PLANETARY ORBITS IN CONSTANT CURVATURE PLANES

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ABSTRACT

A law of gravitation is defined and justified for constant curvature planes and it is demonstrated that Kepler's three laws of planetary motion have natural analogs in this new context.

1 Introduction

Gauss [4] took it for granted that every 2-dimensional surface S in $\Re^3$ has, at least locally, a *geodesic polar parametrization* $\vec{\mathbf{X}}(\rho, \theta)$ wherein the parameter ρ denotes the distance, on S, of the point $\vec{\mathbf{X}}(\rho, \theta)$ from the origin O $= \vec{\mathbf{X}}(\mathbf{0}, \mathbf{0})$ and θ denotes the signed angle between a reference ρ-parameter curve through O and the ρ-parameter curve from O to $\vec{\mathbf{X}}(\rho, \theta)$. Using somewhat more modern terminolgy, we simply assume that S is any 2-dimensional manifold that consists of the plane together with the metric

$$d\rho^2 + G(\rho, \theta)d\theta^2 \tag{1}$$

The ρ-parameter curves of this metric are identical with the geodesics that emanate from O. We follow the convention that

$$\vec{\mathbf{X}}(\rho, \pi + \theta) = \vec{\mathbf{X}}(-\rho, \theta)$$

and note that all these metrics endow the same portion of the straight line $\theta = c$ with the same lengths. These Euclidean straight lines are also geodesics of S. The archtypical example is, of course, the (polar coordinates) metric

$$d\rho^2 + \rho^2 d\theta^2 \tag{2}$$

which defines a manifold that is isometric to the Euclidean plane. The hyperbolic and elliptic planes have the respective metrics

$$d\rho^2 + R^2 \sinh^2(\frac{\rho}{R})d\theta^2 \qquad \text{and} \qquad d\rho^2 + R^2 \sin^2(\frac{\rho}{R})d\theta^2 \tag{3}$$

where R is an arbitrary positive number. These are collectively called *non-Euclidean* planes. The manifolds of (2) and (3) are also collectively known as the *constant curvature* planes since their Gaussian curvatures are constant. When $R = 1$ these are the *unit hyperbolic plane* and the *unit elliptic plane*. The Euclidean, hyperbolic and elliptic planes are all symmertric (homogeneous) so that O can be an arbitrary point.

As is well known, in the Euclidean plane

$$\rho = \frac{1}{k(1 + e\cos(\theta - \alpha))}$$

is the equation of a circle, ellipse, parabola, or hyperbola according as $e = 0, 0 < e < 1, e = 1$, or $e > 1$. In view of the observations in [5, 8] it is therefore reasonable to define the corresponding curves

$$\rho = R\tanh^{-1}\left(\frac{1}{k(1 + e\cos(\theta - \alpha))}\right) \tag{4}$$

as the *hyperbolic circle, ellipse, parabola, or hyperbola* (provided that $k|e - 1| > 1$), and the corresponding curves

$$\rho = R\tan^{-1}\left(\frac{1}{k(1 + e\cos(\theta - \alpha))}\right) \tag{5}$$

as the *elliptic circle, ellipse, parabola, or hyperbola*. These definitions of circles, agree, of course, with the standard one.

In Section 2 it will be shown that these curves describe planetary motion and in Section 3 it will be demonstrated that these curves do indeed possess the same focal properties as their Euclidean namesakes.

GEODESICS

These are curves whose second derivative is 0. Thus, both coefficients of (xxx) vanish. The same substitutions that were used above lead to the equation

$$\frac{d^2u}{d\theta^2} + \frac{G_\rho}{2G} = 0$$

We add the assumption that

$$\frac{G_\rho}{2G} = -\int \frac{1}{G}$$

It follows that

$$\frac{d^2u}{d\theta^2} + u = 0$$

which has the solution

$$u = C\cos(\theta - \alpha)$$

or

$$\rho = \coth^{-1}(C\cos(\theta + \alpha)).$$

Let

$$F\rho = -\int \frac{d\rho}{G\rho}$$

be the miracle function. Then the geodesic with x and y intercepts equal to a and b respectively has

