Gossiping with interference in radio ring networks

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Abstract

In this paper, we study the problem of gossiping with interference constraint in radio networks. Gossiping (or total exchange information) is a protocol where each node in the network has a message and is expected to distribute its own message to every other node in the network. We focus on the case where the transmission network is a ring network that is a node has 2 neighbors and can only transmit to its neighbors (such an action is named a call). We consider synchronous protocols where it takes one unit of time (step) to transmit a unit-length message. During one step we suppose that a node cannot send and receive (half duplex model). Moreover communication is subject to interference constraints. We model them by a fixed integer $d_I \geq 1$, which implies that nodes within distance d_I from a sender in the network cannot receive messages from another node. Here we focus on the case $d_I = 1$, which implies that if a node receives a message from one of its neighbors, its other neighbor cannot send at the same time. A round consists of a set of non-interfering (or compatible) calls and uses one step. We solve completely the problem for ring networks, with unit length messages and $d_I = 1$ by giving the minimum running time (makespan) of a gossiping protocol that is the minimum number of rounds needed to complete the gossiping. We first show lower bounds and then give gossiping algorithms which meet these lower bounds and so are optimal.

Keywords: Gossiping, Radio Networks, Interference, Rings

1 Introduction

Transmission model A radio network consists of communication devices each of which can act at a given time either as a sender or as a receiver but not both. We model the radio network as a symmetric digraph $G = (V, E)$, where V is the set of nodes and E is the set

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of possible communications. There is an arc from u to v if u can transmit a message to v. We suppose that the digraph is symmetric; so if u can transmit a message to v, then v can also transmit a message to u. Some authors use an undirected graph (replacing the two arcs (u, v) and (v, u) by an edge $\{u, v\}$). However *calls* (transmissions) are directed: a call (s, r) is defined as the transmission from the node s to the node r, in which s is the *sender* and r is the *receiver* and s and r are adjacent in G. The distinction of sender and receiver will be important for our interference model.

The network is assumed to be synchronous and the time is slotted into *steps*. We suppose that each device is equipped with an half duplex interface: a node cannot both receive and transmit during a step. This models the near-far effect of antennas: when one is transmitting, its own power prevents any other signal to be properly received.

One important feature of our model is the assumption that a node can transmit or receive at most one message per step (in particular we do not allow concatenation of messages).

Interference model We use a binary asymmetric model of interference based on the distance in the communication digraph like in [1, 3, 8]. Let $d(u, v)$ denote the distance, that is the length of a shortest directed path, from u to v in G and d_I be an non negative integer. We assume that when a node u transmits, all nodes v such that $d(u, v) \leq d_I$ are subject to the interference from u's transmission. We assume that all nodes of G have the same interference range d_I. Two calls (s, r) and (s', r') do not interfere if and only if $d(s, r') > d_I$ and $d(s', r) > d_I$. Otherwise calls interfere (or there is a collision). During a given step only non interfering (or compatible) calls can be done and we will define a round as a set of such compatible calls. We will focus on the cases when $d_I = 1$.

The binary interference model is a simplified version of the reality, where the Signal-to-Noise-and-Interferences Ratio (the ratio of the received power from the source of the transmission to the sum of the noise and the received powers of all other simultaneously transmitting nodes) has to be above a given threshold for a transmission to be successful. However, the values of the completion times that we obtain in the above problem will lead to lower bounds on the corresponding real life values.

Gossiping Many problems related to information dissemination have been considered in the literature (see the survey in [5]). The most studied is broadcasting (One To All communication) where a distinguished node (source) has to distribute its information (message) to all the other nodes of the network. A variant where the distinguished node has a different message to send to all other nodes is know as personalized broadcasting and has been studied in particular for sensor networks on the reverse problem of data gathering.

Another important problem less studied is gossiping (also called All To All communication or total exchange) where each node in the network has a message and is expected to distribute its own message to every other node in the network. The running time or completion time of a gossiping protocol is the number of rounds (steps) needed to achieve the gossiping. The gossiping problem consists in finding the minimum running time (makespan) of a gossiping protocol and efficient algorithms that attain this makespan. It has been mainly studied in both full duplex or half duplex model (i.e. without interferences) and with unbounded size of messages.

Limited size of the messages was considered in [2] where results are obtained in particular for the full duplex model (which can be viewed as the special case $d_I = 0$)). In particular in a ring network with n nodes the minimum gossiping time is $n-1$ if n is even or n if n is odd (in full duplex model). A survey for gossiping in the case $d_I = 1$ has been done in [6] but most of the results concern unbounded size of messages where concatenation is allowed. The case with both $d_I = 1$ and small messages (unit length ones) has been studied in [7]. The authors established that the makespans of gossiping protocols in path and ring networks with n nodes are $3n + \Theta(1)$ and $2n + \Theta(1)$ respectively. They proved for general graphs an upper bound of $0(nlog^2n)$. This bound was improved in [9] to $0(nlogn)$ with the help of probabilistic argument.

In this article we determine exactly the minimum gossiping time in the case of a ring network C_n on n nodes with unit length messages (no concatenation) when the interference distance is $d_I = 1$ (expressed in [7] as follows: " a node can successfully receive a message if and only if exactly one of its neignbors transmits during the same round"). Our initial goal was to get exact results for rings and any d_I, but already the determination of the exact minimum gossiping time when $d_I = 1$ was not easy and this shows the difficulty of the gossiping problem in such a setting. Our results depend on the congruence of n modulo 12.

Theorem 1 *The minimum number of rounds R needed to achieve a gossiping in a ring network C_n $(n \geq 3)$, with the interference model $d_I = 1$ is :*

$$\begin{cases} 2n-3 & \textit{if } n \equiv 0 \pmod{12} \\ 2n-2 & \textit{if } n \equiv 4, 8 \pmod{12} \\ 2n-1 & \textit{if } n \textit{ is odd, except when } n = 3 \textit{ for which } R = 3 \textit{ and } n = 5 \textit{ for which } R = 10 \\ 2n+1 & \textit{if } n \equiv 2, 6, 10 \pmod{12} \textit{ except for } n = 6 \textit{ for which } R = 12 \end{cases}$$

2 Notations

Let a ring network C_n be a cycle of length n with the node set $Z_n : 0, 1, ..., n-1$. With the model above a node receives a message only through one of its two neighbors. We will consider only useful (valid) calls where the sender sends a message to a receiver only if it is unknown to the receiver. In the communication model presented in this paper, a message is transmitted by using two types of sendings :

(a) via a (regular) call $(i, i+1)$ (resp. $(i, i-1)$), where the node i sends to the node $i+1$ (resp. $i-1$) one message which is not known to the node $i+1$ (resp. $i-1$).

(b) via what we name a 2-call $(i : i-i, i+1)$, where the node i sends at the same time to both nodes $i-1$ and $i+1$ one message (the same one) which is not known to both nodes.

A 2-call consists of 2 calls with a same sender i and two receivers $i+1$ and $i-1$. Remark that in each sending of either type, only one message can be sent.

Recall that two calls interfere if the receiver of one call is at the distance less than equal to d_I (a natural number) of a sender of another call. In this paper, we focus on the case $d_I = 1$. So in this case, for example, the calls $(i, i+1)$ and $(i-2, i-1)$ will interfere, and as well as the calls $(i : i-1, i+1)$ and $(i+2, i+3)$.

3 Lower bounds

Alltogether we need $n(n-1)/2$ calls to transmit all the messages; therefore to minimize the number of rounds we have to use rounds with the maximum number of calls. However the number of calls in a round depends on the number of 2-calls it contains. We also have the constraint that each node can be involved in at most one useful 2-call and so the number of 2-calls in any protocol is bounded by n.

Remark In [6] the authors gave a general lower bound, of $2n-2$ which is not correct as when $n = 12p$ we will give a gossiping protocol wth makespan $2n-3$. Indeed, they claimed that in a round three consecutive nodes cannot simultaneously receive a message. This is correct, but they deduce from that fact that the maximum number of receivers in a round is $n/2$ which is exact only if there are no 2-calls. For example, for $n = 6$, we can do the 2-calls $(1 : 0, 2)$ and $(4 : 3, 5)$ and so we have four receivers and not 3. Note that their model (which is the same as ours) includes explicitely as written in their article the fact that "a message transmitted by a node reaches all its neighbors in the same time step".

We first determine the total number of calls a round can contain knowing the number of 2-calls it contains. Let $f_\alpha(n)$ be the maximum number of calls in a round with α 2-calls. Note that $\alpha \leq \lfloor \frac{n}{3} \rfloor$. Let x_α be the number of rounds with α 2-calls.

