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ON A DESIGN FROM PRIMITIVE REPRESENTATIONS OF THE FINITE SIMPLE GROUPS

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Abstract. In this paper we present a design construction from primitive permutation representations of a finite simple group G. The group G acts primitively on the points and transitively on the blocks of the design. T he construction has this property that with some conditions we can obtain t-design for $t \geq 2$. We examine our design for fourteen sporadic simple groups. As a result we found a 2-(176,5,4) design with full automorphism group M_{22}.
Keywords. primitive permutation; group; finite simple group; automorphism group.

1. Introduction

Designs are interesting combinatorial objects. They have important applications in coding theory and information theory. Constructing combinatorial designs by using finite permutation groups is a well-studied subject. Well-known methods to construct 1-design from primitive permutation representations of finite simple groups were introduced by Moori and Key [10, 11]. Also in [5] a generalization of the construction in [10] was described. Here we present a design construction from primitive representations of a finite simple group. The groups we consider are primitive on the points and transitive on the blocks of constructed designs. In some conditions we can construct t-designs for $t \geq 2$. We employ this method on some simple groups and calculate full automorphism groups of constructed designs.

Sporadic simple groups are interesting family of the finite simple groups. Designs that are invariant under sporadic groups or have full automorphism group equal to sporadic groups are very interesting. Some of these designs were presented in [2, 8, 10, 12, 13]. Here we consider fourteen sporadic groups for our purpose. For these sporadic groups we obtained some designs for which the full automorphism groups are the same as the sporadic group or double cover of that. These constructed

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designs from sporadic groups are usually 1-design or 2-design and in some cases groups act primitively on the points and the blocks of the designs.

In Section 2, first we present some preliminary definitions and lemmas which will be used in the proof of our main results. In Theorem 2.1 we give our design construction from primitive permutation representations of a finite simple group. Then some properties of these design are considered. One advantage of our design in comparison with the designs presented in [5, 10, 11] is that we can determine some conditions to construct t-design for $t \geq 2$. In Proposition 2.3 these conditions are determined. Applying this result on some finite simple groups we found some 2-designs from 1-transitive actions of these simple groups. In Section 3 we describ constructed designs from fourteen sporadic simple groups. Especially we make use of large sporadic simple groups Co_3 and Fi_{23}. These groups are full automorphism group of the constructed designs which act primitively on the points and the blocks. In Section 4 we present a 2-(176,5,4) design from Mathieu group M_{22}. With the best of knowledge, this design is new and group M_{22} acts primitively on the points and transitively on the blocks of this design. The full automorphism group of this design is isomorphic to M_{22}.

2. Design Construction

In this paper all groups are assumed to be finite. Our notations are standard and for design are from [3] and for group theory and character theory are from [6, 9]. For the name and structure of finite simple groups we use the Atlas notation [4]. All computations were done with GAP [14] and Magma [1]. All programs are accessible from the author upon request.

Let t, λ, v and k be integers such that $1 \leq t \leq k \leq v$ and $\lambda > 0$. Let X be an v-set. A t-(v, k, λ) design is a pair $D = (X, B)$ such that B is a collection of k-subsets of X called blocks such that every t-subset of X appears in exactly λ blocks. A design is called simple if it has no repeated blocks. All designs in this paper are simple. The design D is called symmetric if the number of points and the blocks are equal.

An automorphism of D is a permutation f on X such that $f(b) \in B$ for each $b \in B$. A group whose elements are automorphism of D is called an automorphism group of D. We use $Aut(D)$ to denote the full automorphism group of D.

Let G be a finite permutation group acting on a set X. The orbit of $x \in X$ is defined as $O(x) = \{x^g | g \in G\}$ and the stabilizer subgroup of x is $G_x = \{g \in G | x^g = x\}$. It is well-known that $|G| = |O(x)|.|G_x|$. For $g \in G$, the conjugacy class of g is $cl(g) = \{a^{-1}ga | a \in G\}$. It is well-known that $|G| = |cl(g)|.|C_G(g)|$ such that $C_G(g) = \{a \in G | ag = ga\}$ is centralizer subgroup of g in G.