$$\alpha = \tan^{-1}\left(-\frac{F(b)}{F(a)}\right) \qquad \text{and} \qquad C = \sqrt{F(a)^2 + F(b)^2}$$

$$\rho = \coth^{-1}(C\cos(\theta + \alpha)).$$

Arclength of geodesic

$$\int \sqrt{d\rho^2 + \sinh^2 \rho d\theta^2} = \int \sqrt{\frac{C^2 \sin^2(\theta+\alpha)}{(C^2\cos^2(\theta+\alpha)-1)^2} + \frac{1}{C^2\cos^2(\theta+\alpha)-1}} d\theta$$

$$= \sqrt{C^2-1}\int \frac{d\theta}{C^2\cos^2(\theta+\alpha)-1} = \tanh^{-1}\left(\frac{\tan(\theta+\alpha)}{\sqrt{C^2-1}}\right)$$

The hyperbolic distance between the points (ρ_1, θ_1) and (ρ_2, θ_2) is

$$\cosh^{-1}[\cosh(\rho_1)\cosh(\rho_2) - \cos(\theta_2 - \theta_1)\sinh(\rho_1)\sinh(\rho_2)]$$

2 Planetary Orbits

It is not unreasonable to speculate on the physics of the non-Euclidean planes. The second half of the 19th century saw some work done on the Archimedian Law of the Lever [1]. More recently, Gal'perin [2, 3] investigated the concept of the center of mass of finite point-mass systems. Lamphere [8] studied uniform circular motion. The non-Euclidean analogs of Kepler's three laws of planetary motion are derived in this section.

Our strategy is based on the derivation of Kepler's classic laws in [7]. We begin by obtaining the connection form ω_{12} for the metric of (1). This is accomplished by analyzing the moving frame.

$$\vec{\mathbf{E}}_{\mathbf{1}} = \vec{\mathbf{X}}_\rho \qquad \vec{\mathbf{E}}_{\mathbf{2}} = \frac{\mathbf{1}}{\sqrt{\mathbf{G}}}\vec{\mathbf{X}}_\theta \tag{6}$$

After the connection form has been obtained, the covariant derivative is used to determine the acceleration of any arbitrary path in general as well as the planetary orbits in particular. The assumption of the centrality of the force of attraction is tantamount to the vanishing of the coefficient of $\vec{\mathbf{E}}_{\mathbf{2}}$ in the acceleration vector and this yields a second order ordinary differential equation which is easily integrated to Eq'n (8) if we stipulate that G is independent of θ. When the metric in question is further specialized to the non-Euclidean geometries of (3) an analog of Kepler's second law is obtained. A generalization of Newton's inverse square gravitational law is then defined and motivated. This law of attraction, in combination with the coefficient of $\vec{\mathbf{E}}_\mathbf{1}$, yields the second order ordinary differential equation (11) which, surprisingly, reduces to Newton's Euclidean equation in the non-Euclidean case as well. Analogs of Kepler's first and third laws are then easily obtained.

When the vectors of (6) are used as a moving frame in the surface defined by (1), they yield the connection [10, p. 277]

$$\omega_{12} = \frac{-(\sqrt{E})_\theta}{\sqrt{G}} d\rho + \frac{(\sqrt{G})_\rho}{\sqrt{E}} d\theta$$

$$= 0 d\rho + (\sqrt{G})_\rho d\theta = (\sqrt{G})_\rho d\theta.$$

Let $\vec{\alpha}(t) = \vec{\mathbf{X}}(\rho(\mathbf{t}), \theta(\mathbf{t}))$ be an arbitrary curve in this plane and let $\vec{\mathbf{v}}(\mathbf{t})$ and $\vec{\mathbf{a}}(\mathbf{t})$ be its velocity and acceleration vectors, respectively. Then, if $'$ denotes differentiation with respect to t, the velocity vector is

