12-12-2019 2:30 PM

Albert forms, Quaternions, Schubert Varieties & Embeddability

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Graduate Program in Mathematics
A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

Abstract

The origin of embedding problems can be understood as an effort to find some minimal datum which describes certain algebraic or geometric objects. In the algebraic theory of quadratic forms, Pfister forms are studied for a litany of powerful properties and representations which make them particularly interesting to study in terms of embeddability. A generalization of these properties is captured by the study of central simple algebras carrying involutions, where we may characterize the involution by the existence of particular elements in the algebra. Extending this idea even further, embeddings are just flags in the Grassmannian, meaning that their study is amenable to tools coming from intersection theory. We show that in each of the preceeding cases, embeddability can be used to obtain new characterizations of some primary information related to the ambient structure.

Keywords: Quadratic forms, Albert forms, Hermitian forms, algebraic groups, algebraic varieties, Schubert varieties, algebraic cycles, involution varieties, descent, central simple algebras, Chow groups.

Summary for lay audience

In 1908, Wedderburn published his foundational paper "On hypercomplex numbers", whose significance can be formalised in a single structure theorem, telling us that in some sense, all algebras look a certain way. It was later discovered that for the same reason a sum of two squares times another sum of two squares is still a sum of two squares, these algebras encode some deep number theoretic properties. Living in between several worlds, from the algebraic theory of quadratic forms to function fields and algebraic varieties, these objects interact intimately with one another.

This thesis explores the interplay between properties of numbers, algebras and geometric objects. The contributions of this work is threefold. Firstly, we discover that some classes of quadratic forms determine other, larger classes. Secondly, we find certain elements inside algebras which summarize important properties of these objects. Lastly, we establish a bridge between an algebraic and geometric view of algebras by considering combinatorial descriptions of how objects filter through space.

Co-Authorship Statement

Chapter 5 of this thesis incorporates material which results from joint research with Professor Nicole Lemire and Dr. Caroline Junkins.

Acknowledgments

I would like to begin by thanking my advisor Nicole Lemire for her constant support and patience during the course of my Ph.D. I can say with utmost certainty that no one will have read this work more times than her. I would also like to extend my gratitude to all the examiners, Dr. Ajneet Dhillon, Dr. David Riley, Dr Hristo Sendov for their commitment to review this work and uphold the standards of academic integrity. In particular, I would like to extend a personal thanks to the external examiner Dr. Adam Chapman for suggesting numerous additions to this thesis which have improved the overall quality of the work.

A little more than 10 years ago, I walked into a grade 12 mathematics classroom taught by Jeremy Klassen. At the end of that year I walked out wanting to become a mathematician. If I chose this path, it was only because I stood on the shoulders of Mr. Klassen, so that I could see what lied ahead.

This work was the culmination of several long years. I want to thank two colleagues of mine, Dinesh Valluri and Chandrasekar Rajamani for both stimulating my mathematical ideas and more importantly, for the good times we shared both in and out of mathematics.

My parents came to Canada as immigrants less than 20 years ago, with nothing but their desire to provide a better life for their children. This thesis is a testament to that desire, and it is from the bottom of my heart that I dedicate this work to the both of them.

I want to give my final thanks to Dr. Marina Palaisti. My colleague, my wife and my partner in crime. Nothing I have achieved in this work would have been possible without her. I owe all the success in my mathematical career to her. She has been there to encourage me when I fail and praise me when I succeed. She is the luck in my life.

Thank you, for dancing with me to the end of love.

Contents

Introduction

The central aim of this thesis is to elucidate an important connection between quadratic forms, central simple algebras and flag varieties through the lens of embeddability. We concern ourselves with these objects only insofar as they relate to one another. There are several prevailing themes which we will highlight throughout this work, touching on various aspects from the algebraic theory of quadratic forms to the structure of central simple algebras with involution and the geometric properties of symplectic Grassmannians. Throughout this work, we assume K is a field of characteristic not 2.

The study of central simple algebras, finite dimensional associative K-algebras with no non-trivial two-sided ideals and center equal to the base field K has been one of the central driving forces in many important areas of mathematics including algebraic geometry, algebra and number theory. An important result of Albert [Alb39], demonstrated that a central simple K-algebra A is of order 2 in the Brauer group if and only if it can be equipped with an involution (of the first kind), i.e. an anti-automorphism of order 2 (fixing the base field). A direct consequence of this result was the ability to relate geometric properties of a central simple algebra with algebraic properties of Hermitian and quadratic forms. The simplest form of this relationship can be understood when considering the first example of a non-commutative K-algebra which does not take the form of a matrix algebra. Introduced by Hamilton in 1843, the quaternion $\mathbb{R}$-algebra $\left(\frac{a,b}{K}\right)$ is generated by $\{1, i, j, k\}$ under the relations

$$i^2 = -a,\ j^2 = -b,\ ij = k = -ji,$$

where $a = b = 1$. We can associate an involution to $\left(\frac{a,b}{K}\right)$, called the canonical involution $\gamma : \left(\frac{a,b}{K}\right) \longrightarrow \left(\frac{a,b}{K}\right)$ defined by

$$\gamma(r_0 \cdot 1 + r_1 \cdot i + r_2 \cdot j + r_3 \cdot k) = r_0 \cdot 1 - r_1 \cdot i - r_2 \cdot j - r_3 \cdot k$$

with

$$(X_0 \cdot 1 + X_1 \cdot i + X_2 \cdot j + X_3 \cdot k) \cdot \gamma(X_0 \cdot 1 + X_1 \cdot i + X_2 \cdot j + X_3 \cdot k)$$

which can also be identified with $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$, a homogeneous polynomial of degree 2. This latter form defines a quadratic form, which carries several structural properties encoding information about the associated algebra (in this case, the quaternion). For instance, the existence of a non-trivial solution to the norm form $X_0^2 - aX_1^2 - bX_2^2 + abX_3^2$ is equivalent to the existence of an isomorphism $(a, b)_K \cong M_2(K)$. As powerful a result as this may seem to be, the fact that it works relies deeply on the structure of the norm form. In fact, quadratic forms with such powerful structural properties and associations have been studied for many years under the name of Pfister forms, first described in detail by Albrecht Pfister in 1965. Pfister forms have several important connections ranging from involutions on tensor products of quaternions and Milnor K-theory [Mil70] to the computation of Chow groups of quadrics [Kar95]. These properties are so strong that it is oftentimes desirable to understand which quadratic forms can be realized as subspaces or subforms of Pfister forms. In [HI00], Hoffmann and Izhboldin offered a characterizations of embeddability which allowed one to understand what field theoretic conditions are necessary to realize a quadratic form as a subform of a Pfister form. A particularly interesting case in their investigation was the Albert form, a quadratic form

$$q = \langle a, b, -ab, -c, -d, cd \rangle$$

associated to the biquaternion K-algebra $\left(\frac{a,b}{K}\right) \otimes \left(\frac{c,d}{K}\right)$ whose form theoretic properties determine whether or not the associated biquaternion K-algebra can be decomposed into matrix algebras.

The connection betweeen involutions on tensor products of quaternion K-algebras and Pfister forms goes even deeper as it was conjuctured by Shapiro [Sha77a] that a totally decomposable involution, i.e. an involution that can be decomposed as a tensor product of involutions on quaternion algebras acting diagonally, corresponds to an m-fold Pfister form (up to similarity) for some natural number m. This conjecture, commonly referred to as the Pfister Factor Conjecture, was proven by Becher [Bec08] in 2008 using several techniques from valuation theory and Hermitian forms. In particular, Becher's proof was non-constructive in the sense that he did not demonstrate how exactly specific involutions on tensor products of quaternions corresponded to specific Pfister forms. The advantage of obtaining an explicit correspondence would lie in our ability to frame decomposability in terms of the coefficents and obtain a more rigid understanding of the algebra theoretic structure in terms of associated Pfister form. In particular, understanding the precise structure of the Pfister form lends itself naturally to a more precise understanding of Pfister involutions [BFPQ03].

From another perspective, the study of Lagrangian involution varieties has become a focal point of several mathematical disciplines concerned with the geometric properties of central simple algebras. Remarkably, these varieties have a natural closed embedding into Severi-Brauer varieties which are amenable to study via three mathematical viewpoints. The first treats these as geometric objects which characterize the solutions to certain equations. In particular, the points of the Severi-Brauer variety $\mathrm{SB}(A)$ are precisely the reduced ideals of dimension 1 lying inside a given central simple algebra A. The second viewpoint treats $\mathrm{SB}(A)$ algebraically, in terms of the structural properties of A. The third examines $\mathrm{SB}(A)$ cohomologically via the second Galois cohomology group of the multiplicative group of a field. These viewpoints have been effective in tackling problems concerning Lagrangian involution varieties and generalized involution varieties.[KMRT98]

For instance, Krashen [Kra10] and McFaddin [McF17] characterized the zero cycles with coefficients for involution varieties in small index. More recently, Junkins, Krashen and Lemire [JKL17] determined torsion in the Chow group for certain algebraic groups of type A_n by determining necessary and sufficent index conditions on a central simple algebra A such that the corresponding generalized Severi-Brauer variety $\mathrm{SB}_m(A)$ contains a certain twisted Schubert variety with points inside the Grassmannian. The motiviation for studying these twisted Schubert varieties arises from the combinatorial nature as reduced ideals of specified dimension satisfying certain containment conditions. These objects, Schubert varieties, form the building blocks of the Chow group of the Grassmannians. In particular, the classical notion of "essential set" originally introduced by W. Fulton [Ful92] and adapted by D. Anderson [And16] has proved to be fundamental in understanding difficult properties concerning Schubert cycles of Lagrangian Grassmannian. Where generalized Severi-Brauer varieties are characterized by ideals of a fixed reduced dimension inside a central simple algebra, involution varieties are defined by *isotropic* ideals of fixed reduced dimension inside a central simple algebra with symplectic involution. The addition of an isotropy condition means reasoning about isotropic ideals must now also include reasoning about the structure of totally isotropic spaces of Hermitian and bilinear forms. This necessitates a natural extension of the techniques introduced by Junkins, Krashen and Lemire [JKL17] incorporating the theory of forms. Moreover, these ideas lend themselves particularly well to computing torsion elements in the Chow group, describing the geometric interactions between subvarieties in terms of intersection theory [Kar95].

0.1 Summary

The main results of this work lie in the crossroads of several distinct and important mathematical areas. The applications include an explicit description of Pfister forms in terms of embeddable Albert forms, a constructive proof of the Pfister Factor Conjecture in small dimensions and a complete description of small index involution varieties via computation of its Chow groups.

In Chapter 1 we introduce the theory of symmetric bilinear forms and its analogue in characteristic $\neq 2$, the theory of quadratic forms. We review the central pillars which make the algebraic theory of quadratic forms an attractive formalism for establishing latter results concerning involutions, Hermitian forms and Chow groups. We view Pfister forms as central to all major developments in the theory of quadratic forms. In particular, in Section 1.3 we consider the problem of determining under what conditions, if any, a quadratic form q can be embedded into an m-fold Pfister form p. The main results of this chapter are contained in Section 1.4 where Theorem 1.4.4 is a novel characterization of 4-fold Pfister forms in terms of embedded Albert forms:

Theorem 1.4.4. Consider an anisotropic Albert form $\langle a, b, -ab, -c, -d, cd\rangle$ over K where $a, b, c \in K' \subset K$, $\mathrm{trdeg}_{K'} K'(d) = 1$. If $\langle 1\rangle \perp q$ is 4-embeddable, that is, $\langle 1\rangle \perp q \subset \langle\langle x, y, z, w\rangle\rangle$ for some anisotropic Pfister form $\langle\langle x, y, z, w\rangle\rangle$ over K, then

$$\langle\langle x, y, z, w\rangle\rangle \cong \langle\langle a, b, -c, -d\rangle\rangle\,.$$

In particular, if $-1 \in (K^\times)^2$ we have that

$$\langle\langle x, y, z, w\rangle\rangle \cong \langle\langle a, b, c, d\rangle\rangle\,.$$

In Chapter 2 we introduce the theory of central simple algebras with a particular emphasis on central simple K-algebras equipped with involution. We review some of the basic results associated to the structure of such algebras in relation to the associated involution. Motivated by the characterization of similitudes in [Sha77a] and [Sha77b], we find an explicit basis of generators for maximally decomposed similitudes. The main result of this chapter is a constructive proof of the Pfister Factor Conjecture for $A = (\otimes_{i=1}^m Q_i, \otimes_{i=1}^m \sigma_i)$ with $m \leq 3$,

Theorem 2.3.2. Let $\otimes_{i=1}^n (Q_i, \sigma_i)$ be a split $K-$algebra and assume $\sigma = \otimes_{i=1}^n \sigma_i$ is an anisotropic involution. If $n \leq 3$, then q_σ is a Pfister form.

To our knowledge this is the first constructive characterization of Pfister forms in terms of the algebraic structure induced by the involution. We believe our result can easily be used in the context of both the structure of algebras and algebraic programming and demonstrate the flexibility of our approach by explicitly determining Albert forms inside of a predetermined 4-fold Pfister form.

Chapters 3 and 4 set up the background for discussing Schubert cycles in a principled manner. Chapter 3 discusses Hermitian forms over arbitrary finite dimensional K-algebras and provides a clarification of the well-known reduction theorem in characteristic $\neq 2$. Chapter 4 introduces algebraic groups and reviews the classification of split semisimple algebraic groups in terms of root systems and Dynkin diagrams. In Section 4.3, we review an alternative characterization of algebraic groups in terms of automorphisms of central simple algebras which references our earlier results of embeddability of Albert forms and Pfister elements. We conclude with a short review of projective homogeneous G-varieties paying special attention to the case of Grassmannian varieties and flags which will be the focus of the final chapter.

In Chapter 5, we study Schubert cycles of complete flag varieties corresponding to maximal symplectic grassmannians. In particular, in Section 5.2 we consider under what structural conditions can Schubert cycles be realized as closed subvarieties of symplectic Grassmannians. We find a new characterization of these results in a direction which extends the work of [JKL17] from algebraic groups of type A_n to algebraic groups of type C_n of small degree.

Theorem 5.3.11. The maximal symplectic Grassmannian variety $\mathrm{SG}(A, \sigma)$ has a closed subvariety P such that $P \otimes_K L \simeq X_\lambda$ for a Schubert subvariety

X_λ if and only if $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ and $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$ where E_λ is the essential set of the partition λ. Moreover, in this case, A contains a flag of isotropic right ideals $I_{a_1} \subset \cdots \subset I_{a_r}$ for $\overline{E_\lambda} = \{a_1, \ldots, a_r\}$ such that for any finite extension L/K,

$$P(L) = \{J \subseteq A_L : \mathrm{rank}(J \cap (I_a)_L) \geq j \text{ for } (j,a) \in E_\lambda\}.$$

We conclude our inquiry in Section 5.4, applying the ideas developed in Section 5.2 to compute torsion in the Chow group corresponding to the involution variety of a central simple algebra of degree 4 with symplectic involution. As we will see, this is simply a twisted form of the Lagrangian Grassmannian variety. Our main result towards this direction is stated here for brevity.

Theorem 5.4.1. Let (A,σ) be a degree 4 central simple K-algebra equipped with a symplectic involution σ. Then the torsion of the topological filtration corresponding to the maximal symplectic Grassmannian, $\mathrm{SG}(A,\sigma)$ is determined as follows:

1. If $\mathrm{ind}(A) = 4$, then, $|\,\mathrm{Tors}(\mathrm{T}^*(\mathrm{SG}(A,\sigma))| = 1$,
2. If $\mathrm{ind}(A) = 2$, and σ is anisotropic then, $|\,\mathrm{Tors}(\mathrm{T}^*(\mathrm{SG}(A,\sigma))| = 2$,
3. If $\mathrm{ind}(A) = 2$, and σ is isotropic then, $|\,\mathrm{Tors}(\mathrm{T}^*(\mathrm{SG}(A,\sigma))| = 1$,
4. If $\mathrm{ind}(A) = 1$, then, $|\,\mathrm{Tors}(\mathrm{T}^*(\mathrm{SG}(A,\sigma))| = 1$.

A corollary of this result, Corollary 5.4.1, determines torsion in the corresponding Chow group using deep results relating the topological filtration with the Chow group.

Chapter 1

Quadratic forms

We briefly introduce the theory of symmetric bilinear forms over fields of characteristic $\neq 2$ to setup the algebraic theory of quadratic forms which will be the central object of interest in this chapter. The first two sections review some classical results coming from these theories along with algebraic techniques necessary to work with them. The last section investigates the relationship between Albert forms and Pfister forms in terms of embeddability and gives a new result connecting these two notions.

1.1 Symmetric Bilinear forms

Let K be a field of characteristic $\neq 2$. By K^{alg} we denote an algebraic closure of the field K, and by K^{sep} a separable closure of K. Let V be a finite dimensional vector space over the field K. A *symmetric bilinear form* on V is a map

$$b : V \times V \longrightarrow K$$

satisfying the following properties for all $v_1, v_2, w_1 \in V$ and $c, d \in K$:

- $b(v_1, w_1) = b(w_1, v_1)$,
- $b(cv_1 + dv_2, w_1) = cb(v_1, w_1) + db(v_2, w_1)$.

We denote a finite dimensional vector space V equipped with a symmetric bilinear form b by (V, b) and refer to such a pairing as a (symmetric) *bilinear space*. A bilinear form is said to be *skew-symmetric* if it is linear in each component and $b(v_1, w_1) = -b(w_1, v_1)$. We say a bilinear form b is *non-degenerate* if $b(v,w)=0$ for every $w \in V$ implies $v=0$. Using this definition, what follows is a classical result in linear algebra which characterizes non-degeneracy in several different forms:

Proposition 1.1.1. *[Lam05, Proposition I.1.2] The following are equivalent:*

1. *(V,b) is non-degenerate,*
2. *The map ev: $V \longrightarrow V^\vee$ defined by*
$$v \longrightarrow ev_v : w \longrightarrow b(v,w)$$
is a K-isomorphism where $V^\vee$ is the dual vector space to V.
3. *The matrix $(b(e_i,e_j))$ associated to b is invertible with $e_1,\dots,e_n$ forming a basis of V.*

We call a morphism between bilinear spaces preserving the structure of the associated bilinear forms an *isometry*. To be precise, an isometry of two bilinear spaces (V,b_1) and (W,b_2) is a K-linear isomorphism $\phi : V \longrightarrow W$ such that
$$b_1(v_1,v_2) = b_2(\phi(v_1),\phi(v_2))$$
for all $v_1,v_2 \in V$. In this sense, isometries are transformations preserving the metrics induced on the space. In the case b_1 and b_2 are isometric we will write $b_1 \cong b_2$. An immediate consequence of this definition is that over the algebraic closure, $\dim_{K^{alg}}(V) = \dim_{K^{alg}}(W)$ implies $(V,b_1) \cong_{K^{alg}} (W,b_2)$. To get a sense of why this is the case we first consider symmetric bilinear forms on a 1-dimensional vector space defined by
$$b_1(x,y) = cxy,\ \forall x,y \in K,$$
$$b_2(x,y) = dxy,\ \forall x,y \in K,$$
for some $c,d \in K^\times$. For convenience we denote (V,b_1) by $\langle c\rangle_{b_1}$ and observe that
$$\langle c\rangle_{b_1} \ \cong\ \langle d\rangle_{b_2}$$
if and only if $d \in D_K(b_1)$ where $D_K(b_1) = \{b_1(v,v) \in K^\times \mid v \in V\}$. Now, if $\dim_{K^{alg}}(V) = \dim_{K^{alg}}(W)$ then it is easy to see that
$$\langle c\rangle_{b_1} \ \cong_{K^{alg}}\ \langle d\rangle_{b_2},$$

since $d = c(\frac{\sqrt{d}}{\sqrt{c}})^2 \in D_{K^{alg}}(b_1)$ with $\frac{\sqrt{d}}{\sqrt{c}} \in K^{alg}$. By orthogonally decomposing the space we can proceed inductively.

We say (V, b) is an *anisotropic* bilinear space if it contains no non-trivial solutions i.e. $b(v, v) = 0$ if and only if $v = 0$. In contrast, a non-zero vector $v \in V$ is *isotropic* if $b(v, v) = 0$ in which case we say the symmetric bilinear form b is *isotropic*. Extending the idea of isotropy to its limit, we define the *hyperbolic form* on a vector space V with subspace W such that $V := W \oplus W^\vee$ to be the symmetric bilinear form $b_{\mathbb{H}(W)}$ defined by the mapping

$$b_{\mathbb{H}(W)}(v_1 + w_1^*, v_2 + w_2^*) = w_2^*(v_1) + w_1^*(v_2)$$

for all $v_1, v_2 \in W$ and $w_1^*, w_2^* \in W^\vee$. We consider a subspace $W \subset V$ to be *totally isotropic* if

$$b|_W = 0.$$

In other words, W is a totally isotropic subspace of (V, b) if $b(w, w) = 0$ for every $w \in W$. We can easily see that a subspace W is totally isotropic if and only if $W \subset W^\perp$. Moreover $\dim_K(W) \leq \frac{1}{2} \dim_K(V)$, since the non-degeneracy assumption implies

$$\dim_K(W) + \dim_K(W^\perp) = \dim_K(V)$$

with $W^\perp = \{v \in V \mid b(v, w) = 0 \text{ for all } w \in W\}$. The notion of a hyperbolic form coincides with that of isotropy in the following sense: (V, b) is hyperbolic if and only if there exists a totally isotropic subspace $W \subset V$ such that

$$\dim_K(W) = \frac{1}{2} \dim_K(V).$$

An important connection between symmetric bilinear forms and symmetric matrices is that we can interpret the diagonalization of invertible symmetric matrices through Gram-Schmidt to give us that any non-degenerate symmetric bilinear form is diagonalizable. In particular, over an algebraically closed field, an even dimensional (non-degenerate) symmetric bilinear form is hyperbolic. To state this result correctly, we must first define additive and multiplicative operations on symmetric bilinear forms in the form of orthogonal sums and tensor products. Let (V, b_1) and (W, b_2) be bilinear spaces with associated symmetric bilinear forms over K. We define the *orthogonal sum* of b_1 and b_2, denoted by $b_1 \perp b_2$, to be the map

$$b_1 \perp b_2 : (V \oplus W) \times (V \oplus W) \to K$$

defined by

$$(b_1 \perp b_2)((v_1, w_1), (v_2, w_2)) = b_1(v_1, v_2) + b_2(w_1, w_2).$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$. It is easy to see that $b_1 \perp b_2$ is indeed a symmetric bilinear form such that $(b_1 \perp b_2)(V, W) = 0$. Alternatively, we define the *Kronecker product* or *tensor product* of b_1 and b_2, denoted by $b_1 \otimes b_2$, to be the map

$$b_1 \otimes b_2 : (V \otimes W) \times (V \otimes W) \to K$$

defined by

$$(b_1 \otimes b_2)(v_1 \otimes v_2, w_1 \otimes w_2) = b_1(v_1, w_1) \cdot b_2(v_2, w_2)$$

for every $v_1, v_2 \in V$ and $w_1, w_2 \in W$. The following two results are classical and of monumental importance. They are foundational to the modern algebraic theory of both symmetric bilinear forms and quadratic forms.

Theorem 1.1.2. *(Witt's decomposition theorem) If b is a non-degenerate symmetric bilinear form on V then there exist subspaces $U, W \subset V$ such that*

$$b = b|_U \perp b|_W$$

with $b|_U$ anisotropic and $b|_W$ hyperbolic. Moreover, $b|_U$ is unique up to isometry.

Proof. See [Lam05, Proposition I.4.1].

Theorem 1.1.3. *(Witt's cancellation theorem) Let b_0, b_1 and b_2 be non-degenerate symmetric bilinear forms over K. If b_1 and b_2 are anisotropic then*

$$b_1 \perp b_0 \cong b_2 \perp b_0 \Longrightarrow b_1 \cong b_2.$$

Proof. See [Lam05, Proposition I.4.2].

Together, these results make it possible to define a ring structure on the class of symmetric bilinear forms via their associated Witt ring. We remark that the isometry classes of nondegenerate symmetric bilinear forms over K, denoted by $M_b(K)$, form a semi-ring under orthogonal sum and tensor product. Define the *Grothendieck-Witt group* of K, denoted with $\widehat{W_b(K)}$, by an equivalence relation $\sim$ on $M_b(K) \times M_b(K)$ such that

$$(b_1, b_2) \sim (d_1, d_2)$$

if and only if

$$b_1 \perp d_2 \cong d_1 \perp b_2$$

with $b_1, b_2, d_1, d_2 \in M_b(K)$. To avoid confusion, we denote the equivalence class of (b_1, b_2) in $\widehat{W_b(K)}$ by $b_1 - b_2$. By the classical results stated above, it can be shown that $\widehat{W_b(K)}$ has the structure of a ring with addition defined by

$$(b_1 - b_2) + (d_1 - d_2) = (b_1 \perp d_1) - (b_2 \perp d_2)$$

and multiplication defined by $\widehat{W_b(K)}$ by:

$$(b_1 - b_2)(d_1 - d_2) = ((b_1 \otimes d_1) \perp (b_2 \otimes d_2)) - ((b_1 \otimes d_2) \perp (b_2 \otimes d_1)).$$

We conclude this section by giving several useful properties of the Grothendieck-Witt ring.

Lemma 1.1.4. *[EKM08, Proposition I.2.4, Theorem I.4.7]*

1. *Let b_1 and b_2 be anisotropic symmetric bilinear forms over K. Then*

$$b_1 = b_2 \in \widehat{W_b(K)} \Longleftrightarrow b_1 \cong b_2.$$

2. *The additive group $(\widehat{W_b(K)}, +)$ is generated by the classes of 1-dimensional symmetric bilinear forms subject to the relation*

$$\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$$

for all $a, b \in K^\times$ such that $a + b \neq 0$.

1.2 Quadratic forms

Let us assume, as before, that all fields discussed are of characteristic $\neq 2$. There is a one-to-one correspondence between symmetric bilinear forms and quadratic forms over a field K given by the map

$$b \longrightarrow q_b : V \longrightarrow K$$

where $q_b(v) = b(v, v)$ for $v \in V$. We denote (V, q) to be a *(n-dimensional) quadratic space* if the following conditions are satisfied:

1. V is an n-dimensional vector space over K,

2. $q(av) = a^2 q(v)$ for any $a \in K$ and $v \in V$,

3. $b_q(v, w) = \frac{1}{2}\left(q(v+w) - q(v) - q(w)\right)$ is symmetric bilinear.

Remark 1.2.1. Notice that 3. gives us the reverse correspondence from quadratic forms to symmetric bilinear forms:

$$b_q(v, v) = \frac{1}{2}\left(q(2v) - q(v) - q(v)\right) = q(v).$$

Clearly, in the case that $\mathrm{char}(K) = 2$ this correspondence no longer holds.

We say that two quadratic spaces (V, q) and (W, p) are isometric, denoted by $(V, q) \cong (W, p)$, if there exists an isomorphism of vector spaces $g : V \longrightarrow W$, such that

$$p(g(v)) = q(v),$$

for all $v \in V$. To shorten notation, we will refer to the quadratic space (V, q) by q and denote an isometry $(V, q) \cong (W, p)$ by $q \cong p$. In the same way that symmetric matrices are diagonalisable by Gram-Schmidt, (non-degenerate) quadratic forms have a diagonal representation.

Theorem 1.2.2. *[Lam05, Criterion I.2.3] Let (V,q) be a quadratic space and $c \in K^\times$. Then $c \in D(q)$ if and only if $(V, q) \cong (K^\times v, \langle c \rangle) \perp (V', p)$ where $q(v) = c$ and (V', p) is a quadratic subspace of (V, q).*

Let $M_q(K)$ denote the set of all isometry classes of (non-singular) quadratic forms over K. The binary operations

$$(V_1, q_1) \perp (V_2, q_2) = (V_1 \oplus V_2, q_1 \perp q_2)$$

with $(q_1 \perp q_2)(v \oplus w) = q_1(v) + q_2(w)$ and

$$(V_1, q_1) \otimes (V_2, q_2) = (V_1 \otimes V_2, q_1 \otimes q_2)$$

with $(q_1 \otimes q_2)(v \otimes w) = q_1(v) \cdot q_2(w)$ give $M_q(K)$ the structure of a semi-ring. In particular, the fact that $M_q(K)$ is a commutative cancellation monoid (by Theorem 1.1.3) allows us to define a ring structure via the Grothendieck construction. We call

$$\widehat{W_q(K)} = \mathrm{Groth}\left(M_q(K)\right) = M_q(K) \times M_q(K) / \sim$$

the Grothendieck-Witt ring, defined by the relations

$$(q_1, q_2) \sim (p_1, p_2) \text{ if and only if } q_1 \perp p_2 \cong p_1 \perp q_2,$$

$$(p_1, p_2) + (q_1, q_2) := (p_1 \perp q_1, p_2 \perp q_2),$$

$$(p_1, p_2) \times (q_1, q_2) := (p_1 \otimes q_1 \perp p_2 \otimes q_2, p_2 \otimes q_1 \perp p_1 \otimes q_2).$$

where we identify $M_q(K) \hookrightarrow \widehat{W_q(K)}$ via $q \mapsto (q, 0)$.

Example 1.2.3. Below, we give examples of the Grothendieck-Witt ring for various fields.

1. $\widehat{W_q(\mathbb{C})} \cong \mathbb{Z}$.

2. $\widehat{W_q(\mathbb{R})} \cong \mathbb{Z}[C_2]$, where C_2 is a cyclic group of order 2.

3. $\widehat{W_q(\mathbb{F}_p)} \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/4$, whenever $p = 3 \mod 4$.

4. $\widehat{W_q(\mathbb{F}_p)} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^3$, whenever $p = 1 \mod 4$.

Consider the ring homomorphism $\dim : \widehat{W_q(K)} \longrightarrow \mathbb{Z}$, given by $\dim((q_1, q_2)) = \dim((q_1, 0) - (q_2, 0)) = \dim(q_1 - q_2)$. Denote the kernel of the dim map by $\widehat{I_q(K)}$. Although the Grothendieck-Witt ring retains much of the structural information pertaining to quadratic forms over a particular field, we wish to reduce our consideration to anisotropic quadratic forms only. To do this, we construct the *Witt ring* of K by applying the decomposition of Theorem 1.1.2: For any quadratic form q,

$$q \cong q_{\text{an}} \perp m\mathbb{H},$$

where q_{an} denotes the anisotropic quadratic form and $m\mathbb{H}$, $m \in \mathbb{Z}$, denotes an orthogonal sum of m *hyperbolic planes*, $\mathbb{H} := \langle 1, -1 \rangle$ which can be identified with $X^2 - Y^2$. To classify the anisotropic quadratic forms it suffices then to quotient by the ideal $\mathbb{Z}\mathbb{H}$:

$$\mathrm{W}_q(K) \cong \widehat{W_q(K)}/\mathbb{Z}\mathbb{H}.$$

The following result demonstrates some useful characterizations of the Witt ring.

Theorem 1.2.4. *[Lam05, Proposition II.I.4]*

1. *The elements of* $\mathrm{W}_q(F)$ *are in one-to-one correspondence with the isometry classes of all anisotropic forms.*
2. *Two forms* q*,* p *represent the same element in* $\mathrm{W}_q(F)$ *if and only if* $q_{an} \cong p_{an}$*.*
3. *If* $\dim(q) = \dim(p)$ *then* $\overline{q} = \overline{p} \in \mathrm{W}_q(F)$ *if and only if* $q \cong p$*.*

Example 1.2.5. We present the Witt rings of the fields discussed in Example 1.2.3.

1. $\mathrm{W}_q(\mathbb{C}) \cong \mathbb{Z}/2$.
2. $\mathrm{W}_q(\mathbb{R}) \cong \mathbb{Z}$.
3. $\mathrm{W}_q(\mathbb{F}_p) \cong \mathbb{Z}/4$, whenever $p = 3 \mod 4$.
4. $\mathrm{W}_q(\mathbb{F}_p) \cong (\mathbb{Z}/2)[F_p^\times/(F_p^\times)^2]$, whenever $p = 1 \mod 4$.