Lemma 1 *For $n \geq 3$, $f_\alpha(n) = 2\lfloor \frac{n+\alpha}{4} \rfloor + \epsilon_\alpha(n)$, where $\epsilon_\alpha(n) = 1$ if $n + \alpha \equiv 3 \pmod 4$ and 0 otherwise.*

Proof. The proof is by induction on the number of nodes $n \geq 3$ and then on α. We first deal with the small cases. We compute the values of $f_\alpha(n)$ for $\alpha = 0$ (resp. $1, 2, 3, 4$) and $3 \leq n \leq 6$ (resp. $3 \leq n \leq 6$, $6 \leq n \leq 9$, $9 \leq n \leq 12$, $12 \leq n \leq 15$). The results are summarized in Table 1 and one can see that the values obtained satisfy the formula in the lemma. One has to be careful that some calls are impossible due to interferences. Indeed at first glance one may think that if we have α 2-calls, we can put the maximum number of single calls between the remaining $n - 3\alpha$ nodes that is $\lfloor \frac{n-3\alpha}{2} \rfloor$. But that is not necessarily true; for example, if $\alpha = 0$ and $n = 6$, then one could expect to have 3 single calls; however we can suppose w.l.o.g. that the first call is $(0, 1)$; then the second call should be $(3, 2)$, but then 0 prevents 5 to be a receiver and 3 prevents 4 to be a receiver and so we cannot have a third call between 4 and 5. Similarly, let $\alpha = 0$ and $n = 5$: here again we could expect to have one 2-call and one single call; however, we can suppose w.l.o.g. that the 2-call is $(1 : 0, 2)$, but here 3 and 4 cannot be sender otherwise there will be interference with 2 and 0 respectively.

Now we use induction. Consider a solution for the cycle C_n with α 2-calls and $f_\alpha(n)$ calls in total. W.l.o.g., by relabeling if needed the nodes or reversing the orientation of the cycle, we can suppose that there is the call $(n-2, n-1)$. This implies that node 0 cannot be a sender, as otherwise there will be an interference. Then we can add the four nodes $n, n+1, n+2, n+3$ and the two calls: $(n+1, n), (n+2, n+3)$ which form a valid round. We have $f_\alpha(n+4) = f_\alpha(n) + 2$. Similarly, we can add the 12 nodes $n, n+1, \ldots, n+11$ and the four 2-calls, $(n+3j+1 : n+3j, n+3j+2)$, $j = 0, 1, 2, 3$. Doing so, we get a valid round and $f_{\alpha+4}(n+12) = f_\alpha(n) + 8$ (we have $\alpha \leq \lfloor \frac{n}{3} \rfloor$ and $n \geq 3$). Now starting with the small values computed above, as for $0 \leq \alpha \leq 4$ we have the first four possible values of n, we get by induction for $0 \leq \alpha \leq 4$ the values of $f_\alpha(n)$ for all n. Then we can obtain the value for $\alpha + 4$ and $n + 12$ from the values for α and n. Note that for a given α, $n \geq 3\alpha$ implies that for $\alpha + 4$, $n + 12 \geq 3(\alpha + 4)$. ⋄

Now we will be able to determine the lower bounds. For that purpose we first state three simple but useful relations:

$$\sum_\alpha x_\alpha = R \tag{1}$$

α \ n	3	4	5	6	7	8	9	10	11	12	13	14	15
0	1	2	2	2									
1	2	2	2	3									
2				4	4	4	5						
3							6	6	6	7			
4										8	8	8	9

Table 1: Computation of $f_\alpha(n)$ for small values

$$\sum_\alpha \alpha x_\alpha \leq n \tag{2}$$

$$\sum_\alpha f_\alpha(n) x_\alpha \geq n(n-1) \tag{3}$$

Equation (1) is by definition; (2) follows from the fact that each node can act as a sender in at most one 2-call and so the number of 2-calls $\sum_\alpha \alpha x_\alpha$ is at most n; (3) follows from the fact that in each round with α 2-calls, there are at most $f_\alpha(n)$ calls and so the number of calls in the protocol is at most $\sum_\alpha f_\alpha(n)x_\alpha$, but it should be $n(n-1)$.

Remark Note that equality in (3) implies that each round has exactly the maximum number $f_\alpha(n)$ of calls.

Theorem 2 *For $n \geq 3$, the lower bound of the minimum number of rounds R is:*

$$R \geq \begin{cases} 2n-3 & \text{if } n \equiv 0 \pmod{12} \\ 2n-2 & \text{if } n \equiv 4, 8 \pmod{12} \\ 2n-1 & \text{if } n \text{ is odd, except when } n = 3 \text{ for which } R = 3 \text{ and } n = 5 \text{ for which } R = 10 \\ 2n+1 & \text{if } n \equiv 2, 6, 10 \pmod{12} \text{ except for } n = 6 \text{ for which } R = 12 \end{cases}$$

Proof.

- Case $n \equiv 0 \pmod 4$:
  In that case, $f_\alpha(n) = \frac{n}{2} + 2\lfloor\frac{\alpha}{4}\rfloor + \epsilon_\alpha$ where $\epsilon_\alpha = 1$ if $\alpha \equiv 3 \pmod 4$ and 0 otherwise.

  Using (1) $\sum_\alpha x_\alpha = R$, inequality (3) becomes $\frac{n}{2}R + \sum_\alpha (2\lfloor\frac{\alpha}{4}\rfloor + \epsilon_\alpha)x_\alpha \geq n(n-1)$.

  So $R \geq (2n-3) + \frac{1}{n}(n - \sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$ where $F(\alpha) = \alpha - 4\lfloor\frac{\alpha}{4}\rfloor - 2\epsilon_\alpha)$.

  For any α, $\alpha \geq 4\lfloor\frac{\alpha}{4}\rfloor + 2\epsilon_\alpha$ and hence $F(\alpha) \geq 0$. As, by (2), $n \geq \sum_\alpha \alpha x_\alpha$, we get $R \geq 2n-3$.

  The equality holds only if we have equality everywhere. However, for $\alpha \not\equiv 0 \pmod 4$, $\alpha > 4\lfloor\frac{\alpha}{4}\rfloor + 2\epsilon_\alpha$. So we should have $x_\alpha = 0$ for $\alpha \not\equiv 0 \pmod 4$.

  Consider the case $\alpha \equiv 0 \pmod 4$. As the nodes used in the α 2-calls are 3α, then $3\alpha \equiv 0 \pmod{12}$. By the above remark concerning equality in (3), each round with α 2-call has exactly $f_\alpha(n) = \frac{n+\alpha}{2}$ calls and so there are $\frac{n-3\alpha}{2}$ single calls. But, when $n \equiv 4, 8 \pmod{12}$ as $3\alpha \equiv 0 \pmod{12}$, the remaining number of nodes are $n - 3\alpha \geq 2$.

Therefore in any round there are at least two single calls. In particular, during the first round, two nodes have sent their messages exactly to one neighbour and so cannot be involved in any 2-call. Then the equality is not possible.

- Case $n \equiv 1 \pmod 4$:
Then $f_\alpha(n) = \frac{n-1}{2} + 2\lfloor \frac{\alpha+1}{4} \rfloor + \epsilon_\alpha$, where $\epsilon_\alpha = 1$ for $\alpha \equiv 2 \pmod 4$, and 0 otherwise.

  Using (1) $\sum_\alpha x_\alpha = R$, inequality (3) becomes $\frac{n-1}{2}R + \sum_\alpha (2\lfloor \frac{\alpha+1}{4} \rfloor + \epsilon_\alpha)x_\alpha \geq n(n-1)$, and so $R \geq 2n-2+\frac{2}{n-1}(n-1-\sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$, where $F(\alpha) = \alpha - 2\lfloor \frac{\alpha+1}{4} \rfloor - \epsilon_\alpha$.

  So, $F(\alpha) = 0$ for $\alpha = 0$, $F(\alpha) = 1$ for $\alpha = 1, 2, 3$ and $F(\alpha) \geq 2$ for $\alpha \geq 4$. Therefore, except if $x_\alpha = 0$ for $\alpha \geq 4$ and $x_1 + x_2 + x_3 \leq 1$, $\sum_\alpha F(\alpha)x_\alpha > 1$ and as by (2), $n - \sum_\alpha \alpha x_\alpha \geq 0$, then $R > 2n-2$. When $x_\alpha = 0$ for $\alpha \geq 4$ and $x_1 + x_2 + x_3 \leq 1$, then $\sum_\alpha \alpha x_\alpha \leq 3$ and as $n \geq 5, n - \sum_\alpha \alpha x_\alpha \geq 2$ and so as $F(\alpha) \geq 0$, $R > 2n-2$.