$$\vec{\mathbf{v}}(\mathbf{t}) = \vec{\mathbf{X}}'(\mathbf{t}) = \vec{\mathbf{X}}_\rho \rho' + \vec{\mathbf{X}}_\theta \theta'$$

$$= \rho' \vec{\mathbf{E}}_\mathbf{1} + \theta' \sqrt{\mathbf{G}} \vec{\mathbf{E}}_\mathbf{2}$$

and the acceleration vector is its covariant derivative [10]

$$\vec{\mathbf{a}}(\mathbf{t}) = \nabla_{\vec{\mathbf{v}}} \vec{\mathbf{v}}$$

$$= [(\rho')' + \theta' \sqrt{G}(-(\sqrt{G})_\rho) d\theta\,(\vec{\mathbf{v}}(\mathbf{t}))] \vec{\mathbf{E}}_\mathbf{1} +$$

$$[(\theta' \sqrt{G})' + \rho' (\sqrt{G})_\rho d\theta\,(\vec{\mathbf{v}}(\mathbf{t}))] \vec{\mathbf{E}}_\mathbf{2}$$

$$= [\rho'' - \theta'^2 \sqrt{G}(\sqrt{G})_\rho] \vec{\mathbf{E}}_\mathbf{1} +$$

$$[\theta'' \sqrt{G} + \theta' (\sqrt{G})' + \rho' \theta' (\sqrt{G})_\rho] \vec{\mathbf{E}}_\mathbf{2}$$

$$= [\rho'' - \theta'^2 \sqrt{G}(\sqrt{G})_\rho] \vec{\mathbf{E}}_\mathbf{1} +$$

$$[\theta'' \sqrt{G} + \theta' [(\sqrt{G})_\rho \rho' + (\sqrt{G})_\theta \theta'] + \rho' \theta' (\sqrt{G})_\rho] \vec{\mathbf{E}}_\mathbf{2}$$

$$= [\rho'' - \theta'^2 \sqrt{G}(\sqrt{G})_\rho] \vec{\mathbf{E}}_\mathbf{1} +$$

$$[\theta''\sqrt{G} + 2\rho'\theta'(\sqrt{G})_\rho + (\sqrt{G})_\theta\theta'^2]\vec{\mathbf{E}}_\mathbf{2} \qquad (7)$$

Since the attraction the sun exerts on the planet is central, that is, directed towards O, it follows that the coefficient of $\vec{\mathbf{E}}_\mathbf{2}$ in Eq'n (7) vanishes. If we now add the assumption

G is independent of θ

then

$$\theta''\sqrt{G} + 2\rho'\theta'(\sqrt{G})_\rho = 0$$

or

$$\frac{1}{\sqrt{G}}\left[\theta''G + \rho'\theta'G_\rho\right] = 0$$

or

$$\frac{1}{\sqrt{G}}[\theta'G]' = 0$$

from which it follows that for some constant h

$$\theta'G = h. \qquad (8)$$

Set

$$H(\rho) = \int \sqrt{G}d\rho$$

and let the *double* of $\vec{\alpha}(t)$ be the curve

$$2\vec{\alpha}(t) = \vec{\mathbf{X}}(t) = (2\rho(t), \theta(t))$$

Then the area of the wedge

$$\theta \le \tau \le \theta + \Delta\theta, \qquad 0 \le r \le 2\rho = 2\rho(\tau)$$

that is swept out by the radius $2\rho(t)$ of the double of $\vec{\alpha}(t)$ is

$$A = \int_\theta^{\theta+\Delta\theta}\int_0^{2\rho}\sqrt{G}drd\tau = \int_\theta^{\theta+\Delta\theta}(H(2\rho) - H(0))d\tau$$

Hence

$$\frac{dA}{d\theta} = H(2\rho) - H(0) \qquad (9)$$

For the general hyperbolic plane

$$H(\rho) = \int \sqrt{G}d\rho = \int R\sinh(\rho/R)d\rho = R^2\cosh(\rho/R) + C$$

and for the general elliptic plane

$$H(\rho) = \int \sqrt{G}d\rho = \int R\sin(\rho/R)d\rho = -R^2\cos(\rho/R) + C$$

The following proposition is the constant curvature analog of Kepler's second law.

