MASS IN HYPERBOLIC 3-SPACE
In progress

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Contents:

1. A hyperbolic Theorem of Pappus.

2. A hyperbolic version of Newton's Theorem that the center of gravity and the center of mass of the uniform sphere are identical.

3. A hyperbolic version of the characterization of concurrent cevians for the n-simplex.

Preliminaries:

The *(unsigned) moment* of the point-mass (X, x) with respect to the hyperplane ν is

$$M_\nu(X, x) = x \sinh d(X, \nu)$$

Proposition 3.1 verbatim

Proposition 3.2: The two point-masses (X, x) and (Y, y) have equal moments with respect to the intersecting hyperplane μ, if and only if μ contains at least one of their centroids. They have equal moments with respect to every straight line if and only if they are identical.

PROOF: ? $\square$

Proposition 3.3 Verbatim Proof: very slight modifications, if any.

A hyperplane is said to be oriented if one of its sides is designated as positive and the other as negative. It is clear that every hyperplane has two orientations.The *(signed) moment* of the finite point-mass system $\mathcal{X} = \{(X_i, x_i), i = 1, 2, 3, ..., n\}$ with respect to the oriented hyperplane μ is

$$M_\mu(\mathcal{X}) = \sum_{i=1}^{n} \sigma_\mu(X_i) M_\mu(X_i, x_i)$$

where $\sigma_\mu(X) = 1, -1, 0$ according as X is in the positive half-space of μ, negative half-space of μ or on μ itself. The finite point-mass system $\mathcal{X}$ is said to be *balanced* with respect to the oriented plane provided

$$M_m(\mathcal{X}) = 0.$$

It is clear that if μ and μ' are reverses of each other, then for every finite system $\mathcal{X}$ we have

$$M_\mu(\mathcal{X}) = -M_{\mu'}(\mathcal{X})$$

and

$$M_\mu(\mathcal{X}) = 0 \quad \text{if and only if} \quad M_{\mu'}(\mathcal{X}) = 0$$

Corollary 0.1 *For every finite point-mass system $\mathcal{X}$ and oriented hyperplane μ*

$$M_\mu(\mathcal{X}) = M_\mu(\mathcal{C}(\mathcal{X}))$$

PROOF: This follows from Theorem 3.4 by induction. Q.E.D.

Theorem 3.7. Just need to replace "line m" with "hyperplane μ" once in proof.
Proposition 3.8: replace "straight line" with "intersecting hyperplane".

1 Examples

In n dimensions(?),

$$ds^2 = d\rho^2 + \sinh^2 \rho d\theta_1^2 + \cdots + \sinh^2 \rho \sin^2 \theta_1 \cdots \sin^2 \theta_{n-2} d\theta_{n-1}^2$$

$$dV = \sinh^{n-1} \rho \sin^{n-2} \theta_1 \cdots \sin \theta_{n-2}$$

3-Mass of solid of revolution

$$\int\int\int_D \cosh \rho \sinh^2 \rho \sin \phi d\rho \wedge d\theta \wedge d\phi$$

$$= 2\pi \int\int_S \cosh \rho \sinh^2 \rho \sin \phi d\rho \wedge d\phi$$

3-Mass of sphere of radius r:

$$2\pi \int_0^r \cosh \rho \sinh^2 \rho \left(\int_0^\pi \sin \phi d\phi \right) d\rho = \frac{4\pi}{3} \sinh^3 r$$

2-Mass of generator

$$\int\int \cosh r \sinh r dr \wedge d\tau$$

Theorems:

It is widely accepted [Refs] that the hyperbolic analog of Newton's Law of Gravitation is

$$F = \frac{m_1 \times m_2}{\sinh^2 d} \tag{1}$$

This can be argued as follows. One of the ways of justifying Newton's inverse square assumption is to observe that the total flux of the gravitational field across any sphere centered at the sun is independent of that sphere's radius. Consequently the gravitational flux arriving at a planet at distance r from the sun should be inversely proportional to the surface area of the sphere of radius r. In the unit hyperbolic case this sphere is known [11, Ex. 3.4.5] to have volume

$$\pi(\sinh 2r - 2r)$$

which, when differentiated, yields a surface area of

$$4\pi \sinh^2 r$$

which lends plausibility to (3). Assuming this Law of Gravitation we prove the hyperbolic version of a famous Theorem of Newton's. A different proof appears in [Velpry].

Theorem 1.1 *The gravitational attraction that a uniform spherical mass exerts on a point mass outside the sphere and at distance R from its center is inversely propostional to* $\sinh^2 R$.

