L^∞ bounds for numerical solutions of noncoercive convection-diffusion equations

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Abstract In this work, we apply an iterative energy method *à la* de Giorgi in order to establish L^∞ bounds for numerical solutions of noncoercive convection-diffusion equations with mixed Dirichlet-Neumann boundary conditions.

Key words: finite volume schemes, uniform bounds, noncoercive elliptic equations
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1 Introduction

The continuous problem. Let Ω be an open bounded polygonal domain of $\mathbb{R}^p$ with $p = 2$ or 3. We denote by $\mathrm{m}(\cdot)$ both the Lebesgue and $p-1$ dimensional Hausdorff measure. We assume that $\partial\Omega = \Gamma^D \cup \Gamma^N$ with $\Gamma^D \cap \Gamma^N = \emptyset$ and $\mathrm{m}(\Gamma^D) > 0$ and we denote by $\mathbf{n}$ the exterior normal to $\partial\Omega$. Let $\mathbf{U} \in C(\bar{\Omega})^2$ be a velocity field, $b \in L^\infty(\Omega)$ assumed to be nonnegative, $f \in L^\infty(\Omega)$ a source term and $v^D \in L^\infty(\Gamma^D)$ a boundary condition.

We consider the following convection-diffusion equation with mixed boundary conditions:

$$\mathrm{div}(-\nabla v + \mathbf{U}v) + bv = f \qquad \text{in } \Omega, \tag{1a}$$

$$(-\nabla v + \mathbf{U}v) \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma^N, \tag{1b}$$

$$v = v^D \qquad \text{on } \Gamma^D. \tag{1c}$$

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This noncoercive elliptic linear problem has been widely studied by Droniou and coauthors, even with less regularity on the data, see for instance [2, 4, 3, 5]. Nevertheless, up to our knowledge, the derivation of explicit L^∞ bounds on numerical solutions has not been done in the literature.

The numerical scheme. The mesh of the domain Ω is denoted by $\mathscr{M} = (\mathscr{T}, \mathscr{E}, \mathscr{P})$ and classically given by: $\mathscr{T}$, a set of open polygonal or polyhedral control volumes; $\mathscr{E}$, a set of edges or faces; $\mathscr{P} = (x_K)_{K\in\mathscr{T}}$ a set of points. In the following, we also use the denomination "edge" for a face in dimension 3. As we deal with a Two-Point Flux Approximation (TPFA) of convection-diffusion equations, we assume that the mesh is admissible in the sense of [6] (Definition 9.1).

We distinguish in $\mathscr{E}$ the interior edges, $\sigma = K|L$, from the exterior edges: $\mathscr{E} = \mathscr{E}_{int} \cup \mathscr{E}_{ext}$. Among the exterior edges, we distinguish the edges included in Γ^D from the edges included in Γ^N: $\mathscr{E}_{ext} = \mathscr{E}^D \cup \mathscr{E}^N$. For a given control volume $K \in \mathscr{T}$, we define $\mathscr{E}_K$ the set of its edges, which is also split into $\mathscr{E}_K = \mathscr{E}_{K,int} \cup \mathscr{E}_K^D \cup \mathscr{E}_K^N$. For each edge $\sigma \in \mathscr{E}$, we pick one cell in the non empty set $\{K : \sigma \in \mathscr{E}_K\}$ and denote it by K_σ. In the case of an interior edge $\sigma = K|L$, K_σ is either K or L.

Let $\mathrm{d}(\cdot,\cdot)$ denote the Euclidean distance. For all edges $\sigma \in \mathscr{E}$, we set $\mathrm{d}_\sigma = \mathrm{d}(x_K, x_L)$ if $\sigma = K|L \in \mathscr{E}_{int}$ and $\mathrm{d}_\sigma = \mathrm{d}(x_K, \sigma)$ if $\sigma \in \mathscr{E}_{ext}$ with $\sigma \in \mathscr{E}_K$ and the transmissibility coefficient is defined by $\tau_\sigma = \mathrm{m}(\sigma)/\mathrm{d}_\sigma$, for all $\sigma \in \mathscr{E}$. We also denote by $\mathbf{n}_{K,\sigma}$ the normal to $\sigma \in \mathscr{E}_K$ outward K. We assume that the mesh satisfies the regularity constraint:

$$\exists \xi > 0 \text{ such that } \mathrm{d}(x_K, \sigma) \geq \xi\, \mathrm{d}_\sigma, \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K. \tag{2}$$