We call the image of $\widehat{I_q(K)}$ under the projection map $\widehat{W_q(K)} \longrightarrow \mathrm{W}_q(K)$ the *fundamental ideal* of $\mathrm{W}_q(K)$ and denote it by $I_q(K)$. By construction, the fundamental ideal $I_q(K)$ consists of all even dimensional quadratic forms in $\mathrm{W}_q(K)$. In other words, $I_q(K)$ is generated as a K-module by all 1-fold *Pfister forms* $\langle 1, -a\rangle$, denoted by $\langle\langle a\rangle\rangle$, with $a \in K^\times$. Moreover, since $\dim(\gtrdot\mathbb{H}) \in 2\mathbb{Z}$ for any $m \in \mathbb{N}$, we have the Cartesian square

$$\begin{tikzcd}
\widehat{W_q(K)} \arrow[r, "\dim"] \arrow[d] & \mathbb{Z} \arrow[d, "\mathrm{mod}\ 2"] \\
\mathrm{W}_q(K) \arrow[r, "\dim_2"'] & \mathbb{Z}/2
\end{tikzcd},$$

where $\dim_2 : W_q(K) \longrightarrow \mathbb{Z}/2$ is defined by $\dim_2(\overline{q}) = \dim(q) \mod 2$.

Now that we have the notion of (the first) fundamental ideal in terms of 1-fold Pfister forms, it turns out we can naturally extend it by considering $I_q^n(K)$ to be the n-th power of the fundamental ideal $I_q(K)$, generated as a K-module by forms $\langle 1, -a_1\rangle \otimes ... \otimes \langle 1, -a_n\rangle$ for some $a_1, \cdots, a_n \in K^\times$. The advantage of this characterization is that it induces a natural filtration. The usefulness of such a filtration relies on an incredibly important classical result in quadratic forms due to Arason and Pfister.

Theorem 1.2.6. *[Lam05, Hauptsatz X.5.1]*

$$\textit{If } p \in I_q^n(K) \textit{ and } p \textit{ is anisotropic, then } \dim p \geq 2^n.$$

Indeed, an immediate consequence of the Arason-Pfister hauptsatz, theorem 1.2.6, is that quadratic forms are bounded in some sense by these fundamental ideals i.e. $\cap_{n=0}^{\infty} I_q^n(K) = 0$ where we identify $I_q^0(K) = \mathrm{W}_q(K)$ for notational convenience. It serves to reason then that a complete classification of quadratic forms necessitates a rigorous understanding of which forms are contained in which ideals. Recalling the map $\dim_2$ defined earlier, we see immediately that $\ker(\dim_2) = I_q(K)$. Identifying the higher fundamental ideals in terms of some particular algebraic properties of the associated quadratic forms turns out to be quite fruitful in the first few cases, but ultimately unsatisfactory. The existence of algebraic realizations to higher fundamental ideals remains an open question. Alternatively, we might ask ourselves if fundamental ideals might be described in terms of some cohomological invariants. Indeed, we might begin by asking if there exists a class of cohomological invariants (of quadratic forms) $\{e_*^q\}$,

$$e_r^q : I_q^r(K) \longrightarrow H^r(G_K, \mu_2) := H^r(Gal(K^{sep}/K), \mu_2),$$

where μ_2 are the roots of unity of order 2 with the appropriate characterizing properties such that

$$q_1 - q_2 = 0 \in I_q^n(K), \text{ for } n > 2^r \geq \dim(q_1) + \dim(q_2).$$

The existence of such invariants would then imply that we can determine whether two quadratic forms q_1, q_2 are isometric via the vanishing conditions on $\{e_i^q\}$ for $0 \leq i \leq r \in \mathbb{N}$. Remarkably, the $\{e_i^q\}$ turn out to appear naturally as structural characterizations of $I^n(K)$ in low-dimensions. For $n = 0$, e_0^q is the *rank* of a quadratic form i.e. $e_0^q = \dim_2 : I_q^0(K) = \mathrm{W}_q(K) \longrightarrow \mathbb{Z}/2 = H^0(G_K, \mu_2)$ and $\ker(e_0^q) = I_q^1(K)$. In the case $n = 1$, we introduce the notion of *discriminant* by considering the following map:

$$\det : \widehat{W_q(K)} \longrightarrow K^\times / K^{\times 2}$$

$$q_1 - q_2 \mapsto \det(q_1) \det(q_2)^{-1}.$$

Although this map is well-defined, it turns out to be inadequate for our purposes. The disadvantage lies in the fact that $\det(\mathbb{H}) = \det(\langle 1, -1 \rangle) = -1$, which prevents us from factoring through $\mathrm{W}_q(K)$. To adjust this, we embed $K^\times/(K^\times)^2$ into a larger group with more flexibility and consider the map

$$\mathrm{disc} : I_q(K) \longrightarrow \mathbb{Z}/2 \times K^\times/(K^\times)^2$$

$$(q) \mapsto (\dim_2 q, d_\pm(q)),$$

where $d_\pm(q) = (-1)^{\frac{1}{2}\dim(q)(\dim(q)-1)} \det(q) \in K^\times/K^{\times 2}$ and the multiplication on $\mathbb{Z}/2 \times K^\times/K^{\times 2}$ is defined by

$$(a, b) \cdot (a', b') = (a + a', (-1)^{aa'} bb').$$

Direct computation shows that

$$e_1^q = \mathrm{disc} : I_q(K) \longrightarrow \mathrm{im}(e_1) = \{0\} \times K^\times/K^{\times 2} \cong H^1(G_K, \mu_2),$$

and $\ker e_1^q = I_q^2(K)$. In terms of the dimension $\dim_2$ and signed determinant $d_\pm$, we can rephrase the result as follows:

$$q \in I_q^2(K) \text{ if and only if } \begin{cases} \dim_2(q) = 0 \\ d_\pm(q) = 1 \end{cases}.$$

The map, $e_2^q : I_q^2(K) \longrightarrow H^2(G_K, \mu_2)$ turns out to be harder to encode algebraically, we will see in Chapter 2 that $H^2(G_K, \mu_2)$ can be identified with the 2-torsion part of the Brauer group. In trying to determine the appropriate algebraic realization of e_2^q, a natural candidate to consider is the *Clifford algebra* associated to a quadratic space (V, p) given by

$$C(V) = C(V, p) = T(V)/(v \otimes v - p(v)),$$

where $T(V)$ is the tensor algebra of V defined by

$$T(V) = \oplus_{n=0}^{\infty} V^{\otimes n},$$

where $V^0 = K$ and $(v \otimes v - p(v))$ is an ideal generated by $v \otimes v - p(v)$ for all $v \in V$. $C(V)$ depends on the isometry classes of (V, p) uniquely and carries with it a natural $\mathbb{Z}/2$-graded algebra structure:

$$C(V) = C_0(V) \oplus C_1(V),$$

where $C_0(V) = \oplus_{i=0}^{\infty} V^{\otimes 2i}$ and $C_1(V) = \oplus_{i=0}^{\infty} V^{\otimes 2i+1}$.

Using classical results in the structure theory of Clifford algebras, it can be shown that

$$\begin{cases} C(V) \in \mathrm{Br}_2(K), \text{ if } V \text{ is even-dimensional} \\ C_0(V) \in \mathrm{Br}_2(K), \text{ if } V \text{ is odd-dimensional} \end{cases}$$

where $\mathrm{Br}(K)$ denotes the Brauer group (defined rigorously in Section 2.1) and $\mathrm{Br}_2(K)$ denotes the 2-torsion elements of $\mathrm{Br}(K)$. In particular, this classification can be used to construct a morphism,

$$c : I_q^2(K) \longrightarrow \mathrm{Br}(K)$$

defined by

$$q \mapsto \begin{cases} [C(q)], \text{ if } \dim q \text{ is even} \\ [C_0(q)], \text{ if } \dim q \text{ is odd} \end{cases} .$$

Here we abuse notation and use $C(q)$ to represent $C(V, q)$. It follows quite easily from this definition that

- $c(I_q^2(K)) \subseteq \mathrm{Br}_2(K)$, and
- $I_q^3(K) \subset \ker(c)$

In fact, $c(I_q^2(K)) = \mathrm{Br}_2(K)$ by a famous result of Merkurjev [Mer81] which shows (among other things) that

$$I_q^2(K)/I_q^3(K) \cong \mathrm{Br}_2(K).$$

1.3 Albert forms and Embeddings

An *n-fold Pfister form* over K is a quadratic form of type

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle ,$$

where $a_i \in K^\times$, and we write $\langle\langle a_1, \ldots, a_n \rangle\rangle$ for short. We say that a quadratic form q over K is *n-embeddable* if there exists an anisotropic n-fold Pfister form ϕ, such that $q \subseteq \phi$ i.e. there exists a quadratic form p over K such that $q \perp p \cong \phi$. Note that $q \subseteq \phi$ is equivalent to the existence of a linear subspace $W \subseteq V$ such that $q = \phi|_W$. The anisotropy of ϕ assures a non-trivial embedding, since any isotropic Pfister form is hyperbolic (see [Lam05, Theorem X.1.7]) and one can always embed such a quadratic form q into $m \cdot \mathbb{H}$, for all m satisfying $\dim q \leq m$. Indeed, since for any $a \in K^\times$ we have $a = (\frac{a+1}{2})^2 - (\frac{a-1}{2})^2 \in D_K(\langle 1, -1\rangle)$ it follows $q = \langle a_1, \cdots, a_n\rangle \subset \dim(q)\mathbb{H}$ which yields the smallest non-specified (hyperbolic) Pfister form containing q.

There are several useful applications of m-embeddability within the theory of quadratic forms. For example, embeddability can in some sense be understood as a minimal datum for constructing a larger Pfister form. If p is a 2-fold Pfister form such that $\langle 1, a\rangle \subset p$, then $p \cong \langle\langle a, c\rangle\rangle$ for some $c \in K^\times$. Similarly, if q is a 3-fold Pfister form such that $\langle 1, a, b\rangle \subset q$ then $q \cong \langle\langle a, b, c\rangle\rangle$ for some $c \in K^\times$. This relationship can be generalized in one direction to get an important class of forms known as Pfister neigbours. A quadratic form σ over K is called a *Pfister neigbour* if σ can be embedded into a Pfister form p (up to multiplication by a scalar multiple) where $\dim_K(p) < 2\dim_K \sigma$. In terms of Milnor K-theory, [HI00, Theorem 5.1] shows that when computing the transfer kernel of the Milnor K-theory of a function field extension the generators must satisfy certain minimality conditions with respect to the degree. In the theory of motives, an understanding of the embeddability into an anisotropic Pfister quadric is necessary in order to define the local motivic cohomology of a point for the isotropic motivic category [Vis19].

Examples 1.3.1.

1. Any anisotropic 3-dimensional quadratic form $\langle a, b, ab\rangle$ is 2-embeddable. Indeed $\langle a, b, ab\rangle \subset \langle 1, a, b, ab\rangle \cong \langle\langle -a, -b\rangle\rangle$, where the anisotropy of $\langle\langle -a, -b\rangle\rangle$ is assured by the anisotropy of $\langle a, b, ab\rangle$.

2. If q is an anisotropic n-fold Pfister form, then q is m-embeddable in some m-fold Pfister form where $m \geq n$.

3. If q is a 2^m dimensional quadratic form, then q is m-embeddable if and only if q is an m-fold Pfister form.

Before we proceed any further it is necessary to define an important class of field extensions called function fields. By viewing quadratic forms as homogeneous polynomials of degree 2, we are naturally led to consider the algebraic varieties corresponding to quadric surfaces, i.e. integral projective quadrics. Under mild assumptions on the quadratic form, such as non-degeneracy and anisotropy, we are induced to studying the class of *function fields* which in the context of quadratic forms, refers to the function field of the projective quadric associated to the form. The pursuit of a deep understanding of function fields has contributed to numerous fundamental results in the theory of quadratic forms [AP71] and central simple algebras [HHK09]. In particular, both the Milnor Conjecture [Vo03] and the Bloch-Kato Conjecture [Vo11] rely on the machinery of function fields. For a $(n+1)$-dimensional quadratic form $\phi = \langle a_0, \cdots, a_n \rangle$ over K, the *function field* $K[\phi]$ of the quadric associated to ϕ is given by solving the polynomial equation with respect to X_0,

$$K[\phi] = K(X_1, \cdots, X_n)(\sqrt{-(a_1X_1^2 + \cdots + a_nX_n^2)/a_0})$$

An important property to notice about the function fields is that if ϕ is anisotropic over K then ϕ is isotropic over $K[\phi]$. Alternatively, if ϕ is isotropic, then the function field $K[\phi]$ is a transcendental extension of K. More generally,

Lemma 1.3.2. *A function field $K[\phi]$ is purely transcendental if and only if the form ϕ is isotropic over K.*

Proof. It is easy to see that isotropy over $K[X]$ implies isotropy over K. Indeed, assume $f_i \in K[X]$ are such that $\phi(f_1, \ldots, f_n) = 0$, and we may assume without loss of generality that $X \nmid f_i$ for some $i = 1, \ldots, n$. We can see in this case that $\phi(f_1(0), \ldots, f_n(0)) = 0$ where $f_i(0) \in K$ and $f_j \neq 0$ for some $j = 1, \ldots, n$. The reverse direction follows by definition, i.e. ϕ is isotropic if and only if $\phi \cong \mathbb{H} \perp \psi$ for some quadratic form ψ over K. In particular, since the transformation $\mathbb{H}(\frac{X_0+X_1}{2}, \frac{X_0-X_1}{2}) = X_0X_1$ is an isometry $G : V \longrightarrow V$ of $\mathbb{H}$ can be extended to ϕ by $\phi(GX) = X_0X_1 + \psi(X_2, \ldots, X_n)$. This implies that $K[\phi] = K[X_1, \ldots, X_n]$. □

Remark 1.3.3. We can find an even smaller "generic" subfield $K(\phi) \subset K[\phi]$ with respect to the condition of obtaining isotropy. Observing that $a_0X_0^2 + \cdots + a_nX_n^2 = 0$ and considering the mapping $Z_i \longrightarrow X_i/X_0$ we see that the field extension,

$$K(\phi) = K(Z_1, \cdots, Z_n) = K(Z_1, \cdots, Z_{n-1})(\sqrt{-(a_0 + a_1Z_1^2 + \cdots + a_{n-1}Z_{n-1}^2)/a_n})$$

satisfies properties similar to $K[\phi]$. In particular, we have that ϕ is isotropic over $K(\phi)$.

We will state an important result concerning function fields which we will rely on shortly.

Theorem 1.3.4. *[Lam05, Remark X.4.8][EKM08, Theorem 22.5] Suppose φ is a quadratic form over K. If q is an anisotropic Pfister form hyperbolic over $K[\varphi]$, then $ab\varphi \subseteq q$ for any $a \in D_K(q)$ and any $b \in D_K(\varphi)$.*

The problem of embeddability was first considered by Hoffmann [Hof95], where it was shown that if $\dim q \leq 2^n + 1$, then there exists a field extension L/K, such that ϕ is $(n+1)$-embeddable over L. If the field extension is required to be purely transcendental, then Hoffmann and Izhboldin [HI00, Theorem 1.1] showed that any anisotropic form q can be embedded after base changing to some purely transcendental extension $K(X)$ over K (preserving the anisotropy of q). The idea is rather simple; we assume that $q_1 \subset q$ is a subform maximal with respect to embeddability in some Pfister form π over K, and we recursively construct higher Pfister forms containing larger subforms. In particular, if $q_1 \perp q_2 \cong q$ and $q_1 \perp \pi_2 \cong \pi$, then $\langle 1, \pi_2(Y) - q_2(X)\rangle \otimes \pi$ is an anisotropic Pfister form containing a subform of q of dimension strictly bigger than $\dim q_1$ over $K(X,Y) = K(X_1, \ldots, X_{\dim q_2}, Y_1, \ldots, Y_{\dim \pi_2})$. To see that this is indeed the case, it suffices to compare the Witt indices of $\pi \perp -q$ and $\langle 1, q_2(X) - \pi_2(Y)\rangle \otimes \pi \perp -q$. We will see a special case of this result in Lemma 1.3.8.

The next natural step is to consider the structure determined by embeddability. In other words, if q is n-embeddable ($q \subset \pi$) with minimal π, to what extent is π determined by q. This generalizes the notion of a Pfister neigbor which determines conditions for a (2^n+1)-dimensional quadratic form to be $(n+1)$-embeddable. Note that over a local or global field, all (2^n+1)-dimensional quadratic forms are $(n+1)$-embeddable. We study the first non-trivial case of embeddability by studying Albert forms and offer a new result connecting Pfister forms with their embedded Albert forms in Theorem 1.4.4. Let us begin by stating a classical result connecting biquaternions and Albert forms.

Theorem 1.3.5. *[Lam05, Theorem III.4.8] Let $\left(\frac{a,b}{K}\right)$ and $\left(\frac{c,d}{K}\right)$ be quaternion algebras over K, and $A = \left(\frac{a,b}{K}\right) \otimes \left(\frac{c,d}{K}\right)$ denote a biquaternion K-algebra. The following are equivalent:*

1. *A is a division algebra.*

2. *$q = (\langle\langle a,b\rangle\rangle \perp -\langle\langle c,d\rangle\rangle)_{an}$ is anisotropic over K.*

3. $\left(\frac{a,b}{K}\right)$ *and* $\left(\frac{c,d}{K}\right)$ *are division algebras which do not share a common quadratic splitting field.*

We call the quadratic form

$$q = (\langle\langle a, b\rangle\rangle \perp - \langle\langle c, d\rangle\rangle)_{\mathrm{an}} = \langle -a, -b, ab, c, d, -cd\rangle\,,$$

with $a, b, c, d \in K^{\times}$, an anisotropic *Albert form*. It follows easily that any 6-dimensional form $\phi \in I_q^2(K)$ is similar to an Albert form [Lam05, Corollary XII.2.13]. We proceed to show that q cannot be a subform of any anisotropic 3-fold Pfister form.

Lemma 1.3.6. *No anisotropic Albert form q is* 3*-embeddable.*

Proof. Assume the contrary. Then the fact that $q \perp \langle x, y\rangle \in I_q^3(K)$ implies

$$\det(q \perp \langle x, y\rangle) = 1.$$

However, since $\det(q) = -1$ we must also have $\det(\langle x, y\rangle) = -1$, the later being equivalent to requiring $\langle x, y\rangle \cong \mathbb{H}$ (see [Lam05, Theorem I.3.2]). We conclude that any 3-fold Pfister form containing q as a subform must be isotropic, hence hyperbolic (see [Lam05, Theorem X.1.7]). □

Following results established in [Hof95, Main Lemma] we see that there exists a field extension L/K with several nice properties such that a prescribed Albert form q is 4-embeddable. We abuse notation and denote such a field by K again, since we are mostly interested in understanding the structure of the Pfister form not its existence.

Remark 1.3.7. The above assumption is necessary since it is not always the case that there even exists an n-embeddable quadratic form for some $n \in \mathbb{N}$ over an arbitrary base field K. Indeed, [HI00] refers to Kahn for providing the following example: Assume $\mathrm{cd}(K) \leq 3$. Then $I_q^m(K) = 0$ for all $m \geq 4$, meaning that there are no non-trivial m-fold Pfister forms to embed into. Alternatively, we could restrict ourselves to considering fields K, such that $\mathrm{cd}(K) \geq 3$, but this is not a sufficient condition to ensure 4-embeddability for Albert forms. Consider $K(X) := K(X_1, \ldots, X_6)$; then

$$q = \langle X_1, X_2, X_3, X_4, X_5, -X_1X_2X_3X_4X_5\rangle \nsubseteq \langle\langle f_1(X), f_2(X), f_3(X), f_4(X)\rangle\rangle\,,$$

for any $f_1(X), \cdots, f_4(X) \in K[X]$.

Henceforth, let us assume that the Albert form is 4-embeddable i.e. $q \subset \pi$, where π is some anisotropic 4-fold Pfister form over K. Now we would like to proceed with as much generality as possible, however not much can be said without one additional assumption: $q \subset \pi'$, where $\pi' \perp \langle 1 \rangle \cong \pi$. It turns out this can be imposed by extending our arguments to a field extension L of K, such that $\langle 1 \rangle \perp q_L \subset \pi_L$ and π_L remains anisotropic.

Lemma 1.3.8. *[Hof95, Proof of Theorem 1.1.] Assume ϕ is a codimension 1 subform of an anisotropic quadratic form ψ. If $\phi \subset \pi$, where π is an anisotropic m-fold Pfister form, then there exists a field extension L/K, such that $\psi_L \subset \pi_L$.*

Proof. By assumption, $\phi \perp \phi_1 \cong \pi$ and $\phi \perp \langle x \rangle \cong \psi$ for some $x \in K^\times$. Consider the function field extension $L = K[\phi_1 \perp -\langle x \rangle]$. Then by base change to L we see that $(\phi_1)_L \perp -\langle x \rangle_L$ is isotropic, implying

$$\langle x \rangle_L \subset (\phi_1)_L.$$

It remains to show that π_L is anisotropic. Indeed, if we assume that the Pfister form π_L is isotropic, it must also be hyperbolic so by Theorem 1.3.4 we have $cd(\phi_1 \perp -\langle x \rangle) \subset \pi$ for any $c \in D_K(\phi_1 \perp -\langle x \rangle))$ and $d \in D_K(\pi)$. In particular, we may choose $d = 1$, since π is a m-fold Pfister form i.e. $\pi = \langle 1, -a_1 \rangle \dots \langle 1, -a_m \rangle$ for some $a_1, \cdots, a_m \in K$. It follows that

$$c(\phi_1 \perp -\langle x \rangle) \perp \phi_2 \cong \pi$$

for some anisotropic quadratic form ϕ_2. Moreover, since Pfister forms are strongly multiplicative [Lam05, Theorem X.2.8] we have that $c\pi \cong \pi$ for any $c \in D_K(\phi_1) \subseteq D_K(\pi)$. In particular, $\pi \cong c\pi \cong c\phi_1 \perp c\phi$ implies $c(\phi_1 \perp -\langle x \rangle) \perp \phi_2 \cong \pi \cong c\phi_1 \perp c\phi$, which by Theorem 1.1.3 lets us conclude that

$$-c\langle x \rangle \perp \phi_2 \cong c\phi.$$

or alternatively (multiplying by c^{-1}),

$$-\langle x \rangle \perp c^{-1}\phi_2 \cong \phi.$$

However, this contradicts the anisotropy of $\phi \perp \langle x \rangle \cong \psi$. □

Henceforth we make the additional assumption that $\langle 1 \rangle \perp q$ is anisotropic. The above Lemma motivates our assumption that $q \subset \pi$ implies $\langle 1 \rangle \perp q \subset$

π_L. We define the *pure subform* ϕ' of ϕ to be a quadratic form over K such that

$$\langle 1\rangle \perp \phi' \cong \phi.$$

We proceed by invoking a fundamental result in Pfister forms also known as the *Pure Subform Theorem*:

Theorem 1.3.9. *[Lam05, Thereom X.1.5] Let $\varphi = \langle\langle a_1, \ldots, a_m\rangle\rangle$ be an m-fold Pfister form with $-b \in D_K(\varphi')$. Then there exist $b_2, \ldots, b_m \in K$ such that*

$$\varphi \cong \langle\langle b, b_2, \ldots, b_m\rangle\rangle .$$

The following is a generalization of Theorem 1.3.9:

Theorem 1.3.10. *[Lam05, Theorem X.1.10] If $\tau = \langle\langle b_1, \ldots, b_r\rangle\rangle$, $r \geq 0$, $\nu = \langle\langle d_1, \ldots, d_s\rangle\rangle$, $s \geq 1$, and $-e_1 \in D_K(\tau\nu')$, then there exist $e_2, \ldots, e_s \in K^\times$, such that*

$$\langle\langle b_1, \ldots, b_r, d_1, \ldots, d_s\rangle\rangle \cong \langle\langle b_1, \ldots, b_r, e_1, \ldots, e_s\rangle\rangle .$$

Remark 1.3.11. There is a slight difference between the Theorems cited here and those appearing in the text. This difference is captured by a preference for alternative notation. In [Lam05], $\langle\langle a\rangle\rangle = \langle 1, a\rangle$ whereas we use $\langle\langle a\rangle\rangle := \langle 1, -a\rangle$. With this difference in mind, the results are equivalent.

Corollary 1.3.12. *[Lam05, Corollary X.1.11] Let q be an anisotropic Pfister form. If $q \cong \langle 1, -b, -c, \ldots\rangle$ with $b, c \in K^\times$, then*

$$q \cong \langle\langle b, c, d_1, \ldots, d_s\rangle\rangle$$

for some suitable $d_1, \ldots, d_s \in K^\times$.

Proof. An immediate consequence of Theorem 1.3.9 is that $q \cong \langle\langle b, b_1, \ldots, b_m\rangle\rangle$ for some $b_1, \ldots, b_m \in K^\times$. Moreover, since $q \cong \langle 1, -b, -c, \ldots,\rangle = \langle\langle b\rangle\rangle \perp \langle -c, \ldots\rangle$ we have that

$$\langle\langle b\rangle\rangle \perp \langle\langle b\rangle\rangle \langle\langle b_1, \ldots, b_m\rangle\rangle' \cong \langle\langle b\rangle\rangle \perp \langle -c, \ldots\rangle$$

which by Theorem 1.1.3 implies $-c \in D_K(\langle\langle b\rangle\rangle \langle\langle b_1, \ldots, b_m\rangle\rangle')$. By applying Theorem 1.3.10 we arrive at our desired result. $\square$

Lemma 1.3.13. *Let $q = \langle -a, -b, ab, c, d, -cd\rangle$ be an anisotropic Albert form with $a, b, c \in K^\times$ and π be a 4-fold Pfister form over K. If $\langle 1\rangle \perp q \subset \pi$, then*

$$\pi \cong \langle\langle a, b, -c, z\rangle\rangle$$

for some $z \in K^\times$.

Proof. By assumption, $\langle 1, -a, -b, ab, c, d, -cd\rangle \subset \pi$. Theorem 1.3.12 implies $\pi \cong \langle\langle a, b, u, v\rangle\rangle$ for suitable $u, v \in K^\times$. In particular,

$$\langle 1, -a, -b, ab, c, d, -cd\rangle \subset \langle\langle a, b, u, v\rangle\rangle$$

can be better understood by decomposing the Pfister form as

$$\langle 1, -a, -b, ab\rangle \perp \langle\langle a, b\rangle\rangle \langle\langle u, v\rangle\rangle' .$$

where we recall that $\langle 1\rangle \perp \langle\langle u, v\rangle\rangle' = \langle\langle u, v\rangle\rangle$. By Theorem 1.1.3 we have

$$\langle c, d, -cd\rangle \subset \langle\langle a, b\rangle\rangle \langle\langle u, v\rangle\rangle' .$$

Now, Theorem 1.3.10 with $c \in D_K(\langle\langle a, b\rangle\rangle \langle\langle u, v\rangle\rangle')$ combine to imply

$$\langle\langle a, b, u, v\rangle\rangle \cong \langle\langle a, b, -c, z\rangle\rangle$$

for suitable $z \in K^\times$. □

Remark 1.3.14. Notice that in the case $-1 \in (K^\times)^2$, it is easy to see that for $a, b, c, d \in K^\times$, an Albert form $q = \langle a, b, ab, c, d, cd\rangle$ embeds canonically into the pure part of a 4-fold Pfister form $\langle\langle a, b, c, d\rangle\rangle$ for a field K of any characteristic (see [CD17, Corollary 5.5] for characteristic 2 case). However, whether $\langle\langle a, b, c, d\rangle\rangle$ is anisotropic, or whether such a 4-fold Pfister form is unique remains unclear. We will show in the following section that even in the case $-1 \notin (K^\times)^2$, we still expect to see a unique embedding under certain additional assumptions which are intuitively motivated at the end.

1.4 Strong Albert forms

Let K' be a field of characteristic $\neq 2$. In this section, we proceed to determine a novel result on the structure of 4-fold Pfister form containing an Albert subform. Assume $a, b, c \in K'$ and $z \in K'(d)$, where $\mathrm{trdeg}_{K'}(K'(d)) = 1$. We make frequent use of the following exact sequence due to Milnor (see [Lam05, Theorem IX.3.1]).

Theorem 1.4.1. *Let $E = K'(d)$ and let i be the functorial map*

$$i : W(K') \longrightarrow W(E).$$

Then the following sequence of abelian groups is split exact:

$$0 \longrightarrow W(K') \xrightarrow{i} W(E) \xrightarrow{\oplus\partial_\pi} \oplus_\pi W(E_\pi) \longrightarrow 0,$$

where the direct sum extends over all monic, irreducible polynomials $\pi \in K'[d]$ and E_π is the residue field of the field completion with respect to π.

Recall the definition of the *second residue homomorphism* ∂_π :

$$\partial_\pi : W(K') \longrightarrow W(E_\pi) = W(K'[d]/\pi)$$

$$q_1 \perp \pi q_2 \mapsto \overline{q_2},$$

where $q_1 = \langle a_1, \ldots, a_n \rangle$, $q_2 = \langle b_1, \ldots, b_m \rangle$,

$$\partial_\pi(a_i) \equiv 0 \bmod \pi,$$

$$\partial_\pi \pi(b_j) \equiv \overline{b_j} \bmod \pi$$

for all i, j. The proof of our result relies on two standard theorems in quadratic form theory:

Theorem 1.4.2. *[Lam05, Theorem VII.2.7] Let $K \subset L$ be a field extension of odd degree. If a K-quadratic form q is anisotropic over L, then q is anisotropic over K.*

Theorem 1.4.3. *[Lam05, Theorem IX.1.3] Let γ be a quadratic form over K', and let $p(x) \in K'[x] \cap D_{K'(x)}(\gamma)$ where x is a transcendental element over K'. Then,*

1. *$p(x)$ is already represented by γ over $K'[x]$.*

2. *If $e \in K'$ is such that $p(e) \neq 0$, then $p(e) \in D_{K'}(\gamma)$.*

Combining the above results, we have discovered the following relationship between Albert forms and 4-fold Pfister forms:

Theorem 1.4.4. *Consider an anisotropic Albert form* $q = \langle a, b, -ab, -c, -d, cd\rangle$ *over* K *where* $a, b, c \in K' \subset K$, $trdeg_{K'} K'(d) = 1$*. If* $\langle 1\rangle \perp q$ *is 4-embeddable, that is,* $\langle 1\rangle \perp q \subset \langle\langle x, y, z, w\rangle\rangle$ *for some anisotropic Pfister form* $\langle\langle x, y, z, w\rangle\rangle$ *over* K*, then*

$$\langle\langle x, y, z, w\rangle\rangle \cong \langle\langle a, b, -c, -d\rangle\rangle .$$

In particular, if $-1 \in (K^\times)^2$ *we have that*

$$\langle\langle x, y, z, w\rangle\rangle \cong \langle\langle a, b, c, d\rangle\rangle .$$

Proof. By lemma 1.3.13, we see that

$$\langle c, d, -cd\rangle \subset \langle\langle a, b\rangle\rangle \langle\langle -c, z\rangle\rangle' ,$$

which reduces down to

$$d \langle 1, -c\rangle \subset c \langle -a, -b, ab\rangle \perp -z \langle\langle a, b, -c\rangle\rangle .$$

Now we will proceed to show that the isotropy of

$$c \langle -a, -b, ab\rangle \perp -z \langle\langle a, b, -c\rangle\rangle \perp \langle -d\rangle$$

implies $z = d$. Note that isotropy over E implies isotropy over E_π for all monic irreducible $\pi \in K'[d]$, since $\langle 1, -1\rangle \cong \pi \langle 1, -1\rangle$.

Using the descent of isotropy and letting $\pi = d$, we have by Milnor's exact sequence that

$$\partial_d \left(c \langle -a, -b, ab\rangle \perp -z \langle\langle a, b, -c\rangle\rangle \perp \langle -d\rangle\right)$$

$$= \partial_d(-z \langle\langle a, b, -c\rangle\rangle) \perp \langle -1\rangle$$

is isotropic. Now, given $\langle\langle a, b, -c\rangle\rangle \in W(K')$ we must have that $z = dz'$ with $z' \in K'[d]$ for the above equation to be isotropic. We write

$$z' = f_n d^n + \ldots + f_0,$$

for some $n \in \mathbb{N}$ and $f_0, \ldots, f_n \in K'$.

We continue by showing n is odd. Indeed, following Theorem 1.4.3,

$$\langle d\rangle \subset c \langle -a, -b, ab\rangle \perp -z \langle\langle a, b, -c\rangle\rangle$$

over $K'(d)$ implies

$$d \in D_{K'[d]}(c\langle -a,-b,ab\rangle \perp -z\langle\langle a,b,-c\rangle\rangle),$$

which means we can represent $d = cu - dz'v$ for some $u \in D_{K'[d]}(\langle -a,-b,ab\rangle)$, $v \in D_{K'[d]}(\langle\langle a,b,-c\rangle\rangle)$.

