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A CHARACTERIZATION OF $U_4(2)$ BY NSE*

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Abstract. Let G be a finite group and $\omega(G)$ be the set of element orders of G. Let $k \in \omega(G)$ and m_k be the number of elements of order k in G. Let $nse(G) = \{m_k | k \in \omega(G)\}$. The aim of this paper is to prove that, if G is a finite group such that nse(G)=nse($U_4(2)$), then $G \cong U_4(2)$.

Keywords. element order; number of elements of the same order; projective special unitary group; simple K_n - group.

1. Introduction

This section contains the relevant definitions, some standard facts on nse, and a brief exposition of nse history. Throughout this paper, G is a finite group. We denote by $\pi(G)$ the set of prime divisors of $|G|$, and by $\omega(G)$, we introduce the set of order of elements from G. Set $m_k = m_k(G) = |\{g \in G | o(g) = k\}|$, and nse($G$)=$\{m_k | k \in \omega(G)\}$. In fact, m_k is the number of elements of order k in G and $nse(G)$ is the set of sizes of elements with the same order in G.

To the world's mathematics and researchers, one of the important problems in group theory is characterization of a group by a given property, that is, to prove there exists only one group with a given property (up to isomorphism). Until now, different characterizations are investigated for finite simple groups. For instance, in [21, 22] motivated by one of the Thompson's problem, the authors introduced a new characterization for the finite simple group G by $nse(G)$ and $|G|$. In fact, they proved that if G is a simple K_i- group ($i = 3, 4$), then G is characterizable by nse(G) and $|G|$ (The simple group G is called simple K_n-group if $|\pi(G)| = n$). Following this result, several groups were characterized by nse and order. For example, in [5, 11], it is proved that Suzuki group, and sporadic groups are characterizable by nse and order.

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We remark here that not all groups can be characterized by their group orders and the set nse. As an illustration, let $H_1 = C_4 \times C_4$ and $H_2 = C_2 \times Q_8$, where C_2 and C_4 are cyclic groups of order 2 and 4 respectively, and Q_8 is a quaternion group of order 8. It is easy to see that $nse(H_1) = nse(H_2) = \{1, 3, 12\}$ and $|H_1| = |H_2| = 16$ but $H_1 \ncong H_2$.

However, it is claimed that some simple groups could be characterized by exactly the set nse without considering the order of group. In fact, a finite nonabelian simple group H is called characterizable by nse, if every finite group G with $nse(G) = nse(H)$ implies that $G \cong H$. In [7, 8, 9, 10, 12, 13, 24] it is proved that the alternating groups A_n, where $n \in \{7, 8\}$, the symmetric groups S_n where $n \in \{3, 4, 5, 6, 7\}$, M_{12}, $L_2(27)$, $L_2(q)$ where $q \in \{16, 17, 19, 23\}$, $L_2(q)$ where $q \in \{7, 8, 11, 13\}$, $L_2(q)$ where $q \in \{17, 27, 29\}$, are uniquely determined by nse(G). Besides, in [1, 14, 15, 16] it is proved that $U_3(4)$, $L_3(4)$, $U_3(5)$, $L_3(5)$, are uniquely determined by nse(G). Recently, in [3, 6, 18, 19], it is proved that the simple groups $U_3(3)$, $L_3(3)$, $G_2(4)$, $L_2(3^n)$, where $|\pi(L_2(3^n))| = 4$, and $L_2(2^m)$, where $|\pi(L_2(2^m))| = 4$, are uniquely determined by nse(G). Therefore, it is natural to ask what happens with other kinds of simple groups.

In an effort to fill some of the empty ground about the characterization of simple groups by nse, in this paper we will prove the following main theorem.
Main Theorem. Let G be a group such that $nse(G) = nse(U_4(2))$. Then G is isomorphic to $U_4(2)$.

