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A NEW PROOF OF A REDUCTION FORMULA FOR THE APPELL SERIES F_3 DUE TO BAILEY*

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Abstract. In this short note, we provide a new proof of an interesting and useful reduction formula for the Appell series F_3 due to Bailey [*On the sum of a terminating* ${}_3F_2(1)$, Quart. J. Math. Oxford Ser. (2) **4** (1953), 237–240].

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1. Introduction

Hypergeometric series and many their generalizations, including multiple series, play an important role in the applied mathematics and mathematical physics. In this short note we are interested only for multiple hypergeometric series of Appell type, precisely for Appell series F_3, which is defined by (cf. [2, 3, 6])

$$F_3(a,a',b,b';c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n}{(c)_{m+n}}\cdot\frac{w^m z^n}{m!\;n!} \tag{1.1}$$

for $|w|<1$, $|z|<1$, $c\neq 0,-1,-2,\ldots$.

As usual $(\lambda)_k$ is the well known Pochhammer symbol (or the shifted factorial or the raised factorial, since $(1)_n=n!$) defined by

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}=\lambda(\lambda+1)\cdots(\lambda+k-1).$$

A survey on multiple hypergeometric series of Appell type with a rich list of references has been recently published by Schlosser [10].

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The corresponding confluent functions or the Humbert functions [6, 7]

$$\Phi_2(a,b;c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\cdot\frac{w^m z^n}{m!\;n!},$$

$$\Phi_3(b;c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(b)_m}{(c)_{m+n}}\cdot\frac{w^m z^n}{m!\;n!},$$

and

$$\Xi_1(a,a',b;c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m}{(c)_{m+n}}\cdot\frac{w^m z^n}{m!\;n!},\quad |w|<1,$$

$$\Xi_2(a,b;c;w,z)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n}{(c)_{m+n}}\cdot\frac{w^m z^n}{m!\;n!},\quad |w|<1,$$

are connected with F_3 by the following confluence (limit) formulas:

$$\lim_{\varepsilon\to 0}F_3\left(a,a',\frac{b}{\varepsilon},\frac{b'}{\varepsilon};c;\varepsilon w,\varepsilon z\right)=\Phi_2(a,a';c;bw,b'z),$$

$$\lim_{\varepsilon\to 0}F_3\left(a,\frac{a'}{\varepsilon},\frac{b}{\varepsilon},\frac{b'}{\varepsilon};c;\varepsilon w,\varepsilon^2 z\right)=\Phi_3(a;c;bw,a'b'z),$$

and

$$\lim_{\varepsilon\to 0}F_3\left(a,a',b,\frac{b'}{\varepsilon};c;w,\varepsilon z\right)=\Xi_1(a,a'b;c;bw,b'z),\quad |w|<1,$$

$$\lim_{\varepsilon\to 0}F_3\left(a,\frac{a'}{\varepsilon},b,\frac{b'}{\varepsilon};c;w,\varepsilon^2 z\right)=\Xi_2(a,b;c;w,a'b'z)\quad |w|<1.$$

For some recent and interesting results on Appell series F_3, including limit formulas, integral representations, differentiation formulas, etc. we refer a paper by Brychkov and Saad [5].

On the other hand, Bailey [4] established the following interesting and useful reduction formula for the Appell series F_3 viz,

$$\begin{aligned}(1.2)\quad & F_3(a,b,1-a,1-b;c;x,-x)\\ &={}_4F_3\left[\begin{matrix}\frac12(a+b),1-\frac12(a+b),\frac12(1+a-b),\frac12(1-a+b)\\ \frac12,\ \ \frac12 c,\ \ \frac12 c+\frac12\end{matrix};x^2\right]\\ &-\frac{(a-b)(a+b-1)x}{c}{}_4F_3\left[\begin{matrix}\frac12(a+b+1),\frac12(3-a-b),1+\frac12(a-b),1-\frac12(a-b)\\ \frac32,\ \ \frac12 c+\frac12,\ \ \frac12 c+1\end{matrix};x^2\right]\end{aligned}$$

by employing the following sums from [4] viz,

$$\sum_{r=0}^{m}(-1)^r\binom{m}{r}(a)_r(1-a)_r(b)_{m-r}(1-b)_{m-r} = (-1)^m 2^{2m}\left(\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}m\right)_m \times\left(\frac{1}{2}-\frac{1}{2}a+\frac{1}{2}b-\frac{1}{2}m\right)_m$$

and

$$\sum_{r=0}^{2m}(-1)^r\binom{2m}{r}(a)_r(b)_r(a)_{2m-r}(b)_{2m-r} = 2^{2m}\left(\frac{1}{2}\right)_m (a)_m(b)_m(a+b+m)_m.$$

Here ${}_4F_3$ is the generalized hypergeometric function [9].

The aim of this short note is to derive Bailey's reduction formula (1.2) for the Appell series F_3 by another method. For this, we require the following results:

$$(1.3)\qquad {}_3F_2\left(\begin{matrix}A,B,C\\E,F\end{matrix};1\right) = \frac{\pi 2^{1-2C}\Gamma(E)\Gamma(F)}{\Gamma(\frac{1}{2}A+\frac{1}{2}E)\Gamma(\frac{1}{2}A+\frac{1}{2}F)\Gamma(\frac{1}{2}B+\frac{1}{2}E)\Gamma(\frac{1}{2}B+\frac{1}{2}F)}$$

provided $A+B=1$, $E+F=2C+1$ and $\mathrm{Re}(C)>0$, and

$$(1.4)\qquad {}_3F_2\left(\begin{matrix}A,B,E+n-1\\D,E\end{matrix};1\right) = \frac{\Gamma(D)\Gamma(D-A-B)}{\Gamma(D-A)\Gamma(D-B)}{}_3F_2\left(\begin{matrix}A,B,1-n\\A+B-D+1,E\end{matrix};1\right)$$

for $n\geqslant 1$.