PROOF: Let the given spherical mass be a sphere S centered at O with radius r and let the given point mass be located at P. Because of the homogeneity of mass and the law of gravitational attraction, it may be assumed that the given sphere has uniform density 1 and that the mass at P is 1. Then the total force with which the sphere attracts P is

$$\int\int\int_S \frac{\cos\alpha}{\sinh^2 s} \cosh\rho \sinh^2\rho \sin\phi d\rho d\phi d\theta$$

$$= 2\pi \int_0^r \sinh^2\rho \cosh\rho \int_0^\pi \sin\phi \cdot \frac{\cos\alpha}{\sinh^2 s} d\phi d\rho$$

By the Law of Cosines

$$\cos\phi = \frac{\cosh R \cosh\rho - \cosh s}{\sinh R \sinh\rho}$$

and hence

$$\sin\phi d\phi = \frac{\sinh s ds}{\sinh R \sinh\rho}.$$

Consequently,

$$\sin\phi\cdot\frac{\cos\alpha}{\sinh^2 s}d\phi = \frac{1}{\sinh R\sinh\rho}\cdot\frac{\tanh(R-b)}{\sinh s\tanh s}ds$$

$$= \frac{1}{\sinh R\sinh\rho}\cdot\frac{\tanh R-\tanh b}{1-\tanh R\tanh b}\cdot\frac{\cosh s ds}{\sinh^2 s}$$

$$= \frac{1}{\sinh R\sinh\rho}\cdot\frac{\tanh R-\cos\phi\tanh\rho}{1-\tanh R\cos\theta\tanh\rho}\cdot\frac{\cosh s ds}{\sinh^2 s}$$

$$= \frac{1}{\sinh R\sinh\rho}\cdot\frac{\tanh R-\tanh\rho\frac{\cosh R\cosh\rho-\cosh s}{\sinh R\sinh\rho}}{1-\tanh R\tanh\rho\frac{\cosh R\cosh\rho-\cosh s}{\sinh R\sinh\rho}}\cdot\frac{\cosh s ds}{\sinh^2 s}$$

$$= \frac{\tanh R\sinh\rho\sinh R-\tanh\rho\cosh\rho\cosh R+\tanh\rho\cosh s}{\cosh s\tanh R\tanh\rho}$$

$$\cdot\frac{1}{\sinh R\sinh\rho}\cdot\frac{\cosh s ds}{\sinh^2 s}$$

$$= \frac{1}{\sinh R\sinh\rho}\cdot\frac{1}{\sinh^2 s}\left(\cosh\rho\sinh R-\frac{\cosh\rho\cosh R}{\tanh R}+\frac{\cosh s}{\tanh R}\right)ds$$

$$= \frac{\coth\rho}{\sinh^2 s\sinh^2 R}\left(\sinh^2 R-\cosh^2 R+\cosh s\frac{\cosh R}{\cosh\rho}\right)ds$$

$$= \frac{1}{\sinh\rho\sinh^2 R}\left(\cosh R\frac{\cosh s}{\sinh^2 s}-\cosh\rho\frac{1}{\sinh^2 s}\right)ds.$$

It follows that

$$\int_0^{\pi}\sin\phi\cdot\frac{\cos\alpha}{\sinh^2 s}d\phi$$

$$= \frac{1}{\sinh\rho\sinh^2 R}\int_{R-\rho}^{R+\rho}\left(\cosh R\frac{\cosh s}{\sinh^2 s}-\cosh\rho\frac{1}{\sinh^2 s}\right)ds$$

$$= \frac{1}{\sinh\rho\sinh^2 R}\left[-\cosh R\frac{1}{\sinh s}+\cosh\rho\frac{\cosh s}{\sinh s}\right]_{R-\rho}^{R+\rho}$$

$$= \frac{1}{\sinh\rho\sinh^2 R}\left[-\frac{\cosh R}{\sin(R+r)}+\cosh\rho\frac{\cosh(R+r)}{\sinh(R+r)}\right.$$

$$\left.+\frac{\cosh R}{\sin(R-r)}-\cosh\rho\frac{\cosh(R-r)}{\sinh(R-r)}\right]$$

$$= \frac{1}{\sinh\rho\sinh^2 R}\cdot\frac{\cosh\rho\sinh(R-\rho-R-\rho)+2\cosh R\cosh R\sinh\rho}{\sinh(R+\rho)\sinh(R-\rho)}$$

$$= \frac{2}{\sinh^2 R}\cdot\frac{\cosh^2 R-\cosh^2\rho}{\sinh^2 R\cosh^2\rho-\cosh^2 R\sinh^2\rho}$$

$$= \frac{2}{\sinh^2 R} \cdot \frac{\cosh^2 R - \cosh^2 \rho}{(\cosh^2 R - 1)\cosh^2 \rho - \cosh^2 R(\cosh^2 \rho - 1)}$$
$$= \frac{2}{\sinh^2 R}$$