First note that both $\deg u, \deg v \equiv 0 \bmod 2$, since they are represented by quadratic forms. We next consider the highest degree terms with respect to d and recall that, by assumption, all quadratic forms appearing here are anisotropic over K'. Since $\deg dz'v = 1 + \deg z' + \deg v$ and $\deg cu = \deg u$, and $\deg v \equiv \deg u \equiv 0 \bmod 2$, we have that $1 + \deg z' + \deg v = \deg u$ if and only if $\deg z'$ is odd (if $u \neq 0$) or $\deg z' = 0$ (if u $= 0$). This is relevant since $d = cu - dz'v$ is possible as a representation with $u, v \in K'[d]$ only when $u \neq 0$ and degree > 1 terms cancel or $u = 0$ and $z'v = -1$ in which case $\deg z' = 0$. Let us assume that $\deg z'$ is odd; then z' has a decomposition

$$z' = \pi_1 \cdots \pi_m$$

into its irreducible components $\pi_1, \ldots, \pi_m \in K'[d]$. Moreover, it must be the case that at least one π_i has odd degree, say π_1. Then, repeating the above argument with $\pi = \pi_1$ we see by descent of isotropy,

$$\partial_{\pi_1}(c\langle -a,-b,ab\rangle \perp -z\langle\langle a,b,-c\rangle\rangle \perp -\langle d\rangle)$$

$$= d\pi_2 \cdots \pi_n \langle\langle a,b,-c\rangle\rangle$$

is isotropic in $K'[d]/\pi_1$. However, $[K'[d]/\pi_1 : K']$ is odd implies that $\langle\langle a,b,-c\rangle\rangle$ is anisotropic over $K'[d]/\pi_1$ by Theorem 1.4.2, which is a contradiction. Hence $\deg z$ cannot be odd and must therefore be 0 i.e. $z' = f_0 \in K'$. Now since $d = cu - df_0v$ we have by degree and component comparison that $cu = 0$ and $f_0v = -1$. In particular, $-f_0 = v^{-1}$ and $v \in D_{K'[d]}(\langle\langle a,b,-c\rangle\rangle)$ implies $-f_0 \in D_{K'[d]}(\langle\langle a,b,-c\rangle\rangle)$. Combining the preceeding results we see that $z = dz' = df_0$ and $\langle\langle a,b,-c,z\rangle\rangle \cong \langle\langle a,b,-c,-d\rangle\rangle$ since $\langle\langle a,b,-c,z\rangle\rangle = \langle\langle a,b,-c\rangle\rangle \perp \langle\langle a,b,-c\rangle\rangle\langle -z\rangle = \langle\langle a,b,-c\rangle\rangle \perp \langle\langle a,b,-c\rangle\rangle\langle -df_0\rangle$ and by roundness of Pfister forms [EKM08, Corollary 6.2] we have

$$\langle\langle a,b,-c\rangle\rangle\langle -df_0\rangle \cong \langle\langle a,b,-c\rangle\rangle\langle d\rangle.$$

Putting this all together yields our desired result. $\square$

In light of Lemma 1.3.8, it suffices to assume $q \subset \langle\langle x, y, z, w\rangle\rangle$ by base change to a function field extension. However, we will need to be extremely careful that the algebraic independence of the coefficients is still respected over the function field (this is not generally the case). In Section 2.3 we will see an alternative method of understanding embeddings of Albert forms into 4-fold Pfister forms by establishing a set of elements which are by their very construction m-embeddable for prescribed m. Moreover, as a consequence of the above result, we will be able to realize embeddability as a decomposition of a central simple algebra in terms of quaternion algebras.

Remark 1.4.5. An interesting consequence of Theorem 1.4.4 follows by using [Mer13, Example 3.7] which, in our interest, defines a non-trivial cohomological invariant of degree 4 central simple algebras. In particular, it shows that the map sending a biquaternion K-algebra to the appropriate 4-fold Pfister form describes a unique inclusion of the underlying Albert form into the pure part of the 4-fold Pfister form. Moreover, if we assume all fields considered contain an algebraically closed field then Theorem 1.4.4 implies that whenever the cohomological invariant of a biquaternion algebra is non-trivial, the embedding of the Albert form into the pure part of an anisotropic Pfister form is unique.

We conclude observing the following result which arises while considering the reduced Whitehead group in algebraic K-theory [KMRT98, §17]. In particular, it allows us to postulate that the condition $q \subset \pi$ is not strong enough to determine π. Indeed, $\langle 1\rangle \perp q \subset \pi$ might be necessary.

Corollary 1.4.6. *[KMRT98, Proposition 17.30] Let p be an Albert form and consider the change of base map*

$$i : I_q^4(K)/I_q^5(K) \longrightarrow I_q^4(K(q))/I_q^5(K(q)).$$

Then,

$$\ker(i) = \{\pi + I_q^5(K) \mid p \subset \pi,\ 4 \text{ fold Pfister}\}.$$

Chapter 2

Associative algebras

In this chapter, we will develop the notion of a central simple algebra (CSA) which forms the basis of several results in subsequent chapters. For instance, the basic objects of the Brauer group introduced in Chapter 1 are central simple algebras while the definition of Hermitian forms in Chapter 3 and the classification of split semisimple algebraic groups fundamentally rely on central simple algebras in their formulation. In particular, we study the properties of anti-automorphisms of order 2 on CSAs insofar as they determine the algebraic structure of the underlying algebra. In the final section, we provide a new constructive proof of the Pfister Factor Conjecture for $n \leq 3$, characterizing the precise relationship between involutions and Pfister forms. We conjecture that this result is limited only by the computational complexity of determining what we call Pfister elements.

2.1 Central simple K-algebras, the Brauer group and quaternions

Let K denote a field of characteristic $\neq 2$. By K^{alg} we mean an algebraic closure of the field K, and by K^{sep} a separable closure of K. All K-algebra are assumed to be finite-dimensional, unital and associative algebras over K.

A *central simple K-algebra* A is a K-algebra, such that A has no proper two-sided ideals and $Z(A) = K \cdot 1_A$, where $Z(A)$ denotes the center of A. The fundamental Theorem of this section is Wedderburn's Structure Theorem for central simple K-algebras:

Theorem 2.1.1. *(Wedderburn's Structure Theorem) The following are equivalent:*

1. *A is a central simple K-algebra.*

2. *There exists a unique pair (n, D), where $n \in \mathbb{N}$ and D is a central division K-algebra, such that*
$$A \cong M_n(D).$$

3. *There is a K-algebra homomorphism*
$$\phi : A \otimes_K L \longrightarrow M_m(L), \quad \textit{for some } m \in \mathbb{N}$$
with a field extension L of K, called a splitting field of A.

4. *The canonical map*
$$A \otimes A^{op} \longrightarrow \operatorname{End}_K A$$
$$a \otimes b^{op} \mapsto (x \mapsto axb)$$
is an isomorphism of K-algebras.

We define A^{op} to be the *opposite algebra* of A consisting of the same elements as that of A with multiplication defined by

$$a^{op}b^{op} = (ba)^{op}$$

for $a, b \in A$. An immediate consequence of conditions 1. and 2. is that both K^{alg} and K^{sep} are splitting fields of A . Moreover,

$$\dim_K A = \dim_{K^{\mathrm{alg}}} A \otimes K^{\mathrm{alg}} = \dim_{K^{\mathrm{alg}}} M_m(K^{\mathrm{alg}}) = m^2;$$

and m is called the *degree* of A, denoted by $\deg A$.

The relationship between A and D in 2. can also be expressed via the endomorphism ring of a (unique) simple left A-module V, in the following way.

$$A = \operatorname{End}_D V, \qquad D = \operatorname{End}_A V \tag{1}$$

and

$$\dim_K V = \deg A \deg D.$$

The algebra D occuring in the decomposition 2. of A is called the *division algebra associated to A*. We define the *index* of A to be the degree of the division algebra D associated to K, i,e.

$$\mathrm{ind}(A) = \deg(D).$$

There are many interesting results in the study of central simple K-algebras concerning the central division K-algebra D associated to a central simple K-algebra A. In favor of this perspective, we define the following equivalence relation, known as *Brauer equivalence*. Let A, B be central simple K-algebras; then

$$A \sim B \text{ if and only if } M_{n_1}(A) \cong_K M_{n_2}(B)$$

for some $n_1, n_2 \in \mathbb{N}$. By 1. we may reformulate the Brauer-equivalence relation above in terms of the simple left A-module V and B-module W in the following way:

$$A \sim B \text{ if and only if } \operatorname{End}_A V \cong \operatorname{End}_B W.$$

Clearly, if A and B are two Brauer-equivalent central simple K-algebras and $\deg A = \deg B$, then $A \cong B$. More than that, the tensor product endows central simple K-algebras under Brauer-equivalence with the structure of an abelian group, called the *Brauer group* of K and denoted by $\mathrm{Br}(K)$.

Proposition 2.1.2. *[Pie82, Theorem 13.3]: For a central simple K-algebra A, the following are equivalent:*

1. *L is a splitting field of A.*
2. *There exists a central simple K-algebra B Brauer-equivalent to A, such that $B \supset L$ and $[L : K] = \deg B$.*

If L is a field extension of K contained in B such that $[L : K] = \deg B$ we say that L is a *maximal subfield* of B. Indeed, if B is a central division K-algebra such that L is a maximal subfield of B then B contains no proper field extension of L. This will be an important point to keep in mind when we construct a cross-product algebra later which will be the first step in realizing the Brauer group of K as the second Galois cohomology group $H^2(G_K, K^{\mathrm{sep}\times})$, where by $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ we mean the absolute Galois group of the field K. In particular, we will need an incredibly powerful theorem due to Skolem-Noether:

Theorem 2.1.3. *(Skolem-Noether) Let A be a central simple K-algebra and B a simple K-subalgebra. If $f, g : B \longrightarrow A$ are K-algebra homomorphisms, then there exists $a \in A^\times$, such that*

$$f(b) = \mathrm{Inn}(a) \circ g(b) = ag(b)a^{-1}, \text{ for all } b \in B.$$

Moreover, every automorphism of A is an inner automorphism.

Now, the general idea is to characterize the Brauer-equivalence classes of A by the internal structure of a subalgebra which should determine the entire class up to a condition which depends on an element in $H^2(G_K, K^{\mathrm{sep}\times})$. To do this, we make use of a standard fact in central division K-algebras which states that any central division K-algebra D contains a maximal separable splitting field L. The argument is trivial in the case of finite fields and inductive otherwise. Now, if L is a maximal separable splitting field of D, then L^{sep} is a Galois splitting field of D by the uniqueness of the central division K-algebra associated to a central simple K-algebra. In view of this remark, Proposition 2.1.2 implies that any central simple K-algebra A is Brauer-equivalent to a central simple K-algebra B containing a Galois splitting field L and such that $\deg B = [L : K]$. This can be rephrased in terms of the functorial properties of the Brauer group of K, namely

$$\mathrm{Br}(K) = \textstyle\bigcup_{E \supset K \text{ Galois}} \mathrm{Br}(E/K), \tag{2}$$

with $\mathrm{Br}(E/K) = \ker(\mathrm{Br}(K) \longrightarrow \mathrm{Br}(E))$ defined by sending $[A]$ to $[A \otimes_K E]$. Assuming (via Brauer-equivalence) that A contains a maximal Galois splitting field L, we obtain a set of linearly independent K-algebra homomorphisms

$$\{\sigma : L \longrightarrow L \subset A \mid \sigma \in \mathrm{Gal}(L/K)\},$$

which can be extended by Skolem-Noether to a set of linearly independent inner K-automorphisms of A given by elements $e_\sigma \in A$ for $\sigma \in \mathrm{Gal}(L/K)$ i.e.

$$\{\mathrm{Inn}(e_\sigma) : A \longrightarrow A \mid \sigma \in \mathrm{Gal}(L/K)\}.$$

As a consequence, we can derive the following structural conditions:

Lemma 2.1.4. *[Pie82, Lemma 14.1] If A satisfies the conditions above, then*

1. $\{e_\sigma \mid \sigma \in \mathrm{Gal}(L/K)\}$ *is an L-basis of $A \otimes_K L$.*

2. *If $\sigma, \tau \in \mathrm{Gal}(L/K)$, then $\phi(\sigma,\tau) = \mathrm{Inn}(e_{\sigma\tau}^{-1}) \circ \mathrm{Inn}(e_\sigma) \circ \mathrm{Inn}(e_\tau) \in L^\times$.*

3. $\phi(\sigma,\tau)\phi(\rho\sigma,\tau)^{-1}\phi(\rho,\sigma\tau)(\tau(\phi(\rho,\sigma)))^{-1} = 1$, *for all $\sigma, \tau, \rho \in \mathrm{Gal}(L/K)$.*

In terms of cohomology, 3. is equivalent to the 2-cocycle condition for cohomology groups. Conversely, if A satisfies 1., 2. and 3. then we say A is a *crossed product* of L and $\mathrm{Gal}(L/K)$ with respect to ϕ, denoted by $A := (L, \mathrm{Gal}(L/K), \phi)$. Moreover, as the following proposition demonstrates, all K-algebras which can be constructed as crossed product algebras are central simple K-algebras containing a maximal Galois splitting field.

Proposition 2.1.5. *[Pie82, Proposition 14.1] Let L be a Galois extension of K and assume*

$$\phi : \mathrm{Gal}(L/K) \times \mathrm{Gal}(L/K) \longrightarrow L^\times$$

satisfies the 2-cocycle condition, i.e. $\phi \in H^2(\mathrm{Gal}(L/K), L^\times)$.

If $\{e_\sigma | \sigma \in \mathrm{Gal}(L/K)\}$ is a basis for the K-space

$$A = \otimes_{\sigma \in Gal(L/K)} e_\sigma L,$$

then we can define a K-linear multiplication $\mu : A \times A \longrightarrow A$ by

$$\mu(\sum_\sigma e_\sigma c_\sigma, \sum_\tau e_\tau d_\tau) = \sum_{\sigma,\tau} e_{\sigma,\tau}\phi(\sigma,\tau)\tau(c_\sigma)d_\tau,$$

where $c_\sigma, d_\sigma \in L$ and A is a central simple K-algebra containing L as maximal splitting Galois extension of K. Moreover,

$$\{e_\sigma | \sigma \in \mathrm{Gal}(L/K)\}$$

satisfies 2. of Lemma 2.1.4 with respect to ϕ.

The above results allow us to relate $\mathrm{Br}(L/K)$ and $H^2(\mathrm{Gal}(L/K), L^\times)$ modulo some compatibility conditions.

Theorem 2.1.6. *[Pie82] If L/K is a Galois extension, then the map*

$$\alpha_{L/K} : H^2(\mathrm{Gal}(L/K), L^\times) \longrightarrow \mathrm{Br}(L/K)$$

$$\phi \mapsto [(L, \mathrm{Gal}(L/K), \phi)]$$

is an isomorphism.

By viewing the absolute Galois group Gal_K of the field K as a *profinite group* (see [Ber10, Section I.1.2.4]) we are able to leverage several nice functorial properties of the latter. In particular, using [Ber10, Theorem III.7.30], we have that

$$H^2(G_K, K^{\mathrm{sep}\times}) = \varinjlim_{E \supset K \text{ finite Galois}} H^2(\mathrm{Gal}(E/K), E^\times). \qquad (3)$$

This leads to the following powerful characterization of the Brauer group in terms of Galois cohomology,

Theorem 2.1.7. *[Pie82, Theorem 14.6][Ber10, Theorem VIII.20.6] The map*

$$\varinjlim_{L \supset K \ Gal} \alpha_{L/K} : H^2(G_K, (K^{sep})^\times) \longrightarrow \mathrm{Br}(K).$$

is an isomorphism.

Let A be a central simple K-algebra and assume that D is a central division K-algebra Brauer-equivalent to A. An incredibly useful consequence of this equivalence is that it allows us to compute the torsion of an element of A in $\mathrm{Br}(K)$ as a divisor of $\deg(D)$. In particular, assuming several facts with respect to the degree of a central simple K-algebra under base change (see [Pie82, Corollary 14.4]), it is easy to show that if D is of degree $p_1^{e_1} \cdots p_m^{e_m}$ then $D \cong D_1 \otimes \cdots \otimes D_m$, with D_i pairwise non-isomorphic central division K-algebras, such that $\deg D_i = p_i^{e_i}$, for $1 \leq i \leq m$

We conclude this section with a non-trivial example and application of a central division K-algebra which is not a matrix algebra. We define a *quaternion K-algebra*, denoted by $\left(\frac{a,b}{K}\right)$, to be the K-algebra on the two generators i, j subject to the relations

$$i^2 = a, j^2 = b, ij = -ji.$$

The first example of a quaternion algebra $\left(\frac{-1,-1}{K}\right)$ was discovered by Hamilton. As it turns out, these algebra have many interesting algebraic connections with quadratic forms, the most basic of which are illustrated below.

Fact 2.1.8. *[Lam05, Proposition III.1.1, Theorem III.2.8, Theorem III.2.11]*

1. $\left(\frac{a,b}{K}\right)$ *is a central simple* K*-algebra, for all* $a, b \in K^\times$.
2. $\left(\frac{a,b}{K}\right) \cong \left(\frac{ax^2,by^2}{K}\right)$, *for all* $a, b, x, y \in K^\times$.
3. $\left(\frac{1,-1}{K}\right) \cong \left(\frac{a,1}{K}\right) \cong \left(\frac{a,-a}{K}\right) \cong M_2(K)$, *for all* $a \in K^\times$.
4. $\left(\frac{a,b}{K}\right) \otimes_K L \cong \left(\frac{a,b}{L}\right)$.
5. $\left(\frac{a,b}{K}\right) \otimes_K \left(\frac{a,c}{K}\right) \sim \left(\frac{a,bc}{K}\right)$.

It is important to note that any central simple K-algebra A with $\deg A = 2$ is isomorphic to a quaternion K-algebra. Moreover, $\left(\frac{a,b}{K}\right)^{\otimes 2} \sim 1 \in \mathrm{Br}(K)$, and the fact that the quaternion K-algebras are minimal among elements of order 2 in $\mathrm{Br}(K)$ suggests that they should play an important role in the characterization of $\mathrm{Br}_2(K)$, the set of 2-torsion elements in $\mathrm{Br}(K)$. This turns out to be the correct intuition:

Theorem 2.1.9. *[Mer81] Let* $[A] \in \mathrm{Br}_2(K)$. *Then*

$$A \sim \otimes_{i=1}^{m} \left(\frac{a_i, b_i}{K}\right)$$

for some $m \in \mathbb{N}$ *and* $a_1, \dots, a_m, b_1, \dots, b_m \in K^\times$.

In fact, we will see in the next section that this Theorem can be reinterpreted in terms of involutions of central simple K-algebras.

2.2 Involutions

The aim of this section is to develop central simple K-algebras with involution as twisted forms of symmetric or skew-symmetric bilinear forms up to a scalar factor. We refer to [KMRT98] for a thorough treatment of the results presented in this section.

Let A be a central simple K-algebra. An *involution* on A is a map $\sigma : A \longrightarrow A$, such that for all $x, y \in A^\times$:

1. $\sigma(x+y) = \sigma(x) + \sigma(y)$,
2. $\sigma(xy) = \sigma(y)\sigma(x)$,
3. $\sigma^2(x) = x$.

A central simple K-algebra A equipped with an involution σ is denoted by (A, σ). The centrality of A implies that $\sigma(K) = K$. Moreover, $\sigma|_K : K \longrightarrow K$ is either the identity automorphism or a non-trivial automorphism of order two. If $\sigma|_K = \mathrm{id}_K$, then σ is called an involution *of the first kind*. Otherwise, if $\sigma|_K \neq \mathrm{id}_K$, then σ is called an involution *of the second kind*. It will be the convention of this thesis that by an involution σ we will always mean an involution of the first kind. We say that two central simple K-algebras with involution $(A, \sigma), (A', \sigma')$ are *isomorphic* if there exists a K-linear homomorphism $f : A \longrightarrow A'$, such that $f \circ \sigma = \sigma' \circ f$.

Let V be a finite dimensional vector space over K. A bilinear form $b : V \times V \longrightarrow K$ is called *non-singular*, if the induced map

$$\hat{b} : V \longrightarrow V^{\vee}$$

$$v \mapsto b(v, -)$$

is an isomorphism of K-vector spaces.

There is a canonical anti-automorphism of $\mathrm{End}_K V$ satisfying 1. and 2. associated to a non-degenerate bilinear form of V defined by

$$b \mapsto \sigma_b = \hat{b}^{-1} \circ (-)^t \circ \hat{b} : \mathrm{End}_K V \longrightarrow \mathrm{End}_K V.$$

with $\sigma_b(f) = \hat{b}^{-1} \circ f^t \circ \hat{b}$, where $f^t \in \mathrm{End}_K V$ is the transpose of f defined by sending g to $g \circ f$. Alternatively, σ_b can be characterized as the adjoint involution of b satisfying the property: $b(v, fw) = b(\sigma_b(f)v, w)$, for all $v, w \in V$.

The correspondence between non-degenerate bilinear forms V over K and involutions on $M_n(K)$ is given below for completeness

Theorem 2.2.1. *[KMRT98, Theorem 0.1] There is a one-to-one correspondence between the set of non-degenerate bilinear forms B over K modulo multiplication by invertible scalar factor and K-linear anti-automorphisms* *of* $\mathrm{End}_K V$*. In particular, under this equivalence, non-singular symmetric or skew-symmetric bilinear forms of* V *over* K *correspond to involutions of the first kind on* $\mathrm{End}_K V$*.*

To be more explicit, we may endow V with a basis $\{e_1, \ldots, e_n\}$ over K and identify $\mathrm{End}_K V$ with $M_n(K)$, so that there is a correspondence between non-degenerate bilinear forms of V over K and elements in $\mathrm{GL}_n(K)$ given by

$$b \mapsto [b(e_i, e_j)] = B$$

$$b(x, y) = x^T M y \leftarrow\!\shortmid [m_{ij}] = M.$$

We say b is *symmetric* if and only if $[b(e_i, e_j)] = [b(e_i, e_j)]^T$ and b is *skew-symmetric* if and only if $[b(e_i, e_j)] = -[b(e_i, e_j)]^T$. In particular, we identify the involution σ_b on $\mathrm{End}_K V$ with the involution σ_B on $M_n(K)$ given by $\sigma_B(M) = B^{-1}M^tB$ for all $M \in M_n(K)$.

Now, ideally, we would like to classify an involution σ on a central simple K-algebra using what we know about involutions on $M_n(K)$. The most natural way to do this is to lift the involution σ on A to an involution $\sigma \otimes_K \mathrm{id}_L$ on $A \otimes_K L \cong M_n(L)$, where L is a splitting field of A, and classify σ based on the symmetric or skew-symmetric properties of the non-degenerate bilinear form associated to the lift. The problem with doing this directly lies in the fact that neither the non-degenerate associated bilinear form b nor the invertible associated matrix form B are determined uniquely by σ. Moreover, both of these choices depend on the choice of the splitting field L which is clearly not unique. Therefore, we must identify the symmetric or skew-symmetric behavior of a non-degenerate bilinear form associated to an involution with the structural properties of the central simple K-algebra which are invariant under base change.

In this regard, we define the sets of symmetric and skew-symmetric elements in a central simple K-algebra A with involution σ as follows:

$$\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\} = \{a + \sigma(a) \mid a \in A\}$$

$$\mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\} = \{a - \sigma(a) \mid a \in A\}.$$

Furthermore, given any embedding

$$A \hookrightarrow A \otimes 1 \subset A \otimes K^{\mathrm{sep}} \cong M_n(K^{\mathrm{sep}}).$$

we can construct the unique characteristic polynomial $\mathrm{char.\,pol.}_{A,a}(X)$ (independent to the chosen embedding) of an element $a \in A$ viewed as an element of $M_n(K^{\mathrm{sep}})$. For more information, we refer to [Pie82, 16.1]. The characteristic polynomial $\mathrm{char.\,pol.}_{A,a}(X)$ is of the form

$$\mathrm{char.\,pol.}_{A,a}(X) = X^n - c_{n-1}(a)X^{n-1} + c_{n-2}(a)X^{n-2} - \ldots + (-1)^n c_0(a) \quad (2.1)$$

where $c_{n-1}(a) = \mathrm{Trd}_A(a)$ denotes the reduced trace and $c_0(a) = \mathrm{Nrd}_A(a)$ denotes the reduced norm. Moreover, we have the following result.

Lemma 2.2.2. *[KMRT98, Lemma 2.3] Let A be a central simple K-algebra with an involution σ. Then,*

$$t_A : A \times A \longrightarrow K$$

$$(x, y) \mapsto \mathrm{Trd}_A(xy)$$

is a non-singular symmetric bilinear form, such that $\mathrm{Sym}(A,\sigma) \perp \mathrm{Skew}(A,\sigma) = A$.

In particular, the structure of the characteristic polynomial is determined by the involution in a peculiar way.

Lemma 2.2.3. *[KMRT98, Proposition 2.2.9] Let A be a central simple K-algebra with involution σ of symplectic type. The minimal polynomial of every element in* $\mathrm{Sym}(A,\sigma)$ *is a square. In particular,* $\mathrm{Nrd}_A(s)$ *is a square in K for all $s \in$* $\mathrm{Sym}(A,\sigma)$.

As a consequence, to find invariants of the minimal polynomial in the symplectic case we will have to instead consider the *pfaffian characteristic polynomial*, $\mathrm{Prp}_{\sigma,s}(X) \in K[X]$ defined by the property that for every $s \in \mathrm{Sym}(A,\sigma)$,

$$\mathrm{char.pol}_{A,s}(X) = (\mathrm{Prp}_{\sigma,s}(X))^2.$$

For $s \in Sym(A,\sigma)$ we define the *pfaffian trace* and *pfaffian norm* as coefficents,

$$\mathrm{Prp}_{\sigma,s}(X) := X^m - \mathrm{Trp}_\sigma(s)X^{m-1} + \cdots + (-1)^m \mathrm{Nrp}_\sigma(s).$$

The invariants captured by the characteristic polynomial are instrumental in determining some key aspects of the algebraic structure determined by an involution on a central simple K-algebra. We proceed to show that the sets

$\mathrm{Sym}(A, \sigma)$ and $\mathrm{Skew}(A, \sigma)$ are stable under base change. Indeed, this follows by the stability of dimension under base change, i.e. $\dim_K \mathrm{Sym}(A, \sigma) = \dim_L \mathrm{Sym}(A \otimes L, \sigma \otimes \mathrm{id}_L)$ and $\dim_K \mathrm{Skew}(A, \sigma) = \dim_L \mathrm{Skew}(A \otimes L, \sigma \otimes \mathrm{id}_L)$. Moreover, assuming L is a splitting field of A and identifying $(A \otimes L, \sigma \otimes \mathrm{id}_L)$ with $(M_n(L), \sigma_B)$ we obtain the following relations:

1. $B = B^T$ if and only if the bilinear form b associated to $\sigma \otimes \mathrm{id}_L$ is symmetric,

2. $B = -B^T$ if and only if the bilinear form b associated to $\sigma \otimes \mathrm{id}_L$ is skew-symmetric.

As a consequence, if $M \in M_n(L)$, then $B^T = B$ implies

$$\sigma_B(M) = M \text{ if and only if } BM = (BM)^T,$$

and $B^T = -B$ implies

$$\sigma_B(M) = -M \text{ if and only if } BM = (BM)^T.$$

Rewriting these into a single statement gives

$$B^{-1} \circ \mathrm{Sym}(M_n(L), t) = \begin{cases} \mathrm{Sym}(A, \sigma), & \text{if } B = B^T \\ \mathrm{Skew}(A, \sigma), & \text{if } B = -B^T \end{cases}.$$

Moreover, in conjunction with the fact that $\dim \mathrm{Sym}(M_n(L), t) = \frac{n(n+1)}{2}$, we can summarize the results as follows

$$b \text{ is symmetric } \text{ if and only if } \dim_K \mathrm{Sym}(A, \sigma) = \frac{n(n+1)}{2}$$

and

$$b \text{ is skew-symmetric } \text{ if and only if } \dim_K \mathrm{Skew}(A, \sigma) = \frac{n(n+1)}{2}.$$

This definition is independent of both our choice of splitting field L and the associated bilinear form b. We may thus classify involutions on a central simple K-algebra A in the following manner:

1. An involution σ is of *orthogonal type* if for any splitting field L there exists an isomorphism $(A \otimes L, \sigma \otimes \mathrm{id}_L) \cong (\mathrm{End}_L(V), \sigma_b)$ with b a symmetric bilinear form.

2. Alternatively, we say an involution σ is of *symmetric type* if for any splitting field L there exists an isomorphism $(A\otimes L, \sigma\otimes \mathrm{id}_L) \cong (\mathrm{End}_L V, \sigma_b)$ with b a skew-symmetric bilinear form.

The relationship between involutions on A of orthogonal type and involutions on A of symplectic type is given in the following classification result.

Proposition 2.2.4. *[KMRT98, Proposition 2.7] Let A be a central simple K-algebra with an involution σ. Then*

1. *For all $u \in A^\times$ such that $\sigma(u) = \pm u$, the map $\mathrm{Inn}(u)\circ\sigma$ is an involution.*

2. *For every involution σ_1, there exists $u \in A^\times$, uniquely determined up to scalar factor such that $\sigma_1 = \mathrm{Inn}(u)\circ\sigma$ and $\sigma(u) = \pm u$. Moreover,*

$$\mathrm{Sym}(A,\sigma_1) = \begin{cases} u\,\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\sigma)u^{-1}, & \text{if } \sigma(u) = u \\ u\,\mathrm{Skew}(A,\sigma) = \mathrm{Skew}(A,\sigma)u^{-1}, & \text{if } \sigma(u) = -u \end{cases},$$

and

$$\mathrm{Skew}(A,\sigma_1) = \begin{cases} u\,\mathrm{Skew}(A,\sigma) = \mathrm{Skew}(A,\sigma)u^{-1}, & \text{if } \sigma(u) = u \\ u\,\mathrm{Sym}(A,\sigma) = \mathrm{Sym}(A,\sigma)u^{-1}, & \text{if } \sigma(u) = -u \end{cases}.$$

3. *Assume that $\sigma' = \mathrm{Inn}(u)\circ\sigma$, where $u \in A^\times$, such that $\sigma(u) = \pm u$. Then σ and σ' are of the same type if and only if $\sigma(u) = u$.*

In fact, the preceding proposition can be used to show the following more general structural result,

Corollary 2.2.5. *[KMRT98, Proposition 2.8] Let A be a central simple K-algebra with involution σ.*

1. *If $\deg A$ is odd, then $A \cong M_{\deg A}(K)$ and σ is necessarily of orthogonal type. In particular, $\mathrm{Skew}(A,\sigma)\cap A^\times = \emptyset$.*

2. *If $\deg A$ is even, then $\mathrm{ind}\, A = 2^k$ for some $k \in \mathbb{N}$ and A has involutions of both types. Moreover, $\mathrm{Skew}(A,\sigma)\cap A^\times \neq \emptyset$ and $\mathrm{Sym}(A,\sigma)\cap (A^\times \setminus \mathrm{Skew}(A,\sigma)) \neq \emptyset$.*

In light of Proposition 2.2.4, consider the central simple K-algebra $M_n(K)$ with involution t denoting the transpose. If σ is an involution on $M_n(K)$ we have that

$$\sigma = \mathrm{Inn}(M) \circ t,$$

for some $M \in \mathrm{GL}_n(K)$ uniquely determined up to a factor in $K^\times$ such that $M = \pm M^T$. Since the transpose involution is orthogonal by definition, we have that the involution σ is orthogonal if and only if $M = M^T$ and σ is symplectic if and only if $M \in \mathrm{Skew}(M_n(K), t)$.

By Corollary 2.2.5, we have thus characterized all involutions which can be equipped on a central simple K-algebras of odd degree. From this perspective, it is natural to consider what happens in the case of central simple K-algebras of even degree. In particular, what are the involutions on the quaternion K-algebras $\left(\frac{a,b}{K}\right)$ and their tensor products. To start we define an involution γ on $\left(\frac{a,b}{K}\right)$ by a map sending

$$r_0 \cdot 1 + r_1 \cdot i + r_2 \cdot j + r_3 \cdot k \mapsto r_0 \cdot 1 - (r_1 \cdot i + r_2 \cdot j + r_3 \cdot k).$$

It follows that γ is a symplectic involution on K, that is, $\mathrm{Sym}(\left(\frac{a,b}{K}\right), \gamma) = 1$. Moreover and perhaps more interestingly, this involution has several characteristic properties which define a structure on the central simple K-algebra in terms of coefficients of the characteristic polynomial Equation 2.1.