- Case $n \equiv 3 \pmod 4$:
Then $f_\alpha(n) = \frac{n-1}{2} + \lfloor \frac{\alpha}{2} \rfloor + \epsilon'_\alpha$, where $\epsilon'_\alpha = 1$ for $\alpha \equiv 1 \pmod 4$ and 0 otherwise. Like in the preceding case we get: $R \geq 2n-2+\frac{2}{n-1}(n-1-\sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$, where $F(\alpha) = \alpha - \lfloor \frac{\alpha}{2} \rfloor - \epsilon'_\alpha$. $F(\alpha) = 0$ for $\alpha = 0, 1$, $F(\alpha) = 1$ for $\alpha = 2$ and $F(\alpha) \geq 2$ for $\alpha \geq 3$. Therefore, except if $x_\alpha = 0$ for $\alpha \geq 3$ and $x_2 \leq 1$, we have $F(\alpha) > 1$ and as $n - \sum_\alpha \alpha x_\alpha \geq 0$ we get $R > 2n-2$.

  For $n \geq 7$, consider the case where $x_\alpha = 0$ for $\alpha \geq 3$ and $x_2 \leq 1$.

  If $x_2 = 1$, there should be equality everywhere otherwise we have $R > 2n-2$. In particular by the remark concerning equality in (3), each round with α 2-calls has exactly $f_\alpha(n)$ calls. Therefore, among the 3 first rounds, there is at most one round with two 2-calls ($x_2 \leq 1$) and two with at most one 2-call. So, there are at least two nodes which send their messages during these 3 rounds to only one neighbour, and cannot be involved in making any 2-call. Hence $n-2 \geq \sum_\alpha x_\alpha$ and so $R > 2n-2$.

  If $x_2 = 0$, then all the rounds have exactly $f_\alpha(n)$ calls except perhaps one round which might have $f_\alpha(n) - 1$ calls; otherwise we will have a gap of 2 in inequality (3) and then $R \geq 2n-2+\frac{2}{n-1}(n-1-\sum_\alpha \alpha x_\alpha + 2) > 2n-2$. Now, if $n \geq 11$, then at least two senders send their messages to only one neighbour. It is also the case when $n = 7$ and the first round has one 2-call and 2 single calls. It remains to deal with the case $n = 7$ and the first round has one 2-call and only one single call. A careful analysis of interference restrictions shows that in the second round there is one sender of a single call which is not a sender in the first round and again at least two senders send their messages to one neighbour and so cannot be involved in any 2-call. In summary, $n - \sum_\alpha x_\alpha \geq 2$ and so $R > 2n-2$.

  For $n = 3$, we can have $x_1 = 3$ and $R = 3$ which is optimal.

- Case $n \equiv 2 \pmod 4$:
$f_\alpha(n) = \frac{n-2}{2} + 2\lfloor \frac{\alpha+2}{4} \rfloor + \epsilon_\alpha$, where $\epsilon_\alpha = 1$ if $\alpha \equiv 1 \pmod 4$ and 0 otherwise.

  By (1) and (3), we get: $\frac{n-2}{2}R + \sum_\alpha (2\lfloor \frac{\alpha+2}{4} \rfloor + \epsilon_\alpha)x_\alpha \geq n(n-1)$ that is $R \geq 2n + \frac{2}{n-2}(n - \sum_\alpha \alpha x_\alpha + \sum_\alpha F(\alpha)x_\alpha)$ where $F(\alpha) = \alpha - 2\lfloor \frac{\alpha+2}{4} \rfloor - \epsilon_\alpha)$. Hence, $F(\alpha) = 0$ for $\alpha = 0, 1, 2$ and $F(\alpha) > 0$ for $\alpha \geq 3$. Therefore, $R > 2n$, except if $x_\alpha = 0$ for $\alpha \geq 3$, but in this case, if $n \geq 10$ and if each round with α 2-calls ($\alpha = 0, 1, 2$), has exactly $f_\alpha(n)$ calls, then at least two senders in the first round send exactly their messages to only one neighbour and these two senders can not be involved in any 2-call. So $n > \sum_\alpha x_\alpha$ and $R > 2n$ or $R \geq 2n+1$.

For $n=6$, we can have equality everywhere with $x_2 = 3$ and so $x_0 = 9$ and $R = 12$. This bound is attained with the following protocol. At the round r, for $r = 1,2,3$, we have two 2-calls: $(r : r-1, r+1)$ and $(r+3 : r+2, r+4)$. At the end of these three rounds each node has exactly 3 messages $\{i-1, i, i+1\}$.Then we complete with the nine rounds that containing two single calls, $r = 4, 5, ..., 12 : (r, r-1)$ and $(r+2, r+1)$.

For $n = 2$, the proof above does not work, as $n - 2 = 0$. In that case, clearly $R = 2$.

$\diamond$

4 Upper bounds

Sketch of the proof: We will design in a first phase a small number of rounds (between 0 and 4 when $n \equiv 0 \pmod 4$ and 5 in the other cases) which will contain mainly 2-calls. Then in a second phase we do sequences of two symmetric rounds associated to a matchings (see formal definition after). More precisely we will do sequences of four rounds associated to two matchings whose union is either a hamilton cycle (case $n \equiv 0 \pmod 4$) or a hamilton path (case n odd) or a path of length $n-2$ (case $n \equiv 2 \pmod 4$. In such a sequence of four rounds, all the nodes covered by the two matchings will receive one message from the left and one from the right. We will design the protocol in such a way each node is uncovered by the same number of matchings (zero, one or two according to the values of n). So at the end of the protocol, the nodes will have all received the same quantity of messages (in fact all).

The section is organized as follows: we first give some definitions and notations which will be useful to describe our protocols and to compute their makespans. We will illustrate them with the example of $n = 8$. Then we will do the case $n \equiv 0 \pmod 4$, where the sequences of consecutive four rounds are easy to describe. Then we will describe the five first rounds of the first phase for the other cases of congruences. Finally, we will describe the sequence of consecutive four rounds successively in the case n odd and $n \equiv 2 \pmod 4$. In these cases, the proofs are more involved and we have in some specific cases to use non standard rounds at the end to attain the lower bounds.

4.1 Definitions

Recall the nodes are the integers modulo n labeled from 0 to $n-1$. We will use i to denote the label of a node, but also to denote its own message.

Symmetric rounds and matching: We will say that two consecutive rounds t and $t+1$ are symmetric if, when (x, y) is a call in the round t, then (y, x) is a call in the round $t+1$. Two symmetric rounds are associated to a matching in C_n. More precisely the matching $M = \{< x_k, x_k + 1 >\}, 1 \leq k \leq p$ corresponds to the two symmetric rounds $\{(x_k, x_k + 1), 1 \leq k \leq p$ and $\{(x_k + 1, x_k), 1 \leq k \leq p\}$.

We will say that a round is obtained by a translation of $+1$ of an other round if it contains the calls $(x+1, y+1)$ where (x, y) is a call of the first round. Similarly a matching denoted $M+1$ is the translated of a matching M, if it contains the edges $< x_k + 1, x_k + 2 >$ where $< x_k, x_k + 1 >$ is an edge of M.

Example 1 For $n = 8$, we have the following optimal protocol with $2n - 2 = 14$ rounds. In the Table 2, columns correspond to the nodes and each line corresponds to a round. An array in a cell indicates the direction of a call, a value indicates the message received by the corresponding node, and a cross indicates that the node is neither a sender nor a receiver.

At the beginning, each node knows only its own message. Let the first round consist of the 2-calls $\{(1 : 0, 2), (5 : 4, 6)\}$, and the second round be $\{(2 : 1, 3), (6 : 5, 7)\}$. One can see that the calls of this second round are obtained from the calls of the first round by a translation of $+1$. We will shortly say that the round 2 is obtained from the round 1 by a translation of $+1$. Similarly the round 3 is obtained from the round 2 by a translation of $+1$ and so consists of the 2-calls $\{(3 : 2, 4), (7 : 6, 0)\}$ and the round 4 is obtained from the round 3 by a translation of $+1$. At the end of the round 4, all 2-calls have been used and each node knows its own message and those of its two neighbors.

