Combinatorics and freeness of hyperplane arrangements and reflection arrangements

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Kurzzusammenfassung

Ein Arrangement von Hyperebenen (kurz Arrangement) ist eine endliche Menge linearer Unterräume der Kodimension eins in einem endlichdimensionalen Vektorraum.

Die Klasse der Spiegelungsarrangements nimmt in der Theorie eine zentrale Rolle ein. Eine weitere wichtige Klasse sind freie Arrangements, die zuerst von Terao in den 1980er Jahren untersucht wurden. Nach seinem bedeutenden Satz sind alle Spiegelungsarrangements frei. Teraos Vermutung besagt, dass die algebraische Eigenschaft der Freiheit über einem fest gewählten Körper in Wirklichkeit eine kombinatorische Eigenschaft ist. Mit Hilfe stärkerer Freiheitsbegriffe lassen sich Klassen von Arrangements definieren, die Teraos Vermutung erfüllen oder wahrscheinlich erfüllen. Daher ist es naheliegend, Spiegelungsarrangements im Zusammenhang mit Teraos Vermutung zu untersuchen, d. h. Spiegelungsarrangements zu klassifizieren, die einer dieser Klassen kombinatorisch freier oder fast kombinatorisch freier Arrangements angehören.

Das erste Hauptresultat dieser Arbeit ist die Klassifikation rekursiv freier Spiegelungsarrangements. Mit Hilfe dieser Klassifikation können wir eine Vermutung von Abe über die von ihm eingeführte Klasse der divisionell freien Arrangements bestätigen.

Nahe verwandt mit reellen Spiegelungsarrangements, auch Coxeterarrangements genannt, ist die Klasse der simplizialen Arrangements. Insbesondere in höheren Rängen ist nicht viel über diese Klasse bekannt. Zumindest in Rang 3 gibt es eine vermutlich vollständige Liste von Grünbaum, (geringfügig erweitert von Cuntz). Wie sich zeigt, besitzen fast alle der bekannten simplizialen Arrangements die stärkste kombinatorische Freiheitseigenschaft: Sie sind überauflösbar. Überauflösbare Arrangements haben besonders schöne algebraische, geometrische, topologische und kombinatorische Eigenschaften.

Das zweite Hauptresultat ist eine vollständige Klassifikation überauflösbarer simplizialer Arrangements in allen Rängen. Dies erklärt Teraos Vermutung für eine große Teilklasse der bekannten simplizialen Arrangements. Erstaunlicherweise impliziert die Klassifikation die starke Ganzzahligkeitseigenschaft kristallographisch zu sein für irreduzible überauflösbare simpliziale Arrangements vom Rang größer 3. Um unser Klassifikationsresultat zu zeigen, führen wir das kombinatorische Werkzeug der Coxetergraphen für simpliziale Arrangements ein, eine Verallgemeinerung von Coxetergraphen für (endliche) Coxetergruppen. Weiterhin beweisen wir einige hilfreiche allgemeine Resultate zur Kombinatorik von simplizialen Arrangements.

Schlagworte: *Arrangements von Hyperebenen, Spiegelungsarrangements, freie Arrangements, Teraos Vermutung, rekursiv freie Arrangements, simpliziale Arrangements, überauflösbare Arrangements, Coxetergraphen*

Abstract

An arrangement of hyperplanes (or just arrangement) is a finite set of codimension one linear subspaces in a finite dimensional vector space.

The class of reflection arrangements plays a pivotal role in the theory. Another important class are the free arrangements first studied by Terao in the 1980s. By his famous theorem all reflection arrangements are free. Terao's conjecture claims that the algebraic property of freeness over a fixed field is actually a combinatorial property. There are stronger notions of freeness giving rise to classes of arrangements in which Terao's conjecture holds or might hold. So it is natural to investigate Terao's conjecture in connection with reflections arrangements, i.e. classify the reflection arrangements which belong to one of these classes of combinatorially free or almost combinatorially free arrangements.

The first main result of this thesis is the classification of recursively free reflection arrangements. With this classification we can confirm a conjecture by Abe about the class of divisionally free arrangements he introduced.

Closely connected to real reflection arrangements, also called Coxeter arrangements, is the class of simplicial arrangements. In particular in higher ranks not much about this class is known. At least in rank 3 there is a list by Grünbaum (slightly updated by Cuntz) conjectured to be complete. One observes that almost all known simplicial arrangements satisfy the strongest combinatorially freeness property: they are supersolvable. Supersolvable arrangements have particularly nice algebraic, geometric, topological and combinatorial properties.

The second main result is a complete classification of supersolvable simplicial arrangements in all ranks. This clarifies Terao's conjecture for a large subclass of the known simplicial arrangements. Surprisingly the classification implies the strong integrality property of being crystallographic for irreducible supersolvable simplicial arrangements of rank greater than 3. To prove this classification result we introduce the combinatorial tool of Coxeter graphs for simplicial arrangements, a generalization of Coxeter graphs for (finite) Coxeter groups. Furthermore, we prove some helpful results about the combinatorics of simplicial arrangements in general.

Keywords: *Hyperplane arrangements, reflection arrangements, free arrangements, Terao's conjecture, recursively free arrangements, simplicial arrangements, supersolvable arrangements, Coxeter graph*

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1 Introduction

An arrangement of hyperplanes (or just arrangement) is a finite set of codimension one linear subspaces in a finite dimensional vector space. Their study is interesting from several points of view involving discrete and algebraic geometry, algebra, topology, group theory and combinatorics.

One of the central questions in the theory of hyperplane arrangements is to what extent the (discrete) geometrical, algebraic, or topological structure of a given arrangement $\mathcal{A}$ is controlled by its combinatorial structure, i.e. its intersection lattice $L(\mathcal{A})$. This thesis is concerned with the discrete geometrical and algebraic properties governed by the combinatorics of an arrangement.

The class of *reflection arrangements*, i.e. arrangements consisting of the reflecting hyperplanes of a finite complex reflection group plays a pivotal role in the theory and gives rise to important examples or counterexamples. Contained in the class of all reflection arrangements are the subclasses of Weyl and Coxeter arrangements, i.e. the (real) reflection arrangements associated to a Weyl group or a finite Coxeter group. Many questions about a reflection group can be formulated in terms of the reflection arrangements attached to it. This provides a very useful geometric point of view. In particular the special properties of Weyl groups and Coxeter groups are deeply intertwined with the combinatorial structure of their associated arrangements.

A further important class are the *free arrangements* (denoted by $\mathcal{F}$). The algebraic property of freeness was first studied by Terao [Ter80a] in the 1980s. It turns out that all reflection arrangements are free. This was shown by Terao [Ter80b] (earlier work of Arnold [Arn79] and Saito [Sai80] implied the freeness of Coxeter arrangements). The concept of free arrangements has proved itself to be very useful, in particular in connection with the class of reflection arrangements. A recent remarkable example is a new classification free proof of the celebrated Shapiro-Steinberg-Kostant-Macdonald formula for the exponents of a Weyl group by Abe, Barakat, Cuntz, Hoge, and Terao [ABC$^+$16].

But what characterizes free arrangements in general is still a mystery and a driving force of ongoing research. The fundamental open problem regarding free arrangements is Terao's conjecture:

Conjecture I ([OT92, Conj. 4.138])**.** *The freeness of an arrangement $\mathcal{A}$ over a fixed field only depends on its intersection lattice $L(\mathcal{A})$, i.e. its combinatorics.*

Ziegler demonstrated that the field of definition in fact should be fixed, [Zie90].

Motivated by this conjecture there are the stronger notions of *inductive freeness* (we denote the corresponding class by $\mathcal{IF}$), first introduced by Terao in [Ter80a], *recursive freeness* (we denote the corresponding class by $\mathcal{RF}$) which was introduced by Ziegler in [Zie87], and *divisional freeness* (the class is denoted by $\mathcal{DF}$) recently introduced by Abe [Abe16]. The first two notions are based on the applicability of Terao's Addition-Deletion Theorem (see Theorem 3.1.5). Divisional freeness is similarly based on Abe's Division Theorem (Theorem 3.1.8), [Abe16]. Terao's conjecture is true for the subclasses $\mathcal{IF}$ and $\mathcal{DF}$, i.e. freeness of arrangements in these classes can be derived combinatorially. Whether recursive freeness is combinatorial is still an open problem. In particular we have the following relation between the different freeness classes:

$$\mathcal{IF} \subsetneq {}^{1}\mathcal{DF} \subsetneq {}^{2}\mathcal{F},$$

$$\mathcal{IF} \subsetneq {}^{3}\mathcal{RF} \subsetneq {}^{4}\mathcal{F},$$

and $\mathcal{DF} \neq \mathcal{RF}$ (see Theorem 4.1 and Section 4.2). Since reflection arrangements are free, this raises the question of whether Terao's conjecture holds for them. This in turn suggests investigating the stronger freeness properties for reflection arrangements.

In [BC12] Barakat and Cuntz proved that all Coxeter arrangements are inductively free and in [HR15] Hoge and Röhrle completed the classification of inductively free reflection arrangements (see Theorem 4.2) by inspecting the remaining complex cases (see also the table below). They gave an easy characterization for all the cases but one, namely if the complex reflection group W admits an irreducible factor isomorphic to G_{31} (Shephard-Todd numbering [ST54]) and handling this case also turns out to be the most difficult part of their paper. That the group G_{31} in connection with the class of free arrangements plays a special role among the exceptional groups will become clearer in Subsection 4.1.4.

In [CH15b] Cuntz and Hoge gave first examples for free but not recursively free arrangements. One of them is the reflection arrangement of the exceptional complex reflection group G_{27}. Then Abe, Cuntz, Kawanoue, and Nozawa [ACKN16] found smaller examples (with 13 hyperplanes, being the smallest possible, and with 15 hyperplanes) for free but not recursively free arrangements in characteristic 0.

Nevertheless, free but not recursively free arrangements seem to be rare. Furthermore, recursive freeness is in general much harder to prove or disprove since in contrast to inductive freeness one might also have to add several new hyperplanes to the arrangement or show that this is not possible while preserving freeness.

A natural question is which other reflection arrangements are free but not recursively free. We answer the question and complete the picture for reflection arrangements by showing which of the not inductively free reflection arrangements are recursively free and which are free but not recursively free. Our first main theorem gives a complete classification of all recursively free reflection arrangements:

[1]First examples are in [Abe16], and [Müc17]
[2]See also [Abe16]
[3]See [OT92, Ex. 4.59]
[4]A first example appeared in [CH15b], see also Chapter 4

Theorem I. *For W a finite complex reflection group, the reflection arrangement $\mathcal{A}(W)$ of W is recursively free if and only if W does not admit an irreducible factor isomorphic to one of the exceptional reflection groups $G_{27}, G_{29}, G_{31}, G_{33}$ and G_{34}.*

If we restrict our view to real reflection arrangements (also called Coxeter arrangements) we observe that they all possess the geometric property of being simplicial, i.e. they cut simplicial cones out of the ambient space. This leads to consider the class of all real arrangements with this property called *simplicial arrangements* as a generalization of Coxeter arrangements.

Simplicial arrangements were first investigated by Melchior [Mel41] in 1941 by the means of triangulations of the real projective plane by a finite set of projective lines. Their study became popular again in the 1970s after Deligne [Del72] proved that the complement of a complexified simplicial arrangements is a $K(\pi, 1)$ space. In [Grü09] Grünbaum gives an extensive list of rank 3 irreducible simplicial arrangements, the slightly extended list by Cuntz [Cun12] is conjectured to be complete. But not much is known about simplicial arrangements of higher rank. In a series of papers Cuntz and Heckenberger investigated a class of objects called finite Weyl groupoids, a generalization of Weyl groups. Their work resulted in a complete classification of these objects, [CH15a]. Since Weyl groupoids are in one to one correspondence with crystallographic arrangements [Cun11a], and these constitute a large subclass of the known simplicial arrangements, this explains a large subset of the arrangements in Grünbaum's list. But there are still many non crystallographic simplicial arrangements lacking a satisfactory explanation.

The list given by Grünbaum contains two infinite series of irreducible simplicial arrangements of rank three parametrized by positive integers. They are denoted $\mathcal{R}(1) = \{\mathcal{A}(2n, 1) \mid n \geq 3\}$ and $\mathcal{R}(2) = \{\mathcal{A}(4m+1, 1) \mid m \geq 3\}$ (see Definition 6.1.6). The irreducible simplicial rank 3 arrangements which do not belong to one of these infinite classes are called *sporadic*. One observes that each of the 94 sporadic arrangements in [Cun12] consists of no more than 37 hyperplanes. So the following is conjectured:

Conjecture IIa (cf. [CG15, Conj. 1.6])**.** *Let $\mathcal{A}$ be an irreducible simplicial arrangement of rank three. If $|\mathcal{A}| > 37$ then $\mathcal{A} \in \mathcal{R}(1) \cup \mathcal{R}(2)$.*

Geis and Cuntz observed that simpliciality is a purely combinatorial property of the intersection lattice of an arrangement [CG15]. This combinatorial characterization suggests a connection of the class of simplicial arrangements with other classes of arrangements which can be defined combinatorially, e.g. combinatorially free arrangements. In fact, many of the known simplicial arrangements belong to the class of free arrangements or even the class of inductively free arrangements, c.f. [Ter80a, Table 1]. This suggests a connection of simpliciality with (inductive) freeness.

Supersolvable arrangements (we denote this class by $\mathcal{SS}$) were first considered by R. Stanley [Sta72]. They are now a well studied class of arrangements. Supersolvable arrangements possess nice algebraic, geometric, topological, and combinatorial properties, cf. [OT92, Theorems 2.63, 3.81, 4.58, 5.113]. In particular all supersolvable arrangements are inductively free. Hence in this class Terao's conjecture holds. One might even say that supersolvable arrangements are “the most combinatorially free” class of arrangements. Looking at the list of all known simplicial arrangements (including the known higher rank cases) one further observes that in fact for each fixed rank almost all of them belong to the class of supersolvable arrangements.

As the list (at least for rank 3) is conjectured to be complete but a conceptional approach towards a general classification is still missing, one might ask if there is an approach for a subclass with additional properties, e.g. supersolvable simplicial arrangements. This approach is chosen in this thesis resulting in our second main theorem (obtained in joint work with M. Cuntz, [CM17]), a complete classification of (irreducible) supersolvable simplicial arrangements:

Theorem II. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement, $(\ell \geq 3)$. Then either*

(1) $\ell = 3$ *and $\mathcal{A}$ is L-equivalent to an arrangement in* $\mathcal{R}(1) \cup \mathcal{R}(2)$*, or*

(2) $\ell \geq 4$ *and $\mathcal{A}$ is isomorphic to one of the reflection arrangements* $\mathcal{A}(A_\ell), \mathcal{A}(B_\ell)$*, or isomorphic to* $\mathcal{A}_\ell^{\ell-1} = \mathcal{A}(B_\ell) \setminus \{\{x_1 = 0\}\}$*. In particular $\mathcal{A}$ is crystallographic.*

As a result of Part (1) of Theorem II we can reformulate Conjecture IIa in the following way:

Conjecture IIb. *Let $\mathcal{A}$ be an irreducible simplicial 3-arrangement. If $|\mathcal{A}| > 37$ then $\mathcal{A}$ is supersolvable.*

Surprisingly, Part (2) of Theorem II implies a strong integrality property for irreducible supersolvable simplicial arrangements of rank greater than 3: they are crystallographic.

Furthermore, inspired by the work of Cuntz and Heckenberger on finite Weyl groupoids, we introduce Coxeter graphs for simplicial arrangements generalizing the notion of the Coxeter graph for a (finite) Coxeter group respectively Coxeter arrangement. These serve as our main tool of investigation to prove Theorem II.

In sum, the question of combinatorial freeness of reflection arrangements and related classes is an active field of ongoing research with many contributions by a number of people which is displayed in Table 2.1. The inclusion symbol displayed in a certain cell means, that all arrangements in the class of this row are already contained in one of the smaller freeness classes to the left. For example all Coxeter arrangements are inductively free by [BC12], there is no recursively free but not inductively free Coxeter arrangement, and hence in this row there is a "$\subseteq$" in the column labeled $\mathcal{RF}$. A question mark (only appearing in the row "simplicial arrangements") means that it is still an open problem which arrangements in this column belong to which freeness class (since there is no classification for simplicial arrangements in general yet).

In this thesis we fill several gaps in Table 2.1 and provide new data on the interplay of combinatorial freeness with the different classes of arrangements all related to reflection groups.

	$\mathcal{SS}$	$\mathcal{IF}$	$\mathcal{DF}$	$\mathcal{RF}$	$\mathcal{F}$
Weyl arrangements	[HR14]	[BC12]	$\subseteq$	$\subseteq$	$\subseteq$
Coxeter arrangements	[HR14]	[BC12]	$\subseteq$	$\subseteq$	$\subseteq$ [Arn79], [Sai80]
(complex) reflection arrangements	[HR14]	[HR15]	[HR15], [Abe16], Ch. 4	Ch. 4	[Ter80b]
crystallographic arrangements	[AHR14b], [BC12]	[BC12]	$\subseteq$	$\subseteq$	$\subseteq$
simplicial arrangements	Ch. 6	?	?	?	?
restrictions of (complex) reflection arrangements	[AHR14b]	[AHR14a]	Ch. 4, [Röh15]	Ch. 4	[OS82], [HR13], [Dou99]

Table 2.1: Combinatorial freeness for the different classes related to reflection arrangements.

The thesis is organized as follows: In Chapter 2 we list all needed notions and some elementary results from the theory of hyperplane arrangements. In Chapter 3 we introduce the different classes of arrangements which play the central role in this thesis. Furthermore, in Section 3.3 we prove some new results about the geometry and combinatorics of simplicial arrangements. In Chapter 4 we prove our first main result, the classification of recursively free reflection arrangements given by Theorem I. In Chapter 5 we introduce the combinatorial tool of Coxeter graphs for simplicial arrangements. This is a direct generalization of Coxeter graphs for finite Coxeter groups respectively Coxeter arrangements to simplicial arrangements. We also derive the main properties of these combinatorial data attached to a simplicial arrangement. Finally, Chapter 6 gives the proof of Theorem II, i.e. the classification of supersolvable simplicial arrangements. The proof highly depends on the tool of Coxeter graphs and their properties introduced in Chapter 5.

The results of Chapter 4 are found in [Müc17]. The results of Section 3.3, Chapter 5, and Chapter 6 are joint work with M. Cuntz, and are found in [CM17].

2 Preliminaries

In this chapter we collect the basic notions and definitions from the theory of hyperplane arrangements needed throughout this thesis. They can all be found in the book by P. Orlik and H. Terao, [OT92] which also serves as a nice introduction to the theory of hyperplane arrangements. Furthermore, at the end of this chapter we list some general combinatorial results for arrangements respectively real arrangements needed later on.

Let $\mathbb{K}$ be a field. An *arrangement of hyperplanes* (or just *arrangement*) is a pair $(\mathcal{A}, V)$, where $\mathcal{A}$ is a finite set of hyperplanes (codimension 1 subspaces) in a finite dimensional vector space V over $\mathbb{K}$. If the vector space V is unambiguous for $(\mathcal{A}, V)$ we simply write $\mathcal{A}$. If $\dim(V) = \ell$ then $\mathcal{A}$ is called an *ℓ-arrangement*. In this thesis $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. We denote the empty ℓ-arrangement by Φ_ℓ.

If $\alpha \in V^*$ is a linear form, we write $\alpha^\perp = \ker(\alpha)$ and interpret α as a normal vector for the hyperplane $H = \alpha^\perp$. Suppose we have chosen a basis $x_1, \ldots, x_\ell$ for V^*. Then if $\alpha = \sum_{i=1}^{\ell} a_i x_i \in V^*$ we occasionally write it as a row vector $(a_1, \ldots, a_\ell)$. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in V. If $\alpha_j = \sum_{i=1}^{\ell} a_{ij} x_i \in V^*$ such that $H_j = \alpha_j^\perp$ then the hole arrangement $\mathcal{A}$ is given explicitly by the matrix $(a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq n} \in \mathbb{K}^{\ell \times n}$.

We can visualize a real 3-arrangement $\mathcal{A}$ by drawing a projective picture. We chose an affine hyperplane $H_0 \in \mathbb{R}^3$ which does not contain the origin and consider all the intersections of hyperplanes from $\mathcal{A}$ with H_0. If there is a hyperplane $\tilde{H} \in \mathcal{A}$ which is parallel to H_0 then in our picture $\tilde{H}$ becomes the line at infinity in the projective plane.

Example 2.1. Let x, y, z be a basis of $(\mathbb{R}^3)^*$, and let $\mathcal{A}$ be the arrangement in $\mathbb{R}^3$ containing the following hyperplanes

$$\begin{aligned}\mathcal{A} =& \{\ker(x), \ker(y), \ker(z), \ker(x+z), \ker(y+z), \ker(x-y)\} \\ =& \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,0,1)^\perp, (0,1,1)^\perp, (1,-1,0)^\perp\}.\end{aligned}$$

The arrangement $\mathcal{A}$ might also be defined by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

with normal vectors of the hyperplanes as columns.

By taking $H_0 = \ker(z-1)$ we obtain the projective picture of $\mathcal{A}$ displayed in Figure 2.1. In particular $\ker(z) \in \mathcal{A}$ (the third column of the matrix) is parallel to H_0; in our picture it corresponds to the line at infinity drawn as an arc in the upper right corner.

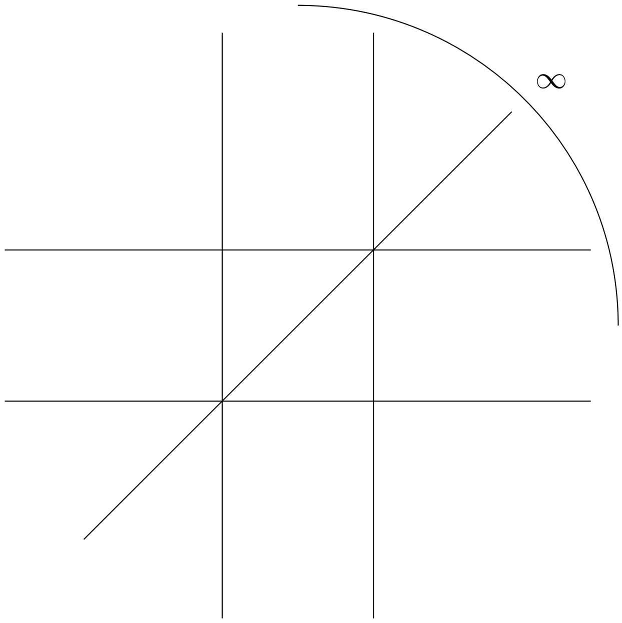


Figure 2.1: A projective picture of $\mathcal{A}$.

The central combinatorial object attached to an arrangement $\mathcal{A}$ is the *intersection lattice* $L(\mathcal{A})$. It is the set of all subspaces X of V of the form $X = H_1 \cap \ldots \cap H_r$ with $\{H_1, \ldots, H_r\} \subseteq \mathcal{A}$, partially ordered by reverse inclusion:

$$X \leq Y \iff Y \subseteq X, \text{ for X,Y } \in L(\mathcal{A}).$$

The bottom element of the intersection lattice is the hole space V as the intersection of the empty set of hyperplanes. The top element is the intersection of all hyperplanes contained in $\mathcal{A}$; it is denoted by $T(\mathcal{A}) = \cap_{H \in \mathcal{A}} H$ and called the *center* of $\mathcal{A}$. For $X \in L(\mathcal{A})$ the *rank* $r(X)$ of X is defined as $r(X) := \ell - \dim X$, and the rank of the arrangement $\mathcal{A}$ is defined as $r(\mathcal{A}) := r(T(\mathcal{A}))$. An ℓ-arrangement $\mathcal{A}$ is called *essential* if $r(\mathcal{A}) = \ell$. For $X \in L(\mathcal{A})$ we define the *localization*

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\}$$

of $\mathcal{A}$ at X, and the *restriction* $(\mathcal{A}^X, X)$ of $\mathcal{A}$ to X by

$$\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}.$$

For, $0 \leq q \leq \ell$ we write $L_q(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid r(X) = q\}$. If X is a subspace of V and $X \subseteq H$ for all $H \in \mathcal{A}$, i.e. $X \subseteq T(\mathcal{A})$, then H/X is a hyperplane in V/X for all $H \in \mathcal{A}$ and we can define the quotient arrangement $(\mathcal{A}/X, V/X)$ by $\mathcal{A}/X := \{H/X \mid H \in \mathcal{A}\}$. If $(\mathcal{A}, V)$ is not essential, i.e. $\dim(T(\mathcal{A})) > 0$, we sometimes identify it with the essential $r(\mathcal{A})$-arrangement $(\mathcal{A}/T(\mathcal{A}), V/T(\mathcal{A}))$.

Given two arrangements $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ (over the same field $\mathbb{K}$) we can form their *product* $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ which is defined by

$$\mathcal{A} := \mathcal{A}_1 \times \mathcal{A}_2 = \{H_1 \oplus V_2 \mid H_1 \in \mathcal{A}_1\} \cup \{V_1 \oplus H_2 \mid H_2 \in \mathcal{A}_2\},$$

see [OT92, Def. 2.13]. In particular $|\mathcal{A}| = |\mathcal{A}_1| + |\mathcal{A}_2|$. If an arrangement $\mathcal{A}$ can be written as a non-trivial product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$, (where “non-trivial” means $\mathcal{A}_i \neq \Phi_0$), then $\mathcal{A}$ is called *reducible*, otherwise *irreducible*.

Lemma 2.2 ([OT92, Prop. 2.14]). *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product. Define a partial order on $L(\mathcal{A}_1) \times L(\mathcal{A}_2)$ by*

$$(X_1, X_2) \leq (Y_1, Y_2) \iff X_1 \leq Y_1 \text{ and } X_2 \leq Y_2,$$

for $(X_1, X_2), (Y_1, Y_2) \in L(\mathcal{A}_1) \times L(\mathcal{A}_2)$. Then there is an isomorphism of lattices

$$\begin{aligned} \pi: \quad L(\mathcal{A}_1) \times L(\mathcal{A}_2) &\to L(\mathcal{A}_1 \times \mathcal{A}_2) \\ (X_1, X_2) &\mapsto X_1 \oplus X_2. \end{aligned}$$

Corollary 2.3. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product and $X = X_1 \oplus X_2 \in L(\mathcal{A})$. Then we have*

$$\mathcal{A}_X = (\mathcal{A}_1)_{X_1} \times (\mathcal{A}_2)_{X_2} \text{ and } \mathcal{A}^X = (\mathcal{A}_1)^{X_1} \times (\mathcal{A}_2)^{X_2}.$$

Let $\mathcal{A}$ and $\mathcal{B}$ be two arrangements in V with $L(\mathcal{A}) \cong L(\mathcal{B})$ as lattices. Then $\mathcal{A}$ and $\mathcal{B}$ are called *lattice equivalent* or L-equivalent and we write $\mathcal{A} \sim_L \mathcal{B}$. If $\mathcal{A}$ and $\mathcal{B}$ are arrangements in V such that there is a $\varphi \in \mathrm{GL}(V)$ with $\mathcal{B} = \varphi(\mathcal{A}) = \{\varphi(H) \mid H \in \mathcal{A}\}$ then we say that $\mathcal{A}$ is *(linearly) isomorphic* to $\mathcal{B}$. Suppose $\mathcal{A}$ and $\mathcal{B}$ are booth arrangements in a $\mathbb{K}$-vector space V and we have chosen a basis of V^* such that $\mathcal{A}$ and $\mathcal{B}$ are given by matrices $A, B \in \mathbb{K}^{\ell \times n}$ respectively. Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if there exists a non-singular matrix $C \in \mathbb{K}^{\ell \times \ell}$ and a non-singular monomial matrix $M \in \mathbb{K}^{n \times n}$ such that $B = CAM$.

