb{Z}[q,q^{-1}] \otimes_{\mathcal{H}^\Theta} \mathcal{H}.$$

This is the *antispherical module* of the Hecke algebra $\mathcal{H}$. Note that in the special case $\Theta = \emptyset$, $\mathcal{N}^\Theta$ is just the Hecke-algebra $\mathcal{H}$. The set $\{N_w := 1 \otimes H_w\}$ for minimal coset representatives $w \in C \in W_\Theta \backslash W$ forms a basis for $\mathcal{N}^\Theta$ as a $\mathbb{Z}[q,q^{-1}]$-module.

Remark 7.2. By instead using the surjection $\mathcal{H}^\Theta \twoheadrightarrow \mathbb{Z}[q,q^{-1}]$ given by $H_{s_\alpha} \mapsto q^{-1}$ to form the $\mathcal{H}^\Theta$-bimodule structure on $\mathbb{Z}[q,q^{-1}]$, it is possible to construct another induced right $\mathcal{H}$-module $\mathcal{M}^\Theta := \mathbb{Z}[q,q^{-1}] \otimes_{\mathcal{H}^\Theta} \mathcal{H}$ [Soe97, §3]. This is the *spherical module* of the Hecke algebra $\mathcal{H}$. This module also has the property that $\mathcal{M}^\emptyset = \mathcal{H}$. By an analogous argument to the one below, one can show that the $\mathcal{H}$-module that appears in the Kazhdan–Lusztig algorithm for generalized Verma modules in [Milb, Ch. 6 §3] is isomorphic to the spherical module.

One can compute [Soe97] that the action of C_s on $\mathcal{N}^\Theta$ for $s \in S$ is given by

$$N_w C_s = \begin{cases} 0 & \text{if } ws \in C \\ qN_w + N_{ws} & \text{if } ws > w \text{ and } ws \notin C \\ q^{-1}N_w + N_{ws} & \text{if } ws < w \text{ and } ws \notin C \end{cases}.$$

From this, we conclude that $\mathcal{H}_\Theta$ is isomorphic as an $\mathcal{H}$-module to the antispherical module $\mathcal{N}^\Theta$ under the isomorphism

$$\begin{aligned} \phi : \mathcal{H}_\Theta &\to \mathcal{N}^\Theta \\ \delta_C &\mapsto N_{w_\Theta w}. \end{aligned}$$

Here w_Θ is the longest element in W_Θ. Note that in the special case $\Theta = \emptyset$, this provides an $\mathcal{H}$-module isomorphism between $\mathcal{H}_\emptyset$ and the Hecke algebra $\mathcal{H}$, viewed as a module over itself with the right regular action.[2] The benefit of viewing $\mathcal{H}_\Theta$ as an induced module

[2]This justifies the notational choice in [Milb, Ch. 5 §2], where the $\mathbb{Z}[q,q^{-1}]$-module $\mathcal{H}_\emptyset$ is referred to as $\mathcal{H}$.

is that it allows us to use the involution d of $\mathcal{H}$ to construct an involution of the induced module, which we can then use to define self-duality in $\mathcal{H}_\Theta$. There is a homomorphism of additive groups

$$\mathcal{N}^\Theta \to \mathcal{N}^\Theta$$
$$a \otimes H \mapsto \overline{a \otimes H} := \overline{a} \otimes \overline{H}.$$

This homomorphism has the property that $\overline{N}_e = N_e$ and

$$\overline{NH} = \overline{N}\,\overline{H} \tag{7.1}$$

for all $N \in \mathcal{N}^\Theta$ and $H \in \mathcal{H}$. We say that an element $E \in \mathcal{H}_\Theta$ is *self-dual* if the corresponding element in $\mathcal{N}^\Theta$ is fixed under this involution; that is, if $\overline{\phi(E)} = \phi(E)$. Since $\phi(T_\alpha(E)) = \phi(E)C_{s_\alpha}$ for any $\alpha \in \Pi$ and $E \in \mathcal{H}_\Theta$ and C_{s_α} is self-dual in $\mathcal{H}$, property (7.1) implies that T_α preserves self-duality.

7.3 The Recursion Relation in Theorem 6.1 Is Equivalent to Self-duality

The main content of this chapter is a proof that condition (ii) in Theorem 6.1 is equivalent to $\varphi(C)$ being self-dual in the sense of the preceding section.