Hence the total force is

$$4\pi \int_0^r \frac{\sinh^2 \rho \cosh \rho}{\sinh^2 R} d\rho = \frac{\frac{4\pi}{3} \sinh^3 \rho}{\sinh^2 R}.$$

Q.E.D.

A simplex $\{A_0, A_1, ..., A_n\}$ of H^n is a set of points no k of which are contained in a $(k-2)$-dimensional subspace of H^n. A *cevian* of this simplex is a line segment $A_k B_k$ where B_k is in the span (convex hull) of $\{A_0, A_1, ..., A_{k-1}, A_{k+1}, ..., A_n\}$. The Euclidean version of the following theorem was proved in [Landy].

Theorem 1.2 *A set $\{A_0B_0, A_1B_1, ..., A_nB_n\}$ of cevians of a simplex $\{A_0, A_1, ..., A_n\}$ in $\Re^n$ is concurrent if and only if each of the vertices can be assigned a weight so that the centroid of each weighted face(t) is located at B_k.*

Proof: Suppose first that each vertex has been assigned a weight so that the centroid of each weighted face is located at its base point. Then, by Prop'n xxx?, each of the cevians balances the simplex. Consequently each of the cevians contains the centroid of the weighted simplex and so the cevians are concurrent.

The converse is proved by induction on n. The case $n = 0$ is trivial. If $n = 1$ and $0 < d < A_0A_1$, set

$$\lambda_0 = \sinh d, \qquad \lambda_1 = \sinh(A_0A_1)$$

Then the centroid of the point-mass system $\{A_0, \lambda_0\}, (A_1, \lambda_1)\}$ is located at the point X of A_0A_1 such that $A_0X = d$.

Assume the theorem holds for $n - 2$. Suppose the cevians $A_iB_i, i = 0, 1, 2, ..., n$, are concurrent at X. The straight lines A_0B_1 and A_1B_0 intersect at X and hence they span a plane, say α.

Let σ be the simplex spanned by $A_2, A_3, ..., A_n$. The straight line A_0B_1 intersects σ in some point F_1 whereas the straight lineA_1B_0 intersects σ in some point F_2. Since $\alpha \cap \sigma$ is necessarily convex, it follows that $F_1 = F_2 = F$.

Let the vertices $A_2, A_3, ..., A_n$ of σ be assigned respective weights λ_2, $\lambda_3, ..., \lambda_n$ so that their centroid is (F, λ_{01}). Let λ_0, λ_1 be weights such that

$$\mathcal{C}\{(A_0, \lambda_0), (A_2, \lambda_2), ..., (A_n, \lambda_n)\}$$
$$= \mathcal{C}\{(A_0, \lambda_0), (F, \lambda_{01})\} = (B_1, .)$$

and

$$\mathcal{C}\{(A_1, \lambda_1), (A_2, \lambda_2), ..., (A_n, \lambda_n)\}$$

$$= \mathcal{C}\{(A_0, \lambda_0), (F, \lambda_{01})\} = (B_0, .)$$

Then, by Prop'n xxx,

$$\mathcal{C}\{(A_0, \lambda_0), (A_1, \lambda_1), ..., (A_n, \lambda_n)\}$$

$$= \mathcal{C}\{(A_0, \lambda_0), (A_1, \lambda_1), (F, \lambda_{01}) = (X, .)$$

Q.E.D.

In the Euclidean case it is known that the barycentric coordinates of a point in the interior of a triangle are proportional to the areas of the triangles formed by that point with each of the three sides of the triangle. This fails to work in hyperbolic geometry with respect to either area or mass.

Proposition 1.3 *Let α and β be asymptotically parallel hyperplanes of H^n and let P be a point on neither. If the $(n-1)$-dimensional solid S is a subset of α and*

$$T = \text{proj}_{P,\alpha,\beta}(S)$$

then

$$C(T) = \text{proj}_{P,\alpha,\beta}(C(S))$$

Proof: The validity of this proposition for the case when S is a two-point mass system follows from Lemma 6.3 (plane paper). The case where S is an arbitrary finite point mass system follows by a straightforward induction on the number of points. Finally, when S is a solid, the proposition follows from Proposition 4.3 (xxxplane paper). Q.E.D.