For an arrangement $\mathcal{A}$ the *Möbius function* $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is defined by:

$$\mu(X) = \begin{cases} 1 & \text{if } X = V, \\ -\sum_{V \supseteq Y \supsetneq X} \mu(Y) & \text{if } X \neq V. \end{cases}$$

We denote by $\chi_\mathcal{A}(t)$ the *characteristic polynomial* of $\mathcal{A}$ which is defined by:

$$\chi_\mathcal{A}(t) = \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim(X)}.$$

Remark 2.4. If $\mathcal{A}$ is a 3-arrangement then the characteristic polynomial is given by

$$\chi_\mathcal{A}(t) = t^3 + \mu_1 t^2 + \mu_2 t + \mu_3,$$

with

$$\mu_1 = -|\mathcal{A}|, \quad \mu_2 = \sum_{X \in L(\mathcal{A})} (|\mathcal{A}_X| - 1), \quad \mu_3 = 1 - \mu_1 - \mu_2.$$

The characteristic polynomial behaves well with respect to the product construction:

Lemma 2.5 ([OT92, Lem. 2.50]). *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product of two arrangements. Then*

$$\chi_\mathcal{A}(t) = \chi_{\mathcal{A}_1}(t) \chi_{\mathcal{A}_2}(t).$$

We state the following geometric theorem generalizing the well known Silverster-Gallai theorem for real arrangements in its dual version for arrangements. It was first proved by Motzkin [Mot51] for $\ell = 4$ and later by Hansen [Han65] for all ℓ.

Theorem 2.6 (Hansen-Motzkin)**.** *Let $\mathcal{A}$ be a real ℓ-arrangement, $\ell \geq 2$. Then there is an $X \in L_{\ell-1}(\mathcal{A})$ and an $H \in \mathcal{A}$ such that $X = H \cap Y$ for a $Y \in L_{\ell-2}(\mathcal{A})$, and $\mathcal{A}_X = \mathcal{A}_Y \cup \{H\}$. In particular $\mathcal{A}_X/X$ is reducible with $\mathcal{A}_X/X \cong \mathcal{A}_Y/Y \times \{\{0\}\}$.*

This theorem will be an important ingredient for the proof of one of our main results in Chapter 6.

3 Different classes of arrangements

In this chapter we list the different classes of arrangements that play the main role in this thesis.

The first section recalls the class of free arrangements and their several subclasses including Abe's recently introduced class of divisionally free arrangements.

In Section 3.2 we recall the notion of reflection arrangement given by a finite complex reflection group and their main properties.

In Section 3.3 we consider simplicial arrangements. They are classical objects in discrete geometry and may be seen as a generalization of real reflection arrangements (also called Coxeter arrangements). This will become clearer in Chapter 5, where we generalize the notion of a Coxeter graph for a (finite) Coxeter group to simplicial arrangements. Furthermore, Section 3.3 contains some new results on simplicial arrangements which we will apply in Chapter 5 and Chapter 6.

In Section 3.4 we look at supersolvable arrangements which are now a well studied class of arrangements. They possess particularly nice geometric, algebraic and combinatorial properties as becomes apparent.

3.1 Free arrangements

Let V be an ℓ-dimensional vector space over $\mathbb{C}$ and let $S = S(V)$ be the symmetric algebra of the dual space V^* of V. If $x_1, \ldots, x_\ell$ is a basis of V^*, then we identify S with the polynomial ring $\mathbb{C}[x_1, \ldots, x_\ell]$ in ℓ variables. The algebra S has a natural $\mathbb{Z}$-grading: Let S_p denote the $\mathbb{C}$-subspace of S of the homogeneous polynomials of degree p ($p \in \mathbb{N}_{\geq 0}$), then $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ for $p < 0$.

Let $\operatorname{Der}(S)$ be the S-module of $\mathbb{C}$-derivations of S. It is a free S-module with basis $D_1, \ldots, D_\ell$ where D_i is the partial derivation $\partial/\partial x_i$. We say that $\theta \in \operatorname{Der}(S)$ is *homogeneous of polynomial degree* p provided $\theta = \sum_{i=1}^{\ell} f_i D_i$, with $f_i \in S_p$ for each $1 \leq i \leq \ell$. In this case we write $\operatorname{pdeg} \theta = p$. With this definition we get a $\mathbb{Z}$-grading for the S-module $\operatorname{Der}(S)$: Let $\operatorname{Der}(S)_p$ be the $\mathbb{C}$-subspace of $\operatorname{Der}(S)$ consisting of all homogeneous derivations of polynomial degree p, then $\operatorname{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \operatorname{Der}(S)_p$.

Definition 3.1.1. Let $\mathcal{A}$ be an arrangement of hyperplanes in V. Then for $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. A *defining polynomial* $Q(\mathcal{A})$ is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

The *module of $\mathcal{A}$-derivations* of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := D(Q(\mathcal{A})) := \{\theta \in \operatorname{Der}(S) \mid \theta(Q(\mathcal{A})) \in Q(\mathcal{A})S\}.$$

We say that $\mathcal{A}$ is *free* if the module of $\mathcal{A}$-derivations is a free S-module.

If $\mathcal{A}$ is an arrangement over a subfield $\mathbb{K} \leq \mathbb{C}$ then we say $\mathcal{A}$ is free provided the complex arrangement $\mathcal{A} \otimes_{\mathbb{K}} \mathbb{C}$ is free.

Example 3.1.2. All 2-arrangements are free, c.f. [OT92, Ex. 4.20].

If $\mathcal{A}$ is a free arrangement, let $\{\theta_1, \dots, \theta_\ell\}$ be a homogeneous basis for $D(\mathcal{A})$. Then the polynomial degrees of the θ_i, $i \in \{1, \dots, \ell\}$, are called the *exponents* of $\mathcal{A}$. We write $\exp(\mathcal{A}) := \{\{\operatorname{pdeg} \theta_1, \dots, \operatorname{pdeg} \theta_\ell\}\}$, where the notation $\{\{*\}\}$ emphasizes the fact, that $\exp(\mathcal{A})$ is a multiset in general. The multiset $\exp(\mathcal{A})$ is uniquely determined by $\mathcal{A}$, see also [OT92, Def. 4.25].

If $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = \{\{b_1, \dots, b_\ell\}\}$, then by [OT92, Thm. 4.23]:

$$\sum_{i=1}^{\ell} b_i = |\mathcal{A}|. \tag{3.1.3}$$

The following proposition shows that the product construction mentioned before is compatible with the notion of freeness:

Proposition 3.1.4 ([OT92, Prop. 4.28]). *Let $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ be two arrangements. The product arrangement $(\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ is free if and only if both $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ are free. In this case*

$$\exp(\mathcal{A}_1 \times \mathcal{A}_2) = \exp(\mathcal{A}_1) \cup \exp(\mathcal{A}_2).$$

The following theorem provides a useful tool to prove the freeness of arragnements.

Theorem 3.1.5 (Addition-Deletion [OT92, Thm. 4.51]). *Let $\mathcal{A}$ be a hyperplane arrangement and $\mathcal{A} \neq \Phi_\ell$. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple. Any two of the following statements imply the third:*

$$\begin{aligned} \mathcal{A} \text{ is free with } \exp(\mathcal{A}) &= \{\{b_1, \dots, b_{l-1}, b_l\}\}, \\ \mathcal{A}' \text{ is free with } \exp(\mathcal{A}') &= \{\{b_1, \dots, b_{l-1}, b_l - 1\}\}, \\ \mathcal{A}'' \text{ is free with } \exp(\mathcal{A}'') &= \{\{b_1, \dots, b_{l-1}\}\}. \end{aligned}$$

Choose a hyperplane $H_0 = \ker \alpha_0 \in \mathcal{A}$. Let $\bar{S} = S/\alpha_0 S$. If $\theta \in D(\mathcal{A})$, then $\theta(\alpha_0 S) \subseteq \alpha_0 S$. Thus we may define $\bar{\theta} : \bar{S} \to \bar{S}$ by $\bar{\theta}(f + \alpha_0 S) = \theta(f) + \alpha_0 S$. Then $\bar{\theta} \in D(\mathcal{A}'')$, [OT92, Def. 4.43, Prop. 4.44].

Theorem 3.1.6 ([OT92, Thm. 4.46]). *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are free arrangements with $\mathcal{A}' := \mathcal{A} \setminus \{H_0\}$, $H_0 := \ker \alpha_0$. Then there is a basis $\{\theta_1, \dots, \theta_\ell\}$ for $D(\mathcal{A}')$ such that*

(1) $\{\theta_1, \dots, \theta_{i-1}, \alpha_0\theta_i, \theta_{i+1}, \dots, \theta_\ell\}$ is a basis for $D(\mathcal{A})$,

(2) $\{\bar{\theta}_1, \dots, \bar{\theta}_{i-1}, \bar{\theta}_{i+1}, \dots, \bar{\theta}_\ell\}$ is a basis for $D(\mathcal{A}'')$.

One of the most important results about free arrangements in general is Terao's Factorization theorem:

Theorem 3.1.7 (Factorization [OT92, Thm. 4.137])**.** *If $\mathcal{A}$ is a free arrangement with* $\exp(\mathcal{A}) = \{\{b_1, \ldots, b_\ell\}\}$*, then*

$$\chi_{\mathcal{A}}(t) = \prod_{i=1}^{\ell}(t - b_i).$$

A very recent and remarkable result is due to Abe which connects the division of characteristic polynomials with freeness:

Theorem 3.1.8 (Division theorem [Abe16, Thm. 1.1])**.** *Let $\mathcal{A}$ be a hyperplane arrangement and $\mathcal{A} \neq \Phi_\ell$. Assume that there is a hyperplane $H \in \mathcal{A}$ such that $\chi_{\mathcal{A}^H}(t)$ divides $\chi_{\mathcal{A}}(t)$ and $\mathcal{A}^H$ is free. Then $\mathcal{A}$ is free.*

3.1.1 Inductively, recursively and divisionally free arrangements

Theorem 3.1.5 motivates the following two definitions of classes of free arrangements.

Definition 3.1.9 ([OT92, Def. 4.53])**.** The class $\mathcal{IF}$ of *inductively free* arrangements is the smallest class of arrangements which satisfies

(1) The empty arrangement Φ_ℓ of rank ℓ is in $\mathcal{IF}$ for $\ell \geq 0$,

(2) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A}$ also belongs to $\mathcal{IF}$.

Example 3.1.10. All supersolvable arrangements (see Section 3.4) are inductively free by [OT92, Thm. 4.58].

Definition 3.1.11 ([OT92, Def. 4.60])**.** The class $\mathcal{RF}$ of *recursively free* arrangements is the smallest class of arrangements which satisfies

(1) The empty arrangement Φ_ℓ of rank ℓ is in $\mathcal{RF}$ for $\ell \geq 0$,

(2) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A}' \in \mathcal{RF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A}$ also belongs to $\mathcal{RF}$,

(3) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{RF}$, $\mathcal{A} \in \mathcal{RF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A})$, then $\mathcal{A}'$ also belongs to $\mathcal{RF}$.

Note that we have:

$$\mathcal{IF} \subsetneq \mathcal{RF} \subsetneq \{ \textit{ free arrangements } \},$$

where the properness of the last inclusion was established by Cuntz and Hoge in [CH15b].

Furthermore, similarly to the class $\mathcal{IF}$ of inductively free arrangements, Theorem 3.1.8 motivates the following class of free arrangements:

Definition 3.1.12 ([Abe16, Def. 4.3]). The class $\mathcal{DF}$ of *divisionally free* arrangements is the smallest class of arrangements which satisfies

(1) If $\mathcal{A}$ is an ℓ-arrangement, $\ell \leq 2$, or $\mathcal{A} = \Phi_\ell$, $\ell \geq 3$, then $\mathcal{A}$ belongs to $\mathcal{DF}$,

(2) if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{DF}$ and $\chi_{\mathcal{A}^{H_0}}(t) \mid \chi_{\mathcal{A}}(t)$, then $\mathcal{A}$ also belongs to $\mathcal{DF}$.

Abe showed that the new class of divisionally free arrangements properly contains the class of inductively free arrangements:

$$\mathcal{IF} \subsetneq \mathcal{DF},$$

by [Abe16, Thm. 1.6]. He conjectured that there are arrangements which are divisionally free but not recursively free. Our classification of recursively free reflection arrangements in this paper provides examples to confirm his conjecture (see Theorem 4.1 and Section 4.2).

The next easy lemma will be useful to disprove the recursive freeness of a given arrangement:

Lemma 3.1.13. *Let $\mathcal{A}$ and $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ be free arrangements and $L := L(\mathcal{A}')$. Let $P_H := \{X \in L_2 \mid X \subseteq H\} = \mathcal{A}'' \cap L_2$, then $\sum_{X \in P_H}(|\mathcal{A}'_X| - 1) \in \exp(\mathcal{A}')$, and if $\mathcal{A}'$ is irreducible then $\sum_{X \in P_H}(|\mathcal{A}'_X| - 1) \neq 1$.*

Proof. By Theorem 3.1.6 and (3.1.3) there is a $b \in \exp(\mathcal{A}')$, such that $|\mathcal{A}^H| = |\mathcal{A}'| - b$ and if $\mathcal{A}'$ is irreducible, then $b \neq 1$. Since $|\mathcal{A}^H| = |\mathcal{A}'| - \sum_{X \in P_H}(|\mathcal{A}'_X| - 1)$, the claim directly follows. □

The next two results are due to Hoge, Röhrle, and Schauenburg, [HRS15].

Proposition 3.1.14 ([HRS15, Thm. 1.1]). *Let $\mathcal{A}$ be a recursively free arrangement and $X \in L(\mathcal{A})$. Then $\mathcal{A}_X$ is recursively free.*

Hoge and Röhrle have shown in [HR15, Prop. 2.10] that the product construction is compatible with the notion of inductively free arrangements.

The following refines the statement for recursively free arrangements:

Proposition 3.1.15 ([HRS15, Thm. 1.2]). *Let $(\mathcal{A}_1, V_1), (\mathcal{A}_2, V_2)$ be two arrangements. Then $\mathcal{A} = (\mathcal{A}_1 \times \mathcal{A}_2, V_1 \oplus V_2)$ is recursively free if and only if both $(\mathcal{A}_1, V_1)$ and $(\mathcal{A}_2, V_2)$ are recursively free and in that case the multiset of exponents of $\mathcal{A}$ is given by* $\exp(\mathcal{A}) = \exp(\mathcal{A}_1) \cup \exp(\mathcal{A}_2)$.

3.2 Reflection arrangements

Let $V = \mathbb{C}^\ell$ be a finite dimensional complex vector space. An element $s \in \mathrm{GL}(V)$ of finite order with fixed point set $V^s = \{x \in V \mid sx = x\} = H_s$ a hyperplane in V is called a *reflection*. A finite subgroup $W \leq \mathrm{GL}(V)$ which is generated by reflections is called a *finite complex reflection group*.

The (irreducible) finite complex reflection groups were classified by Shephard and Todd, [ST54]. There is one infinite series $G(r, e, \ell)$, parametrized by positive integers $r, e, \ell \in \mathbb{N}$ with $e \mid r$. The series $G(r, e, \ell)$ includes the Coxeter groups of type A, BC, and D, i.e. $A_\ell = G(1, 1, \ell + 1)$, $BC_\ell = G(2, 1, \ell)$ and $D_\ell = G(2, 2, \ell)$. Furthermore, there are 34 "exceptional" groups $G_4, \ldots, G_{37}$.

Let $W \leq \mathrm{GL}(V)$ be a finite complex reflection group acting on the vector space V. The *reflection arrangement* $(\mathcal{A}(W), V)$ is the arrangement of hyperplanes consisting of all the reflecting hyperplanes of reflections of W.

Terao [Ter80b, Thm. 2] has shown that each reflection arrangement is free, see also [OT92, Prop. 6.59].

The complex reflection group W is called *reducible* if $V = V_1 \oplus V_2$ where V_i are stable under W. Then the restriction W_i of W to V_i is a reflection group in V_i. In this case the reflection arrangement $(\mathcal{A}(W), V)$ is the product of the two reflection arrangements $(\mathcal{A}(W_1), V_1)$ and $(\mathcal{A}(W_2), V_2)$. The complex reflection group W is called *irreducible* if it is not reducible, and then the reflection arrangement $\mathcal{A}(W)$ is irreducible.

For later purposes, we now look at the action of a finite complex reflection group W on its associated reflection arrangement $\mathcal{A}(W)$ and (reflection) subarrangements $\mathcal{A}(W') \subseteq \mathcal{A}(W)$ corresponding to reflection subgroups $W' \leq W$.

Let W be a finite complex reflection group and $\mathcal{A} := \mathcal{A}(W)$. Then W acts on the set $\mathscr{A} := \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\}$ of subarrangements of $\mathcal{A}$ by $w.\mathcal{B} = \{w.H \mid H \in \mathcal{B}\}$ for $\mathcal{B} \in \mathscr{A}$. The *(setwise) stabilizer* $S_\mathcal{B}$ of $\mathcal{B}$ in W is defined by $S_\mathcal{B} = \{w \in W \mid w.\mathcal{B} = \mathcal{B}\}$. We denote by $W.\mathcal{B} = \{w.\mathcal{B} \mid w \in W\} \subseteq \mathscr{A}$ the orbit of $\mathcal{B}$ under W.

The following lemma is similar to a statement from [OT92, Lem. 6.88].

Lemma 3.2.1. *Let W be a finite complex reflection group, $\mathcal{A} := \mathcal{A}(W)$, and $\mathscr{A} = \{\mathcal{B} \mid \mathcal{B} \subseteq \mathcal{A}\}$. Let $\mathcal{B} := \mathcal{A}(W') \in \mathscr{A}$ be a reflection subarrangement for a reflection subgroup $W' \leq W$. Then $S_\mathcal{B} = N_W(W')$ and $|W.\mathcal{B}| = |W : S_\mathcal{B}| = |W : N_W(W')|$.*

Proof. Let W, W', $\mathcal{A}$, and $\mathcal{B}$ be as above. Let $S_\mathcal{B}$ be the stabilizer of $\mathcal{B}$ in W. It is clear by the Orbit-Stabilizer-Theorem that $|W.\mathcal{B}| = |W : S_\mathcal{B}|$. Let $H_r \in \mathcal{B}$ for a reflection $r \in W'$, then $w.H_r = H_{w^{-1}rw}$. So we have

$$\begin{aligned} S_\mathcal{B} = \ & \{w \in W \mid w.H_r \in \mathcal{B} \text{ for all reflections } r \in W'\} \\ = \ & \{w \in W \mid w^{-1}rw \in W' \text{ for all reflections } r \in W'\} \\ = \ & N_W(W'). \end{aligned}$$

The last equality is because W' is by definition generated by the reflections it contains and the group normalizing all generators of W' is the normalizer of W'. □

If $W \leq \mathrm{GL}(V)$ with V a real vector space is a finite real reflection group, i.e. W is a finite Coxeter group, then we call $(\mathcal{A}(W), V)$ a *Coxeter arrangement*. Similarly if W is a Weyl group, i.e. a crystallographic Coxexter group, $\mathcal{A}(W)$ is called a *Weyl arrangement*. Finite Coxeter groups are completely classified by their Coxeter graphs, a combinatorial datum attached to each such group, see [Hum90]. In Chapter 5 we will generalize this notion to all simplicial arrangements.

3.3 Simplicial arrangements

Many of the notions in this section were introduced in the more general setting of simplicial arrangements on convex cones and Tits arrangements in [CMW16].

We firstly recall the definition of a simplicial arrangement.

Definition 3.3.1. Let $\mathcal{A}$ be an arrangement in a finite dimensional real vector space V. Then $\mathcal{A}$ is called *simplicial* if every connected component of $V \setminus \bigcup_{H \in \mathcal{A}} H$ is an open simplicial cone. We denote by $\mathcal{K}(\mathcal{A})$ the set of connected components of $V \setminus \bigcup_{H \in \mathcal{A}} H$; a $K \in \mathcal{K}(\mathcal{A})$ is called a *chamber*.

Note that the only simplicial 1-arrangement is the arrangement $\mathcal{A} = \{\{0\}\}$, i.e. the non empty one, and that every real 2-arrangement with more than one hyperplane is simplicial.

There are the following classical examples for simplicial arrangements.

Example 3.3.2. Let $W \leq \mathrm{GL}(V)$ be a finite real reflection group acting on the real vector space V, i.e. a finite Coxeter group (see Section 3.2). Suppose that W has full rank, i.e. $\mathrm{rk}(W) = \dim(V)$. Then the reflection arrangement $(\mathcal{A}(W), V)$, (also called Coxeter arrangement) is simplicial.

We will frequently consider examples of simplicial 3-arrangements from Grünbaum's list, in particular in Chapter 5 and Chapter 6. We then use his notation and numbering where an arrangement is denoted by $\mathcal{A}(n, k)$ (c.f. Definition 6.1.6) where n is the number of hyperplanes and k some natural number, see [Grü09].

Example 3.3.3. For $0 \leq k \leq \ell$ let $\mathcal{A}_\ell^k$ be the ℓ-arrangement defined as follows

$$\begin{aligned} \mathcal{A}_\ell^k := \quad & \{\ker(x_i - x_j) \mid 1 \leq i < j \leq \ell\} \\ & \cup \{\ker(x_i) \mid 1 \leq i \leq k\}. \end{aligned}$$

The arrangements $\mathcal{A}_\ell^k$ are simplicial, cf. [CH15a]. In particular $\mathcal{A}_\ell^0 = \mathcal{A}(D_\ell)$ and $\mathcal{A}_\ell^\ell = \mathcal{A}(B_\ell)$ are the reflection arrangements of the finite reflection groups of type D_ℓ and B_ℓ respectively. Figure 3.1 displays $\mathcal{A}_3^0 = \mathcal{A}(6, 1)$ up to $\mathcal{A}_3^3 = \mathcal{A}(9, 1)$.

We recall the following combinatorial characterization of simplicial 3-arrangements.

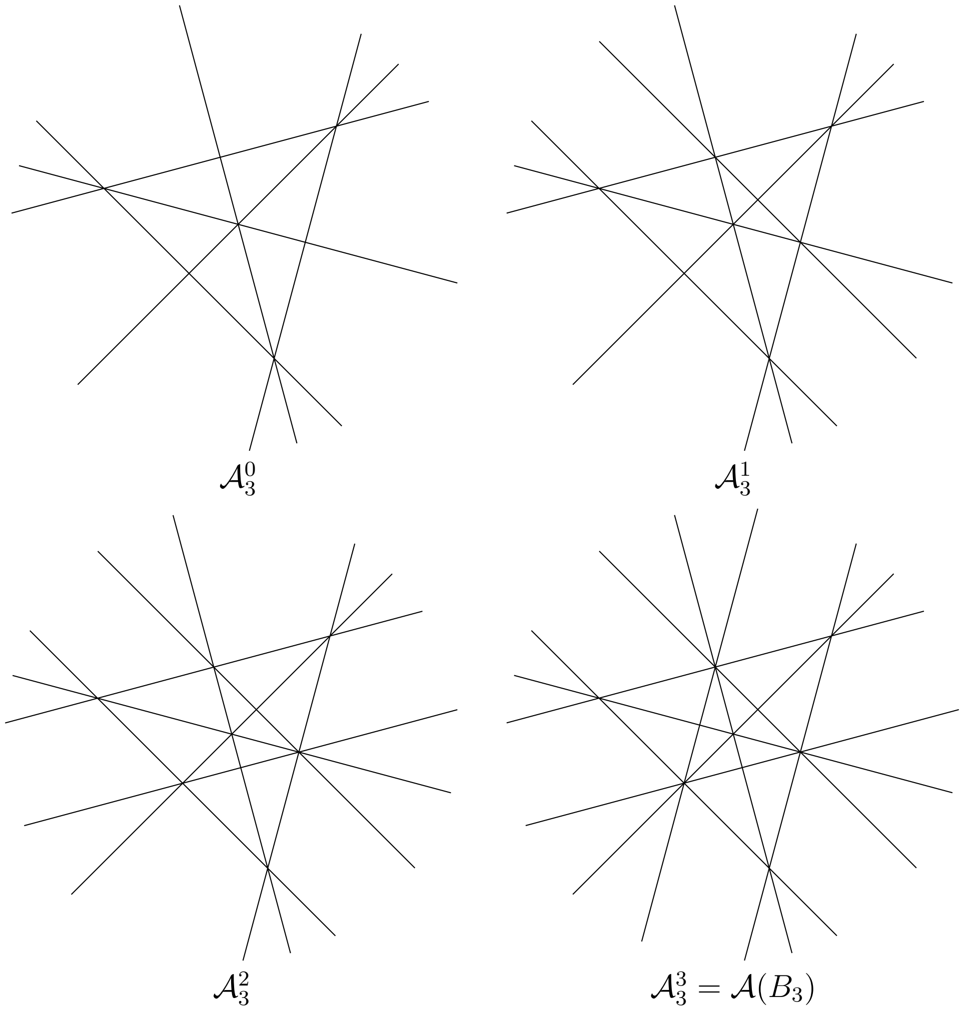


Figure 3.1: Projective pictures of $\mathcal{A}_3^0$, $\mathcal{A}_3^1$, $\mathcal{A}_3^2$, and $\mathcal{A}_3^3 = \mathcal{A}(B_3)$.

Lemma 3.3.4. *[CG15, Cor. 2.7] Let $\mathcal{A}$ be a 3-arrangement. Then $\mathcal{A}$ is simplicial if and only if*

$$\mu_2 := \sum_{X \in L_2(\mathcal{A})} (|\mathcal{A}_X| - 1) = 2|L_2(\mathcal{A})| - 3.$$

More generally real simplicial ℓ-arrangements are characterized by the next combinatorial property.