Let the round 5 be $\{(1, 0), (2, 3), (5, 4), (6, 7)\}$ and the round 6 be $\{(0, 1), (3, 2), (4, 5), (7, 6)\}$. Note that these two rounds are symmetric and correspond to the perfect matching $M = \{< 2j, 2j + 1 >, 0 \leq j \leq 3\}$. We also have another two symmetric rounds represented by $M' = M + 1 = \{< 2j + 1, 2j + 2 >, 0 \leq j \leq 3\}$ obtained from M by a translation of $+1$. Note that these four rounds associated to M and M' corresponds to a hamilton cycle. Therefore, at the end of these four rounds, each node i will have received a message from the left (precisely that of node $i - 2$) and one message from the right (precisely that of node $i + 2$). Then we do again two pairs of symmetric rounds associated to M and M'. We finish with a pair of symmetric rounds associated to M. The protocol is therefore completed within $2n - 2 = 14$ rounds.

round \ node	matching	0	1	2	3	4	5	6	7
1	(2-calls)	1	$\longleftrightarrow$	2	$\times$	5	$\longleftrightarrow$	5	$\times$
2		$\times$	2	$\longleftrightarrow$	2	$\times$	6	$\longleftrightarrow$	6
3		7	$\times$	3	$\longleftrightarrow$	3	$\times$	7	$\longleftrightarrow$
4		$\longleftrightarrow$	0	$\times$	4	$\longleftrightarrow$	4	$\times$	0
5	M	2	$\leftarrow$	$\rightarrow$	1	6	$\leftarrow$	$\rightarrow$	5
6		$\rightarrow$	7	4	$\leftarrow$	$\rightarrow$	3	0	$\leftarrow$
7	M'	6	3	$\leftarrow$	$\rightarrow$	2	7	$\leftarrow$	$\rightarrow$
8		$\leftarrow$	$\rightarrow$	0	5	$\leftarrow$	$\rightarrow$	4	1
9	M	3	$\leftarrow$	$\rightarrow$	0	7	$\leftarrow$	$\rightarrow$	4
10		$\rightarrow$	6	5	$\leftarrow$	$\rightarrow$	2	1	$\leftarrow$
11	M'	5	4	$\leftarrow$	$\rightarrow$	1	0	$\leftarrow$	$\rightarrow$
12		$\leftarrow$	$\rightarrow$	7	6	$\leftarrow$	$\rightarrow$	3	2
13	M	4	$\leftarrow$	$\rightarrow$	7	0	$\leftarrow$	$\rightarrow$	3
14		$\rightarrow$	5	6	$\leftarrow$	$\rightarrow$	1	2	$\leftarrow$

Table 2: An optimal protocol for $n = 8$

Sets L_i^t and R_i^t and their sizes l_i^t and r_i^t: We define L_i^t (resp. R_i^t) as the set of messages that have already been received at the end of the round t at the node i from the left that is via the call $(i-1,i)$ (resp. from the right that is via the call $(i+1,i)$). By convention, as at the beginning each node knows its own message, we will define $L_i^0 = R_i^0 = \{i\}$.

We will use l_i^t and r_i^t to denote the sizes of L_i^t and R_i^t, respectively. Note that the messages in L_i^t (resp. R_i^t) arrive only via a call $(i-1,i)$ (resp. $(i+1,i)$ and so $L_i^t - \{i\} \subseteq L_{i-1}^t$ (resp. $R_i^t - \{i\} \subseteq R_{i+1}^t$. Furthermore as we consider only useful calls $L_i^t \cap R_i^t = i$. Finally, a call $(i-1,i)$ is valid in round $t+1$ if and only if $l_{i-1}^t \geq l_i^t$, otherwise node i already knows all messages of node $i-1$. Similarly a call $(i+1,i)$ is valid in round $t+1$ if and only if $r_{i+1}^t \geq r_i^t$.

Order to send the messages: It might happen that a node has more than one message (unknown by the receiver) to transmit to its neighbor, situation where $l_{i-1}^t > l_i^t$ or $r_{i+1}^t > r_i^t$. In the protocols we design we will send the one who arrives the earliest (FIFO). More precisely, in the call $(i-1,i)$, the node $i-1$ will send the message unknown to i with the largest index. Similarly for the call $(i+1,i)$, the node $i+1$ will send the message unknown to i with the smallest index.

Note that in such protocols, the values of l_i^t (resp. r_i^t) precisely determine the sets L_i^t (resp. R_i^t) and so provide all the messages acquired by the node i at the end of t rounds. Indeed, if $l_i^t = k_1$ and $r_i^t = k_2$, then at the end of t rounds, $L_i^t = \{i-k_1+1, i-k_1+2, \ldots, i-1, i\}$ (resp. $R_i^t = \{i, i+1, \ldots, i+k_2-1\}$) and the set of messages acquired at the node i is $\{i-k_1+1, i-k_1+2, \ldots, i-1, i, i+1, \ldots, i+k_2-1\}$.

Table 3 gives the couple of values (l_i^t, r_i^t) for node i at the end of round t in the case $n = 8$.

round \ node	0	1	2	3	4	5	6	7
initial state	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
1	(1,2)		(2,1)		(1,2)		(2,1)	
2		(1,2)		(2,1)		(1,2)		(2,1)
3	(2,2)		(2,2)		(2,2)		(2,2)	
4		(2,2)		(2,2)		(2,2)		(2,2)
5	(2,3)			(3,2)	(2,3)			(3,2)
6		(3,2)	(2,3)			(3,2)	(2,3)	
7	(3,3)	(3,3)			(3,3)	(3,3)		
8			(3,3)	(3,3)			(3,3)	(3,3)
9	(3,4)			(4,3)	(3,4)			(4,3)
10		(4,3)	(3,4)			(4,3)	(3,4)	
11	(4,4)	(4,4)			(4,4)	(4,4)		
12			(4,4)	(4,4)			(4,4)	(4,4)
13	(4,5)			(5,4)	(4,5)			(5,4)
14		(5,4)	(4,5)			(5,4)	(4,5)	

Table 3: Values of (l_i^t, r_i^t) for $n = 8$

Balanced protocols and labels: We will design "balanced protocols" where, at the end of each round, the nodes have received almost the same number of messages from left and right. In particular at the end of any round t, the values of l_i^t and r_i^t at each node i will consist of at most 3 consecutive values denoted $\alpha-1$, α and $\alpha+1$ In order to facilitate the proof of the protocol presented, we will attach some labels to the nodes which reflect how the values of l_i^t and r_i^t behave comparing to the value α.

We will mainly use 6 types of labels assigned to the nodes: $B, B^+, R^+, R^-, L^+, L^-$. The labels are defined as below. We use memo-technic letters where a B stands for balanced (l_i^t and r_i^t are equal), and L (resp. R) when the value of the l_i^t (resp. r_i^t) is different from α. A superscript $+$ (resp. $-$) stands for the value concerned being $\alpha+1$ (resp. $\alpha-1$).

Let the label be B, if $l_i^t = r_i^t = \alpha$, B^+ if $l_i^t = r_i^t = \alpha+1$, L^+ if $l_i^t = \alpha+1, r_i^t = \alpha$, R^+ if $l_i^t = \alpha, r_i^t = \alpha+1$, L^- if $l_i^t = \alpha-1, r_i^t = \alpha$, and R^- if $l_i^t = \alpha, r_i^t = \alpha-1$.

Table 4 gives for $n = 8$ the labels of the nodes at the end of each round t and one can see there the regularity in the pattern of the labels.

round \ node	0	1	2	3	4	5	6	7	α
initial state	B	B	B	B	B	B	B	B	1
1	R^+	B	L^+	B	R^+	B	L^+	B	1
2	R^+	R^+	L^+	L^+	R^+	R^+	L^+	L^+	1
3	B^+	R^+	B^+	L^+	B^+	R^+	B^+	L^+	1
4	B	B	B	B	B	B	B	B	2
5	R^+	B	B	L^+	R^+	B	B	L^+	2
6	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	2
7	B^+	B^+	R^+	L^+	B^+	B^+	R^+	L^+	2
8	B	B	B	B	B	B	B	B	3
9	R^+	B	B	L^+	R^+	B	B	L^+	3
10	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	3
11	B^+	B^+	R^+	L^+	B^+	B^+	R^+	L^+	3
12	B	B	B	B	B	B	B	B	4
13	R^+	B	B	L^+	R^+	B	B	L^+	4
14	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	4

Table 4: Labels for n=8

We can express the condition of validity of a call in terms of labels. Recall that a call $(i, i+1)$ is valid in round $t+1$ if and only if $l_i^t \geq l_{i+1}^t$, and a call $(i+1, i)$ is valid in round $t+1$ if and only if $r_{i+1}^t \geq r_i^{t-1}$.

The following proposition shows this condition of validity of a call by using labels.

Proposition 1 *A call $(i, i+1)$ is valid for round $t+1$, if the labels of nodes i and $i+1$ are in one of the following situations at the end of round t:*

- *Node i is labeled B^+ or L^+,*
- *Node i is labeled B, R^+ or R^- and node $i+1$ is labeled B, R^+, L^- or R^-.*
- *Node i is labeled L^- and node $i+1$ is labeled L^-.*

Similarly, a call $(i+1, i)$ is valid for round $t+1$, if the labels of nodes $i+1$ and i are in one of the following situations at the end of round t:

- *Node $i+1$ is labeled B^+ or R^+,*

- *Node i is labeled B, L^+, L^- or R^-, and node $i+1$ is labeled B, L^+ or L^-.*
- *Node i is labeled R^- and node $i+1$ is labeled R^-.*

The proposition covers 25 cases among the 36 possibilities. In the 11 other cases the call is not valid. We will use intensively the Proposition 1 to prove that two symmetric rounds associated to a matching are valid. We list only the cases that we will use:

Proposition 2 *The two symmetric rounds associated to a matching $M = \{< x_k, x_k + 1 >, 1 \leq k \leq p\}$ are valid if the end nodes x_k and $x_k + 1$ of an edge of the matching have the following pair of labels:*
L^+R^+; BB or B^+B^+; L^+B or L^+B^+; BR^+ or B^+R^+; R^-L^-.