An *n-simplex* of H^n is a set of $n+1$ points (vertices) $\sigma = \{A_0, A_1, ..., A_n\}$ that are not contained in any $n-1$ dimensional hyperplane. The convex hull of σ is the *solid simplex* denoted by $|\sigma|$. We assume that each point of a solid simplex is assigned mass density 1. A $facet$ σ^i of σ is the $(n-1)$-simplex obtained by deleting A_i from σ. A *median* from a vertex A_i of a simplex is the line segment joining A_i to the centroid of the solid facet $|\sigma^i|$.

Theorem 1.4 *The centroid of every solid n-simplex is located at the intersection of its medians.*

Proof: This is obvious if $n = 1$ and the case $n = 2$ was proved in xxx. We proceed by induction on n and assume that the theorem holds for all simplices of dimension less than $k \geq 2$. Let $\{A_0, A_1, ..., A_k\}$ be a k-simplexσ in a hyperbolic space of dimension k. Let α be the hyperplane containing $A_1, A_2, ..., A_k$ and let β be any hyperplane that is asymptotically parallel to α and intersects m_0 in one of its interior points. If

$$A'_i = \text{proj}_{P,\alpha,\beta}(A'_i), \quad \text{and} \quad C' = \text{proj}_{P,\alpha,\beta}(C)$$

then it follows from the previous proposition that C' is the location of the centroid of the cross section $\{A'_1, A'_2, \cdots, A'_k\}$. By xxx

$$M_{m_0}(\{A'_1, A'_2, \cdots, A'_k\}) = 0$$

and hence by the above lemma

$$M_{m_0}(\{A'_0, A'_1, \cdots, A'_k\}) = 0$$

Proposition 1.5 *For each* $i = 0, 1, \cdots, n$ *, let* σ^i *denote the facet of the simplex* $\sigma = \{A_0, A_1, \cdots, A_n\}$ *opposite the vertex* A_i*. Then*

$$\mathrm{mass}(\sigma) = \frac{1}{n}\sum_{i=0}^{n} \sinh[d(O, \sigma^i] Vol(\sigma^i)$$

Proof???: To find the mass of the simplex, we assume without loss of generality that its centroid is located at the origin O of a Gaussian parametrization of H^n. Then

$$\mathrm{mass}(\sigma) = \int_{\sigma}^{(n)} \cosh \rho dV$$
$$= \sum_{i=0}^{n} \int_{K(O,\sigma^i O}^{(n)} \cosh \rho dV.$$

Let

$$\rho_i = \rho_i(\theta_i) = \coth^{-1}(C_i \cos(\theta - \alpha_i))$$

be the equation of the hyperplane that contains σ^i. If,

$$d\Theta = \sin^{n-2}\theta_1 \cdots \sin\theta_{n-2} d\rho d\theta_1 d\theta_2 \cdots d\theta_{n-1}$$

then

$$\int_{K(O,\sigma^i)}^{(n)} \cosh \rho dV$$
$$= \int_{\sigma^i}^{(n-1)} \int_0^{\rho_i(\theta_1)} \cosh \rho \sinh^{n-1} \rho d\theta$$
$$= \frac{1}{n} \int_{\sigma^i}^{(n-1)} \sinh^n \rho_i d\Theta$$
$$= \int_{\sigma^i}^{(n-1)} [\sinh(\coth^{-1}(C_i \cos(\theta_1 - \alpha_i)]^n d\Theta$$
$$= \int_{\sigma^i}^{(n-1)} \frac{d\Theta}{[C_i^2 \cos^2(\theta_1 - \alpha_i) - 1]^{n/2}}.$$

On the other hand, the volume of the simplex σ^i is

$$\int_{\sigma^i}^{(n-1)} dV^{(n-1)}$$

□

Lemma 1.6 *Let $\mathcal{C} = (C, c)$ be the centroid of the point-mass system $\mathcal{X} = \{(X, x_i)\}_{i=1}^n$ and let $\mathcal{A} = (A, a)$ be an arbitrary point-mass. Then*

$$a \cosh[d(A, C)] = \sum_{i=1}^{n} x_i \cosh[d(A, X_i)]$$

PROOF: By induction on n. The case $n = 1$ is clear. Assume the lemma holds for n and let

$$\mathcal{X} = \{(X, x_i)\}_{i=1}^n \quad \text{and} \quad \mathcal{X}' = \{(X, x_i)\}_{i=1}^{n+1}$$