As a consequence, we obtain that

$$\sum_{\sigma\in\mathscr{E}_K} \mathrm{m}(\sigma)\mathrm{d}_\sigma \leq \frac{p}{\xi}\mathrm{m}(K) \quad \forall K \in \mathscr{T}. \tag{3}$$

The size of the mesh is defined by $h = \max\{\mathrm{diam}\,(K) : K \in \mathscr{T}\}$.

Let us define

$$\begin{aligned}
f_K &= \frac{1}{\mathrm{m}(K)}\int_K f, \quad b_K = \frac{1}{\mathrm{m}(K)}\int_K b \quad \forall K \in \mathscr{T},\\
U_{K,\sigma} &= \frac{1}{\mathrm{m}(\sigma)}\int_\sigma \mathbf{U}\cdot\mathbf{n}_{K,\sigma}, \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K,\\
v_\sigma^D &= \frac{1}{\mathrm{m}(\sigma)}\int_\sigma v^D, \quad \forall \sigma \in \mathscr{E}^D.
\end{aligned}$$

Given a Lipschitz-continuous function on $\mathbb{R}$ which satisfies

$$B(0) = 1, \quad B(s) > 0 \quad \text{and} \quad B(s) - B(-s) = -s \quad \forall s \in \mathbb{R}, \tag{4}$$

we consider the B-scheme defined by

$$\sum_{\sigma\in\mathscr{E}_K} \mathscr{F}_{K,\sigma} + \mathrm{m}(K) b_K v_K = \mathrm{m}(K) f_K, \quad \forall K \in \mathscr{T}, \tag{5}$$

where the numerical fluxes are defined by

$$\mathscr{F}_{K,\sigma} = \begin{cases} 0, \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K^N, \\ \tau_\sigma \Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) v_K - B(U_{K,\sigma}\mathrm{d}_\sigma) v_{K,\sigma} \Big), \quad \forall K \in \mathscr{T}, \forall \sigma \in \mathscr{E}_K \setminus \mathscr{E}_K^N, \end{cases} \tag{6}$$

with the convention $v_{K,\sigma} = v_L$ if $\sigma = K|L$ and $v_{K,\sigma} = v_\sigma^D$ if $\sigma \in \mathscr{E}_K^D$. Let us recall that the upwind scheme corresponds to the case $B(s) = 1 + s^-$ (s^- is the negative part of s, while s^+ is its positive part) and the Scharfetter-Gummel scheme to the case $B(s) = s/(e^s - 1)$. They both satisfy (4). The centered scheme which corresponds to $B(s) = 1 - s/2$ does not satisfy the positivity assumption. It can however be used if $|U_{K,\sigma}|\mathrm{d}_\sigma \leq 2$ for all $K \in \mathscr{T}$ and $\sigma \in \mathscr{E}_K$. Thanks to the hypotheses (4), we notice that the numerical fluxes through the interior and Dirichlet boundary edges rewrite

$$\mathscr{F}_{K,\sigma} = \tau_\sigma B(|U_{K,\sigma}|\mathrm{d}_\sigma)(v_K - v_{K,\sigma}) + \mathrm{m}(\sigma) \left(U_{K,\sigma}^+ v_K - U_{K,\sigma}^- v_{K,\sigma} \right). \tag{7}$$

Main result. The scheme (5)-(6) defines a linear system of equations $\mathbb{M}\mathbf{v} = \mathbf{S}$ whose unknown is $\mathbf{v} = (v_K)_{K\in\mathscr{T}}$; It is well-known that $\mathbb{M}$ is an M-matrix, which ensures existence and uniqueness of a solution to the scheme. Moreover, we may notice that, if v^D and f are nonnegative functions, then $\mathbf{S}$ has nonnegative values and therefore $v_K \geq 0$ for all $K \in \mathscr{T}$. Our purpose is now to establish L^∞ bounds on $\mathbf{v}$ as stated in Theorem 1.