Lemma 3.3.5. *[CG15, Cor. 2.4] Let $\mathcal{A}$ be an ℓ-arrangement. Then $\mathcal{A}$ is simplicial if and only if*

$$\ell|\chi_{\mathcal{A}}(-1)| - 2\sum_{H \in \mathcal{A}} |\chi_{\mathcal{A}^H}(-1)| = 0.$$

Definition 3.3.6. Let $\mathbb{K}$ be any field and $\mathcal{A}$ an arrangement in $V \cong \mathbb{K}^\ell$. Define

$$s(\mathcal{A}) := \ell|\chi_{\mathcal{A}}(-1)| - 2\sum_{H \in \mathcal{A}} |\chi_{\mathcal{A}^H}(-1)|.$$

If $\mathcal{A}$ satisfies $s(\mathcal{A}) = 0$ then $\mathcal{A}$ is called *combinatorially simplicial*, see [CG15].

Simpliciality, at least geometrically for real arrangements, is compatible with taking localizations and restrictions, compare with the more general statements in [CMW16].

Lemma 3.3.7. *Let $\mathcal{A}$ be a simplicial arrangement over $\mathbb{R}$ and $X \in L(\mathcal{A})$. Then we have*

(1) $(\mathcal{A}_X/X, V/X)$ is simplicial,

(2) $(\mathcal{A}^X, X)$ is simplicial.

Proof. The walls $H_1, \dots, H_{r(X)}$ of a chamber K_X in $\mathcal{A}_X$ are a subset of the walls of a chamber $K \in \mathcal{K}(\mathcal{A})$ so by Remark 3.3.14 we obtain the first statement. If $\alpha_1, \dots, \alpha_{r(X)}$ are corresponding normals of these walls pointing to the inside of K and also K_X then they are linearly independent, hence K_X/X is a simplicial cone and $\mathcal{A}_X/X$ is simplicial.

Since every face of a simplicial cone is a simplicial cone, Statement (2) follows directly. □

Example 3.3.8. Let $\mathcal{A} = \mathcal{A}(W)$ be the Coxeter arrangement of the finite real reflection group W in V and let $X \in L(\mathcal{A})$. Then $\mathcal{A}_X/X$ is a reflection arrangement, namely the Coxeter arrangement of a parabolic subgroup of W. The arrangement $\mathcal{A}_X/X$ is simplicial in accordance with Lemma 3.3.7(1).

In the next example we see that the bigger class of combinatorially simplicial arrangements defined over arbitrary fields is neither closed under taking localizations nor closed under taking restrictions.

Example 3.3.9. Let $V = \mathbb{C}^4$, $\zeta = -\frac{1}{2}(1-\sqrt{3}i)$ be a primitive third root of unity and $(\mathcal{A}, V)$ the complex 4-arrangement containing 18 hyperplanes and defined by

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & -\zeta & -\zeta^2 \end{pmatrix}.$$

Note that $\mathcal{A}$ is a subarrangement of the reflection arrangement of the complex reflection group $G(3,1,4)$, see [OT92, Ch. 6.4] for a definition of these reflection arrangements. This is to say if

$$\begin{aligned} \mathcal{B} &:= \mathcal{A}(G(3,1,4)) \\ &= \{\ker(x_i - \zeta^k x_j) \mid 1 \le i < j \le 4,\, 0 \le k \le 2\} \\ &\quad \cup \{\ker(x_i) \mid 1 \le i \le 4\}, \end{aligned}$$

then we obtain $\mathcal{A}$ by removing 4 hyperplanes,

$$\mathcal{A} = \mathcal{B} \setminus \{\ker(x_1), \ker(x_2), \ker(x_3), \ker(x_3 - x_4)\}.$$

A quick calculation shows that $\mathcal{A}$ satisfies $s(\mathcal{A}) = 0$ so it is combinatorially simplicial. While for the reflection arrangement $\mathcal{B}$ all localizations and restrictions are again combinatorially simplicial, localizing $\mathcal{A}$ at the rank 3 intersection $X = H_1 \cap H_2 \cap H_3 \in L(\mathcal{A})$, where the hyperplane H_i corresponds to the i-th column of the defining matrix above, yields the 3-arrangement $\mathcal{C} = \mathcal{A}_X/X$. It contains 10 hyperplanes and is given by

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -\zeta & -\zeta^2 & -1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & -\zeta & -\zeta^2 & -1 \end{pmatrix}.$$

For $\mathcal{C}$ we have $s(\mathcal{C}) = 4$, so it is not combinatorially simplicial.

Now let $H = H_8 = (1, 0, 0, -\zeta)^\perp \in \mathcal{A}$. Then $\mathcal{D} := \mathcal{A}^H$ contains 10 hyperplanes and may be defined by

$$\begin{pmatrix} 1 & \zeta & 1 & 0 & 0 & 0 & 0 & -1 & \zeta & 1 \\ 0 & 0 & 0 & 1 & \zeta & 1 & 0 & \zeta & -1 & -1 \\ 0 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

For $\mathcal{D}$ we have $s(\mathcal{D}) = 4$, thus it is also not combinatorially simplicial.

The product construction described above is compatible with simpliciality.

Proposition 3.3.10. *Let $\mathcal{A}_1$, $\mathcal{A}_2$ be combinatorially simplicial arrangements in $\mathbb{K}^{\ell_1}$ and $\mathbb{K}^{\ell_2}$ respectively. Then the product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is combinatorially simplicial.*

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be combinatorially simplicial. Then by Lemma 3.3.5 we have

$$\ell_1|\chi_{\mathcal{A}_1}(-1)| - 2\sum_{H\in\mathcal{A}_1} |\chi_{\mathcal{A}_1^H}(-1)| = 0,$$

and

$$\ell_2|\chi_{\mathcal{A}_1}(-1)| - 2\sum_{H\in\mathcal{A}_2} |\chi_{\mathcal{A}_2^H}(-1)| = 0.$$

By Lemma 2.5 we have $\chi_{\mathcal{A}}(t) = \chi_{\mathcal{A}_1}(t)\chi_{\mathcal{A}_2}(t)$. By Corollary 2.3 we get

$$\begin{aligned} \ell|\chi_{\mathcal{A}}(-1)| - 2\sum_{H\in\mathcal{A}} |\chi_{\mathcal{A}^H}(-1)| &= (\ell_1+\ell_2)|\chi_{\mathcal{A}_1}(-1)\chi_{\mathcal{A}_2}(-1)| \\ &\quad -2\sum_{H\in\mathcal{A}_1} |\chi_{\mathcal{A}_2}(-1)\chi_{\mathcal{A}_1^H}(-1)| - 2\sum_{H\in\mathcal{A}_2} |\chi_{\mathcal{A}_1}(-1)\chi_{\mathcal{A}_2^H}(-1)| \\ &= |\chi_{\mathcal{A}_2}(-1)|(\ell_1|\chi_{\mathcal{A}_1}(-1)| - 2\sum_{H\in\mathcal{A}_1} |\chi_{\mathcal{A}_1^H}(-1)|) \\ &\quad + |\chi_{\mathcal{A}_1}(-1)|(\ell_2|\chi_{\mathcal{A}_2}(-1)| - 2\sum_{H\in\mathcal{A}_2} |\chi_{\mathcal{A}_2^H}(-1)|) \\ &= 0 + 0 = 0. \end{aligned}$$

Hence $\mathcal{A}$ is combinatorially simplicial. □

Proposition 3.3.11. *Let $\mathcal{A}_1$ be an arrangement in $\mathbb{R}^{\ell_1}$ and let $\mathcal{A}_2$ be an arrangement in $\mathbb{R}^{\ell_2}$. Then the product $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is simplicial if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both simplicial.*

Proof. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are simplicial, then $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ is simplicial by Proposition 3.3.10.

Conversely, let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be simplicial. Then $\mathcal{A}_i$ is isomorphic to $\mathcal{A}_{X_i}/X_i$ for $i = 1, 2$ as $r(X_i)$-arrangements in V/X_i where $X_i = \{0\} \oplus V_{3-i}$. But these localizations regarded as essential arrangements in quotient spaces are simplicial by Lemma 3.3.7. □

Combinatorial simpliciality of $\mathcal{A}_1 \times \mathcal{A}_2$ does not imply combinatorial simpliciality of $\mathcal{A}_1$ and $\mathcal{A}_2$ in general:

Example 3.3.12. Let ζ, $\mathcal{A}$ and $\mathcal{D}$ be as in Example 3.3.9. Let $\mathcal{A}_1 = \mathcal{D}$ and $\mathcal{A}_2 = \mathcal{A}^H$ where $H = H_5 = (1, 0, -\zeta, 0)^\perp$ as in Example 3.3.9. Define $\omega := \frac{1}{3}(1 - \zeta)$. Then $\mathcal{A}_2$ is given by

$$\begin{pmatrix} 1 & 0 & \omega & \omega & \omega & \omega & \omega & \omega & 0 & 0 \\ 0 & 0 & 1 & \zeta & \zeta^2 & 0 & 0 & 0 & \zeta & 1 \\ 0 & 1 & 0 & 0 & 0 & -\zeta & -\zeta^2 & -1 & 1 & 1 \end{pmatrix}.$$

Recall that for the non combinatorially simplicial arrangement $\mathcal{A}_1$ we have $s(\mathcal{A}_1) = 4$. Furthermore, $\chi_{\mathcal{A}_1}(t) = (t-1)(t-4)(t-5) = \chi_{\mathcal{A}_2}(t)$, and $s(\mathcal{A}_2) = -4$. So $\mathcal{A}_2$ is neither combinatorially simplicial. But, similar to the proof of Proposition 3.3.11, for $\mathcal{A}_1 \times \mathcal{A}_2$ we have

$$\begin{aligned} s(\mathcal{A}_1 \times \mathcal{A}_2) &= |\chi_{\mathcal{A}_2}(-1)| s(\mathcal{A}_1) + |\chi_{\mathcal{A}_1}(-1)| s(\mathcal{A}_2) \\ &= |\chi_{\mathcal{A}_1}(-1)| s(\mathcal{A}_1) + |\chi_{\mathcal{A}_1}(-1)| s(\mathcal{A}_2) \\ &= |\chi_{\mathcal{A}_1}(-1)| (s(\mathcal{A}_1) + s(\mathcal{A}_2)) \\ &= |\chi_{\mathcal{A}_1}(-1)| (4 - 4) \\ &= 0. \end{aligned}$$

So the product $\mathcal{A}_1 \times \mathcal{A}_2$ is combinatorially simplicial.

Now, we introduce some further technical notions.

Definition 3.3.13. For $\alpha \in V^*$ we write $\alpha^+ = \alpha^{-1}(\mathbb{R}_{>0})$ and $\alpha^- = (-\alpha)^+$ for the positive respectively negative open half-space defined by α.

For $K \in \mathcal{K}(\mathcal{A})$ define the *walls* of K as

$$W^K := \{H \in \mathcal{A} \mid \dim(H \cap \overline{K}) = \ell - 1\}.$$

If $R \subseteq V^*$ is a finite set such that $\mathcal{A} = \{\alpha^\perp \mid \alpha \in R\}$ and $\mathbb{R}\alpha \cap R = \{\pm\alpha\}$ for all $\alpha \in R$ then R is called a *(reduced) root system* for $\mathcal{A}$.

If $B^K \subseteq V^*$ such that $|B^K| = |W^K|$, $W^K = \{\alpha^\perp \mid \alpha \in B^K\}$ and $K = \cap_{\alpha \in B^K} \alpha^+$ then B^K is called a *basis* for K.

If R is a root system for $\mathcal{A}$ we obtain a basis for K as

$$B^K_R := \{\alpha \in R \mid \alpha^\perp \in W^K \text{ and } \alpha^+ \cap K = K\}.$$

Furthermore, for $\gamma \in B^K$ let $K\gamma$ be the unique adjacent chamber in $\mathcal{K}(\mathcal{A})$, such that $\langle \overline{K} \cap \overline{K\gamma} \rangle = \gamma^\perp$. If there is a chosen numbering of $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$ then we simply write $K_i = K\alpha_i$.

Remark 3.3.14. The notions W^K, R and B^K make also sense for a not necessarily simplicial real arrangement $\mathcal{A}$. Since the normals of the facets of a cone constitute a basis if and only if the cone is simplicial, we observe that B^K is indeed a basis of V^* for all $K \in \mathcal{K}(\mathcal{A})$ if and only if $\mathcal{A}$ is simplicial.

The following notion was first introduced in [Cun11a, Def. 2.3].

Definition 3.3.15. Let $\mathcal{A}$ be a simplicial arrangement. If there exists a root system $R \subseteq V^*$ for $\mathcal{A}$ such that for all $K \in \mathcal{K}(\mathcal{A})$ we have

$$R \subseteq \sum_{\alpha \in B_R^K} \mathbb{Z}\alpha,$$

then $\mathcal{A}$ is called *crystallographic* and in this case we call R a *crystallographic root system* for $\mathcal{A}$.

Example 3.3.16. Let W be a Weyl group, i.e. a crystallographic finite real reflection group with (reduced) root system $\Phi(W)$. Then the Weyl arrangement $\mathcal{A}(W) = \{\alpha^\perp \mid \alpha \in \Phi(W)\}$ is a crystallographic arrangement with crystallographic root system $R = \Phi(W)$.

A complete classification of crystallographic arrangements by finite Weyl groupoids was obtained in [CH15a], see also [Cun11a]. It is worth mentioning that the class of crystallographic arrangements is much bigger than the class of Weyl arrangements with many more (74) sporadic cases. However, it turns out that irreducible crystallographic arrangements of rank greater or equal to 4 are all restrictions of (irreducible) Weyl arrangements (see for example [CL17, Thm. 3.7]):

Theorem 3.3.17. *Let $\mathcal{A}$ be an irreducible simplicial ℓ-arrangement with $\ell \geq 4$. Then it is crystallographic if and only if it is isomorphic to a restriction of some (irreducible) Weyl arrangement.*

Investigating the geometry of adjacent chambers we firstly obtain the following lemma.

Lemma 3.3.18. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement and $K \in \mathcal{K}(\mathcal{A})$ with basis $B^K = \{\alpha_1, \dots, \alpha_\ell\}$. Then for $1 \leq i, j \leq \ell$ there are $c_{ij}^K \in \mathbb{R}$ such that*

$$\{\beta_j^i = \alpha_j - c_{ij}^K \alpha_i \mid j = 1, \dots, \ell\}$$

is a basis for K_i. If $i \neq j$ then $c_{ij}^K \leq 0$ and c_{ij}^K is uniquely determined by B^K. If $i = j$ then $c_{ij}^K > 0$.

Proof. Since B^K is a basis for V^*, the uniqueness of the c_{ij}^K for $i \neq j$ directly follows. By [Cun11a, Lem. 2.2] we have $\beta_j^i \in \pm \sum_{\alpha \in B^K} \mathbb{R}_{\geq 0}\alpha$. Hence $c_{ij}^K \leq 0$ for $i \neq j$ and $c_{ii}^K > 0$. □

Definition 3.3.19. Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$ with basis $B^K = \{\alpha_1, \dots, \alpha_\ell\}$ for K. For $i \neq j$ let c_{ij}^K be the uniquely determined coefficients from Lemma 3.3.18. For $1 \leq i \leq \ell$ we set $c_{ii}^K = 2$ and define the linear map $\sigma_{\alpha_i}^K := \sigma_i^K$ by

$$\sigma_i^K(\alpha_j) := \alpha_j - c_{ij}^K \alpha_i$$

for $1 \leq j \leq \ell$. With respect to the basis B^K this map is represented by the matrix

$$S_i^K := \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & \ddots & & & & \\ -c_{i1}^K & \cdots & -c_{i(i-1)}^K & -1 & -c_{i(i+1)}^K & \cdots & -c_{i\ell}^K \\ & & & & \ddots & & \\ & & & & & 1 & 0 \\ & & & & & 0 & 1 \end{pmatrix}.$$

Remark 3.3.20. We observe that σ_i^K is a reflection at the hyperplane $\alpha_i^\perp$. In particular $\det(S_i^K) = -1$. From the preceding definition we observe that $c_{ij}^K \neq 0$ if and only if $c_{ji}^K \neq 0$ (cf. [Cun11a]).

Definition 3.3.21. Let $\mathcal{A}$ be an arrangement with chambers $\mathcal{K}(\mathcal{A})$. A sequence $(K^0, K^1, \dots, K^{n-1}, K^n)$ of distinct chambers in $\mathcal{K}(\mathcal{A})$ is called a *gallery* if for all $1 \leq i \leq n$ we have $\langle \overline{K^{i-1}} \cap \overline{K^i} \rangle = H \in \mathcal{A}$, i.e. if K^i and K^{i-1} are adjacent with common wall H. We denote by $\mathcal{G}(\mathcal{A})$ the set of all galleries of $\mathcal{A}$.

We say that $G \in \mathcal{G}(\mathcal{A})$ has length n if it is a sequence of $n+1$ chambers. For $G = (K^0, \dots, K^n) \in \mathcal{G}(\mathcal{A})$ we denote by $b(G) = K^0$ the first chamber and by $e(G) = K^n$ the last chamber in G.

Definition 3.3.22. Let $\mathcal{A}$ be a simplicial ℓ-arrangement. We fix a chamber $K^0 \in \mathcal{K}(\mathcal{A})$. Let $\mathcal{G}(K^0, \mathcal{A}) = \{G \in \mathcal{G}(\mathcal{A}) \mid b(G) = K^0\}$ be the set of galleries starting with K^0.

Let $B^{K^0} = \{\alpha_1^0, \dots, \alpha_\ell^0\}$ be a basis for K^0. For $(K^0, \dots, K^n) = G \in \mathcal{G}(K^0, \mathcal{A})$ we denote by $B_G^{K^n} = B_G$ the basis for K^n induced by G and B^{K^0}, i.e. such that

$$B^{K^{i+1}} = \sigma_{\mu_i}^{K^i}(B^{K^i}) = \{\alpha_j^{i+1} = \sigma_{\mu_i}^{K^i}(\alpha_j^i) = \alpha_j^i - c_{\mu_i j}^{K^i} \alpha_{\mu_i}^i \mid 1 \leq j \leq \ell\},$$

where $K^{i+1} = K_{\mu_i}^i$, $\mu_i \in \{1, \dots, \ell\}$, and $0 \leq i \leq n-1$.

Definition 3.3.23. Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$. We call a basis $B^K = \{\alpha_1, \dots, \alpha_\ell\}$ *locally crystallographic* if the c_{ij}^K are all integers.

If B^K is a locally crystallographic basis then we call the matrix $C^K = (c_{ij}^K)_{i,j=1,\dots,\ell}$ the Cartan matrix of B^K.

Example 3.3.24. Let $\mathcal{A} = \mathcal{A}_\ell^k$. Then $\mathcal{A}$ is crystallographic with crystallographic root system R. In particular for $K \in \mathcal{K}(\mathcal{A})$ the basis B_R^K is a locally crystallographic basis for K and the corresponding Cartan matrix is (up to simultaneous permutation of columns and rows) one of the matrices displayed in Table 3.1, see [CH15a, Prop. 3.8].

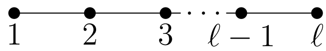
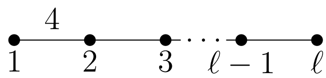
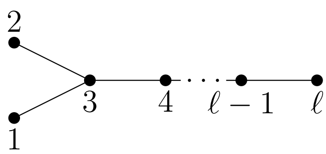
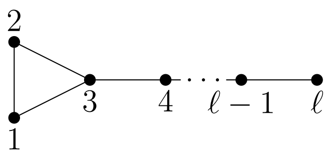
C^K	$\Gamma(K)$
$A_\ell : \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & \cdots & -1 & 2 \end{pmatrix}$	
$C_\ell : \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -2 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$	
$D_\ell : \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$	
$D'_\ell : \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$	

Table 3.1: Cartan matrices and Coxeter graphs.

Definition 3.3.25. Let B^K be a locally crystallographic basis with Cartan matrix C^K. If (up to simultaneous permutation of columns and rows) C^K is one of the matrices shown in the left column of Table 3.1 then we say C^K is of type A, C, D, or D' respectively.

If B^K is a locally crystallographic basis with Cartan Matrix of type A, C, D, or D' then the corresponding Coxeter graph $\Gamma(K)$ (see Section 5) is displayed in the right column of Table 3.1.

Lemma 3.3.26. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$ with basis $B^K = \{\alpha_1, \dots, \alpha_\ell\}$, and K_i an adjacent chamber. Then for all $1 \leq j \leq \ell$ we have $c_{ij}^{K_i} = c_{ij}^K$ and in particular $\sigma_i^K \circ \sigma_i^{K_i} = \sigma_i^{K_i} \circ \sigma_i^K = \mathrm{id}$.*

Proof. We have $\sigma_i^K(\alpha_j)^\perp = \beta_j^{i\perp} = (\alpha_j - c_{ij}^K \alpha_i)^\perp \in W^{K_i}$ but similarly $\sigma_i^{K_i}(\beta_j^i)^\perp = \alpha_j^\perp = (\beta_j^i - c_{ij}^{K_i} \beta_i^i)^\perp = (\alpha_j - c_{ij}^K \alpha_i - c_{ij}^{K_i}(-\alpha_i))^\perp \in W^K$. Thus $c_{ij}^K = c_{ij}^{K_i}$. □

Similarly to the crystallographic case we have the following.

Lemma 3.3.27 (cf. [CH09, Lem. 4.5])**.** *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, K, B^K, and K_i as before. Let $i \neq j$ and suppose $c_{ij}^K = 0$. Then $c_{jk}^{K_i} = c_{jk}^K$ for all $k \in \{1, \dots, \ell\}$.*

Proof. The proof is the same as in [CH09].

If $k = i$ then by Lemma 3.3.26 $c_{jk}^K = c_{kj}^K = 0 = c_{kj}^{K_i} = c_{jk}^{K_i}$. And if $k = j$ then all the coefficients are equal to 2. So let $k \in \{1, \dots, \ell\} \setminus \{i, j\}$. Since $c_{ij}^K = 0$ we have $|\mathcal{A}_{\alpha_i^\perp \cap \alpha_j^\perp}| = 2$. So application of $\sigma_j^{K_i} \circ \sigma_i^K$ and $\sigma_i^{K_j} \circ \sigma_j^K$ on α_k should yield a normal of the same wall of the chamber $K\alpha_i\sigma_i^K(\alpha_j) = K\alpha_j\sigma_j^K(\alpha_i)$. Now

$$\begin{aligned}
\sigma_j^{K_i}(\sigma_i^K(\alpha_k)) &= \sigma_i^K(\alpha_k) - c_{jk}^{K_i}\sigma_i^K(\alpha_j) \\
&= \alpha_k - c_{ik}^K\alpha_i - c_{jk}^{K_i}(\alpha_j - c_{ij}^K\alpha_i) \\
&= \alpha_k - c_{ik}^K\alpha_i - c_{jk}^{K_i}\alpha_j,
\end{aligned}$$

and similarly

$$\sigma_i^{K_j}(\sigma_j^K(\alpha_k)) = \alpha_k - c_{jk}^K\alpha_j - c_{ik}^{K_j}\alpha_i.$$

Since i, j, k are pairwise different and $\{\alpha_1 \dots, \alpha_\ell\}$ are linearly independent, comparing the coefficients of α_j in both terms gives $c_{jk}^{K_i} = c_{jk}^K$. □

We will continue the investigation of the combinatorial and geometric properties of simplicial arrangements in Chapter 5.

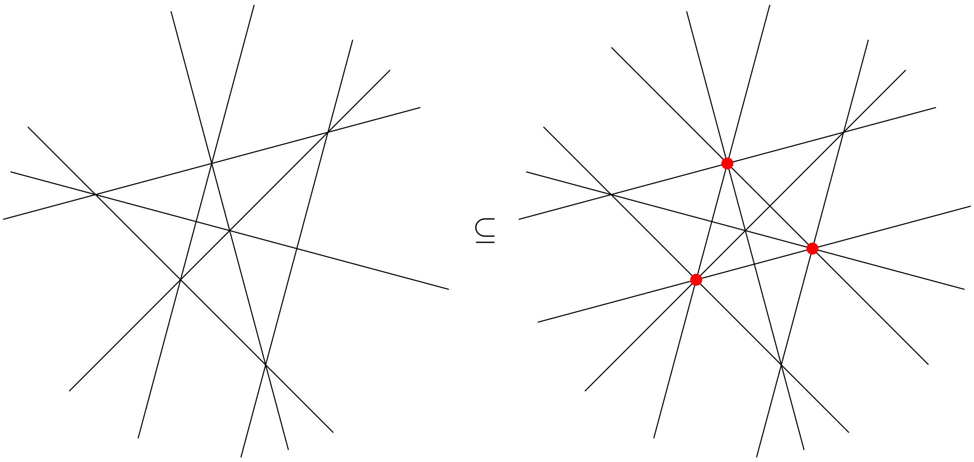


Figure 3.2: A non-supersolvable arrangement and the supersolvable (reflection) arrangement $\mathcal{A}(B_3)$.

3.4 Supersolvable arrangements

An element $X \in L(\mathcal{A})$ is called *modular* if $X + Y \in L(\mathcal{A})$ for all $Y \in L(\mathcal{A})$. An arrangement $\mathcal{A}$ with $r(\mathcal{A}) = \ell$ is called *supersolvable* if the intersection lattice $L(\mathcal{A})$ is supersolvable, i.e. there is a maximal chain of modular elements

$$V = X_0 < X_1 < \ldots < X_\ell = T(\mathcal{A}),$$

$X_i \in L(\mathcal{A})$ modular.

Example 3.4.1. The hyperplanes itself, $T(\mathcal{A})$, and V are always modular elements in $L(\mathcal{A})$. Hence all 1-arrangements and 2-arrangements are supersolvable.

Example 3.4.2. Let $\mathcal{A}$ be an essential 3-arrangement. Then $\mathcal{A}$ is supersolvable if there exists an $X \in L_2(\mathcal{A})$ which is connected to all other $Y \in L_2(\mathcal{A})$ by a suitable hyperplane $H \in \mathcal{A}$, i.e. $X + Y \in \mathcal{A}$.

Consider Figure 3.2 displaying a projective picture of a supersolvable arrangement and a non-supersolvable subarrangement. The three red points are modular. The arrangement on the right-hand side is actually a reflection arrangement (see 3.2).

Supersolvability is preserved by taking localizations and restrictions, see [AHR14b, Lem. 2.6], and [Sta72, Prop. 3.2]:

Lemma 3.4.3. *Let $\mathcal{A}$ be an arrangement , $X \in L(\mathcal{A})$ and $Y \in L(\mathcal{A})$ a modular element with $X \subseteq Y$. Then Y is modular in $L(\mathcal{A}_X)$. In particular if $\mathcal{A}$ is supersolvable, so is $\mathcal{A}_X$ for all $X \in L(\mathcal{A})$.*

Lemma 3.4.4. *Let $\mathcal{A}$ be an arrangement , $X \in L(\mathcal{A})$ and $Y \in L(\mathcal{A})$ a modular element. Then $Y \cap X$ is modular in $L(\mathcal{A}^X)$. In particular if $\mathcal{A}$ is supersolvable so is $\mathcal{A}^X$ for all $X \in L(\mathcal{A})$.*

Combining the previous two lemmas with Lemma 3.3.7 we obtain the following.