Proposition 2.2.6. *For γ defined as above, the following hold:*

1. $\mathrm{Trd}_{\left(\frac{a,b}{K}\right)}(x) = \gamma(x) + x$, *for all* $x \in \left(\frac{a,b}{K}\right)^\times$.

2. $\mathrm{Nrd}_{\left(\frac{a,b}{K}\right)}(x) = \gamma(x)x$, *for all* $x \in \left(\frac{a,b}{K}\right)^\times$,

3. *If σ is an involution on a central simple K-algebra A, such that*

$$\sigma(x) + x \in K, \text{ and } \sigma(x)x \in K, \text{ for all } x \in K,$$

then A is a quaternion algebra.

The involution γ on $\left(\frac{a,b}{K}\right)^\times$ is called the (canonical) *sympletic* involution on $\left(\frac{a,b}{K}\right)$. It is easy to see that every orthogonal involution σ on $\left(\frac{a,b}{K}\right)^\times$ is of the form $\sigma = \mathrm{Inn}(u) \circ \gamma$, where $u \in \mathrm{Skew}(\left(\frac{a,b}{K}\right)^\times, \gamma) \backslash K$ is uniquely determined up to a scalar factor in $K^\times$. To generalize the above toward the tensor product of quaternion K-algebras with involutions, we have the following:

Proposition 2.2.7. *[KMRT98, Proposition 2.23] Let (A_i, σ_i) be central simple K-algebras with involutions σ_i, $1 \leq i \leq n$. Then $(\otimes_{i=1}^n A_i, \otimes_{i=1}^n \sigma_i)$ is a central simple K-algebra with involution. Moreover, $\otimes_{i=1}^n \sigma_i$ is of symplectic type if and only if an odd number of involutions σ_i are of symplectic type.*

Thus we have defined a correspondence between involutions on $M_n(K)$ and symmetric/skew-symmetric bilinear forms. As a consequence we were able to characterize the symmetry conditions of the associated bilinear form as a structural property of a central simple K-algebra with involution. In addition, this classification made it possible to obtain results which have implications toward both the structure of the central simple K-algebra and all other possible involutions on it. However, due to the scope of this work, we have made no comment on the existence of such involutions nor on how these involutions compare via Brauer-equivalence. In order to remedy this, we will state several results which are based on a natural generalization of the results established thus far. The first of these is due to Albert [Alb39] in his seminal work *Structure of Algebras* which developed much of the modern structure theory of central simple K-algebras we have today.

Albert provided a simple criterion to determine whether or not a central simple K-algebra carries an involution. We will state his result without proof.

Theorem 2.2.8. *[Alb39, Theorem 10.19] Let A be a central simple K-algebra. Then*

$$A \text{ has an involution } \sigma \text{ if and only if } [A] \in \mathrm{Br}_2(K).$$

In particular, if A has an involution σ, then any central simple K-algebra B such that $A \sim B$ has an induced involution denoted by σ_B.

In particular, if (A, σ) is a central simple algebra equipped with an involution of the first kind, then $\mathrm{ind}(A) = 2^n$. Since A carries an involution we can see that the order of A in the Brauer group is either 1 if $\mathrm{ind}(A) = 1$ or 2 if $\mathrm{ind}(A) > 1$. Moreover, given that $A^{\mathrm{ind}(A)} = 1 \in Br(F)$, any prime dividing $\mathrm{ind}(A)$ must also divide the order of A in $\mathrm{Br}(K)$ (which we denote by by $\mathrm{ord}(A)$). Succinctly, this can be summarized as $\mathrm{ord}(A) = 2$ implies $\mathrm{ind}(A) = 2^n$. In fact, algebras which carry an involution have a particular representation in the Brauer group. This follows from a result of [Mer81] connecting Milnor K-theory, Galois cohomology and quadratic form theory together. We rephrase the result using our established notation,

Theorem 2.2.9. *[Mer81, Theorem 2.2] Let A be a central simple K-algebra with involution σ. Then*

$$A \sim \otimes_{i=1}^{m} \left(\frac{a_i, b_i}{K} \right),$$

for some $m \in \mathbb{N}$ and $a_1, \ldots, a_m, b_1, \ldots, b_m \in K^{\times}$.

We can rephrase the above results in terms of Hermitian forms as follows. Let A be a central simple K-algebra with an involution σ and D is a central division K-algebra such that $A \sim D$ with induced involution σ_D. A *Hermitian form* on D (with respect to an involution σ_D) is a bi-additive map

$$h : V \times V \longrightarrow D,$$

such that,

1. $h(av, bw) = \sigma_D(a) \cdot h(v, w) \cdot b$, for all $a, b \in D$ and $v, w \in V$,
2. $h(v, w) = \sigma_D(h(w, v))$, for all $v, w \in V$.

If 2. is replaced by $h(v, w) = -\sigma_D(h(w, v))$, for all $v, w \in V$, we say that h is *skew-hermitian*. Similar to the case of non-degenerate bilinear forms,

$$D = K \text{ and } \sigma_D = \sigma \mid_D .$$

We define a hermitian form to be non-degenerate by the condition that $h(v, w) = 0$ for all $w \in V$ implies $v = 0 \in V$. As was to be expected, the existence of a non-singular hermitian or skew-hermitian form on D implies the existence of an involution σ_h on A:

Proposition 2.2.10. *[KMRT98, Proposition 4.1] For every non-singular hermitian or skew-hermitian form h on M, there exists a unique involution σ_h on $A = \operatorname{End}_D V$, such that $\sigma_h|_K = \sigma|_K$, and*

$$h(x, f(y)) = h(\sigma_h(f)(x), y), \text{ for all } x, y \in V.$$

If $D = K$, Hermitian forms can be realized as quadratic forms. We conclude this section with the following generalization of the correspondence given at the beginning of the section.

Theorem 2.2.11. *[KMRT98, Theorem 4.2] If σ is an involution on D, the map $h \mapsto \sigma_h$ defines a one-to-one correspondence between non-singular hermitian and skew-hermitian forms on V up to a factor in $K^{\times}$ and involutions on $A = \operatorname{End}_D V$. In particular, the involutions σ_h on A and σ on D are of the same type if h is hermitian and of the opposite type if h is skew-hermitian.*

2.3 Pfister elements

In this section we outline a new approach to realizing the connection between involutions on central simple K-algebras and quadratic forms. All forms used henceforth are assumed to be non-degenerate. We begin by stating a powerful representation Theorem for quadratic forms.

Theorem 2.3.1. *[Lam05, Theorem IX.2.8] For any quadratic form φ and any anisotropic form γ over K, the following are equivalent:*

1. $\varphi(f_1(X_1,\ldots,X_n),\ldots,f_n(X_1,\ldots,X_n)) = \gamma(X_1,\ldots,X_n)$ *for some* $f_1,\ldots,f_n \in K(X_1,\ldots,X_n)$.
2. φ *is a subform of* γ *over* K.
3. $D_L(\varphi) \subseteq D_L(\gamma)$ *for any field extension* L/K.

In particular, if $\varphi(X_1,\ldots,X_n) \in D_{K(X_1,\ldots,X_n)}(\gamma)$ *then* $\dim_K(\varphi) \leq \dim_K(\gamma)$.

This characterization gives us a means of identifying Pfister forms in terms of the values they can assume over a transcendental extension. We say that two quadratic forms p and q over K are *similar* if $p \cong cq$ for some $c \in K^\times$. By [Lam05, Theorem X.2.8] we see that for an anisotropic quadratic form q and transcendental field extension $K(X) = K(X_1,\cdots,X_{2^n})$,

$$q(X)q \cong_{K(X)} q \text{ if and only if } q \text{ is similar to a Pfister form.} \tag{2.2}$$

Assume $\otimes_{i=1}^n (Q_i,\sigma_i)$ is a split K-algebra with orthogonal involution $\sigma = \otimes_{i=1}^n \sigma_i$ adjoint to a $n-$fold Pfister form q_σ. Since $\otimes_{i=1}^n Q_i \cong M_{2^n}(K)$ are isomorphic as K-algebras with involution we see by Equation 2.2 that there must exist $\theta \in \otimes_{i=1}^n Q_i$, satisfying

$$b_{q_\sigma}(\theta v, \theta w) = q_\sigma(X) b_{q_\sigma}(v,w)$$

for all $v,w \in K^{2^n}$. Moreover, given that σ is adjoint to b_{q_σ}, we see that

$$b_{q_\sigma}((\sigma(\theta)\theta)v, w) = b_{q_\sigma}(q_\sigma(X)v, w)$$

and non-degeneracy of b_{q_σ} implies $\sigma(\theta)\theta = q_\sigma(X)$. In order to determine $\theta \in \otimes_{i=1}^n Q_i$ explicitly, we introduce some notation.

Let $Q_m = \left(\dfrac{a_m, b_m}{K}\right)$ denote the quaternion $K-$algebra such that $\{1, i_m, j_m, k_m\}$ generate Q_m as a K-algebra subject to the relations

$$i_m^2 = a_m, j_m^2 = b_m,\ k_m^2 = -a_m b_m$$

$$i_m j_m = -j_m i_m,\ i_m k_m = -k_m i_m,\ j_m k_m = -k_m j_m$$

Assuming that $\otimes_{i=1}^m \sigma_i$ is an orthogonal involution we can, without loss of generality, further suppose that for $p = 1, \ldots, m$ the σ_p are also orthogonal involutions defined by

$$\sigma_p(i_p) = -i_p, \sigma_p(j_p) = j_p, \sigma_p(k_p) = k_p.$$

Now that we have defined the appropriate notation, we focus our attention on constructing $\theta_m \in \otimes_{i=1}^m (Q_i, \sigma_i)$ such that $\sigma(\theta)\theta = q_\sigma(X)$ for $\sigma = \otimes_{i=1}^m \sigma_i$. In some sense, θ_m can be understood as an element which structures the norm form of $\otimes_{i=1}^m (Q_i, \sigma_i)$. Note that the tensor structure between elements of different quaternion algebras will be implied, i.e.

$$i_1 i_2 j_3 := (i_1 \otimes 1 \otimes 1)(1 \otimes i_2 \otimes 1)(1 \otimes 1 \otimes j_3) = i_1 \otimes i_2 \otimes j_3$$

with the involution $\sigma = \otimes_{i=1}^3 \sigma_i$ acting diagonally.

Theorem 2.3.2. *Let $\otimes_{i=1}^n (Q_i, \sigma_i)$ be a split $K-$algebra and assume $\sigma = \otimes_{i=1}^n \sigma_i$ is an anisotropic involution. If $n \leq 3$, then q_σ is a Pfister form.*

Proof. We proceed in a case-by-case basis:

1. Let $K(X) = K(X_1, X_2)$ and $\theta_1 = X_1 + i_1 X_2$. Then, since $(\sigma(\theta_1)\theta_1)$ $= X_1^2 - a_1 X_1^2$, we have that $b_{q_\sigma}(\theta_1 v, \theta_1 w) = (X_1^2 - a_1 X_2^2) b_{q_\sigma}(v, w)$ for all $v, w \in V \otimes_K K(X)$. Hence $q_{\theta_1}(X_1, X_2) = X_1^2 - a_1 X_2^2 \in K(X)$ is a similarity factor for $q_{\sigma_{K(X)}}$ which implies $q_{\theta_1} = \langle 1, -a_1 \rangle \subset q_\sigma$ (see [Lam05] Ch IX. Corollary 2.10). Since $\dim(q_\sigma) = 2$ we have

$$\langle\langle a_1 \rangle\rangle = \langle 1, -a_1 \rangle \cong q_\sigma.$$

2. Let $K(X) = K(X_1, \cdots, X_4)$ and $\theta_2 = X_1 + i_1 X_2 + j_1 i_2 X_3 + k_1 i_2 X_4$. Then. since $\sigma(\theta_2)\theta_2 = X_1^2 - a_1 X_2^2 - b_1 a_2 X_3^2 + a_1 b_1 a_2 X_4^2$, we see that

$$b_{q_\sigma}(\theta_2 v, \theta_2 w) = (X_1^2 - a_1 X_2^2 - b_1 a_2 X_3^2 + a_1 b_1 a_2 X_4^2) b_{q_\sigma}(v, w)$$

for all $v, w \in V_{K(X)}$. By the same reasoning as in the previous case and vice-versa. However, if Q_1 is split this means that $\langle 1, -a_1, -b_1, a_1b_1\rangle$ is hyperbolic meaning that $b_1 \in \langle\langle a_1\rangle\rangle$. We conclude that $q_{\theta_2} = \langle\langle a_1, b_1a_2\rangle\rangle \cong q_\sigma$. Moreover, if either Q_1 or Q_2 is split we must have the other is split as well since, by assumption, $Q_1 \otimes Q_2$ is split., It follows that $b_1 \in \langle\langle a_1\rangle\rangle$ implies that

$$\langle\langle a_1, a_2\rangle\rangle \cong q_\sigma$$

3. Let $K(X) = K(X_1, \cdots, X_8)$ and $\theta_3 = X_1 + i_1X_2 + j_1(i_2X_3 + j_2i_3X_4 + k_2i_3X_5) + k_1(i_3X_6 + i_2j_3X_7 + i_2k_3X_8)$. As in the preceeding cases, computing $\sigma(\theta_3)\theta_3$, we conclude that

   $$q_\sigma \cong \langle\langle a_1\rangle\rangle \perp -b_1b_2a_3 \langle\langle a_2\rangle\rangle \perp a_1b_1a_2b_3 \langle\langle a_3\rangle\rangle \perp -b_1a_2 \langle\langle a_1a_2a_3\rangle\rangle .$$

   It may seem surprising at first, but q_σ is, in fact, a Pfister form. To see this we can simply compute the dimension, discriminant and Clifford invariants of q_σ i.e.

   $$\dim(q_\sigma) = 1, (q_\sigma) = 1, w(q_\sigma) = Q_1 \otimes Q_2 \otimes Q_3 = 1.$$

   See Appendix A for a more detailed calculation of the Clifford invariant.

□

Remark 2.3.3. Although we verified the algebraic invariants necessary for a quadratic form to be a Pfister form, this was redundant as the characterization of Pfister form described in Equation 2.2 implies that the involution norm $\sigma(\theta_m)\theta_m$ describes a Pfister form for $m \leq 3$. Moreover, we did not need to explicitly assume anisotropy in the cases we have discussed thus far, as $q_{\theta_m}(X)$ being a similarity factor for an isotropic form $(q_\sigma)_{K(X)} = (q_{an})_{K(X)} \perp r\mathbb{H}$ implies that $q_{\theta_m}(X)$ is a similarity factor for $(q_{an})_{K(X)}$. Indeed, if q_σ is isotropic then $\dim(q_{an}) < \dim(q_\sigma)$ but given that $q_{\theta_m}(X)$ is a similarity factor for q_{an} this means $\dim(q_{\theta_m(X)}) < \dim(q_{an})$ which is a contradiction unless $q_{an} \cong q$ or q_σ is hyperbolic. Using Theorem 2.3.1 and considering the dimensions of q_{θ_m} for $m = 1, 2, 3$ we can use these facts to reduce to the case that either

1. q is hyperbolic.
2. $q_{an} \cong q_\sigma \cong q_{\theta_m}$.

This is just the restatement that isotropic strongly multiplicative forms are hyperbolic (see [Lam05, Theorem X.2.9]).

A particularly important work towards this direction was carried out by Shapiro in [Sha77a] through his analysis of determining quadratic forms by means of studying its similarity factors. In the language of [Sha77a], the claim that θ_m is a Pfister element is equivalent to the claim that (V, q_σ) is a Pfister form given that q_{σ_m} is admissible in $\mathrm{Sim}(V)$. A key distinction between our work and that of [Sha77a] and [Sha77b] is that our approach determines both the space of similarities along with the associated Pfister form both completely and constructively using only properties of the involution. In contrast, [Sha77a] argues by induction on a small set of similarity factors to induce additional structure on the quadratic form.

One motivation in studying similarity factors is their relationship to quaternion algebras equipped with involutions, a connection which forms the basis of the Pfister factor conjecture [Sha77b]. A proof of the Pfister Factor Conjecture was first given by Becher,

Theorem 2.3.4. *[Bec08] Let $n \in \mathbb{N}$ and let (A, σ) be a K-algebra with involution such that $\deg(A) = 2^n$. There exist K-quaternion algebras with involution (Q_i, σ_i) such that $(A, \sigma) \cong \otimes_{i=1}^n (Q_i, \sigma_i)$ if and only if, for every field extension L/K, the L-algebra with involution $(A, \sigma)_L$ is either anisotropic or hyperbolic.*

Remark 2.3.5. In the course of the proof, Becher offered several analogous characterizations of his result. However, none of the characterizations addressed the explicit structure of the Pfister form associated to the involution in terms of the involutions on quaternion K-algebras.

2.4 An example: Pfister elements and embeddability

In this section, we give an example of using Pfister elements to determine embeddability from a computational point of view. In other words, what forms lend themselves to embed into m-fold Pfister forms naturally and what forms impose conditions on the coefficents of the underlying algebra. We briefly consider the example of Albert forms once more.

We proceed to show that elements which belong to a tensor product of m split quaternion K-algebras satisfying

$$b_{q_\sigma}((\sigma(\theta)\theta)v, w) = b_{q_\sigma}(q_\sigma(X)v, w)$$

where q_σ is assumed to be anisotropic are m-embeddable. Firstly, we observe that if $(Q_i, \sigma_i) \cong (M_2(K), \sigma_{q_i})$ with $q_i = \langle\langle a_i \rangle\rangle$, $a_i \in K^\times$ then $\otimes_{i=1}^m (M_2(K), \sigma_i) \cong (M_{2^m}(K), \sigma_q)$ where $q = \langle\langle a_1, \ldots, a_m \rangle\rangle$. We denote $(X_1, \ldots, X_{2^m})$ by X and suppose there exists $\theta \in \otimes_{i=1}^m M_2(K(X))$ satisfying

$$\sigma(\theta)\theta = p(X) \in K(X)^\times,$$

where $p(X)$ is a homogeneous polynomial of degree 2. Then it follows by Theorem 2.3.1 that $p \subseteq q = \langle\langle a_1, \ldots, a_m \rangle\rangle$, i.e. p is m-embeddable. As a consequence of this reformulation, we are better able to understand the necessary conditions which allow us to embed a quadratic form p into an m-fold Pfister form q. Indeed, to demonstrate the advantage of our approach, we revisit the problem of embeddability for Albert forms in Section 1.3.

Following the notation introduced in the previous section, let $Q_m = \left(\dfrac{a_m, b_m}{K}\right)$ be the quaternion K-algebra generated by $\{1, i_m, j_m, k_m\}$ with the associated orthogonal involution $\sigma_m : Q_m \longrightarrow Q_m$ defined by the mapping $\sigma_m(i_m) = -i_m$. We would like to explicitly construct the element in $\otimes_{i=1}^3 Q_i$ corresponding to the Albert form. Consider the element

$$\alpha := j_1(i_2X_3 + j_2i_3X_4 + k_2i_3X_5) + k_1(i_3X_6 + i_2j_3X_7 + i_2k_3X_8) \in \otimes_{i=1}^3 Q_i$$

and let σ denote the associated orthogonal involution $\otimes_{i=1}^3 \sigma_i$ on $\otimes_{i=1}^3 Q_i$. By the construction of α, we have

$$\sigma(\alpha)\alpha = -b_1a_2X_3^2 - b_1b_2a_3X_4^2 + b_1a_2b_2a_3X_5^2 + a_1b_1a_3X_6^2 + a_1b_1a_2b_3X_7^2 - a_1b_1a_2a_3b_3X_8^2$$

Assume $\otimes_{i=1}^3 Q_i \cong M_{2^3}(K)$ (this is equivalent to the condition that σ be adjoint to a symmetric bilinear form), if σ is an anisotropic involution then it follows from Theorem 2.3.1 that

$$q_\alpha := \langle -b_1a_2, -b_1b_2a_3, -b_1a_2b_2a_3 \rangle \perp \langle a_1b_1a_3, a_1b_1a_2b_3, -a_1b_1a_2a_3b_3 \rangle \subseteq q_\sigma.$$

Note that, despite the fact that it seems universal, we know by Lemma 1.3.6 that it will ultimately fail. In particular, since $\dim(q_\alpha) = 6$ and $\det(q_\alpha) = -a_1$ we have that by [Lam05, Corollary XII.2.13] q_α is an Albert form if and only if $\det(q_\alpha) = -1$, i.e. $a_1 = 1$. However, $a_1 = 1$ implies $(Q_1, \sigma_1) \cong (M_2(K), \mathbb{H})$, which means that q_σ is isotropic hence hyperbolic, since by [Lam05, Theorem X.1.7], isotropic Pfister forms are (necessarily) hyperbolic.

To demonstrate the usefulness of our approach, we repeat the above arguments with a slight modification. In what follows, we highlight the flexibility inherent in the constructibility of Pfister elements along with their power to reveal deeper structure. Let us choose

$$\alpha := j_1 j_4(i_2 X_3 + j_2 i_3 X_4 + k_2 i_3 X_5) + k_1(i_3 X_6 + i_2 j_3 X_7 + i_2 k_3 X_8) \in \otimes_{i=1}^{3} Q_i$$

then $\sigma(\alpha)\alpha$ is equivalent to

$$-b_1a_2b_4X_3^2 - b_1b_2a_3b_4X_4^2 + b_1a_2b_2a_3b_4X_5^2 + a_1b_1a_3X_6^2 + a_1b_1a_2b_3X_7^2 - a_1b_1a_2a_3b_3X_8^2$$

and q_α is given by

$$\langle -b_1a_2b_4, -b_1b_2a_3b_4, -b_1a_2b_2a_3b_4 \rangle \perp \langle a_1b_1a_3, a_1b_1a_2b_3, -a_1b_1a_2a_3b_3 \rangle \subseteq q_\sigma.$$

In similar fashion as before, it follows that, $\dim(q_\alpha) = 6$ and $\det(q_\alpha) = -a_1 b_4$. Therefore, by choosing $b_4 = a_1 (K^\times)^2$ we have q_α is an Albert form that is 4-embeddable in q_σ if and only if q_σ is anisotropic. The condition that b_4 lie in the same square class as a_1 is one of many different conditions imposed by our choice of α. The precise choice of α makes arguing in general difficult given the intractability of the associated computation. In particular, we conjecture the following:

Conjecture 2.4.1. Consider $(A, \sigma) = \otimes_{i=1}^{n} (Q_i, \sigma_i)$ where (Q_i, σ_i) is a quaternion $\mathbb{Q}$-algebra equipped with orthogonal involution σ_i. The computational complexity of determining all Pfister elements $\theta_m \in A$ is $\Theta(2^{p(n)})$ where $p(n)$ is a polynomial function of n.

Chapter 3

Hermitian forms

In this section, we will clarify some ambiguity in the literature over the reduction theorem for ϵ-Hermitian forms over rings in the case that the base ring contains $\frac{1}{2}$. The topics covered here are largely background material and the reader is encouraged to consult with either [Knu91] or [Sch85] for a more detailed exposition.

3.1 Sesquilinear forms

Let A be a associative unital ring such that $2 \in A^\times$. An *involution* on A is a map $\sigma : A \longrightarrow A$ such that σ is an anti-automorphism of order 2, i.e. $\sigma(x+y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(y)\sigma(x)$, $\sigma(\sigma(x)) = x$ and $\sigma(1) = 1$ for all $x, y \in A$. Note that the definition given here coincides with that of an involution on a central simple K-algebra given in Section 2.2, where the last condition ensures trivial action over a base field by K-linearity. We will retain prior conventions and denote a ring A carrying an involution σ by (A, σ).

Example 3.1.1. We give some examples of rings carrying an involution.

1. Let K be a field of characteristic $\neq 2$. The central simple K-algebra $M_n(K)$ equipped with the transpose involution is a ring with involution. Moreover, any central simple K-algebra equipped with an involution is a ring with involution.

2. Consider $A \times A^{op}$ equipped with a map

$$\epsilon : A \times A^{op} \longrightarrow A \times A^{op}$$

defined by sending $(x, y^{op}) \mapsto (y, x^{op})$. We call $(A \times A^{op}, \epsilon)$ the *hyperbolic ring* of A, denoted by $H(A)$.

3. Observe the matrix ring $M_2(A)$ equipped with a map $\sigma : M_2(A) \longrightarrow M_2(A)$ defined by

$$\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

   It is easy to see that $(M_2(A), \sigma)$ is a ring with involution. In particular, observe that $\sigma(x)x = \det(x)I_2$, for all $x \in M_2(A)$. It turns out that σ is the only linear map on $M_2(A)$, such that $\sigma(1) = 1$ and $\sigma(x)x \in A$ [KMRT98, Exercise I.5.2].

We begin by considering a right A-module M, denoted by M_A. A *sesquilinear form* on M_A is a bi-additive map

$$s : M \times M \longrightarrow A$$

such that $s(xa, yb) = \sigma(a)s(x, y)b$ for all $x, y \in M$ and $a, b \in A$. Alternatively, we may also say that (M, s) is a sesquilinear form on (A, σ). Let us proceed by defining a duality for M_A with respect to σ which will allow us to define a correspondence akin to that between bilinear and quadratic forms in previous chapters.

We start by considering the *dual module* of M, denoted by M^*, which we identify with the right A-module $(M^\vee)_A = \mathrm{Hom}_A(M_A, A)$ defined by the action

$$(fa)(m) := \sigma(a)f(m),$$

for $a \in A, m \in M$. Notice that, without the existence of an involution σ we would have to restrict ourselves to only considering the natural left A-module structure on $M^\vee$ with action given by $(af)(x) := af(x)$. Using the module structure induced by the involution we will be able to define a duality giving us a correspondence between Hermitian forms and sesequilinear forms. Now, taking the dual module of $M = M^*$ once again, we obtain the *double dual* M^{**} and a natural transformation $\mathrm{can_M} : M \longrightarrow M^{**}$ defined by sending $m \mapsto m^{**}$ with $m^{**}(f) := \sigma(f(m))$ for all $f \in M^*$.

Lemma 3.1.2. *[Knu91, Proposition I.3.1.2] If M_A is a finitely generated projective (right) A-module, then*

$$\mathrm{can}_M : M \longrightarrow M^{**}$$

is an isomorphism.

Proof. We reduce to the case where $M = A$. Recall that by construction, $M^{**} \cong (M^\vee)^\vee$ (see [Knu91, Lemma I.2.1.1]). By projectivity of M we may assume that $M \oplus N \cong A^k$ for some $k \in \mathbb{N}$. Since $\mathrm{Hom}_A(-, A)$ preserves direct sums i.e. $\mathrm{Hom}_A(M_A \oplus N_A, A) \cong \mathrm{Hom}_A(M_A, A) \oplus \mathrm{Hom}_A(N_A, A)$ we have that can_{A^k} is an isomorphism if and only if can_M and can_N are both K-isomorphisms. Following this line of reasoning we show can_{A^k} is an isomorphism by reducing to the case of showing can_A is an isomorphism which follows trivially by construction. □

Let (M, s) be a sesquilinear form on (A, σ). We define the *left adjoint* of (M, s) to be the A-linear homomorphism $s_l : M \longrightarrow M^*$ defined by $s_l(x)(y) := s(x, y)$ for all $x, y \in M$. Similarly, we define the *right adjoint* of (M, s) to be the A-linear homomorphism $s_r : M \longrightarrow M^*$ defined by $s_r(x)(y) = s(x, y)^* := \sigma(s(y, x))$ for all $x, y \in M$. In particular, A-linear homomorphisms naturally induce sesquilinear forms. Indeed, given an A-linear homomorphism $h : M \longrightarrow M^*$, we define a map

$$s_h : M \times M \longrightarrow A$$

by setting $s_h(x, y) := h(x)(y)$ for all $x, y \in M$. It is then easy to check that h induces a sesquilinear form (M, s_h) on (A, σ). We denote the set of sesquilinear forms on (A, σ) by $\mathrm{Sesq}_A(M)$ and observe that the left (or right) adjoints induce a correspondence

$$\mathrm{Sesq}_A(M) \longleftrightarrow \mathrm{Hom}_A(M, M^*).$$

Moreover, this correspondence can be understood as an isomorphism of $Z(A)$-modules where $Z(A)$ denotes the center of A carrying a natural left action on both $\mathrm{Sesq}_A(M)$ and $\mathrm{Hom}_A(M, M^*)$. For $\epsilon = \pm 1$, we say that a sesquilinear form $s \in \mathrm{Sesq}_A(M)$ is *ϵ-Hermitian* if $s_r = \epsilon s_l$, i.e.

$$s(x, y) = \epsilon s(x, y)^* = \epsilon \sigma(s(y, x)) \text{ for all } x, y \in M.$$

An equivalent formulation in terms of corresponding elements over $\mathrm{Hom}_A(M, M^*)$ is obtained by first considering the adjoint s^* of a sesquilinear form s. We know that $(s^*)_l(x)(y) = s^*(x, y) = \sigma(s(y, x))$ by construction and recalling that $m^{**}(f) := \sigma(f(m))$ we get $\sigma(s(y, x)) = x^{**}s_l(y)$. Now, $x^{**}s_l(y) = (s_l)^*(x^{**})(y)$ since $\phi^*(f) = f\phi$ whenever $\phi \in \mathrm{Hom}_A(M, M^*)$ and $f \in M^{**}$. By definition of can_M we can rewrite the latter term $(s_l)^*(x^{**})(y)$ as $(s_l)^*\mathrm{can}_M$. Combining the above relations we obtain $(s^*)_l = (s_l)^*can_M$ and by extending the earlier definition of ϵ-Hermitian forms to the adjoint we have that an element $h \in \mathrm{Hom}_A(M, M^*)$ is called *ϵ-Hermitian* if $h = \epsilon h^*\mathrm{can}_M$.

Now let us consider the $Z(A)$-module homomorphisms $S_\epsilon : \mathrm{Sesq}_A(M) \longrightarrow \mathrm{Sesq}_A(M)$ given by

$$S_\epsilon(s) = s + \epsilon s^*.$$

Firstly, we notice that since $2 \in A^\times$ we have $\ker(S_{-\epsilon}) = \mathrm{im}(S_\epsilon)$. Indeed, it is easy to see that $\mathrm{im}(S_\epsilon) \subset \ker(S_{-\epsilon})$ and the reverse containment is a consequence of the fact that $s = \frac{1}{2}s + \epsilon\frac{1}{2}s^*$ for any ϵ-Hermitian form $s \in Sesq_A(M)$. We remark that $\ker(S_{-\epsilon}) = \mathrm{im}(S_\epsilon) \cong \mathrm{Sesq}_A(M)/\ker(S_\epsilon)$ and note that elements of $\mathrm{im}(S_\epsilon)$ are generally referred to as *even ϵ-Hermitian forms*. However, in our considerations ϵ-Hermitian forms and even ϵ-Hermitian forms are one and the same, so we will use the terms interchangably.

To define ϵ-Hermitian forms in terms of $\mathrm{Hom}_A(M, M^*)$ we first observe a $Z(A)$-module homomorphisms $S_\epsilon : \mathrm{Hom}_A(M, M^*) \longrightarrow \mathrm{Hom}_A(M, M^*)$ given by

$$S_\epsilon(h) = h + \epsilon h^*\mathrm{can}_M.$$

and notice that $\ker(S_{-\epsilon}) = \mathrm{im}(S_\epsilon)$. To see this, observe that $\mathrm{im}(S_\epsilon) \subset \ker(S_{-\epsilon})$ since $(h + \epsilon h^*\mathrm{can}_M) - \epsilon((h + \epsilon h^*\mathrm{can}_M)^*\mathrm{can}_M$ is equivalent to $(h - (h^*\mathrm{can}_M)^*\mathrm{can}_M) + \epsilon(h^*\mathrm{can}_M - h^*\mathrm{can}_M)$, which is precisely the additive identity given that $(h^*\mathrm{can}_M)^*\mathrm{can}_M) = \mathrm{can}_M^* h^{**}\mathrm{can}_M = \mathrm{can}_M^*\mathrm{can}_{M^*}h = h$. The reverse equality follows similarly to the previous case, $h = \frac{1}{2}h + \frac{1}{2}h = \frac{1}{2}h + \frac{1}{2}\epsilon h^*\mathrm{can}_M$, since $h \in \ker(S_{-\epsilon})$ is equivalent to $h = \epsilon h^*\mathrm{can}_M$.