Theorem 1. *Assume that* $\mathbf{U} \in C(\bar{\Omega})^2$*,* $b \in L^\infty(\Omega)$ *with* $b \geq 0$ a.e.*,* $f \in L^\infty(\Omega)$ *and* $v^D \in L^\infty(\Gamma^D)$*. There exists non-negative constants* $\overline{M}$ *(*resp. $\underline{M}$*) depending only on* Ω*,* ξ*, the function* B*,* $\|\mathbf{U}\|_{L^\infty}$*,* $\|f^+\|_{L^\infty}$ *and* $\|(v^D)^+\|_{L^\infty}$ *(*resp. $\|f^-\|_{L^\infty}$ *and* $\|(v^D)^-\|_{L^\infty}$*) such that the solution* $\mathbf{v}$ *to the scheme* (5)-(6) *verifies*

$$-\underline{M} \;\leq\; v_K \;\leq\; \overline{M}, \quad \forall K \in \mathscr{T}.$$

The rest of this paper is dedicated to the proof of Theorem 1. It relies on a De Giorgi iteration method (see [7] and references therein). In Section 2, we start by studying a particular case where the data is normalized. Then, we give the proof of the theorem in Section 3.

Let us mention that from the bounds of Theorem 1, it is possible to establish global-in-time L^∞ bounds for the corresponding evolution equation by using an entropy method (see [1, Theorem 2.7]).

2 Study of a particular case

In this section, we consider the particular case where the source f is non-negative and the boundary condition v^D is non-negative and bounded by 1.

Let us start with some notations. Given $m \geq 1$, we denote the m-th truncation threshold by

$$C_m = 2(1-2^{-m}), \tag{8}$$

Then, we introduce the m-th energy

$$E_m(\mathbf{v}) = \sum_{\sigma\in\mathscr{E}_{int}\cup\mathscr{E}^D} \tau_\sigma \left[\log(1+(v_{K,\sigma}-C_m)^+) - \log(1+(v_K-C_m)^+)\right]^2. \tag{9}$$

When there is no ambiguity we write $E_m = E_m(\mathbf{v})$. The first proposition is a fundamental estimate of the energy.

Proposition 1. *Assume that $f_K \geq 0$ for all $K \in \mathscr{T}$ and $v_\sigma^D \in [0,1]$ for all $\sigma \in \mathscr{E}^D$, so that the solution $\mathbf{v}$ to* (5)-(6) *satisfies $v_K \geq 0$ for all $K \in \mathscr{T}$. Then one has for all $m \geq 1$ that*

$$E_m \;\leq\; \frac{4p}{\beta_{\mathbf{U}}^2}\left(\|\mathbf{U}\|_{L^\infty}^2 + \|f\|_{L^\infty}\right) \sum_{\substack{K\in\mathscr{T}\\ v_K>C_m}} \mathrm{m}(K). \tag{10}$$

where $\beta_{\mathbf{U}} := \inf_{x\in[-\|\mathbf{U}\|_{L^\infty},\|\mathbf{U}\|_{L^\infty}]} B(\mathrm{diam}(\Omega)\,x)$ (because of (4)*, $\beta_{\mathbf{U}} \in (0,1]$).*