Let G be a finite group and H be a subgroup of G. Assume that Ω is the set of all conjugates of H in G. Let $\chi_H = \chi(G|H)$ be the permutation character corresponding to the action of G on Ω. For $g \in G$ if $cl(g) \cap H = \emptyset$ then $\chi_H(g) = 0$.

In what follows we present some lemmas that are used in constructing design.

Lemma 2.1. *[6, Corollary 1.5A] Let G be a group acting transitively on a set Ω with at least two points. Then action of G on Ω is primitive if and only if for each $x \in \Omega$, G_x is a maximal subgroup of G.*

Lemma 2.2. *[7, Corollary 3.1.3] Let G be a finite group and H a subgroup of G containing a fixed element x. Then the number h of conjugates of H in G containing x is given by*

$$h = [N_G(H) : H]^{-1} \sum_{i=1}^{m} \frac{|C_G(x)|}{|C_H(x_i)|}$$

where $x_1, ..., x_m$ are representatives of H-conjugacy classes that fuse to the G-class $cl(x)$.

Lemma 2.3. *[9, Corollary 5.14] Let G be a finite group and H be a subgroup of G. Let Ω be the set of all conjugates of H in G. Then for all $g \in G$, $\chi_H(g)$ is equal to the number of points in Ω fixed by g.*

Let G be a finite simple group and M be a maximal subgroup of G. Let Ω be the set of all conjugates of M in G. In the action of G on Ω, the stabilizer subgroup of each point of Ω is conjugate to M. For $g \in G$ let $cl(g) \cap M \neq \emptyset$. Then by Lemma 2.2 each $g \in G$ is in $r = \sum_{i=1}^{m} \frac{|C_G(g)|}{|C_M(g_i)|}$ conjugates of M, where $g_1, ..., g_m$ are representatives of the M-conjugacy classes that fuse to the G-class $cl(g)$. Then by Lemma 2.3, $\chi_M(g) = r$. Let nX be a conjugacy class of the elements of order n such that $nX \cap M \neq \emptyset$. Consider $g \in nX$. Then g is contained in r conjugates of M. We define $B_{g,M} = \{M_1, ..., M_r\}$ such that M_i for $i \in \{1, 2, ..., r\}$, is a conjugate of M for which $g \in M_i$. Also set $S_{g,M} = \cap_{i=1}^{r} M_i$. Clearly for $g, g' \in nX$ if the subgroups $S_{g,M}$ and $S_{g',M}$ are conjugate, then $|S_{g,M} \cap nX| = |S_{g',M} \cap nX|$.

Lemma 2.4. *Let G be a finite simple group and M be a maximal subgroup of G. Let nX be a conjugacy class of the elements of order n such that $nX \cap M \neq \emptyset$. Then for each $g, g' \in nX$ the following hold:*

1. *$B_{g,M} = B_{g',M}$ if and only if $S_{g,M} \cap nX = S_{g',M} \cap nX$,*
2. *if $S_{g,M} \cap S_{g',M} \cap nX \neq \emptyset$ then $S_{g,M} \cap nX = S_{g',M} \cap nX$,*
3. *$\frac{|nX|}{|S_{g,M} \cap nX|}$ is a positive integer,*
4. *$\frac{|M \cap nX|}{|S_{g,M} \cap nX|}$ is a positive integer.*