Figure 1:

Theorem 2.1 *In a constant curvature plane, let $\vec{\alpha}(t)$ denote a curve whose acceleration vector is constantly directed at the origin O. Then, if t is interpreted as time, the radius of the double of $\vec{\alpha}(t)$ sweeps equal areas in equal times.*

Proof: Let A denote the area swept out by the doubled radius. In the hyperbolic case, by (9),

$$\frac{dA}{d\theta} = R^2 \cosh(2\rho/R) - R^2 = 2R^2 \sinh^2(\rho/R)$$

so that

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = 2R^2 \sinh^2(\rho/R)\theta' = 2G\theta' = 2h$$

which means that the area swept by the doubled radius is proportional to the elapsed time.

In the elliptic case, by (9),

$$\frac{dA}{d\theta} = R^2 - R^2 \cos(2\rho/R) = 2R^2 \sin^2(\rho/R)$$

or

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = 2R^2 \sin^2(\rho/R)\theta' = 2G\theta' = 2h$$

which means that here too the area swept by the doubled radius is proportional to the elapsed time.

In the Euclidean plane the area swept out by the double radius is four times that swept out by the radius. Therefore the statement of the theorem is equivalent to the classical Kepler's second law.

□

We next turn to Kepler's first law. This calls for a law of gravitation for which we propose an attraction of

$$\frac{k}{G} \tag{10}$$

where $k > 0$. Note that in the Euclidean case (2) this reduces to Newton's law of gravitation. This observation, together with Occam's razor, could be sufficient grounds for the proposed attraction of (10), but we offer an additional heuristic rationale. One of the ways of justifying Newton's inverse square assumption is to observe that the total flux of the gravitational field across any sphere centered at the sun is independent of that sphere's radius. Consequently the gravitational flux arriving at a planet at distance ρ from the sun should be inversely proportional to the surface area of the sphere of radius ρ. In the unit hyperbolic case this sphere is known [11, Ex. 3.4.5] to have volume

$$\pi(\sinh 2\rho - 2\rho)$$

which, when differentiated, yields a surface area of

$$4\pi \sinh^2 \rho = 4\pi G$$

Thus, assumption (10) is reasonable in this well known plane as well.

Since the assumption of the centrality of the attraction resulted in the vanishing of the coefficient of $\vec{\mathbf{E}}_{\mathbf{2}}$ of Eq'n (7), it follows that the magnitude of the force is proportional to the coefficient of $\vec{\mathbf{E}}_{\mathbf{1}}$ and so the gravitational equation is

$$\rho'' - \theta'^2\sqrt{G}(\sqrt{G})_\rho = -\frac{k}{G} \tag{11}$$

The substitution

$$u = -\int \frac{1}{G}d\rho \tag{12}$$

yields

$$\frac{du}{d\theta} = -\frac{1}{G}\frac{d\rho}{d\theta}$$

or

$$\frac{d\rho}{d\theta} = -G\frac{du}{d\theta}$$

Hence, by two applications of (8)

$$\frac{d\rho}{dt} = \frac{d\rho}{d\theta}\frac{d\theta}{dt} = -G\frac{du}{d\theta}\frac{h}{G} = -h\frac{du}{d\theta}$$

$$\frac{d^2\rho}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h\frac{h}{G}\frac{d^2u}{d\theta^2} = -\frac{h^2}{G}\frac{d^2u}{d\theta^2}$$

