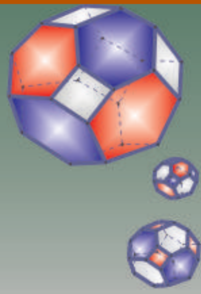
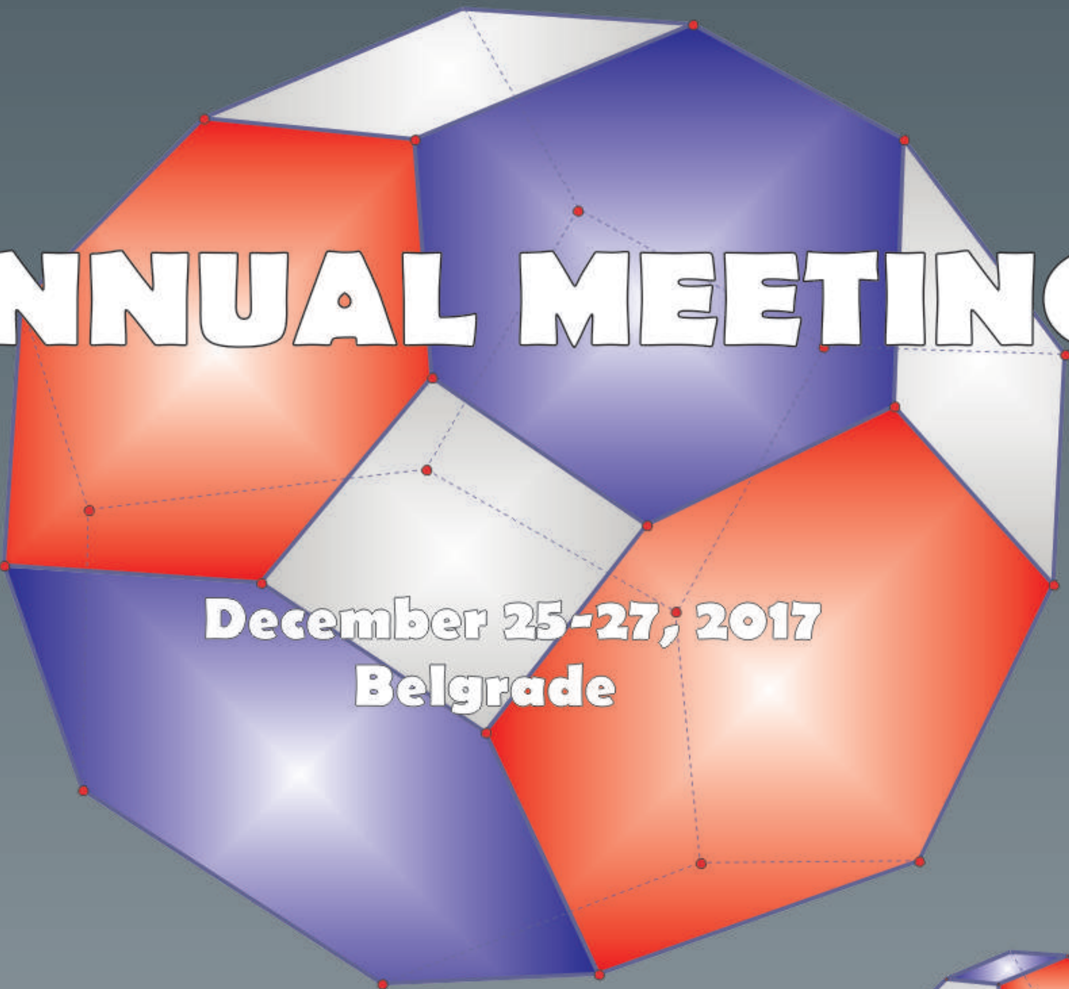
SEMINAR
TOPOLOGY OF CONFIGURATION SPACES
of the Mathematical Institute
of the Serbian Academy of Sciences & Arts