Proof. (1) Let $B_{g,M} = B_{g',M}$. Then obviously $S_{g,M} \cap nX = S_{g',M} \cap nX$. Now suppose $S_{g,M} \cap nX = S_{g',M} \cap nX$. Then $g, g' \in S_{g,M}$ and $g, g' \in S_{g',M}$. If $B_{g,M} \neq B_{g',M}$ without lose of generally, there exists M', a conjugate of M such that $M' \in B_{g,M}$ but $M' \notin B_{g',M}$. Now since $g' \in S_{g,M}$ then $g' \in M'$ and $M' \in B_{g',M}$ which is a contradiction. (2) Suppose $x \in S_{g,M} \cap S_{g',M} \cap nX$. So $x \in S_{g,M}$

and $x \in S_{g',M}$. Then we have $B_{x,M} = B_{g,M} = B_{g',M}$ and by (1) the result is achieved. (3) By (2) each element of nX is contained in a unique $S_{g,M} \cap nX$. Then $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$, which is a disjoint union of the elements of class nX for some $S_{g_i,M}$ for $i \in \{1, 2, ..., k\}$. Therefore $k = \frac{|nX|}{|S_{g,M} \cap nX|}$ is positive integer. (4) According to the proof of (3) we have $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$. For each $i \in \{1, 2, ..., k\}$, $S_{g_i,M} \cap M \cap nX = S_{g_i,M} \cap nX$ or $S_{g_i,M} \cap M \cap nX = \emptyset$. Therefore $nX \cap M = \cup_{j=1}^{h}(S_{g_j,M} \cap nX)$ such that for each $j \in \{1, 2, ..., h\}$ we have $S_{g_j,M} \cap M \cap nX = S_{g_j,M} \cap nX$ hence $h = \frac{|M \cap nX|}{S_{g,M} \cap nX}$ is positive integer. □

Now we are ready to present a design construction from primitive permutation representations of a finite simple group.

Theorem 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. Set $B = \{B_{x,M} | x \in nX\}$. Then $D = (\Omega, B)$ is a,*

$$1 - ([G : M], \chi_M(g), \frac{|M \cap nX|}{|S_{g,M} \cap nX|})$$

design which has $\frac{|nX|}{|S_{g,M} \cap nX|}$ blocks. Also G is an automorphism group of D which acts primitively on the points and transitively on the blocks of D.

Proof. In the action of G on Ω, the stabilizer of each point of Ω is conjugate to M. Since $B_{g,M}$ is the collection of all conjugates of M that contain g, by Lemma 2.3 the length of a block is $\chi_M(g)$. Clearly $|S_{g,M} \cap nX| \geq 1$. If $|S_{g,M} \cap nX| = 1$ then by Lemma 2.4(1) for every $g', g'' \in nX$ we have $B_{g',M} \neq B_{g'',M}$ and so $|B| = |nX|$. If $|S_{g,M} \cap nX| = m > 1$ then by Lemma 2.4(1) for m elements $g_1, ..., g_m$ of $S_{g,M} \cap nX$ we have $B_{g_1,M} = B_{g_2,M} = ... = B_{g_m,M}$. So according to the proof of Lemma 2.4(3), $nX = \cup_{i=1}^{k}(S_{g_i,M} \cap nX)$ for $i \in \{1, 2, ..., k\}$, which is a disjoint union of the elements of $S_{g_i,M} \cap nX$. Then for each $S_{g_i,M} \cap nX$ there exists a unique block. Hence the number of different blocks is equal to $\frac{|nX|}{|S_{g,M} \cap nX|}$ which by Lemma 2.4(3) is a positive integer. By the proof of Lemma 2.4(4) $M \cap nX = \cup_{j=1}^{h}(S_{g_j,M} \cap nX)$ for $j \in \{1, 2, ..., h\}$, which is a disjoint union of the elements of $S_{g_j,M} \cap nX$. Then for each $j \in \{1, 2, ..., h\}$, subgroup M is in a unique block. Then each point appears in exactly $\frac{|M \cap nX|}{|S_{g,M} \cap nX|}$ blocks which by Lemma 2.4(4) is a positive integer. The group G acts on the points and the blocks with conjugation. Since M is maximal then by Lemma 2.1, G acts primitively on the points. Each block is corresponding to an element of nX and since G is transitive on nX then is transitive on the blocks. □

We denote the constructed design in Theorem 2.1 by $D(G, M, nX)$. The design $D(G, M, nX)$ is not necessary symmetric and the action of G on the blocks is not necessary primitive. In the following propositions we consider some conditions to achieve these properties.

