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On Optimal Layer Reinsurance Model

Antonella Campana

Department of Economics
University of Molise
Campobasso, Italy

Paola Ferretti

Department of Economics
Ca' Foscari University of Venice
Venice, Italy

Abstract

In this paper, we consider the class of non-proportional reinsurance contracts known as layer reinsurance model or limited stop-loss treaty. With the aim of finding an optimal layer reinsurance, we make the choice of considering an optimization criteria preserving stop-loss order: we derive some conditions of optimality by minimizing insurer risk exposure under a generic concave distortion risk measure.

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1 Introduction

Optimal reinsurance has been treated in the literature under different perspectives, nevertheless the topic is still of interest both for researchers and practitioners. Some studies have been devoted to find and to investigate the optimal reinsurance model by extending both the considered premium principles and the risk measures.

In the work of Chi and Tan [5] some optimal reinsurance models based on two specific risk-measure are considered and the robustness of the optimal reinsurance over a prescribed class of premium principles is analysed.

After a brief introductory paragraph on the properties of distorted risk measurements, in this paper the analysis focuses on the results proposed by Chi and Tan [5] with reference to the layer reinsurance and for this class of reinsurance contracts we derive some optimality conditions.

In Section 2, we present some notations and preparatory results, in order to illustrate the model of reinsurance introduced in [5] and here generalised. In Section 3 the risk exposure of the insurer, in the presence of reinsurance and according to a distortion risk measure, is examined and discussed. In Section 4 optimality of stop-loss reinsurance is recalled and some conditions of optimality are derived by comparing layer reinsurance models. In Section 5 some concluding remarks are proposed.

2 Preparatory settings and results

Some notations, abbreviations and conventions used throughout the paper are the following. F_X denotes the one-dimensional cumulative distribution function (cdf) of the real-valued random variable (r.v.) X, with $F_X(x) = Pr\{X \le x\}$. It is assumed $E[X] < \infty$.

In reinsurance literature, often the proposed models make reference to a distortion risk measure defined on a distortion function g, that is a non-decreasing function $g : [0,1] \to [0,1]$ where $g(0) = 0$ and $g(1) = 1$. The distortion risk measure of a non-negative random variable X associated to a distortion function g is denoted by $\rho_g(\cdot)$ and is defined by (see [7]):

$$\rho_g(X) = \int_0^\infty g(1 - F_X(t))dt. \tag{1}$$

In particular, distortion risk measures such as *Value-at-Risk* (VaR) and *Conditional Value at Risk* (CVaR) are extensively used within banking and insurance sectors for quantifying market risks.

In general, a distortion risk measure $\rho_g(\cdot)$ obeys the properties of translation invariance, additivity for comonotonic risks and positive homogeneity. In particular, we will use the known result (see Theorem 7 in [7]) for which it is:

For any random pair (X, Y), X is smaller than Y in stochastic order, write $X \le_{st} Y$, if and only if their respective distortion risk measures are ordered:

$$X \le_{st} Y \Leftrightarrow \rho_g(X) \le \rho_g(Y) \quad \textit{for all distortion functions } g. \tag{2}$$

A subclass of distortion functions that is often considered in the literature is the class of concave distortion functions; a risk measure with a concave distortion function is called a *concave distortion risk measure*. Note that the very-well known distortion risk measures VaR and CVaR perform in different ways with respect to concavity, in fact VaR is not a concave distortion risk measure whereas CVaR is a concave distortion risk measure.

Moreover, the stop-loss order can be characterized in terms of ordered concave distortion risk measures. In fact, it is (see Theorem 8 in [7])

For any random pair (X, Y), X is smaller than Y in the stop-loss order, write $X \leq_{sl} Y$, i.e.

$$E[(X-d)_+] \leq E[(Y-d)_+], \quad \forall d \in \mathfrak{R}$$

(where $(x-a)_+ = \max(x-a, 0)$), if and only if their respective concave distortion risk measures are ordered:

$$X \leq_{sl} Y \Leftrightarrow \rho_g(X) \leq \rho_g(Y) \text{ for all concave distortion functions } g. \tag{3}$$

3 Insurer risk exposure

In the insurance framework, let X be a non-negative r.v. denoting the loss initially assumed by an insurer in the absence of reinsurance.