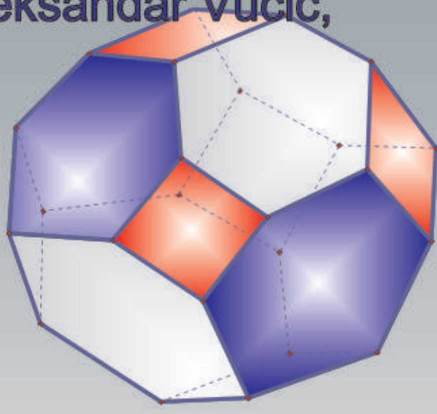


ANNUAL MEETING

December 25-27, 2017
Belgrade

Confirmed speakers

Đorđe Baralić, Jelena Grbić, Vladimir Grujić, Jelena Ivanović, Jelena Katić, Aleksandra Kostić, Edin Liđan, Nela Milošević, Luka Milićević, Jovana Nikolić, Zoran Petrović, Zoran Pucanović, Marko Radovanović, Sonja Telebaković, Aleksandar Vučić, Đorđe Žikelić



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ПРОГРАМ ГОДИШЊЕГ СУСРЕТА
СЕМИНАРА ЗА КОНФИГУРАЦИОНЕ ПРОСТОРЕ
МАТЕМАТИЧКОГ ИНСТИТУТА САНУ

Понедељак 25. децембар

10.00 Отварање Годишњег сусрета
10.15-11.00 Владимир Грујић, *Symmetric and quasisymmetric enumerators*
11.00-11.15 Пауза
11.15-12.00 Марко Радовановић, *Recurrence formulas for Kostka and inverse Kostka numbers*
12.00-14.00 Пауза
14.00-15.00 Јелена Грбић, **заједнички састанак са Одељењем за математику**
15.15-16.00 Зоран Пуцановић, *On the connection between the topological graph theory and the theory of commutative rings*

Уторак, 26. децембар

9.30-10.15 Јелена Катић, Дарко Милинковић, Јована Николић, *Spectral invariants in Floer theory*
10.20-10.50 Лука Милићевић, *Blocking points in general position*
10.50-11.05 Пауза
11.05-11.35 Александар Вучић, *Orthogonal shadows and index of Grassmann manifolds*
11.40-12.10 Един Лиђан, *Homology groups of generalized polyomino type tilings*

Среда 27. децембар

9.30-10.15 Зоран Петровић, *Associating simplicial complexes to commutative rings*
10.20-10.50 Јелена Ивановић, *A simple permutoassociahedron*
10.50-11.05 Пауза
11.05-11.35 Соња Телебаковић, *On the Brauerian Representation and 1-dimensional Topological Quantum Field Theories*
11.40-12.10 Ђорђе Баралић, *Universal simplicial complexes inspired by toric topology*

Затварање скупа и коктел

Апстракти-Abstracts

Ђорђе Баралић, Математички институт САНУ
Universal simplicial complexes inspired by toric topology

Let $\boldsymbol{k}$ be the field $\mathbf{F}_p$ or the ring $\mathbf{Z}$..We study combinatorial and topological properties of the universal complexes X $(\boldsymbol{k}^n)$ and K $(\boldsymbol{k}^n)$ whose simplices are certain unimodular subsets of $\boldsymbol{k}^n$. We calculate their $\mathbf{f}$ -vectors, show that they are shellable but not shifted, and find their applications in toric topology and number theory. Using discrete Morse theory, we detect that X$(\boldsymbol{k}^n)$, K$(\boldsymbol{k}^n)$ and the links of their simplicies are homotopy equivalent to a wedge of spheres specifying the exact number of spheres in the corresponding wedge decompositions. This is a generalisation of Davis and Januszkiewicz's result that K $(\mathbf{Z}^n)$ and K $(\mathbf{Z}_\mathbf{2}^\mathbf{n})$ are $(n-2)$-connected simplicial complexes. This is joint work with Jelena Grbić and Aleksandar Vučić.

Јелена Грбић, Универзитет у Саутхемптону, Велика Британија
Toric Topology from homotopy theory point of view

At the beginning of this millennium, Toric Topology has been recognised as a new branch of Topology closely related to Algebraic Geometry, Combinatorics and Algebra. Initially problems of Toric Topology were motivated by the study of toric geometry. The approach I take departs from geometry and brings in the tools and techniques of homotopy theory. That allows one to generalise the fundamental concepts of Toric Topology which will further have applications to geometric group theory, robotics and applied mathematics.