Theorem 7.3. *Let $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ be a function satisfying*

$$\varphi(C) = \delta_C + \sum_{D<C} P_{CD}\delta_D \text{ for } P_{CD} \in q\mathbb{Z}[q] \tag{7.2}$$

for all $C \in W_\Theta \backslash W$. Then the following are equivalent.

(i) If $\alpha \in \Pi$ and $C \in W_\Theta \backslash W$ are such that $Cs_\alpha < C$, then there exist $m_D \in \mathbb{Z}$ such that

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D\le C} m_D \varphi(D).$$

(ii) All $\varphi(C)$ are self-dual.

Proof. Assume that (i) holds. Using the definition of T_α, we compute

$$\begin{aligned} T_\alpha(\varphi(Cs_\alpha)) &= T_\alpha(\delta_{Cs_\alpha} + \sum_{E<Cs_\alpha} P_{Cs_\alpha E}\delta_E) \\ &= \delta_C + q\delta_{Cs_\alpha} + \sum_{E<Cs_\alpha} P_{Cs_\alpha E} T_\alpha(\delta_E) \\ &= \delta_C + \sum_{D<C} Q_{CD}\delta_C \end{aligned}$$

for some $Q_{CD} \in \mathbb{Z}[q]$. Therefore, $m_C = 1$. Thus, for any $\alpha \in \Pi$ such that $Cs_\alpha < C$ (for $C \neq W_\Theta$ such an α must exist by Theorem 5.9),

$$\varphi(C) = T_\alpha(\varphi(Cs_\alpha)) - \sum_{D<C} m_D \varphi(D). \tag{7.3}$$

Now we show that all $\varphi(C)$ are self-dual by induction in $\ell(w^C)$. If $C = W_\Theta$, then $\varphi(W_\Theta) = \delta_{W_\Theta}$ is self-dual because $\phi(\delta_{W_\Theta}) = 1 \otimes H_e$ and $\overline{H}_e = H_e$ in $\mathcal{H}$. Assume $\varphi(D)$ is self-dual for all $D < C$. Then because T_α preserves self-duality, equation (7.3) implies that $\varphi(C)$ is self-dual. We conclude that (i) implies (ii).

Assume that (ii) holds. In [Soe97, §3 Thm. 3.1], Soergel constructs such a $\varphi(C)$ for each $C \in W_\Theta \backslash W$ and proves that φ must be unique. His construction goes as follows. We prove existence of a self-dual $\varphi(C)$ satisfying equation (7.2) by induction in $\ell(w^C)$. For $C = W_\Theta$, equation (7.2) can only be satisfied by $\varphi(W_\Theta) = \delta_{W_\Theta}$, which is self-dual by the argument above. Assume inductively that for $D < C$, a self-dual $\varphi(D)$ satisfying equation (7.2) exists. Then, there exists some $\alpha \in \Pi$ such that $Cs_\alpha < C$ (Theorem 5.9). For this α, we have

$$T_\alpha(\varphi(Cs_\alpha)) = \delta_C + \sum_{D<C} Q_{CD}\delta_D$$

for appropriately chosen $Q_{CD} \in \mathbb{Z}[q]$. Define

$$\varphi(C) := T_\alpha(\varphi(Cs_\alpha)) - \sum_{D<C} Q_{CD}(0)\varphi(D).$$

This satisfies equation (7.2) and is self-dual by the induction assumption and the fact that T_α preseves self-duality. This proves the existence of a φ satisfying equation (7.2) and (ii). The uniqueness of such a φ follows from the observation that for any $E \in \sum_{C \in W_\Theta \backslash W} q\mathbb{Z}[q]\delta_C$, self-duality implies $E = 0$. Indeed, if $E = \sum_{C \in W_\Theta \backslash W} R_C \delta_C$ and we let C be maximal such that $R_C \neq 0$, then $\overline{\phi(E)} = \phi(E)$ implies that $\overline{R}_C = R_C$, which is impossible because $R_C \in q\mathbb{Z}[q]$. This fact immediately implies uniqueness of the φ constructed above, because for any other $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying equation (7.2) and (ii), $\varphi(C) - \varphi'(C) \in \sum_{C \in W_\Theta \backslash W} q\mathbb{Z}[q]\delta_C$ is self-dual, so $\varphi(C) - \varphi'(C) = 0$.