Proof. In order to shorten some expressions hereafter, let us introduce $w_K^m = v_K - C_m$ for all $K \in \mathscr{T}$ and $w_\sigma^{m,D} = v_\sigma^D - C_m$ for all $\sigma \in \mathscr{E}^D$. Let us note that we identify $\mathbf{w}^m = (w_K^m)_{K\in\mathscr{T}}$ and the associate piecewise constant function. Therefore, we can write

$$\mathrm{m}(\{\mathbf{w}^m > 0\}) = \sum_{w_K^m>0} \mathrm{m}(K).$$

First, observe that E_m is the discrete counterpart of

$$\int_\Omega |\nabla \log(1+w^m)|^2 \mathbf{1}_{\{w^m>0\}} = \int_\Omega \nabla w^m \cdot \frac{\nabla w^m}{(1+w^m)^2} \mathbf{1}_{\{w^m>0\}}, \;\text{ with } w^m = v - C_m\,,$$

where $\mathbf{1}_A$ is the indicator function of A. Let us define $\varphi : s \mapsto s/(1+s)\mathbf{1}_{\{s\geq 0\}}$, which satisfies $\varphi'(s) = 1/(1+s)^2\mathbf{1}_{\{s\geq 0\}}$ and let us introduce F_m another discrete counterpart of the preceding quantity

$$F_m = \sum_{\sigma\in\mathscr{E}_{int}\cup\mathscr{E}^D} \tau_\sigma \left((w_{K,\sigma}^m)^+ - (w_K^m)^+\right)\left(\varphi(w_{K,\sigma}^m) - \varphi(w_K^m)\right).$$

It is clear that $E_m \leq F_m$ for all $m \geq 1$, as for all $x, y \in \mathbb{R}$ we have

$$\left(\log(1+x^+) - \log(1+y^+)\right)^2 \leq (x^+ - y^+)\,(\varphi(x) - \varphi(y))\,.$$

Let us now multiply the scheme (5) by $\varphi(w_K^m)$ and sum over $K \in \mathscr{T}$. Due to the non-negativity of b and $\mathbf{v}$, we obtain, after a discrete integration by parts,

$$\sum_{\sigma\in\mathscr{E}_{int}\cup\mathscr{E}^D} \mathscr{F}_{K,\sigma}(\varphi(w_K^m) - \varphi(w_{K,\sigma}^m)) \leq \sum_{K\in\mathscr{T}} \mathrm{m}(K) f_K \varphi(w_K^m).$$

Using that φ is bounded by 1 and vanishes on $\mathbb{R}_-$, we deduce that

$$\sum_{\sigma\in\mathcal{E}_{int}\cup\mathcal{E}^D} \mathcal{F}_{K,\sigma}(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m)) \le \|f\|_{L^\infty}\,\mathrm{m}(\{\mathbf{w}^m>0\}). \tag{11}$$

We focus now on the left-hand-side of (11). Due to (7) and the definition of w_K^m, we can rewrite $\mathcal{F}_{K,\sigma}$ as

$$\mathcal{F}_{K,\sigma} = \tau_\sigma B(|U_{K,\sigma}|\mathrm{d}_\sigma)(w_K^m - w_{K,\sigma}^m) + \mathrm{m}(\sigma)\left(U_{K,\sigma}^+(w_K^m+C_m) - U_{K,\sigma}^-(w_{K,\sigma}^m+C_m)\right).$$

Observe that since φ is a non-decreasing function, one has

$$(x-y)\,(\varphi(x)-\varphi(y)) \ge (x^+-y^+)(\varphi(x)-\varphi(y)), \quad \forall x,y\in\mathbb{R}.$$

Therefore, using the definition of $\beta_{\mathbf{U}}$ we obtain that

$$\sum_{\sigma\in\mathcal{E}_{int}\cup\mathcal{E}^D} \mathcal{F}_{K,\sigma}(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m)) \ge \beta_{\mathbf{U}}F_m - G_m, \tag{12}$$

with

$$G_m = -\sum_{\sigma\in\mathcal{E}_{int}\cup\mathcal{E}^D} \mathrm{m}(\sigma)\left(U_{K,\sigma}^+(w_K^m+C_m) - U_{K,\sigma}^-(w_{K,\sigma}^m+C_m)\right)(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m)).$$