2. Notation and Preliminaries

Before we get started, let us fix some notations that will be used throughout the paper. For a natural number n by $\pi(n)$, we mean the set of all prime divisors of n, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow r-subgroup of G is denoted by P_r and by $n_r(G)$, we mean the number of Sylow r- subgroup of G. Also the largest element order of P_r is signified by $exp(P_r)$. Moreover, we denote by ϕ the Euler function. In the following, we bring some useful lemmas which be used in the proof of the main theorem.

Lemma 2.1. *[25]. Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and $|G| \leqslant s(s^2 - 1)$.*

Lemma 2.2. *[24]. Let G be a group. If $1 \neq n \in nse(G)$ and $2 \nmid n$, then the following statements hold:*
(1) $2 \,|\, |G|$;
(2) $m_2 = n$;
(3) *for any* $2 < t \in \omega(G)$, $m_t \neq n$.

Lemma 2.3. *[2]. Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \,|\, |L_m(G)|$.*

Lemma 2.4. *[23]. Let G be a group and P be a cyclic Sylow p-group of G of order p^α. If there is a prime r such that $p^\alpha r \in \omega(G)$, then $m_{p^\alpha r} = m_r(C_G(P))m_{p^\alpha}$. In particular $\phi(r)m_{p^\alpha}|m_{p^\alpha r}$, where $\phi(r)$ is the Eular function of r.*

Lemma 2.5. *[17]. Let G be a finite group and $p \in \pi(G)$ be odd. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic group and $s > 1$, then the number of elements of order n is always a multiple of p^s.*

We say that a group G acts semi regularly on set X if G acts on X in such a way that $G_x = 1$ for all $x \in X$.

Lemma 2.6. *[20]. Let the finite group G acts on the finite set X. If the action is semi regular, then $|G| \mid |X|$.*

Let us mention the structure of simple K_3-groups, that will be needed in Section 3.

Lemma 2.7. *[4]. If G is a simple K_3-group, then G is isomorphic to one of the following groups:*

$$A_5,\ A_6,\ L_2(7),\ L_2(8),\ L_2(17),\ L_3(3),\ U_3(3),\ U_4(2).$$

Lemma 2.8. *[22]. Let G be a group and M a simple K_3-group. Then $G \cong M$ if and only if the following hold: (1) $|G| = |M|$, (2) $nse(G) = nse(M)$.*

3. Main Theorem and its Proof

Suppose G is a group such that $nse(G) = nse(U_4(2))$. By Lemma 2.1, we can assume that G is finite. Let m_n be the number of elements of order n. We notice that $m_n = k\phi(n)$, where k is the number of cyclic subgroups of order n in G. In addition, we notice that if $n > 2$, then $\phi(n)$ is even. If $n \in \omega(G)$, then by Lemma 2.3 and the above argument, we have

$$\text{(3.1)} \qquad \begin{cases} \phi(n)|m_n \\ n|\sum_{d|n} m_d \end{cases}$$

In the proof of the main theorem, we often apply formula (3.1) and the above comments.

Proof of the Main Theorem. Let G be a group with

$$nse(G) = nse(U_4(2)) = \{1, 315, 800, 3780, 4320, 5184, 5760\}$$

where $U_4(2))$ is the projective special unitary group of degree 4 over field of order 2. We have divided the proof into a sequence of lemmas.

Remark 3.1. Let $2 \neq p \in \pi(G)$, by formula (3.1), $p|(1+m_p)$ and $(p-1)|m_p$, which implies that $p \in \{3, 5, 7, 17, 19\}$.

In the following lemma, we prove some basic properties of group G:

Lemma 3.1. *If $p \in \pi(G)$ and $p \in \{2, 3, 5\}$, then*

(1) $2 \in \pi(G)$ *and* $m_2 = 315$*;*

(2) $m_3 = 800$*,* $m_5 = 5184$*;*

(3) $\{5^2, 3^6, 2^9\} \bigcap \omega(G) = \emptyset$*;*

(4) $|P_2||2^9$*.*

Proof. The proof is straightforward according to Lemma 2.2, Lemma 2.3, and formula (3.1). $\square$

Lemma 3.2. $\{17, 19\} \bigcap \pi(G) = \emptyset$.