Therefore, any φ satisfying equation (6.1) and (ii) must be of the form

$$\varphi(C) := T_\alpha(\varphi(Cs_\alpha)) - \sum_{D<C} Q_{CD}(0)\varphi(D)$$

for some $\alpha \in \Pi$, so

$$T_\alpha(\varphi(Cs_\alpha)) = \sum_{D \le C} m_D \varphi(D) \text{ for } m_D = \begin{cases} Q_{CD}(0) & \text{if } D < C \\ 1 & \text{if } D = C \end{cases}.$$

We conclude that (ii) implies (i). $\square$

This establishes the relationship between the results in this dissertation and the results in [Soe97, §2 §3]. In particular, it establishes the relationship between the Whittaker Kazhdan–Lusztig polynomials P_{CD} and polynomials that have shown up elsewhere in the combinatorics literature under the name "parabolic Kazhdan–Lusztig polynomials." We explicitly list these relationships now.

Remark 7.4.

1. The Whittaker Kazhdan–Lusztig polynomials P_{CD} are equal to the polynomials n_{xy} in [Soe97] for $x = w_\Theta w^C$ and $y = w_\Theta w^D$.

2. A normalization of P_{CD} gives the parabolic Kazhdan–Lusztig polynomials in [Deo87]. Indeed, the polynomials
$$(q^{\ell(w_\Theta w^D)} - q^{\ell(w_\Theta w^C)})P_{CD}$$
are polynomials in the variable $v := q^{-2}$, and they are precisely the polynomials $P^I_{(w_\Theta w^D)^{-1},(w_\Theta w^D)^{-1}}$ in [Deo87] for $u = v$ and $W_\Theta = W_I$.

3. In the special case where $\Theta = \emptyset$, the polynomials
$$(q^{\ell(v)} - q^{\ell(w)})P_{wv}$$
are the Kazhdan–Lusztig polynomials as defined in [KL79].

7.4 Combinatorial Duality of Whittaker Modules and Generalized Verma Modules

We conclude this chapter by listing some results established in [Soe97] relating the Whittaker Kazhdan–Lusztig polynomials P_{CD} to the polynomials arising in the Kazhdan–Lusztig algorithm for generalized Verma modules established in [Milb, Ch. 6 §3]. These results recover the Kazhdan–Lusztig inversion formulas of [KL79] as a special case. We also establish a formula relating the Whittaker Kazhdan–Lusztig polynomials to the Kazhdan–Lusztig polynomials in [Milb]. We refer the reader to [Soe97] for omitted proofs.

We start by discussing inversion formulas. Let

$$\mathcal{H}_\Theta^* = \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(\mathcal{H}_\Theta, \mathbb{Z}[q,q^{-1}])$$
$$\mathcal{N}^{\Theta *} = \mathrm{Hom}_{\mathbb{Z}[q,q^{-1}]}(\mathcal{N}^\Theta, \mathbb{Z}[q,q^{-1}])$$

be the dual $\mathbb{Z}[q,q^{-1}]$-modules to $\mathcal{H}_\Theta$ and $\mathcal{N}^\Theta$, respectively. The isomorphism $\phi : \mathcal{H}_\Theta \to \mathcal{N}^\Theta$ induces an isomorphism of

$$\phi^* : \mathcal{N}^{\Theta *} \to \mathcal{H}_\Theta^*$$
$$N \mapsto \phi \circ N.$$

We refer to the inverse of this isomorphism by ϕ'.

We can extend the involution on $\mathcal{N}^\Theta$ to a $\mathbb{Z}[q,q^{-1}]$-skew linear involution on $\mathcal{N}^{\Theta *}$ in the following way. For $F \in \mathcal{N}^{\Theta *}$, define

$$\overline{F}(N) := \overline{F(\overline{N})}.$$

We say an element $E \in \mathcal{H}_\Theta^*$ is *self-dual* if the corresponding element in $\mathcal{N}^{\Theta *}$ is self-dual; that is, if $\overline{\phi'(E)} = \phi'(E)$. We define a basis $\{\delta^C\}$ for $\mathcal{H}_\Theta^*$ as a $\mathbb{Z}[q,q^{-1}]$-module by the formula

$$\delta^C(\delta_D) = \begin{cases} (-1)^{\ell(w_\Theta w^C)} & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}.$$

The following theorem guarantees the existence of inverse Whittaker Kazhdan–Lusztig polynomials.