Let $\mathrm{Proj}(A)$ denote the category of finitely generated projective right A-modules and $\mathrm{Sesq}^\epsilon(M)$ the set of ϵ-Hermitian sesquilinear forms over M. We call a pair (M, s) with $M \in \mathrm{Proj}(A)$ and $s \in \mathrm{Sesq}^\epsilon(M)$ an ϵ-Hermitian module. Furthermore, we say that a sesquilinear form s is *non-degenerate* if s_l and s_r are A-module isomorphisms. Combining these notions, we write $\mathrm{herm}_\epsilon(A)$

to denote the category whose objects are non-degenerate ϵ-Hermitian modules and whose morphisms consist of A-module homomorphisms $f : M \longrightarrow N$ such that $s_2(f(x), f(y)) = s_1(x, y)$ for all $x, y \in M$ with $(M, s_1), (N, s_2) \in \mathrm{Ob}(\mathrm{herm}_\epsilon(A))$. Similarly to the case of quadratic forms, we refer to the morphisms in the category $\mathrm{herm}_\epsilon(A)$ as *isometries.*

Remark 3.1.3. We now make an extended but crucial remark regarding *unitary* or (ϵ, Λ)-*quadratic forms.* First introduced by Bak in [Bak81] as a quotient of $\mathrm{Sesq}_A(M)$, its theoretical contribution was meant to generalize most of the preceding arguments to characteristic 2. In keeping with this generality, both [Knu91] and [Sch85] develop important results regarding transfer and reduction in terms of these unitary forms. Since our considerations are limited to characteristic $\neq 2$, we wish to avoid the unnecessary addition of technicalities and notation. However, to reference results from the aforementioned authors it will be necessary to at the very least demonstrate the equivalence of definitions in our setting. For a projective left A-module M and $\epsilon = \pm 1$ we define

$$\mathrm{Sesq}^{-\epsilon}(M) := \{f \in \mathrm{Hom}(M, M^*) \mid f + \epsilon f^* can_M = 0\}.$$

$$\mathrm{Sesq}_{-\epsilon}(M) := \{f - \epsilon f^* can_M \mid f \in \mathrm{Hom}(M, M^*)\}.$$

A *unitary* $(\epsilon, \Lambda_M) - $*module* is an element of $\mathrm{Hom}(M, M^*)/\Lambda_M$, where M is a projective left A-module and Λ_M satisfies

1. Λ_M is an additive subgroup of $Hom(M, M^*)$ such that

$$\mathrm{Sesq}_{-\epsilon}(M) \subset \Lambda_M \subset \mathrm{Sesq}^{-\epsilon}(M).$$

2. $f^*\Lambda_N f \subset \Lambda_M$ for all $f \in \mathrm{Hom}(M, N)$ where N is a projective left A-module.

We can see that by definition, any (ϵ, Λ)-unitary form $[h] \in Hom(M, M^*)/\Lambda_M$ is uniquely determined by its even ϵ-Hermitian form representation, $h + \epsilon h^* \mathrm{can}_M$. From our considerations regarding S_ϵ, we see there is a one-to-one correspondence between even ϵ-Hermitian forms and elements of

$$\mathrm{Hom}(M, M^*)/\ker(S_\epsilon) = \mathrm{Hom}(M, M^*)/\,\mathrm{Sesq}^{-\epsilon}.$$

In particular, the isometries (morphisms) of even ϵ-Hermitian forms and unitary forms coincide as well (see [Knu91, Remark I.5.3.3] and [Sch85, Lemma 7.3.6]). As a consequence, we note that the theory of unitary forms coincides completely with that of even ϵ-Hermitian forms in our context [Sch85, Remark 7.3.4]. This concludes our remark.

Just like in the case of bilinear forms, we have a natural addition operation in $\mathrm{herm}_\epsilon(A)$ given by taking orthogonal sums. To be precise, we define the *orthogonal sum* of two ϵ-Hermitian spaces (M, s_1) and (N, s_2) as

$$(M, s_1) \perp (N, s_2) := (M \oplus N, s_1 \oplus s_2),$$

where $(s_1 \oplus s_2)(m_1 \oplus n_1, m_2 \oplus n_2) = s_1(m_1, m_2) + s_2(n_1, n_2)$ for $m_1, m_2 \in M$ and $n_1, n_2 \in N$. Additionally, for $P \in \mathrm{Proj}(A)$ we have the notion of a hyperbolic ϵ-Hermitian space $H^\epsilon(P)$ given by:

$$H^\epsilon := (P \oplus P^*, \mathbb{H}_\epsilon),$$

where $\mathbb{H}_\epsilon(p_1 \oplus q_1^*, p_2 \oplus q_2^*) = q_1^*(p_2) + \epsilon\sigma(q_2^*(p_1))$ for all $p_1, p_2 \in P$ and $q_1^*, q_2^* \in P^*$. Any ϵ-Hermitian space isometric to H^ϵ for some $P \in \mathrm{Proj}(A)$ is called *ϵ-hyperbolic*. This naming scheme is appropriate as it turns out that ϵ-hyperbolic forms share much in common with their counterpart in the classical theory of quadratic forms over fields. That is to say, over a splitting field, such as the algebraic closure, the ϵ-Hermitian forms associated to an orthogonal involution are precisely the hyperbolic forms in the theory of quadratic forms.

Lemma 3.1.4. *Properties of ϵ-hyperbolic forms,*

1. *If (M, s) is a ϵ-hyperbolic Hermitian space then, $(M, \lambda s)$ is also an ϵ-hyperbolic Hermitian space for any $\lambda \in Z(A)^\times$.*

2. *For $P, Q \in \mathrm{Proj}(A)$, $H^\epsilon(P \oplus Q) = H^\epsilon(P) \oplus H^\epsilon(Q)$.*

Proof. These results follow immediately by applying the definition of $H^\epsilon(P)$. □

We have seen that the structure of nonsingular ϵ-Hermitian modules behaves well in terms of hyperbolicity and orthogonal sum. This extends to decomposition in the following sense: Let $(M, s) \in \mathrm{Ob}(\mathrm{herm}_\epsilon(A))$. If $M = P \oplus Q$ and $s(P, Q) = 0$, then $(M, s) \cong (P, s\mid_{P \oplus P}) \oplus (Q, s\mid_{Q \oplus Q})$. In this case we say that $(P, s\mid_{P \oplus P})$ is a subspace of (M, s).

3.2 Reduction and equivalence

We begin by restating the reduction criterion demonstrated in [Knu91, Theorem II.4.6.1] making use of the fact that $2 \in A^{\times}$ to express the result in terms of the category of ϵ-Hermitian spaces.

Theorem 3.2.1 (Reduction criterion)**.** *Let* A *be a finite dimensional* K-*algebra with involution* σ *and denote by* rad(A) *the radical of A. The canonical reduction functor*

$$\overline{F} : \mathrm{herm}_{\epsilon}(A) \longrightarrow \mathrm{herm}_{\epsilon}(A/\operatorname{rad}(A))$$

has the following properties:

1. $\overline{F}$ *is essentially surjective and preserves orthogonal sums.*
2. *Any isometry in* $\mathrm{herm}_{\epsilon}(A/\operatorname{rad}(A))$ *can be lifted to an isometry in* $\mathrm{herm}_{\epsilon}(A)$.

Proof. The proof follows [Knu91, Theorem II.4.6.1] and [Sch85, Theorem 7.4.4] with the following modifications. We reduce the general case of invariant ideals to that of the radical, which is in fact invariant by [Knu91, II.4.6] (it is easy to see $\sigma(\operatorname{rad}(A)) = \operatorname{rad}(A)$). Furthermore, completeness in the I-adic topology (see [Knu91, §II.4.5]) follows trivially since all ideals contained in the radical are nilpotent in a finite dimensional algebra by Nakayama's Lemma. Finally, following Remark 3.1.3, we can view unitary (ϵ, Λ)-unitary spaces as ϵ-Hermitian spaces and arrive at our result. □

The reduction criterion allows us to transfer arguments concerning the structure of ϵ-Hermitian forms over finite dimensional algebras to division algebras. Firstly, we observe it is not necessarily that case that a finite dimensional K-algebra A is semisimple. However, semisimplicity is guaranteed once we quotient out the radical i.e.

$$\mathrm{A/rad(A)} \cong A_1 \times \cdots \times A_n,$$

for some $n \in \mathbb{N}$ and finite-dimensional simple algebras $A_1, \ldots, A_n$. Since our algebra A carries an involution σ and rad(A) is invariant under such an involution, we would like to have a decomposition which is also invariant under σ. By [Knu91, $I.I.$2.8] such a decomposition exists and is structured as follows:

$$\mathrm{A/rad(A)} \cong A_1 \cdots \times A_n \times H(B_1) \times \cdots \times H(B_m),$$

where A_i are finite dimensional simple $Z(A_i)^\times$-algebras, such that $\sigma(A_i) = A_i$ and $H(B_i)$ is the hyperbolic ring of a simple finite dimensional $Z(B_i)^\times$-algebra B_i invariant under σ. What we have effectively shown by the arguments above is that every ϵ-Hermitian $A/\operatorname{rad}(A)$-module decomposes as a product of ϵ-Hermitian A_i modules where A_i can be realized as a finite dimensional central simple $Z(A_i^\times)$-algebra. A more detailed explanation following this line of reasoning is given in [Knu91, Example II.5.2.5 and Proposition II.5.2.6].

We can now trace the ideas leading up to this point as follows (see [Knu91, §II.6.6]). We wish to study ϵ-Hermitian forms over arbitrary rings A carrying an involution σ. To focus our considerations we restricted our rings by requiring that A is a finite dimensional K-algebra such that $2 \in \mathrm{A}^\times$. As a consequence of these restrictions we observed a reduction functor $\overline{F}$ with the property that isometries of finitely generated A/rad(A) modules carrying an ϵ-Hermitian form with respect to σ can be lifted to $\operatorname{herm}_\epsilon(A)$. Moreover, by the decomposition of semisimple modules respecting the involution σ, we can reduce our considerations even further to $\operatorname{herm}_\epsilon(A_i)$, where A_i is a finite dimensional simple $Z(A_i^\times)$-algebra with involution σ. By Wedderburn's Theorem such algebras are isomorphic to $M_l(D)$ for some $l \in \mathbb{N}$ and D is a $Z(A_i^\times)$-division algebra carrying an involution σ'. We may describe D in terms of an endomorphism $\operatorname{End}_{A_i}(M)$ where M is a simple A_i-module. This alternative form is useful as we are able to apply techniques from Morita theory to reduce our considerations even further down from $\operatorname{herm}_\epsilon(A_i)$ to $\operatorname{herm}_\epsilon(D)$.

Let P be a faithfully projective right A-module. Recall that P is *faithfully projective* if P is finitely generated as an A-module and $P \otimes_A N = 0$ implies $N = 0$ for any $N \in \mathrm{Proj(A)}$. We wish to define an equivalence of categories between $\operatorname{herm}_\epsilon(A)$ and $\operatorname{herm}_\epsilon(\operatorname{End}_A(P))$. The first necessary step to take in this direction is to define an involution on $\operatorname{End}_A(P)$ induced by the involution on A. In this regard, let us assume $(P, s) \in \operatorname{herm}_\epsilon(A)$ and consider the (left) adjoint s_l. We define an involution s_0 on $\operatorname{End}_A(P)$ by the mapping:

$$f \mapsto s_l^{-1} f^* s_l.$$

Furthermore, we may define the ϵ-Hermitian form

$$s_0' : P \times P \longrightarrow \operatorname{End}_A(P)$$

where $s_0'(x,y)$ is the map $(x \otimes_A s_l(y)) : p \mapsto x \cdot s_l(y)(p)$ (see [Knu91, I.9.2]). Then we may state Morita equivalence for Hermitian modules as follows,

Theorem 3.2.2. *[Knu91, Theorem I.9.3.5] Let (P, b) be a ϵ_0-Hermitian module over (A, σ) and $B = End_A(P)$ with $\epsilon_0 \pm 1$. If P is faithfully projective, then the functor*

$$F : \mathrm{herm}_\epsilon(A) \longrightarrow \mathrm{herm}_{\epsilon\epsilon_0}(End_A(P))$$

defines an equivalence of categories given by

$$F : (M, s) \mapsto (M \otimes_A P, s_0's).$$

An important Corollary of this result applies to central simple k-algebras. Let A be a central simple K-algebra with involution σ, by Wedderburn's theorem, we have $A \cong M_n(D)$ for some $n \in \mathbb{N}$ and central division k-algebra D. Since every projective module P over A is faithfully projective we obtain the following as a consequence,

Corollary 3.2.3. *Let (A, σ) be a central simple K-algebra with involution. If $A \cong M_n(D)$, then D has an involution τ such that*

$$F : \mathrm{herm}_\epsilon(A) \longrightarrow \mathrm{herm}_{\epsilon\epsilon_0}(D)$$

is an equivalence of categories for some $\epsilon_0 = \pm 1$. In particular, hyperbolic spaces of A correspond to hyperbolic spaces of D.

We conclude this chapter by stating and proving the following result which parallels [Sch85, Corollary II.7.11.4] and [Knu91, Proposition II.7.1.1] in describing all ϵ-Hermitian forms of equal rank as twisted forms of the same element over the algebraic closure.

Proposition 3.2.4. *Let A be a finite dimensional K-algebra with involution σ. If $(M, h_1), (N, h_2) \in \mathrm{herm}_\epsilon(A)$ such that $\mathrm{rank}_A(M) = \mathrm{rank}_A(N)$, then $(M \otimes K^{alg}, h_1) \cong (N \otimes K^{alg}, h_2)$ as $A \otimes K^{alg}$-modules.*

Proof. We start by observing that by the Reduction criterion Theorem 3.2.1 it suffices to consider the case where $\mathrm{rad}(A) = 0$. In particular, we may without loss of generality, assume that

$$A \cong A_1 \times \cdots \times A_n$$

where A_i are finite dimensional simple $Z(A_i^\times)$-algebras such that $\sigma(A_i) = A_i$. Furthermore, since every ϵ-Hermitian module over a direct product decomposes as a direct product of ϵ-Hermitian modules, we may once again reduce our considerations to considering A such that $(A \otimes L, \sigma \otimes id_L) \cong (M_n(L), \tau)$, where L is a separable Galois extension of $Z(A_i^\times)$. Consider the faithfully projective right $M_n(L)$-module $(L^n, n\langle 1\rangle) \in \text{herm}_\epsilon(M_n(L))$, where $n\langle 1\rangle = \langle 1, ..., 1\rangle$. Using the Double Centralizer Theorem (see [KMRT98, Theorem I.1.5]), we see that $\text{End}_{M_n(L)}(L^n) \cong L$ and by Morita equivalence we see

$$F : \text{herm}_\epsilon(M_n(L)) \longrightarrow \text{herm}_\epsilon(L)$$

is an equivalence of algebras. Now since the theory of 1-hermitian spaces coincides with that of symmetric bilinear forms over K^{alg} and the theory of -1-Hermitian spaces coincides with that of alternating forms, in each case we may leverage our understanding of the latter to show:

$$(M \otimes K^{alg}, h_1) \cong (N \otimes K^{alg}, h_2).$$

In other words, since any two symmetric bilinear forms (or skew-symmetric bilinear forms) of equal dimension are isometric it suffices to let $L = K^{alg}$. In this case we see that by equivalence of categories $F(M, h_1) \cong F(N, h_2)$ if and only if $\text{rank}_A(M) = \text{rank}_A(N)$. $\square$

Chapter 4

Algebraic groups

In this chapter we give a brief overview of algebraic groups and introduce only the notions necessary for our later use. There are several classic resources for the study of algebraic groups. For a more detailed investigation, the reader is encouraged to visit any of [Bor91], [Hum75] or [KMRT98].

4.1 Preliminaries

An *affine algebraic group* over a field K is an affine variety G over K, with a group structure such that the maps,

$$\mu : G \times G \longrightarrow G,$$

$$\mu(g,h) = gh$$

and

$$\iota : G \longrightarrow G,$$

$$\iota(g) = g^{-1},$$

are morphisms between algebraic varieties corresponding to multiplication and inversion respectively.

Examples 4.1.1. Examples of affine algebraic groups.

1. The additive group $\mathbb{G}_a = \operatorname{Spec} K[X]$ is a 1-dimensional, irreducible, affine algebraic group.

2. The multiplicative group $\mathbb{G}_m = \operatorname{Spec} K[X, X^{-1}]$ is a 1-dimensional, irreducible, affine algebraic group.

3. The general linear group $GL_n = \operatorname{Spec} K[X_{11}, \ldots, X_{nn}, D]$ with $D = \det(X_{ij})$ is a n^2-dimensional affine algebraic group.

The last example is of particular importance since it follows by [Hum75, Theorem 8.6] that G is an affine algebraic group over K if and only if G is a closed subgroup of GL_n for some $n \in \mathbb{N}$. We will sometimes refer to affine varieties which can be realized as subgroups of GL_n for some $n \in \mathbb{N}$ as *linear algebraic groups*. For example, the subgroups SL_n (matrices of determinant 1), $\mathbb{U}_n$ (upper triangular matrices), D_n (diagonal matrices) of GL_n are all examples of affine (linear) algebraic groups.

Consider the affine algebraic groups G, G'. A map $\varphi : G \longrightarrow G'$ is a morphism of algebraic groups if it is both a group homomorphism and a morphism of affine varieties. As a consequence of this definition, it is easy to see that $\ker(\varphi)$ and $\operatorname{Im}(\varphi)$ are closed subgroups of G and G' respectively and

$$\dim(G) = \dim(\ker(\varphi)) + \dim(\operatorname{Im}(\varphi)).$$

We say an affine algebraic group G over K is *connected* if it is irreducible as an algebraic variety over K. In other words, if $G = G_1 \cup \cdots \cup G_n$ is a decomposition of G into its irreducible components (as a algebraic variety) and $1 \in G_1$ where 1 is the identity component of G then we say that G_1 is the connected component of G denoted by G^0. In other words, G is connected if and only if $G = G^0$.

Example 4.1.2. Below we present some classical examples of connected algebraic groups.

1. The additive and multiplicative groups $\mathbb{G}_a$ and $\mathbb{G}_m$ are both connected algebraic groups.

2. The orthogonal group $\mathbb{O}_n$ is a linear algebraic group with $\mathbb{O}_n^0 = S\mathbb{O}_n$.

3. The general linear group GL_n is a connected algebraic group, i.e $GL_n = GL_n^0$.

Note that the linear algebraic group $\mathbb{O}_n$ corresponds to the group of all isometry classes of the quadratic form $n\langle 1\rangle$. Indeed,

$$\mathbb{O}_n(K) = \{M \in GL_n(K) \mid MI_nM^T = I_n\}.$$

A non-trivial, connected, algebraic group G is called *simple* if it contains no non-trivial connected normal closed subgroups over K^{alg}. Analogously, we say that a non-trivial connected algebraic group G is *semisimple* if G contains no non-trivial connected normal solvable subgroups over K^{alg}. As a consequence, a direct product of simple algebraic groups is by definition semisimple.

Examples 4.1.3. Below we present examples of simple and semisimple algebraic groups

1. The group SL_n is a simple algebraic group.
2. The group SO_n is a simple algebraic group.
3. The group PGL_2 is a semisimple algebraic group.

The maximal connected, normal, solvable subgroup of an algebraic group G is called the *radical* of G and is denoted by $R(G)$. It follows from this definition that a connected linear algebraic group G is semisimple if $R(G) = 1$. For instance, SL_n is semisimple since $R(\mathrm{SL}_n) = 1$. Similarly, we call the maximal connected, normal, unipotent subgroup of an algebraic group G the *unipotent radical*, denoted by $R_u(G)$, and say that an algebraic group G is *reductive* if $R_u(G) = 1$. For example, GL_n is a reductive group. It follows from the preceeding definitions that for any algebraic group G, $G/R(G)$ is semisimple and $G/R_u(G)$ is reductive. In particular, since $R_u(G) \subseteq R(G)$ we have that every semisimple group is automatically reductive.

An affine algebraic group T over K which decomposes as a direct product of copies of $\mathbb{G}_m$ over the algebraic closure is called a *torus* i.e.

$$T_{K^{alg}} \cong (\mathbb{G}_m)^n$$

for some $n \in \mathbb{N}$. We say that a torus is *split* if $T \cong (\mathbb{G}_m)^n$ for some $n \in \mathbb{N}$ over the base field K. Along the same lines, we declare a reductive algebraic group G as *split* if G contains a split torus which is maximal with respect to inclusion among tori in G. It turns out that tori play a central role in the structure and classification of algebraic groups. In this direction, we define the *rank* of an algebraic group in terms of the dimension of the split torus i.e. $\mathrm{rank}(G) = n$ if and only if $T \subset G_{K^{alg}}$ where T is a split maximal torus such that $T \cong (G_m)^n$. This is well-defined as all maximal tori are conjugate by [Bor91, Theorem III.10.6]. We now proceed to set-up the machinery necessary for the classification of certain types of algebraic groups.

4.2 Root systems

In this section we present the classification of all split semisimple affine algebraic groups. See [Bor91, §24.A], [KMRT98, §25] or [Hum75, §35] for a more in depth discussion of the classification.

Let V be a vector space over $\mathbb{R}$ equipped with the (non-degenerate) symmetric bilinear form $(\cdot,\cdot)$ given by the Euclidean inner product. For any given non-zero vector $v \in V$, we define the *reflection* $s_v : V \longrightarrow V$ to be the linear transformation on V given by

$$s_v(w) = w - 2\frac{(v,w)}{(w,w)}w.$$

A finite subset $\phi \subset V$ of non-zero vectors is called a *root system* if the following axioms are satisfied:

1. ϕ spans V as an $\mathbb{R}$-vector space;
2. If $\alpha \in \phi$, then $\mathbb{R}\alpha \cap \phi = \{\pm\alpha\}$;
3. If $\alpha, \beta \in \phi$, then $s_\alpha(\beta) \in \phi$;
4. If $\alpha, \beta \in \phi$, then $2\frac{(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$.

We define the *rank* of the root system ϕ to be the dimension of the associated vector space V over $\mathbb{R}$. Along these lines, we say that $\{\alpha_1, \cdots, \alpha_n\} \subseteq \phi$ is a *basis of the root system* ϕ if $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of V over $\mathbb{R}$ and for any $\alpha \in \phi$ we have that

$$\alpha = \sum_{i=1}^{n} r_i \alpha_i$$

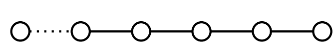
where r_i are all integer coefficients of the same sign. A root is called *positive* if all coefficents in terms of the basis of the root system are positive integers. Alternatively, a root is *negative* if all coefficents are negative. We define the *Weyl group* of ϕ, denoted by $W(\phi)$, to be the subgroup of $GL(V)$ generated by all reflections s_α for $\alpha \in \phi$. It follows from the definition of a root system that the Weyl group $W(\phi)$ must be finite, since ϕ is finite, and that $W(\phi)$ must permute the roots of ϕ, as it is generated by reflections along roots $\alpha \in \phi$.

Fix a basis $\Pi \subset \phi$ of the root system ϕ. We define the *Dynkin diagram* to be a directed graph with vertices indexed by elements of Π and the number of edges between two vertices $\alpha_i, \alpha_j \in \Pi$ given by $4\frac{(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}\frac{(\alpha_j,\alpha_i)}{(\alpha_i,\alpha_i)}$. That is to say, the number of edges $n_{\alpha_i\alpha_j}$, is determined by the angle between α_i and α_j. We choose the edges in the Dynkin diagrams of Theorem 4.2.1 to represent the order of $s_\alpha s_\beta$ in the Weyl group, whenever an edge joins two simple (indecomposable elements) roots α and β. Although there may, in general, be multiple different choices of basis $\Pi \subset \phi$, it turns out that the Dynkin diagram of ϕ does not depend on the choice of basis. Moreover, it preserves the structure of the root system associated to an algebraic group G, so that we can recover the Weyl group $W(\phi)$ of ϕ from its Dynkin diagram. To see this let $m_{ij} = 2, 3, 4, 6$ correspond to the number of edges joining roots α_i and α_j i.e. $0, 1, 2, 3$ then,

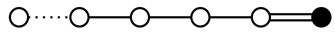
$$\{s_{\alpha_1}, \ldots, s_{\alpha_n} |\ (s_{\alpha_i})^2 = 1,\ (\alpha_i\alpha_j)^{m_{ij}} = 1\}.$$

We say that a root system ϕ of V is called *irreducible* if there does not exist a partition $\phi_1, \phi_2 \subset \phi$ such that $\phi_1 \cap \phi_2 = \varnothing$ and $(\alpha_1, \alpha_2) = 0$ for all $\alpha_1 \in \phi_1$ and $\alpha_2 \in \phi_2$. In fact, the irreducibility of a root system ϕ is precisely equivalent to the connectedness of the associated Dynkin diagram of ϕ. The formal language we have thus far established puts us in a position to state the main result of this section that all Dynkin diagrams of irreducible root systems can be classified into one of 9 types,

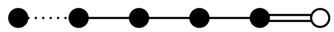
Theorem 4.2.1. *[Hum75, Appendix. Root Systems] Dynkin diagrams can be classified into four classical types $A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 2), D_n(n \geq 3)$ and five exceptional types, E_6, E_7, E_8, F_4 and G_2 where the subscript denotes the rank of root systems:*

(A_n) 

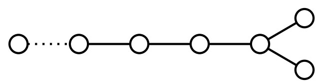
(B_n)



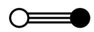
(C_n)



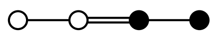
(D_n)



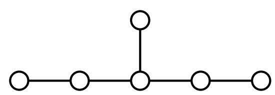
(G_2)



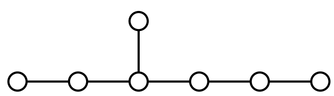
(F_4)



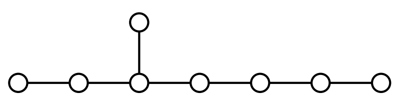
(E_6)



(E_7)



(E_8)



So how do we get a root system out of a split affine algebraic group? We use the character group of the torus. Without loss of generality, choose a split maximal torus $T \subset G$ (note that all maximal tori are conjugate so this choice is well-defined) and denote by $T^* := \mathrm{Hom}(T, G_m)$ the *character group* of T. Note that since T is a split maximal torus, we have that

$$T^* \cong \mathbb{Z}^n$$

where $n = \mathrm{rank}(T)$. We will show that by combining the *Lie algebra* associated to G and a particular representation of G, it is possible to exhibit a root system $\phi \subset T^*$ which will allow us to classify G (up to isomorphism) by the Dynkin diagram associated to the root system ϕ. Firstly, consider the adjoint representation

$$\mathrm{Ad} : G \longrightarrow GL(\mathrm{Lie}(G))$$

where $\mathrm{Lie}(G)$ is the Lie algebra associated to G over K. We say that a character $\alpha \in T^*$ is called a *weight* of the adjoint representation with a *weight subspace* L_α, given by

$$L_\alpha(K) := \{l \in \mathrm{Lie}(G) \mid Ad(t)l = \alpha(t)l \text{ for all } t \in T(K)\}.$$

It follows we can use the weight subspaces (see [KMRT98, §22]) to decompose $\mathrm{Lie}(G)$ i.e.

$$\mathrm{Lie}(G) = \oplus_\alpha L_\alpha,$$

where we sum over all non-zero weights α. This latter fact gives us the representation to determine the root system of G.

Theorem 4.2.2. *[KMRT98, Theorem VI.25.1] Let G be a split semisimple algebraic group and let $T \subset G$ be a split maximal torus. The set of non-zero weights α of the decomposition* $\mathrm{Lie}(G) = \oplus_\alpha L_\alpha$ *is a root system in T^*.*

We briefly recall the context of the previous sections. Let G be a split semisimple algebraic group and $T \subset G$ be a split maximal torus. We define the root system $\phi(G) = \{0 \neq \alpha \in T^*\}$ consisting of non-zero weights α coming from the representation of $\mathrm{Lie}(G) = \oplus_\alpha L_\alpha$. We proceed by associating two lattices associated to $\phi(G)$ in order to setup our next classification result. We define Λ_r to be a *root lattice* of ϕ if

$$\Lambda_r := \mathrm{span}_{\mathbb{Z}}(\alpha_1, \dots, \alpha_n),$$

where α_i are the roots of ϕ. Alternatively, we call Λ_w the *weight lattice* of ϕ whenever

$$\Lambda_w := \{t \in T \mid \alpha^\vee(t) \in \mathbb{Z} \text{ for all } \alpha \in \phi\},$$

where $\alpha^\vee : T \longrightarrow \mathbb{R}$ is defined by $\alpha^\vee(t) = 2\frac{(t,\alpha)}{(\alpha,\alpha)}$. We say that G is *simply connected* if $T^* = \Lambda_w$ and G is *adjoint* if $T^* = \Lambda_r$. In particular, notice that $\Lambda_r \subseteq T^* \subseteq \Lambda_w$.

We conclude this section with the classification of split simple affine algebraic groups of classical type over a field K,

Theorem 4.2.3. *Let G be a split simple affine algebraic group of classical type. Then we may classify G according to its type as follows,*

(A_n): If G is simply connected, then

$$G = \mathrm{SL}_{n+1}(K),$$

otherwise, $G = \mathrm{SL}_{n+1}/\mu_k$ where k divides $n+1$.

(B_n): If G is simply connected, then

$$G = \mathrm{Spin}_{2n+1}(K).$$

If G is adjoint, then

$$G = \mathrm{SO}_{2n+1}(K).$$

(C_n): If G is simply connected, then

$$G = \mathrm{Sp}_{2n}(K),$$

otherwise if G is adjoint, then

$$G = \mathrm{PGSp}_{2n}(K).$$

(D_n): In this case, $n = 2m$. If G is simply connected, then

$$G = \mathrm{Spin}_{2n}(K).$$

If G is adjoint, then

$$G = \mathrm{PGSO}_{2n}(K).$$

Alternatively, in the case G is neither simply connected nor adjoint we have $G = \mathbb{O}_{2n}(K)$ or $G = \mathrm{Spin}^{\pm}_{2n}(K)$.

(D_n): In this case, $n = 2m+1$. If G is simply connected, then

$$G = \mathrm{Spin}_{2n}.$$

If G is adjoint, then

$$G = \mathrm{PGSO}_{2n}(K).$$

Alternatively, in the case G is neither simply connected nor adjoint then $G = \mathbb{O}_{2n}(K)$.

4.3 Multipliers of similitudes

In this section we summarize a classical result of Weil [W60] which shows that there is a deep connection between algebras with involution and classical groups of adjoint type. In particular, we will see how the notions of embeddability in Pfister forms and strong Pfister elements relate to those of similitudes. For a more detailed discussion of these results see [KMRT98, Chapter III] or [Sha77a].

We begin by introducing some preliminary definitions which will allow us to state this connection more precisely. Let (A, σ) denote a central simple K-algebra equipped with involution σ. Recall that by Skolem-Noether, we know that every automorphism of A is an inner automorphism, in other words

$$\mathrm{Aut}(A, \sigma) = \{\mathrm{Inn}(a) : A \longrightarrow A \mid \sigma(a)a \in K^{\times}\}.$$

To see why this equality holds, observe that a map $f : (A, \sigma) \longrightarrow (A, \sigma)$ is an automorphism if

$$f \circ \sigma = \sigma \circ f.$$

Since f is an inner automorphism, this definition is equivalent to

$$a\sigma(-)a^{-1} = \sigma(a)^{-1}\sigma(-)\sigma(a),$$

where $f = \mathrm{Inn}(a) = a(-)a^{-1}$, implying the desired equivalence. We denote the *group of similitudes* of (A, σ) by $\mathrm{Sim}(A, \sigma)$ where

$$\mathrm{Sim}(A, \sigma) := \{a \in A^{\times} \mid \sigma(a)a \in K^{\times}\}.$$

Associated to the involution σ we define the *multiplier map* $\mu : \mathrm{Sim}(A, \sigma) \longrightarrow A^{\times}$ by $\mu(a) = \sigma(a)a$.

Remark 4.3.1. Recall the construction of strong Pfister elements θ_m in Section 2.4 and Section 2.3.2. We see that by necessity $\theta_m \in \mathrm{Sim}(A, \sigma)$ when $A = \otimes_{i=1}^{m} Q_i$. Moreover, by the computation of maximal admissible subspaces in [Sha77a] we have an upper bound on the decomposition of elements coming from $\mathrm{Sim}(A, \sigma)$ determined by the degree of A, whenever A is a tensor product of quaternion K-algebras. Alternatively, the elements of $\mathrm{Sim}(A, \sigma)$ which are maximal in terms of the number of elements they decompose into are precisely the strong Pfister elements whenever A is a tensor product of quaternion K-algebras.