Lemma 3.4.5. *Let $\mathcal{A}$ be a supersolvable simplicial arrangement and $X \in L(\mathcal{A})$. Then we have*

(1) $(\mathcal{A}_X/X, V/X)$ *is supersolvable and simplicial,*

(2) $(\mathcal{A}^X, X)$ *is supersolvable and simplicial.*

Furthermore, supersolvability is compatible with products.

Lemma 3.4.6 ([HR14, Prop. 2.5])**.** *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product. Then $\mathcal{A}$ is supersolvable if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both supersolvable.*

So together with Proposition 3.3.11 we get the following.

Proposition 3.4.7. *Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product. Then $\mathcal{A}$ is supersolvable and simplicial if and only if $\mathcal{A}_1$ and $\mathcal{A}_2$ are both supersolvable and simplicial.*

Because of the previous proposition, to classify supersolvable and simplicial arrangements, it suffices to classify the irreducible ones.

The following property of the characteristic polynomial of a supersolvable arrangement is due to Stanly [Sta72], see also [OT92, Thm. 2.63].

Theorem 3.4.8. *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with*

$$V = X_0 < X_1 < \ldots < X_\ell = T(\mathcal{A})$$

a maximal chain of modular elements. Let $b_i := |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}|$ for $1 \leq i \leq \ell$. Then

$$\chi_{\mathcal{A}}(t) = \prod_{i=1}^{\ell} (t - b_i).$$

Theorem 3.4.9 ([OT92, Thm. 4.58])**.** *Let $\mathcal{A}$ be a supersolvable ℓ-arrangement with*

$$V = X_0 < X_1 < \ldots < X_\ell = T(\mathcal{A})$$

a maximal chain of modular elements. Let $b_i := |\mathcal{A}_{X_i} \setminus \mathcal{A}_{X_{i-1}}|$ for $1 \leq i \leq \ell$. Then $\mathcal{A}$ is inductively free with exponents

$$\exp(\mathcal{A}) = \{\{b_1, \ldots, b_\ell\}\}.$$

A helpful result is due to Amend, Hoge and Röhrle who classified the supersolvable restrictions of irreducible reflection arrangements, [AHR14b, Thm. 1.3]. Here we only need the following weaker version for real reflection arrangements of rank greater or equal to 4.

Theorem 3.4.10. *Let $\mathcal{A} = \mathcal{A}(W)$ be an irreducible real reflection arrangement of rank $\ell \geq 4$ associated to the finite reflection group W and $X \in L(\mathcal{A})$ with $m := \dim(X) \geq 4$. Then $\mathcal{A}^X$ is supersolvable if and only if one of the following holds:*

(1) $W = A_\ell$ *and then* $\mathcal{A}^X = \mathcal{A}(A_m)$

(2) $\mathcal{A}^X = \mathcal{A}_m^k$ *with* $k \in \{m, m-1\}$.

Together with Theorem 3.3.17 this gives us the following classification of irreducible supersolvable crystallographic arrangements of rank ≥ 4.

Theorem 3.4.11. *Let* $\mathcal{A}$ *be an irreducible supersolvable crystallographic* ℓ*-arrangement with* $\ell \geq 4$. *Then* $\mathcal{A}$ *is isomorphic to one of the reflection arrangements* $\mathcal{A}(A_\ell)$, $\mathcal{A}(B_\ell)$ *or isomorphic to* $\mathcal{A}_\ell^{\ell-1} = \mathcal{A}(B_\ell) \setminus \{\{x_1 = 0\}\}$.

We will continue the investigation of supersolvable simplicial arrangements in the last chapter.

4 Recursively free reflection arrangements

In this chapter we prove Theorem I giving a classification of recursively free reflection arrangements.

For the special case $W \cong G_{31}$, we obtain a (with respect to "Addition" and "Deletion") isolated cluster of free but not recursively free subarrangements of $\mathcal{A}(W)$ in dimension 4.

In Section 4.2 equipped with Theorem I and results from the previous section, we are able to positively settle a conjecture by Abe [Abe16, Conj. 5.11] about his new class of divisionally free arrangements, which we state as the next theorem.

Theorem 4.1. *There is an arrangement $\mathcal{A}$ such that $\mathcal{A} \in \mathcal{DF}$ and $\mathcal{A} \notin \mathcal{RF}$.*

In the last section we will comment on the situation of a restriction of a reflection arrangement.

In order to compute the different intersection lattices of the reflection arrangements in question, to obtain the respective invariants, and to recheck our results we used the functionality of the GAP computer algebra system, [GAP14].

The following theorem proved by Barakat, Cuntz, Hoge and Röhrle, which provides a classification of all inductively free reflection arrangements, is our starting point for inspecting the recursive freeness of reflection arrangements:

Theorem 4.2 ([HR15, Thm. 1.1], [BC12, Thm. 5.14])**.** *For W a finite complex reflection group, the reflection arrangement $\mathcal{A}(W)$ is inductively free if and only if W does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geq 3$, G_{24}, G_{27}, G_{29}, G_{31}, G_{33}, or G_{34}.*

Thus, to prove Theorem I, we only have to check the non-inductively free cases from Theorem 4.2 since inductive freeness implies recursive freeness.

4.1 Proof of Theorem I

Thanks to Proposition 3.1.15, the proof of Theorem I reduces to the case when $\mathcal{A}(W)$ respectively W are irreducible. We consider the different irreducible reflection arrangements provided by Theorem 4.2 which are not inductively free in turn.

4.1.1 The reflection arrangements $\mathcal{A}(G(r,r,\ell))$, $r,\ell \geq 3$

For an integer $r \geq 2$ let $\theta = \exp(2\pi i/r)$, and $C(r)$ the cyclic group generated by θ. The reflection arrangement $\mathcal{A}(W)$ with $W = G(r,r,\ell)$ contains the hyperplanes

$$H_{i,j}(\zeta) := \ker(x_i - \zeta x_j),$$

with $i,j \leq \ell$ and $i \neq j$, $\zeta \in C(r)$, and if W is the full monomial group $G(r,1,\ell)$, then $\mathcal{A}(G(r,1,\ell)$ additionally contains the coordinate hyperplanes $E_i := \ker(x_i)$, [OT92, Ch. 6.4].

To show that the reflection arrangements $\mathcal{A}(G(r,r,\ell))$ for $r,\ell \geq 3$ are recursively free, we need the intermediate arrangements $\mathcal{A}_\ell^k(r)$ with $\mathcal{A}(G(r,r,\ell)) \subseteq \mathcal{A}_\ell^k(r) \subseteq \mathcal{A}(G(r,1,\ell))$. They are defined as follows:

$$\mathcal{A}_\ell^k(r) := \mathcal{A}(G(r,r,\ell)) \dot{\cup} \{E_1,\ldots,E_k\},$$

and their defining polynomial is given by

$$Q(\mathcal{A}_\ell^k(r)) = x_1 \cdots x_k \prod_{\substack{1 \leq i < j \leq \ell \\ 0 \leq n < r}} (x_i - \zeta^n x_j).$$

The following result by Amend, Hoge and Röhrle immediately implies the recursive freeness of $\mathcal{A}(G(r,r,\ell))$, for $r,\ell \geq 3$.

Theorem 4.1.1 ([AHR14a, Thm. 3.6])**.** *Suppose $r \geq 2$, $\ell \geq 3$ and $0 \leq k \leq \ell$. Then $\mathcal{A}_\ell^k(r)$ is recursively free.*

Corollary 4.1.2. *Let W be the finite complex reflection group $W = G(r,r,\ell)$. Then the reflection arrangement $\mathcal{A} := \mathcal{A}(W)$ is recursively free.*

Proof. We have $\mathcal{A} \cong \mathcal{A}_\ell^0(r)$ and by Theorem 4.1.1, $\mathcal{A}_\ell^0(r)$ is recursively free. □

4.1.2 The reflection arrangement $\mathcal{A}(G_{24})$

We show that the reflection arrangement of the finite complex reflection group G_{24} is recursively free by constructing a so called supersolvable resolution for the arrangement, (see also [Zie87, Ch. 3.6], and making sure that in each addition-step of a new hyperplane the resulting arrangements and restrictions are free with suitable exponents. As a supersolvable arrangement is always inductively free (Example 3.1.10), it follows that $\mathcal{A}(G_{24})$ is recursively free.

Lemma 4.1.3. *Let W be the complex reflection group $W = G_{24}$. Then the reflection arrangement $\mathcal{A} = \mathcal{A}(W)$ is recursively free.*

j	$\exp(\mathcal{A}_j)$	$\exp(\mathcal{A}_j^{H_j})$
1	1,10,11	1,11
2	1,11,11	1,11
3	1,11,12	1,11
4	1,11,13	1,11
5	1,12,13	1,13
6	1,13,13	1,13
7	1,13,14	1,13
8	1,13,15	1,13
9	1,14,15	1,15
10	1,15,15	1,15
11	1,15,16	1,15
12	1,15,17	1,15

Table 4.1: The exponents of the free arrangements $\mathcal{A}_j$ and $\mathcal{A}_j^{H_j}$.

Proof. Let $\omega := -\frac{1}{2}(1 + i\sqrt{7})$, then the reflecting hyperplanes of $\mathcal{A}$ can be defined by the following linear forms (see also [LT09, Ch. 7, 6.2]):

$$\begin{aligned}\mathcal{A} = \quad & \{ (1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,1,0)^\perp, (-1,1,0)^\perp,\\ & (1,0,1)^\perp, (-1,0,1)^\perp, (0,1,1)^\perp, (0,-1,1)^\perp, (\omega,\omega,2)^\perp,\\ & (-\omega,\omega,2)^\perp, (\omega,-\omega,2)^\perp, (-\omega,-\omega,2)^\perp, (\omega,2,\omega)^\perp,\\ & (-\omega,2,\omega)^\perp, (\omega,2,-\omega)^\perp, (-\omega,2,-\omega)^\perp, (2,\omega,\omega)^\perp,\\ & (2,-\omega,\omega)^\perp, (2,\omega,-\omega)^\perp, (2,-\omega,-\omega)^\perp \}.\end{aligned}$$

The exponents of $\mathcal{A}$ are $\exp(\mathcal{A}) = \{\{1, 9, 11\}\}$.

If we define

$$\begin{aligned}\{H_1,\dots,H_{12}\} := \quad & \{ (\omega^2,\omega,0)^\perp, (-\omega^2,\omega,0)^\perp, (\omega,\omega^2,0)^\perp\\ & (-\omega,\omega^2,0)^\perp, (2-\omega,\omega,0)^\perp, (-2+\omega,\omega,0)^\perp,\\ & (\omega,2-\omega,0)^\perp, (-\omega,2-\omega,0)^\perp, (\omega,2,0)^\perp,\\ & (-\omega,2,0)^\perp, (2,\omega,0)^\perp, (-2,\omega,0)^\perp \},\end{aligned}$$

and the arrangements $\mathcal{A}_j := \mathcal{A}\dot{\cup}\{H_1,\dots,H_j\}$ for $1 \le j \le 12$, then

$$X = (1,0,0)^\perp \cap (0,1,0)^\perp \cap (1,1,0)^\perp \cap (-1,1,0)^\perp \cap_{j=1}^{12} H_j \in L(\mathcal{A}_{12})$$

is a rank 2 modular element, and $\mathcal{A}_{12}$ is supersolvable. In each step, $\mathcal{A}_j$ is free, $\mathcal{A}_j^{H_j}$ is inductively free (since $\mathcal{A}_j^{H_j}$ is a 2-arrangement), and $\exp(\mathcal{A}_j^{H_j}) \subseteq \exp(\mathcal{A}_j)$. The exponents of the arrangements $\mathcal{A}_j$ and $\mathcal{A}_j^{H_j}$ are listed in Table 4.1.

Since by Example 3.1.10 a supersolvable arrangement is inductively free, $\mathcal{A}$ is recursively free. □

We found the set of hyperplanes $\{H_1, \dots, H_{12}\}$ by "connecting" a suitable $X \in L_2(\mathcal{A})$ to other $Y \in L_2(\mathcal{A})$ via addition of new hyperplanes such that X becomes a modular element in the resulting intersection lattice, subject to each addition of a new hyperplane results in a free arrangement, (compare with [OT92, Ex. 4.59]).

4.1.3 The reflection arrangement $\mathcal{A}(G_{27})$

In [CH15b, Remark 3.7] Cuntz and Hoge have shown that the reflection arrangement $\mathcal{A}(G_{27})$ is not recursively free. In particular they have shown that there is no hyperplane which can be added or removed from $\mathcal{A}(G_{27})$ resulting in a free arrangement.

4.1.4 The reflection arrangements $\mathcal{A}(G_{29})$ and $\mathcal{A}(G_{31})$

In [HR15, Lemma 3.5] Hoge and Röhrle settled the case that the reflection arrangement $\mathcal{A}(G_{31})$ of the exceptional finite complex reflection group G_{31} is not inductively free by testing several cases with the computer.

In this part we will see, that the reflection arrangement $\mathcal{A}(G_{31})$ is additionally not recursively free and as a consequence the closely related reflection subarrangement $\mathcal{A}(G_{29})$ is also not recursively free. Furthermore, we obtain a new computer-free proof, that $\mathcal{A}(G_{31})$ is not inductively free.

Theorem 4.1.4. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement with W isomorphic to one of the finite complex reflection groups G_{29}, G_{31}. Then $\mathcal{A}$ is not recursively free.*

We will prove the theorem in two parts.

In the first part, we will characterize certain free subarrangements of $\mathcal{A}(G_{31})$ which we can obtain from $\mathcal{A}(G_{31})$ by successive deletion of hyperplanes such that all the arrangements in between are also free. We call such arrangements *free filtration subarrangements*. Then we will investigate the relation between the two reflection arrangements $\mathcal{A}(G_{29})$ and $\mathcal{A}(G_{31})$, and obtain that $\mathcal{A}(G_{29})$ is the smallest of these free filtration subarrangements of $\mathcal{A}(G_{31})$. This yields a new proof, that $\mathcal{A}(G_{31})$ is not inductively free (since inductive freeness implies that the empty arrangement is a free filtration subarrangement).

In the second part, we will show that if $\tilde{\mathcal{A}}$ is a free filtration subarrangement of $\mathcal{A}(G_{31})$, there is no possible way to obtain a free arrangement out of $\tilde{\mathcal{A}}$ by adding a new hyperplane which is not already contained in $\mathcal{A}(G_{31})$.

This will conclude the proof of Theorem 4.1.4.

Definition 4.1.5. Let $i = \sqrt{-1}$. The arrangement $\mathcal{A}(G_{31})$ can be defined as the union of the following collections of hyperplanes:

$$\begin{aligned}\mathcal{A}(G_{31}) := &\{\ker(x_p - i^k x_q) \mid 0 \leq k \leq 3, 1 \leq p < q \leq 4\} \dot\cup \\ &\{\alpha^\perp \mid \alpha \in G(4,4,4).(1,1,1,1)\} \dot\cup \\ &\{(1,0,0,0)^\perp, (0,1,0,0)^\perp, (0,0,1,0)^\perp, (0,0,0,1)^\perp\} \dot\cup \\ &\{\alpha^\perp \mid \alpha \in G(4,4,4).(-1,1,1,1)\}.\end{aligned} \tag{4.1.6}$$

The first set contains the hyperplanes of the reflection arrangement $\mathcal{A}(G(4,4,4))$. The second and the last set contain the hyperplanes defined by the linear forms in orbits of the group $G(4,4,4)$. The union of the first and the second set gives the 40 hyperplanes of the reflection arrangement $\mathcal{A}(G_{29})$. In particular, $\mathcal{A}(G_{29}) \subseteq \mathcal{A}(G_{31})$, compare with [LT09, Ch. 7, 6.2].

The free filtration subarrangements of $\mathcal{A}(G_{31})$

In this subsection we characterize certain free subarrangements of $\mathcal{A}(G_{31})$ which we can obtain by successively removing hyperplanes from $\mathcal{A}(G_{31})$, the so called *free filtration subarrangements*. We will use this characterization in Subsection 4.1.4 to prove Theorem 4.1.4. Furthermore, along the way, we obtain another (computer-free) proof that the arrangement $\mathcal{A}(G_{31})$ cannot be inductively free (recall Definition 3.1.9) without checking all the cases for a possible inductive chain but rather by examining the intersection lattices of certain subarrangements and using the fact, that $\mathcal{A}(G_{29})$ plays a “special” role among the free filtration subarrangements of $\mathcal{A}(G_{31})$.

Definition 4.1.7. Let $\mathcal{A}$ be a free ℓ-arrangement and $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ a free subarrangement. A strictly decreasing sequence of free arrangements

$$\mathcal{A} = \mathcal{A}_0 \supsetneq \mathcal{A}_1 \supsetneq \ldots \supsetneq \mathcal{A}_{n-1} \supsetneq \mathcal{A}_n = \tilde{\mathcal{A}}$$

is called a *(finite) free filtration* from $\mathcal{A}$ to $\tilde{\mathcal{A}}$ if $|\mathcal{A}_i| = |\mathcal{A}| - i$ for each i. If there exists a (finite) free filtration from $\mathcal{A}$ to $\tilde{\mathcal{A}}$, we call $\tilde{\mathcal{A}}$ a *free filtration subarrangement*.

The notion of free filtration was first introduced by Abe and Terao in [AT16].

Note that, since all the subarrangements $\mathcal{A}_i$ in the definition are free, with Theorem 3.1.6 the restrictions $\mathcal{A}_{i-1}^{H_i}$ are free and we automatically have $\exp(\mathcal{A}_{i-1}^{H_i}) \subseteq \exp(\mathcal{A}_{i-1})$ and $\exp(\mathcal{A}_{i-1}^{H_i}) \subseteq \exp(\mathcal{A}_i)$.

If $\mathcal{A}$ is an inductively free ℓ-arrangement, then Φ_ℓ is a free filtration subarrangement.

The main result of this subsection is the following proposition which we will prove in several steps divided into some lemmas.

Proposition 4.1.8. *Let $\mathcal{A} := \mathcal{A}(G_{31})$ be the reflection arrangement of the finite complex reflection group G_{31}. Let $\tilde{\mathcal{A}}$ be a smallest (w.r.t. the number of hyperplanes) free filtration subarrangement. Then $\tilde{\mathcal{A}} \cong \mathcal{A}(G_{29})$. In particular $\mathcal{A}$, $\mathcal{A}(G_{29})$ and all other free filtration subarrangements $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ are not inductively free.*

To prove Proposition 4.1.8, we will characterize all free filtration subarrangements of $\mathcal{A}(G_{31})$ by certain combinatorial properties of their intersection lattices.

The next lemma gives a sufficient condition for $\tilde{\mathcal{A}} \subseteq \mathcal{A}(G_{31})$ being a free filtration subarrangement. With an additional assumption on $|\tilde{\mathcal{A}}|$, this condition is also necessary.

Lemma 4.1.9. *Let $\mathcal{N} \subseteq \mathcal{A} := \mathcal{A}(G_{31})$ be a subcollection of hyperplanes and $\tilde{\mathcal{A}} := \mathcal{A} \setminus \mathcal{N}$. If $\mathcal{N}$ satisfies*

$$\bigcup_{X \in L_2(\mathcal{N})} X \subseteq \bigcup_{H \in \tilde{\mathcal{A}}} H, \qquad (*)$$

then $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is a free filtration subarrangement, with exponents $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$.

If furthermore $|\mathcal{N}| \leq 13$, then $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is a free filtration subarrangement if and only if $\mathcal{N}$ satisfies $(*)$.

Proof. We proceed by induction on $|\mathcal{N}|$.

We use the fact, that G_{31} acts transitively on the hyperplanes of $\mathcal{A}$. In particular, all the 3-arrangements $\mathcal{A}^H$ for $H \in \mathcal{A}$ are isomorphic and furthermore, they are free with exponents $\exp(\mathcal{A}^H) = \{\{1, 13, 17\}\}$ (see [OT92, App. C and App. D]).

First let $\mathcal{N} = \{H\}$ consist of only a single hyperplane. Since $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = \{\{1, 13, 17, 29\}\}$, the arrangement $\tilde{\mathcal{A}} = \mathcal{A}'$ is just a deletion with respect to H, hence free by Theorem 3.1.5, and $\tilde{\mathcal{A}}$ is a free filtration subarrangement with $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 28\}\}$.

With $\mathcal{N}$, each subcollection $\mathcal{N}' = \mathcal{N} \setminus \{K\}$, for a $K \in \mathcal{N}$, fulfills the assumption of the lemma. By the induction hypotheses $\mathcal{B} = \mathcal{A} \setminus \mathcal{N}'$ is a free filtration subarrangement with $\exp(\mathcal{B}) = \{\{1, 13, 17, 29 - |\mathcal{N}'|\}\} = \{\{1, 13, 17, 29 - |\mathcal{N}| + 1\}\}$. Now condition $(*)$ just means that $|\mathcal{B}^K| = 31$, so $\mathcal{B}^K \cong \mathcal{A}^H$ for any $H \in \mathcal{A}$ and is free with $\exp(\mathcal{B}^K) = \{\{1, 13, 17\}\}$. Hence, again by Theorem 3.1.5, the deletion $\mathcal{B}' = \mathcal{B} \setminus \{K\}$ is free and thus $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{N} = \mathcal{B}'$ is a free filtration subarrangement with $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$.

Finally, let $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{N}$ be a free filtration subarrangement with $|\mathcal{N}| \leq 13$. For an associated free filtration $\mathcal{A} = \mathcal{A}_0 \supsetneq \ldots \supsetneq \mathcal{A}_n = \tilde{\mathcal{A}}$ with say $\mathcal{A}_i = \mathcal{A}'_{i-1} = \mathcal{A}_{i-1} \setminus \{H_i\}$ for $1 \leq i \leq n$, we have $|\mathcal{A}_{i-1}^{H_i}| = 31$. So $H_i \cap H_j \subseteq K$, $j < i$, for a $K \in \mathcal{A}_i$ and for $i = n$ this is condition $(*)$. □

Before we continue with the characterization of the free filtration subarrangements, we give a helpful partition of the reflection arrangement $\mathcal{A}(G_{31})$:

Lemma 4.1.10. *Let $\mathcal{A} = \mathcal{A}(G_{31})$. There are exactly 6 subcollections $M_1, \ldots, M_6 \subseteq \mathcal{A}$, such that $\mathcal{A} \setminus M_i \cong \mathcal{A}(G_{29})$, $M_i \cap M_j \cong \mathcal{A}(A_1^4)$ and $M_i \cap M_j \cap M_k = \emptyset$ for $1 \leq i < j < k \leq 6$. Thus we get a partition of $\mathcal{A}$ into 15 disjoint subsets $\{M_i \cap M_j \mid 1 \leq i < j \leq 6\} =: \mathcal{F}$ on which G_{31} acts transitively.*

Proof. Let $W := G_{31}$ and $W' := G_{29} \leq W$. Then $N_W(W') = W'$ and $|W : W'| = 6$, so with Lemma 3.2.1 there are exactly 6 subarrangements, say $\mathcal{B}_1, \ldots, \mathcal{B}_6$ with $\mathcal{B}_i \cong \mathcal{A}(W') \subseteq \mathcal{A}$, (respectively 6 conjugate reflection subgroups of W isomorphic to W'). Now we get the M_i as $M_i = \mathcal{A} \setminus \mathcal{B}_i$ and in particular the corresponding action of W on the subcollections is transitive.

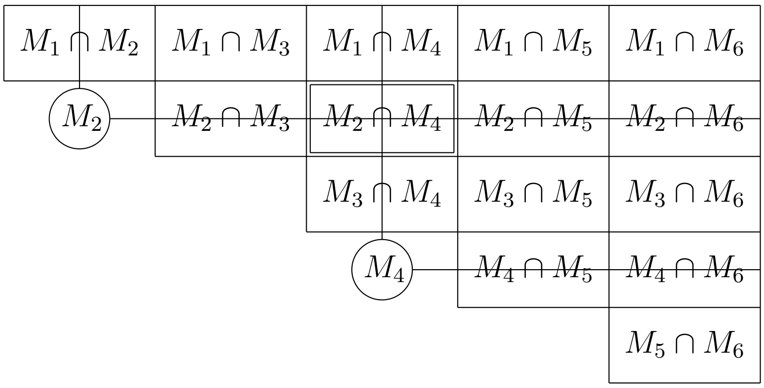


Figure 4.1: The partition of $\mathcal{A}$ into 15 disjoint subsets $\mathcal{F} = \{M_i \cap M_j,\, 1 \leq i < j \leq 6\}$, each consisting of 4 hyperplanes.

To see the claim about their intersections we look at the different orbits of reflection subgroups of W on $\mathcal{A}$ acting on hyperplanes. First W' has 2 orbits $\mathcal{O}_1 = \mathcal{A}(W')$, and $\mathcal{O}_2 = \mathcal{A} \setminus \mathcal{A}(W') = M_i$ for an $i \in \{1, \ldots, 6\}$. Similarly a subgroup $\tilde{W}' = g^{-1}W'g \neq W'$ conjugate to W' has also 2 orbits $\tilde{\mathcal{O}}_1 = \mathcal{A}(\tilde{W}')$, and $\tilde{\mathcal{O}}_2 = \mathcal{A} \setminus \mathcal{A}(\tilde{W}') = M_j$ and $j \in \{1, \ldots, 6\} \setminus \{i\}$. Now the intersection $W' \cap \tilde{W}'$ of these two conjugate subgroups is isomorphic to $G(4,4,4) \leq W$ and $G(4,4,4)$ has two orbits $\mathcal{O}_{21}$, $\mathcal{O}_{22}$ on $\mathcal{O}_2$ of size 16 and 4, respectively two orbits $\tilde{\mathcal{O}}_{21}$, $\tilde{\mathcal{O}}_{22}$ on $\tilde{\mathcal{O}}_2$ of size 16 and 4 (see Definition 4.1.5). Because of the cardinalities of $\mathcal{A}(W')$ and $\mathcal{A}(\tilde{W}')$ we have $M_i \cap M_j = \mathcal{O}_2 \cap \tilde{\mathcal{O}}_2 \neq \emptyset$, and $M_i \cap M_j = \mathcal{O}_{22} = \tilde{\mathcal{O}}_{22}$. Since the collection $M_i \cap M_j$ is stabilized by $G(4,2,4) \geq G(4,4,4)$, the lines orthogonal to the hyperplanes in $M_i \cap M_j$ are the unique system of imprimitivity $G(4,2,4)$. Hence we get $M_i \cap M_j \cong \mathcal{A}(A_1^4) = \{\ker(x_i) \mid 1 \leq i \leq 4\}$.