The substitution of this value into the gravitational equation (11) yields

$$-\frac{h^2}{G}\frac{d^2u}{d\theta^2} - \frac{h^2}{G^{3/2}}(\sqrt{G})_\rho = -\frac{k}{G}$$

or

$$\frac{d^2u}{d\theta^2} + \frac{(\sqrt{G})_\rho}{\sqrt{G}} = \frac{k}{h^2}$$

or

$$\frac{d^2u}{d\theta^2} + \frac{G_\rho}{2G} = \frac{k}{h^2} \tag{13}$$

In all three cases, the Euclidean, the hyperbolic and the elliptic, the fact that

$$\frac{G_\rho}{2G} = -\int \frac{1}{G}d\rho = u$$

converts Eq'n (13) into the second order linear equation

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{h^2}$$

whose general solution can be expressed as

$$u = \frac{k}{h^2}(1 + e\cos(\theta - \alpha)) \tag{14}$$

for some positive real number e and arbitrary real number α. In the Euclidean case this yields

$$\rho = \frac{h^2/k}{1 + e\cos(\theta - \alpha)}$$

which, for $0 < e < 1$ describes a Euclidean ellipse.

In the general hyperbolic case, Eq'n (14) yields the hyperbolic ellipse

$$\rho = R\tanh^{-1}\left(\frac{h^2}{kR(1 + e\cos(\theta - \alpha))}\right)$$

and in the general elliptic case Eq'n (14) yields the elliptic ellipse

$$\rho = R\tan^{-1}\left(\frac{h^2}{kR(1 + e\cos(\theta - \alpha))}\right)$$

These considerations prove the following theorem which is the constant curvature analog of Kepler's first law.

Theorem 2.2 *The planetary orbits in the non-Euclidean geometries are ellipses.*

□

This section concludes with an analog of Kepler's third law which states that in the Euclidean case the squares of the return times of the planets is proportional to the cubes of their semi major axes. Let E denote either of the non-Euclidean ellipses

$$\tanh(\rho/R) = \frac{1}{k(1 + e\cos(\theta - \alpha))} \tag{15}$$

or

$$\tan(\rho/R) = \frac{1}{k(1 + e\cos(\theta - \alpha))} \tag{16}$$

The *associated Euclidean ellipse* E' is defined to be

$$E' : \quad \rho = \frac{1}{k(1 + e\cos(\theta - \alpha))}, \tag{17}$$

drawn in the same polar coordinate system as E.

Figure 2:

Theorem 2.3 *The squares of the return times of the non-Euclidean planetary motion about a fixed mass is proportional to the cubes on the semi major axes of the associated Euclidean ellipses.*

Proof: Let P be an arbitrary point on the orbit E of (15, 16) and let P' be the intersection of the radius OP with the auxiliary ellipse E' (Fig. 1).

As P traces out its orbit E, P' traces out the Euclidean ellipse E'. Let T be their common return time. Since P' traces out an ellipse it follows from [9 Book I Proposition XI, 7] that its acceleration vector is directed towards the origin O and has a magnitude that is inversely proportional to the OP'^2. It therefore follows from the Euclidean Kepler's third law that T^2 is proportional to the cube on the semi major axis of the Euclidean ellipse E' of (17).

□

3 Geometric Properties

In this section we discuss the geometric properties of the curves of (4, 5). It turns out that in this respect they are quite similar to their Euclidean analogs.

Theorem 3.1 *In a constant curvature plane a curve is an ellipse (hyperbola) if and only if has two foci such that the sum (difference) of the distances of the arbitrary point on the curve from the foci is constant.*

Proof: This is, of course, well known in the Euclidean case. Since rotations about the origin are isometries of all the constant curvature manifolds, it may be assumed that $\alpha = 0$. In the hyperbolic case of (4) set

$$A = (\rho(0), 0), \qquad B = (\rho(\pi), \pi), \qquad F = (\rho(0) - \rho(\pi), 0)$$

where

$$\rho(0) = \tanh^{-1}\left(\frac{1}{k + ke}\right), \qquad \rho(\pi) = \tanh^{-1}\left(\frac{1}{k - ke}\right)$$

