EXPLICIT SPECTRUM OF A CIRCULANT-TRIDIAGONAL MATRIX WITH APPLICATIONS

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ABSTRACT. We consider a circulant-tridiagonal matrix and compute its determinant by using generating function method. Then we explicitly determine its spectrum. Finally we present applications of our results for trigonometric factorizations of the generalized Lucas sequences.

1. INTRODUCTION

Tridiagonal matrices have been used in many different fields, especially in applicative fields such as numerical analysis (e.g., orthogonal polynomials), engineering, telecommunication system analysis, system identification, signal processing (e.g., speech decoding, deconvolution), special functions, partial differential equations and naturally linear algebra (see [1, 3, 4, 12, 16, 17]). Some authors consider a general tridiagonal matrix of finite order and then compute its LU factorizations, determinant and inverse (see [2, 5, 8, 13]).

A tridiagonal Toeplitz matrix of order n has the form:

$$A_n = \begin{bmatrix} a & b & & & 0 \\ c & a & b & & \\ & c & a & & \\ & & \ddots & \ddots & b \\ 0 & & & c & a \end{bmatrix},$$

where a, b and c's are nonzero complex numbers.

A tridiagonal 2-Toeplitz matrix has the form:

$$T_n = \begin{bmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots \\ c_1 & a_2 & b_2 & 0 & 0 & \cdots \\ 0 & c_2 & a_1 & b_1 & 0 & \cdots \\ 0 & 0 & c_1 & a_2 & b_2 & \cdots \\ 0 & 0 & 0 & c_2 & a_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where a, b and c's are nonzero complex numbers.

Let a_1, a_2, b_1 and b_2 be real numbers. The period two second order linear recurrence system is defined to be the sequence $f_0 = 1,\ f_1 = a_1$, and

$$f_{2n} = a_2 f_{2n-1} + b_1 f_{2n-2} \text{ and } f_{2n+1} = a_1 f_{2n} + b_2 f_{2n-1}$$

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for $n \geq 1$. Let $D = a_1 a_2 + b_1 + b_2$ and $D^2 - 4b_1 b_2 \neq 0$.

Gover [6] and Marcellán and Petronilho [15] showed that the eigenvalues of matrix T_{2n+1} are a_1 and

$$\frac{a_1 + a_2}{2} \pm \sqrt{\left(\frac{a_1 - a_2}{2}\right)^2 + b_1 c_1 + b_2 c_2 + 2\sqrt{b_1 b_2 c_1 c_2} \cos \frac{k\pi}{n+1}}$$

for $1 \leq k \leq n$.

They also gave a closed equation for the eigenvalues of T_{2n} : They are the solutions of the following quadratic equations

$$(\lambda - a_1)(\lambda - a_2) - \left[b_1 c_1 + b_2 c_2 + \sqrt{b_1 b_2 c_1 c_2} z_{nk}\right] = 0, \ k = 1, 2, \ldots, n,$$

where $z_{nk}, \ k = 1, 2, \ldots, n$, are the zeros of the polynomial $R_n(z)$ defined by

$$R_{n+1}(x) = x R_n(x) - R_{n-1}(x), \quad n \geq 1$$

with initials $R_0(x) = 1, \ R_1(x) = x + \beta$ where $\beta^2 = b_2 c_2 / b_1 c_1$.

Meanwhile, a matrix C_n is called a circulant matrix if it has the form

$$C_n = \begin{bmatrix} a_1 & a_2 & a_3 & & a_{n-1} & a_n \\ a_n & a_1 & a_2 & & a_{n-2} & a_{n-1} \\ a_{n-1} & a_n & a_1 & & a_{n-3} & a_{n-2} \\ & & & & & \\ a_3 & a_4 & a_5 & & a_1 & a_2 \\ a_2 & a_3 & a_4 & & a_n & a_1 \end{bmatrix}.$$

Circulant matrices are a special type of Toeplitz matrix and have many interesting properties. Circulant matrices have been used in many areas such as physics, differential equations and digital image processing. Also circulant and skew circulant matrices have become an important tool in networks engineering.