Proposition 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. If $C_G(g)$ is maximal subgroup then the action of G on the blocks of $D(G, M, nX)$ is primitive.*

Proof. Since $C_G(g)$ is maximal, by Lemma 2.1 the action of G on right cosets of $C_G(g)$ is primitive. Each block is corresponding to an element of nX and respectively to a right coset of $C_G(g)$. Then the action of G on the blocks of $D(G, M, nX)$ is primitive. □

Let $g \in nX$ and consider $D(G, M, nX)$. According to the definition, for every $x \in C_G(g)$ we have $B_{g,M} = B_{g^x,M} = (B_{g,M})^x$. Therefore $C_G(g)$ is a subgroup of stabilizer of the block $B_{g,M}$.

Proposition 2.2. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. Then the following hold,*

1. *if $C_G(g)$ is conjugate to M and $|S_{g,M} \cap nX| = 1$ then $D(G, M, nX)$ is symmetric and G acts on the blocks primitively,*
2. *if $|S_{g,M} \cap nX|.|C_G(g)| = |M|$ then $D(G, M, nX)$ is symmetric.*

Proof. (1) The number of points and blocks in $D(G, M, nX)$ are $\frac{|G|}{|M|}$ and $\frac{|nX|}{|S_{g,M} \cap nX|} = \frac{|G|}{|S_{g,M} \cap nX|.|C_G(g)|}$, respectively. Since $C_G(g)$ is conjugate to M then $|C_G(g)| = |M|$ and the number of points and blocks are equal. Also by Proposition 2.1 the action of G on blocks is primitive. (2) In this case clearly the number of points and blocks are equal and $D(G, M, nX)$ is symmetric. □

In Table 2.1, 2.2 and 2.3 the columns from left are: number of row, the considered finite simple group G, maximal subgroup M of G, a conjugacy class of G, properties of the constructed design from G, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

Example 2.1. In Table 2.1 $D(G, M, nX)$ for some finite simple groups was constructed. By [4] in $L_2(11)$ centralizer of an element of class $2A$ is maximal subgroup D_{12}. The design $D(L_2(11), D_{12}, 2A)$ in row 3 of the table is a 1-(55,7,7) design with 55 blocks then is symmetric and $L_2(11)$ acts on the blocks primitively. This design is an example such that satisfies Lemma 2.2(1). In $G_2(3)$ for maximal subgroup $M = (3^{1+2} \times 3^2) : 2S_4$, consider $D(G_2(3), M, 3A)$. Let $g \in 3A$, in this case $|S_{g,M} \cap 3A| = 2$ and $|C_{G_2(3)}(g)| = \frac{|M|}{2}$ then $D(G_2(3), M, 3A)$ in row 2 of the table is a 1-(364,13,13) symmetric design. This is an example that satisfies Lemma 2.2(2).

Always t-designs with $t \geq 2$ are interesting. In two following propositions we consider some conditions to construct t-designs for $t \geq 2$.

Table 2.1: $D(G, M, nX)$ for some finite simple groups

No.	G	M	nX	$D(G, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	A_9	$L_2(8):3$	$2B$	1-(120,8,9)	135	A_9	NO
2	$G_2(3)$	$(3^{1+2} \times 3^2):2S_4$	$3A$	1-(364,13,13)	364	$G_2(3)$	YES
3	$L_2(11)$	D_{12}	$2A$	1-(55,7,7)	55	$L_2(11):2$	YES
4	$S_6(2)$	$2.[2^6]:(S_3 \times S_3)$	$2B$	1-(315,43,43)	315	$S_6(2)$	YES
5	$U_5(2)$	$(2^{1+6} \times 3^{1+2}):2A_4$	$2A$	1-(165,37,37)	165	$U_5(2)$	YES
6	$U_3(3)$	$4^2:S_3$	$3B$	1-(63,3,16)	336	$S_6(2)$	NO
7	$O_8^+(2)$	$(3 \times U_4(2)):2$	$6A$	1-(1120,8,360)	50400	$O_8^+(2)$	NO
8	$O_8^+(2)$	$(3 \times U_4(2)):2$	$3A$	1-(1120,40,40)	1120	$O_8^+(3):4$	YES