Now let $W' = G(4,2,4)$. Here we also have $N_W(W') = W'$, so $|W : W'| = 15$, and hence again with Lemma 3.2.1 there are 15 distinct subarrangements isomorphic to $\mathcal{A}(W') \subseteq \mathcal{A}$. Since each to W' conjugate reflection subgroup of W has a unique system of imprimitivity consisting of the lines orthogonal to the hyperplanes in $M_i \cap M_j$ for $i, j \in \{1, \ldots, 6\}, i \neq j$ and they are distinct, the $M_i \cap M_j$ are distinct and disjoint.

Finally, each hyperplane in $\mathcal{A}$ belongs to a unique intersection $M_i \cap M_j$, so they form a partition $\mathcal{F}$ of $\mathcal{A}$. Since W acts transitively on $\mathcal{A}$, and interchanges the systems of imprimitivity corresponding to the reflection subarrangements isomorphic to $\mathcal{A}(G(4,2,4))$, it acts transitively on $\mathcal{F}$. □

The partition $\mathcal{F}$ in Lemma 4.1.10 can be visualized in a picture, see Figure 4.1.

In the above proof we used some facts about the actions and orders of complex reflection (sub)groups from the book by Lehrer and Taylor, [LT09] in particular [LT09, Ch. 8, 10.5]).

Furthermore, it will be helpful to know the distribution of the $\mathcal{A}_X, X \in L_2(\mathcal{A})$ with respect to the partition given by Lemma 4.1.10:

Lemma 4.1.11. *Let $H \in \mathcal{A}$, $X \in \mathcal{A}^H$, and $H \in \mathcal{B}_{ij} := M_i \cap M_j \in \mathcal{F}$ for some $1 \leq i < j \leq 6$. For $\mathcal{A}_X$ there are 3 cases:*

(1) $A_X = \{H, K_1, \dots, K_5\} \cong \mathcal{A}(G(4,2,2))$ with $K_1 \in \mathcal{B}$, $\{K_2, K_3\} \subseteq \mathcal{B}_{km} = M_k \cap M_m$, and $\{K_4, K_5\} \subseteq \mathcal{B}_{pq} = M_p \cap M_q$, with $\{i,j,k,m,p,q\} = \{1, \dots, 6\}$.

(2) $A_X = \{H, K_1, K_2\} \cong \mathcal{A}(A_2)$ with $K_1 \in \mathcal{B}_{ik} = M_i \cap M_k$, and $K_2 \in \mathcal{B}_{jK} = M_j \cap M_k$ for some $k \in \{1, \dots, 6\} \setminus \{i,j\}$.

(3) $A_X = \{H, K\} \cong \mathcal{A}(A_1^2)$ with $K \in \mathcal{B}_{km} = M_k \cap M_m$ for some $k, m \in \{1, \dots, 6\} \setminus \{i,j\}$.

Proof. This is by explicitly writing down the partition $\mathcal{F}$ from Lemma 4.1.10 with respect to definition 4.1.5 and a simple computation. □

The following lemma provides the next step towards a complete characterization of the free filtration subarrangements of $\mathcal{A}(G_{31})$.

Lemma 4.1.12. *Let $\mathcal{M} \subseteq \mathcal{A} := \mathcal{A}(G_{31})$ be a subcollection, such that $\mathcal{B} = \mathcal{A} \setminus \mathcal{M} \cong \mathcal{A}(G_{29})$. Then for all $\mathcal{N} \subseteq \mathcal{M}$, $\tilde{\mathcal{A}} := \mathcal{A} \setminus \mathcal{N}$ is a free filtration subarrangement with exponents* $\exp(\mathcal{A}^{(\mathcal{N})}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$.

Proof. Let $\mathcal{M} \subseteq \mathcal{A}$ such that $\mathcal{B} = \mathcal{A} \setminus \mathcal{M} \cong \mathcal{A}(G_{29})$. We claim that $\mathcal{M}$ satisfies condition $(*)$, so with Lemma 4.1.9, $\mathcal{B}$ is a free filtration subarrangement. Furthermore, if $\mathcal{M}$ satisfies condition $(*)$, so does every subcollection $\mathcal{N} \subseteq \mathcal{M}$ and $\tilde{\mathcal{A}} := \mathcal{A} \setminus \mathcal{N}$ is a free filtration subarrangement with exponents $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$.

Now let $H \in \mathcal{M}$ be an arbitrary hyperplane in $\mathcal{M}$ and let $X \in \mathcal{A}^H$. Then by Proposition 4.1.11 there are three different cases:

$$\begin{aligned}
&(1)\ |\mathcal{A}_X| &&= 2, \quad \mathcal{A}_X = \{H, K\},\\
&(2)\ |\mathcal{A}_X| &&= 3, \quad \mathcal{A}_X = \{H, H', K\},\\
&(3)\ |\mathcal{A}_X| &&= 6, \quad \mathcal{A}_X = \{H, H', K_1, \dots, K_4\},
\end{aligned}$$

with $H' \in \mathcal{M}$ and $K, K_i \in \mathcal{B} \cong \mathcal{A}(G_{29})$. For arbitrary $H, H' \in \mathcal{M}$ there is a hyperplane $K \in \mathcal{B}$ such that $H \cap H' = X \subseteq K$. Hence $\mathcal{M}$ satisfies condition $(*)$ and as mentioned before with Lemma 4.1.9 $\tilde{\mathcal{A}}$ is a free filtration subarrangement with exponents $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$. □

The next lemma completes the characterization of the free filtration subarrangements $\tilde{\mathcal{A}} \subseteq \mathcal{A}(G_{31})$ and enables us to prove Proposition 4.1.8.

Lemma 4.1.13. *Let $\mathcal{A} = \mathcal{A}(G_{31})$. A subarrangement $\mathcal{A} \setminus \mathcal{N} = \tilde{\mathcal{A}} \subseteq \mathcal{A}$ is a free filtration subarrangement if and only if*

(1) $\mathcal{A}(G_{29}) \subseteq \tilde{\mathcal{A}}$
or

(2) $|\mathcal{N}| \leq 13$ and $\mathcal{N}$ satisfies $(*)$ from Lemma 4.1.9.

In both cases the exponents of $\tilde{\mathcal{A}}$ are $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - |\mathcal{N}|\}\}$.

Proof. Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be a subarrangement. If $\tilde{\mathcal{A}}$ satisfies (1) then by Lemma 4.1.12 it is a free filtration subarrangement and if $\tilde{\mathcal{A}}$ satisfies (2) then by Lemma 4.1.9 it is also a free filtration subarrangement. This gives one direction.

The other direction requires more effort. The main idea is to use the partition $\mathcal{F}$ of $\mathcal{A}$ from Lemma 4.1.10, the distribution of the localizations $\mathcal{A}_X$ with respect to this parson given by Lemma 4.1.11, and some counting arguments.

So let $\mathcal{A} \setminus \mathcal{N}' = \tilde{\mathcal{A}}' \subseteq \mathcal{A}$ be a subarrangement such that $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}'$, $|\mathcal{N}'| \geq 14$, and suppose that $\tilde{\mathcal{A}}'$ is a free filtration subarrangement. Since $\tilde{\mathcal{A}}'$ is a free filtration subarrangement there has to be another free filtration subarrangement say $\tilde{\mathcal{A}} \supseteq \tilde{\mathcal{A}}'$, $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{N}$ such that $|\mathcal{N}| = 13$. By Lemma 4.1.9 we then have $\bigcup_{X \in L_2(\mathcal{N})} X \subseteq \bigcup_{H \in \tilde{\mathcal{A}}} H$ and $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 16, 17\}\}$. We claim that there is no $H \in \tilde{\mathcal{A}}$ such that $|\tilde{\mathcal{A}}^H| \in \{30, 31\}$, so by Theorem 3.1.6 contradicting the fact that $\tilde{\mathcal{A}}'$ is a free filtration subarrangement.

If $\mathcal{A}(G_{29}) \subseteq \tilde{\mathcal{A}}$ then by Lemma 4.1.10 there is an $1 \leq i \leq 6$ such that $\mathcal{N} \subseteq M_i$. With respect to renumbering the M_i we may assume that $\mathcal{N} \subseteq M_1$. Let $\mathcal{B}_{1j} = M_1 \cap M_j$, $2 \leq j \leq 6$ be the blocks of the partition of M_1 from Lemma 4.1.10. Since $|\mathcal{N}| = 13$ we have $\mathcal{B}_{1j} \cap \mathcal{N} \neq \emptyset$, and there is a k such that $|\mathcal{B}_{1k} \cap \mathcal{N}| \geq 3$. By $\tilde{\mathcal{A}}' \supsetneq \mathcal{A}(G_{29})$, we have $H \notin M_1$. But then, using Lemma 4.1.11, we see that $|\tilde{\mathcal{A}}^H| < 30$ (because $\mathcal{N}$ completely contains at least two localizations as in Lemma 4.1.11(2), and (3)), so $\tilde{\mathcal{A}}'$ is not free by Theorem 3.1.6 and in particular it is not a free filtration subarrangement contradicting our assumption.

If $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}$ we claim that for such a free filtration subarrangement $\tilde{\mathcal{A}}$ with $|\mathcal{N}| = 13$ there is a $H \in \mathcal{A}$, $H \in \mathcal{B} \in \mathcal{F}$ (see Lemma 4.1.10), such that

$$\mathcal{N} = \bigcup_{H' \in \mathcal{B} \setminus \{H\}} \mathcal{A}_{H \cap H'} \setminus \{H'\}, \tag{4.1.14}$$

which enables us to describe $\tilde{\mathcal{A}}^K$ for each $K \in \tilde{\mathcal{A}}$.

So let $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{N}$ be a free filtration subarrangement with $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}$ and $|\mathcal{N}| = 13$. By Lemma 4.1.9 $\mathcal{N}$ has to satisfy condition $(*)$. Let $\mathcal{F}_{\mathcal{N}} := \{\mathcal{B} \in \mathcal{F} \mid \mathcal{N} \cap \mathcal{B} \neq \emptyset\}$ be the blocks in the partition $\mathcal{F}$ of $\mathcal{A}$ containing the hyperplanes from $\mathcal{N}$ and let $\mathcal{B}_{ab} := M_a \cap M_b \in \mathcal{F}$ $(a \neq b,\ a, b \in \{1, \dots, 6\})$. First we notice that $|\mathcal{F}_{\mathcal{N}}| \geq 4$ because $|\mathcal{N}| = 13$. Since $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}$, by Lemma 4.1.10 we have one of the following cases

(1) there are $\mathcal{B}_{ij}, \mathcal{B}_{kl} \in \mathcal{F}_{\mathcal{N}}$, such that $|\{i, j, k, l\}| = 4$,

(2) there are $\mathcal{B}_{ij}, \mathcal{B}_{ik}, \mathcal{B}_{jk} \in \mathcal{F}_{\mathcal{N}}$, such that $|\{i, j, k\}| = 3$.

But since $|\mathcal{F}_{\mathcal{N}}| \geq 4$, in case (2) there is a $\mathcal{B}_{ab} \in \mathcal{F}_{\mathcal{N}}$ with $a \in \{i, k, l\}$ and $b \notin \{i, j, k\}$, so we are again in case (1), (compare with Figure 4.1). Hence (with possibly renumbering the M_i) we have $\mathcal{B}_{12}, \mathcal{B}_{34} \in \mathcal{F}_{\mathcal{N}}$. By the distribution of the simply intersecting hyperplanes in $\mathcal{A}$ with respect to $\mathcal{F}$ (Lemma 4.1.11(3)) and by condition $(*)$ we further have $|\mathcal{N} \cap \mathcal{B}_{12}| \leq 2$, $|\mathcal{N} \cap \mathcal{B}_{34}| \leq 2$ resulting in $|\mathcal{F}_{\mathcal{N}}| \geq 5$. Next, suppose for all $\mathcal{B}_{ab} \in \mathcal{F}$ we have $\{a, b\} \subseteq \{1, 2, 3, 4\}$, so in particular $\mathcal{N} \subseteq \mathcal{A}(G(4, 4, 4))$ (see Figure 4.1, Definition 4.1.5 and Lemma 4.1.10). Then because of $|\mathcal{N} \cap \mathcal{B}_{12}| \leq 2$, $|\mathcal{N} \cap \mathcal{B}_{34}| \leq 2$, $|\mathcal{N}| = 13$, and $|\mathcal{F}_{\mathcal{N}}| \geq 5$ we find $\mathcal{B}_{1a}, \mathcal{B}_{2b} \in \mathcal{F}_{\mathcal{N}}$, $a, b \in \{3, 4\}$, such that $|(\mathcal{B}_{1a} \cup \mathcal{B}_{2b}) \cap \mathcal{N}| \geq 5$.

But this contradicts condition ($*$) by Lemma 4.1.11(2). So there is a $\mathcal{B}_{ab} \in \mathcal{F}_{\mathcal{N}}$ with $\{a,b\} \nsubseteq \{1,2,3,4\}$. Now for $\mathcal{B}_{ab}$ there are again two possible cases

(1) $a = 5$ and $b = 6$,

(2) $a \in \{1,2,3,4\}$ and $b \in \{5,6\}$.

In the first case, by Lemma 4.1.11(3) and condition ($*$), we then have $|\mathcal{N} \cap \mathcal{B}| \leq 2$ for all $\mathcal{B} \in \mathcal{F}_{\mathcal{N}}$ so $|\mathcal{F}_{\mathcal{N}}| \geq 7$. So in this (after renumbering the M_i once more) we may assume that we are in the second case. In the second case, again by Lemma 4.1.11(3) and condition ($*$) we then have $|\mathcal{B}_{ij} \cap \mathcal{N}| \leq 2$ for $i \neq a, j \neq a$. We may assume that $a = 1$ (the other cases are similar), then only $|(\mathcal{B}_{13} \cup \mathcal{B}_{14}) \cap \mathcal{N}| \leq 4$ by Lemma 4.1.11(2) and condition ($*$). So in this case we also have $|\mathcal{F}_{\mathcal{N}}| \geq 7$ and further $|\mathcal{B}_{34} \cap \mathcal{N}| = 1$ by Lemma 4.1.11(3).

We remark that for a subarrangement $\mathcal{C} \subseteq \mathcal{A}$ with $\mathcal{C} \cong \mathcal{A}(G(4,2,4))$ there is a $\mathcal{B}_{ij} \in \mathcal{F}$, such that $\mathcal{C} = \mathcal{B}_{ij} \cup (\mathcal{A} \setminus (M_i \cup M_j)) = \mathcal{B}_{ij} \cup \bigcup_{a,b \in \{1,\ldots,6\} \setminus \{i,j\}} \mathcal{B}_{ab}$ (compare again with Figure 4.1, Definition 4.1.5 and Lemma 4.1.10). If $\mathcal{N}$ is of the claimed form (4.1.14), by Lemma 4.1.11(1) we have $\mathcal{N} \subseteq \mathcal{A}(G(4,2,4))$ and furthermore, since $|\mathcal{N}| = 13$ and $\mathcal{N}$ has to satisfy condition ($*$), with Lemma 4.1.11 one easily sees, that if $\mathcal{N} \subseteq \mathcal{A}(G(4,2,4))$, it has to be of the form (4.1.14).

To finally prove the claim, we want to show that $\mathcal{N} \subseteq \mathcal{A}(G(4,2,4))$ (for one possible realization of $\mathcal{A}(G(4,2,4))$ inside $\mathcal{A}$ given by $\mathcal{F}$).

So far we have that there are $\mathcal{B}_{12}, \mathcal{B}_{34}, \mathcal{B}_{1b} \in \mathcal{F}_{\mathcal{N}}$ ($b \in \{5,6\}$). This can be visualized in the following picture (Figure 4.2(a), compare also with Figure 4.1), where the boxes represent the partition $\mathcal{F}$, a double circle represents a hyperplane already fixed in $\mathcal{N}$ by the above considerations, a solid circle a hyperplane which can not belong to $\mathcal{N}$ without violating condition ($*$), and a non solid circle a hyperplane which may or may not belong to $\mathcal{N}$.

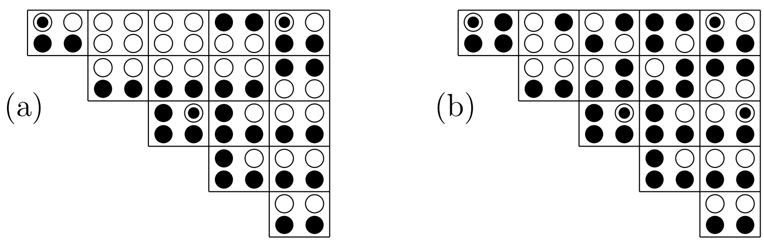


Figure 4.2: Possible choices for hyperplanes in $\mathcal{N}$.

Suppose that there is a $\mathcal{B}_{cd} \in \mathcal{F}_{\mathcal{N}}$ such that $\{c,d\} \cap \{3,4\} \neq \emptyset$. This is the case if and only if $\mathcal{N} \subseteq \mathcal{A}(G(4,2,4))$ by our remark before.

Then the hyperplanes left to be chosen for $\mathcal{N}$ reduce considerably (see Figure 4.2(b)).

If we proceed in this manner using the same arguments as above we arrive at a contradiction to $|\mathcal{N}| = 13$, condition ($*$), and Lemma 4.1.11.

To finish the proof, let $\tilde{\mathcal{A}} = \mathcal{A} \setminus \mathcal{N}$ for an $\mathcal{N}$ of the form (4.1.14). Then by Lemma 4.1.11(3) and the distribution of the $H \in \tilde{\mathcal{A}}$ with respect to $\mathcal{F}$ we have $|\tilde{\mathcal{A}}^H| \leq 29$

since for H there are at least two hyperplanes in $\mathcal{N}$ simply intersecting H and we are done. □

Example 4.1.15. We illustrate the change of the set of hyperplanes which can be added to $\mathcal{N}$ along a free filtration from $\mathcal{A}$ to $\mathcal{A} \setminus \mathcal{N} = \tilde{\mathcal{A}}$ with $|\tilde{\mathcal{A}}| = 47$, $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}$, by the following sequence of pictures (Figure 4.3). Each circle represents a hyperplane in the free filtration subarrangement $\mathcal{A}_i$, a solid circle represents a hyperplane which we can not add to $\mathcal{N}$ without violating condition $(*)$ from Lemma 4.1.9. A non-solid circle represents a hyperplane, which can be added to $\mathcal{N}$, such that $(*)$ ist still satisfied. The different boxes represent the partition $\mathcal{F}$ of $\mathcal{A}$ into subsets of 4 hyperplanes given by Lemma 4.1.13:

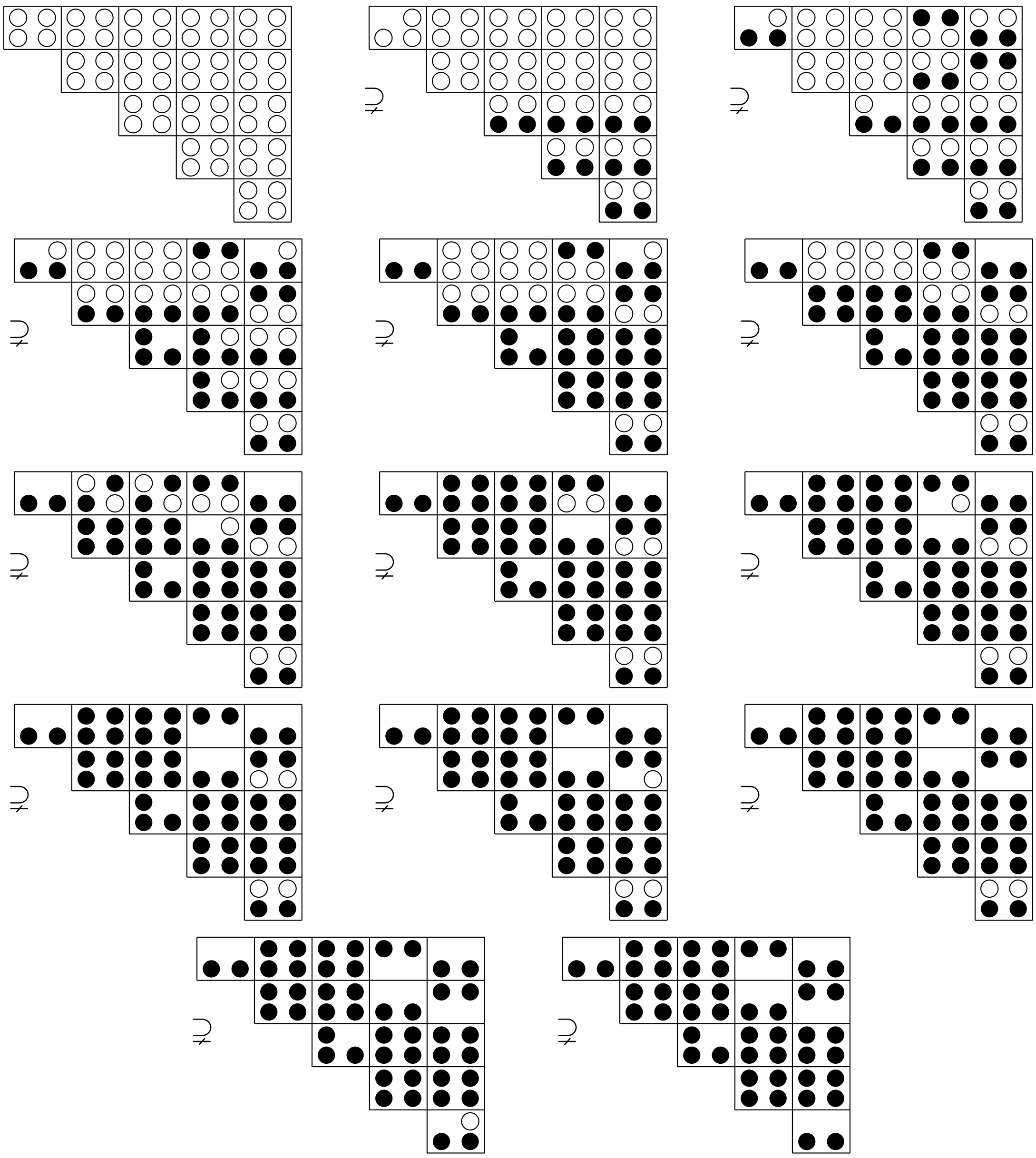


Figure 4.3: The change of hyperplanes which can be removed along a free filtration from $\mathcal{A}$ to $\tilde{\mathcal{A}}$.

Now we can prove Proposition 4.1.8:

Proof of Proposition 4.1.8. Let $\tilde{\mathcal{A}}$ be a free filtration subarrangement.

If $\mathcal{A}(G_{29}) \nsubseteq \tilde{\mathcal{A}}$, then with Lemma 4.1.13, $|\tilde{\mathcal{A}}| \geq 47$.

Now assume that $\tilde{\mathcal{A}} \cong \mathcal{A}(G_{29})$. In Lemma 4.1.12 we saw, that $\tilde{\mathcal{A}}$ is a free filtration subarrangement.

In [HR15, Remark 2.17] it is shown that one cannot remove a single hyperplane from $\mathcal{A}(G_{29}) = \mathcal{B}$ resulting in a free arrangement $\mathcal{B}'$, so there is no smaller free filtration subarrangement of $\mathcal{A}$. □

The reflection arrangements $\mathcal{A}(G_{29})$ and $\mathcal{A}(G_{31})$ are not recursively free

Let $\mathcal{A} := \mathcal{A}(W)$ be the reflection arrangement of the complex reflection group $W = G_{31}$ and $\mathcal{B} := \mathcal{A}(W)$ the reflection arrangement of the complex reflection group $W = G_{29}$. As we saw in the previous section $\mathcal{B} \subsetneq \mathcal{A}$ is a free filtration subarrangement.

We use the characterization of all free filtration subarrangements $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ from Lemma 4.1.13 and show that for all these subarrangements there exists no hyperplane H outside of $\mathcal{A}$ we can add to $\tilde{\mathcal{A}}$ such that the resulting arrangement $\tilde{\mathcal{A}}\dot{\cup}\{H\}$ is free.

Firstly we show that it is not possible for $\tilde{\mathcal{A}} = \mathcal{A}$:

Lemma 4.1.16. *There is no way to add a new hyperplane H to $\mathcal{A}$ such that the arrangement $\tilde{\mathcal{A}} := \mathcal{A}\dot{\cup}\{H\}$ is free.*

Proof. The exponents of $\mathcal{A}$ are $\exp(\mathcal{A}) = \{\{1, 13, 17, 29\}\}$. Inspection of the intersection lattice $L := L(\mathcal{A})$ gives the following multisets of invariants:

$$\{\{|\mathcal{A}_X| \mid X \in L_2\}\} = \{\{2^{360}, 3^{320}, 6^{30}\}\}. \tag{4.1.17}$$

Now assume that there exists a new hyperplane H which we can add to $\mathcal{A}$ such that $\tilde{\mathcal{A}} := \mathcal{A}\dot{\cup}\{H\}$ is free. Then by Lemma 3.1.13 we have $\sum_{X \in P_H}(|\mathcal{A}_X| - 1) \in \exp(\mathcal{A})$ where $P_H = \{X \in L_2 \mid X \subseteq H\}$. Hence with (4.1.17) H contains at least 4 different rank 2 subspaces (e.g. $13 = (6-1) + (6-1) + (3-1) + (2-1)$) from the intersection lattice.

But up to symmetry there are no more than 5 possibilities to get a hyperplane H with $|\{X \in L_2 \mid X \subseteq H\}| \geq 3$ such that $\chi_{\tilde{\mathcal{A}}}(t)$ factors over the integers, but in each case $\chi_{\tilde{\mathcal{A}}}(t) = (t-1)(t-15)(t-16)(t-29)$, so with Theorem 3.1.6 $\tilde{\mathcal{A}}$ can not be free. □

Now we will prove that for all free filtration subarrangements $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ (see definition 4.1.7) there exists no other hyperplane $H \notin \mathcal{A}$ we can add to $\tilde{\mathcal{A}}$ such that $\tilde{\mathcal{A}}\dot{\cup}\{H\}$ is free.

Lemma 4.1.18. *Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be a free filtration subarrangement. Let H be a new hyperplane such that $\tilde{\mathcal{A}}\dot{\cup}\{H\}$ is free. Then $H \in \mathcal{A}$.*

Proof. In Lemma 4.1.13 we have shown, that $\tilde{\mathcal{A}}$ is free with exponents $\exp(\tilde{\mathcal{A}}) = \{\{1, 13, 17, 29 - n\}\}$, $n \leq 20$. Let $L = L(\mathcal{A})$ and $\tilde{L} = L(\tilde{\mathcal{A}}) \subseteq L$. We once more use the following multiset of invariants:

$$\{\{|\mathcal{A}_X| \mid X \in L_2\}\} = \{\{2^{360}, 3^{320}, 6^{30}\}\}.$$

Thus for $X \in \tilde{L}_2$ we have $2 \leq |\tilde{\mathcal{A}}_X| \leq 6$.