Let f be a non-negative function defined for all possible outcomes of X, which represents the reinsured amount; this function is known as ceded loss function and is such that $0 \leq f(x) \leq x$. In this way the insurance company exposure to loss is reduced by passing part of the risk of loss to a reinsurer (or a group of reinsurers). Let $R_f(x)$ be the retained loss function, that is $R_f(x) = x - f(x)$.

A simple reinsurance contract can be represented by the risk sharing scheme $(f(X), R_f(X))$, where $f(X)$ denotes the amount assured by the reinsurer and $R_f(X)) = X - f(X)$ is the residual loss covered by the ceding company.

Under the reinsurance arrangement, the risk exposure of the ceding company is no longer captured by X but it is the sum of the retained loss and the incurred reinsurance premium.

Let π denote the reinsurance premium principle: $\pi : \chi \to \mathfrak{R}_+$, where χ is the set of all non-negative random variables with finite expectation. The reinsurance premium is a function of the loss ceded to the reinsurer and it is defined by $\pi(f(X))$.

The total risk exposure $T_f(X)$ of the insurer is given by:

$$T_f(X) = R_f(X) + \pi(f(X)) \tag{4}$$

Recently, the problem of optimal reinsurance have been addressed in the hypothesis of different risk measures of the risk exposure of the insurer and under different premium principles. The optimal ceded loss function is the one that minimizes an appropriately chosen risk measure ρ_g on $T_f(X)$ under a given premium principle for the reinsurance premium $\pi(f(X))$: the ρ_g-based optimal reinsurance model.

The optimal reinsurance model first considered by Cai and Tan [2, 3] and Chi et al. [4, 5] and recently generalised in [1]), is based on VaR and CVaR. The Authors considered a reinsurance premium computed under a principle satisfying three basic axioms, that is, *distribution invariance, risk loading* and *stop-loss ordering preserving*. The Authors assumed that $f(x)$ and $R_f(x)$ are non-decreasing functions on the set of all the possible outcomes x of X and so they considered the following set of admissible ceded loss functions:

$$\mathcal{C} = \{f : 0 \leq f(x) \leq x \wedge \text{both } R_f(x) \text{ and } f(x) \text{ are non-decreasing functions}\}$$

For any ceded loss function $f \in \mathcal{C}$, a layer reinsurance contract can be defined as (Chi et al. [5]):

$$h_f(x) = \min\{(x-a)_+, b\} \tag{5}$$

with the deductible $a \geq 0$ and the upper limit $b \geq 0$. The parametric values a and b can be computed for VaR and CVaR by setting the condition:

$$\rho_g(h_f(X)) = \rho_g(f(X)). \tag{6}$$

Chi and Tan in [5] proved that the layer reinsurance (5), with parameters a and b computed by (6), satisfies the inequality $h_f(X) \leq_{sl} f(X)$ and minimises VaR and CVaR of $T_f(X)$ (that is the total exposure to risk) in the set $\mathcal{C}$:

$$\rho_g(T_{h_f}(X)) \leq \rho_g(T_f(X)) \quad \forall f \in \mathcal{C}. \tag{7}$$

Note that the properties of ρ_g ensure

$$\begin{aligned} \rho_g(T_{h_f}(X)) &= \rho_g(R_{h_f}(X)) + \pi(h_f(X)) & (8)\\ &= \rho_g(X) - \rho_g(h_f(X)) + \pi(h_f(X)), & (9) \end{aligned}$$

and the inequality (7) results to be equivalent to

$$\rho_g(h_f(X)) - \pi(h_f(X)) \geq \rho_g(f(X)) - \pi(f(X)). \tag{10}$$

Given (6), the inequality (7) is equivalent to

$$\pi(f(X)) \geq \pi(h_f(X)). \tag{11}$$

The inequality (11) follows from the stop-loss preserving property satisfied by the reinsurance premium principle π.

4 Comparing layer reinsurance models

Let us consider the set $\mathbf{C}$ of the loss functions so defined:

$$\mathbf{C} = \{f \in C^1 : f(0) = 0 \quad \wedge \quad 0 \le f'(x) \le 1\} \tag{12}$$

In the set $\mathbf{C}$, the *crossing condition for reinsurance contracts* is satisfied (see [8]):

Theorem 4.1. *Let $f_1(x)$ and $f_2(x)$ be two elements of $\mathbf{C}$ with $E[f_1(X)] \ge E[f_2(X)]$. If there exists a value $s \ge 0$ such that $f_1(x) \le f_2(x)$ for $0 \le x \le s$ and $f_1(x) \ge f_2(x)$ for $x > s$, then $R_{f_1}(X) \le_{sl} R_{f_2}(X)$.*

By Theorem 4.1, it is $R_{f_1}(X) \le_{sl} R_{f_2}(X)$ and by (3), the inequality

$$\rho_g(R_{f_1}(X)) \le \rho_g(R_{f_2}(X))$$

can be deduced. Then, the next result can be stated.