Define generalized Fibonacci and Lucas sequences by the recursion for $n > 1$

$$U_n = P U_{n-1} - Q U_{n-2} \quad \text{and} \quad V_n = P V_{n-1} - Q V_{n-2},$$

with the initials $U_0 = 0, U_1 = 1$, and, $V_0 = 2, V_1 = P$, resp. When $P = 1$ and $Q = -1$, $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

Recently some authors have studied various interesting combinatorial matrices defined by terms of certain sequences. We could refer to the works [10, 14, 11, 18] for details about combinatorial matrix examples: In [9], the authors consider skew circulant type matrices with any continuous Fibonacci numbers. Then they discuss the invertibility of the skew circulant type matrices and present explicit determinants and inverse matrices of them.

In this paper, we consider a circulant-tridiagonal matrix and then compute its determinant by using generating function method. Then we explicitly determine its all eigenvalues. We show that determinant of the matrix satisfies a period second order recurrence system. We give applications for trigonometric factorizations of the Lucas sequences.

2. The main results

We define a tridiagonal matrix $H_n = [h_{ij}]$ of order n with $h_{11} = a$, $h_{i+1,i} = h_{i,i+1} = b$ for odd i, $h_{i,i+1} = h_{i+1,i} = a$ for even i and $h_{nn} = a$ if n is even and $h_{nn} = b$ if n is odd.

Then the matrix H_{2n} takes the form

$$H_{2n}=\begin{bmatrix} a & b & & & & \\ b & 0 & a & & & \\ & a & 0 & b & & \\ & & b & \ddots & \ddots & \\ & & & \ddots & 0 & b \\ & & & & b & a \end{bmatrix}.$$

We define a period two second order linear recurrence system $\{h_n\}$ given by $h_0=0,\ h_1=a+b$, and the recursions

$$\begin{aligned} h_{2n} &= (a-b)\,h_{2n-1}-abh_{2n-2}, \\ h_{2n+1} &= (-a+b)\,h_{2n}-abh_{2n-1} \end{aligned}$$

for $n\geq 1$.

We give relationships between the period two second order linear recurrence system $\{h_n\}$ and the determinant of H_n. Then we determine the eigenvalues of matrix H_n.

By expanding the determinant of H_n with respect to the first row, we have the following result without proof.

Lemma 1. *For $n>1$,*

$$\det H_{2n}=(-1)^{n+1}\left(a^{2n}-b^{2n}\right)\ \textit{and}\ \det H_{2n+1}=(-1)^{n}\left(a^{2n+1}+b^{2n+1}\right).$$

Let

$$H(x)=\sum_{n\geq 0}h_nx^n.$$

Also let

$$H_1(x)=\sum_{n\geq 0}h_{2n+1}x^{2n+1}\ \text{and}\ H_2(x)=\sum_{n\geq 0}h_{2n}x^{2n}.$$

By these equations, we get the equation system

$$\begin{aligned} H_2(x) &= (a-b)\,xH_1(x)-abx^2H_2(x), \\ H_1(x)-(a+b)\,x &= (-a+b)\,xH_2(x)-abx^2H_1(x). \end{aligned}$$

By Cramer solution of the system, we obtain

$$H_1(x)=\frac{(a+b)\,abx^3+(a+b)\,x}{1+(a^2+b^2)\,x^2+a^2b^2x^4}$$

and

$$H_2(x)=\frac{\left(a^2-b^2\right)x^2}{1+(a^2+b^2)\,x^2+a^2b^2x^4}.$$

Thus we get

$$H(x)=H_1(x)+H_2(x)=\frac{(b+a)\,x+\left(a^2-b^2\right)x^2+\left(a^2b+ab^2\right)x^3}{1+(a^2+b^2)\,x^2+a^2b^2x^4}.$$

Here note that

$$\begin{aligned} 1+\left(a^2+b^2\right)x^2+a^2b^2x^4 &= \left(1-\alpha^2x^2\right)\left(1-\beta^2x^2\right) \\ &= (1-\alpha x)(1+\alpha x)(1-\beta x)(1+\beta x), \end{aligned}$$

where $\alpha=\mathbf{i}a$, $\beta=\mathbf{i}b$ and $\mathbf{i}=\sqrt{-1}$.