Proof. We prove that $17 \notin \pi(G)$. Conversely, suppose that $17 \in \pi(G)$. Then formula (3.1) implies $m_{17} = 5184$. On the other hand, by formula (3.1), we conclude that if $2.17 \in \omega(G)$, then $m_{2.17} \in \{800, 4320, 5184, 5760\}$ and $2.17|1 + m_2 + m_{17} + m_{2.17}(= 6300, 9820, 10684, 11260)$, which is a contradiction, and hence $2.17 \notin \omega(G)$. Since $2.17 \notin \omega(G)$, the group P_{17} acts fixed point freely on the set of elements of order 2 of G and by Lemma 2.6, $|P_{17}||m_2$, which is a contradiction. Hence, $17 \notin \pi(G)$. Similarly, we can prove that $19 \notin \pi(G)$. $\square$

To remove the prime 7, let us first show that $5 \in \pi(G)$.

Lemma 3.3. $\{5\} \bigcap \pi(G) = \{5\}$.

Proof. Assume that $5 \notin \pi(G)$.

• If $3, 7 \notin \pi(G)$, then G is a 2-group. Since $2^9 \notin \omega(G)$, we have $\omega(G) \subseteq \{1, 2, 2^2, \cdots, 2^8\}$. Hence $|G| = 2^m = 20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$, where k_1, k_2, k_3, k_4, k_5 and m are non-negative integers, and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 2$. It is obvious that $20160 \leqslant |G| \leqslant 20160 + (k_1 + k_2 + k_3 + k_4 + k_5)5760$ and so $20160 \leqslant |G| \leqslant 20160 + 2.5760$. Now, it is easily seen that the equation has no solution.

Hence 3 or 7 belongs to $\pi(G)$, and the following cases are considered.

• If $7 \in \pi(G)$, by formula (3.1) $m_7 = 5760$, then as $exp(P_7) = 7$, $|P_7||1 + m_7$ and so $|P_7| = 7$. Since $n_7 = \frac{m_7}{\phi(7)} = 2^6.3.5||G|$, it follows that $5 \in \pi(G)$, which is a contradiction.

• If $3 \in \pi(G)$, then $exp(P_3) = 3, 3^2, 3^3, 3^4, 3^5$.

⋆ If $exp(P_3) = 3$, then by Lemma 2.3, $|P_3||(1 + m_3)$ and so $|P_3||3^2$. We will consider two cases for $|P_3|$.

Case 1 If $|P_3| = 3$, then since $n_3 = \frac{m_3}{\phi(3)} = 2^3.5^3||G|$, $5 \in \pi(G)$ which is a contradiction.

Case 2 If $|P_3| = 3^2$, then since $5, 7 \notin \pi(G)$ and $\pi(G) \subseteq \{2, 3, 5, 7\}$, we can assume that $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$, and so we have

$$\omega(G) \subseteq \{1, 2, \cdots 2^8\} \cup \{3, 3.2, 3.2^2, 3.2^3, \cdots, 3.2^7\}$$

($2^8.3 \notin \omega(G)$ by formula (3.1)) and $|\omega(G)| \leqslant 17$. Therefore $20160+800k_1+3780k_2+4320k_3+5184k_4+5760k_5 = |G| = 2^a.9$ where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1+k_2+k_3+k_4+k_5 \leqslant 10$. Since $20160 \leqslant 2^a.9 \leqslant 20160+10.5760$, we have $a = 12$, or $a = 13$. If $a = 12$, then since $|P_2||2^9$, we have a contradiction. Similarly, we can rule out $a = 13$.