Example 4.3.2. Let $(A, \sigma) = (\mathrm{End}_K(V), \sigma_q)$ where $\dim_K(V) = n$. Then the following hold:

1. $\mathrm{Iso}(A, \sigma) := \{a \in A^* \mid \sigma(a)a = 1\}$ can be identified with $\mathrm{O}(q)$ where

$$\mathrm{O}(q) := \{M \in M_n(K) \mid q(Mv) = q(v) \text{ for every } v \in V\},$$

2. $\mathrm{Sim}(A, \sigma)$ can be identified with $\mathrm{GO}(q)$ where

$$\mathrm{GO}(q) := \{M \in M_n(K) \mid q(Mv) = cq(v) \text{ for every } v \in V\}$$

with $c \in K^\times$.

Considering the group of isometries in terms of μ, it is easy to see that

$$\mathrm{Iso}(A, \sigma) = \ker(\mu).$$

This induces the following exact sequences

$$1 \longrightarrow \mathrm{Iso}(A, \sigma) \hookrightarrow \mathrm{Sim}(A, \sigma) \xrightarrow{\mu} K^\times \longrightarrow 1,$$

$$1 \longrightarrow K^\times \hookrightarrow \mathrm{Sim}(A, \sigma) \xrightarrow{Inn} \mathrm{Aut}(A, \sigma) \longrightarrow 1$$

and an isomorphism $\mathrm{Sim}(A, \sigma)/K^\times \cong \mathrm{Aut}(A, \sigma)$ where we denote the quotient $\mathrm{Sim}(A, \sigma)/K^\times$ by $\mathrm{PSim}(A, \sigma)$ and say that $\mathrm{PSim}(A, \sigma)$ is the group of *projective similitudes* of (A, σ).

Example 4.3.3. Consider $a, b \in K^\times$ such that (a, b) is a quaternion K-algebra equipped with an involution σ.

1. If σ is orthogonal, such that $\sigma(i) = -i$, with $i^2 = -a$, then

$$\mathrm{Sim}(A, \sigma) = K(i)^\times \cup K(i) \cdot v$$

for any v with the property that $v \in (a, b)^\times$ and $iv = -vi$.

2. If σ is symplectic i.e. σ is the cannonical involution, then

$$\mathrm{Sim}(A, \sigma) = (a, b)^\times.$$

Recall that for $a \in A$,

$$\text{char. pol.}_{A,a}(X) = X^n - c_{n-1}(a)X^{n-1} + c_{n-2}(a)X^{n-2} - \ldots + (-1)^n c_0(a),$$

where $c_{n-1}(a) = \text{Trd}_A(a)$ denotes the reduced trace and $c_0(a) = \text{Nrd}_A(a)$ denotes the reduced norm. Then for any $a \in A$, $\sigma(a)a = k \in K^\times$ implies that $\text{Nrd}_A(\sigma(a)a) = \text{Nrd}_A(\sigma(a))\,\text{Nrd}_A(a) = \text{Nrd}_A(k)$ which can be reduced to

$$\text{Nrd}_A(a)^2 = k^{2n}$$

since $\text{Nrd}_A(a) = \text{Nrd}_A(\sigma(a))$ and $\text{Nrd}_A(k) = k^{2n}$ for every $k \in K^\times$. We classify similitudes into two groups: proper and improper similitudes. We say a similitudes $a \in \text{Sim}(A, \sigma)$ is *proper* if $Nrd_A(a) = k^n$ and *improper* if $Nrd_A(a) = -k^n$. Along these lines, we denote the group of proper similitudes is by $PSim_+(A, \sigma)$ while we denote the group of improper similitudes by $PSim_-(A, \sigma)$. We have now established all the necessary terminology to state the celebrated result of Weil.

Theorem 4.3.4. *[W60] If G is an adjoint absolutely simple affine algebraic group of classic type over a field K, then there exists a central simple K-algebra A equipped with an involution σ such that*

$$G \cong \text{Sim}(A, \sigma)^0,$$

where $\text{Sim}(A, \sigma)^0$ *is the connected component of* $\text{Sim}(A, \sigma)$ *and can be identified with the set of proper similitudes,* $\text{PSim}_+(A, \sigma)$.

We conclude this section by classifying all absolutely simple affine algebraic groups of classical type over K motivated by the result of Weil,

Theorem 4.3.5. *[KMRT98, §26] Let G be an absolutely simple affine (linear) algebraic group over K. The following hold:*

1. *If G is simply connected, then*

   ${}^1A_n$: $G = \text{SL}_1(A)$, *where A is a central simple algebra of degree $n+1$.*

   ${}^2A_n$: $G = \text{SU}(A, \sigma)$, *where A is a central simple algebra of degree $n+1$ over L and L/K is a quadratic field extension with unitary involution σ such that $\sigma|_K = id_K$.*

   B_n: $G = \text{Spin}_{2n+1}(q)$, *where q is a quadratic form over K.*

C_n: $G = \mathrm{Iso}(A, \sigma)$, *where A is a central simple algebra of degree $2n$ and σ is a symplectic involution.*

D_n: $G = \mathrm{Spin}_{2n}(A, \sigma)$, *where A is a central simple algebra of degree $2n$ and σ is an orthogonal involution.*

2. *If G is adjoint, then*

^1A_n: $G = \mathrm{PGL}_1(A)$, *where A is a central simple algebra of degree $n+1$.*

^2A_n: $G = \mathrm{PSim}(A, \sigma)$, *where A is a central simple algebra of degree $n+1$ over L and L/K is a quadratic field extension with unitary involution σ such that $\sigma|_K = id_K$.*

B_n: $G = \mathrm{SO}_{2n+1}(q)$, *where q is a quadratic form over K.*

C_n: $G = \mathrm{PSim}(A, \sigma)$, *where A is a central simple algebra of degree $2n$ and σ is a symplectic involution.*

D_n: $G = \mathrm{PSim}_+(A, \sigma)$, *where A is a central simple algebra of degree $2n$ and σ is an orthogonal involution.*

4.4 Projective homogeneous varieties

Let G be a split algebraic group over a field K with a split maximal torus $T \subset G$. We say X is a *homogeneous G-variety* if X is a variety over K such that there is a G-action $\phi : G \times X \longrightarrow X$ on X defined by $(g, x) \mapsto g \cdot x$ with the following properties:

1. $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G_{K^{alg}}$, $x \in X_{K^{alg}}$, and
2. For every $x, y \in X_{K^{alg}}$ there exists $g \in G_{K^{alg}}$ such that $g \cdot x = y$.

Note that the first condition is precisely the definition of a group action and the second condition is the definition of a transitive group action. If G is a smooth algebraic group, we then say a homogeneous G-variety is implicitly projective. In fact, as we will see, there is a correspondence between projective homogeneous G-varieties and quotients of G. In order to state this correspondence appropriately it is necessary to define two important classes of subgroups of G.

We say a a subgroup H of G is *Borel* if H is a connected, solvable and closed subgroup of G which is maximal with respect to these properties. Further, we say a subgroup $P \subset G$ is *parabolic* if G/P is projective. It turns out that if G is smooth, then for any Borel subgroup $H \subset G$, G/H is projective, which implies that P is parabolic if and only if it contains a Borel subgroup (see [Hum75, Theorem 29.3]). For example, if $G = \mathrm{GL}(V)$ for some K-vector space V then it can be easily confirmed from the above characterization that the set of upper triangular matrices in G is a Borel subgroup.

Now consider the root system $\phi(G,T)$ of G with a basis of non-decomposable (simple) positive roots $\Pi \subset \phi(G,T)$. For any subset $\Theta \subset \Pi$ we define the parabolic subgroup P_Θ to be the group generated by a Borel subgroup $B \supset T$ and the set of reflections s_α for $\alpha \in \Theta$, in particular any parabolic subgroup is of this form by [Hum75, Theorem 29.3]. We have the following well-known correspondence:

Theorem 4.4.1. *[Bor91, § 24.A] Let G be a smooth split affine algebraic group with a split maximal torus T and root system $\phi(G,T)$. Then X is a homogeneous G-variety if and only if $X \cong G/P_\Theta$, where $\Theta \subset \Pi$ and Π is a basis of positive roots of $\phi(G,T)$.*

We can restate the above result by saying that all projective homogeneous G-varieties can be classified by subsets $\Theta \subset \phi(G,T)$ consisting of vertices in the corresponding Dynkin diagram of G. This allows us to define several useful varieties which we encounter in practice as quotients of G with respect to subsets of simple positive roots.

We conclude this section by introducing another way of understanding both Borel and parabolic subgroups of G is in terms of their actions on flags of vector spaces. Consider an n-dimensional K-vector space V and a sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim_K(V_m) = m$ for all m present in the flag. If the sequence of subspaces is full, i.e. each V_i for $i = 1, \ldots, m$ is realized in the sequence, we say the flag is *complete*, in all other cases we refer to the flag as *partial*. By fixing a basis, say $\{e_1, \ldots, e_n\} \subset V$, it is easy to see that the Borel subgroup of upper triangular matrices can be realized as the stabilizer of the full flag,

$$0 \subset \mathrm{span}(e_1) \subset \mathrm{span}(e_1, e_2) \subset \cdots \subset \mathrm{span}(e_1, \ldots, e_n) = V.$$

In this sense, we say that $\mathrm{GL}_n / \mathrm{U}_n$ is the complete flag variety where U_n is the set of upper triangular matrices. The parabolic subgroups enter this picture when we extend our considerations to partial flag varieties. Indeed, the stabilizer of any partial flag must contain the upper triangular matrices, a Borel subgroup, which is precisely the definition of a parabolic subgroup. For example, we define the *Grassmannian variety*, denoted by $\mathrm{Gr}(m, n+1)$, to be the variety of all m-dimensional linear subspaces of a $n+1$-dimensional K-vector space V. It is not too hard to see that the Grassmannian variety is a homogeneous G-variety, with $\mathrm{Gr}(m, n+1) \cong G/P_{\Theta_m}$ where G is a split algebraic group of type A_n and $\Theta_m := \{\alpha_1, \dots, \alpha_{m-1}, \alpha_{m+1}, \dots, \alpha_n\}$. In particular,

$$\mathrm{Gr}(1, n+1) \cong \mathrm{Gr}(n, n+1) \cong \mathbb{P}^n,$$

since all $(n+1)$-dimensional and 1-dimensional subspaces of V can be identified with the n-dimensional projective space.

Remark 4.4.2. The connection between projective homogeneous G-varieties and groups of similitudes is much deeper than what we have covered here. For example, in [McF19], it was shown that for a second generalized involution variety X of a degree 4 central simple algebra A equipped with a symplectic involution σ, the group of K_1-zero-cycles of X is determined by $\mu(\mathrm{Sim}(A, \sigma))$.

We conclude by giving a description of the maximal symplectic Grassmannian in terms of an algebraic group G and a parabolic subgroup P. Let G be a split semisimple group of classical type C_n, that is to say, let $G = \mathrm{Sp}_{2n}(K)$. It is not hard to see that for a skew-symmetric bilinear form on a vector space of dimension 2n, $\mathrm{Sp}(V)$ acts transitively on the space of n-dimensional totally isotropic subspaces of V. The parabolic subgroup P_θ which stabilizes a fixed n-dimensional totally isotropic subspace can be described is given by the set of simple roots $\{\alpha_1, \dots, a_{n-1}\}$ with the last root omitted.

For a more general overview of the connection between Grassmannians, orthogonal Grassmannians and symplectic Grassmannians see [BL00, §3].

Chapter 5

Schubert cycles

In this chapter, we define the notion of an involution variety and use it as a proxy for studying the algebraic structure of isotropic ideals contained in central simple K-algebras with involution. We use the language of Schubert cycles to prove a lifting criteria which allows us to realize twisted forms of Schubert varieties by some nice combinatorial description. Working in the language of algebraic groups covered earlier, we compute the torsion Chow group for classical groups of type C_n corresponding to Lagrangian Grassmannians associated to central simple K-algebras of degree 4 with symplectic involution. To our understanding, this is not treated anywhere in the existing literature. We refer the reader to [Kra10] for more background on involution varieties, [BL00] for a review of Schubert cycles and [EKM08] for an exposition on Chow groups of quadrics.

5.1 Involution varieties

Let (A, σ) be an algebra with involution (of the first kind) over a field K with $\mathrm{char}(K) \neq 2$. We define the *reduced dimension* of a (right) ideal $I \subset A$ by

$$\mathrm{rdim}(I) := \dim_K(I)/\deg(A).$$

For each (right) ideal $I \subset A$, we define its *orthogonal ideal* $I^\perp$, by

$$I^\perp = \{x \in A \mid \sigma(x)y = 0 \text{ for } y \in I\}.$$

We say that a right ideal I is *isotropic* if $I \subseteq I^\perp$. We observe that since $\mathrm{rdim}(I) + \mathrm{rdim}(I^\perp) = \deg(A)$, we can bound the reduced dimension of an isotropic ideal by $\frac{1}{2}\deg(A)$. Moreover, we can make use of this bound to define the notion of index for algebras with involution. In this regard, we define the *index* of (A, σ) by

$$\operatorname{ind}(A, \sigma) = \{\operatorname{rdim}(I) \mid I \subseteq I^{\perp}\}.$$

Note that, since we know $\operatorname{ind}(A) \mid \operatorname{rdim}(I)$ and $(A, \sigma) \cong (\operatorname{End}_D(V), \sigma_h)$ for some symmetric or skew-symmetric hermitian space (V, h) over D, we see that $\operatorname{ind}(A, \sigma) = \{n \operatorname{ind}(A) \mid 0 \leq n \leq w(h)\}$, where $w(h)$ denotes the Witt index of (V, h). This highlights a correspondence between isotropic ideals of (A, σ_h) and totally isotropic subspaces of (V, h) (see [KMRT98, Proposition II.6.2]). For any isotropic ideal I in (A, σ) there exists a totally isotropic subspace $W \subset V$, such that

$$I = \operatorname{Hom}_D(V, W).$$

We say an isotropic ideal $I \in A$ is *hyperbolic* if and only if

$$\max(\operatorname{ind}(A, \sigma)) = \frac{1}{2}\deg(A).$$

In general, hyperbolic ideals encode properties of the involution. We consider two examples of hyperbolic ideals.

1. Assume (A, σ) is a central simple K-algebra equipped with a symplectic involution. Then every ideal of reduced dimension 1 is isotropic. Indeed, if $\operatorname{rdim}(I) = 1$ for some $I \in A$ then $\operatorname{ind}(D) = 1$ which means that σ is adjoint to a skew-symmetric bilinear form, containing all lines through the origin as solutions.

2. Let $(A, \sigma) = (M_2(K), \sigma_q)$ where q is a 2-dimensional quadratic form over K. It follows by the hyperbolicity of $q \otimes_K K(\sqrt{-\det(q)})$ that $(A, \sigma) \otimes_K K(\sqrt{-\det(q)})$ is a central simple K-algebra of degree 2 with hyperbolic involution.

A result of Bayer-Fluckiger, Shapiro and Tignol [BFST93, Theorem 2.2] shows that a central simple algebra (A, σ) with orthogonal involution is hyperbolic if and only if $(A, \sigma) \cong (M_2(K) \otimes B, \sigma_q \otimes \tau)$ where (B, τ) is some central simple algebra with orthogonal involution and q is a hyperbolic plane.

We are now ready to consider involution varieties, which are determined by conditions on the reduced dimension of isotropic ideals inside (A, σ), our definitions follows [Kra10, §8]. If X is a variety over K we will abuse notation and denote the covariant functor from the category of commutative K-algebras to the category of sets given by

$$R \mapsto \mathrm{Mor}_{\mathrm{Sch}}(\mathrm{Spec}(R), X),$$

as X, and call $X(R)$ the R-points of X (see [Kra08, §2] for more details).

The *(k-th) generalized involution variety* is the subvariety of the Grassmannian representing a functor of points given by isotropic reduced dimension k right ideals inside of (A, σ) i.e.

$$\mathrm{IV}_k(A, \sigma)(R) = \{I \in \mathrm{Gr}(n^2 - nk, A)(R) \mid I \text{ is a right ideal, } \sigma(I)I = 0\}.$$

Remark 5.1.1. The generalized involution variety is an analogue of the generalized Severi-Brauer variety of a central simple K-algebra A, $\mathrm{SB}_k(A)$, defined to be the variety of ideals whose L-points are right ideals in A with reduced dimension n,

$$\mathrm{SB}_k(A)(R) = \{I \in \mathrm{Gr}(n^2 - nk, A)(R) \mid I \text{ is a right ideal}\}.$$

It is easy to see that in the case $k > \frac{1}{2}\deg(A)$ we have the set of L-points of $IV_k(A, \sigma)(L)$ is empty since $\mathrm{rdim}(I) + \mathrm{rdim}(I^{\perp}) = \deg(A)$ and $\mathrm{rdim}(I) \leq \mathrm{rdim}(I^{\perp})$ implies $\mathrm{rdim}(I) \leq \frac{1}{2}\deg(A)$ for all isotropic ideals I of (A, σ). Following [KMRT98, §1.C], we realize $SB_r(A)$ as a closed subvariety of $\mathrm{Gr}(rn, A)(K)$ which we can identify with $\mathbb{P}(\wedge^{rn} A)$. The argument is identical for $IV_r(A)$ by replacing $\mathrm{Gr}(rn, A)(K)$ with $\mathrm{LG}(rn, A)(K)$ which has points corresponding to the rn-dimensional subspaces of A that form a totally isotropic subspace of the hermitian form associated to (A, σ).

The aim of the next section is to understand how isotropic ideals interact geometrically by studying $\mathrm{IV}_n(A, \sigma)$ for $n \leq \frac{1}{2}\deg(A)$. As a first approach, our intuition suggests we consider the case that $n = 1$ and σ is symplectic. However, it is not hard to see that this is precisely equivalent to studying the Severi-Brauer variety of A, which are treated extensively in [KMRT98]. Indeed, if we realize an ideal of reduced dimension 1 as a rank 1 vector space over some division algebra we know that it must be totally isotropic with respect to the associated skew-Hermitian form. In other words, the Severi-Brauer variety associated to A is equivalent to a generalized involution variety, i.e. $\mathrm{SB}(A) = \mathrm{IV}_1(A, \sigma)$. The central focus of this chapter is directed toward understanding Schubert cycles of maximal symplectic Grassmannians i.e. $\mathrm{SG}(n, 2n) = \mathrm{IV}_n(A, \sigma)$ in the less understood case where $n = \frac{1}{2}\deg(A)$. We obtain a lifting property for these cycles and use this as a motivation to determine the torsion elements in the corresponding Chow group. A brief overview of current results for the maximal orthogonal Grassmannian case is given in the Appendix.

In the next section, we attempt to extend the computation of torsion elements in a Chow group obtained by [JKL17] in type A_n and those obtained by [Kar16] in type B_n to the type C_n case corresponding to maximal symplectic Grassmannians.

5.2 Maximal Symplectic Grassmannians

Let (A, σ) be a central simple K-algebra carrying a symplectic involution with $\deg(A) = 2n$. For a field extension L/K we identify the maximal symplectic Grassmannian (also known as the *Lagrangian involution variety*) $\mathrm{SG}(A, \sigma) := IV_n(A, \sigma)$ where the set of L-points can be identified with

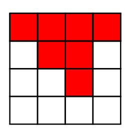
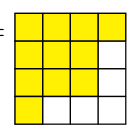
$$\mathrm{SG}(A, \sigma)(L) = \{I \subset A_L \mid I = I^{\perp}\}$$

In the case that (A, σ) is split, i.e. $(A, \sigma) = (\mathrm{End}_K(V), \sigma_h)$, the rational points of $\mathrm{SG}(A, \sigma)$ correspond to the maximal isotropic subspaces of V as discussed in Section 5.1. It follows that for a non-split central simple K-algebra (A, σ), the variety $\mathrm{SG}(A, \sigma)$ is a twisted form of the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$, defined by the set of maximal totally isotropic vector subspaces associated to the quadratic form adjoint to the involution.

The correspondence between isotropic subspaces and isotropic right ideals is also used to construct a *complete isotropic flag* for (A, σ) in the split case, i.e. when $(A, \sigma) = (\mathrm{End}_K(V), \sigma_h)$, with $\dim_K(V) = 2n$. First, fix a maximal isotropic subspace $V_n \subset V$ (which necessarily has dimension n), and extend to a complete flag of subspaces $V_1 \subset V_2 \subset \cdots \subset V_n$ such that $\dim_K(V_j) = j$. This flag then corresponds to a complete flag of isotropic right ideals of $(\mathrm{End}_K(V), \sigma_h)$ by setting $I_j := \mathrm{Hom}_F(V, V_j)$. We will use this flag to define the Schubert subvarieties of $\mathrm{SG}(\mathrm{End}_K(V), \sigma_h)$ in terms of isotropic right ideals.

Consider a *strict* partition $a = [a_1, \dots, a_n]$ with $n \geq a_1 > a_2 > \cdots > a_n > 0$. Such a partition can be represented by an *upper shifted* Young diagram with a_j boxes in the j-th row, starting at the jth column of an $n \times n$ box. Alternatively, this diagram can be described by a partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ with $a_j = \lambda_j - j + 1$ for $j = 1, \dots, \lfloor \frac{n}{2} \rfloor$ and $\lambda_{\lfloor \frac{n}{2} \rfloor + 1}, \dots, \lambda_n$ determined such that λ is self-dual. In this context, the term *self-dual* comes from Lakshmibai and Weyman, and refers to a diagram inside an $n \times n$ box which is symmetric about the north-west to south-east diagonal of the box [LW90].

Example 5.2.1. Suppose $n = 4$. The strict partition $a = [4, 2, 1, 0]$, corresponds to the upper shifted diagram on the left. We associate to a the self-dual partition $\lambda = [4, 3, 3, 1]$, which corresponds to the self-dual diagram on the right. Note that the upper shifted diagram is obtained from the self-dual diagram by removing all coloured blocks below the North-West to South-East diagonal of the box.

$a =$  $\lambda =$ 

These partitions, or their associated Young diagrams (of either upper shifted or self-dual type), are used to define the Schubert subvarieties of the Lagrangian grassmannian $\mathrm{SG}(n, 2n)$. Given a split algebra with symplectic involution $(A, \sigma) = (\mathrm{End}_K(V), \sigma_h)$ with $\deg(A) = 2n$, we fix a full chain of isotropic right ideals

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset A \qquad \text{such that } \mathrm{rdim}(I_j) = j$$

Let $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n]$ be a self-dual partition with $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, and let $a(\lambda) = [a_1, \dots, a_n]$ be the corresponding strict partition.

The Schubert variety $X_\lambda \subseteq \mathrm{SG}(\mathrm{End}_K(V), \sigma_h)$ is then defined by intersection conditions with respect to the above isotropic chain:

$$X_\lambda = \{J \in \mathrm{SG}(\mathrm{End}_K(V), \sigma_h) \mid \mathrm{rdim}(J \cap I_{n+1-a_j}) \geq j \text{ for } j = 1, \dots, n\}$$

Using the correspondence between self-dual Young diagrams and upper shifted Young diagrams, we can also define X_λ in terms of λ itself:

$$X_\lambda = \{J \in \mathrm{SG}(\mathrm{End}_K(V), \sigma_h) \mid \mathrm{rdim}(J \cap I_{n+j-\lambda_j}) \geq j \text{ for } 1 \leq j \leq n \text{ and } \lambda_j \geq j\}$$

Lemma 5.2.2. *Let A be a central simple K-algebra of degree n and let $1 \leq d \leq n$. Given an ideal $I_a \subset A$ of reduced dimension a and a fixed integer $1 \leq r \leq a$, the subset*

$$X_{(r,a)} = \{J \in \mathrm{SB}(d, A) \mid \mathrm{rdim}(J \cap I_a) \geq r\},$$

is a closed subscheme of $\mathrm{SB}(d, A)$.

Sketch of the Proof. We proceed by expressing $X_{(r,a)}$ as a degeneracy locus for the scheme $\mathrm{SB}(d, A)$. Recall that the kth degeneracy locus of a morphism $\phi : E \longrightarrow F$ of vector bundles over a scheme X is

$$D_k(\phi) = \{x \in X \mid \mathrm{rank}(\phi(x)) \leq k\},$$

where $\phi(x) : E(x) \longrightarrow F(x)$ is the induced map on fibers over a point $x \in X$. Now, $D_k(X)$ is a closed subscheme of X as the zero scheme of the induced morphism $\Lambda^{k+1}(\phi) : \Lambda^{k+1}(E) \longrightarrow \Lambda^{k+1}(F)$, denoted by $\mathrm{Z}(\Lambda^{k+1}(\phi))$. In other words,

$$D_k(\phi) = \mathrm{Z}(\Lambda^{k+1}).$$

The scheme structure of $D_k(\phi)$ is encoded in this latter description. That is to say, locally, its ideal is generated by $(k+1)$-minors of a matrix for ϕ. See [Fu98, Chapter 14, p. 243] for more discussion on degeneracy loci.

Npw, let us consider the tautological sequence of vector bundles for $S = \mathrm{SB}(d, A)$:

$$0 \to \mathcal{I} \to A \otimes \mathcal{O}_S \to \mathcal{Q},$$

where $\mathcal{I}$ is the universal subbundle of $S = \mathrm{SB}(d, A)$ and $\mathcal{Q}$ is the universal quotient bundle of S. Over a point $J \in \mathrm{SB}(d, A)$, the induced sequence of fibers over J is

$$0 \to J \to A \to A/J \to 0.$$

In particular, the right ideal I_a of A induces a morphism of vector bundles determined by the composite of the inclusion and the quotient morphism

$$\phi : I_a \otimes \mathcal{O}_S \longrightarrow A \otimes \mathcal{O}_S \longrightarrow \mathcal{Q}.$$

On the fiber over a point $J \in \mathrm{SB}(d,A)$, we have that $\phi(J) : I_a \to A \to A/J$. Note that, $\mathrm{im}(\phi(J)) = (I_a + J)/J \cong I_a/(J \cap I_a)$ so that, in particular, $\ker(\phi(J)) = J \cap I_a$. It follows that for the morphism $\phi : I_a \otimes \mathcal{O}_S \longrightarrow \mathcal{Q}$, we see that

$$D_{(a-r)n}(\phi) = \{J \in \mathrm{SB}(d,A) \mid \dim(I_a/(J \cap I_a)) \leq (a-r)n\}$$

which is equivalent to

$$D_{(a-r)n}(\phi) = \{J \in \mathrm{SB}(d,A) \mid \mathrm{rdim}(J \cap I_a) \geq r\} = X_{(r,a)}.$$

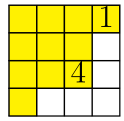
□

For an algebra with symplectic involution (A,σ) over a field K, and vector space V such that $\mathrm{End}_{K^{alg}}(V) \cong A_{K^{alg}}$, we say that a closed subvariety $P \subset \mathrm{SG}(A,\sigma)$ is a *twisted form* of the Schubert variety X_λ if we can defined an isomorphism of varieties between $\mathrm{SG}(A,\sigma)$ and $\mathrm{SG}(n,2n)$ such that $P_{K^{alg}}$ is identified with X_λ. Recall, in this context $\mathrm{SG}(n,2n)$ is the variety of points consisting of totally isotropic subspaces of dimension n in V where $\dim_{K^{alg}}(V) = 2n$. Thus, our first main goal will be to determine which twisted forms of Schubert varieties appear in the Lagrangian involution varieties of a given algebra with symplectic involution (A,σ).

To do this, we must first determine which of the intersection conditions in the definition of X_λ are essential to determine X_λ uniquely, and which can be removed without changing the variety itself. We introduce a variation on Fulton's *essential set*, defined in [Ful92]. For each self-dual partition λ, we define a set of pairs $E_\lambda := \{(j, n+j-\lambda_j) \mid \lambda_j > \lambda_{j+1} \text{ and } \lambda_j \geq j\}$ with the convention that $\lambda_{n+1} = 0$ for any partition λ.

We note that the first condition $\lambda_j > \lambda_{j+1}$ describes the situation that the right-most box in the jth row of the corresponding Young diagram lies at the south-eastern edge of a "corner". The second condition $\lambda_j \geq j$ guarantees that such a corner lies on or above the north-west to south-east diagonal of the $n \times n$ square. The second coordinate of these pairs records the reduced rank of the right ideal of the flag which defines the intersection condition.

Example 5.2.3. Suppose $n = 4$, and consider the self-dual partition $\lambda = [4,3,3,1]$. The corresponding diagram has two corners above or on the diagonal, corresponding to the set $E_\lambda = \{(1,1),(3,4)\}$.

$\lambda =$ 

Let $\overline{E_\lambda} = \{a \mid (j,a) \in E_\lambda\}$. Note that for any $a \in \overline{E_\lambda}$, there exists a unique $1 \le j \le d$ such that $(j,a) \in E_\lambda$. Using $\overline{E_\lambda}$ we can identify a unique subchain in E_λ, $I_{a_1} \subset I_{a_2} \subset \cdots \subset I_{a_m}$, $a_i \in \overline{E_\lambda}$ with the property that

$$X_\lambda = \{J \in \mathrm{SG}(\mathrm{End}_K(V), \sigma_h)(K) \mid \mathrm{rdim}(J \cap I_a) \ge j \text{ for } (j,a) \in E_\lambda\}.$$

This E_λ uniquely defines X_λ and is minimal in the sense that if any pairs in E_λ are changed or removed, the variety defined by this new set of conditions will not be equal to X_λ (see [And16, Corollary 4.2]). Note that the Young diagram for λ can also be reconstructed from E_λ (keeping in mind that the diagram must be self-dual).

Proposition 5.2.4. *Let X_λ be a Schubert subvariety of $\mathrm{SG}(\mathrm{End}_K(V), \sigma_h)$ with $\dim_K(V) = 2n$ and let (A,σ) be an algebra with symplectic involution over K of degree $2n$. If $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ and $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$, then $\mathrm{SG}(A,\sigma)(K)$ contains a closed subvariety over K which is a twisted form of X_λ.*

Proof. Since A has an isotropic right ideal of reduced rank k if and only if $k \in \mathrm{ind}(A,\sigma)$, it follows that if $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ and $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$, then A contains a partial flag of isotropic ideals $I_{a_1} \subset \cdots \subset I_{a_r} \subset A$ for $\overline{E_\lambda} = \{a_1, \ldots, a_r\}$ (note that $\mathrm{ind}(A) \mid a_i$ implies $a_i \in \mathrm{ind}(A,\sigma)$). We may then define a closed subvariety $P_\lambda \subseteq \mathrm{SG}(A,\sigma)(K)$ by

$$P_\lambda(L) = \{J \in \mathrm{SG}(A,\sigma)(L) \mid \mathrm{rdim}(J \cap (I_a \otimes_K L)) \ge j \text{ for } (j,a) \in E_\lambda\},$$

where L/K is a field extension. We will proceed by demonstrating that $P_\lambda(L)$ is, in fact, a closed subvariety of $\mathrm{SG}(A,\sigma)$. To do this, we aim to show that the Schubert cycle

$$X_\lambda = \{J \in \mathrm{SG}(A,\sigma) \ \mid \ \mathrm{rdim}(J \cap I_a) \ge r,\ (r,a) \in E_\lambda\}$$

defines a closed subvariety in $\mathrm{SG}(A,\sigma)$. In particular, let us begin by considering the case for generalized Severi-Brauer varieties, since intersecting the consequent Schubert cycles with $\mathrm{SG}(A,\sigma)$ will give us the desired claim. Let λ be a partition fitting in a $d \times (n-d)$ box. Assume that $\mathrm{ind}(A)$ divides $\gcd(\overline{E_\lambda})$. For any $(k,a) \in E_\lambda$, there exists an ideal I_a of A defined over K with reduced dimension a. Thus, we may define

$$Q_\lambda = \{J \in \mathrm{SB}(d,A) \mid \mathrm{rdim}(J \cap I_a) \geq r, (r,a) \in E_\lambda\} = \cap_{(r,a)\in E_\lambda} X_{(r,a)},$$

where Q_λ is realized as a closed subscheme of $\mathrm{SB}(d,A)$ via the intersection of closed subschemes $X_{(r,a)}$ defined in Lemma 5.2.2. For a splitting field L/K of A, it is clear that

$$(Q_\lambda)_L = \{J \in \mathrm{SB}(\mathrm{End}_K(V),\sigma_h)(K) \mid \mathrm{rdim}(J \cap I_a) \geq j \text{ for } (j,a) \in E_\lambda\}.$$

Indeed, since $A_L = \mathrm{End}_L(V)$ for some n-dimensional L-vector space V, and all ideals of reduced dimension r, take the form $\mathrm{Hom}_L(V,W)$ for some r-dimensional subspace $W \subset V$.