For an interior edge, w_K^m and $w_{K,\sigma}^m$ play a symmetric role in the preceding sum. As $w_\sigma^{m,D}\le 0$ for all $\sigma\in\mathcal{E}^D$ and φ vanishes on $\mathbb{R}_-$, we can always assume that $w_K^m \ge w_{K,\sigma}^m$ and an edge has a contribution in the sum if at least $w_K^m>0$. Then, under these assumptions one has

$$\begin{aligned}-\mathrm{m}(\sigma)\left(U_{K,\sigma}^+(w_K^m+C_m) - U_{K,\sigma}^-(w_{K,\sigma}^m+C_m)\right)(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m))\\ \le \|\mathbf{U}\|_{L^\infty}\mathrm{m}(\sigma)(w_{K,\sigma}^m+C_m)(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m)).\end{aligned}$$

But, $w_{K,\sigma}^m+C_m \le 2(1+(w_{K,\sigma}^m)^+)$ and applying the definition of φ, we get

$$\begin{aligned}(w_{K,\sigma}^m+C_m)(\varphi(w_K^m)-\varphi(w_{K,\sigma}^m)) &\le 2\frac{(w_K^m)^+-(w_{K,\sigma}^m)^+}{1+(w_K^m)^+}\\ &\le 2\frac{(w_K^m)^+-(w_{K,\sigma}^m)^+}{\sqrt{1+(w_K^m)^+}\sqrt{1+(w_{K,\sigma}^m)^+}}.\end{aligned}$$

Therefore,

$$G_m \le 2\|\mathbf{U}\|_{L^\infty}\sum_{\sigma\in\mathcal{E}_{int}\cup\mathcal{E}^D} \mathrm{m}(\sigma)\frac{|(w_K^m)^+-(w_{K,\sigma}^m)^+|}{\sqrt{1+(w_K^m)^+}\sqrt{1+(w_{K,\sigma}^m)^+}}.$$

We apply now Cauchy-Schwarz inequality in order to get

$$G_m \leq 2\|\mathbf{U}\|_{L^\infty}(F_m)^{1/2}\left(\sum_{\sigma\in\mathscr{E}^{sp}} \mathrm{m}(\sigma)\mathrm{d}_\sigma\right)^{1/2}, \tag{13}$$

where $\mathscr{E}^{sp}$ is the set of interior and Dirichlet boundary edges on which $(w_K^m)^+ - (w_{K,\sigma}^m)^+ \neq 0$. It appears that, due to (3),

$$\sum_{\sigma\in\mathscr{E}^{sp}} \mathrm{m}(\sigma)\mathrm{d}_\sigma \leq \sum_{K\in\mathscr{T};w_K^m>0}\left(\sum_{\sigma\in\mathscr{E}_{K,int}\cup\mathscr{E}_K^D} \mathrm{m}(\sigma)\mathrm{d}_\sigma\right) \leq \frac{p}{\xi}\mathrm{m}(\{\mathbf{w}^m>0\}). \tag{14}$$

We deduce from (11), (12), (13) and (14) that

$$\beta_{\mathbf{U}}F_m \leq 2\|\mathbf{U}\|_{L^\infty}(F_m)^{1/2}(\frac{p}{\xi}\mathrm{m}(\{\mathbf{w}^m>0\}))^{1/2} + \|f\|_{L^\infty}\mathrm{m}(\{\mathbf{w}^m>0\}),$$

which yields (10) using Young's inequality and the bounds $E_m \leq F_m$ and $\beta_{\mathbf{U}} \leq 1$.

Before stating the main result of the section, we need a technical lemma.

Lemma 1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real numbers and let $K,\rho>0$ and $\alpha>1$. Then if for all $n\in\mathbb{N}$*

$$u_{n+1} \leq K\rho^n u_n^\alpha,$$

one has

$$0 \leq u_n \leq \left(u_0\,\rho^{\frac{1}{(\alpha-1)^2}}K^{\frac{1}{\alpha-1}}\right)^{\alpha^n}\rho^{-\frac{n(\alpha-1)+1}{(\alpha-1)^2}}K^{-\frac{1}{\alpha-1}}$$

for all $n\in\mathbb{N}$ and the bound is optimal. In particular, if $u_0 \leq \rho^{-\frac{1}{(\alpha-1)^2}}K^{-\frac{1}{\alpha-1}}$, then $\lim u_n = 0$.