The result (1.3) is the well known Whipple's summation theorem [1, 3] and the result (1.4) is recorded in [1, p. 93, Eq. (2.4.12)].

2. Derivation of Bailey's Result (1.2)

In order to establish Bailey's result (1.2), we proceed as follows.

Denoting the left hand side of (1.2) by S and using (1.1), we have

$$S = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(b)_n(1-a)_m(1-b)_n(-1)^n x^{m+n}}{(c)_{m+n}m!\ n!}.$$

Now replacing m by $m-n$ and using a known result recorded in Rainville [9, Lemma 10, p. 56]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m) = \sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n),$$

as well as the elementary identities

$$(a)_{m-n} = \frac{(-1)^n (a)_m}{(1-a-m)_n}, \quad (1-a)_{m-n} = \frac{(-1)^n (1-a)_m}{(a-m)_n}$$

and

$$(m-n)! = \frac{(-1)^n m!}{(-m)_n},$$

we obtain

$$S = \sum_{m=0}^{\infty} \frac{(a)_m (1-a)_m}{(c)_m} \frac{x^m}{m!} \sum_{n=0}^{m} \frac{(-m)_n (b)_n (1-b)_n}{(1-a-m)_n (a-n)_n n!},$$

i.e.,

$$S = \sum_{m=0}^{\infty} \frac{(a)_m (1-a)_m}{(c)_m} \frac{x^m}{m!} \, {}_3F_2 \left(\begin{matrix} -m, \quad b, \quad 1-b \\ 1-a-m, \quad a-m \end{matrix} ; 1 \right),$$

after summing up the inner sum.

At this step it is interesting to observe that in the ${}_3F_2$, we cannot apply the classical summation theorem due to Whipple (1.3), because here $m \in \mathbb{N}_0$. So, in order to evaluate ${}_3F_2$, we have to first make use of the transformation formula (1.4) by letting $A \to b$, $B \to 1-b$, $1-n \to -m$, $E \to a-m$ and $D \to 1+a+m$, we have

$$\begin{aligned} S \quad = \quad & \sum_{m=0}^{\infty} \frac{(a)_m (1-a)_m}{(c)_m} \frac{x^m}{m!} \frac{\Gamma(1+a-b+m)\Gamma(a+b+m)}{\Gamma(1+a+m)\Gamma(a+m)} \\ & \times {}_3F_2 \left(\begin{matrix} b, 1-b, a \\ 1+a+m, a-m \end{matrix} ; 1 \right). \end{aligned}$$

Now, the ${}_3F_2$ can be evaluated with the help of classical Whipple's summation theorem (1.3) and making use of Legendre's duplication formula for the gamma function (cf. [8, p. 110])

$$\Gamma(2z) = \frac{2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})}{\sqrt{\pi}}$$

for $z = (a+b+m)/2$ and $z = (a-b+1+m)/2$. After some algebra, we have

$$S = \sum_{m=0}^{\infty} \frac{(-1)^m 2^{2m}}{(c)_m} \cdot \frac{x^m}{m!} \cdot \frac{\Gamma(\frac{1}{2}(a+b+m))\Gamma(\frac{1}{2}(a-b+m+1))}{\Gamma(\frac{1}{2}(a+b-m))\Gamma(\frac{1}{2}(a-b-m+1))}.$$

Now, separating into even and odd powers of x and making use of the elementary identities viz,

$$(c)_{2m} = 2^{2m} \left(\frac{1}{2}c\right)_m \left(\frac{1}{2}c + \frac{1}{2}\right)_m,$$

$$(c)_{2m+1} = 2^{2m} c \left(\frac{1}{2}c + \frac{1}{2}\right)_m \left(\frac{1}{2}c + 1\right)_m,$$

$$(2m)! = 2^{2m} \left(\frac{1}{2}\right)_m m!\,,$$

$$(2m+1)! = 2^{2m} \left(\frac{3}{2}\right)_m m!\,,$$

as well as

$$\frac{\Gamma(z+m)}{\Gamma(z-m)} = (-1)^m (z)_m (1-z)_m, \quad \frac{\Gamma(z+m+1)}{\Gamma(z-m)} = (-1)^m (z+1)_m (1-z)_m,$$

we have

$$S = \sum_{m=0}^{\infty} \frac{\left(\frac{1}{2}(a+b)\right)_m \left(1-\frac{1}{2}(a+b)\right)_m \left(\frac{1}{2}(1+a-b)\right)_m \left(\frac{1}{2}(1-a+b)\right)_m}{\left(\frac{1}{2}\right)_m \left(\frac{1}{2}c\right)_m \left(\frac{1}{2}c+\frac{1}{2}\right)_m} \cdot \frac{x^{2m}}{m!}$$

$$- Cx \sum_{m=0}^{\infty} \frac{(\frac{1}{2}(a+b+1))_m (\frac{1}{2}(3-a-b))_m (1+\frac{1}{2}(a-b))_m (1-\frac{1}{2}(a-b))_m}{\left(\frac{3}{2}\right)_m (\frac{1}{2}c+1)_m (\frac{1}{2}c+\frac{1}{2})_m} \frac{x^{2m}}{m!},$$

where the constant C is given by

$$C = \frac{(a-b)(a+b-1)}{c}.$$

Finally, summing up the series, we arrive at the right hand side of the desired result.

This completes the proof of (1.2).

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