Proof. In all the case mentioned in this proposition the validity conditions of Proposition 1 are both satisfied for the calls $(x_k, x_k + 1)$ and $(x_k + 1, x_k)$. ◇

4.2 Case $n \equiv 0 \pmod 4$

Now we construct protocols which will match the lower bounds given in the previous section when $n \equiv 0 \pmod 4$. The proof will use the following lemma:

Lemma 2 *Let $n \equiv 0 \pmod 4$ and let $M = \{< 2j, 2j + 1 >, 0 \leq j \leq (n-2)/2\}$ and $M' = M + 1 = \{< 2j + 1, 2j + 2 >, 0 \leq j \leq (n-2)/2\}$ be the two perfect matchings of the ring on n nodes. Suppose that at the end of some round, the labels of the nodes are all B. Then if we perform the two pairs of symmetric rounds associated to M and M' at the end of these four rounds all the labels will still be B.*

Proof. We first do the two symmetric rounds associated to the matching M. These rounds are valid by Proposition 2, as the labels of the end nodes of an edge of the matching M are BB. After these two rounds, node $2j$ (resp. $2j + 1$) will have received a message from node $2j + 1$ (resp. $2j$) and so its label will be R^+ (resp. L^+). Then we perform the two symmetric rounds associated to the matching $M' = M + 1$. Again these two rounds are valid by Proposition 2, as the labels of the end nodes of an edge of the matching M' are L^+R^+. After these two rounds, node $2j + 1$ (resp. $2j + 2$) will have received a message from node $2j + 2$ (resp. $2j + 1$) and so its label will become B (resp. B). The label list after the four rounds is again $BB \dots BB$ proving the lemma. Note that we could have perform first the two symmetric calls associated to the matching M' then those associated to M. The reader can follow the proof either in the Example 1 or in the Table 4 or in Table 5 where the matchings are indicated. ◇

4.2.1 Subcases $n = 12p + 4, n = 12p + 8$

Here we give a simple protocol using only single calls and no 2-calls. At the beginning the label list is $BB \dots B$ (with $\alpha = 1$). We repeat $\frac{n-2}{2}$ times the sequence of four rounds associated to the matchings M and $M' = M + 1$. By Lemma 2, at the end of rounds $4h, 0 \leq h \leq \frac{n-2}{2}$, the label list remains unchanged and is $BB \dots B$ with $\alpha = h + 1$. Thus, at the end of round $2n - 4$, each node misses one message (of the diagonal opposite); more precisely, the missing message at node i is $i + \frac{n}{2}$, but this message was already acquired by both nodes $i - 1$ and

node / round, matching	0	1	2	3	4	5	6	7
initial state	B	B	B	B	B	B	B	B
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_2	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_4	B	B	B	B	B	B	B	B
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_6	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_8	B	B	B	B	B	B	B	B
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{10}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+
M'	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_{12}	B	B	B	B	B	B	B	B
M	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{14}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+

Table 5: Matchings and labels for the case $n = 8$ without 2-calls

$i + 1$. Then it suffices to do two more symmetric rounds associated to the matching M in order node i receive it from one of its neighbors ($i+1$ if i is even or $i-1$ if i is odd). In total, $2n-2$ rounds are used in this protocol, which is optimal as the number of rounds match the lower bound.

The protocol is described in Table 5 for $n = 8$. Note that this protocol is different from that given in Example 1 (see also Table 4) where we first do four rounds with the 2-calls. More generally for $n = 12p + 8$, we can first do four rounds containing all the 2-calls. The first round consists of the calls $\{(4j + 1 : 4j, 4j + 2), j = 0, 1, 2, \ldots, 4p + 1\}$ the second, third and fourth rounds being obtained by successive translations of $+1$. At the end of round 4 the label list is $BB \ldots B$ (with $\alpha = 2$). Then we do the same protocol as above, that is $6p + 2$ sequences of four rounds consisting of two pairs of two symmetric rounds associated to the matchings M and M' and two symmetric rounds associated with M.

4.2.2 Subcase $n = 12p$

In the first three rounds we will use up all n 2-calls. The first round consists of the calls: $\{(3j + 1 : 3j, 3j + 2), j = 0, 1, 2, \ldots, 4p - 1\}$. The second round is obtained from the first one by a translation of $+1$ and the third one from the second by a translation of $+1$. At the end of these three rounds the label list is $BB \ldots B$ (with $\alpha = 2$). Then we do the same protocol as in the preceeding case. We repeat $\frac{n-4}{2}$ times the sequence of four rounds associated to the matchings M and $M' = M + 1$. By Lemma 2, at the end of rounds $4h + 3, 0 \leq h \leq \frac{n-2}{2}$ the label list remains unchanged and is $(BB \ldots B)$ (with $\alpha = h + 2$). Thus, at the end of round $2n - 5$, each node misses one message. Then it suffices to do two more symmetric rounds associated to the matching M in order node i receive it from one of its neighbors. In total, $2n - 3$ rounds are used in this protocol, which is optimal as the number of rounds match the lower bound.

4.3 Construction of the 5 first rounds for n odd and $n \equiv 2 \pmod 4$

In the cases n odd and $n \equiv 2 \pmod 4$, the first phase of the protocol consists in designing 5 rounds in such a way that the label list obtained after these 5 rounds is as indicated in the following Proposition 3.

Proposition 3 *We can design the first five rounds of the protcol such that the label list of the nodes consists of a sequence (starting at node* 0*) of*

$$\begin{cases} \frac{n-3}{2}\ R^+L^+ \text{ followed by } BBB & \text{if } n \equiv 1,3 \pmod 6 \\ \frac{n-7}{2}\ R^+L^+ \text{ followed by } BR^+L^+BL^+R^+B & \text{if } n \equiv 5 \pmod 6 \\ \frac{n-4}{2}\ R^+L^+ \text{ followed by } R^+BBB & \text{if } n \equiv 6,10 \pmod{12} \\ \frac{n-4}{2}\ R^+L^+ \text{ followed by } BBBB & \text{if } n \equiv 2 \pmod{12} \end{cases}$$

We give in the next paragraph the tables with the first five rounds for the values of $n_0 = 7, 9, 11, 13, 15, 17$ and $n_0 = 10, 14, 18$. Then we show how to extend the construction for $n = 12p + n_0$.

4.3.1 Table of the first five rounds for $n_0 = 7, 9, 11, 13, 15, 17$ and $n_0 = 10, 14, 18$

The nodes are labeled from 0 to $n_0 - 1$ and we have built the rounds in such a way they have the following properties

- Property 1 : Round 1 contains the 2-call emitted by node 1 : $\{(1 : 0, 2)\}$. Therefore, node 0 is receiving from node 1 and so node $n_0 - 1$ is not emitting (either it receives from node $n_0 - 2$ or it does nothing).
- Property 2 : Round 2 contains the 2-call emitted by node $n_0 - 1$: $\{(n_0 - 1 : 0, n_0 - 2)\}$.
- Property 3 : Round 3 contains the 2-call emitted by node 0 : $\{(0 : 1, n_0 - 1)\}$.
- Property 4 : In round 4, node 0 receives a message from node 1 (namely message 2), and node $n_0 - 1$ does not send any message.
- Property 5 : In round 5, node 0 sends a message to node 1 namely message $n_0 - 1$ and node $n_0 - 1$ does not receive any message.
- Property 6 : the label list consists of a sequence of R^+L^+ followed by BBB if $n_0 = 7, 9, 13, 15$ or $BR^+L^+BL^+R^+B$ if $n_0 = 11, 17$ or R^+BBB if $n_0 = 10, 18$ or $BBBB$ if $n_0 = 14$.

4.3.2 Extension of the first five rounds for $n = 12p + n_0$

Let $n = 12p + n_0$. The nodes are labeled from 0 to $n - 1$. We add to the nodes of the examples for n_0, the $12p$ nodes $n_0 + i$ for $0 \leq i \leq 12p - 1$. We now describe the calls added in the first five rounds (one can follow the extension on the Table 15 which shows how the rounds for $n = 19$ are obtained from those for $n_0 = 7$).