Proof. Just observe that the sequence $v_n = u_n\,\rho^{\frac{n(\alpha-1)+1}{(\alpha-1)^2}}K^{\frac{1}{\alpha-1}}$ satisfies $0 \leq v_{n+1} \leq v_n^\alpha$ for all $n \geq 0$ which directly yields the result.

Proposition 2. *Assume that $f_K \geq 0$ for all $K\in\mathscr{T}$ and $v_\sigma^D \in [0,1]$ for all $\sigma\in\mathscr{E}^D$, so that $v_K \geq 0$ for all $K\in\mathscr{T}$. Then, there exists $\eta>0$ depending only on Ω, p and ξ such that one has the implication*

$$E_1 \leq \eta\,\frac{\beta_{\mathbf{U}}^4}{(\|\mathbf{U}\|_{L^\infty}^2+\|f\|_{L^\infty})^2} \quad\Rightarrow\quad (v_K \leq 2,\ \forall K\in\mathscr{T}). \tag{15}$$

Proof. The proof consists in establishing an induction property on E_m which guarantees that if E_1 is small enough then $\lim E_m = 0$. Then, as $\lim C_m = 2$ and thanks to the discrete Poincaré inequality, we deduce that

$$\sum_{K\in\mathscr{T}} \mathrm{m}(K)\left(\log(1+(v_K-2)^+)\right)^2 = 0,$$

which implies $v_K \le 2$ for all $K \in \mathscr{T}$.

For establishing the induction, first observe that as $C_m = C_{m-1} + 2^{-m+1}$, for any $q > 0$ we have:

$$\mathbf{1}_{\{\mathbf{w}^m > 0\}} \le \frac{\left(\log(1+(\mathbf{w}^{m-1})^+)\right)^q}{(\log(1+2^{-m+1}))^q} \mathbf{1}_{\{\mathbf{w}^{m-1}>0\}}, \tag{16}$$

and thus

$$\mathrm{m}(\{\mathbf{w}^m > 0\}) \le \frac{1}{(\log(1+2^{-m+1}))^q} \sum_{K \in \mathscr{T}} \mathrm{m}(K) \left(\log(1+(w_K^{m-1})^+)\right)^q.$$

We may choose for instance $q = 3$ and apply a discrete Poincaré-Sobolev inequality (whose constant $C_{\Omega,p}$ depends only on Ω and p), which leads to

$$\mathrm{m}(\{\mathbf{w}^m > 0\}) \le \frac{1}{(\log(1+2^{-m+1}))^3} \frac{C(\Omega)}{\xi^{3/2}} E_{m-1}^{3/2}. \tag{17}$$

Noticing that for $x \in [0,1]$, $(\log(1+x))^3 \ge (\log 2)^3 x^3$, we deduce from (10) and (17) that

$$E_m \le \frac{4}{\beta_{\mathbf{U}}^2} \left(\|\mathbf{U}\|_{L^\infty}^2 + \|f\|_{L^\infty}\right) \frac{\tilde{C}_{\Omega,p}}{\xi^{3/2}} 8^{m-1} E_{m-1}^{3/2}.$$

Thus the sequence $(E_m)_{m\ge 0}$ satisfies the hypothesis of Lemma 1 with $\alpha = 3/2$ and K proportional to $(\|\mathbf{U}\|_{L^\infty}^2 + \|f\|_{L^\infty})/\beta_{\mathbf{U}}^2$. We deduce the upper bound for E_1 under which $\lim E_m = 0$.

Remark: The arguments developed in this section still hold, up to minor adaptation, for $f \in L^r(\Omega)$ with $r > p/2$.

3 Proof of Theorem 1

First observe that if one replaces the data f and v^D by either f^+ and $(v^D)^+$, or f^- and $(v^D)^-$, in the scheme (5)-(6), then the corresponding solutions, say respectively $\mathbf{P} = (P_K)_{K\in\mathscr{T}}$ and $\mathbf{N} = (N_K)_{K\in\mathscr{T}}$, are non-negative and such that $\mathbf{v} = \mathbf{P} - \mathbf{N}$ is the solution to (5)-(6) in the original framework.