Suppose we add a new hyperplane H such that $\tilde{\mathcal{A}}\dot{\cup}\{H\}$ is free. Then by Lemma 3.1.13 we have $\sum_{X \in P_H}(|\tilde{\mathcal{A}}_X| - 1) \in \exp(\tilde{\mathcal{A}})$ where $P_H = \{X \in \tilde{L}_2 \mid X \subseteq H\}$.

We immediately see that $|P_H| \geq 3$ and if $|P_H| \in \{3, 4\}$ there must be at least two $X \in P_H$ with $|\tilde{\mathcal{A}}_X| \geq 4$ or $|\mathcal{A}_X| = 6$. But for $X, Y \in L_2$, $X \neq Y$, with $|\mathcal{A}_X| = |\mathcal{A}_Y| = 6$ we either have $X + Y = V$ or $X \subseteq K$ and $Y \subseteq K$ for a $K \in \mathcal{A}$. Hence in this case $H \in \mathcal{A}$.

Now assume that $|P_H| \geq 5$ and there is at most one $X \in P_H$ with $|\tilde{\mathcal{A}}_X| \geq 4$ or $|\mathcal{A}_X| = 6$. Then there are either at least three $X \in P_H$ with $|\tilde{\mathcal{A}}_X| = 3$ or at least four $X \in P_H$ with $|\tilde{\mathcal{A}}_X| = 2$. But in both cases with the same argument as above we must have $H \in \mathcal{A}$.

This finishes the proof. □

We close this section with the following corollary which completes the proof of Theorem 4.1.4.

Corollary 4.1.19. *Let $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ be a free filtration subarrangement of $\mathcal{A} = \mathcal{A}(G_{31})$. Then $\tilde{\mathcal{A}}$ is not recursively free and in particular $\mathcal{A}(G_{31})$ and $\mathcal{A}(G_{29})$ are not recursively free.*

Proof. The statement follows immediately from Lemma 4.1.18 and Proposition 4.1.8. □

4.1.5 The reflection arrangement $\mathcal{A}(G_{33})$

In this section we will see, that the reflection arrangement $\mathcal{A}(W)$ with W isomorphic to the finite complex reflection group G_{33} is not recursively free.

Lemma 4.1.20. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement with $W \cong G_{33}$. Then $\mathcal{A}$ is not recursively free.*

Proof. With Theorem 4.2 the reflection arrangement $\mathcal{A}$ is not inductively free.

In [HR15, Remark 2.17] it is shown that one cannot remove a single hyperplane from $\mathcal{A}$ resulting in a free arrangement $\mathcal{A}'$

Thus to prove the lemma, we have to show, that we also cannot add a new hyperplane H such that the arrangements $\tilde{\mathcal{A}} := \mathcal{A}\dot{\cup}\{H\}$ and $\tilde{\mathcal{A}}^H$ are free with suitable exponents.

The exponents of $\mathcal{A}$ are $\exp(\mathcal{A}) = \{\{1, 7, 9, 13, 15\}\}$.

Now suppose that there is a hyperplane H such that $\tilde{\mathcal{A}}$ is free. Looking at the intersection lattice $L := L(\mathcal{A})$ we find the following multiset of invariants:

$$\{\{|\mathcal{A}_X| \mid X \in L_2\}\} = \{\{2^{270}, 3^{240}\}\}.$$

With Lemma 3.1.13 and the same argument as in the proof of Lemma 4.1.16 for H we must have:

$$|P_H| = |\{X \in L_2 \mid X \subseteq H\}| \geq 4.$$

If we look at all the possible cases for an H such that $|P_H| \geq 2$ (there are only 2 possible cases up to symmetry) we already see that in none of these cases the characteristic polynomial of $\tilde{\mathcal{A}}$ splits into linear factors over $\mathbb{Z}$ and by Theorem 3.1.7 $\tilde{\mathcal{A}}$ is not free.

Hence we cannot add a single hyperplane H to $\mathcal{A}$ and obtain a free arrangement $\mathcal{A}\dot{\cup}\{H\} = \tilde{\mathcal{A}}$ and $\mathcal{A}$ is not recursively free. □

4.1.6 The reflection arrangement $\mathcal{A}(G_{34})$

In this part we will see, that the reflection arrangement $\mathcal{A}(W)$ with W isomorphic to the finite complex reflection group G_{34} is free but not recursively free.

Lemma 4.1.21. *Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement with $W \cong G_{34}$. Then $\mathcal{A}$ is not recursively free.*

Proof. To prove the lemma, we could follow the same path as in the proof of Lemma 4.1.20.

But since the arrangement of $\mathcal{A}(G_{33})$ is a parabolic subarrangement (localization) $\mathcal{A}_X$ of the reflection arrangement $\mathcal{A} = \mathcal{A}(G_{34})$ for a suitable $X \in L(\mathcal{A})$ (see e.g. [OT92, Table C.15.] or [LT09, Ch. 7, 6.1]). Since this localization is not recursively free by Lemma 4.1.20, $\mathcal{A}$ cannot be recursively free by Proposition 3.1.14. □

This completes the proof of Theorem I.

4.2 Abe's conjecture

In this section we give the proof of Theorem 4.1, which settles [Abe16, Conj. 5.11].

The following result by Abe gives the divisional freeness of the reflection arrangement $\mathcal{A}(G_{31})$.

Theorem 4.2.1 ([Abe16, Cor. 4.7])**.** *Let W be a finite irreducible complex reflection group and $\mathcal{A} = \mathcal{A}(W)$ its corresponding reflection arrangement. Then $\mathcal{A} \in \mathcal{IF}$ or $W = G_{31}$ if and only if $\mathcal{A} \in \mathcal{DF}$.*

With results from the previous section we can now state the proof of the theorem.

Proof of Theorem 4.1. Let $\mathcal{A} = \mathcal{A}(G_{31})$ be the reflection arrangement of the finite complex reflection group G_{31}. Then on the one hand by Theorem 4.2.1 we have $\mathcal{A} \in \mathcal{DF}$, but on the other hand by Theorem 4.1.4 we have $\mathcal{A} \notin \mathcal{RF}$. □

Remark 4.2.2. Furthermore, with Corollary 4.1.19, we see that every free filtration subarrangement $\tilde{\mathcal{A}} \subseteq \mathcal{A}(G_{31})$ still containing a hyperplane $H \in \tilde{\mathcal{A}}$ such that $|\tilde{\mathcal{A}}^H| = 31$ is in $\mathcal{DF}$.

4.3 Restrictions

In [AHR14a] Amend, Hoge and Röhrle showed, which restrictions of (irreducible) reflection arrangements are inductively free. Despite the free but not inductively free reflection arrangements them self investigated in this paper, by [AHR14a, Thm. 1.2] there are four restrictions of reflection arrangements which remain to be inspected, namely

(1) the 4-dimensional restriction $(\mathcal{A}(G_{33}), A_1)$,

(2) the 5-dimensional restriction $(\mathcal{A}(G_{34}), A_1)$,

(3) the 4-dimensional restriction $(\mathcal{A}(G_{34}), A_1^2)$, and

(4) the 4-dimensional restriction $(\mathcal{A}(G_{34}), A_2)$,

which are free but not inductively free (compare with [OT92, App. C.16, C.17]).

Using similar techniques as for the reflection arrangements $\mathcal{A}(G_{31})$, and $\mathcal{A}(G_{33})$, we can say the following about the remaining cases:

Proposition 4.3.1.

(1) $(\mathcal{A}(G_{33}, A_1)$ *is recursively free,*

(2) $(\mathcal{A}(G_{34}, A_1)$ *is not recursively free,*

(3) $(\mathcal{A}(G_{34}), A_1^2)$ *is not recursively free, and*

(4) $(\mathcal{A}(G_{34}), A_2)$ *is not recursively free.*

Proof. Let $\mathcal{A}$ be as in (1). The arrangement may be defined by the following linear forms:

$$\begin{aligned}\mathcal{A} = \{&(1,0,0,0)^\perp,(1,1,0,0)^\perp,(1,1,1,0)^\perp,(1,1,1,1)^\perp,(0,1,0,0)^\perp,\\ &(0,1,1,0)^\perp,(0,1,1,1)^\perp,(0,0,1,0)^\perp,(0,0,1,1)^\perp,(0,0,0,1)^\perp,\\ &(\zeta^2,0,-1,\zeta^2)^\perp,(1,0,-1,\zeta^2)^\perp,(2\zeta,2\zeta+\zeta^2,\zeta,-\zeta^2)^\perp,\\ &(-1,\zeta+2\zeta^2,\zeta^2,-1)^\perp,(\zeta,0,-1,\zeta^2)^\perp,(2,-2\zeta-\zeta^2,1,-\zeta^2)^\perp,\\ &(\zeta,\zeta-\zeta^2,2\zeta,\zeta)^\perp,(\zeta^2,\zeta-2\zeta^2,-1,\zeta^2)^\perp,\\ &(\zeta^2,-\zeta+\zeta^2,2\zeta^2,\zeta^2)^\perp,(\zeta^2,0,-\zeta,\zeta^2)^\perp,(\zeta^2,0,-\zeta^2,1)^\perp,\\ &(\zeta^2,0,-1,\zeta)^\perp,(2\zeta,\zeta-\zeta^2,-2\zeta^2,-\zeta^2)^\perp,(\zeta,2\zeta+\zeta^2,-1,\zeta^2)^\perp,\\ &(-2\zeta^2,\zeta-\zeta^2,2\zeta,\zeta)^\perp,(-1,2\zeta+\zeta^2,\zeta,-\zeta^2)^\perp,\\ &(2\zeta,\zeta-\zeta^2,\zeta,-\zeta^2)^\perp,(2\zeta,2\zeta+\zeta^2,\zeta,-1)^\perp\}\\ =\{&H_1,\dots,H_{28}\},\end{aligned}$$

where $\zeta = \frac{1}{2}(-1+i\sqrt{3})$ is a primitive cube root of unity.

We can successively remove 6 hyperplanes

$$\{H_5,H_6,H_7,H_{13},H_{25},H_{28}\} =: \{K_1,\dots,K_6\} =: \mathcal{N},$$

with respect to this order such that $\mathcal{A}\setminus\mathcal{N} = \tilde{\mathcal{A}}$ is a free filtration subarrangement with a free filtration $\mathcal{A} = \mathcal{A}_0 \supsetneq \mathcal{A}_1 \supsetneq \cdots \supsetneq \mathcal{A}_6 = \tilde{\mathcal{A}}$, $\mathcal{A}_i = \mathcal{A}\setminus\{K_1,\dots,K_i\}$. Moreover, all the restrictions $\mathcal{A}_{i-1}^{K_i}$, $(1\le i\le 6)$, are inductively free. Then we can add 2 new hyperplanes

$$\{I_1,I_2\} := \{(-2\zeta-3\zeta^2,3,2,1)^\perp,(\zeta,0,2,1)^\perp\},$$

such that $\tilde{\mathcal{A}}_j := \tilde{\mathcal{A}}\cup\{I_1,\dots,I_j\}$, $(j=1,2)$ are all free and in particular the arrangement $\tilde{\mathcal{A}}_2 = \tilde{\mathcal{A}}\cup\{I_1,I_2\}$ is inductively free. Furthermore, the $\tilde{\mathcal{A}}_j^{I_j}$ are inductively free. Hence $\mathcal{A}$ is recursively free.

The arrangement in (2) is isolated which can be seen similarly as for the arrangement $\mathcal{A}(G_{33})$.

To show that the restrictions $(\mathcal{A}(G_{34}),A_1^2),(\mathcal{A}(G_{34},A_2)$ from (3) and (4) are not recursively free, we look at the exponents of their minimal possible free filtration subarrangements computed by Amend, Hoge, and Röhrle in [AHR14a, Lemma 4.2, Tab. 11,12] and then use Lemma 3.1.13 and a similar argument as in the proof of Lemma 4.1.18.

Let $\mathcal{A}$ be as in (3). Then Amend, Hoge, and Röhrle showed that the multiset of exponents of a minimal possible free filtration subarrangement $\tilde{\mathcal{A}}\subseteq\mathcal{A}$ are $\exp(\tilde{\mathcal{A}}) = \{\{1,13,15,15\}\}$, (see [AHR14a, Tab. 11]). Now, as in the proof of Lemma 4.1.18, suppose $\tilde{\mathcal{A}}\subseteq\mathcal{A}$ is a free filtration subarrangement, and there is a hyperplane H, such that $\tilde{\mathcal{A}}\cup\{H\}$ is free. Then by Lemma 3.1.13 we have $\sum_{X\in P_H}(|\tilde{\mathcal{A}}_X|-1)\ge 13$, where $P_H = \{X\in L_2(\tilde{\mathcal{A}}) \mid X\subseteq H\}$. Now $L_2(\tilde{\mathcal{A}})\subseteq L_2(\mathcal{A})$ and we have the following multiset if invariants of $\mathcal{A}$:

$$\{\{|\mathcal{A}_X| \mid X\in L_2(\mathcal{A})\}\} = \{\{2^{264},3^{304},4^{34},5^{16}\}\}.$$

So in particular we should have $\sum_{X \in P_H}(|\mathcal{A}_X| - 1) \geq 13$, and $|P_H| \geq 4$. If $|P_H| = 4$ then there are at least two $X, Y \in P_H$ with $|\mathcal{A}_X| = |\mathcal{A}_Y| = 5$ But for all such X, Y we either have $X + Y = V$ or $X + Y \in \mathcal{A}$. So there is at most one $X \in P_H$ such that $|\mathcal{A}_X| = 5$. If $|P_H| = 4$ we must have at least $X, Y \in P_H$ with $|\mathcal{A}_X| = 5$, $|\mathcal{A}_Y| = 4$. But again for all such X, Y we either have $X + Y = V$ or $X + Y \in \mathcal{A}$. Considering the other cases (giving a number partition of the smallest exponent not equal to 1) similarly we get that $H \in \mathcal{A}$. Hence $\mathcal{A}$ is not recursively free.

Finally, let $\mathcal{A}$ be as in (4). Then Amend, Hoge, and Röhrle showed that the multiset of exponents of a minimal possible free filtration subarrangement $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ are $\exp(\tilde{\mathcal{A}}) = \{\{1, 9, 10, 11\}\}$ or $\exp(\tilde{\mathcal{A}}) = \{\{1, 10, 10, 10\}\}$, (see [AHR14a, Tab. 12]). Suppose $\tilde{\mathcal{A}} \subseteq \mathcal{A}$ is a free filtration subarrangement, and there is a hyperplane H, such that $\tilde{\mathcal{A}} \cup \{H\}$ is free. Then inspecting the intersection lattice of $\mathcal{A}$ analogously to case (3) we again get $H \in \mathcal{A}$. Hence $\mathcal{A}$ is not recursively free. □

Since the restrictions $(\mathcal{A}(G_{34}), A_1^2)$ and $(\mathcal{A}(G_{34}), A_2)$ behave somehow similar to the reflection arrangement $\mathcal{A}(G_{31})$, they also give examples for divisionally free but not recursively free arrangement, (compare with Theorem 4.1 and Section 4.2). For further details on divisional freeness of restrictions of reflection arrangements see the recent note by G. Röhrle, [Röh15].

5 Coxeter graphs for simplicial arrangements

In this chapter arrangements are always assumed to be real.

We introduce Coxeter graphs of chambers of simplicial arrangements and use the results from Subsection 3.3 to derive their properties.

5.1 Definition and examples

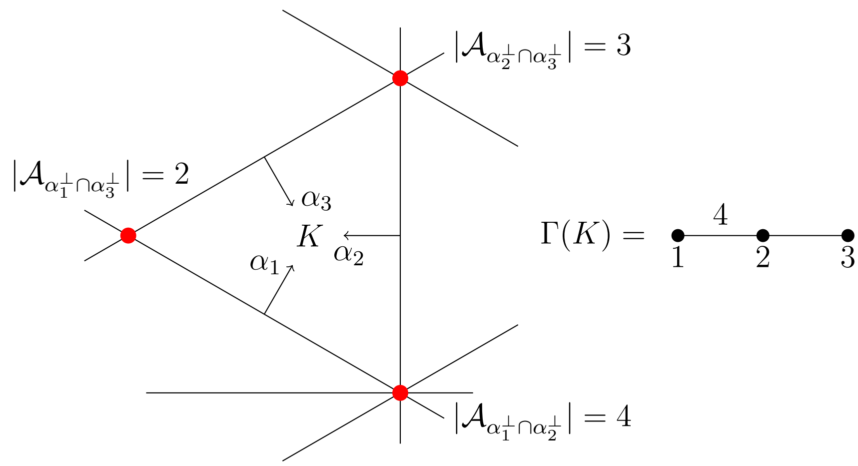
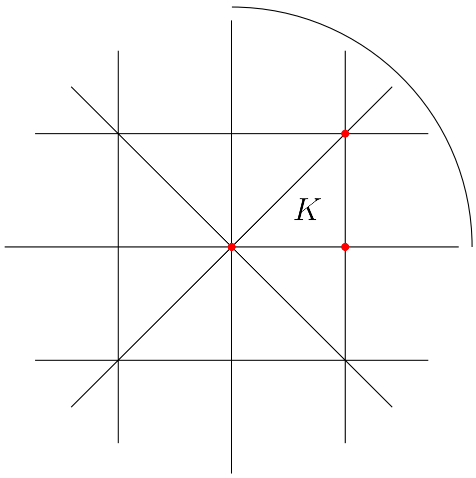
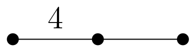


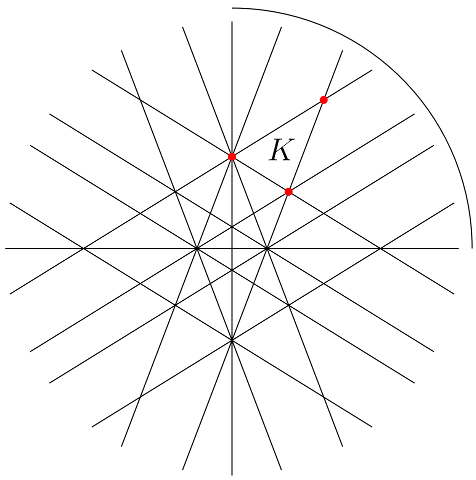
Figure 5.1: The Coxeter graph $\Gamma(K)$ of a chamber K.

Definition 5.1.1. Let $K \in \mathcal{K}(\mathcal{A})$ be a chamber of the simplicial ℓ-arrangement $\mathcal{A}$ and B^K some basis for K. We define a labeled non directed simple graph $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} = B^K$ and edges $\mathcal{E} = \{\{\alpha, \beta\} \mid |\mathcal{A}_{\alpha^\perp \cap \beta^\perp}| \geq 3\}$. An edge $e = \{\alpha, \beta\} \in \mathcal{E}$ is labeled with $m^K(e) = m^K(\alpha, \beta) = |\mathcal{A}_{\alpha^\perp \cap \beta^\perp}|$. Since the label $m(\alpha, \beta) = 3$ appears more often we omit it in drawing the graph. We call $\Gamma(K)$ the *Coxeter graph* of K. If we have chosen a numbering $B^K = \{\alpha_1, \dots, \alpha_\ell\}$ then $\{\alpha_i, \alpha_j\} \in \mathcal{E}$ is simply denoted by $\{i, j\}$ and $\mathcal{V} = \{1, \dots, \ell\}$, see Figure 5.1.

Example 5.1.2. Let $\mathcal{A}(W)$ be the Coxeter arrangement of the Coxeter group W. Then $\mathcal{A}$ is a simplicial arrangement (c.f. Example 3.3.2) and for all $K \in \mathcal{K}(\mathcal{A})$ the Coxeter graph $\Gamma(K)$ is indeed the Coxeter graph of W, see for example [Hum90, Ch. 2]. Figure 5.2 displays projective pictures of the Coxeter arrangements $\mathcal{A}(B_3)$ and $\mathcal{A}(H_3)$.



$\{\Gamma(K) \mid K \in \mathcal{K}(\mathcal{A}(B_3))\} = \{$  $\}$



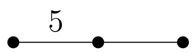
$\{\Gamma(K) \mid K \in \mathcal{K}(\mathcal{A}(H_3))\} = \{$  $\}$

Figure 5.2: Coxeter graphs and Coxeter arrangements of the Coxeter groups B_3 and H_3.

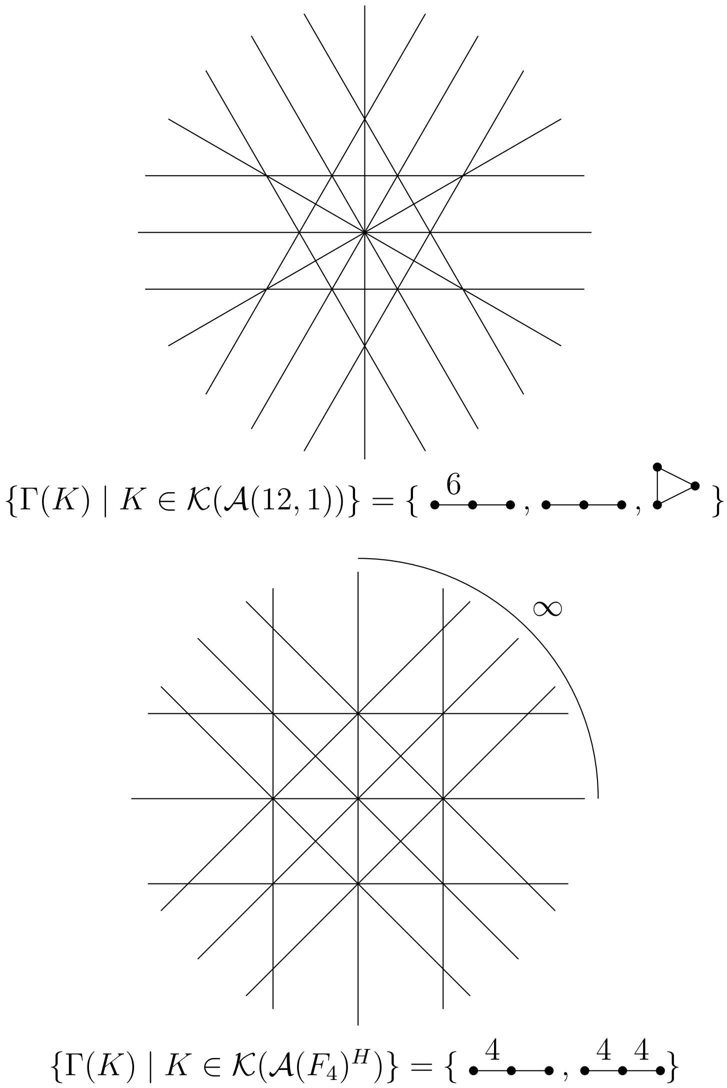


Figure 5.3: The simplicial arrangements $\mathcal{A}(12,1)$ and $\mathcal{A}(F_4)^H = \mathcal{A}(13,2)$ and their sets of Coxeter graphs.

Example 5.1.3. In general there is not only one Coxeter graph for all the chambers of a simplicial arrangement. Figure 5.3 displays two simplicial arrangements and their corresponding sets of Coxeter graphs.

Now one might ask if a simplicial arrangement is determined by its set of Coxeter graphs (similar to a finite Coxeter group which is determined by its Coxeter graph). This is actually not true for simplicial arrangements in general as the list in [CH15a] shows. But we will see in Chapter 6 that the set of Coxeter graphs suffices to classify supersolvable simplicial arrangements.

5.2 Properties of Coxeter graphs

Lemma 5.2.1. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$ with basis $B^K = \{\alpha_1, \dots, \alpha_\ell\}$, K_i an adjacent chamber, and c_{ij}^K as in Remark 3.3.20. If $m^K(i,j) = 3$ and $c_{ij}^K \neq 0$ for $i \neq j$ then $c_{ji}^K = 1/c_{ij}^K$. In particular if $c_{ij}^K = -1$ then $c_{ji}^K = c_{ij}^K$.*

Proof. This is directly clear from the definition of the c_{ij}^K since

$$\mathcal{A}_{\alpha_i^\perp \cap \alpha_j^\perp} = \{\alpha_i^\perp, \alpha_j^\perp, (\alpha_j - c_{ij}^K \alpha_i)^\perp\}.$$

□

Lemma 3.3.27 gives us the following property of the Coxeter graphs of two adjacent chambers.

Lemma 5.2.2. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$ a chamber, $B^K = \{\alpha_1, \dots, \alpha_\ell\}$, $\Gamma(K) = (\mathcal{V}, \mathcal{E})$, and K_i an adjacent chamber with $B^{K_i} = \{\sigma_i^K(\alpha_1), \dots, \sigma_i^K(\alpha_\ell)\}$ and $\Gamma(K_i) = (\mathcal{V}_i, \mathcal{E}_i)$. Then if $\{i,j\} \notin \mathcal{E}$ $(i \neq j)$ but $\{j,k\} \in \mathcal{E}$ then $\{j,k\} \in \mathcal{E}_i$ (disregarding the labels).*

The next Lemma is a direct generalization of [CH09, Prop. 4.6] from crystallographic arrangements to general simplicial arrangements. It may be proved completely analogously but here we give a more geometric proof.

Lemma 5.2.3. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement with chambers $\mathcal{K}(\mathcal{A})$. Then the following are equivalent.*

(1) $\mathcal{A}$ is an irreducible arrangement.

(2) $\Gamma(K)$ is connected for all $K \in \mathcal{K}(\mathcal{A})$.

(3) $\Gamma(K)$ is connected for some $K \in \mathcal{K}(\mathcal{A})$.

Proof. We may assume that ℓ is at least 2 since otherwise the statement of the theorem is trivial.

The implication (2)⇒(3) is trivial.