Proposition 2.3. *Let G be a finite simple group, M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. Let $m \in \{1, 2, ..., \chi_M(g)\}$. If intersection of every m different conjugates of M has $f \geq 1$ elements of the class nX then $D(G, M, nX)$ is an $m - ([G : M], \chi_M(g), \frac{f}{|S_{g,M} \cap nX|})$ design. Also G is an automorphism group of $D(G, M, nX)$ such that acts primitively on the points and transitively on the blocks of $D(G, M, nX)$.*

Proof. Consider an m-set of different conjugates of M, set S intersection of these subgroups. By the proof of Lemma 2.4(4) we have this partition $S \cap nX = \cup_{j=1}^{h}(S_{g_j,M} \cap nX)$. Therefore for each $j \in \{1, 2, ..., h\}$ these m conjugates of M are in a unique block. Then every m conjugates of M appears in exactly $\frac{|S \cap nX|}{|S_{g,M} \cap nX|} = \frac{f}{|S_{g,M} \cap nX|}$ blocks and result is concluded. □

Example 2.2. In Table 2.2 we consider some finite simple groups in their 1-transitively action such that satisfy conditions of Proposition 2.3.

Table 2.2: Constructed $D(G, M, nX)$ from Proposition 2.3

No.	G	M	nX	$D(G, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	$S_4(3)$	$3^{1+2}:2A_4$	3A	2-(40,13,4)	40	$L_4(3):2$	YES
2	$S_4(4)$	$2^6:(3 \times A_5)$	2A	2-(85,21,5)	85	$L_4(4):2$	YES
3	$S_4(5)$	$5^{1+2}:4A_5$	5A	2-(156,31,6)	156	$L_4(5):4$	YES
4	$L_2(8)$	D_{18}	2A	2-(28,4,1)	63	$L_2(8):3$	NO
5	$L_2(16)$	D_{34}	2A	2-(120,8,1)	255	$L_2(16):4$	NO

Corollary 2.1. *Let G be a finite simple group. Let M be a maximal subgroup of G and Ω be the set of all conjugates of M in G. Let nX be a conjugacy class of the elements of order n such that $M \cap nX \neq \emptyset$ and $g \in nX$. For $t > 1$ let the action of G on Ω be t-transitive. For $m \in \{1, 2, ..., t\}$ consider an m-set of different conjugates of M and set S intersection of these subgroups. If $S \neq \langle 1 \rangle$ and $|S \cap nX| = k \geq 1$ then $D(G, M, nX)$ is an $m - ([G : M], \chi_M(g), \frac{k}{|S_{g,M} \cap nX|})$ design*

and G is an automorphism group of $D(G, M, nX)$ that acts t-transitively on the points and transitively on the blocks.

Proof. Since G is t-transitive on Ω then intersection of any m-set of different conjugates of M is conjugate to S and result is concluded by Proposition 2.3. □

Example 2.3. In Table 2.3 we consider some finite simple groups in their 2-transitively action such that satisfy conditions of Corollary 2.1.

Table 2.3: Constructed $D(G, M, nX)$ from Corollary 2.1

No.	G	M	nX	$D(G, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	$L_2(11)$	A_5	2A	2-(11,3,3)	55	$L_2(11)$	NO
2	$L_3(3)$	$3^2:2S_4$	2A	2-(13,5,15)	117	$L_3(3)$	NO
3	$L_3(3)$	$3^2:2S_4$	3A	2-(13,4,1)	13	$L_3(3)$	YES
4	$L_3(5)$	$5^2:GL_2(5)$	2A	2-(31,7,35)	775	$L_3(5)$	NO
5	$L_3(5)$	$5^2:GL_2(5)$	5A	2-(31,6,1)	31	$L_3(5)$	YES
6	$S_6(2)$	$U_4(2):2$	2A	2-(28,16,20)	63	$S_6(2)$	NO

3. Constructed Designs from Sporadic Groups

In this section we construct some designs from fourteen sporadic simple groups. For each considered sporadic group, we present one or two designs that their full automorphism groups are as the same sporadic group. These results are presented in Table 3.1. For information on the sporadic simple groups and their maximal subgroups we use Atlas [4].