Coming back to our case, we have a central simple K-algebra A of degree $2n$ with a symplectic involution σ and $\mathrm{SG}(A,\sigma)$ is a closed subscheme of $\mathrm{SB}(n,A)$ consisting of the isotropic ideals of $\mathrm{SB}(n,A)$ with respect to σ. Considering an isotropic ideal $I_a \in A$ of reduced dimension a,

$$X_{(r,a)} \cap \mathrm{SG}(A,\sigma) = \{J \in \mathrm{SG}(A,\sigma) \mid \mathrm{rdim}(J \cap I_a) \geq r\},$$

is a closed subvariety of $\mathrm{SG}(A,\sigma)$. In particular, if we let λ be a self-dual partition fitting in a $n \times n$ box and assume the conditions of Proposition 5.2.4 i.e. $\mathrm{ind}(A) | \gcd(\overline{E_\lambda})$ and $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$ then for any $(r,a) \in E_\lambda$, there exists a reduced dimension a isotropic ideal $I_a \in A$ defined over K such that

$$P_\lambda = \{J \in \mathrm{SG}(A,\sigma) \mid \mathrm{rdim}(J \cap I_a) \geq r, (r,a) \in E_\lambda\} = \cap_{(r,a)\in E_\lambda} \mathrm{SG}(A,\sigma) \cap X_{(r,a)}$$

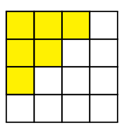
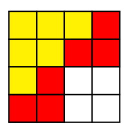
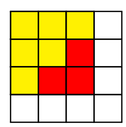
and P_λ is a closed subscheme of $\mathrm{SG}(A,\sigma)$. For a splitting field L/K of (A,σ), it is clear that $(P_\lambda)_L \cong X_\lambda$. Indeed, this is follows since $(A_L,\sigma_L) \cong (\mathrm{End}_L(V),\sigma_h)$ for some n-dimensional L-vector space V and some hermitian form h on V associated to σ where all isotropic ideals of reduced dimension r take the form $\mathrm{Hom}_L(V,W)$ for some r-dimensional subspace $W \subset V$ which is totally isotropic with respect to h. $\square$

5.3 The singular locus of a Schubert variety

The goal of this section is to prove the converse of Proposition 5.2.4. That is, we want to show that if $\mathrm{SG}(A,\sigma)$ has a closed subvariety P defined over K which is a twisted form of a Schubert variety X_λ, then we must have $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$ and $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ (which is equivalent to $\overline{E_\lambda} \subset \mathrm{ind}(A,\sigma)$). More specifically, we would like to show that P must be defined via intersection conditions with right ideals of A.

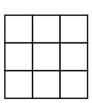
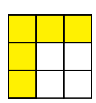
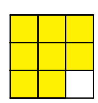
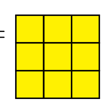
Fixing n, consider a self-dual partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ which corresponds to a singular Schubert variety $X_\lambda \subset \mathrm{SG}(n,2n)$. The singular locus $\mathrm{Sing}(X_\lambda)$ of X_λ, consists of a union of Schubert subvarieties $X_\mu \subset X_\lambda$ such that μ is a partition obtained from λ obtained by adding either a pair of dual South-East hooks, or a single self-dual South-East hook, to the Young diagram of λ. For a more precise version of this statement, we refer the reader to Section 9.3 of [BL00].

Example 5.3.1. Suppose $n = 4$. For the self-dual partition $\lambda = [3,2,1,0]$, $\mathrm{Sing}(X_\lambda)$ consists of two subvarieties X_μ and $X_{\mu'}$ with $\mu = [4,4,2,2]$ and $\mu' = [3,3,3,0]$.

$\lambda =$  $\mu =$  $\mu' =$ 

The set of smooth Schubert varieties consists of those X_λ for which $\mathrm{Sing}(X_\lambda) = \emptyset$. In terms of self-dual Young diagrams, X_λ is smooth if and only if the uncoloured boxes in the $n \times n$ square form a $k \times k$ square, for some $0 \le k \le n$.

Example 5.3.2. Suppose $n = 3$. The smooth Schubert subvarieties of $LG(3,6)$ are given by:

$[0,0,0] =$  $[3,1,1] =$  $[3,3,2] =$  $[3,3,3] =$ 

As it turns out, similar to the case of generalized Severi-Brauer varieties [JKL17, Theorem 2.8], we want to show that the smooth Schubert subvarieties of $\mathrm{SG}(A,\sigma)$ can be defined by inclusions. We start by considering the split case $(A,\sigma) = (\mathrm{End}_K(V), \sigma_h)$ with $\dim_K(V) = 2n$.

Suppose $\lambda = [n^k, k^{n-k}]$ is the self-dual partition whose corresponding Young diagram leaves a $k \times k$ square un-coloured for some $0 \leq k \leq n$ (i.e. λ consists of k copies of n followed by $n-k$ copies of k). If $k > 0$, this diagram has exactly one corner on or above the North-West to South-East diagonal, with $E_\lambda = \{(k,k)\}$ (for $k = 0$, we have $X_\lambda = \mathrm{SG}(\mathrm{End}_K(V), \sigma_h)$). Thus, the smooth variety X_λ with $E_\lambda = \{(k,k)\}$ can be defined by a single intersection condition:

$$\begin{aligned} X_\lambda &= \{J \in \mathrm{SG}(\mathrm{End}_K(V), \sigma_h) \mid \mathrm{rdim}(J \cap I_k) \geq k\} \\ &= \{J \in \mathrm{SG}(\mathrm{End}_K(V), \sigma_h) \mid I_k \subseteq J\} \end{aligned}$$

Using this description, we can classify twisted forms of smooth Schubert subvarieties of $\mathrm{SG}(n, 2n)$.

Proposition 5.3.3. *Consider the maximal symplectic Grassmannian* $\mathrm{SG}(A,\sigma)$ *with* $\deg(A) = 2n$*, together with a Galois splitting field* L/K*. For any* $1 \leq k \leq n$*, there exists a closed subvariety* $P \subset \mathrm{SG}(A,\sigma)$ *defined over* K *such that* $P \otimes_K L = X_\lambda$ *for* $\lambda = [n^k, k^{n-k}]$ *if and only if* $k \in \mathrm{ind}(A,\sigma)$*.*

Proof. Assume P is a closed subvariety of $\mathrm{SG}(A,\sigma)$ such that $P_L \cong X_\lambda$, for a self-dual partition $\lambda = [n^k, k^{n-k}]$ which means that $E_\lambda = \{(k,k)\}$. We remark that X_λ can also be described as the set of maximal isotropic right ideals of $A_L \simeq \mathrm{End}_L(V)$ containing the right ideal $\mathrm{Hom}_L(V,W)$ for some isotropic vector subspace $W \subseteq V$ with $\dim_L(W) = k$. Now, consider the right ideal of $\mathrm{End}_L(V)$ defined as

$$\overline{I} := \bigcap_{J \in P_L} J$$

First, notice that P is fixed by $\mathrm{Gal}(L/K)$, since the collection of J in the indexing set above is permuted by the Galois action. By descent it follows that $\overline{I} = I_L$ for some right ideal I of A. Moreover, using the definition of X_λ we see that $I_k \subseteq J$ implies $I_k \subseteq \overline{I}$ which for us means $\mathrm{rdim}(\overline{I}) \geq k$ implies $\mathrm{rdim}(I) \geq k$. Similarly, the isotropy of $\overline{I}$ descends to the isotropy of I since $\sigma(I \otimes_K 1)(I \otimes_K 1) \subset \sigma(\overline{I})\overline{I} = 0$ where we identify I with $I \otimes_K 1 \subset I_L$. This gives $\mathrm{rdim}(I) \in \mathrm{ind}(A,\sigma)$ and since $\mathrm{rdim}(I) > k$ we have $k \in \mathrm{ind}(A,\sigma)$. Indeed, by the definition of reduced dimension, all elements of $\mathrm{ind}(A,\sigma)$ are multiples of $\mathrm{ind}(A)$ which means that if I is an isotropic ideal in (A,σ) then $\mathrm{rdim}(I) = r \,\mathrm{ind}(A)$ and $k = m \,\mathrm{ind}(A)$ where $r \geq m$. In fact, by applying

Lemma 5.3.4 (4.) (which we will see shortly) it follows that $\mathrm{rdim}(I) = k$.

To see the reverse direction, if $k \in \mathrm{ind}(A, \sigma)$, then we have by Proposition 5.2.4 that $\mathrm{SG}(A, \sigma)$ has a closed subvariety X_λ with $\lambda = [n^k, k^{n-k}]$, $E_\lambda = \{(k, k)\}$, and $\overline{E_\lambda} = k$. $\square$

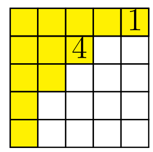
We define a set $S_\lambda := \{(j, a) \in E_\lambda \mid \ j < a\}$. It can be easily shown that a pair (j, a) is in S_λ if and only if $(j + 1, a) \in E_\mu$ for some $X_\mu \subseteq \mathrm{Sing}(X_\lambda)$. By this reasoning, we refer to S_λ as the *essential singular set* of X_λ. As before, we set $\overline{S_\lambda} := \{a \mid (j, a) \in S_\lambda\}$. While E_λ determines all corners of the Young diagram of λ above or on the North-West to South-East diagonal, S_λ picks up only the "inside" corners.

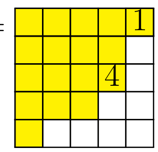
Lemma 5.3.4. *Let X_λ be a Schubert subvariety of $\mathrm{SG}(\mathrm{End}_K(V), \sigma_h)$ with respect to the flag of isotropic right ideals $I_1 \subset I_2 \subset \cdots \subset I_n \subset \mathrm{End}_K(V)$, with $\mathrm{rdim}(I_r) = r$. The following are equivalent:*

1. *λ_k defines an outside corner of the self-dual Young diagram of λ, i.e. $n = \lambda_k > \lambda_{k+1}$.*
2. *$(k, k) \in E_\lambda$.*
3. *$k \in \overline{E_\lambda} \setminus \overline{S_\lambda}$.*
4. *$I_k \subseteq J$ for all $J \in X_\lambda$ and there exists some $J' \in X_\lambda$, such that $I_{k+1} \not\subseteq J'$.*

Proof. (1) $\iff$ (2): It follows from the definition of E_λ that $n = \lambda_k > \lambda_{k+1}$ if and only if $(k, n + k - \lambda_k) = (k, k) \in E_\lambda$.
(2) $\Longrightarrow$ (3): It suffices to show that $k \notin \overline{S_\lambda}$, which follows immediately from the definition of S_λ, as $(j, k) \in S_\lambda$ implies $j < k$.
(3) $\Longrightarrow$ (2): If $k \in \overline{E_\lambda} \setminus \overline{S_\lambda}$, then $(j, k) \in E_\lambda$ for some $1 \leq j \leq k$. If $j < k$, this contradicts the assumption that $k \notin \overline{S_\lambda}$. So, we must have $j = k$ and hence $(k, k) \in E_\lambda$.
(2) $\Longrightarrow$ (4): If $(k, k) \in E_\lambda$, then for all $J \in X_\lambda$ we must have $\mathrm{rdim}(J \cap I_k) \geq k$, hence $I_k \subseteq J$. Suppose that for all $J \in X_\lambda$, $I_{k+1} \subseteq J$ or, equivalently $\mathrm{rdim}(J \cap I_{k+1}) \geq k + 1$. This implies $\lambda_{k+1} = n = \lambda_k$, contradicting the assumption that $\lambda_k > \lambda_{k+1}$.
(4) $\Longrightarrow$ (1): Suppose $\mathrm{rdim}(J \cap I_k) = k$ for all $J \in X_\lambda$, but there exists some $J' \in X_\lambda$ such that $\mathrm{rdim}(J' \cap I_{k+1}) < k + 1$. It follows that $\lambda_k = n$ but $\lambda_{k+1} < n$. $\square$

Example 5.3.5. Suppose $n = 5$ and consider the self-dual partition $\lambda = [5, 3, 2, 1, 1]$. In this case, $E_\lambda = \{(1, 1), (2, 4)\}$, and $S_\lambda = \{(2, 4)\}$. Furthermore, $\mathrm{Sing}(X_\lambda) \supset X_\mu$ with $\mu = [5, 4, 4, 3, 1]$.

$\lambda =$ 

$\mu =$ 

We obtain the following corollary to Proposition 5.3.3,

Corollary 5.3.6. *Let λ be a self-dual partition, L/K a field extension and k be the unique element in $\overline{E_\lambda} \setminus \overline{S_\lambda}$. If $\mathrm{SG}(A, \sigma)$ has a closed subvariety P such that $P \otimes_K L \simeq X_\lambda$, then $k \in \mathrm{ind}(A, \sigma)$. Moreover, there exists an isotropic right ideal $I_k \subset A$ such that for any $J \in P(L)$, $(I_k \otimes_K L) \subseteq J$.*

Proof. This is a straight-forward application of Lemma 5.3.4 to the proof of Proposition 5.3.3. $\square$

In order to provide the full converse to Proposition 5.2.4 for an arbitrary Schubert variety X_λ, it remains to show that if $\mathrm{SG}(A, \sigma)(K)$ contains an K-form P of X_λ, then $\mathrm{ind}(A) \mid \gcd(\overline{S_\lambda})$ and $\max(\overline{S_\lambda}) \in \mathrm{ind}(A, \sigma)$. To do this, we construct closed subvarieties of P which are defined over K and to which we can apply Corollary 5.3.6. These subvarieties will be obtained from the structure of $\mathrm{Sing}(X_\lambda)$. We rely on the fact for a K-variety X, the singular locus of $X_{K^{alg}}$ defines a Zariski-closed subset Z of X. By equipping Z with the reduced induced scheme structure, we may view Z as a subvariety of X defined over K. In particular, if $\mathrm{SG}(A, \sigma)(K)$ has a closed subvariety P defined over K such that $P(L) \simeq X_\lambda$ for a splitting field L/K of A, then P has a closed subvariety $Z \subset P$ defined over K such that $Z(L) \simeq \mathrm{Sing}(X_\lambda)$.

5.3.1 An iterative process

In general, the singular locus of a Schubert variety may have many irreducible components, none of which are required to be smooth. We deal with this by recursively considering "the singular locus of a component of the singular locus" until we achieve subvarietes of P which are K-forms of smooth Schubert varieties.

Consider the variety P defined over K with the property that $P \otimes_K L \simeq X_\lambda$, we repeat the subvariety construction to achieve a closed subvariety $Z \subset P$ defined over K with $Z \otimes_K L \simeq X_\mu$, provided that μ can be obtained from λ by adding a finite number of hooks. We have the following lemma which provides a combinatorial description of some particular partitions which can be formed by adding hooks to a given partition λ. In essence, it describes the effect of adding self-dual hooks on the Young diagram in terms of the essential set. This will be key in proving Theorem 5.3.11 which is a new result describing the existence of twisted Schubert varieties in terms of combinatorial information pertaining to the essential set and the index.

Lemma 5.3.7. *Consider a partition $\lambda = [\lambda_1, \ldots, \lambda_n]$ and suppose λ_j corresponds to an inside corner. That is, $(j, a) \in E_\lambda$ with $j < a$.*

1. *If $a < n - a$, adding $a - j$ hooks (or pairs of self-dual hooks) to λ will result in a partition $\mu = [\mu_1, \ldots, \mu_n]$ such that*

$$\mu_i = \begin{cases} n & \text{if } i \leq a \\ \lambda_i & \text{if } n - a \geq i > a \\ a & \text{if } i > n - a \end{cases}$$

2. *If $a \geq n - a$, adding $a - j$ hooks (or pairs of self-dual hooks) to λ will result in a partition $\mu = [\mu_1, \ldots, \mu_n]$ such that*

$$\mu_i = \begin{cases} n & \text{if } i \leq a \\ a & \text{if } i > a \end{cases}$$

3. *In both cases, $a \in \overline{E_\mu} \setminus \overline{S_\mu}$.*

Proof. The basic idea is that we begin adding a hook to j with a corresponding dual hook, if necessary, to ensure the partition is self-dual. Repeating this procedure inductively on the new partition $\overline{\lambda}$ will yield the desired results.

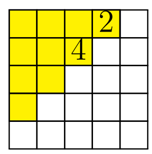
To be more explicit, in the first case, we observe that $(j, a) \in E_\lambda$ if and only if $\lambda_j = n - a + j$. It is easy to see that adding $a - j$ hooks to the corner (j, λ_j) implies that the resulting partition μ_j has the following relationship with λ_j:

$$\mu_j = \lambda_j + a - j = n$$

which gives that $\mu_i = n$ for all $i \leq a$. By duality, $(i,k) \in E_\lambda$ with $i \leq a$ and $n - a + 1 \leq k \leq n$ implies that $(k,i) \in E_\lambda$ with with $i \leq a$ and $n - a + 1 \leq k \leq n$. In diagramatic terms, this establishes both a top right $a \times a$ box and a bottom left $a \times a$ box. Now, since $(i,k) \notin E_\lambda$ for $i > a$ and $n - a + 1 \leq k \leq n$ we have again by duality $(k,i) \notin E_\lambda$ for $i > a$ and $n - a + 1 \leq k \leq n$ which gives us that the last a rows of μ are precisely a.

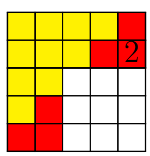
The proof of 2. follows similarly to 1. Indeed, the first a rows are n as before and since there are less than a rows remaining we have that the $n - a$ rows must be determined by duality. □

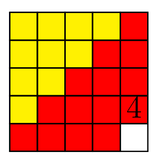
Example 5.3.8. Consider $n = 5$ and $\lambda = [4,3,2,1,0]$. The diagram of λ has 2 inside corners with $E_\lambda = \{(1,2),(2,4)\}$ and $\overline{S_\lambda} = \{2,4\}$.

$\lambda =$ 

- For $(1,2)$, we have $a = 2 < 5 - 2$, so applying part 1 of Lemma 5.3.7, we obtain the partition $\mu = [5,5,2,2,2]$ after adding a pair of self-dual hooks.

- For $(2,4)$, we have $a = 4 \geq 5 - 4$, so applying part 2 of Lemma 5.3.7 may be applied to obtain the partition $\alpha = [5,5,5,5,4]$ after adding 2 pairs of self-dual hooks.

$\mu =$ 

$\alpha =$ 

Note that X_μ and X_α are both smooth.

5.3.2 Galois action on the singular locus

We desire a stronger claim than the existence of an K-form of $\mathrm{Sing}(X_\lambda)$. In particular, we would like to say that for any Schubert variety $X_\mu \subseteq \mathrm{Sing}(X_\lambda)$, if P is a twisted form of X_λ defined over F, then P has a closed subvariety $Z \subset P$, also defined over F, such that Z is a twisted form of X_μ. More precisely, we have:

Lemma 5.3.9. *Let K/F be a Galois splitting field for A, and suppose that we have a subvariety P of* $\mathrm{SG}(A,\sigma)$*, such that $P_L = X_\lambda$. If $X_\mu \subset \mathrm{Sing}(X_\lambda)$ is an irreducible component of the singular locus of X_λ, defined by the addition of a hook to the Young diagram for λ, then there exists a subvariety $Z \subset P$ such that $Z_L = X_\mu$.*

Proof. By the geometric description of the irreducible components of the singular set, it is automatic that the Galois action, which acts via elements of $\mathrm{PSp}(V_L)$ cannot nontrivially permute the components of the singular set. Hence, considered as points on the Hilbert scheme of $\mathrm{SG}(A,\sigma)$, these irreducible components are fixed by the Galois action, and hence correspond to K-rational subvarieties $Z \subset P$ as claimed. □

Proposition 5.3.10. *Let λ and μ be partitions defining Schubert subvarieties of* $\mathrm{SG}(n,2n)$ *such that μ is obtained from λ by adding finitely many self-dual hooks. For a central simple K-algebra A of degree n, if* $\mathrm{SG}(A,\sigma)$ *contains a closed subvariety P defined over K such that $P_{K^{alg}} \simeq X_\lambda$, then P contains a closed subvariety Z defined over K such that $Z_{K^{alg}} \simeq X_\mu$.*

Proof. If μ is obtained from λ by adding finitely many hooks, we may form a sequence $\alpha_1, \ldots, \alpha_k$ such that $\lambda = \alpha_1$, $\mu = \alpha_k$ and for each $2 \le i \le k$, α_i is obtained from α_{i-1} by adding precisely one hook or pair of dual hooks. It follows from the definition of the singular locus that for each $2 \le i \le k$, $X_{\alpha_i} \in \mathrm{Sing}(X_{\alpha_{i-1}})$. Under the assumption that $\mathrm{SG}(A,\sigma)$ contains a twisted form of X_{α_1} over K, the above argument implies that $\mathrm{SG}(A,\sigma)$ must also contain a twisted form of X_{α_2} over K. By induction on i, we obtain the result that $\mathrm{SG}(A,\sigma)$ must finally contain a twisted form of $X_{\alpha_k} = X_\mu$ defined over K. □

This process yields the desired converse to Proposition 5.2.4.

Theorem 5.3.11. *The maximal symplectic Grassmannian* $\mathrm{SG}(A,\sigma)$ *has a closed subvariety P such that $P \otimes_K L \simeq X_\lambda$ for a Schubert subvariety X_λ if and only if* $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ *and* $\max(\overline{E_\lambda}) \in \mathrm{ind}(A,\sigma)$. *Moreover, in this case, A contains a flag of isotropic right ideals $I_{a_1} \subset \cdots \subset I_{a_r}$ for $\overline{E_\lambda} = \{a_1, \ldots, a_r\}$ such that for any finite extension L/K,*

$$P(L) = \{J \subseteq A_L : \mathrm{rank}(J \cap (I_a)_L) \ge j \textit{ for } (j,a) \in E_\lambda\}$$

Proof. If X_λ is smooth, then the result follows immediately from Proposition 5.3.3. Suppose $S_\lambda \neq \emptyset$ and that $\mathrm{Sing}(X_\lambda) = X_{\mu_1} \cup \cdots \cup X_{\mu_k}$. Suppose $(j, a) \in S_\lambda$ for some $j < a$. Using the replacement process described in Lemma 5.3.7 together with Proposition 5.3.10, P has a closed subvariety Z defined over K such that $Z_{K^{alg}} \simeq X_\mu$ where μ is obtained from λ by adding hooks implying $a \in \overline{E_\mu} \setminus \overline{S_\mu}$ where λ_a is an insider corner of λ. Applying Corollary 5.3.6 we must have $a \in \mathrm{ind}(A, \sigma)$. In particular, this gives that $\mathrm{ind}(A) \mid \gcd(\overline{S_\lambda})$ (by 5.3.6 it follows that $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$). Now, recall that outside corners are unique if they exist. Indeed, suppose $(j, n) \in E_\lambda$ is an outside corner i.e. $\lambda_j = n$, then $(j, n - n + j) = (j, j) \in E_\lambda$. Similarly, if (k, λ_k) is an inside corner then $\lambda_k < n$ and $k > j$ so $a_k = (n - \lambda_k) + k > k > j$. Combining these facts we see that inside corners must be strictly bigger than outside corners which gives us that $\max(\overline{E_\lambda}) \leq \max(\overline{S_\lambda}) \in \mathrm{ind}(A, \sigma)$.

Now, suppose $\mathrm{ind}(A) \mid \gcd(\overline{E_\lambda})$ and $\max(\overline{E_\lambda}) \in \mathrm{ind}(A, \sigma)$. This is equivalent to the claim that A contains a flag of isotropic right ideals $I_{a_1} \subset \cdots \subset I_{a_r}$ with $\overline{E_\lambda} = \{a_1, \ldots, a_r\}$. For a splitting field L/K of A, fix a full flag of isotropic right ideals $I_1{}' \subset I_2{}' \subset \cdots \subset I_n{}'$ such that $I_{a_j} \otimes_K L = I_{a_j}{}'$ for all $a_j \in \overline{E_\lambda}$. Let X_λ be the Schubert subvariety of $\mathrm{SG}(n, 2n)$ defined by λ with respect to this flag.

Denote by P_λ the closed K-subvariety of $LG(A, \sigma)$. Recall that P_λ is a closed subvariety bya the arguments provided in the proof of Proposition 5.2.4). Now, for any field extension L/K,

$$P_\lambda(L) := \{J \subseteq A_L : \mathrm{rank}(J \cap (I_a)_L) \geq j \text{ for } (j, a) \in E_\lambda\}$$

If P is a twisted form of X_λ defined over K, the goal is to show that $P = P_\lambda$. Let L/K be an arbitrary finite field extension and let $J \in P(L)$. After extending to a splitting field L'/L, we find that for any $(j, a) \in E_\lambda$, we have

$$\mathrm{rank}(J \cap (I_a)_L) = \mathrm{rank}((J \cap (I_a)_L)_{L'}) = \mathrm{rank}(J_{L'} \cap (I_a)_{L'}) \geq j$$

since $I_{L'} \in P'_L(L') = X_\lambda(L')$. So $P(L) \subseteq P_\lambda(L)$. We have that $i : P \hookrightarrow P_\lambda$ is an inclusion of K-varieties since $i_L : P(L) \hookrightarrow P_\lambda(L)$ for all finite field extensions L/K. If L'/K is a splitting field for P, then $i_{L'}$ induces the identity map. So $\mathrm{coker}(i)_{L'} = \mathrm{coker}(i_{L'}) = 0$. Thus, $\mathrm{coker}(i)$ is a form of the zero variety and so $\mathrm{coker}(i) = 0$. It follows that $P = P_\lambda$ as required. $\square$

5.4 Bounds for torsion in the topological filtration

The remainder of this chapter is dedicated to computing properties of the torsion Chow group corresponding to maximal symplectic Grassmannians of degree 4 algebras with symplectic involution. For details and deeper results on the Chow group please see [EKM08, Chapter X] We begin by consider a smooth projective variety X over K, a field of characteristic 0. Consider the Grothendieck group, the abelianization of a commutative monoid (see Grothendieck-Witt group defined in Chapter 1),

$$\mathrm{K}_0(X) = \langle [\mathcal{O}_V] \mid V \subseteq X \text{ closed subvariety}\rangle.$$

We define a topological filtration on $\mathrm{K}_0(X)$ by setting

$$\mathrm{K}_0(X)^{(i)} := \langle [\mathcal{O}_V] \mid \mathrm{codim}(V) \geq i\rangle \quad \text{and then} \quad T^i(X) := \mathrm{K}_0(X)^{(i)}/\mathrm{K}_0(X)^{(i+1)}$$

where $\mathcal{O}_V$ is the sheaf of K-algebras mapping to the be the ring of regular functions on an open set. In particular, we will avoid explicitly defining the Chow group by observing the following identifications in the case that X is a projective quadric (see [Kar91, §3.1, Corollary 4.4 and Corollary 4.5]):

$$T^1(X) \cong \mathrm{CH}^1(X),$$
$$T^2(X) \cong \mathrm{CH}^2(X).$$

The product in $\mathrm{K}_0(X)$ induces a graded ring structure on $T^*(X)$, and taking the class of a subvariety $W \subset X$ to the corresponding coherent sheaf $\mathcal{O}_W$ induces a natural surjection of graded rings $\mathrm{CH}^d(X) \to \mathrm{T}^d(X)$ which becomes an isomorphism when tensored with $\mathbb{Z}[\frac{1}{(d-1)!}]$ (see [SGA6, Corollary 1 – Theorem 2.12 in Appendix (RRR) and Example 0 of Chapter 2, §4])

We denote by $\overline{X}$ the variety X over the algebraic closure of K. For the case $\overline{X} = \mathrm{LG}(n, 2n)$, a $\mathbb{Z}$ basis of $\mathrm{T}^i(\mathrm{LG}(n, 2n))$ is given by

$$\{\Sigma_\lambda : |\lambda| = \Sigma_{i=1}^n \lambda_i, n \geq \lambda_1 \geq \cdots \geq \lambda_n \geq 0\}$$

where $\lambda = [\lambda_1, \ldots, \lambda_n]$ is a partition of $|\lambda| = \sum_{j=1}^n \lambda_j$ and $\Sigma_\lambda = [\mathcal{O}_V]$ for $V = X_\lambda$ (see [KT02, §2])

The remainder of this section is devoted to proving the main result of this Chapter, the computation of torsion elements in the Chow group of a maximal symplectic Grassmannian for a central simple K-algebra of $\deg(A) = 4$ equipped with a symplectic involution. We first make some general remarks on the key ideas used in our proof.

Recall that $X = \mathrm{SG}(A, \sigma) = \mathrm{IV}_2(A, \sigma)$. To compute $|\operatorname{Tors}(T^*(X))|$ with $\deg(A) = 4$ and $\operatorname{ind}(A) \mid 4$ we sketch the basic ideas here, for a more detailed exposition on the background refer to Appendix C. First, we observe that A decomposes as:

$$(A, \sigma) \cong (Q_1 \otimes Q_2, \sigma),$$

where Q_1, Q_2 are quaternion algebras. By [Lou78, Theorem B], a symplectic involution σ on a central simple algebra of degree 4 fixes both Q_1 and Q_2. Thus we may decompose σ into a diagonal action given by $\tau \otimes \gamma$ where τ is an orthogonal involution on Q_1 (resp Q_2) and γ is the canonical involution on Q_2 (resp Q_1). We shift our attention slightly and consider the pfaffian norm associated to the symplectic involutions Nrp_σ given by $\mathrm{Nrp}_\sigma(a) = \sigma(a)a$ for all $a \in \mathrm{Sym}(A, \sigma)$. By [KMRT98, Proposition 16.8] it follows that Nrp_σ is, in fact, an Albert quadric. If $(A, \sigma) \cong (Q_1, \tau) \otimes_K (Q_2, \gamma)$ then we have an explicit description of the Albert quadric (see [KMRT98, Example 16.15]) given by,

$$\mathrm{Nrp}_\sigma \cong \mathrm{Nrd}_{Q_1}(v)(\mathrm{Nrd}'_{Q_1} \perp -\mathrm{Nrd}'_{Q_2})$$

where Nrd'_Q denotes the pure part of the norm form Nrd_Q. In either case, using [KMRT98, Proposition 15.20] we are able to realize the set-theoretic isomorphism of Nrp_σ and a codimension 1 subform s_σ as an isomorphism of varieties using [Kra10, Proposition 8.10] and [McF19, Remark 3.5, Theorem 4.1]. In particular, we can summarize these results as saying that, geometrically speaking, $IV_2(A, \sigma)(R)$ is a hyperplane of $\mathrm{SB}_2(A)(R)$ given by $\{Trd_A = 0\}$ and there is an isomorphism of varieties representing the functors of points corresponding to ideals of reduced dimension 2 and isotropic ideals of reduced dimension 2:

$$\mathrm{SB}_2(A, \sigma) \cong X_{\mathrm{Nrp}_\sigma},$$

$$\mathrm{SG}(A, \sigma) \cong X_{s_\sigma},$$

where X_q denotes a projective quadric associated to some quadratic form q. We proceed to compute $|\operatorname{Tors}(T^*(X_{s_\sigma}))|$. The main idea is to use the some key ideas from [Kar91], especially Theorem 3.8 and Corollary 4.5, which can be restated for our purposes as:

$$|\operatorname{Tors}(T^*(X_{s_\sigma}))| = 2^{s(s_\sigma)},$$

for $q \notin I^2(K)$ and $s(q)$ is given by $C_0(q) \cong M_{2^{s(q)}}(D)$, with $C_0(q)$ denoting the even Clifford algebra corresponding to q and D the central simple division algebra Brauer-equivalent to $C_0(q)$.

Theorem 5.4.1. *Let (A, σ) be a degree 4 central simple K-algebra equipped with a symplectic involution σ. Then the torsion of the topological filtration corresponding to the maximal symplectic Grassmannian,* $\mathrm{SG}(A, \sigma)$ *is determined as follows:*

1. *If* $\operatorname{ind}(A) = 4$ *then* $|\operatorname{Tors}(\mathrm{T}^*(\mathrm{SG}(A, \sigma))| = 1$
2. *If* $\operatorname{ind}(A) = 2$ *and σ is anisotropic then* $|\operatorname{Tors}(\mathrm{T}^*(\mathrm{SG}(A, \sigma))| = 2$
3. *If* $\operatorname{ind}(A) = 2$ *and σ is isotropic then* $|\operatorname{Tors}(\mathrm{T}^*(\mathrm{SG}(A, \sigma))| = 1$
4. *If* $\operatorname{ind}(A) = 1$ *then* $|\operatorname{Tors}(\mathrm{T}^*(\mathrm{SG}(A, \sigma))| = 1$

Proof.