(1)⇒(2). Suppose there is a $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ is not connected. Then there is a partition $\mathcal{V} = B^K = \Delta_1 \dot\cup \Delta_2$ such that $|\mathcal{A}_{\alpha^\perp \cap \beta^\perp}| = 2$ for $\alpha \in \Delta_1$, and $\beta \in \Delta_2$. Without loss of generality let $\alpha \in \Delta_1$. Then

$$B^{K\alpha} = \{-\alpha\} \dot\cup \{\alpha' + c_{\alpha'}\alpha \mid \alpha' \in \Delta_1 \setminus \{\alpha\}\} \dot\cup \Delta_2$$

is a basis for $K\alpha$ for certain $c_{\alpha'} \geq 0$, c.f. Lemma 3.3.18. Assume that there are $\alpha' + c\alpha \in B^{K\alpha}$ and $\beta \in \Delta_2 \subseteq B^{K\alpha}$ with $|\mathcal{A}_{(\alpha'+c\alpha)^\perp \cap \beta^\perp}| \geq 3$. Then there is a $b > 0$ such that $\alpha' + c\alpha + b\beta \in B^{K\alpha\beta}$. Note that $K\alpha\beta(-\alpha) = K\beta$ since $|\mathcal{A}_{\alpha^\perp \cap \beta^\perp}| = 2$. Then there is a $d \geq 0$ such that $\alpha' + c\alpha + b\beta + d(-\alpha) = \alpha' + (c-d)\alpha + b\beta \in B^{K\beta}$. But $B^{K\beta} = \Delta_1 \dot\cup \{-\beta\} \dot\cup \{\beta' + c_{\beta'}\beta \mid \beta' \in \Delta_2 \setminus \{\beta\}\}$ which gives a contradiction. So for all $\alpha' + c_{\alpha'}\alpha \in B^{K\alpha}$ and $\beta \in \Delta_1$ we have $|\mathcal{A}_{(\alpha'+c\alpha)^\perp \cap \beta^\perp}| = 2$. We conclude that for all $\gamma \in B^K$, for the corresponding adjacent chamber $K\gamma$ there is a partition $B^{K\gamma} = \tilde\Delta_1 \dot\cup \tilde\Delta_2$ with $\tilde\Delta_i \subset \sum_{\lambda \in \Delta_i} \mathbb{R}_{\geq 0}\lambda$ and $|\mathcal{A}_{\tilde\alpha^\perp \cap \tilde\beta^\perp}| = 2$ for all $\tilde\alpha \in \tilde\Delta_1$, $\tilde\beta \in \tilde\Delta_2$. Hence for all $H \in \mathcal{A}$ we either have $H = (\sum_{\alpha \in \Delta_1} c_\alpha \alpha)^\perp$ with $c_\alpha \in \mathbb{R}_{\geq 0}$, or $H = (\sum_{\beta \in \Delta_2} c_\beta \beta)^\perp$ with $c_\beta \in \mathbb{R}_{\geq 0}$ which means that $\mathcal{A}$ is reducible.

(3)⇒(1). Suppose that $\mathcal{A}$ is reducible. Then there exists a basis $\{x_1, \ldots, x_r\} \dot\cup \{y_1, \ldots, y_s\}$ of V^* with $r, s \geq 1$ such that for $H \in \mathcal{A}$ and $H = \gamma^\perp$ for some $\gamma \in V^*$ we either have $\gamma \in \sum_{i=1}^r \mathbb{R}x_i$ or $\gamma \in \sum_{j=1}^s \mathbb{R}y_j$. Let $K \in \mathcal{K}(\mathcal{A})$ be chamber of $\mathcal{A}$. Then $B^K = \Delta_1 \dot\cup \Delta_2$ with $\Delta_1 = B^K \cap \sum_i \mathbb{R}x_i$ and $\Delta_2 = B^K \cap \sum_j \mathbb{R}y_j$. Since $\mathcal{A}$ is simplicial, B^K is a basis of V^* and we have $\Delta_i \neq \emptyset$ for $i = 1, 2$. Furthermore, $\mathcal{A}_{\alpha^\perp \cap \beta^\perp} = \{\alpha^\perp, \beta^\perp\}$ for $\alpha \in \Delta_1$, $\beta \in \Delta_2$ and hence $\Gamma(K)$ is not connected. □

Lemma 5.2.4. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $K \in \mathcal{K}(\mathcal{A})$ with $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$ and $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V} = \{1, \ldots, \ell\}$. Suppose that $\{i, j\} \in \mathcal{E}$ with label $m^K(i, j)$ and there is a $k \in \mathcal{V} \setminus \{i, j\}$ such that $\{k, i\} \notin \mathcal{E}$ and $\{k, j\} \notin \mathcal{E}$. Then $\{i, j\}$ is an edge in $\Gamma(K_k)$ with the same label $m^{K_k}(i, j) = m^K(i, j)$.*

Proof. That $\{i, j\}$ is an edge in $\Gamma(K_k)$ is simply Lemma 5.2.2. The second statement holds because $\sigma_k^K(\alpha_i) = \alpha_i$ and $\sigma_k^K(\alpha_j) = \alpha_j$ and thus

$$m^{K_k}(i, j) = |\mathcal{A}_{\sigma_k^K(\alpha_i)^\perp \cap \sigma_k^K(\alpha_j)^\perp}| = |\mathcal{A}_{\alpha_i^\perp \cap \alpha_j^\perp}| = m^K(i, j).$$

□

Lemma 5.2.5. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement, $X \in L_q(\mathcal{A})$ for $1 \leq q \leq \ell$, and $K_X \in \mathcal{K}(\mathcal{A}_X/X)$ be a chamber of the localization $\mathcal{A}_X/X$. Let $K \in \mathcal{K}(\mathcal{A})$ with $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$ such that $X = \bigcap_{j=1}^q \alpha_{i_j}^\perp$, $K_X = \bigcap_{j=1}^q \alpha_{i_j}^+/X$, and $\Gamma(K)$ with corresponding vertices $\mathcal{V} = \{1, \ldots, \ell\}$. Then $\Gamma(K_X)$ is the induced subgraph on the q vertices $\{i_1, \ldots, i_q\} \subseteq \mathcal{V}$ of $\Gamma(K)$ including the labels.*

Proof. For $q = 1$ the statement is trivially true. For $q \geq 2$ this is easily seen as the intersection lattice $L(\mathcal{A}_X)$ is an interval in the intersection lattice $L(\mathcal{A})$, i.e. $L(\mathcal{A}_X) = L(\mathcal{A})_X = [V, X] = \{Z \in L(\mathcal{A}) \mid Z \leq X\}$. □

With the correspondence from the previous lemma and Lemma 5.2.3 we obtain the following corollary for irreducible simplicial arrangements.

Corollary 5.2.6. *Let $\mathcal{A}$ be an irreducible simplicial ℓ-arrangement and $K \in \mathcal{K}(\mathcal{A})$. Then there is an $X \in L_{\ell-1}(W^K) \subseteq L(\mathcal{A})$ such that $(\mathcal{A}_X/X, V/X)$ is an irreducible simplicial $(\ell-1)$-arrangement.*

To describe the connection between restrictions of simplicial arrangements and Coxeter graphs we need a bit more notation.

Definition 5.2.7. Let $\mathcal{A}$ be a simplicial arrangement, $K \in \mathcal{K}(\mathcal{A})$, $\alpha \in B^K$ and $H = \alpha^\perp \in W^K$. Then we denote the induced chamber in the restriction $\mathcal{A}^H$ by

$$K^H = \Big(\bigcap_{\beta \in B^K \setminus \{\alpha\}} \beta^+ \Big) \cap H,$$

and a basis for K^H is given by

$$B^{K^H} = \{\beta^H \mid \beta^H := \beta|_{H^*} \text{ and } \beta \in B^K \setminus \{\alpha\}\}.$$

Let $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ be the Coxeter graph of K and suppose that there is an edge $\{\alpha, \beta\} \in \mathcal{E}$ connecting the vertices α and β. Define $\Gamma^{\alpha\beta} := (\mathcal{V}^{\alpha\beta}, \mathcal{E}^{\alpha\beta})$ to be the (unlabeled) graph with vertices

$$\mathcal{V}^{\alpha\beta} := \mathcal{V} \setminus \{\alpha, \beta\} \cup \{\alpha\beta\},$$

and edges

$$\begin{aligned} \mathcal{E}^{\alpha\beta} := \quad & \{\{\gamma, \delta\} \in \mathcal{E} \mid \{\gamma, \delta\} \cap \{\alpha, \beta\} = \emptyset\} \cup \\ & \{\{\alpha\beta, \gamma\} \mid \{\alpha, \gamma\} \in \mathcal{E} \text{ or } \{\beta, \gamma\} \in \mathcal{E}\}, \end{aligned}$$

i.e. the *contraction* of $\Gamma(K)$ along the edge $\{\alpha, \beta\}$.

It is convenient to use the following notation: If $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, \ell\}$ corresponding to $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$, $I \subseteq \mathcal{V}$ with $I = \{i_1, \ldots, i_r\}$ and $X = \cap_{i \in I} \alpha_i^\perp$ then for the localization $\mathcal{A}_X$ at the intersection adjacent to the chamber K we simply write $\mathcal{A}^K_{i_1 i_2 \cdots i_r}$, e.g. for $\mathcal{A}_{\alpha_1^\perp \cap \alpha_2^\perp \cap \alpha_4^\perp}$ we write $\mathcal{A}^K_{124}$.

Lemma 5.2.8. *Let $\mathcal{A}$ be a simplicial ℓ-arrangement and $K \in \mathcal{K}(\mathcal{A})$ with Coxeter graph $\Gamma(K) = (\mathcal{V}, \mathcal{E})$. Suppose $\{\alpha, \beta\} \in \mathcal{E}$ is an edge. Let $H \in \mathcal{A}_{\alpha^\perp \cap \beta^\perp}$ be the wall of $K\alpha$ with $H \neq \alpha^\perp$, i.e. $H = \sigma_\alpha^K(\beta)^\perp$. Then $\Gamma^{\alpha\beta}$ is a subgraph of the Coxeter graph $\Gamma^H := \Gamma((K\alpha)^H)$ of the chamber $(K\alpha)^H$.*

If $\{\alpha, \gamma\} \in \mathcal{E}$ is labeled with $m(\alpha, \gamma)$ then for the corresponding label in Γ^H we have $m^H(\alpha\beta, \gamma) \geq m(\alpha, \gamma)$ (see Figure 5.4(a)).

If $\{\alpha, \beta\}, \{\alpha, \gamma\}$, and $\{\beta, \gamma\}$ are edges in $\mathcal{E}$, then for the label of the edge $\{\alpha\beta, \gamma\}$ in Γ^H we have $m^H(\alpha\beta, \gamma) \geq m(\alpha, \gamma) + m(\beta, \gamma) - 2$ (see Figure 5.4(b)).

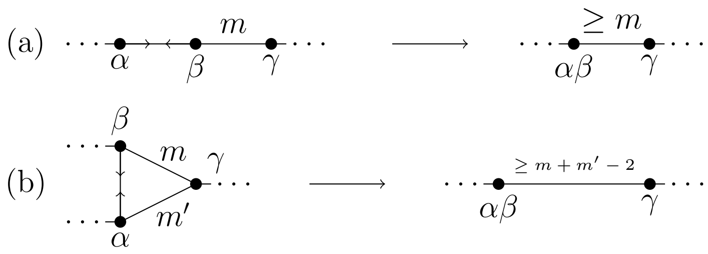


Figure 5.4: Labels and contraction of Coxeter graphs.

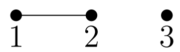


Figure 5.5: Coxeter graph of a reducible 3-arrangement.

Proof. It suffices to prove the statements for 3-arrangements (the statements are trivial for 2-arrangements). The general case then follows by taking localizations, the fact that $(\mathcal{A}^H)_X = (\mathcal{A}_X)^H$, and Lemma 5.2.5. Let $B^K = \{\alpha_1, \alpha_2, \alpha_3\}$ and denote the corresponding vertices of $\Gamma(K)$ by $\{1, 2, 3\}$.

If $\Gamma(K)$ is not connected, i.e. $\mathcal{A}$ is reducible, then either there is no edge in $\Gamma(K)$ and there is nothing to show, or it is the graph of Figure 5.5. In this case, the statement holds, since for all $H \in \mathcal{A}_{12}$ we then have $|\mathcal{A}^H| = 2$, so $\mathcal{A}^H$ is reducible and the Coxeter graph of every chamber of $\mathcal{A}^H$ is the graph with 2 vertices which are not connected.

So assume $\Gamma(K)$ is connected. Without loss of generality let $H = \sigma_1^K(\alpha_2)^\perp \in \mathcal{A}_{12}$. Since $(\sigma_1^K(\alpha_1)^H)^\perp = (-\alpha_1^H)^\perp = \alpha_1^\perp \cap \alpha_2^\perp$ in $\mathcal{A}^H$, $(\sigma_1^K(\alpha_3)^H)^\perp = \sigma_1^K(\alpha_2)^\perp \cap \sigma_1^K(\alpha_3)^\perp$ and so $(\sigma_1^K(\alpha_1)^H)^\perp \cap (\sigma_1^K(\alpha_3)^H)^\perp = \{0\}$, we have to show that $|\mathcal{A}^H| \geq (|\mathcal{A}_{13}| - 1) + (|\mathcal{A}_{23}| - 1)$ to obtain both statements. Let $\mathcal{B} = \mathcal{A}_{13} \cup \mathcal{A}_{23}$. Then $|\mathcal{A}^H| \geq |\mathcal{B}^H|$ and $|\mathcal{B}| = |\mathcal{A}_{13}| + |\mathcal{A}_{23}| - 1$ (since $\mathcal{A}_{13} \cap \mathcal{A}_{23} = \{\alpha_3^\perp\}$). We now deduce that $|\mathcal{B}^H| = |\mathcal{B}| - 1$:

We have $W^K \subset \mathcal{B}$ and $|(W^K)^H| = 2$. Now let $H_1, H_2 \in \mathcal{B} \setminus W^K$ with $H_1 \neq H_2$. We first observe that $H \cap H_1 \neq H \cap \tilde{H}$ for any $\tilde{H} \in W^K$. But we also have $H_1 \cap H \neq H_2 \cap H$. Hence all $H' \in \mathcal{B} \setminus W^K$ give different intersections with H. Thus we obtain

$$\begin{aligned} |\mathcal{A}^H| &\geq |\mathcal{B}^H| = |(W^K)^H| + |(\mathcal{B} \setminus W^K)^H| \\ &= 2 + |(\mathcal{B} \setminus W^K)| = 2 + |\mathcal{B}| - 3 \\ &= |\mathcal{A}_{13}| + |\mathcal{A}_{23}| - 2. \end{aligned}$$

From this inequality by translating back to the corresponding Coxeter graphs all statements from the lemma directly follow. □

Lemma 5.2.9. *Let $\mathcal{A}$ be an irreducible simplicial ℓ-arrangement and $X \in L_q(\mathcal{A})$. Then the restriction $\mathcal{A}^X$ is an irreducible simplicial $(\ell - q)$-arrangement.*

Proof. It suffices to show the statement for $X = H \in \mathcal{A}$.

Since $\mathcal{A}$ is irreducible, there is an $X \in L_2(\mathcal{A})$ with $X \subseteq H$ and $|\mathcal{A}_X| \geq 3$. So there is a chamber $K \in \mathcal{K}(\mathcal{A})$ with $\Gamma(K) = (\mathcal{V}, \mathcal{E})$, $\{\alpha, \beta\} \in \mathcal{E}$ such that $X = \alpha^\perp \cap \beta^\perp$, and H the wall of $K\alpha$ not equal to $\alpha^\perp$. Since $\mathcal{A}$ is irreducible, the Coxeter graph $\Gamma(K)$ is connected by Lemma 5.2.3, and by Lemma 5.2.8 the Coxeter graph $\Gamma(K\alpha^H)$ of the

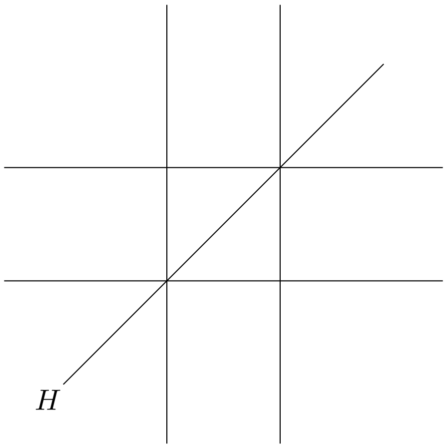


Figure 5.6: An irreducible arrangement $\mathcal{A}$ with a reducible restriction $\mathcal{A}^H$.

chamber $K\alpha^H$ of $\mathcal{A}^H$ contains a subgraph on $\ell - 1$ vertices which is connected (as it is a contraction). So $\Gamma(K\alpha^H)$ is also connected and hence again by Lemma 5.2.3 the restriction $\mathcal{A}^H$ is irreducible. □

This is not true for irreducible arrangements in general:

Example 5.2.10. Let $\mathcal{A}$ be the 3-arrangement give by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

Then $\mathcal{A}$ is clearly irreducible. But for $H = (1, -1, 0)^\perp$ we have $|\mathcal{A}^H| = 2$. Hence $\mathcal{A}^H$ is reducible. See Figure 5.6.

6 Supersolvable simplicial arrangements

In this chapter we prove Theorem II.

In Section 6.1 we prove the first part of the theorem, that is the classification of supersolvable simplicial 3-arrangements.

In Section 6.2 we derive the second part of the theorem for 4-arrangements. This is then the key to describe all the remaining cases of higher rank by an induction in Section 6.3.

In this chapter all arrangements are assumed to be real.

6.1 The rank 3 case

We firstly collect some useful lemmas for supersolvable simplicial 3-arrangements.

Lemma 6.1.1. *Let $\mathcal{A}$ be a supersolvable 3-arrangement with two modular elements $X, Y \in L_2(\mathcal{A})$ and $|\mathcal{A}_X| \neq |\mathcal{A}_Y|$. Then for all $Z \in L_2(\mathcal{A}) \setminus \{X, Y\}$ we have $|\mathcal{A}_Z| = 2$.*

Proof. By Theorem 3.4.8 two different roots of $\chi_\mathcal{A}(t)$ are given by $|\mathcal{A}_X| - 1$ and $|\mathcal{A}_Y| - 1$. So we have

$$\chi_\mathcal{A}(t) = (t-1)(t-(|\mathcal{A}_X|-1))(t-(|\mathcal{A}_Y|-1)),$$

and by Remark 2.4 we get

$$|\mathcal{A}| = -\mu_1 = |\mathcal{A}_X| + |\mathcal{A}_Y| - 1 \leq |\mathcal{A}_X \cup \mathcal{A}_Y|.$$

Hence there is a hyperplane $H \in \mathcal{A}$ with $\mathcal{A}^H = \{X, Y\}$. For every other $Z \in L_2(\mathcal{A})$ the localization $\mathcal{A}_Z$ may only contain exactly one hyperplane from $\mathcal{A}_X$ and $\mathcal{A}_Y$ (otherwise Z would be equal to X or Y). □

Corollary 6.1.2. *Let $\mathcal{A}$ be a supersolvable simplicial 3-arrangement with two modular elements $X, Y \in L_2(\mathcal{A})$ and $|\mathcal{A}_X| \neq |\mathcal{A}_Y|$. Then $\mathcal{A}$ is reducible.*

Proof. For such an arrangement by the previous lemma we can easily find a $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K)$ is not connected and hence by Lemma 5.2.3 it is reducible. □

Lemma 6.1.3 ([Toh14, Lemma 2.1])**.** *Let $\mathcal{A}$ be a supersolvable 3-arrangement. Then all elements $X \in L_2(\mathcal{A})$ with $|\mathcal{A}_X|$ maximal are modular.*

Combining Corollary 6.1.2 and Lemma 6.1.3 we get the following lemma.

Lemma 6.1.4. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement. Then $X \in L_2(\mathcal{A})$ is modular if and only if $|\mathcal{A}_X|$ is maximal under all localizations of elements of rank two.*

Lemma 6.1.5. *Let $\mathcal{A}$ be an irreducible simplicial 3-arrangement such that $\chi_{\mathcal{A}} = (t-1)(t-a)(t-b)$ factors over $\mathbb{Z}$. If $|\mathcal{A}|$ is even, then exactly one of the numbers a, b is even. If $|\mathcal{A}|$ is odd, then a, b are also odd.*

Proof. Compare the coefficient of t, i.e.

$$ab + |\mathcal{A}| - 1 = ab + a + b = \mu_2 = 2|L_2(\mathcal{A})| - 3.$$

By Lemma 3.3.4 the last equation is equivalent to $\mathcal{A}$ being simplicial. Observe that the expression on the right is odd. Thus $ab \equiv |\mathcal{A}| \pmod 2$; the claims now follow from $a + b + 1 = |\mathcal{A}|$. □

We now give a definition of the supersolvable simplicial arrangements this section is concerned with.

Definition 6.1.6. Let $n \in \mathbb{N}$ and $\zeta := \exp(\frac{2\pi i}{2n})$ be a primitive $2n$-th root of unity. We write

$$c_n(m) := \cos \frac{2\pi m}{2n} = \frac{1}{2}(\zeta^m + \zeta^{-m}),$$

and

$$s_n(m) := \sin \frac{2\pi m}{2n} = \frac{1}{2i}(\zeta^m - \zeta^{-m}).$$

The arrangements $\mathcal{A}(2n, 1)$ of the infinite series $\mathcal{R}(1)$ from [Grü09] may be defined by

$$\begin{pmatrix} -s_n(0) & -s_n(1) & \dots & -s_n(n-1) & c_n(1) & c_n(3) & \dots & c_n(2n-1) \\ c_n(0) & c_n(1) & \dots & c_n(n-1) & s_n(1) & s_n(3) & \dots & s_n(2n-1) \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 \end{pmatrix}$$

The arrangements $\mathcal{A}(4n+1, 1)$ of the series $\mathcal{R}(2)$ are constructed as

$$\mathcal{A}(4n+1, 1) = \mathcal{A}(4n, 1) \cup \{(0, 0, 1)^{\perp}\}.$$

Some examples are displayed as projective pictures of the arrangements in Figure 6.1.

Lemma 6.1.7. *Let $\mathcal{A}$ be an irreducible simplicial 3-arrangement, $X \in L_2(\mathcal{A})$ a modular element, $n = |\mathcal{A}_X|$, and $K \in \mathcal{K}(\mathcal{A})$ a chamber with $\langle \overline{K} \cap X \rangle = X$. Then the Coxeter graph $\Gamma(K)$ is the graph of Figure 6.2.*

Proof. Let $B^K = \{\alpha_1, \alpha_2, \alpha_3\}$, and $\mathcal{V} = \{1, 2, 3\}$ the corresponding vertices of $\Gamma(K)$. Since $\mathcal{A}$ is irreducible by Lemma 5.2.3 the graph $\Gamma(K)$ is connected. We may assume that $\{1, 2\}, \{2, 3\} \in E$ and that $m(1, 2) = n$.

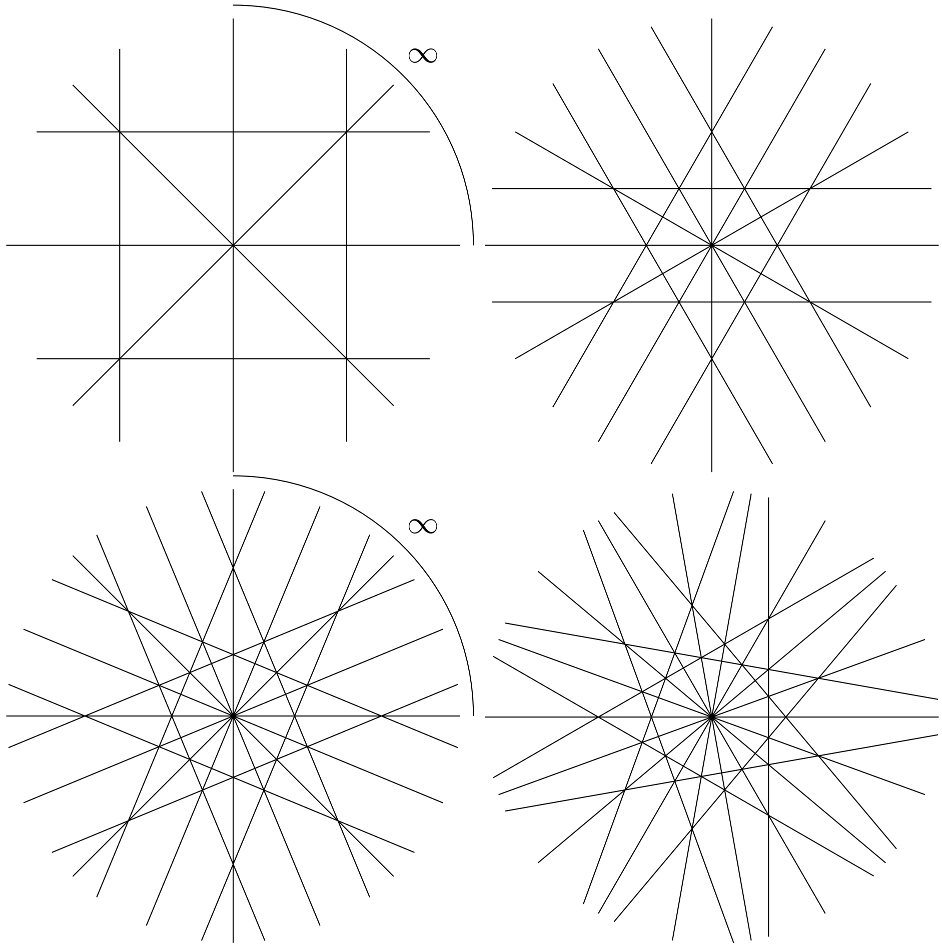


Figure 6.1: Projective pictures of $\mathcal{A}(9,1)$, $\mathcal{A}(12,1)$, $\mathcal{A}(17,1)$, and $\mathcal{A}(18,1)$.

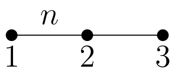


Figure 6.2: The Coxeter graph of a chamber adjacent to the modular element X

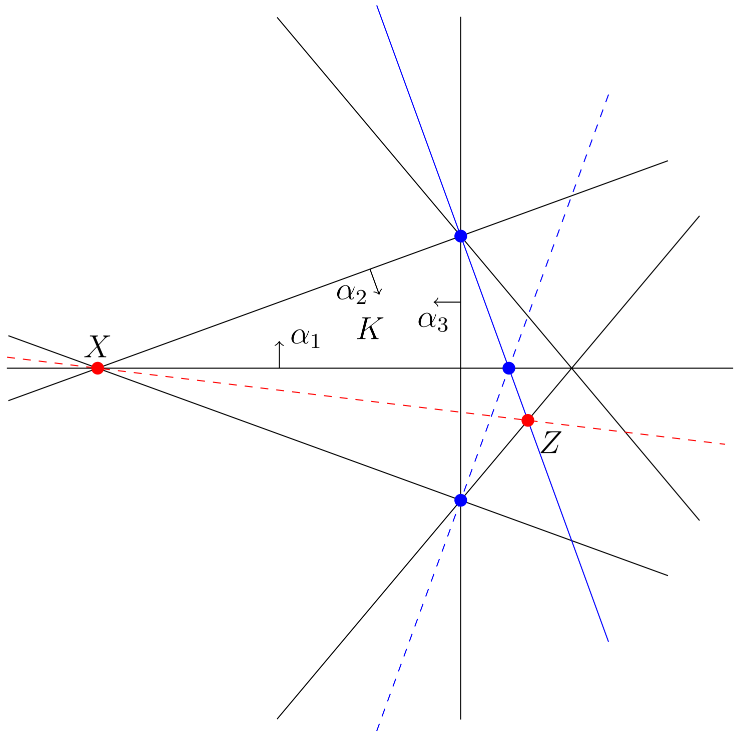


Figure 6.3: Proof of Lemma 6.1.7

First suppose that $\{1,3\} \in E$ and let $H = \sigma_1^K(\alpha_3)^\perp$. Then in particular $H \in \mathcal{A} \setminus \mathcal{A}_X$ so $|\mathcal{A}^H| = n$. But by the last statement of Lemma 5.2.8 we find that $|\mathcal{A}^H| \geq n+1$ which is absurd.