Владимир Грујић, Математички факултет Београд
Symmetric and quasisymmetric enumerators

We present classical and new enumerator functions that appear in algebraic combinatorics and topology. The most famous is the Stanley chromatic function of a graph.

Јелена Ивановић, Архитектонски факултет у Београду
A simple permutoassociahedron

In the early 1990s, a family of combinatorial CW-complexes named permutoassociahedra was introduced by Kapranov, and it was realized by Reiner and Ziegler as a family of convex polytopes. The polytopes in this family are `hybrids' of permutohedra and associahedra. Since permutohedra and associahedra are simple, it is natural to search for a family of simple permutoassociahedra, which is still adequate for a topological proof of Mac Lane's coherence. Such a family was presented in the paper \textit{A simple permutoassociahedron} co-authored with Zoran Petrić and Đorđe Baralić.

Јелена Катић, Дарко Милинковић, Јована Николић, Математички факултет Београд
Spectral invariants in Floer theory

We will present the construction, properties and applications of spectral invariants in Floer theory. We will also describe the construction of spectral invariants for an open subset of the base in a cotangent bundle.

Един Лиђан, Универзитет у Бихаћу, Босна и Херцеговина
Homology groups of generalized polyomino type tilings

A polyomino is a plane geometric figure formed by joining one or more equal squares edge to edge and it may be regarded as a finite subset of the regular square tiling with a connected interior. Polyomino tilling problem asks is it possible to properly cover a finite region M consisting of cells with polyomino shapes from a given set $\boldsymbol{T}$. There are a numerous generalizations of this question towards symmetrical and asymmetrical tillings, higher dimension analogs, polyomino types in other regular lattice grids (triangular, hexagonal), etc. However, the problem in all cases in general is NP-hard and we can give definite answer only in limited number of cases.
In the talk we study problem of tilling a surface S subdivided in finite `combinatorial' grid which may fail to be regular with finite set of polyomino like shapes T and define the homology group $H_S(T)$. We present some new results based on results of Conway, Lagarias and Reid, together with illustrating examples explaining the application of the homology group of generalized polyomino type tillings in combinatorial and topological context. This is joint work with Đorđe Baralić.

Лука Милићевић, Математички институт САНУ
Blocking points in general position

Erdős and Purdy asked the following question: given a set P of n points in the plane, no three collinear, how many new points do we need to take so that each line spanned by P contains a new point? It is easy to see that we always need at least n new points for odd n, and $n-1$ new points for even n. Erdős and Purdy remarked that there are examples which require less than n new points. In this talk, we show that this remark is in fact false; for all $n \geq 5$, we need at least n new points. The proof is based on a classification theorem for some related arrangements of lines in the plane, which is the other main result presented in this talk.

Зоран Петровић, Математички факултет Београд
Associating simplicial complexes to commutative rings

In order to better understand structure of a commutative ring, it is sometimes convenient to associate a simplicial complex to this ring. Some methods of doing this will be presented. In orderto establish topological properties of the associated complexes some basic methods of algebraic topology are used as well as discrete Morse theory. This is joint work with Nela Milošević.

Зоран Пуцановић, Грађевински факултет Београд

On the connection between the topological graph theory and the theory of commutative rings

Let R be a commutative ring with identity and $I^*(R)$ the set of its nontrivial ideals. The intersection graph of ideals $G(R)$ is defined as follows:

$$\mathrm{V}(G(R)) := I^*(R), \quad \mathrm{E}(G(R)) := \{\{I_1, I_2\}: I_1 \cap I_2 \neq \emptyset\},$$

where $\mathrm{V}(G(R))$ and $\mathrm{E}(G(R))$ denotes the set of the vertices (edges) of the graph $G(R)$. We try to establish some connections between commutative ring theory and topological graph theory, by study of the genus of the intersection graph of ideals and classify all graphs of genus 1 and genus 2 that are intersection graphs of ideals of some commutative rings.