1. Assume $\operatorname{ind}(A) = 4$. Note firstly that s_σ is a 5-dimensional anisotropic subform of Nrp_σ, or, in other words

   $$Nrp_\sigma \cong \langle d \rangle \perp s_\sigma$$

   for some $d \in K^\times$. Moreover, $d_\pm(Nrp_\sigma) = 1$ implies $d = -d_\pm(s_\sigma)$. We observe the following well-known result (see [Lam05, V.3.13]) for all odd-dimensional quadratic forms q:

   $$[C_0(q)] = [C(q \perp -d_\pm(q))].$$

   In particular,

   $$[C_0(s_\sigma)] = [C(s_\sigma \perp -d_\pm(s_\sigma))] = [C(Nrp_\sigma)] = [A].$$

   where the last equality follows due to the fact that Nrp_σ is the Albert form of A by [KMRT98, Proposition 16.8]. Since $\dim(C_0(s_\sigma)) = 2^{n-1}$ where $n = \dim(s_\sigma)$ we conclude that $C_0(s_\sigma) \cong A$, whence $s(s_\sigma) = 0$ and $|\operatorname{Tors}(T^*(X_{s_\sigma}))| = 1$.

2. Assume $\mathrm{ind}(A) = 2$. We can assume without loss of generality that $(Q_1, \tau) \cong (M_2(K), ad_q)$. By the remarks preceding [KLST95, Proposition 3.5] we know that

$$Nrp_\sigma \cong \langle 1, -1 \rangle \perp \left\langle -h_1 h_2^{-1} \right\rangle Nrd_{Q_2} \cong \langle 1 \rangle \perp s_\sigma$$

where $h : V \longrightarrow K$ is a Hermitian form over Q_2 corresponding to σ and $h(e_i, e_i) = h_i$ form an orthogonal basis $\{e_1, e_2\}$ of V. The key insight we will need to proceed is that h is isotropic/hyperbolic if and only if q_h is isotropic/hyperbolic (see [Sch85, Theorem 10.1.7]), where q_h is the trace form of h given by

$$q_h \cong \langle h_1 \rangle Nrd_{Q_2} \perp \langle h_2 \rangle Nrd_{Q_2}.$$

Applying this to

$$s_\sigma \cong \langle -1 \rangle \perp \left\langle -h_1 h_2^{-1} \right\rangle Nrd_{Q_2},$$

we have that s_σ is isotropic if and only if either Nrd_{Q_2} is hyperbolic or $-h_1 h_2^{-1} \in D_K(Nrd_{Q_2})$. This can be rephrased as s_σ is isotropic if and only if q_h is hyperbolic if and only if σ is hyperbolic.

Assume σ is anisotropic, then s_σ is anisotropic. Moreover, since

$$[C_0(s_\sigma)] = [M_2(F) \otimes Q_2] = [Q_2]$$

implies $C_0(s_\sigma) \cong M_2(Q_2)$, we have that $s(s_\sigma) = 1$. By a direct application of [Kar91, Theorem 3.8] we see that

$$\mathrm{Tors}(\mathrm{T}^*(X_{s_\sigma})) = \mathbb{Z}/2.$$

In particular $\mathrm{Tors}(\mathrm{T}^2(X_{s_\sigma})) = \mathbb{Z}/2$. If, on the other hand, we assume σ is hyperbolic (since $\deg(A) = 4$ we know that σ can only be either anisotropic or hyperbolic in our case), then s_σ is isotropic, i.e.

$$s_\sigma \cong \langle 1, -1 \rangle \perp q_\sigma$$

with $\dim(q_\sigma) = 3$. It follows easily that

$$|\,\mathrm{Tors}(\mathrm{T}^*(X_{s_\sigma}))| = 1.$$

3. Lastly assume $\operatorname{ind}(A) = 1$, the hyperbolicity of σ implies s_σ is isotropic and so we can conclude by the same reasoning as above that

$$|\operatorname{Tors}(\mathrm{T}^*(X_{s_\sigma}))| = 1$$

□

Using the fact that $T^i(X) \cong \mathrm{CH}^i(X)$ for $i = 0, 1, 2, 3$ (see [Kar91, §3.1]), we have the following corollary:

Corollary 5.4.2. *Let (A, σ) be a degree 4 central simple K-algebra equipped with a symplectic involution σ. Then the torsion of the Chow group corresponding to the maximal symplectic Grassmannian, $\mathrm{SG}(A, \sigma)$ is determined as follows:*

1. *If $\operatorname{ind}(A) = 4$ then $|\operatorname{Tors}(\mathrm{CH}^*(\mathrm{SG}(A, \sigma))| = 1$*
2. *If $\operatorname{ind}(A) = 2$ and σ is anisotropic then $|\operatorname{Tors}(\mathrm{CH}^*(\mathrm{SG}(A, \sigma))| = 2$*
3. *If $\operatorname{ind}(A) = 2$ and σ is isotropic then $|\operatorname{Tors}(\mathrm{CH}^*(\mathrm{SG}(A, \sigma))| = 1$*
4. *If $\operatorname{ind}(A) = 1$ then $|\operatorname{Tors}(\mathrm{CH}^*(\mathrm{SG}(A, \sigma))| = 1$*

In particular, if a non-trivial torsion element exists, then it must be in $\mathrm{CH}^2(\mathrm{SG}(A, \sigma))$.

Proof. We know that $CH^d(X) \cong T^d(X)$ for $d = 0, 1, 2, 3$ by the remarks following [Kar91, §3.1, Theorem 4.5]. The result then follows directly from Theorem 5.4.1. In particular, keeping the notation introduced in the proof of Theorem 5.4.1, we know by [EKM08, Lemma 70.2] that $\mathrm{CH}^i(X_{s_\sigma}) \cong \mathrm{CH}^{i-1}(X_{q_\sigma})$ for all $i > 0$. We can see that, in the case $\operatorname{ind}(A) = 2$ equipped with an anisotropic involution σ implies that torsion must occur in the second graded component of the Chow group. Alternatively, this follows directly by [Kar91, Theorem 4.5]. □

This concludes our investigation into embeddability through various different disciplines, approaches and techniques.

Appendix A

Computations of Pfister elements

In this chapter we give a computational verification that the quadratic form

$$q_{\theta_3} \cong \langle\langle a_1\rangle\rangle \perp -b_1b_2a_3 \langle\langle a_2\rangle\rangle \perp a_1b_1a_2b_3 \langle\langle a_3\rangle\rangle \perp -b_1a_2 \langle\langle a_1a_2a_3\rangle\rangle\,,$$

is in fact similar to a 3-fold Pfister form. It suffices to check that $c(q_{\theta_3}) = 1$ by [Lam05, Theorem X.5.6]. For simplicity, we will use the *Hasse invariant*,

$$s : W(K) \longrightarrow \mathrm{Br}_2(K)$$

defined by $s(\langle a_1, \ldots, a_n\rangle) := \prod_{i<j} \left(\frac{a_i, a_j}{K}\right)$. The Clifford invariant and Hasse invariant are related by [Lam05, Proposition V.3.20] as follows:

$$c(q) = s(q) \cdot \left(\frac{-1, \det(q)}{K}\right),$$

whenever $\dim(q) = 7, 8 \bmod 8$. To show that q_{θ_3} is a Pfister form we need to show that $c(q_{\theta_3}) = 1 \in \mathrm{Br}_2(K)$ by [Lam05, Theorem V.6.11]. Once we compute $s(q_{\theta_3}) = 1$ we are done since $\det(q) = 1$ implies $\left(\frac{-1,\det(q)}{K}\right) = 1 \in Br_2(K)$ which means in our case, computing $c(q)$ is equivalent to computing $s(q)$. We begin by expanding q_{θ_3},

$$q_{\theta_3} := \langle 1, -a_1, -b_1b_2a_3, b_1a_2b_2a_3, a_1b_1a_2b_3, -a_1b_1a_2a_3b_3, -b_1a_2, a_1b_1a_3\rangle\,.$$

Then

$$s(q_{\theta_3}) = \left(\frac{1,1}{K}\right) \otimes \left(\frac{-a_1,-a_1}{K}\right) \otimes \left(\frac{-b_1b_2a_3, a_1b_1b_2a_3}{K}\right)$$

$$\otimes\left(\frac{b_1a_2b_2a_3, a_1a_2}{K}\right)\otimes\left(\frac{a_1b_1a_2b_3, b_1b_3}{K}\right)\otimes\left(\frac{-a_1b_1a_2a_3b_3, -a_1a_2a_3}{K}\right)\otimes\left(\frac{-b_1a_2, a_1b_1a_3}{K}\right).$$

The argument now boils down to the fact that the tensor product of all the expanded terms will reduce to

$$\left(\frac{a_1, b_1}{K}\right)\left(\frac{a_2, b_2}{K}\right)\left(\frac{a_3, b_3}{K}\right)$$

which is trivial in the Brauer group by the assumption of Theorem 2.3.2 that $\otimes_{i=1}^3(Q_i, \sigma_i)$ is split. To arrive at this decomposition it suffices to take the product of the above terms and reduce the result using the following rules (see Fact 2.1.8):

$$\left(\frac{a, b}{K}\right)\left(\frac{a, b}{K}\right) = 1 \in \mathrm{Br}_2(K),$$
$$\left(\frac{1, c}{K}\right) = \left(\frac{-c, c}{K}\right) = 1 \in \mathrm{Br}_2(K),$$
$$\left(\frac{a, b}{K}\right)\left(\frac{a, -b}{K}\right) = \left(\frac{a, -1}{K}\right) \in \mathrm{Br}_2(K).$$

The diligent reader will see that we arrive at our desired result using this simple, albeit lengthy line of reasoning. To verify this claim, we simply expand the computation of all quaternions which have coefficents that are composites of other terms using the properties of quaternions to reduce $s(q_{\theta_3}$ to 1.

1.
$$\left(\frac{-b_1b_2a_3, a_1b_1b_2a_3}{K}\right) = \left(\tfrac{-b_1,a_1}{K}\right)\left(\tfrac{-b_1,b_1}{K}\right)\left(\tfrac{-b_1,b_2}{K}\right)\left(\tfrac{-b_1,a_3}{K}\right)\left(\tfrac{b_2,a_1}{K}\right)\left(\tfrac{b_2,b_1}{K}\right)\left(\tfrac{b_2,b_2}{K}\right)$$
$$\left(\tfrac{b_2,a_3}{K}\right)\left(\tfrac{a_3,a_1}{K}\right)\left(\tfrac{a_3,b_1}{K}\right)\left(\tfrac{a_3,b_2}{K}\right)\left(\tfrac{a_3,a_3}{K}\right) \in \mathrm{Br}_2(K)$$

2.
$$\left(\frac{b_1a_2b_2a_3, a_1a_2}{K}\right) = \left(\tfrac{b_1,a_1}{K}\right)\left(\tfrac{a_2,a_1}{K}\right)\left(\tfrac{b_2,a_1}{K}\right)\left(\tfrac{a_3,a_1}{K}\right)\left(\tfrac{b_1,a_2}{K}\right)\left(\tfrac{a_2,a_2}{K}\right)\left(\tfrac{b_2,a_2}{K}\right)\left(\tfrac{a_3,a_2}{K}\right) \in \mathrm{Br}_2(K)$$

3.
$$\left(\frac{a_1b_1a_2b_3, b_1b_3}{K}\right) = \left(\tfrac{a_1,b_1}{K}\right)\left(\tfrac{a_1,b_3}{K}\right)\left(\tfrac{b_1,b_1}{K}\right)\left(\tfrac{b_1,b_3}{K}\right)\left(\tfrac{a_2,b_1}{K}\right)\left(\tfrac{a_2,b_3}{K}\right)\left(\tfrac{b_3,b_1}{K}\right)\left(\tfrac{b_3,b_3}{K}\right) \in \mathrm{Br}_2(K)$$

4.

$$\left(\frac{-a_1b_1a_2a_3b_3, -a_1a_2a_3}{K}\right) = \left(\frac{-a_1,-a_1}{K}\right)\left(\frac{-a_1,a_2}{K}\right)\left(\frac{-a_1,a_3}{K}\right)\left(\frac{b_1,-a_1}{K}\right)\left(\frac{b_1,a_2}{K}\right)\left(\frac{b_1,a_3}{K}\right)\left(\frac{a_2,-a_1}{K}\right)$$
$$\left(\frac{a_2,a_2}{K}\right)\left(\frac{a_2,a_3}{K}\right)\left(\frac{a_3,-a_1}{K}\right)\left(\frac{a_3,a_2}{K}\right)\left(\frac{a_3,a_3}{K}\right)\left(\frac{b_3,-a_1}{K}\right)$$
$$\left(\frac{b_3,a_2}{K}\right)\left(\frac{b_3,a_3}{K}\right) \in \mathrm{Br}_2(K).$$

5.

$$\left(\frac{-b_1a_2, a_1b_1a_3}{K}\right) = \left(\frac{-b_1,a_1}{K}\right)\left(\frac{-b_1,b_1}{K}\right)\left(\frac{-b_1,a_3}{K}\right)\left(\frac{a_2,a_1}{K}\right)\left(\frac{a_2,b_1}{K}\right)\left(\frac{a_2,a_3}{K}\right) \in \mathrm{Br}_2(K).$$

Appendix B

Maximal Orthogonal Grassmannians

Let (A, σ) be an algebra with orthogonal involution over a field K, $\mathrm{char}(K) \neq 2$. We define the *orthogonal Grassmannian* of (A, σ), denoted using $\mathrm{OG}(A, \sigma)$, by

$$\mathrm{OG}(A, \sigma)(L) = \{I \subset A_L \mid I = I^{\perp}\}.$$

Note that $\mathrm{OG}(A, \sigma)$ is a specification of $\mathrm{IV}_n(A, \sigma)$ with the implicit assumption that σ is orthogonal. We proceed by considering the case where A is split and $\deg(A) = 2n$. In this instance, we have that $(A, \sigma) \cong (M_n(K), \sigma_{b_q})$ where (K^n, q) is a $K-$vector space equipped with a quadratic form q (induced by the involution σ_{b_q}) defined by $q(v) := b_q(v, v)$ for $v \in K^n$.

Lemma B.0.1. *There is a one-to-one correspondence between isotropic ideals of* $(\mathrm{End}_K(V), \sigma_{b_q})$ *and totally isotropic* $K-$*vector subspaces of* (V, q) *given by,*

$$W \subset V \longrightarrow \mathrm{Hom}_K(V, W)$$

Proof. It suffices to show that $\mathrm{Hom}_K(V, W)^{\perp} = \mathrm{Hom}_K(V, W^{\perp})$. To see that this is indeed sufficient we observe that if $\mathrm{Hom}_K(V, W)$ is an isotropic ideal in $\mathrm{End}_K(V)$ then $\mathrm{Hom}_K(V, W) \subset \mathrm{Hom}_K(V, W)^{\perp} = \mathrm{Hom}_K(V, W^{\perp})$ and $W \subset W^{\perp}$, which is precisely the definition of a totally isotropic subspace W in V. The reverse direction follows easily assuming $W \subset W^{\perp} \subset V$ and the above equality. We proceed to show $\mathrm{Hom}_K(V, W)^{\perp} = \mathrm{Hom}_K(V, W^{\perp})$. Assume $f \in \mathrm{Hom}_K(V, W)$ and $g \in \mathrm{Hom}_K(V, W)^{\perp}$. Then we see that

$$b_q(f(u), g(v)) = b_q(\sigma(g)f(u), v) = 0 \text{ for u, v} \in V.$$

Since $\mathrm{im}(f) = W$ for some $f \in \mathrm{Hom}_K(V, W)$ we have that $\mathrm{im}(g) \in W^\perp$ and $g \in \mathrm{Hom}(V, W^\perp)$. If we repeat this line of reasoning with $g \in \mathrm{Hom}_K(V, W^\perp)$ we can see that $\sigma(g)(f(u)) = 0$ for all $u \in V$ which implies $\sigma(g)f = 0$ i.e. $g \in \mathrm{Hom}_K(V, W)^\perp$. $\square$

In general, it is difficult to determine the structure of elements in this correspondence explicitly. We consider the simplest non-trivial example. Let us take isotropic ideals in the split quaternion $K-$algebra $\left(\frac{x, y}{K}\right)$ equipped with an orthogonal involution σ_{b_q}. Since $\dim_K(V) = 2$, we see that (V, q) is isotropic if and only if (V, q) is isometric to the hyperbolic plane i.e. $q \cong \langle 1, -1 \rangle$. Alternatively, given that σ_{b_q} is an orthogonal involution on a degree 2 algebra, we know that it can be characterized by the generator $i_x \in \mathrm{Skew}\left(\left(\frac{x, y}{K}\right), \sigma_{b_q}\right)$ by the considerations in Section 2.3. Assume, in this case, that $i_x^2 = 1$. Then letting $I = (1 + i_x)$ denote a right ideal in $\left(\frac{x, y}{K}\right)$ we obtain $\sigma(I)I = 0$ and hence I is isotropic. An important note to make is that our choice of i_x such that $i_x^2 = 1$ is equivalent to choosing a particular orthogonal involution on $\left(\frac{x, y}{K}\right)$.

Proposition B.0.2. *There is a one-to-one correspondence between isotropic ideals of (A, σ) and totally isotropic subspaces $W \subset V$, where V is a D-vector space such that $A \cong_K \mathrm{End}_D(V)$.*

Proof. Assume, without loss of generality, that $(A, \sigma) \cong (End_D(V), \sigma_h)$ where $h : V \longrightarrow D$ is a Hermitian form. Arguing as above, we have a correspondence between totally isotropic Hermitian subspaces and isotropic right ideals:

$$W \subset V \longleftrightarrow Hom_D(V, W)$$

$\square$

We will use this correspondence to construct a complete isotropic flag for (A, σ). This will be necessary in order to introduce Schubert varieties of twisted flag varieties, which are a central focus of this section. So what is a flag? Loosely, it can be thought of as a filtration of the space. To see what this means in our case, let us first consider $(A, \sigma)_{K^{alg}} = (\mathrm{End}_{K^{alg}}(V), \sigma_b)$ and observe that $\mathrm{W}_q(K^{alg}) = \mathbb{Z}/2$ (see discussion preceeding Example 1.2.3) implies the quadratic form p associated to the symmetric bilinear form b can be decomposed as follows,

$$p \cong \begin{cases} m\mathbb{H}, & \text{if dim(p) = 2m} \\ m\mathbb{H}+ < 1 >, & \text{if dim(p) = 2m + 1} \end{cases}$$

for some $m \in \mathbb{N}$. Note that for symmetric bilinear forms (adjoint to orthogonal involutions), the dimension of a totally isotropic subspace $W \subset V$ is not necessarily $\frac{1}{2}\dim(V)$. This lies in contrast to skew-symmetric bilinear forms (adjoint to symplectic involutions) which are always even-dimensional. It turns out that this subtle difference requires special attention.

Consider $\dim(p) = 2m$ such that the Witt-index denoting the number of hyperbolic planes is m i.e. $w(V,p) = m$ and denote the maximal totally isotropic subspace $W \subset V$ (of dimension m) by V_m. Taking any maximal chain of strictly descending subspaces i.e.

$$\{0\} = V_0 \subset \cdots \subset V_m$$

such that $\dim_K(V_n) = n$ for $n = 0, \ldots, m$ we get precisely the notion of a complete (totally) isotropic flag of $(\mathrm{End}_K(V), \sigma_p)$. Using the correspondence between isotropic subspaces and isotropic ideals established earlier, we have a complete chain of isotropic ideals:

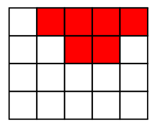
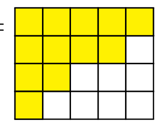
$$(0) = I_0 \subset \cdots \subset I_n = \mathrm{Hom}_K(V, V_n).$$

We will make use of this flag to define the Schubert varieties of $\mathrm{IV}_n(\mathrm{End}_K(V), \sigma_p)$. Consider a *strict* partition defined by the condition $a = [a_1, \ldots, a_m]$ with

$$n \geq a_1 > a_2 > \cdots > a_m > 0.$$

Such a partition can be represented by an *upper shifted* Young diagram with a_j boxes in the j-th row, starting at the j-th column of an $n \times n$ box. Alternatively, this diagram can be described by a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ with $\lambda_j = a_j + j$ for $j = 1, \ldots, m$ and $\lambda_{m+1}, \ldots, \lambda_n$ determined such that the Young diagram for λ is a doubled partition. We use the term *doubled partition* to refer to the Young diagram inside an $n \times (n + 1)$ box which is symmetric about the north-west to south-east diagonal of the box shifted to the right by 1 column (see [Gil18] for more detail).

Example B.0.3. Suppose $n = 4$. The strict partition $a = [4, 2]$, corresponds to the upper shifted diagram on the left. We associate to a the doubled partition $\lambda = [5, 4, 2, 1]$, which corresponds to the diagram on the right. Note that the upper shifted diagram is obtained from the double partition by removing all coloured blocks on and below the NW to SE diagonal of the box.

$a =$  $\lambda =$ 

These partitions, or their associated Young diagrams (of either upper shifted or self-dual type), are used to define the Schubert subvarieties of the maximal orthogonal Grassmannian $\mathrm{OG}(n, 2n + 1)$.

Given a $K^{alg}-$algebra with orthogonal involution $(A, \sigma) = (\mathrm{End}_{K^{alg}}(V), \sigma_h)$ such that $\deg(A) = 2n + 1$, we fix a full chain of isotropic right ideals

$$I_1 \subset I_2 \subset \cdots \subset I_n \subset A \text{ such that } \mathrm{rdim}(I_j) = j$$

Let $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ be a doubled partition with

$$n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0,$$

and let $a(\lambda) = [a_1, \ldots, a_m]$ be the corresponding strict partition.

The *Schubert variety* $X_\lambda \subseteq \mathrm{OG}(\mathrm{End}_{K^{alg}}(V), \sigma_h)$ is defined by intersection conditions with respect to the above isotropic chain:

$$X_\lambda = \{J \in \mathrm{OG}(\mathrm{End}_{K^{alg}}(V), \sigma_h) \mid \mathrm{rdim}(J \cap I_{n+1-a_j}) \geq j \text{ for } j = 1, \ldots, m\}$$

Using the correspondence between self-dual Young diagrams and upper shifted Young diagrams, we can also define X_λ in terms of λ itself:

$$X_\lambda = \{J \in \mathrm{OG}(\mathrm{End}_{K^{alg}}(V), \sigma_h) \mid \mathrm{rdim}(J \cap I_{n+1+j-\lambda_j}) \geq j \text{ for } 1 \leq j \leq n \text{ and } \lambda_j \geq j\}$$

It turns out that the cohomology of Schubert varieties is closely connected to the Chow ring of the ambient space. Much work has been done in understanding this relationship in terms of the underlying quadratic form structure of the involution. For instance, in the case of maximal orthogonal Grassmnaninas (which we have concerned ourselves with thus far) [EKM08, Chapter

XVI] gives a comprehensive treatment of the relationship between Schubert varieties and Chow groups. Although the computation of the torsion Chow group in the general case has remained elusive, several significant steps toward this direction have been made. In particular, [Kar16, Corollary 1.6 and Proposition 4.2] give a characterization of the torsion Chow group using [Kar95, Proposition 2], which is a result describing the torsion in the topological filtration in terms of the restriction map.

Appendix C

Details of theorem 5.4.1

Let us begin by reviewing our assumptions and notation. Let (A, σ) denote a central simple K-algebra A of degree 4 with a symplectic involution σ.

Lemma C.0.1. *[Lou78, Theorem B] A degree 4 central simple K-algebra A with symplectic involution σ decomposes as*

$$(A, \sigma) \cong (Q_1 \otimes_K Q_2, \sigma_1 \otimes_K \sigma_2)$$

where σ_1 is an orthogonal involution and σ_2 is a symplectic involution.

Proof. Assume $\operatorname{ind}(A) = 4$ since it will be shown shortly that the claim follows easily otherwise. We will begin by identifying a σ stable Q_1 inside A which by the Double Centralizer Theorem will give us a decomposition $A \cong Q_1 \otimes C_A(Q_1)$ where $\deg(C_A(Q_1)) = 2$ implies that $C_A(Q_1)$ is a σ-stable quaternion K-algebra. The involution type of σ_1 and σ_2 follows by process of elimination, all other type pairs produce an orthogonal involution.

To construct Q_1 it suffices to define anti-commutative, order 2 elements $i, j \in A$ such that $\sigma(i) = i$ and $\sigma(j) = j$ (and $\sigma(ij) = -ij$ implicitly). In fact, once we can find $i \in A$ such that $\sigma(i) = i$ we can construct $j \in A$ by applying the Skolem-Noether to the inner automorphism $K(i) \longrightarrow K(i)$ defined by $i \mapsto -i$. The problem thus reduces to finding an element $i \in A$ with the desired properties. As it turns out, every $s \in \mathrm{Sym}(A, \sigma)$ has order 2.

Let L/K be a Galois splitting field of A. We proceed to show that for every $s \in \mathrm{Sym}(A \otimes_K L, \sigma_L) \backslash K^\times \cong \mathrm{Sym}(M_4(L), \sigma_L) \backslash I_4 \cdot K^\times$,

$$\det(s - X \cdot I_4) = (X^2 - l)^2$$

for some $l \in L$. By Skolem-Noether, every automorphism is inner which means $\sigma_L \circ (-)^t = \mathrm{Inn}(u)$ or, in other words, $\sigma_L = \mathrm{Inn}(u) \circ (-)^t$. By Proposition 2.2.4 we have that $u \in \mathrm{Skew}(M_4(K), (-)^t)$ and $\mathrm{Sym}(M_4(K), \sigma_L) \cong u \cdot \mathrm{Skew}(M_4(K), (-)^t)$. As a consequence, $s = ua$ where $a \in \mathrm{Skew}(M_4(K), (-)^t)$. Using this decomposition, we have

$$\det(s - X \cdot I_4) = \det(u) \det(a - u^{-1}X),$$

where both u and $a - Xu^{-1}$ are alternating matrices. From a classical result concerning Pfaffians, the determinant of an alternating matrix can be represented as a square. Indeed, by choosing a basis such that an alternating matrix can be written as a direct sum of alternating block matrices, the claim follows directly. This line of reasoning gives us $\det(s - X \cdot I_4) = f(X)^2$ where $f(X) \in L[X]$. In particular, $f(X) = X^2 - l$ for $l \in L$ which follows by noting that since $a - u^{-1}X$ is alternating, both the degree 3 and degree 1 terms must vanish.

Since L/K is Galois and $\det(s - X \cdot I_4)$ is stable under Galois action, we see that $l \in K[X]$ (indeed consider the degree 2 term of $det(s - X \cdot I_4) = (X^2 - l)^2$. We thus conclude that all $s \in \mathrm{Sym}(A, \sigma) \backslash K \cdot 1$ are of order 2. This concludes our proof. $\square$

A natural generalization of the preceeding argument can be phrased as follows. For every $s \in \mathrm{Sym}(A, \sigma) \backslash K$, $\mathrm{charpoly}_s(X) = (X^2 - \mathrm{Trp}_\sigma(s)X + \mathrm{Nrp}_\sigma(s))^2$ where $\mathrm{Trp}_\sigma(s) = 0$ implies $Nrp_\sigma(s) = -s^2 \in K^\times$. We will focus our argument and only refer to $(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2)$ instead of (A, σ) from now on. Notice that $\dim_K(\mathrm{Sym}(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2)) = 6$ where $(\mathrm{Sym}(Q_1 \otimes Q_2), Nrp_{\sigma_1 \otimes \sigma_2})$ is a 6 dimensional quadratic space. We fix some notation,

$$\mathrm{Sym}(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2)^\circ = \{s \in \mathrm{Sym}(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2) \mid Trp_{\sigma_1 \otimes \sigma_2}(s) = 0\}.$$

Now, since $Nrp_{\sigma_1 \otimes \sigma_2}$ is an Albert form using [KMRT98, Example 16.15] or directly calculating $Nrp_{\sigma_1 \otimes \sigma_2}$ with respect to the basis elements in $\mathrm{Sym}(Q_1 \otimes Q_2, \sigma_1 \otimes \sigma_2)$ (which we will do shortly). Notice that from the proof of Lemma C.0.1, $Nrp_{\sigma_1 \otimes \sigma_2}(s) = -s^2$. By decomposing the symmetric elements of $Q_1 \otimes Q_2$ as the $\mathrm{Sym}(Q_1, \sigma_1) \otimes \mathrm{Sym}(Q_2, \sigma_2)$ and $\mathrm{Skew}(Q_1, \sigma_1) \otimes \mathrm{Skew}(Q_2, \sigma_2)$ we can see that

$$Nrp_{\sigma_1 \otimes \sigma_2} = \mathrm{Nrd}_{Q_1}(v)(\mathrm{Nrd}'_{Q_1} \perp -\mathrm{Nrd}'_{Q_2}).$$

In the case where $\text{ind}(A) = 2$ we can reduce the above argument significantly and deduce even more about Nrp_σ. Indeed, by Wedderburns theorem $\text{ind}(A) = 2$ and $\deg(A) = 4$ implies that $A \cong M_2(K) \otimes Q$. Considering the 1-Hermitian form (see [KMRT98, Corollary 4.2 (1)]) associated to $(\text{End}_Q(V), \sigma_h) \cong (M_2(K) \otimes Q), ad_q \otimes \sigma)$ where we have that $h : V \longrightarrow Q$ is a Hermitian form and V is a rank 2 Q-module. Consider am orthogonal Q-basis $\{e_1, e_2\}$, by skew symmetry of h_σ we have $h(e_l, e_l) = h_l \in K$. In particular, by considering the K-vector space V_0 given by the K span of $\{e_1, e_2\}$ we can identify $h|_{V_0}$ with a quadratic map q, meaning that $A \cong End_K(V_0) \otimes Q$ where each components is σ invariant by construction.

We will define the action of σ on an element in $M_2(Q)$ using what we know of h. Consider $\sigma(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. We will compute the b_{21} term, the other computations follow similarly. The main idea is to consider the equality

$$h_\sigma(\sigma(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix})e_1, e_2) = h_\sigma(e_1, \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} e_2)$$

coming from the definition of the involution associated to a Hermitian form. This evaluates to

$$h_\sigma(b_{11}e_1 + b_{21}e_2, e_2) = h_\sigma(e_1, a_{12}e_1 + a_{22}e_2)$$

which can be further reduced to

$$\sigma(b_{21})h_2 = h_1 a_{12}.$$

In other words, $b_{21} = h_2^{-1} h_1 \sigma(a_{12})$. The other coefficients follow from similar calculations, giving us

$$\sigma(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}) = \begin{pmatrix} \sigma(a_{11}) & h_2^{-1} h_1 \sigma(a_{21}) \\ h_2^{-1} h_1 \sigma(a_{12}) & \sigma(a_{22}) \end{pmatrix}.$$

Following [KMRT98, Example 16.15], we can describe Nrp_σ explicitly by computing $-s^2$ for the elements which are symmetric with respect to σ defined above. In particular $Nrp_\sigma = \langle 1, -1 \rangle \perp h_1 h_2^{-1} Nrd_Q$.

To describe Nrp_σ in terms of the isotropy/hyperbolicity of the associated Hermitian form it suffices to consider the trace form $q_h(v) := h(v, v)$. This is a quadratic form which has the property that q_h is isotropic/hyperbolic iff h is isotropic/hyperbolic (see [Sch85, Theorem 10.1.7]). We can explicitly compute the trace form q_h using our construction of h over the rank 2 Q-module V, doing so would give us that $q_h \cong h_1 Nrd_Q \perp h_2 Nrd_Q$. This concludes our remarks on Theorem 5.4.1.

Lemma C.0.2. *Consider a central simple K-algebra A of degree $2m$ equipped with a symplectic involution σ. If m is odd then*

$$(A, \sigma) \cong (M_m(K), ad_p) \otimes (Q, \gamma)$$

for some quadratic form p over K.

Proof. By Wedderburn's theorem, $A \cong M_k(D)$ where $k \in \mathbb{N}$ and D is a division K-algebra. Since the exponent of D is 2 we must have $\text{ind}(D) = 2^l$ for osme $l \in \mathbb{N}$. By assumptions on the degree of A we have that $l = 1$ and $k = m$. Now consider the Hermitian module structure on V a rank m Q-module. Let $\{e_1, \ldots, e_m\}$ be a Q-basis, by considering the vector space V_0 corresponding to the K-span $\{e_1, \ldots, e_m\}$ we see that h restricted to V_0 is a quadratic space. In particular, since $End_K(V_0) \cong M_k(K)$ and $\sigma(End_K(V_0)) = End_K(V_0)$ we have that

$$(A, \sigma) \cong (M_m(K), ad_p) \otimes (Q, \gamma)$$

for some quadratic form p over K.

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