In Table 3.1 the columns from left are: number of row, group G, maximal subgroup M of G, a conjugacy class of G, properties of the constructed design from G, number of the blocks of the design, full automorphism group of the design and symmetric property of the design.

For instance we study properties of the designs in row 15 and 18 of Table 3.1.

The Conway group Co_3 has order $495766656000 = 2^{10}.3^7.5^3.7.11.23$. The group Co_3 has forty two conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The centralizer subgroup of an elements of class $2A$ in Co_3 is maximal subgroup isomorphic to $2.S_6(2)$. The group $McL:2$ is maximal subgroup of index 276 and Co_3 acts 2-transitive on conjugates of $McL:2$. We consider design $D = D(Co_3, McL:2, 2A)$. Intersection of every two different maximal subgroups conjugate to $McL:2$ have 2835 elements of class $2A$ therefore by Corollary 2.1 D is a 2-(276,36,2835) design. The design D has 170775 blocks and since centralizer of an element of the class $2A$ is maximal hence by Proposition 2.1 group Co_3 acts primitively on the blocks of this design. Also Co_3 acts 2-transitively on the points of D. The full automorphism group of D is Co_3.

Table 3.1: Designs constructed from some sporadic groups

No.	G	M	nX	$D(M_{11}, M, nX)$	No. Blocks	$Aut(D)$	Symm
1	M_{11}	$L_2(11)$	2A	3-(12,4,3)	165	M_{11}	NO
2	M_{11}	$2.S_4$	2A	1-(165,13,13)	165	M_{11}	YES
3	M_{12}	$M_9 : S_3$	3A	1-(220,4,4)	220	M_{12}	YES
4	M_{22}	A_7	4B	1-(176,4,315)	13860	M_{22}	NO
5	M_{23}	M_{22}	3A	4-(23,5,16)	28336	M_{23}	NO
6	M_{23}	$L_3(4) : 2$	2A	1-(253,29,435)	3795	M_{23}	NO
7	J_1	$2 \times A_5$	2A	1-(1463,31,31)	1463	J_1	YES
8	J_1	$D_6 \times D_{10}$	2A	1-(2926,46,23)	1463	J_1	NO
9	J_3	$L_2(19)$	2A	1-(14688,96,171)	26163	J_3	NO
10	HS	$U_3(5) : 2$	2B	2-(176,12,66)	15400	HS	NO
11	McL	M_{22}	2A	1-(2025,105,1155)	22275	McL	NO
12	McL	$2^4 : A_7$	2A	1-(22275,435,435)	22275	McL	YES
13	He	$2^6 : 3.S_6$	$2A$	1-(29155,651,558)	24990	He	NO
14	Suz	$U_5(2)$	3A	1-(32760,252,176)	22880	$Suz : 2$	NO
15	Co_3	$McL : 2$	$2A$	2-(276,36,2835)	170775	Co_3	NO
16	Fi_{22}	$O_7(3)$	$2A$	1-(14080,1408,351)	3510	Fi_{22}	NO
17	Co_2	$U_6(2) : 2$	$2A$	1-(2300,284,7029)	56925	Co_2	NO
18	Fi_{23}	$2.Fi_{22}$	$2A$	1-(31671,3511,3511)	31671	Fi_{23}	YES

The Fischer sporadic simple group Fi_{23} has order $4089470473293004800 = 2^{18}.3^{13}.5^2.7.11.13.17.23$. The group Fi_{23} has ninety eight conjugacy classes of elements and fourteen conjugacy classes of maximal subgroups. The group $2.Fi_{22}$ is maximal subgroup of index 31671 and also is centralizer subgroup of an element of the class $2A$. By Proposition 2.2(1) $D(Fi_{23}, 2.Fi_{22}, 2A)$ is a symmetric $1-(31671, 3511, 3511)$ design. The group Fi_{23} acts primitively on the points and the blocks of $D(Fi_{23}, 2.Fi_{22}, 2A)$. The full automorphism group of $D(Fi_{23}, 2.Fi_{22}, 2A)$ is isomorphic to Fi_{23}.