By partial fraction decomposition, we find $A_1,\ A_2,\ B_1$ and B_2 such that

$$H(x) = \sum_{n\geq 0} h_n x^n = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1+\alpha x)} + \frac{B_1}{(1-\beta x)} + \frac{B_2}{(1+\beta x)}.$$

Solving the equation above, we get the coefficients have the forms:

$$A_1 = -\frac{(1+i)}{2},\ A_2 = -\frac{1-i}{2},\ B_1 = \frac{1-i}{2} \text{ and } B_2 = \frac{1+i}{2}.$$

Therefore

$$h_n = \frac{1}{2}\left(-(1+i)\alpha^n - (1-i)(-\alpha)^n + (1-i)\beta^n + (1+i)(-\beta)^n\right).$$

Especially we have that

$$h_{2n+1} = (-1)^n\left(a^{2n+1} + b^{2n+1}\right) \text{ and } h_{2n} = (-1)^{n+1}\left(a^{2n} - b^{2n}\right).$$

By the results above, we have the following result:

Corollary 1. *For $n > 1$,*

$$\det H_n = h_n.$$

If we choose a and b as the roots of the characteristic equation of the general Lucas sequences, $x^2 - Px + Q = 0$, then we have

$$h_{2n+1} = (-1)^n V_{2n+1} \quad \text{and} \quad h_{2n} = (-1)^{n+1} U_{2n}\sqrt{\Delta},$$

where $\Delta = P^2 - 4Q$.

Lemma 2. *For $n > 1$, $\det H_n = 0$ if and only if*

$$\begin{cases} a = \pm b \text{ or } a^2 + b^2 = 2ab\cos\frac{2k\pi}{n} & \text{for } 1 \leq k \leq \frac{n-2}{2} \quad \text{if } n \text{ is even,} \\ a + b = 2\sqrt{ab}\cos\frac{(2k-1)\pi}{2n} & \text{for } 1 \leq k \leq n \quad \text{if } n \text{ is odd.} \end{cases}$$

Proof. Since $h_n = \det H_n$ and by our result mentioned before, $h_n = \det H_n = 0$ if and only if

$$\frac{1}{2}\left(-(1+i)\alpha^n - (1-i)(-\alpha)^n + (1-i)\beta^n + (1+i)(-\beta)^n\right) = 0.$$

Thus

$$(1+i)\alpha^n + (1-i)(-1)^n\alpha^n = (1-i)\beta^n + (1+i)(-1)^n\beta^n$$

or

$$\frac{\alpha^n}{\beta^n} = \frac{(1-i) + (1+i)(-1)^n}{(1+i) + (1-i)(-1)^n}.$$

For even n such that $n = 2m$, we find the all solution of the equation

$$\frac{\alpha^{2m}}{\beta^{2m}} = \frac{(1-i) + (1+i)}{(1+i) + (1-i)} = 1,$$

which, by $\alpha = \mathbf{i}a$ and $\beta = \mathbf{i}b$, satisfies

$$a^{2m} = b^{2m}$$

or

$$a^{2m} - b^{2m} = 0.$$

From (pp. 34, formula 1.396.2, [7]), we recall the known result

$$\prod_{k=1}^{m-1}\left(x^2 + 1 - 2x\cos\frac{k\pi}{m}\right) = \frac{x^{2m} - 1}{x^2 - 1}.$$

By taking $x = a/b$, we get

$$\left(a^2 - b^2\right) \prod_{k=1}^{m-1} \left(a^2 + b^2 - 2ab\cos\frac{k\pi}{m}\right) = a^{2m} - b^{2m}.$$

Thus $a^{2m} - b^{2m} = 0$ if and only if

$$a = \pm b \text{ or } a^2 + b^2 = 2ab\cos\frac{k\pi}{m}$$

for some $1 \leq k \leq m-1$ where $n = 2m$.