$\star$ If $exp(P_3) = 3^2$, then, by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2})$ and so $|P_3||3^8$. (for example, when $m_9 = 5760$). We will consider seven cases for $|P_3|$.
Case 1. If $|P_3| = 3^2$, then $n_3 = \frac{m_9}{\phi(9)}$, since $m_9 \in \{3780, 4320, 5184, 5760\}$, $n_3 = 3^2.2.5.7$, $n_3 = 2^4.3^2.5$, or $n_3 = 2^6.3.5$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^3$, since $n_3 \not\equiv 1\ (mod\, 3)$, we have a contradiction.
Case 2. If $|P_3| = 3^3$, then since $5, 7 \notin \pi(G)$, we can assume that $\{2\} \subseteq \pi(G) \subseteq \{2, 3\}$ and so we have $\omega(G) \subseteq \{1, 2, \cdots 2^8\} \cup \{3, 3.2, 3.2^2, \cdots, 3.2^7\} \cup \{3^2, 3^2.2, 3^2.2^2, \cdots, 3^2.2^7\}$ ($2^8.3 \notin \omega(G)$, $2^8.3^2 \notin \omega(G)$ by formula (3.1)) and $|\omega(G)| \leqslant 25$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.27$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 18$. Since $20160 \leqslant 2^a.27 \leqslant 20160 + 18.5760$, we have $a = 10$, $a = 11$, or $a = 12$.
If $a = 10$, then since $|P_2||2^9$, we have a contradiction. Similarly, we can rule out $a = 11$ and $a = 12$.
Case 3. If $|P_3| = 3^4$, then since $exp(P_3) = 3^2$ and $2^8.3, 2^8.9 \notin \omega(G)$, $\omega(G) \subseteq \{1, \cdots, 2^8\} \cup \{3, \cdots, 3.2^7\} \cup \{3^2, \cdots, 3^2.2^7\}$. On the other hand, if $2^8 \in \omega(G)$ since $2^8.3 \notin \omega(G)$, the group P_3 acts fixed point freely on the set of elements of order 2^8. Hence $|P_3||m_{2^8} = 5760$, which is a contradiction. Hence $2^8 \notin \omega(G)$ and $|\omega(G)| \leqslant 24$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1+k_2+k_3+k_4+k_5 \leqslant 17$. Since $20160 \leqslant 2^a.81 \leqslant 20160 + 17.5760$, we have $a = 8$, $a = 9$, or $a = 10$.
If $a = 8$, then $576 = 800k_1 + 3760k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 17$. By a computer calculation, it is easy to see this equation has no solution.
If $a = 9$, then $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 17$. The only solution of this equation is $(0, 0, 0, 3, 1)$. We show this is impossible. Since $|\omega(G)| = 11$ and $2^8 \notin \omega(G)$, $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$. Hence, if $exp(P_2) = 2^i$ where $3 \leqslant i \leqslant 7$, then $|P_2||(1+m_2+m_4+\cdots+m_{2^i})$ by Lemma 2.3. In fact $|P_2||(1+315+800t_1+3780t_2+4320t_3+5184t_4+5760t_5)$ where t_1, t_2, t_3, t_4, t_5, are non-negative integers and $0 \leqslant t_1 + t_2 + t_3 + t_4 + t_5 \leqslant 6$. Because $k_1 = 0$ and $m_3 = 800$, $m_{2^i} \neq 800$ for $1 \leqslant i \leqslant 7$, $t_1 = 0$. Since $k_2 = 0$, $0 \leqslant t_2 \leqslant 1$. We claim $t_2 = 0$. Suppose, contrary to our claim, $t_2 = 1$. If $m_4 = 3780$, then since $m_9 \in \{3780, 4320, 5184, 5760\}$, we have a contradiction and so $t_2 = 0$. If $m_4 \neq 3780$, then by a computer calculation $m_8 = 3780$, since $m_9 \in \{3780, 4320, 5184, 5760\}$, we have a contradiction and so $t_2 = 0$. Also $k_3 = 0, k_4 = 3$, and $k_5 = 1$, thus $0 \leqslant t_3 \leqslant 1$, $0 \leqslant t_4 \leqslant 4$, and $0 \leqslant t_5 \leqslant 2$. By an easy computer calculation, this is impossible.