Марко Радовановић, Математички факултет Београд

Recurrence formulas for Kostka and inverse Kostka numbers

In the algebra of symmetric functions the change from the basis given by Schur functions to the basis given by elementary symmetric functions involves Kostka numbers. These numbers are known to be hard to compute. Alternatively, these numbers may be seen in the cohomology of Grassmannians in the change from the basis given by Schubert classes to the one given by products of Chern classes. Therefore, obtaining suitable formulas for calculating in these bases produces relations between (inverse) Kostka numbers. In this talk we use this approach toward (inverse) Kostka numbers using quantum cohomology of Grassmannian. To be more precise,we constract a Grőbner basis for the ideal that determines quantum
cohomology of Grassmannians as given by Siebert and Tian, and use it to obtain some recurrence formulas for (inverse) Kostka numbers. Some applications of these formulas will alsobe presented.
This is joint work with Zoran Petrović.

Соња Телебаковић, Математички факултет Београд

On the Brauerian Representation and 1-dimensional Topological Quantum Field Theories

In this lecture we show that every 1-dimensional topological quantum field theory, regarded as a symmetric monoidal functor between the category of 1-cobordisms and the category of matrices, coincides with the Brauerian representation up to multiplication by invertible matrices. Since the Brauerian functor is faithful, we extend our faithfulness result to all 1-**TQFT**. This means that different 1-cobordisms correspond with distinct matrices.

Александар Вучић, Математички факултет Београд
Orthogonal shadows and index of Grassmann manifolds

In this paper we study the $\mathbf{Z}/2$ action on real Grassmann manifolds $G_n(R^{2n})$ and $\tilde{G}_n(R^{2n})$ given by taking (appropriately oriented) orthogonal complement. We completely evaluate the related $\mathbf{Z}/2$ Fadell--Husseini index utilizing a novel computation of the Stiefel-Whitney classes of the wreath product of a vector bundle. These results are used to establish the following geometric result about the orthogonal shadows of a convex body: For $n=2a(2b+1)$, $k=2^{a+1}-1$, C a convex body in R^{2n}, and k real valued functions $\alpha_1,\dots,\alpha_k$ continuous on convex bodies in R^{2n}, with respect to the Hausdorff metric, there exists a subspace $V\subseteq R^{2n}$ such that projections of C to V and its orthogonal complement $V^\perp$ have the same value with respect to each function α_i, which is $\alpha_i\ (pV(C))=\alpha_i\ (pV^\perp(C))$ for all $1\leq i\leq k$. This is joint work with Đorđe Baralić, Pavle Blagojević and Roman Karašev.

On the connection between the topological graph theory and the theory of commutative rings

Zoran S. Pucanović

University of Belgrade, Faculty of Civil Engineering

Joint work with Zoran Petrović and Marko Radovanović

December 25, 2017

Basic definitions and motivation

Definition 1

Let R be a commutative ring with identity and $I^*(R)$ the set of its nontrivial ideals. The intersection graph of ideals $G(R)$ is defined as follows:

$$V(G(R)) := I^*(R), \quad E(G(R)) := \{\{I_1, I_2\} : I_1 \cap I_2 \neq 0\},$$

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We primarily concentrate on the question of its genus.

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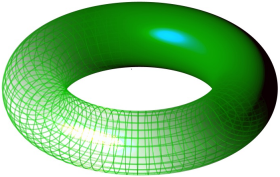

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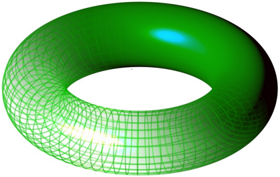

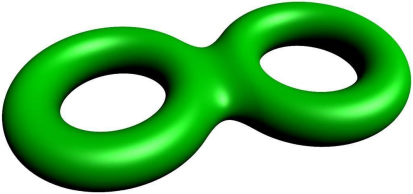

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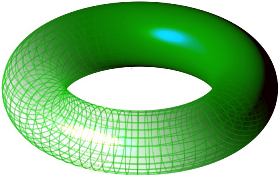
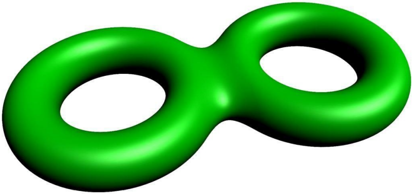