Theorem 4.2. *Let $f_1(x)$ and $f_2(x)$ be two elements of $\mathbf{C}$ with $E[f_1(X)] \ge E[f_2(X)]$. If there exists $s \ge 0$ such that $f_1(x) \le f_2(x)$ for $0 \le x \le s$ and $f_1(x) \ge f_2(x)$ for $x > s$, then for any concave distortion risk measure ρ_g it is*

$$\rho_g(R_{f_1}(X)) \le \rho_g(R_{f_2}(X)). \tag{13}$$

It is then possible to generalise to a concave distortion risk measure the well-known result about optimality of stop-loss reinsurance among the reinsurance treaties with equal expected reinsurance benefit.
Let us define the following subset of $\mathbf{C}$:

$$\mathbf{C}_\mu = \{f \in \mathbf{C} : E(f(X)) = \mu\}.$$

Let f_d be the stop-loss reinsurance of the form $f_d(x) = (x-d)_+$ where $d \ge 0$ is such that $E(f_d(X)) = \mu$, i.e. $f_d \in \mathbf{C}_\mu$. Then, it is

Theorem 4.3. *If $\rho_g(\cdot)$ is a concave distortion risk measure then*

$$\rho_g(R_{f_d}(X)) \le \rho_g(R_f(X)) \qquad \forall f \in \mathbf{C}_\mu \tag{14}$$

Proof. For every $f \in \mathbf{C}_\mu$ it is $f'_d(x) \le f'(x)$ when $x \le d$ and $f'_d(x) \ge f'(x)$ when $x > d$. Then, there is no more than one point where $f_d(x)$ and $f(x)$ cross. By Theorem 4.1, it is necessarily

$$R_{f_d}(X) \le_{sl} R_f(X)$$

and by theorem 4.2, it follows that

$$\rho_g(R_{f_d}(X)) \le \rho_g(R_f(X)).$$

□

Moreover, given the reinsurance premium principle π, let us consider the subset of $\mathbf{C}_\mu$ so defined:

$$\mathbf{C}_{\mu,\theta} = \{f \in \mathbf{C}_\mu : \pi(f(X)) = \theta\}$$

where $\theta \geq \mu$. Then the next result can be deduced.

Theorem 4.4. *For the ρ_g-based optimal reinsurance model, the stop-loss reinsurance of the form $f_d(x) = (x-d)_+ \in \mathbf{C}_{\mu,\theta}$, where $d \geq 0$, is optimal, namely it is:*

$$\rho_g(T_{f_d}(X)) \leq \rho_g(T_f(X)) \qquad \forall f \in \mathbf{C}_{\mu,\theta}. \tag{15}$$

Proof. By Theorem 4.3 and definition (4), by translation invariance property of ρ_g, inequality (15) follows.

□

In the set $\mathbf{C}_\mu$ it is possible to consider layer reinsurance functions, that is

$$h_{a,b}(x) = \min\{(x-a)_+, b\}, \tag{16}$$

where parameters $a \geq 0$ and $b > 0$ must satisfy the condition

$$E(h_{a,b}(X)) = \int_a^{a+b} (1 - F(x))dx = \mu \tag{17}$$

in order to ensure that $h_{a,b} \in \mathbf{C}_\mu$. By comparing two layer reinsurances in $\mathbf{C}_\mu$, it is possible to obtain the next result.

Theorem 4.5. *Given h_{a_1,b_1}, $h_{a_2,b_2} \in \mathbf{C}_\mu$ with $a_1 \leq a_2$, it is*

$$R_{h_{a_2,b_2}}(X) \leq_{sl} R_{h_{a_1,b_1}}(X) \tag{18}$$

Proof. By condition (17) characterising layer reinsurances in $\mathbf{C}_\mu$ and by non decreasing monotonicity of $F(x)$, if $a_1 \leq a_2$ then $b_1 \leq b_2$. Moreover, it is $h_{a_1,b_1}(x) \geq h_{a_2,b_2}(x)$ for $x \leq a_1 + b_1$ and $h_{a_1,b_1}(x) \leq h_{a_2,b_2}(x)$ for $x \geq a_1 + b_1$. By Theorem 4.1, inequality (18) follows.