Theorem 7.5. *[Soe97, §3 Thm. 3.6] There exists a unique function $\psi : W_\Theta \backslash W \to \mathcal{H}_\Theta^*$ satisfying*

(i) $\psi(C) = \delta^C + \sum_{D>C} P^{CD}\delta^D$ for $P^{CD} \in q\mathbb{Z}[q]$, and

(ii) $\psi(C)$ is self-dual.

Proof. We define ψ by the formula

$$\psi(C)(\varphi(D)) = \begin{cases} (-1)^{\ell(w_\Theta w^C)} & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}$$

where $\varphi : W_\Theta \backslash W \to \mathcal{H}_\Theta$ is the unique function from Theorem 6.1. The self-duality of $\varphi(C)$ (Theorem 7.3) implies that $\psi(C)$ is self-dual. Since $\{\delta^D\}$ form a basis for $\mathcal{H}_\Theta^*$, we can express $\psi(C) = \sum_{D \in W_\Theta \backslash W} P^{CD}\delta^D$ for some $P^{CD} \in \mathbb{Z}[q,q^{-1}]$, and the relationship

$$\psi(C)(\varphi(D)) = \left(\sum_{E \in W_\Theta \backslash W} P^{CE}\delta^E \right) \left(\delta_D + \sum_{F<D} P_{DF}\delta_F \right) = \begin{cases} (-1)^{\ell(w_\Theta w^C)} & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}$$

implies that $P^{CC} = 1$, $P^{CE} = 0$ for $E < C$ and $P^{CE} \in q\mathbb{Z}[q]$. This completes the proof of the theorem. □

From the construction of ψ, it follows that the polynomials P^{CD} and P_{CD} are "inverse polynomials" in the following sense[3].

$$\sum_{E \in W_\Theta \backslash W} (-1)^{\ell(w^E)+\ell(w^C)} P^{CE} P_{DE} = \begin{cases} 1 & \text{if } C = D \\ 0 & \text{if } C \neq D \end{cases}. \tag{7.4}$$

Note that by Lemma 5.12, $\ell(w_\Theta w^C) = \ell(w^C) - \ell(w_\Theta)$, so $\ell(w_\Theta w^E) + \ell(w_\Theta w^C) = \ell(w^E) + \ell(w^C)$.

In [Milb, Ch. 6 §3], Miličić establishes a Kazhdan–Lusztig algorithm for generalized Verma modules. We review his results here to establish their relationship with the Whittaker Kazhdan–Lusztig algorithm of this document. Let $\mathcal{H}_\Theta = \bigoplus_{C \in W_\Theta \backslash W} \mathbb{Z}[q, q^{-1}]\delta_C$ be the $\mathbb{Z}[q, q^{-1}]$-module from the preceding section. We can realize $\mathcal{H}_\Theta$ as a $\mathbb{Z}[q, q^{-1}]$-submodule of the $\mathbb{Z}[q, q^{-1}]$-module[4] $\mathcal{H}_\emptyset = \bigoplus_{w \in W} \mathbb{Z}[q, q^{-1}]\delta_w$ by setting

$$\delta_C = \sum_{w \in W_\Theta} q^{\ell(v)} \delta_{vw^C}.$$

For $\alpha \in \Pi$, let $T_\alpha^\emptyset : \mathcal{H}_\emptyset \to \mathcal{H}_\emptyset$ be the endomorphism defined by

$$T_\alpha^\emptyset(\delta_w) = \begin{cases} q\delta_w + \delta_{ws_\alpha} & \text{if } ws_\alpha > w \\ q^{-1}\delta_w + \delta_{ws_\alpha} & \text{if } ws_\alpha < w \end{cases},$$

as in Section 7.2. We introduce $\emptyset$ into the notation here to emphasize that $T_\alpha^\emptyset$ is an endomorphism of $\mathcal{H}_\emptyset$. A computation shows that the endomorphism $\mathcal{T}_\alpha^\emptyset$ transforms δ_C in the following way:

$$T_\alpha^\emptyset(\delta_C) = \begin{cases} (q + q^{-1})\delta_C & \text{if } Cs_\alpha \\ q\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha < C \\ q^{-1}\delta_C + \delta_{Cs_\alpha} & \text{if } Cs_\alpha > C \end{cases}.$$

Miličić showed that $\mathcal{H}_\Theta$ is stable under $T_\alpha^\emptyset$, so $\mathcal{H}_\Theta$ is an $\mathcal{H}$-submodule of $\mathcal{H}_\emptyset$. With this $\mathcal{H}$-module structue, $\mathcal{H}_\Theta$ is isomorphic to Soergel's spherical module (Remark 7.2). In [Milb, Ch. 6 §3], Miličić proves the following Kazhdan-Lusztig algorithm for generalized Verma modules.