$\mathbb{S}_1$



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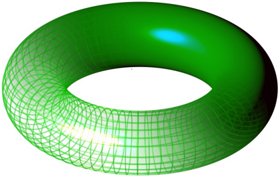

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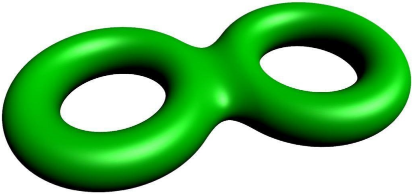

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The idea is to completely classify all graphs of genus 1 and genus 2 that are intersection graphs of ideals of some commutative rings.

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- An embedding in which all faces have boundary consisting of exactly 3 edges is called a *triangulation*.
- Since G is a simple graph, every face has at least 3 boundary edges and every edge is a boundary of 2 faces; so, $2e \geq 3f$, with equality if and only if G is a triangulation of the surface.

Some formulas

- Euler's formula: If n, e and f, are the number of vertices, edges, and faces in a cellular embedding of G in $\mathbb{S}_g$, then

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- Ringel & Jungerman: Let $\delta(\mathbb{S}_g)$ be the number of triangles in a minimal triangulation of $\mathbb{S}_g$. Then

$$\delta(\mathbb{S}_g) = 2\left\lceil \frac{7+\sqrt{1+48g}}{2} \right\rceil + 4(g-1),\ (g \neq 2), \quad \delta(\mathbb{S}_2) = 24.$$

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- $\gamma(G(R)) < \infty$ and (R, M) is local ring $\implies R/M$ is a finite field.
- (R, M) is local and M is minimally generated with k elements
  $\implies \dim(M/M^2) = k$ (over R/M).

Toroidality of $G(R)$

Theorem 2

Let R be a commutative ring with identity. Then, $\gamma(G(R)) = 1$ if and only if it is isomorphic to one of the following graphs:

$$K^5,\ K^6,\ K^7,\ \Gamma[1,2,3,4,5,a],\ \Gamma[1,2,3,4,5,6,7]-\{36,37,46,47\},$$

$$\Gamma[1,2,3,4,5,6,7,d],\ \Gamma[1,2,3,4,5,a,b],\ \Gamma[1,2,3,4,5,6,a,b,c],\ \Gamma-\{3d\}.$$

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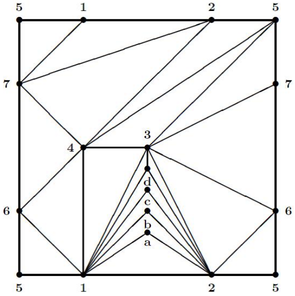

Graph Γ

Sketch of Proof

- $n \geq 3 \implies \gamma(G(R)) \neq 1$. If at least one of the rings R_i is not a field, then $\gamma(G(R)) > 1$ ($G(R)$ contains forbidden subgraph K^8 or is non-consistent with Euler's formula). Otherwise, $G(R)$ is planar.

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- $n = 2$ i.e. $R \cong R_1 \times R_2$.
  - If R_1 and R_2 are fields, then $G(R)$ is planar.

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Let us prove the following: if R_1 is not a field, then it is a PIR. Assume that its maximal ideal M_1 is not principal. Let $x \in M_1$ and $y \in M_1 \setminus \langle x \rangle$. Look at the ideals $R_1 \times 0$, $M_1 \times 0$, $\langle x \rangle \times 0$, $\langle y \rangle \times 0$, $\langle x + y \rangle \times 0$, $M_1 \times R_2$, $\langle x \rangle \times R_2$, $\langle y \rangle \times R_2$, $\langle x + y \rangle \times R_2$, and $0 \times R_2$. Estimating their degrees (for example, the degree of $\langle x \rangle \times 0$ is at least 4), we get that at the minimal case there is a subgraph of $G(R)$ with 10 vertices and 31 edges. If this subgraph were embedded into a torus, we would get $f = 21$, but this contradicts the fact that one must have $2e \geq 3f$.