4. A 2-design invariant under M_{22}

The Mathieu sporadic group M_{22} has order $443520 = 2^7.3^2.5.7.11$. The group M_{22} has twelve conjugacy classes of elements and eight conjugacy classes of maximal subgroups [4].

Consider permutation representation of M_{22} on 176 conjugates of maximal subgroup A_7. The group M_{22} has one conjugacy class of elements of order 3. The centralizer of an element of the class $3A$ is isomorphic to $3 \times A_4$. Each elements of class $3A$ is contained in 5 conjugates of maximal subgroup A_7.

Proposition 4.1. *The $D(M_{22}, A_7, 3A)$ is a 2-(176,5,4) design with full automorphism group isomorphic to M_{22}.*

Proof. Let M_1 and M_2 be two different maximal subgroups isomorphic to A_7. Then $M_1 \cap M_2$ is isomorphic to $3^2{:}4$ or S_4. Both subgroups $3^2{:}4$ and S_4 have 8 elements of order 3. Also for each $g \in 3A$, S_{g,A_7} is subgroup of order 3. Hence by Proposition

2.3 $D(M_{22}, A_7, 3A)$ is a 2-(176,5,4) design. The automorphism group is calculated by GAP [14]. □

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REFERENCES

1. W. Bosma and J. J. Cannon: *Handbook of Magma functions*. School of Mathematics and Statistics, University of Sydney, Sydney (1995).
2. P. J. Cameron and A. Rudvalis: *A design and a geometry for the group Fi_{22}*. Des. Codes Cryptogr. **44** (2007), no. 1-3, 11-14.
3. P. J. Cameron and J. H. van Lint: *Designs, Graphs, Codes and their Links*. Cambridge University Press, 1991.
4. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson: *An Atlas of Finite Groups*. Oxford: Oxford University Press, 1985.
5. D. Crnković, V. Mikulić and B. G. Rodrigues: *Designs, strongly regular graphs and codes constructed from some primitive groups*. In Information Security, Coding Theory and Related Combinatorics (D. Crnkovic & V. Tonchev, Eds.), IOS Press, Amsterdam, 2011, pages 231-252.
6. J.D. Dixon and B. Mortimer: *Permutation Groups*. Graduate Texts in Mathematics 163, Springer-Verlag, New York, 1996.
7. M.S. Ganief: *2-Generations of the Sporadic Simple Groups*. PhD Thesis, University of Natal, 1997.
8. W.H. Haemers, C. Parker, V. Pless and V. D. Tonchev: *A design and a code invariant under the simple group Co_3*. J. Combin. Theory Ser. A, **62** (2) (1993), 225-233.
9. I.M. Isaacs: *Character Theory of Finite Groups*. Academic Press, San Diego, 1976.
10. J.D. Key and J. Moori: *Designs, codes and graphs from the Janko groups J_1 and J_2*. J. Combin. Math. and Combin. Comput., **40** (2002), 143–159.
11. J. Moori: *Finite groups, designs and codes*. Information security, coding theory and related combinatorics, 202-230, NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., **29**, IOS, Am- sterdam, 2011.
12. J. Moori and B. G. Rodrigues: *Some designs and codes invariant under the simple group Co_2*. J. of Algerbra, **316** (2007), 649-661.
13. J. Moori and B. G. Rodrigues: *On some designs and codes invariant under the Higman-Sims group*. Util. Math. **86** (2011), 225-239.
14. The GAP Team: *GAP - Groups, Algorithms, and Programming*. Version 4.5.5, 2012, (`http://www.gap-system.org`).

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