[3] If $\Theta = \emptyset$, the P^{CD} are the inverse Kazhdan–Lusztig polynomials as in [Soe97].

[4] Recall that this module is isomorphic as an $\mathcal{H}$-module to the Hecke algebra $\mathcal{H}$, with module structure given by the right regular action. In [Milb, Ch. 5 §2], this module is referred to as $\mathcal{H}$, but here we choose to use the notation $\mathcal{H}_\emptyset$ as a reminder that it is a special case of $\mathcal{H}_\Theta$.

Theorem 7.6. *[Milb, Ch. 6 §3 Thm. 3.5] There exists a unique function $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying the following.*

(i) For $C \in W_\Theta \backslash W$,

$$\varphi'(C) = \delta_C + \sum_{D<C} P'_{CD}\delta_D$$

for $P'_{CD} \in q\mathbb{Z}[q]$, and

(ii) for $\alpha \in \Pi$ such that $Cs_\alpha < C$, there exist integers m'_D such that

$$T_\alpha^{\emptyset}(\varphi'(Cs_\alpha)) = \sum_{D \le C} m'_D \varphi'(D).$$

Furthermore, the polynomials P'_{CD} are given by the Kazhdan–Lusztig polynomials for (W, S) by

$$P'_{CD} = P_{w^C w^D}.$$

In [Milb, Ch. 6 §3], Miličić establishes that the unique function $\varphi' : W_\Theta \backslash W \to \mathcal{H}_\Theta$ satisfying Theorem 7.6 is the function $\varphi'(D) := \varphi(w^D)$, where $\varphi : W \to \mathcal{H}_\emptyset$ is the unique function guaranteed by Theorem 6.1 in the special case $\Theta = \emptyset$. The Kazhdan–Lusztig polynomials P'_{CD} of Theorem 7.6 describe the multiplicities of irreducible highest weight modules in generalized Verma modules [Milb, Ch. 6 §3 Cor. 3.7].

For arbitrary $\Theta \subset \Pi$, the inverse Whittaker Kazhdan–Lusztig polynomials are equal to the polynomials appearing in Theorem 7.6. The following theorem appears in [Soe97], where it is originally attributed to Douglass [Dou90].

Proposition 7.7. *[Soe97, §3 Prop. 3.9] Let $\Theta \subset \Pi$ be arbitrary, and $C, D \in W_\Theta \backslash W$. Then*

$$P^{CD} = P'_{Cw_0 Dw_0}$$

where w_0 is the longest element of W and $P'_{Cw_0Dw_0}$ for $C, D \in W_\Theta \backslash W$ are the unique polynomials from Theorem 7.6.

If we specialize to $\Theta = \emptyset$, this recovers the Kazhdan–Lusztig inversion formulas.

$$\sum_{u \in W} (-1)^{\ell(u)+\ell(w)} P_{wu} P_{vw_0\, uw_0} = \begin{cases} 1 & \text{if } v = w \\ 0 & \text{if } v \neq w \end{cases}. \tag{7.5}$$

We complete this chapter by describing the relationship between the Whittaker Kazhdan–Lusztig polynomials P_{CD} and the Kazhdan–Lusztig polynomials in [Milb]. If $\Theta = \emptyset$, then

$W_\Theta \backslash W = W$, and each coset contains a single Weyl group element. In this setting, Theorem 6.1 specializes the algorithm in [Milb, Ch. 5 §2 Thm. 2.1], and the polynomials P_{wv} are the Kazhdan–Lusztig polynomials in the sense of [Soe97] and [Milb][5]. The following formula relates Whittaker Kazhdan–Lusztig polynomials to Kazhdan–Lusztig polynomials.