If $a = 10$, then since $|P_2||2^9$, we have a contradiction.
Similarly, we can rule out the other cases.

$\star$ If $exp(P_3) = 3^3$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3})$ and so $|P_3||3^4$ (for example when $m_9 = 5184$ and $m_{27} = 5760$). We will consider two cases for $|P_3|$.
Case 1. If $|P_3| = 3^3$, then $n_3 = \frac{m_{27}}{\phi(27)}$, since $m_{27} \in \{3780, 4320, 5184, 5760\}$, $n_3 = 2.3.5.7$, $n_3 = 2^4.3.5$, or $n_3 = 2^6.5$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3^2$, since $n_3 \not\equiv 1 \ (mod\, 3)$, we have a contradiction.
Case 2. If $|P_3| = 3^4$, then by Lemma 2.5, $27|m_{27}$. Since $(27 \not| 5760)$, it is understood that $m_{27} \in \{3780, 4320, 5184\}$). Since $2^8.3 \notin \omega(G)$, $2^8.3^2 \notin \omega(G)$, $2^8.3^3 \notin \omega(G)$, and $2^8 \notin \omega(G)$, $|\omega(G)| \leqslant 32$. Therefore $20160 + 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5 = |G| = 2^a.81$, where k_1, k_2, k_3, k_4, k_5, and a are non-negative integers and $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. Since $20160 \leqslant 2^a.81 \leqslant 20160 + 25.5760$, we have $a = 8$, $a = 9$, or $a = 10$.
If $a = 8$, then $576 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. By a computer calculation, it is easily seen that the equation has no solution.
If $a = 9$, then $21312 = 800k_1 + 3780k_2 + 4320k_3 + 5184k_4 + 5760k_5$ where $0 \leqslant k_1 + k_2 + k_3 + k_4 + k_5 \leqslant 25$. By a computer calculation, the only solution of this equation is $(0, 0, 0, 3, 1)$. We show this is impossible. Since $|\omega(G)| = 11$ and $2^8 \notin \omega(G)$, $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$. Hence, if $exp(P_2) = 2^i$, where $3 \leqslant i \leqslant 7$ then $|P_2||(1 + m_2 + m_4 + \cdots + m_{2^i})$ by Lemma 2.3. In fact $|P_2||(1 + 315 + 800t_1 + 3780t_2 + 4320t_3 + 5184t_4 + 5760t_5)$ where t_1, t_2, t_3, t_4, t_5, are non-negative integers and $0 \leqslant t_1 + t_2 + t_3 + t_4 + t_5 \leqslant 6$. Because $k_1 = 0$ and $m_3 = 800$, $m_{2^i} \neq 800$ for $1 \leqslant i \leqslant 7$, $t_1 = 0$. Since $k_2 = 0$, $0 \leqslant t_2 \leqslant 1$. We claim $t_2 = 0$. Suppose, contrary to our claim, $t_2 = 1$. If $m_4 = 3780$, then since $m_{27} \in \{3780, 4320, 5184\}$, we have a contradiction and so $t_2 = 0$. If $m_4 \neq 3780$, then by computer calculation $m_8 = 3780$, since $m_{27} \in \{3780, 4320, 5184\}$, we have a contradiction and so $t_2 = 0$. Also $k_3 = 0, k_4 = 3$, and $k_5 = 1$, thus $0 \leqslant t_3 \leqslant 1$, $0 \leqslant t_4 \leqslant 4$, and $0 \leqslant t_5 \leqslant 2$. By an easy computer calculation, this is impossible.
If $a = 10$, then since $|P_2||2^9$, we have a contradiction.