be point-mass systems with respective centroids (X_n^*, x_n^*) and (X_{n+1}^*, x_{n+1}^*). When the hyperbolic Law of Cosines is applied to Figure 6 we obtain

$$\cos \delta = \frac{\cosh r_1 \cosh[d(A, X_{n+1}^*)] - \cosh[d(A, X_n^*)]}{\sinh r_1 \sinh[d(A, X_{n+1}^*)]}$$

and

$$\cos(\pi - \delta) = \frac{\cosh r_2 \cosh[d(A, X_{n+1}^*)] - \cosh[d(A, X_{n+1})]}{\sinh r_2 \sinh[d(A, X_{n+1}^*)]}.$$

The addition of these two equations yields

$$\sinh r_2 (\cosh r_1 \cosh[d(A, X_{n+1}^*)] - \cosh[d(A, X_n^*)])$$

$$+ \sinh r_1 (\cosh r_2 \cosh[d(A, X_{n+1}^*)] - \cosh[d(A, X_{n+1})]) = 0$$

from which it follows that

$$\cosh[d(A, X_{n+1}^*)] = \frac{\sinh r_2 \cosh[d(A, X_n^*)] + \sinh r_1 \cosh[d(A, X_{n+1})]}{\sinh r_2 \cosh r_1 + \sinh r_1 \cosh r_2}.$$

It follows from the fact that (X_{n+1}^*, x_{n+1}^*) is the centroid of (X_n^*, x_n^*) and (X_{n+1}, x_{n+1}) that

$$x_n^* \sinh r_1 = x_{n+1} \sinh r_2$$

and hence

$$\cosh[d(A, X_{n+1}^*)] = \frac{x_n^* \cosh[d(A, X_n^*)] + x_{n+1} \cosh[d(A, X_{n+1})]}{x_n^* \cosh r_1 + x_{n+1} \cosh r_2}.$$

Proposition xxx and the induction hypothesis now yield

$$\cosh[d(A, X_{n+1}^*)] = \frac{\sum_{i=1}^{n+1} x_i \cosh[d(A, X_i)]}{x_{n+1}^*}$$

which completes both the induction step and the proof. Q.E.D.

Theorem 1.7 *Let $\mathcal{C} = (C, c)$ be the centroid of the finite mass-system $\mathcal{X} = \{(X, x_i)\}_{i=1}^n$. Then*

$$c^2 = \sum_{i,j=1}^{n} x_i x_j \cosh[d(X_i, X_j)].$$

PROOF: By induction on n. The theorem is clearly valid for $n = 1$. Assume it is valid for n and let $\{(X_i, x_i)\}_{i=1}^n$ and $\{(X_i, x_i)\}_{i=1}^{n+1}$ be point-mass systems with centroids (C_n, c_n) and $(C_{n+1}, c_{n+1}$ respectively (Fig. 7). Then

$$c_{n+1}^2 = (c_n \cosh r_1 + x_{n+1} \cosh r_2)^2$$

$$= c_n^2 + c_n^2 \sinh^2 r_1 + x_{n+1}^2 + x_{n+1}^2 \sinh^2 r_2 + 2c_n x_{n+1} \cosh r_1 \cosh r_2$$

$$= \sum_{i,j=1}^{n} x_i x_j \cosh[d(X_i, X_j)]$$

$$+x_{n+1}^2 + c_n^2 \sinh^2 r_1 + x_{n+1}^2 \sinh^2 r_2 + 2c_n x_{n+1} \cosh r_1 \cosh r_2$$

$$= \sum_{i,j=1}^{n} x_i x_j \cosh[d(X_i, X_j)]$$

$$+x_{n+1}^2 + (c_n \sinh r_1 - x_{n+1} \sinh r_2)^2 + 2c_n x_{n+1} \cosh(r_1 + r_2)$$

$$= \sum_{i,j=1}^{n} x_i x_j \cosh[d(X_i, X_j)] + x_{n+1}^2 + 2c_n x_{n+1} \cosh[d(X_{n+1}, C_n)].$$

Hence, by the lemma above,

$$c_{n+1}^2 = \sum_{i,j=1}^{n} x_i x_j \cosh[d(X_i, X_j)] + x_{n+1}^2 + 2 \sum_{i=1}^{n} x_{n+1} x_i \cosh d[(X_{n+1}, X_i)]$$

$$= \sum_{i,j=1}^{n+1} x_i x_j \cosh[d(X_i, X_j)].$$

This completes both the induction step and the proof. Q.E.D.

2 References

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