Proposition 7.8. *For $\Theta \subset \Pi$ arbitrary,*

$$P_{CD} = \sum_{v \in W_\Theta} (-q)^{\ell(v)} P_{w_\Theta w^C \; v w_\Theta w^D}.$$

Proof. Pick a total order compatible with the partial order on $W_\Theta \backslash W$. From Theorem 7.6 we see that $P'_{CD} = 0$ for $D > C$ and $P_{CD} = 1$ if $C = D$, so the matrix $P = (P'_{CD})$ of polynomials with respect to our total order is lower triangular with 1's on the diagonal and coefficients in $\mathbb{Z}[q]$. The inverse matrix $Q = (Q_{CD})$ is also lower triangular with 1's on the diagonal and coefficients in $\mathbb{Z}[q]$. From equation (7.4) and Proposition 7.7 we see that the coefficients Q_{CD} of the inverse matrix are related to Whittaker Kazhdan–Lusztig polynomials in the following way:

$$Q_{CD} = (-1)^{\ell(w^C) + \ell(w^D)} P_{Dw_0 Cw_0}. \tag{7.6}$$

Then, if $\varphi : W \to \mathcal{H}_\emptyset$ is the unique function from Theorem 6.1 in the special case $\Theta = \emptyset$, we have

$$\begin{aligned}
\sum_{D \in W_\Theta \backslash W} Q_{CD} \varphi(w^D) &= \sum_{D \in W_\Theta \backslash W} Q_{CD} \left(\sum_{E \in W_\Theta \backslash W} P'_{DE} \delta_E \right) \\
&= \sum_{E \in W_\Theta \backslash W} \left(\sum_{D \in W_\Theta \backslash W} Q_{CD} P'_{DE} \right) \delta_E \\
&= \delta_C.
\end{aligned}$$

In the special case $\Theta = \emptyset$, this implies

$$\sum_{v \in W} Q_{wv} \varphi(v) = \delta_w. \tag{7.7}$$

Then, because

$$\delta_C = \sum_{v \in W_\Theta} q^{\ell(v)} \delta_{vw^C},$$

[5]These polynomials differ in normalization from the Kazhdan–Lusztig polynomials appearing in [KL79]. See Remark 7.4.

we have the following relationship:

$$\begin{aligned}\sum_{D\in W_\Theta\backslash W} Q_{CD}\varphi(w^D) &= \sum_{v\in W_\Theta} q^{\ell(v)}\delta_{vw^C}\\ &= \sum_{v\in W_\Theta} q^{\ell(v)}\left(\sum_{u\in W} Q_{vw^C\ u}\varphi(u)\right)\\ &= \sum_{u\in W}\left(\sum_{w\in W_\Theta} q^{\ell(v)}Q_{vw^C\ u}\right)\varphi(u).\end{aligned}$$

Here the second equality follows from equation (7.7). Since $\{\varphi(u) : u \in W\}$ form a basis for $\mathcal{H}_\emptyset$ by Theorem 6.1, this implies that

$$Q_{CD} = \sum_{v\in W_\Theta} q^{\ell(v)}Q_{vw^C\ w^D}.$$

Thus, since $\ell(vw^C) = \ell(w^C) - \ell(v)$ by Theorem 5.12, an application of equation (7.6) for the special case $\Theta = \emptyset$ results in the following formula:

$$Q_{CD} = (-1)^{\ell(w^C)+\ell(w^D)} \sum_{v\in W_\Theta} (-1)^{\ell(v)} q^{\ell(v)} P_{w^D w_0\ vw^C w_0}.$$

The proposition then follows by combining this formula with equation (7.6) and using the fact that $w^C w_0$ is the shortest element of the coset Cw_0, so it is equal to $w_\Theta w^{Cw_0}$ by Theorem 5.6. $\square$