Note that there are two possible dispositions for O, A, B and F according as $0 < e < 1$ or $e > 1$ (Fig. 2). Moreover, the branch H_1 consists of those points (ρ, θ) of the hyperbola $H_1 \cup H_2$ for which $\rho > 0$, whereas the branch H_2 consists of the points for which $\rho < 0$. We define a and c by means of the equations

$$2a = \tanh^{-1}\left(\frac{1}{k(1-e)}\right) + \tanh^{-1}\left(\frac{1}{k(1+e)}\right) \tag{18}$$

$$2c = \tanh^{-1}\left(\frac{1}{k(1-e)}\right) - \tanh^{-1}\left(\frac{1}{k(1+e)}\right) \tag{19}$$

Assume first that

$$R = 1$$

It is easily verified in the unit hyperbolic case that if we set

$$\Delta_1 = \frac{k - ke + 1}{k - ke - 1}\frac{k + ke + 1}{k + ke - 1}$$

$$\Delta_2 = \frac{k - ke + 1}{k - ke - 1}\frac{k + ke - 1}{k + ke + 1}$$

then

$$2a = \frac{1}{2}\ln \Delta_1 \tag{20}$$

$$2c = \frac{1}{2}\ln \Delta_2 \tag{21}$$

If we also set

$$\Delta_0 = (k - ke + 1)(k - ke - 1)(k + ke + 1)(k + ke - 1)$$

and

$$\Delta_\theta = \frac{k(1 + e\cos\theta) + 1}{k(1 + e\cos\theta) - 1}$$

then it follows from Eq'ns (20, 21) that

$$\cosh 2a = \frac{1}{2}\left(\sqrt{\Delta_1} + \frac{1}{\sqrt{\Delta_1}}\right) = \frac{k^2 + 1 - k^2e^2}{\sqrt{\Delta_0}}$$

$$\sinh 2a = \frac{1}{2}\left(\sqrt{\Delta_1} - \frac{1}{\sqrt{\Delta_1}}\right) = \frac{2k}{\sqrt{\Delta_0}}$$

$$\cosh\rho = \cosh\left(\tanh^{-1}\left[\frac{1}{k(1 + e\cos\theta)}\right]\right)$$

$$= \cosh\left(\frac{1}{2}\ln\Delta_\theta\right) = \frac{1}{2}\left(\sqrt{\Delta_\theta} + \frac{1}{\sqrt{\Delta_\theta}}\right)$$

$$\sinh\rho = \frac{1}{2}\left(\sqrt{\Delta_\theta} - \frac{1}{\sqrt{\Delta_\theta}}\right)$$

$$\cosh 2c = \frac{1}{2}\left(\sqrt{\Delta_2} + \frac{1}{\sqrt{\Delta_2}}\right) = \frac{k^2 - 1 - k^2e^2}{\sqrt{\Delta_0}}$$

$$\sinh 2c = \frac{1}{2}\left(\sqrt{\Delta_2} - \frac{1}{\sqrt{\Delta_2}}\right) = \frac{2ke}{\sqrt{\Delta_0}}$$

For the points P on the ellipse E of Figure 2, the hyperbolic law of cosines states that

$$\cosh\tilde{\rho} = \cosh\rho\cosh 2c - \cos(\pi-\theta)\sinh\rho\sinh 2c$$

We verify that

$$\tilde{\rho} + \rho = 2a$$

by simplifying

$$\cosh\tilde{\rho} - \cosh(2a-\rho)$$

$$= \cosh\rho\cosh 2c + \cos\theta\sinh\rho\sinh 2c - \cosh 2a\cosh\rho + \sinh 2a\sinh\rho$$

$$= \cosh\rho[\cosh 2c - \cosh 2a] + \sinh\rho[\cos\theta\sinh 2c + \sinh 2a]$$

$$= \frac{1}{2}\left(\sqrt{\Delta_\theta} + \frac{1}{\sqrt{\Delta_\theta}}\right)\left(\frac{-2}{\sqrt{\Delta_0}}\right) + \frac{1}{2}\left(\sqrt{\Delta_\theta} - \frac{1}{\sqrt{\Delta_\theta}}\right)\left(\cos\theta\frac{2ke}{\sqrt{\Delta_0}} + \frac{2k}{\sqrt{\Delta_0}}\right)$$