For odd n such that $n = 2m+1$, $\det H_n = 0$ if and only if

$$\frac{\alpha^{2m+1}}{\beta^{2m+1}} = -1$$

or

$$\alpha^{2m+1} + \beta^{2m+1} = 0,$$

which, by $\alpha = \mathbf{i}a$ and $\beta = \mathbf{i}b$, is equivalent to

$$a^{2m+1} + b^{2m+1} = 0.$$

By the product form of Chebyshev polynomials of the first kind, we have that

$$a^m + b^m = \prod_{k=1}^{m} \left((a+b) - 2\sqrt{ab}\cos\frac{(2k-1)\pi}{2m}\right)$$

and so

$$a^{2m+1} + b^{2m+1} = \prod_{k=1}^{2m+1} \left((a+b) - 2\sqrt{ab}\cos\frac{(2k-1)\pi}{2(2m+1)}\right).$$

Finally $a^{2m+1}+b^{2m+1} = 0$ if and only if $a+b = 2\sqrt{ab}\cos\frac{(2k-1)\pi}{2(2m+1)}$ for $1 \leq k \leq 2m+1$. Thus the claim is proven. □

Now we can determine eigenvalues of the matrix H_n. For this, we define a new period second order recurrence system $\{g_n\}$ by the recursion

$$\begin{aligned} g_{2n} &= (a-b-x)\, g_{2n-1} - (ab)\, g_{2n-2}, \\ g_{2n+1} &= (-a+b-x)\, g_{2n} - (ab)\, g_{2n-1} \end{aligned}$$

with $g_0 = 0$ and $g_1 = a+b-x$ for $n > 0$.

On the other hand, by straightforward computations gives us the characteristic equation of matrix H_n as

$$\det\left(H_{2n} - xI_{2n}\right) = g_{2n}$$

and

$$\det\left(H_{2n+1} - xI_{2n+1}\right) = g_{2n+1},$$

where I_n is the identity matrix of order n.

Combining them, we determine the eigenvalues of matrix H_n :

Theorem 1. *The eigenvalues of H_{2n} are*

$$a \pm b \text{ and } \pm\sqrt{a^2 + b^2 - 2ab\cos\frac{k\pi}{n}}, \quad 1 \leq k \leq n-1,$$

and the eigenvalues of H_{2n+1} *are*

$$a+b \text{ and } \pm\sqrt{(a^2+b^2)-2ab\cos\frac{(2k-1)\pi}{2n+1}},\ 1\leq k\leq n.$$

3. Two Applications

Now we will give two applications of our results on trigonometric factorizations of the second order recurrences $\{U_n\}$ and $\{V_n\}$. If we choose the entries of the matrix H_n as $a=\left(P+\sqrt{P^2-4Q}\right)/2$ and $b=\left(P-\sqrt{P^2-4Q}\right)/2$, then we will obtain the following results :

$$V_{2n+1}=V_1\prod_{k=1}^{n}\left(V_2-2Q\cos\frac{2k-1}{2n+1}\pi\right)$$

and

$$U_{2n}=U_2\prod_{k=1}^{n-1}\left(V_2-2Q\cos\frac{k\pi}{n}\right).$$

Especially when $P=1$ and $Q=-1$, then we get

$$L_{2n+1}=\prod_{k=1}^{n}\left(3+2\cos\frac{2k-1}{2n+1}\pi\right) \text{ and } F_{2n}=\prod_{k=1}^{n-1}\left(3+2\cos\frac{k\pi}{n}\right).$$

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