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- R_1 and R_2 are PIRs with maximal ideals $M_1 = \langle x \rangle$, $M_2 = \langle y \rangle$, such that $x^2 = 0$, $y^2 = 0$. The intersection graph $G(R)$ is isomorphic to $\Gamma[1, 2, 3, 4, 5, 6, 7] - \{36, 37, 46, 47\}$ and $\gamma(G(R)) = 1$.

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- If R_1 is PIR with maximal ideal $M = \langle x \rangle$, such that $x^2 = 0$ and R_2 is a field, then $G(R)$ is planar.

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- If R_1 is PIR with maximal ideal $M = \langle x \rangle$, such that $x^k = 0$, $k \geq 5$ and R_2 is a field, then $G(R)$ contains K^8, so $\gamma(G(R)) > 1$.

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From the previous analysis we get the following proposition.

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Proposition 3

Suppose that $R \cong R_1 \times R_2$ is a product of two local Artinian rings. Then $\gamma(G(R)) = 1$ in exactly one of the following cases:

1. *One of the rings is a PIR (principal ideal ring) with maximal ideal M such that $M^4 = 0$ and the other ring is a field;*
2. *Both of the rings are PIRs and squares of their maximal ideals are zero.*

Sketch of Proof

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If R is a local ring with maximal ideal M with two generators and $G(R)$ is toroidal, then M^2 is a principal ideal.

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Proof.

Let us suppose that $M^2(\neq 0)$ is not principal. So, there exist some elements $u, v \in M^2$ such that $u \notin \langle v \rangle$ and $v \notin \langle u \rangle$. It is clear that the ideals $\langle u \rangle$, $\langle v \rangle$, $\langle u+v \rangle$, and M^2 are different. We know that M/M^2 is a union of one-dimensional subspaces. Since $|M/M^2| = |F|^2$, we conclude that there are $|F|+1$ one-dimensional subspaces of M/M^2; so there are at least 3 of them. We get three ideals I_1, I_2, and I_3 which contain $M^2(\neq 0)$. So, we have eight ideals: M, I_1, I_2, I_3, M^2, $\langle u \rangle$, $\langle v \rangle$, $\langle u+v \rangle$. The first five ideals all have degree 7 and the last three have degree (at least) 5. Looking at the subgraph of $G(R)$ induced by these ideals, we conclude that $e \geq 25$. So, $2e - 3f = 2e - 3e + 24 = 24 - e < 0$, which is impossible. □

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  - If $n = 2$ i.e. $M = \langle x, y \rangle$ and $G(R)$ is toroidal, then $M^3 = 0$ and one can choose generators x, y for M in such a way that $M^2 = \langle xy \rangle$, where $x^2 = y^2 = 0$, or $M^2 = \langle x^2 \rangle$, where $xy = 0$.

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  - If $M = \langle x, y \rangle$, then $\gamma(G(R)) = 1$ if and only if M^2 is a principal ideal and $|R/M| \leq 4$. The graph $G(R)$ is isomorphic to one of the graphs K^5, K^6, K^7, $\Gamma[1, 2, 3, 4, 5, a, b]$, $\Gamma[1, 2, 3, 4, 5, 6, a, b, c]$, or $\Gamma - \{3d\}$.

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Genus two intersection graphs

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Theorem 5

Let R be a commutative ring with identity. Then, $\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to one of the following graphs:

$K^8,\ \Gamma',\ \Gamma''[1,\dots,8,v_5,v_6]-\{13,17\},\ \Gamma''[1,\dots,8,v_1,\dots,v_5]-\{7v_5,3v_5\}.$

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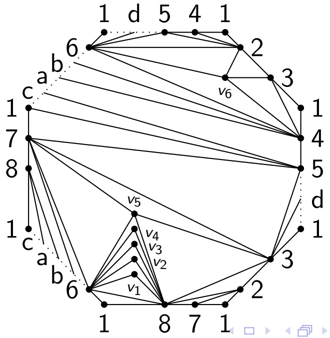
Graph Γ'' :



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- Some of these results were obtained by study of face-size distribution of graph embeddings, which is in general very difficult.
- As indicated, for example by graph Γ', obtaining an embedding of the intersection graph is not always a straight-forward task.