CHAPTER 8

CONCLUSION

The results of Section 6.1 (in particular Corollary 6.11) accomplish the main goal of this dissertation by computing the multiplicity of an irreducible Whittaker module in the composition series of a standard Whittaker module. Furthermore, in arriving to Corollary 6.11 by means of the algorithm in Theorem 6.1, we have computed these multiplicities without reference to the established Kazhdan–Lusztig algorithm for Verma modules. This method is distinct from the methods for computing multiplicities established in [MS97] and [Bac97], and it has several advantages. The primary advantage of this approach is the establishment of Whittaker Kazhdan–Lusztig polynomials. These polynomials reveal the underlying combinatorics of the category of Whittaker modules and provide a concrete combinatorial description that can be used to compare Whittaker modules to other mathematical objects. The explicit conversions between Whittaker Kazhdan–Lusztig polynomials and other types of Kazhdan–Lusztig polynomials from the combinatorics literature in Chapter 7 lay the groundwork for such future combinatorial comparisons. In addition to this, the algorithm of Theorem 6.1 is more efficient than the methods for calculating multiplicities that were established in [MS97] and [Bac97]. These methods compared Whittaker modules to singular blocks of category $\mathcal{O}$, so computing multiplicities involved performing the Kazhdan–Lusztig algorithm for Verma modules with non-singular central character, then using translation functors to determine which irreducible subquotients vanish when λ pairs singularly with certain roots. This amounts to computing the $|W| \times |W|$ (upper triangular) matrix of Kazhdan–Lusztig polynomials and deleting certain rows and columns. In contrast, the method for computing multiplicities established in this manuscript directly computes the $|W_\Theta \backslash W| \times |W_\Theta \backslash W|$ matrix of polynomials to determine multiplicities without providing any superfluous information, so its implementation is more efficient.

In addition to answering multiplicity questions in the category of Whittaker modules, another accomplishment of this project is the systematic development of the geometric category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ of η-twisted Harish-Chandra sheaves. Ever since Beilinson–Bernstein's [BB81] and Brylinski–Kashiwara's [BK81] celebrated proofs of the Kazhdan–Lusztig conjectures introduced the possibility of using algebraic geometry to address questions in representation theory, geometric descriptions of categories of $\mathfrak{g}$-modules have been recognized as a powerful tool. However, geometric descriptions of $\mathfrak{g}$-modules are often in terms of perverse sheaves instead of $\mathcal{D}$-modules. For example, in [BB81] and [BK81], the main geometric arguments were performed using categories of perverse sheaves and the $\mathcal{D}$-module description of these categories was primarily used as an intermediate tool for comparing categories of perverse sheaves to categories of $\mathfrak{g}$-modules. Because the category of $\mathcal{D}$-modules which is obtained by localizing highest weight modules (the category of "Harish-Chandra sheaves" [Milb]) consists of holonomic $\mathcal{D}$-modules with regular singularities, the Riemann–Hilbert correspondence makes it possible to convert that category of $\mathcal{D}$-modules to a category of perverse sheaves. However, the category $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ which we obtain by localizing Whittaker modules contains holonomic $\mathcal{D}$-modules with *irregular* singularities, which means that the Riemann–Hilbert correspondence does not apply. Therefore, geometric descriptions of Whittaker modules must be in terms of $\mathcal{D}$-modules instead of perverse sheaves, and this document establishes the structure and tools necessary for working within this category of $\mathcal{D}$-modules.

The results established in Chapters 1-7 of this manuscript open the door to many other questions about the structure theory of the category of Whittaker modules. In the final section of this document, we outline some natural future directions of this project.

8.1 Future Directions

This manuscript develops tools for analyzing the category $\mathcal{N}$ both algebraically and geometrically, and these tools can be used to address other questions about this category. The most obvious of these questions is computation of multiplicities in the case of singular and non-integral central character.

Question 8.1. Can Corollary 6.11 be generalized to standard Whittaker modules with arbitrary central character?

For Verma modules, the Kazhdan–Lusztig algorithm can be extended to arbitrary $\lambda \in \mathfrak{h}^*$ by reducing the algorithm to the integral Weyl group, which is constructed from the roots with which λ pairs integrally. (See, for example, [Soe90, KT00].) However, the Whittaker setting requires a subtler approach which relates the sub-Weyl group determined by η and the integral Weyl group. By using translation functors and Theorem 5.23, one can establish combinatorial conditions that dictate when the global sections of irreducible η-twisted Harish-Chandra sheaves with singular central character vanish, and this can be used to compute multiplicities for singular λ. To address the non-integral case, a parabolic set of roots can be constructed by taking the union of the root system determined by η and the positive roots. When intersected with the integral root system, this yields a smaller parabolic set of roots, which determines a subset of simple roots depending on both η and λ. I conjecture that the Kazhdan–Lusztig polynomials for non-integral Whittaker modules are the polynomials corresponding to this subset. The conditions on irreducibility of standard Whittaker modules established in [Luk04] support this conjecture.