□

Note that for any concave distortion risk measure ρ_g , by inequality (18) it follows that $\rho_g(R_{h_{a_2,b_2}}(X)) \leq_{sl} \rho_g(R_{h_{a_1,b_1}}(X))$.

By comparing stop-loss reinsurance $f_d \in \mathbf{C}_\mu$ with layer reinsurance and by referring to Theorem 4.3 it is $R_{f_d}(X) \leq_{sl} R_{h_{a,b}}(X)$, $\forall h_{a,b} \in \mathbf{C}_\mu$. Moreover, if the upper limit $b > 0$ is finite, in order to have $E[h_{a,b}(X)] = E[f_d(X)] = \mu$, necessarily the deductible a must satisfy $a \leq d$. Then, by Theorem 4.5, the next result follows:

For an upper finite limit $0 < \bar{b} < \infty$, the ρ_g-based optimal layer reinsurance $h_{a,\bar{b}} \in \mathbf{C}_\mu$ is the one with the deductible $\hat{a} = \max\{a \leq d : E[h_{a,\bar{b}}] = \mu\}$. For a deductible $\bar{a} \leq d$, the ρ_g-based optimal layer reinsurance $h_{\bar{a},b} \in \mathbf{C}_\mu$ is the one with the upper limit $\hat{b} = \max\{b < \infty : E[h_{\bar{a},b}] = \mu\}$.

5 Concluding remarks

In the comparison between the reinsurance treaties, the stochastic stop loss system plays "of course" a fundamental role: a reinsurance contract is preferred to another when it produces a lower withheld risk in the order of stop-loss (see [8]). Consequently, the optimization criteria that preserve stop-loss ordering must be preferred. From this point of view, among risk measures, the concave risk measures result to be appealing: they characterise the stop-loss ordering. For these reasons, in this work we focused on the study of layer reinsurance treaties generally used in reinsurance practice. Therefore, we have obtained optimality conditions by minimizing the risk exposure of insurer under a concave distortion risk measure. The choice of a layer reinsurance model is based on deductible and upper bound present in the corresponding ceded loss function. Following this approach, different layer reinsurance models can be subsequently analysed.

References

[1] Campana A., Ferretti P., On optimal reinsurance with stochastic premium, *Applied Mathematical Sciences*, **13** (18) (2019), 859-867. https://doi.org/10.12988/ams.2019.97102

[2] Cai, J., Tan, K.S., Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures, *Astin Bulletin*, **37** (1) (2007), 93-112. https://doi.org/10.1017/s0515036100014756

[3] Cai, J., Tan, K.S., Weng, C., Zhang, Y., Optimal reinsurance under VaR and CTE risk measures, *Insurance: Mathematics and Economics* **43** (1) (2008), 185-196. https://doi.org/10.1016/j.insmatheco.2008.05.011

[4] Chi, Y., Tan, K. S., Optimal reinsurance under VaR and CVaR Risk measures: a simplified approach, *Astin Bulletin*, **41** (2) (2011), 487-509.

[5] Chi, Y., Tan, K. S., Optimal reinsurance with general premium principles, *Insurance: Mathematics and Economics,* **52** (2013), 180-189. https://doi.org/10.1016/j.insmatheco.2012.12.001

[6] Chi, Y., Weng, C., Optimal reinsurance subject to Vajda condition, *Insurance: Mathematics and Economics,* **53** (1) (2013), 179-189. https://doi.org/10.1016/j.insmatheco.2013.05.002

[7] Dhaene, J., Vanduffel, S., Goovaerts, M.J., Kaas, R., Tang, Q., Vyncke, D., Risk measures and comonotonicity: A review, *Stochastic Models,* **22** (2006), 573-606. https://doi.org/10.1080/15326340600878016

[8] Kaas, R., van Heerwaarden A.E., Goovaerts M.J., *Ordering of Actuarial Risks. Insurance: Mathematics and Economics*, CAIRE, **1** (1994).

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