- Round 1 consists of the calls of the example for n_0, plus the 2-calls $\{(n_0 + 3j + 1 : n_0 + 3j, n_0 + 3j + 2), j = 0, 1, 2, \ldots, 4p - 1\}$.

round \ node	0	1	2	3	4	5	6
1	1		1	4		4	×
2	6	2		2	×	6	
3		0	3		3	×	0
4	2			1	5		5
5		6	4		×	×	×
label	R^+	L^+	R^+	L^+	B	B	B

Table 6: First five rounds for $n_0 = 7$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1		1	4		4	7		7	10		10	×
2	12	2		2	5		5	8		8	×	12	
3		0	3		3	6		6	9		9	×	0
4	2			1	6			5	10		×		11
5		12	4			3	8			7	11		×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Table 7: First five rounds for $n_0 = 13$, no 2-call from the node 11

round \ node	0	1	2	3	4	5	6	7	8
1	1		1	4		4	7		7
2	8	2		2	5		5	8	
3		0	3		3	6		6	0
4	2			1	6		×	×	×
5		8	4			3	×	×	×
label	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Table 8: First five rounds for $n_0 = 9$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1		1	4		4	7		7	10		10	13		13
2	14	2		2	5		5	8		8	11		11	14	
3		0	3		3	6		6	9		9	12		12	0
4	2			1	6			5	10			9	×	×	×
5		14	4			3	8			7	12		×	×	×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B

Table 9: First five rounds for $n_0 = 15$

round \ node	0	1	2	3	4	5	6	7	8	9	10
1	1		1	4		4	7		7	×	×
2	10	2		2	5		5	×	×	10	
3		0	4		×		4	8		8	0
4	2			1	×	6		6	9		9
5		10	3		3	7			6	0	
label	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B

Table 10: First five rounds for $n_0 = 11$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1		1	4		4	7		7	10		10	13		13	×	×
2	16	2		2	5		5	8		8	11		11	×	×	16	
3		0	3		3	6		6	10		×		10	14		14	0
4	2			1	6			5	9		9	12		12	15		15
5		16	4			3	8			7	×	13			12	0	
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B

Table 11: First five rounds for $n_0 = 17$

round \ node	0	1	2	3	4	5	6	7	8	9
1	1		1	4		4	7		7	×
2	9	2		2	5		5	×	9	
3		0	3		3	6		6	×	0
4	2			1	6		×	8		8
5		9	4			3	8		×	×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	B	B	B

Table 12: First five rounds for $n_0 = 10$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1		1	4		4	7		7	10		10	×	×
2	13	2		2	5		5	8		8	×	×	13	
3		0	3		3	6		6	9		9	×	×	0
4	2			1	6			5	10		×	12		12
5		13	4			3	8			7	11		11	×
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B

Table 13: First five rounds for $n_0 = 14$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1		1	4		4	7		7	10		10	13		13	16		16
2	17	2		2	5		5	8		8	11		11	14		14	17	
3		0	3		3	6		6	9		9	12		12	15		15	0
4	2			1	6			5	10			9	14		$\times$	$\times$	$\times$	$\times$
5		17	4			3	8			7	12			11	16		$\times$	$\times$
label	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	B	B	B

Table 14: First five rounds for $n_0 = 18$

- Round 2 consists of the calls of the example for n_0, except we delete the 2-call $\{(n_0 - 1 : 0, n_0 - 2)\}$, which exists by Property 2, and replace it by the 2-call $\{(n_0 - 1 : n_0, n_0 - 2)\}$. Then we add the 2-calls $\{(n_0 + 3j + 2 : n_0 + 3j + 1, n_0 + 3j + 3), j = 0, 1, 2, \ldots, 4p - 1\}$ (recall the node $n_0 + 12p$ means the node 0).
- Round 3 consists of the calls of the example for n_0, except we delete the 2-call $\{(0 : 1, n_0 - 1)\}$ which exists by Property 3, and replace it by the 2-call $\{(0 : 1, n_0 + 12p - 1)\}$. Then we add the 2-calls $\{(n_0 + 3j : n_0 + 3j - 1, n_0 + 3j + 1), j = 0, 1, 2, \ldots, 4p - 1\}$.
- Round 4 consists of the calls of the example for n_0, plus the calls $\{(n_0 + 4k + 1, n_0 + 4k), (n_0 + 4k + 2, n_0 + 4k + 3), k = 0, 1, 2, \ldots, 3p - 1\}$.
- Round 5 consists of the calls of the example for n_0, plus the calls $\{(n_0 + 4k, n_0 + 4k + 1), (n_0 + 4k + 3, n_0 + 4k + 2), k = 0, 1, 2, \ldots, 3p - 1\}$. Also we have to impose that node 0 is now sending to node 1 the message $n_0 + 12p - 1$ (and not $n_0 - 1$).

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1		1	4		4	$\times$	8		8	11		11	14		14	17		17
2	18	2		2	$\times$	6		6	9		9	12		12	15		15	18	
3		0	3		3	$\times$	7		7	10		10	13		13	16		16	0
4	2			1	5		5	9			8	13			12	17			16
5		18	4		$\times$	$\times$	$\times$		6	11			10	15			14	18	
label	R^+	L^+	R^+	L^+	B	B	B	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+

Table 15: First five rounds for $n = 19$ (from $n_0 = 7$)

First we note that these five rounds are valid. There is no interference between the calls added and not between these calls and the ones used for n_0 due to Properties 1,4,5. Indeed in rounds 1 and 4, nodes n_0 and $n_0 + 12p - 1$ are both receiving while by Property 1 or 4 node 0 is receiving and node $n_0 - 1$ is not emitting. In round 5, nodes n_0 and $n_0 + 12p - 1$ are both emitting while, by Property 5, nodes 0 and $n_0 - 1$ do not receive any message.

We also note that in this construction the number of messages received from the left and from the right by the nodes from 0 to $n_0 - 1$ remains the same. Indeed the calls are the same, except in round 2 where node 0 is receiving from $n_0 + 12p - 1$ instead of $n_0 - 1$ and in round 3 where node $n_0 - 1$ is receiving from n_0 instead of 0. Furthermore, during the first three rounds the $12p$ nodes added receive one message from the left and another from the right.

During the round 4, nodes $n_0+2j(0 \leq j \leq 6p-1)$ have received a message from the right and so have label R^+, while during the round 5, nodes $n_0+2j+1(0 \leq j \leq 6p-1)$ have received a message from the left and so have label L^+. So using these labels and the Property 6, the label list starting at n_0 satifies the following Proposition. However as we are in a cycle the starting node of a sequence can be chosen arbitrarily. If we relabel node n_0: 0 and n_0+i: i, we get the following proposition now starting node 0 (as announced at the beginning of the subsection.

Proposition 4 *We can design the first five rounds of the protcol such that the label list of the nodes consists of a sequence (starting at node 0) of*

$$\begin{cases} \frac{n-3}{2}\ R^+L^+ \textit{ followed by } BBB & \textit{if } n \equiv 1,3 \pmod 6 \\ \frac{n-7}{2}\ R^+L^+ \textit{ followed by } BR^+L^+BL^+R^+B & \textit{if } n \equiv 5 \pmod 6 \\ \frac{n-4}{2}\ R^+L^+ \textit{ followed by } R^+BBB & \textit{if } n \equiv 6,10 \pmod{12} \\ \frac{n-4}{2}\ R^+L^+ \textit{ followed by } BBBB & \textit{if } n \equiv 2 \pmod{12} \end{cases}$$

4.4 Construction for n odd

We will now give the second phase of the construction. It will be based on another lemma (Lemma 3) similar to Lemma 2; but now we use two near-perfect matchings whose union is an hamilton path. In what follows we will denote by M_i the near-perfect matching which does not contain node i. So $M_i = \{< i+2j+1, i+2j+2 >, 0 \leq j \leq (n-3)/2\}$.

Lemma 3 *Let n be odd and let the label list be such that nodes i has label R^+ and node $i+1$ has label L^+. If we perform the two pairs of symmetric rounds associated to the near-perfect matchings M_i and M_{i+1}, then if the rounds are valid at the end of these four rounds the labels of the nodes will be unchanged except for nodes i and $i+1$ which will get label B.*

Proof. Let us apply the sequence of four rounds consisting of the two pairs of symmetric rounds associated to the near-perfect matchings M_i and M_{i+1}. Then, at the end of these four rounds a node $\neq i, i+1$ will have received exactly one message from the left and one from the right and so its label remains unchanged.

Node $i+1$ will have received only a message from node $i+2$ in the matching M_i and so its label changed from L^+ to B. Node i will have received only a message from node $i-1$ in the matching M_{i+1} and so its label changed from R^+ to B. $\diamond$

Applying repeatedly sequences of rounds we will be able to transform all the R^+L^+ of the label list of Proposition 4 into BB and so complete the protocol. The main difficulty will be to choose the suitable nodes i and $i+1$ and the order of the near-perfect matchings M_i and M_{i+1} when applying the Lemma 3. We first deal with the case $n \equiv 1$ or $3 \pmod 6$, which is regular. Then we will do the case $n \equiv 5 \pmod 6$, where the 4 last rounds are irregular.