$$= \frac{1}{2\sqrt{\Delta_\theta}\sqrt{\Delta_0}}[-(\Delta_\theta+1) + (\Delta_\theta-1)k(1+e\cos\theta)]$$

$$= \frac{1}{2\sqrt{\Delta_\theta}\sqrt{\Delta_0}}\left[-\frac{2k(1+e\cos\theta)}{k(1+e\cos\theta)-1} + \frac{2k(1+e\cos\theta)}{k(1+e\cos\theta)-1}\right]$$

$$= 0$$

For the points P on the branch H_1 of the hyperbola of Figure 2, the hyperbolic law of cosines yields

$$\cosh\tilde{\rho} = \cosh\rho\cos 2c - \cos\theta\sinh\rho\sinh 2c$$

We verify that

$$\rho - \tilde{\rho} = 2a$$

by noting that

$$\cosh\tilde{\rho} - \cosh(\rho - 2a) = \cosh\tilde{\rho} - \cosh(2a-\rho)$$

which, by the above calculations for the ellipse equals 0. Finally, for the points P on the branch H_2 of the hyperbola of Figure 2, the hyperbolic law of cosines yields

$$\cosh\tilde{\rho} = \cosh\rho\cosh 2c - \cos(\theta\ -\pi)\sinh\rho\sinh 2c$$

$$= \cosh\rho\cosh 2c + \cos\theta\sinh\rho\sinh 2c$$

We verify that

$$\rho - \tilde{\rho} = 2a$$

by noting that

$$\cosh\tilde{\rho} - \cosh(\rho - 2a) = \cosh\tilde{\rho} - \cosh(2a - \rho)$$

which, by the above calculations for the ellipse, also equals 0.

Conversely, let $a > c > 0$ and let loop E (resp. curve $H_1 \cup H_2$) of Figure 2 be the locus of all the points the sum (resp. differenece) of whose distances from O and F equals $2a$, where $OF = 2c$. Set

$$e = \frac{\tanh(a+c) - \tanh(a-c)}{\tanh(a+c) + \tanh(a-c)} \tag{22}$$

and

$$k = \frac{\tanh(a+c) + \tanh(a-c)}{2\tanh(a+c)\tanh(a-c)} \tag{23}$$

Then $0 < e < 1$ (resp. $1 < e$) and e and k satisfy E'qns (18, 19). It follows that the corresponding graph of (4) and loop E (resp. curve $H_1 \cup H_2$) are identical sets.

For the unit elliptic case the reader is reminded that

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{x^2+1}} \qquad \sin(\tan^{-1} x) = \frac{x}{\sqrt{x^2+1}}$$

Set

$$x_1 = \frac{1}{k(1-e)}, x_2 = \frac{1}{k(1+e)}, x_\theta = \frac{1}{k(1+e\cos\theta)}$$

and

$$2a = \tan^{-1} x_1 + \tan^{-1} x_2$$

$$2c = \tan^{-1} x_1 - \tan^{-1} x_2$$

Then

$$\cos 2a = \cos\left(\tan^{-1} x_1 + \tan^{-1} x_2\right)$$

$$= \frac{1}{\sqrt{x_1^2+1}}\frac{1}{\sqrt{x_2^2+1}} - \frac{x_1}{\sqrt{x_1^2+1}}\frac{x_2}{\sqrt{x_2^2+1}} = \frac{1 - x_1x_2}{\sqrt{(x_1^2+1)(x_2^2+1)}}$$

and

$$\sin 2a = \frac{x_1 + x_2}{\sqrt{(x_1^2+1)(x_2^2+1)}}$$

$$\tan\rho = x_\theta, \cos\rho = \frac{1}{\sqrt{x_\theta^2+1}}, \sin\rho = \frac{x_\theta}{\sqrt{x_\theta^2+1}}$$

$$\cos 2c = \frac{1 + x_1x_2}{\sqrt{(x_1^2+1)(x_2^2+1)}}, \qquad \sin 2c = \frac{x_1 - x_2}{\sqrt{(x_1^2+1)(x_2^2+1)}}$$