Note on graph Γ'

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Let R be a commutative Artinian ring. If in decomposition one has $n \geq 3$, then $\gamma(G(R)) = 2$ if and only if $G(R)$ is isomorphic to Γ'.

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- $n = 3$ *and exactly two local rings, say R_1, R_2 are not fields. Then $G(R)$ contains a forbidden subgraph K^9.*

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- $n = 3$ *and exactly two local rings, say R_1, R_2 are not fields. Then $G(R)$ contains a forbidden subgraph K^9.*
- $n = 3$, *exactly two local rings, say R_2, R_3 are fields and $M_1^2 \neq 0$. Then $G(R)$ contains a forbidden subgraph K^9.*

Note on graph Γ'

Sketch of Proof. 2

- $n = 3$, R_2, R_3 *are fields,* $M_1^2 = 0$, $M_1 = \langle x_1, \ldots, x_k \rangle$ *and* $k \geq 2$. *Then* $G(R)$ *contains a forbidden subgraph* K^9.

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Sketch of Proof. 2

- *$n = 3$, R_2, R_3 are fields, $M_1^2 = 0$, $M_1 = \langle x_1, \ldots, x_k \rangle$ and $k \geq 2$. Then $G(R)$ contains a forbidden subgraph K^9.*
- *Therefore, M_1 is a principal ideal, say $M_1 = \langle x \rangle$, and $x^2 = 0$.*

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- *The intersection graph $G(R)$ contains ten vertices:*
- *$v_1 = \langle x \rangle \times 0 \times 0$, $v_2 = \langle x \rangle \times R_2 \times 0$, $v_3 = \langle x \rangle \times 0 \times R_3$, $v_4 = \langle x \rangle \times R_2 \times R_3$, $v_5 = R_1 \times 0 \times 0$, $v_6 = R_1 \times R_2 \times 0$, $v_7 = R_1 \times 0 \times R_3$, $v_8 = 0 \times R_2 \times 0$, $v_9 = 0 \times 0 \times R_3$ and $v_{10} = 0 \times R_2 \times R_3$.*

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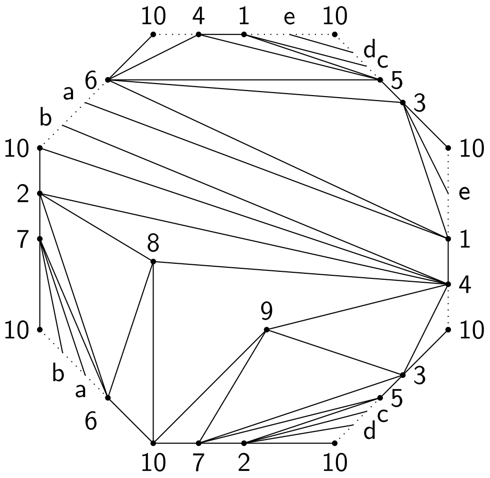
- *We obtain that $G(R)$ is isomorphic to Γ' (isomorphism is given by $v_i \mapsto i$), which can be embedded in $\mathbb{S}_2$ as shown on next figure.*

Note on graph Γ'



Graph Γ'

Some additional results

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On lower bounds for the genus of the intersection graphs of nonlocal rings

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Theorem 7

Let R be a commutative Artinian ring and R_i local Artinian rings such that $R \cong R_1 \times \cdots \times R_k$, where $k \geq 2$. Genus of the intersection graph of a nonlocal ring R is at least

$$\min\left\{\frac{\alpha}{8} \cdot N^{\frac{2k-2}{k}} \cdot (N^{1/k} - \alpha) - \frac{N}{2} + 1, \beta \cdot N^2 - \frac{N}{2} + 1, \frac{(N-6)(N-8)}{48}\right\},$$

where $N = |V(G(R))|$, $\alpha = 2k\left(\frac{1}{3}\right)^{\frac{k-1}{k}}$ and $\beta = \frac{3^k - 2^k - 1}{4\cdot(2\cdot 3^k - 2^{k+1} - 1)^2}$.

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Theorem 8

For every $g > 0$, there are only finitely many nonisomorphic graphs of genus g that are intersection graphs of some rings.

Thank you for your attention