4.4.1 Case $n \equiv 1$ or $3 \pmod 6$

The construction is illustrated for $n = 9$ in Table 16. The idea consists in doing $(n-3)/2$ sequences of four rounds using successively in Lemma 3 for i and $i+1$ the last two nodes with labels R^+L^+ and by doing first the rounds associated to M_{i+1} and then those associated to

round \ node	0	1	2	3	4	5	6	7	8
R_5	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B
M_5	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\leftarrow$
R_7	B^+	B^+	B^+	B^+	B^+	L^+	R^+	L^+	R^+
M_4	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	B	B	B	B	B
M_3	$\rightarrow$	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_{11}	B^+	B^+	B^+	L^+	R^+	L^+	R^+	L^+	R^+
M_2	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	B	B	B	B	B	B	B
M_1	$\rightarrow$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_{15}	B^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+
M_0	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{17}	B	B	B	B	B	B	B	B	B

Table 16: Call matchings and labels for $n = 9$

M_i. We show now in details how the protocol works. By Proposition 4 the label list at round 5 consists of $\frac{n-3}{2}$ pairs of R^+L^+ followed by BBB. We will say that the label list satisfies at the end of the round $5+4h$ the property P_h if it consists of $(n-3-2h)/2$ pairs R^+L^+ followed by $(3+2h)$ B. More precisely nodes $2j$ $(0 \leq j \leq (n-5-2h)/2)$ have labels R^+ and nodes $2j+1$ $(0 \leq j \leq (n-5-2h)/2)$ have labels L^+ and nodes i, $(n-3-2h \leq i \leq n-1)$ have labels B. Note that, at round 5, the label list satisfies property P_0. We will prove by induction in the following claim that we can design the protocol in such a way the label list satisfies at the end of the round $5+4h$ the property P_h for $0 \leq h \leq (n-3)/2$ and in particular at round $2n-1$ the label list is $BB\dots B$. Therefore the gossiping has been completed in $2n-1$ rounds.

Claim 3 Inductive step *Suppose that at the end of the round $5+4h$ the label list satisfies property P_h. Then if we perform successively the two symmetric rounds associated to the near-perfect matching M_{n-4-2h} and then those associated to the near-perfect matching M_{n-5-2h} these rounds are valid and so by Lemma 3 at the end of the round $5+4(h+1)$ the label list satisfies property P_{h+1}.*

Proof. The matching M_{n-4-2h} consists of the edges $\{< n-4-2h+2j+1, n-4-2h+2j+2 >, 0 \leq j \leq (n-3)/2\}$. The labels of the end nodes of the edges of the matching are successively BB for $0 \leq j \leq h$, then BR^+ for $j = h+1$, then L^+R^+ for $h+2 \leq j \leq (n-3)/2$. Therefore, the two rounds associated to matching M_{n-4-2h} are valid by Proposition 2. Note that now nodes $n-5-2h+2j+1$, $(0 \leq j \leq h+1)$ have labels L^+ (for $j=0$ the node $n-5-2h+1$ has kept its label L^+ as it was not in the matching while for the other nodes they have received a message from the left and so their label has changed from B to L^+). Nodes $n-5-2h+2j+2$ $(0 \leq j \leq h+1)$ have received a message from the right and so their labels has changed to R^+. Nodes i $(0 \leq i \leq n-4-2h)$ have now labels B^+. That implies that the labels of the end nodes of the edges of the matching M_{n-5-2h} are either L^+R^+ or B^+B^+ and so the two rounds associated to matching M_{n-5-2h} are valid by Proposition 2. $\diamond$

In the example for $n = 9$, we do first the two rounds (R_6 and R_7) associated to the matching M_5 (here $n-4-2h = 5$ as $n = 9$ and $h = 0$); then the two rounds (R_8 and R_9) associated to the matching M_4 (here $n-5-2h = 4$ as $n = 9$ and $h = 0$); then the two rounds associated to the matching M_3, M_2, M_1, M_0 altogether we have $5 + 4 \cdot 3 = 17$ rounds. In general our protocol uses $2n-1$ rounds, which match the lower bound and so the protocol is optimal.

4.4.2 Case $n \equiv 5 \pmod 6$

The constructions for $n = 11$ (resp. $n = 17$) are indicated on Tables 17 (resp. 18). In that case by Proposition 4, the label list consists of $(n-7)/2$ pairs of R^+L^+ followed by $BR^+L^+BL^+R^+B$.

The construction is similar to that of the preceding case, but first we do the two symmetric rounds associated to the matching M_{n-6} and then those associated to M_{n-5}. By lemma 2, the nodes $n-6$ and $n-5$ (which correspond to the nodes with labels R^+L^+ between the two B in the last part of the label list) will have at round 9 their labels changed to BB if the rounds are valid. That is the case: indeed the rounds associated to the matching M_{n-6} are valid by Proposition 2, as the labels of the end nodes of the edges of this matching are successively L^+B, L^+R^+, BR^+, $(n-9)/2$ times L^+R^+ and L^+B. At the end of these two rounds the label list consists of $(n-7)$ B^+ followed by $L^+R^+B^+L^+B^+B^+R^+$. The rounds associated to the matching M_{n-5} are also valid by Proposition 2, as the labels of the end nodes of the edges of this matching are successively L^+B^+, B^+R^+, $(n-7)/2$ B^+B^+ and L^+R^+.

Similarly to the preceding case, we say that a label list satisfies property P'_h if at the end of round $9+4h$ it consists of $(n-7-2h)/2$ pairs of R^+L^+ followed by $BBBBL^+R^+B$. Then we have the following result (inductive step): if we perform successively the two symmetric rounds associated to the matching M_{n-8-2h} and then those associated to the matching M_{n-9-2h} these rounds are valid and so by Lemma 3 at the end of the round $9+4(h+1)$ the label list satisfies property P'_{h+1}. We can apply the inductive step $(n-7)/2$ times and the end of round $2n-5$ the label list consists of $(n-3)$B followed by L^+R^+B.

Unfortunately we can no more use two near-perfect matchings as the end nodes of the hamilton path which should be used have labels are L^+R^+ and not R^+L^+. Therefore we need to finish the protocol with four specific rounds. We give these rounds in tables 17 and 18. One can check that each node has received once from the left and once from the right except two nodes; the node $n-3$ with label L^+, which receives only from the right, and the node $n-2$ with label R^+, which receives only from the left. At the end of the round $2n-1$, all labels are B.

It is easy to extend these four specific rounds for any $n = 12p + n_0$ where $n_0 = 11, 17$. Nodes i, $0 \le i \le n_0 - 1$ are renumbered $12p + i$ and we add nodes 0 to $12p-1$. It suffices to:

- in round $2n-4$, add the calls $\{(4k, 4k+1), (4k+3, 4k+2), k = 0, 1, \ldots, 3p-1\}$.
- in round $2n-3$, delete the call $(n_0-1, 0)$, add the call $(n-1, 0)$ and the calls $\{(4k+2, 4k+1), (4k+3, 4k+4), k = 0, 1, \ldots, 3p-1\}$.
- in round $2n-2$, add the calls $\{(4k+1, 4k), (4k+2, 4k+3), k = 0, 1, \ldots, 3p-1\}$.
- in round $2n-1$, delete the call $(0, n_0-1)$, add the call $(0, n-1)$ and the calls $\{(4k+1, 4k+2), (4k+4, 4k+3), k = 0, 1, \ldots, 3p-1\}$.

round \ node	0	1	2	3	4	5	6	7	8	9	10
R_5	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B
M_5	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
R_7	B^+	B^+	B^+	B^+	L^+	R^+	B^+	L^+	B^+	B^+	R^+
M_6	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	B	B	B	B	L^+	R^+	B
M_3	$\rightarrow$	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_2	$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	B	B	B	B	B	B	L^+	R^+	B
M_1	$\rightarrow$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_0	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{17}	B	B	B	B	B	B	B	B	L^+	R^+	B
18		7	7			0	0			4	$\times$
19	6	6			10	10		$\times$	2		
20	5			9	9			2	$\times$		5
21			8	8			1	1		$\times$	4
R_{21}	B	B	B	B	B	B	B	B	B	B	B

Table 17: Construction for $n = 11$

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
R_5	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	R^+	L^+	B	L^+	R^+	B
M_{11}	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_{12}	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_9	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B	L^+	R^+	B
M_9	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
M_8	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B	B	B	L^+	R^+	B
								$\vdots$									
R_{29}	B	B	B	B	B	B	B	B	B	B	B	B	B	B	L^+	R^+	B
30		10	10			14	14			1	1			5	5		$\times$
31	9	9			13	13			0	0			4	4		$\times$	
32	8			12	12			16	16			3	3		$\times$		8
33			11	11			15	15			2	2		$\times$		7	7
R_{33}	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B	B

Table 18: Construction for $n = 17$

4.5 Case $n \equiv 2 \pmod 4$

We will now give the second phase of the construction for this case. It will be based on another lemma similar to Lemmas 2 and 3, except now we use two matchings whose union is a path of length $n-2$. In what follows we will denote by $M_{i,i+1}$ the matching which does not contain nodes i and $i+1$. So $M_{i,i+1} = \{< i+2j+2, i+2j+3 >, 0 \le j \le (n-4)/2\}$.