$\star$ If $exp(P_3) = 3^4$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4})$ and so $|P_3||3^4$ (for example when $m_9 = 5760$, $m_{27} = 3780$,and $m_{81} = 4320$).
If $|P_3| = 3^4$, then $n_3 = \frac{m_{81}}{\phi(81)}$, since $m_{81} \in \{3780, 4320, 5184\}$, $n_3 = 2^4.5$, or $n_3 = 2.5.7$, and so $5 \in \pi(G)$, which is a contradiction, and if $n_3 = 2^5.3$, since a cyclic group of order 81 has two elements of order 3, $m_3 \leqslant 2^5.3.2 = 192$, which is a contradiction.

$\star$ If $exp(P_3) = 3^5$, then by Lemma 2.3, $|P_3||(1 + m_3 + m_{3^2} + m_{3^3} + m_{3^4} + m_{3^5})$ and so $|P_3||3^5$ (for example when $m_9 = 5184$, $m_{27} = 5760$,and $m_{81} = m_{243} = 5184$). In a similar way we have a contradiction. Therefore, $5 \in \pi(G)$. □

Lemma 3.4. $\{7\} \bigcap \pi(G) = \emptyset$.

Proof. By Lemma 2.3 $|P_5||1 + m_5$ and so $|P_5| = 5$. In the following, that the prime 7 do not belong to $\pi(G)$ is proved. Let $7 \in \pi(G)$. Then formula (3.1) implies

$m_{7.5} \in \{4320, 5184, 5760\}$ and $7.5|1+m_5+m_7+m_{5.7}(= 15256, 16129, 16705)$, which is a contradiction, and hence $5.7 \notin \omega(G)$. It follows that the Sylow 7-subgroup of G acts fixed point freely on the set of elements of order 5 and so $|P_7||m_5$, which is a contradiction. Hence $7 \notin \pi(G)$. □

From what has already been proved, we conclude $2{,}5 \in \pi(G)$, so the following cases will be considered $\{2, 5\}, \{2, 3, 5\}$.

Lemma 3.5. $\pi(G) = \{2, 3, 5\}$.

Proof. If $\pi(G) = \{2, 5\}$, since $exp(P_5) = 5$, then by Lemma 2.3, $|P_5||1 + m_5$, and so $|P_5| = 5$. Since $n_5 = \frac{m_5}{\phi(5)} = 2^4.3^4$, it follows that 3 belongs to $\pi(G)$, which is a contradiction. Hence $\pi(G) = \{2, 3, 5\}$. The proof is completed by showing that $|G| = |U_4(2)|$. □

Lemma 3.6. $G \cong U_4(2)$.

Proof. First, we show that $|G| = |U_4(2)|$. From the above arguments, we have $|P_5| = 5$. Now, we prove $10 \notin \omega(G)$. Conversely, suppose that $10 \in \omega(G)$. Then formula (3.1) implies $m_{10} \in \{800, 3780, 4320, 5760\}$. On the other hand, if $2.5 \in \omega(G)$, then by Lemma 2.4, $m_{2.5} = m_5.\phi(2).t$ for some integer t, which is a contradiction and hence $2.5 \notin \omega(G)$. Since $2.5 \notin \omega(G)$, the group P_2 acts fixed point freely on the set of elements of order 5, and so $|P_2||m_5$, hence $|P_2||3^4.2^6$. In fact $|P_2||2^6$. In the same way, since $15 \notin \omega(G)$, $|P_3||m_5$ and hence $|P_3||3^4.2^6$. In fact $|P_3||3^4$. Therefore we have $|G| = 2^m.3^n.5$. Since $20160 = 2^6.3^2.5.7 \leqslant |G| = 2^m.3^n.5$, $|G| = 2^6.3^4.5$. Hence $|G| = 2^6.3^4.5 = |U_4(2)|$ and by assumption $nse(G) = nse(U_4(2))$, so by Lemma 2.8, $G \cong U_4(2)$ and the proof is completed. □

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