Hence, by the elliptic (or spherical) law of cosines,

$$
\begin{aligned}
\cos\tilde{\rho} - \cos(2a-\rho) &= \cos\rho\cos 2c - \cos\theta\sin\rho\sin 2c \\
&\quad - \cos 2a\cos\rho - \sin 2a\sin\rho \\
&= \frac{(1+x_1x_2) - x_\theta(x_1-x_2)\cos\theta - (1-x_1x_2) - (x_1+x_2)x_\theta}{\sqrt{(x_\theta^2+1)(x_1^2+1)(x_2^2+1)}} \\
&= \frac{\frac{2}{k^2-k^2e^2} - \frac{\cos\theta}{k(1+e\cos\theta)}\frac{2ke}{k^2-k^2e^2} - \frac{2k}{(k^2-k^2e^2)k(1+e\cos\theta)}}{\sqrt{(x_\theta^2+1)(x_1^2+1)(x_2^2+1)}} \\
&= 0
\end{aligned}
$$

from which it follows that

$$\tilde{\rho} + \rho = 2a.$$

The unit elliptic hyperbola is disposed of by the same argument that was used for the unit hyperbolic hyperbola. The converse follows from an argument similar to that given in the unit hyperbolic case.

The proof of the theorem for general R is obtained by replacing the quantities

$$\rho_A, \rho_B, a, c, \rho, \tilde{\rho}$$

of the foregoing arguments with

$$\frac{\rho_A}{R}, \frac{\rho_B}{R}, \frac{a}{R}, \frac{c}{R}, \frac{\rho}{R}, \frac{\tilde{\rho}}{R}$$

respectively.

□

It is well known that Euclidean parabolas can be given a two-foci definition by fixing one of the foci of the ellipse and letting the other diverge to infinity. Similarly, the hyperbolic parabola (4) is the limiting configuration of hyperbolic ellipses. This can be justified by examining the effect on the ellipse E of Figure 2 of letting c diverge to infinity while holding $a - c$ constant. It follows from Eq'n (22) that e converges to 1 so that the limiting configuration is indeed a hyperbolic parabola. It is clear that such is also the case for the elliptic parabola. We note in passing that the curve of the hyperbolic plane defined as the locus of all points that are equidistant from a given point (*focus*) to given straight line (*directrix*) is **not** a hyperbolic parabola.

Theorem 3.2 *Ellipses, hyperbolas, and parabolas are conic sections.*

Figure 3:

Figure 4:

Proof: For the hyperbolic ellipse we use a slight modification of the well-known diagram and argument that appear in pp. 7-9 of [6]. In Figure 3 the curve E is the intersection of a plane and a cone. We then inscribe two spheres that are tangent to both the cone (along K and L) and the intersecting plane (at F and G). Let $SQBP$ be a generating line of the cone. It is clear that the length of PQ is independent of the position of P on K. Moreover,

$$PQ = BP + BQ = BF + BG$$

so that the fixed points F and G are indeed the foci of the ellipse E.

Figure 10 of [6] can be used for the hyperbola. As for the hyperbolic parabola, suppose the smaller sphere that is tangent to the cone along L is fixed while the plane containing the ellipse E pivots so that A moves closer to S while C recedes to infinity. Let a, c, e, k be as defined in the proof of Theorem 2.1. Then $GA = a - c$ remains bounded while $c = GF/2$ diverges to infinity. It follows from Eq'ns (22, 23) that the limiting cross section is indeed a parabola. In Figures 4 and 5 elliptic 3-space is visualized as a ball in 3-space in which antipodal points are identified. Figure 4 displays an ellipse and a parabola, whereas Figure 5 displays a hyperbola. Note that the parabola still has two foci. With this understanding the above argument in hyperbolic space still works in this space as well.

□

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Figure 5:

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