From there let us show that there is $\overline{M} > V_+^D := \max(\|(v^D)^+\|_{L^\infty}, 1)$ such that for all $K \in \mathscr{T}$ one has $0 \le P_K \le \overline{M}$. The bound for $\mathbf{N}$, which is denoted by $\underline{M}$, can be obtained in the same way.

Let $M > V_+^D$. First observe that $\mathbf{P}^M := \mathbf{P}/M$ satisfies the scheme (5)-(6) where the source term and boundary data have been replaced by f^+/M and $(v^D)^+/M$ respectively. Moreover, one can apply Proposition 1, which yields

$$E_1(\mathbf{P}^M) \le \frac{4p}{\beta_{\mathbf{U}}^2} \left(\|\mathbf{U}\|_{L^\infty}^2 + \frac{\|f^+\|_{L^\infty}}{M}\right) \mathrm{m}(\{\mathbf{P}^M > 1\}). \tag{18}$$

Now observe that $\mathbf{P} = M\,\mathbf{P}^M = V_+^D\,\mathbf{P}^{V_+^D}$. Therefore,

$$\begin{aligned}
E_1(\mathbf{P}^M) &\leq \frac{4p}{\beta_{\mathbf{U}}^2}\Big(\|\mathbf{U}\|_{L^\infty}^2\,\mathrm{m}(\{\mathbf{P}^{V_+^D} > M/V_+^D\}) + \frac{\|f^+\|_{L^\infty}}{M}\mathrm{m}(\Omega)\Big)\\
&\leq \frac{4p}{\beta_{\mathbf{U}}^2}\Big(\|\mathbf{U}\|_{L^\infty}^2 \sum_{K\in\mathscr{T}} \mathrm{m}(K)\,\frac{\log(1+(P_K^{V_+^D}-1)^+)^2}{\log(M/V_+^D)^2} + \frac{\|f^+\|_{L^\infty}}{M}\mathrm{m}(\Omega)\Big)\\
&\leq \frac{C_{\Omega,p}}{\xi\beta_{\mathbf{U}}^2}\|\mathbf{U}\|_{L^\infty}^2\,\frac{E_1(\mathbf{P}^{V_+^D})}{\log(M/V_+^D)^2} + \frac{4p\,\mathrm{m}(\Omega)}{\beta_{\mathbf{U}}^2}\,\frac{\|f^+\|_{L^\infty}}{M}\ ,
\end{aligned}$$

where we used an argument similar to (16) in the second inequality and a discrete Poincaré inequality in the third one. Then, by using (18) again we get

$$E_1(\mathbf{P}^{V_+^D}) \;\leq\; \frac{4\,p\,\mathrm{m}(\Omega)}{\beta_{\mathbf{U}}^2}\left(\|\mathbf{U}\|_{L^\infty}^2 + \frac{\|f^+\|_{L^\infty}}{V_+^D}\right)$$

Therefore, the smallness condition of Proposition 2 is satisfied by $E_1(\mathbf{P}^M)$ if

$$\left[\|\mathbf{U}\|_{L^\infty}^2\left(\|\mathbf{U}\|_{L^\infty}^2 + \frac{\|f^+\|_{L^\infty}}{V_+^D}\right) + \frac{\|f^+\|_{L^\infty}}{M}\log\left(\frac{M}{V_+^D}\right)^2\right]\left(\|\mathbf{U}\|_{L^\infty}^2 + \frac{\|f^+\|_{L^\infty}}{M}\right)^2 \leq\; C_{\Omega,\xi,p}\,\beta_{\mathbf{U}}^4\log\left(\frac{M}{V_+^D}\right)^2. \quad (19)$$

It is clear that (19) is satisfied for M large enough, which permits to define $\overline{M}$. Observe that if $v_+^D = 0$ $(V+^D = 1)$ and $\mathbf{U} = 0$, $\overline{M} = \widetilde{C}_{\Omega,\xi,p}\|f^+\|_{L^\infty}$ works as expected.

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