Lemma 4 *Let $n \equiv 0 \pmod 4$ and let the label list be such that node i has label B, node $i+1$ has label R^+ and node $i+2$ has label L^+. If we perform the two pairs of symmetric rounds associated to the matchings $M_{i,i+1}$ and $M_{i+1,i+2}$, then if the rounds are valid at the end of these 4 rounds the labels of the nodes will be unchanged except for nodes i, $i+1$ and $i+2$ which will get respectively labels R^-, L^- and B.*

Proof. Let us apply the sequence of four rounds consisting of the two pairs of symmetric rounds associated to the matchings $M_{i,i+1}$ and $M_{i+1,i+2}$, whose union forms a path of length $n-2$ between $i+2$ and i. Then, at the end of these four rounds a node $\neq i, i+1, i+2$ will have received exactly one message from the left and one from the right and so its label remains unchanged.

Node $i+1$ is not involved in a call but its label changed from R^+ to L^-. Indeed the common value which was α at the end of round t is now $\alpha+1$ at the end of round $t+4$ ($l_i^t = \alpha, r_i^t = \alpha+1$, but $l_i^{t+4} = \alpha' - 1, r_i^{t+4} = \alpha'$ with $\alpha' = \alpha+1$). Node $i+2$ has received only a message from node $i+3$ in the matching $M_{i,i+1}$ and so its label changed from L^+ to B. Node i has received only a message from node $i-1$ in the matching $M_{i+1,i+2}$ and so its label changed from B to R^-. $\diamond$

We will apply repeatedly Lemma 4 to the unique triple of nodes with successive labels BR^+L^+. Like in the case n odd we have to choose the right order to do the matchings. The last four rounds are different for $n \equiv 6, 10 \pmod{12}$ and $n \equiv 2 \pmod{12}$. The reader can follow the constructions for $n = 10$ (Table 19) and $n = 14$ (Table 20) as examples.

round \ node	0	1	2	3	4	5	6	7	8	9
R_5	R^+	L^+	R^+	L^+	R^+	L^+	R^+	B	B	B
$M_{9,0}$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$
R_7	R^+	B^+	B^+	B^+	B^+	B^+	B^+	R^+	L^+	B
$M_{0,1}$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_9	L^-	B	R^+	L^+	R^+	L^+	R^+	B	B	R^-
$M_{1,2}$	$\rightarrow$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
$M_{2,3}$	$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	L^-	R^-	L^-	B	R^+	L^+	R^+	B	B	R^-
$M_{3,4}$	$\rightarrow$	$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
$M_{4,5}$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$	
R_{17}	L^-	R^-	L^-	R^-	L^-	B	R^+	B	B	R^-
$M_{5,6}$	$\rightarrow$	$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\leftarrow$
$M_{7,8}$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$	
R_{21}	L^-	R^-	L^-	R^-	L^-	R^-	L^-	L^-	R^-	R^-

Table 19: Construction for $n = 10$

4.5.1 Subcase $n \equiv 6, 10 \pmod{12}$

In that case at round 5 the label list consists by Proposition 4 of $(n-4)/2$ pairs of R^+L^+ followed by R^+BBB. We first do the two rounds associated to the matching $M_{n-1,0}$; they are valid as by Proposition 2, the labels of the end nodes of the edges of this matching are L^+R^+ and one BB. Then we do the two rounds associated to the matching $M_{0,1}$ also valid as the labels of the end nodes of the edges of this matching are B^+B^+ or B^+R^+ or L^+B. By Lemma 4the labels of nodes $n-1, 0, 1$ have changed to R^-L^-B. Therefore, at round 9 the label list starting at node $n-1$ (be careful)consists of one pair of L^-R^- then a B followed by $(n-6)/2$ pairs of R^+L^+ and R^+BB. Then we apply at round $5+4h$ $(0 \leq h \leq (n-4)/2)$ the two rounds associated to the matchings $M_{2h-1,2h}$ and then those associated to $M_{2h,2h+1}$. One can check that the rounds are valid and that at the end of round $2n-3$, the label list starting at the node $n-1$ consists of $\frac{n-4}{2}$ pairs R^-L^- followed by BR^+BB. We do once more the two rounds associated to the matchings $M_{n-5,n-4}$ and then those associated to $M_{n-4,n-3}$. The labels of nodes $n-5$, $n-4$, and $n-3$ are BR^+B and these labels become $R^-L^-L^-$; indeed node $n-5$ (resp. $n-3$) received from the left (resp. right) and node $n-4$ did nothing and the commun value α increased by one. Therefore the label list starting at the node $n-1$ consists of $(n-2)/2$ pairs of R^-L^- followed by L^-B. Note that the node $n-2$ should have formally label B after these four rounds, but this node has received twice the same last message and so the last call was useless and its label is as indicated in the table R^-.

4.5.2 Subcase $n \equiv 2 \pmod{12}$

In the case $n \equiv 2 \pmod 4$ (see for $n = 14$, Table 20), the construction is similar to that of the preceding case (the final 4 labels being $BBBB$ instead of R^+BBB). So at the end of round $2n-3$, the label list starting at node $n-1$ consists of $(n-4)/2$ pairs R^-L^- followed by $BBBB$. To finish the protocol, we have to use a tricky construction. We still do two pairs of symmetric rounds as the following. For the rounds $2n-2$ and $2n-1$, the calls are $\{< 2,1 >< 3,4 >, \ldots, < n-7, n-6 >, < n-3, n-4 >, < n-1, 0 >\}$ (the nodes $n-5$ and $n-2$ do nothing), and for the rounds $2n$ and $2n+1$, the calls are $\{< 0,1 >< 3,2 >, \ldots < n-6, n-5 >, < n-1, n-2 >\}$ (the nodes $n-4$ and $n-3$ do nothing). We get the final label list starting at node $n-1$ which consists of $n/2$ pairs R^-L^-.

5 Conclusion

In this article we have determined the exact minimum gossiping time in a ring network with n nodes under the hypothesis of unit length messages and an interference distance $d_I = 1$. One can also try to determine the exact gossiping time for other simple topologies. We have been able to determine the exact value for the path network (easier than for the ring network due to the bottleneck in the middle of the path). The case of grids might be solvable. It will also be interesting to consider stronger interferences. For example, for $d_I = 2$ we will have to use instead of matchings induced matchings (in the case of rings it is a matching such that between two edges of the matching there is at least one uncovered node). In this case in a round without 2-calls we can have at most $n/3$ calls. So, we get a general lower bound of $3n-5$. For the protocols we can use the union of 3 matchings each with at most $n/3$ edges and so the gossiping time is $3n + 0(1)$. In order to determine the exact value we need to consider many cases according to the congruences modulo 24. Similarly for any d_I, we will

round \ node	0	1	2	3	4	5	6	7	8	9	10	11	12	13
R_5	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	B
$M_{13,0}$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\times$
R_7	R^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	B^+	L^+	R^+	L^+	B
$M_{0,1}$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_9	L^-	B	R^+	L^+	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	R^-
$M_{1,2}$	$\rightarrow$	$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\leftarrow$
$M_{2,3}$	$\longleftrightarrow$		$\times$	$\times$	$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$		$\longleftrightarrow$	
R_{13}	L^-	R^-	L^-	B	R^+	L^+	R^+	L^+	R^+	L^+	B	B	B	R^-
						$\vdots$								
R_{25}	L^-	R^-	L^-	R^-	L^-	R^-	L^-	R^-	L^-	B	B	B	B	R^-
26	8	7			12	11			2	$\times$	3		$\times$	
27			10	9			0	13		$\times$		4	$\times$	5
28		8	9			12	13			2	$\times$	$\times$	5	
29	7			10	11			0	1		$\times$	$\times$		6
R_{29}	L^-	R^-	L^-	R^-	L^-	R^-	L^-	R^-	L^-	R^-	L^-	R^-	L^-	R^-

Table 20: $n = 14$

have a gossiping time of $(d_I + 2)n + 0(1)$.

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