An interesting consequence of the algorithm in Theorem 6.1 is its relationship to the Kazhdan–Lusztig algorithm for generalized Verma modules in [Milb] as described in Section 7.4. This relationship leads to the following question.

Question 8.2. Is the combinatorial duality present in the Kazhdan–Lusztig polynomials for generalized Verma modules and the Kazhdan–Lusztig polynomials for Whittaker modules a shadow of a deeper duality between these two categories?

The equivalence between blocks of $\mathcal{N}$ and singular blocks of category $\mathcal{O}$ established in [MS97] and the "parabolic-singular" duality between singular blocks of $\mathcal{O}$ and regular blocks of parabolic category $\mathcal{O}^{\mathfrak{p}}$ described in [BGS96] indicate that a duality between Whittaker modules and generalized Verma modules should exist, but this relationship has not yet been made precise in the literature. A first step in formalizing this idea is to examine the role of projective objects in $\mathcal{N}$ in an attempt to realize blocks of $\mathcal{N}$ as module categories over the endomorphism ring of some projective generator, following the approach of Soergel in [BGS96]. This introduces a Koszul ring, and we can calculate the corresponding Koszul dual ring, with the final goal of realizing blocks of parabolic category $\mathcal{O}^{\mathfrak{p}}$ as modules over the Koszul dual ring.

In addition to these questions, it is natural to ask whether other well-established results of the category of highest weight modules extend to $\mathcal{N}$. In [Jan79], Jantzen introduced a canonical filtration of Verma modules which provided a beautiful conceptual proof of BGG reciprocity for highest weight modules, and he conjectured that this filtration is compatible in a natural way with embeddings of Verma modules. This became known as the *Jantzen Conjecture*. It was discovered by Gabber and Joseph [GJ81] that this conjecture establishes detailed information about the coefficients of the Kazhdan–Lusztig polynomials, and a proof of it implies the Kazhdan–Lusztig conjectures. The standard Whittaker modules in $\mathcal{N}$ have similar structural properties to Verma modules, so it is reasonable to consider the following problem.

Question 8.3. Can one define Jantzen filtrations in $\mathcal{N}$ in order to develop and prove a conjecture analogous to the Jantzen conjecture for $\mathcal{N}$?

In [BB93], Beilinson and Bernstein provide a proof of the Jantzen conjectures using weight filtrations on the corresponding perverse sheaves. However, as described above, the irregular singularities of the $\mathcal{D}$-modules in $\mathcal{M}_{coh}(\mathcal{D}_\lambda, N, \eta)$ make methods of perverse sheaves intractable for $\mathcal{N}$. Therefore, the geometric development of Jantzen filtrations for Whittaker modules would require detailed analysis of holonomic $\mathcal{D}$-modules with irregular singularities, building on the structure established in [Moc11]. In this setting, the interactions between Hodge theory and representation theory have not yet been thoroughly studied, and there are many possibilities for future development. Furthermore, there is an alternate algebraic approach to this problem using Soergel bimodules, which is described below.

The category of Whittaker modules could also be approached using a different set of tools. Inspired by the celebrated algebraic proof of the Kazhdan-Lusztig conjectures by Elias-Williamson in [EW14], one could examine category $\mathcal{N}$ using Soergel bimodules with the goal of providing a purely algebraic proof of Corollary 6.11 that does not appeal to geometry. Indeed, Theorem 6.1 can be reformulated in terms of the anti-spherical module for the Hecke algebra associated to W by the results in Chapter 7. In [LW17], Libedinsky–Williamson use a diagrammatic category of Soergel bimodules (which they refer to as the 'anti-spherical category') to categorify the anti-spherical module, and this categorification establishes positivity of coefficients of parabolic Kazhdan–Lusztig polynomials. This leads to the following question.

Question 8.4. Can the anti-spherical category of Libedinsky-Williamson be used to provide a purely algebraic proof of Corollary 6.11?

In [RW15], Riche–Williamson relate the anti-spherical category of the affine Weyl group to representations of algebraic groups, establishing the importance of this category in modular representation theory. Therefore, an answer to Question 8.4 would also illuminate the role of Whittaker modules in modular representation theory. Additionally, this approach could present an alternate avenue to Question 8.3 by building on Williamson's recent algebraic proof of the Jantzen Conjecture for Verma modules using Soergel bimodules [Wil16].

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SUBJECT INDEX