Now suppose that $m(2,3) \geq 4$. Then $(\sigma_3^K(\alpha_2))^\perp$ (the blue line in Figure 6.3) intersects $(\sigma_2^{K_1}(\sigma_1^K(\alpha_3)))^\perp$ in Z. But Z must lie in $(-\alpha_1)^+ \cap (\sigma_1^K(\alpha_2))^+$ or $\alpha_1^+ \cap \alpha_2^+$ since otherwise $m(2,3) \leq 3$, see Figure 6.3. This implies $Z + X \notin L(\mathcal{A})$ which contradicts the modularity of X. □

We now prove the main result of this section. Notice that if $\mathcal{A}$ is not assumed to be finite, then one also obtains an infinite arrangement described in [CG17].

Theorem 6.1.8. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement. Then $\mathcal{A}$ is lattice equivalent to an arrangement in $\mathcal{R}(1) \cup \mathcal{R}(2)$.*

Proof. The proof is in two steps. First we show that there is a subarrangement $\mathcal{B} \subseteq \mathcal{A}$ with $\mathcal{B} \cong \mathcal{A}(2n,1)$. Then we use Lemma 6.1.5 to see that $\mathcal{A}$ might only contain 1 more hyperplane if n is even.

Let $X \in L_2(\mathcal{A})$ be modular and $n := |\mathcal{A}_X|$. Since $\mathcal{A}$ is irreducible we have $n \geq 3$. We define the subarrangement

$$\mathcal{B} := \bigcup_{\substack{K \in \mathcal{K}(\mathcal{A}),\\ \langle \overline{K} \cap X \rangle = X}} W^K.$$

Then by Lemma 6.1.7 we have $|\mathcal{B}| = 2n$. In the following we consider the projective picture of $\mathcal{A}$ respectively $\mathcal{B}$. Then the n lines $H \in \mathcal{B} \setminus \mathcal{A}_X$ are the edge-lines of

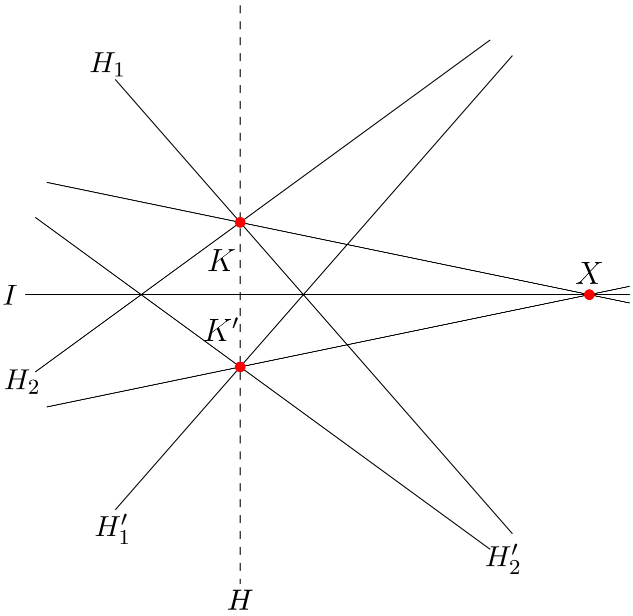


Figure 6.4: The structure of $L(\mathcal{B})$ yields only one possibility for $H \in \mathcal{A} \setminus \mathcal{B}$.

a convex n-gon. By Lemma 6.1.7 all chambers $K \in \mathcal{K}(\mathcal{A})$ adjacent to X have the Coxeter graph of Figure 6.2 and for those we have $\mathcal{B}_Y = \mathcal{A}_Y$ for all $Y \in L_2(W^K)$, i.e. no line of $\mathcal{A} \setminus \mathcal{B}$ intersects the convex n-gon. In particular $\mathcal{A}_X = \mathcal{B}_X$. Furthermore, we have $|\{Y \in L_2(\mathcal{B}) \mid |\mathcal{B}_Y| = 2\}| \geq n$, since each edge of the n-gon contains one such point by the given shape of the Coxeter graphs $\Gamma(K)$ for $\langle \overline{K} \cap X \rangle = X$. The subarrangement $\mathcal{B}$ clearly is supersolvable with modular element X. Since exactly 2 edge-lines of the convex n-gon intersect in a common point we further have $|\mathcal{B}_Y| \leq 3$ for all $Y \in L_2(\mathcal{B}) \setminus \{X\}$. Suppose there is a $Y \in L_2(\mathcal{B})$ with $|\mathcal{B}_Y| = 2$ and $Y \notin L_2(W^K)$ for any chamber K adjacent to X, i.e. Y is an intersection outside of the n-gon. By the supersolvability of $\mathcal{B}$ we have $Y = H_1 \cap H_2$ with $H_1 \in \mathcal{B}_X$ and $H_2 \in \mathcal{B} \setminus \mathcal{B}_X$, i.e. H_2 is an edge-line of the n-gon. But then $|\mathcal{B}^{H_2}| \geq n+1$ contradicting the supersolvability. Thus all intersections Y outside the n-gon are of size 3, i.e. $\mathcal{B}_Y = \{H_1, H_2, H_3\}$ with $H_1 \in \mathcal{B}_X$, H_2, H_3 are edge-lines of the n-gon, and we obtain the following multiset of invariants of the intersection lattice of $\mathcal{B}$:

$$\{\{|\mathcal{B}_Y| \mid Y \in L_2(\mathcal{B})\}\} = \{\{2^n, 3^{|L_2(\mathcal{B})|-n-1}, n^1\}\}.$$

Now

$$\sum_{X \in L_2(\mathcal{B})} (|\mathcal{B}_X| - 1) = n - 1 + n + 2(|L_2(\mathcal{B})| - n - 1) = 2|L_2(\mathcal{B})| - 3,$$

and by Lemma 3.3.4 the supersolvable arrangement $\mathcal{B}$ is simplicial. A projective picture of the arrangement $\mathcal{B}$ is given (after a possible coordinate change) by the edge-lines of a regular convex n-gon and its lines of reflection symmetry, hence $\mathcal{B}$ is L-equivalent to $\mathcal{A}(2n, 1)$. We may assume after an appropiate choice of coordinates that $\mathcal{B} = \mathcal{A}(2n, 1)$ and is given as in Definition 6.1.6 (see [Cun11b, Section 3]).

Assume there is an $H \in \mathcal{A} \setminus \mathcal{B}$. Then for all $H' \in \mathcal{B} \setminus \mathcal{A}_X$, $Y = H \cap H'$ we have $|\mathcal{B}_Y| = 3$ otherwise $|\mathcal{A}^H| \geq n+1$ contradicting the modularity of X. In particular if $K \in \mathcal{K}(\mathcal{B})$ such that $K \cap H \neq \emptyset$, $W^K = \{I, H_1, H_2\}$ (since $\overline{K} \cap X = \{0\}$) with $I \in \mathcal{B}_X$,

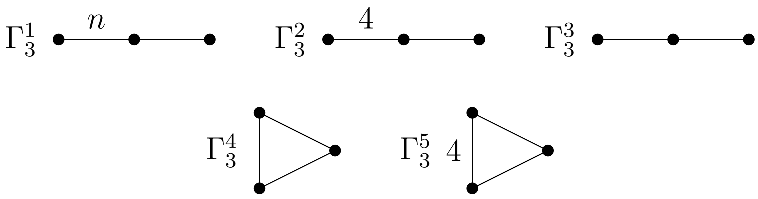


Figure 6.5: Possible Coxeter graphs for an irreducible supersolvable simplicial 3-arrangement.

$H_1, H_2 \in \mathcal{B} \setminus \mathcal{B}_X$, then $H_1 \cap H_2 \subseteq H$. Furthermore, for the adjacent chamber K' with $\langle \overline{K} \cap \overline{K'} \rangle = I$, $W^{K'} = \{I, H_1', H_2'\}$ we also have $H \cap K' \neq \emptyset$ so similarly $H_1' \cap H_2' \subseteq H$. Since H_1, H_2, H_1', H_2' are pairwise different, $H = H_1 \cap H_2 + H_1' \cap H_2'$, see Figure 6.4. Let $\tilde{H} \in \mathcal{A} \setminus \mathcal{B}$ be another hyperplane. Then there exists a chamber $K \in \mathcal{K}(\mathcal{B})$ such that $H \cap K \neq \emptyset$ and $\tilde{H} \cap K \neq \emptyset$ (otherwise there is an $H' \in \mathcal{B} \setminus \mathcal{B}_X$ such that $\tilde{H} \cap H' \notin L_2(\mathcal{B})$ which contradicts the modularity of X). Hence $H = \tilde{H}$. So there is only one possibility for such an H and we obtain $|\mathcal{A} \setminus \mathcal{B}| \leq 1$.

Now suppose $n = |\mathcal{A}_X|$ is odd. Since $\mathcal{A}$ is supersolvable with modular element $X \in L_2(\mathcal{A})$ by Lemma 3.4.8 we have

$$\chi_{\mathcal{A}}(t) = (t-1)(t-a)(t-b),$$

with $a = n-1$ and $b = |\mathcal{A}| - n$. By Lemma 6.1.5 the first root a is even so b has to be odd, i.e. $|\mathcal{A}|$ is even and hence $\mathcal{A} = \mathcal{B}$. If n is even then with a similar argument either $\mathcal{A} = \mathcal{B}$ or there is one more hyperplane $H \in \mathcal{A} \setminus \mathcal{B}$ which has to be $H = (0,0,1)^\perp$ after a possible coordinate change and $\mathcal{A} = \mathcal{A}(4\frac{n}{2}+1, 1)$. □

Remark 6.1.9. Let $\mathcal{A} \in \mathcal{R}(1) \cup \mathcal{R}(2)$. Then by [Cun11b, Thm. 3.6] there exists a minimal subfield $\mathbb{L} \leq \mathbb{R}$ such that there is an arrangement $\mathcal{B}$ in $\mathbb{L}^3$ with $L(\mathcal{B}) \cong L(\mathcal{A})$. Furthermore, if $\mathcal{B}'$ is another arrangement in $\mathbb{L}^3$ which is L-equivalent to $\mathcal{B}$, then there is a semi-linear map $\varphi \in \Gamma\mathrm{L}(\mathbb{L}^3)$ with $\mathcal{B}' = \varphi(\mathcal{B}) = \{\varphi(H) \mid H \in \mathcal{B}\}$. Hence, by the fundamental theorem of projective geometry (see e.g. [Art88, Sec. II.9]) there is a field automorphism μ of $\mathbb{L}$ and an $\psi \in \mathrm{GL}(\mathbb{R}^3)$ such that $\psi(\mu(\mathcal{B}) \otimes_{\mathbb{L}} \mathbb{R}) = \mathcal{A}$. So any (real) arrangement $\mathcal{A}'$ which is L-equivalent to $\mathcal{A}(2n,1)$ or $\mathcal{A}(4m+1,1)$ is essentially this arrangement.

From the proof of Theorem 6.1.8 we obtain the following corollaries.

Corollary 6.1.10. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement and $X \in L_2(\mathcal{A})$ a modular element. Then for all $X' \in L_2(\mathcal{A}) \setminus \{X\}$ we have $|\mathcal{A}_{X'}| \leq 4$.*

Corollary 6.1.11. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-ararrangement, $X \in L_2(\mathcal{A})$ modular with $n = |\mathcal{A}_X|$, and $K \in \mathcal{K}(\mathcal{A})$. Then $\Gamma(K)$ is one of the Coxeter graphs of Figure 6.5. In particular, if $|\mathcal{A}|$ is even or $n \leq 5$, then there is no chamber $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K) = \Gamma_3^5$ and if $n > 4$ and $|\mathcal{A}|$ is even then there is also no chamber $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K) = \Gamma_3^2$.*

Lemma 6.1.12. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 3-arrangement and $H \in \mathcal{A}$. Then for all $H \in \mathcal{A}$ we have*

$$|\mathcal{A}^H| \geq \lceil \frac{|\mathcal{A}|}{4} \rceil + 1.$$

Proof. Let $X \in L_2(\mathcal{A})$ be modular, $n = |\mathcal{A}_X|$, and $H \in \mathcal{A}$. If $H \in \mathcal{A} \setminus \mathcal{A}_X$ then H is a complement of X in $L(\mathcal{A})$, so $\mathcal{A}^H \cong \mathcal{A}_X/X$ and in particular $|\mathcal{A}^H| = n \geq \frac{|\mathcal{A}|}{2} \geq \lceil \frac{|\mathcal{A}|}{4} \rceil + 1$.

Let $t_r^H := |\{X \in \mathcal{A}^H \mid |\mathcal{A}_X| = r\}|$. Then we always have the identity $\sum_{r \geq 2}(r-1)t_r^H = |\mathcal{A}| - 1$. By Corollary 6.1.10 for $H \in \mathcal{A}_X$ we see that $t_r^H = 0$ for $r \notin \{2, 3, 4, n\}$, and $t_n^H = 1$. Furthermore, by Theorem 6.1.8 we have $t_2^H \in \{0, 1, 2\}$ and $t_4^H = 1$ if and only if $|\mathcal{A}| = 2n + 1$ and n is even. So we obtain

$$t_3^H = \frac{|\mathcal{A}| - 1 - t_2^H - 3t_4^H - (n-1)t_n^H}{2} = \frac{|\mathcal{A}| - n - t_2^H - 3t_4^H}{2},$$

and hence

$$|\mathcal{A}^H| = t_2^H + t_3^H + t_4^H + t_n^H = \frac{n + t_2^H}{2} + 1 \geq \lceil \frac{|\mathcal{A}|}{4} \rceil + 1.$$

□

6.2 The rank 4 case

The following proposition and its immediate corollary are the key for the classification of irreducible supersolvable simplicial arrangements of rank $\ell \geq 4$.

Proposition 6.2.1. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement. Then for all $X \in L_2(\mathcal{A})$ we have $|\mathcal{A}_X| \leq 4$.*

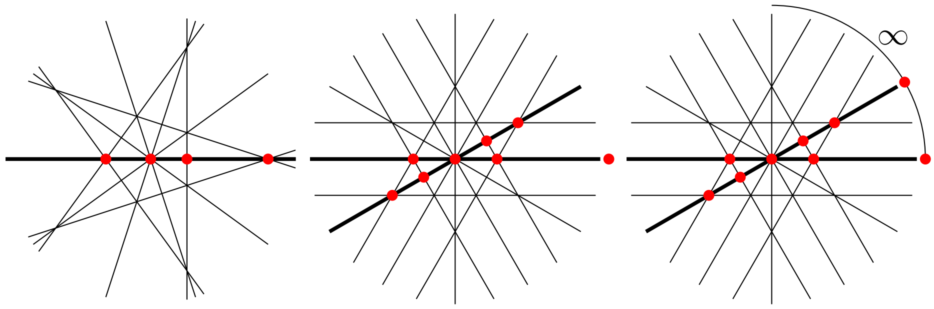
Proof. The proof is in three steps. First we show that if $X \in L_2(\mathcal{A})$ with $|\mathcal{A}_X| \geq 5$ then X necessarily has to be the only rank 2 modular element in $L(\mathcal{A})$. From this we derive that $|\mathcal{A}_X| \leq 6$. Finally by some geometric arguments and using the classification in dimension 3 we exclude the cases $|\mathcal{A}_X| = 5, 6$.

Let $X \in L_2(\mathcal{A})$ be fixed and suppose $|\mathcal{A}_X| \geq 5$.

First assume that there is a modular $X' \in L_2(\mathcal{A}) \setminus \{X\}$. By the irreducibility of $\mathcal{A}$ there is an $H \in \mathcal{A}$ transversal to X and X', i.e. such that $X \nsubseteq H$, $X' \nsubseteq H$, and also $X \cap X' \nsubseteq H$ if $X \cap X' \in L_3(\mathcal{A})$. Let $Y = H \cap X$ and $Y' = H \cap X'$. By Lemma 3.4.4 and Lemma 5.2.9 the restriction $\mathcal{A}^H$ is an irreducible supersolvable simplicial 3-arrangement. Furthermore, $Y \neq Y'$ and $5 \leq |\mathcal{A}_Y^H| \leq |\mathcal{A}_{Y'}^H|$ for the 3-arrangement $\mathcal{A}^H$ by Lemma 6.1.4 since Y' is a modular element in $L_2(\mathcal{A}^H)$. But this contradicts Corollary 6.1.10, the irreducible supersolvable simplicial 3-arrangement $\mathcal{A}^H$ cannot have two distinct rank 2 intersections of size greater or equal to 5, one of them modular. Hence X is the only modular element in $L_2(\mathcal{A})$ and also the one single element in $L_2(\mathcal{A})$ with $|\mathcal{A}_X| \geq 5$.

From now on to the end of the proof let $Y \in L_3(\mathcal{A})$ be a fixed modular intersection of rank 3 with $Y > X$.

Suppose that $|\mathcal{A}_X| \geq 7$. Then since $\mathcal{A}$ is irreducible, by Lemma 3.4.3 the localization $\mathcal{A}_Y/Y$ regarded as an essential 3-arrangement in V/Y is an irreducible supersolvable simplicial 3-arrangement with modular element $X/Y \in L_2(\mathcal{A}_Y/Y))$. So by Theorem



$\mathcal{A}(10,1)$ $\mathcal{A}(12,1)$ $\mathcal{A}(13,1)$

Figure 6.6: $|\mathcal{A}_Y^H| = 4, 5, 6$ respectively for $H \in \mathcal{A}_X$.

6.1.8 we have $|\mathcal{A}_Y| \geq 14$. Let $H \in \mathcal{A}_X$. By Lemma 5.2.9 the restriction $\mathcal{A}^H$ is irreducible and by Corollary 5.2.6 there is a $Y' \in L_2(\mathcal{A}^H) \setminus \{Y\}$ with $Y' \subseteq X$ such that $|(\mathcal{A}^H)_{Y'}| \geq 3$. Since $\mathcal{A}_{Y'}/Y'$ is an irreducible supersolvable simplicial 3-arrangement with modular element X/Y', as for $\mathcal{A}_Y$ we have $|\mathcal{A}_{Y'}| \geq 14$. By Lemma 3.4.4 the rank 3 intersection $Y \cap H = Y$ is modular in $L(\mathcal{A}^H)$ for $H \in \mathcal{A}_X$. By Lemma 6.1.12 we further have $|(\mathcal{A}^H)_Y| = |(\mathcal{A}_Y)^H| \geq 5$ and similarly $|(\mathcal{A}^H)_{Y'}| \geq 5$. Because of the choice of $Y' \in L_2(\mathcal{A}^H) \setminus \{Y\}$ the irreducible supersolvable simplicial 3-arrangement $\mathcal{A}^H$ has two distinct rank 2 intersections of size greater or equal to 5 which contradicts Corollary 6.1.10. Hence $|\mathcal{A}_X| \leq 6$.

To exclude the cases $|\mathcal{A}_X| \in \{5, 6\}$ first assume that $|\mathcal{A}_X| = 6$. We may assume that there is an $Y' \in L_3(\mathcal{A})$, $Y' \neq Y$, and $Y' > X$ such that $\mathcal{A}_{Y'}/Y'$ is an irreducible supersolvable simplicial 3-arrangement. So by Theorem 6.1.8 we have $\mathcal{A}_{Y'}/Y' \sim_L \mathcal{A}(12,1)$ or $\mathcal{A}_{Y'}/Y' \sim_L \mathcal{A}(13,1)$. But then there is an $H \in \mathcal{A}_X$ such that $|\mathcal{A}_{Y'}^H| \geq 5$ which is immediately clear by Figure 6.6. Since by Lemma 3.4.4 the $Y = Y \cap H$ is a rank 2 modular element in $L(\mathcal{A}^H)$ different from $Y' \cap H = Y' \in L_2(\mathcal{A}^H)$, with Corollary 6.1.10 we get a contradiction.

Finally, suppose $|\mathcal{A}_X| = 5$. Then we have $\mathcal{A}_Y/Y \sim_L \mathcal{A}(10,1)$. Again we may assume that there is an $Y' \in L_3(\mathcal{A})$, $Y' \neq Y$, and $Y' > X$ such that $\mathcal{A}_{Y'}/Y'$ is an irreducible supersolvable simplicial 3-arrangement. So $\mathcal{A}_{Y'}/Y' \sim_L \mathcal{A}(10,1)$. Let $H \in \mathcal{A}_X$. Then $|\mathcal{A}_Y^H| = |\mathcal{A}_{Y'}^H| = 4$, see Figure 6.6. Since by Lemma 5.2.9 $\mathcal{A}^H$ is an irreducible supersolvable simplicial 3-arrangement with modular element Y by Theorem 6.1.8 we have $\mathcal{A}^H \sim_L \mathcal{A}(9,1) \cong \mathcal{A}(B_3)$. For the other restrictions $\mathcal{A}^{H'}$ with $H' \in \mathcal{A} \setminus \mathcal{A}_X$ we have $\mathcal{A}^{H'} \sim_L \mathcal{A}(10,1)$. The arrangement $\mathcal{A}$ is supersolvable and by Theorem 3.4.8 the characteristic polynomial factors as follows over the integers

$$\chi_{\mathcal{A}}(t) = (t-1)(t-4)(t-5)(t-(|\mathcal{A}|-10)).$$

Similarly for $H \in \mathcal{A}_X$ by Theorem 3.4.8 we have

$$\chi_{\mathcal{A}^H}(t) = (t-1)(t-3)(t-5),$$

and for $H \in \mathcal{A} \setminus \mathcal{A}_X$

$$\chi_{\mathcal{A}^H}(t) = (t-1)(t-4)(t-5).$$

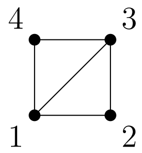


Figure 6.7: Forbidden subgraph.

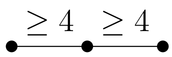


Figure 6.8: Forbidden subgraph of a chamber in $\mathcal{A}^H$ by Corollary 6.1.11.

Now we use Lemma 3.3.5 and insert the numbers:

$$\begin{aligned} 0 &= \ell|\chi_{\mathcal{A}}(-1)| - 2\sum_{H\in\mathcal{A}} |\chi_{\mathcal{A}^H}(-1)| \\ &= \ell|\chi_{\mathcal{A}}(-1)| - 2(\sum_{H\in\mathcal{A}_X} |\chi_{\mathcal{A}^H}(-1)| + \sum_{H\in\mathcal{A}\setminus\mathcal{A}_X} |\chi_{\mathcal{A}^H}(-1)|) \\ &= (4\cdot 2\cdot 5\cdot 6)(|\mathcal{A}| - 9) - 2(5\cdot 2\cdot 4\cdot 6 + (|\mathcal{A}| - 5\cdot 2\cdot 5\cdot 6) \\ &= 2|\mathcal{A}| - 18 - 4 - |\mathcal{A}| + 5 \\ &= |\mathcal{A}| - 17. \end{aligned}$$

Thus $|\mathcal{A}| = 17$. Since $|\mathcal{A}_Y \cup \mathcal{A}_{Y'}| = 15$ there are exactly 2 other hyperplanes H_1, H_2 not contained in either $\mathcal{A}_Y$ or in $\mathcal{A}_{Y'}$. But then there is a $Z \in L_2(\mathcal{A})$, $Z \subsetneq H_i$ for an $i = 1, 2$ such that $Z \notin \mathcal{A}_Y^{H_i}$. This contradicts the modularity of Y and finishes the proof. □

From the previous proposition, by taking localizations and Lemma 5.2.5 we immediately obtain the following theorem.

Theorem 6.2.2. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement with $\ell \geq 4$. Then for all $X \in L_2(\mathcal{A})$ we have $|\mathcal{A}_X| \leq 4$.*

After establishing this strong constraint, in a sequence of lemmas we will decimate the number of possible Coxeter graphs for irreducible supersolvable simplicial 4-arrangements. We will use this to derive the crystallographic property at the end of this section.

Lemma 6.2.3. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ has no subgraph of the form shown in Figure 6.7.*

Proof. Suppose there exists a chamber $K \in \mathcal{K}(\mathcal{A})$ with $B^K = \{\alpha_1, \ldots, \alpha_4\}$ such that $\Gamma(K)$ has a subgraph of this form. Then by Lemma 5.2.8 for $H = \sigma_1^K(\alpha_3)^\perp \in \mathcal{A}_{13}$ and the chamber $K_1^H \in \mathcal{K}(\mathcal{A}^H)$ we find that the graph of figure 6.8 is a subgraph of $\Gamma(K^H)$.

But this is a contradiction since by Lemma 3.4.4 and Lemma 5.2.9 the restricted arrangement $\mathcal{A}^H$ is an irreducible supersolvable simplicial 3-arrangement and such a graph is not contained in the list of Corollary 6.1.11. □

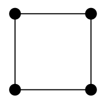


Figure 6.9: No 4-circle.

Lemma 6.2.4. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ has no subgraph of the form shown in Figure 6.9.*

Proof. It is convenient to denote the graph of Figure 6.9 by $\tilde{\Gamma}$.

Suppose there is a chamber such that $\tilde{\Gamma}$ is a subgraph of $\Gamma(K)$ and let K' be an adjacent chamber. By Lemma 6.2.3 the graph $\Gamma(K)$ cannot have a chord. But then by Lemma 5.2.2 the Coxeter graph $\Gamma(K')$ of the adjacent chamber also has a subgraph of the form shown in Figure 6.9 and hence, disregarding the labels, $\Gamma(K')$ is the same graph as $\Gamma(K)$. Thus by induction for all chambers $K \in \mathcal{K}(\mathcal{A})$ the graph $\tilde{\Gamma}$ is a subgraph of $\Gamma(K)$. Now let $X \in L_3(\mathcal{A})$ and $K \in \mathcal{K}(\mathcal{A})$ be some chamber adjacent to X, i.e. $X \in L_3(W^K)$. Then by Lemma 5.2.5 the Coxeter graph $\Gamma(K_X)$ for a chamber $K_X \in \mathcal{K}(\mathcal{A}_X/X)$ contains an induced subgraph on 3 vertices of $\tilde{\Gamma}$ and thus is connected. So $\mathcal{A}_X$ is irreducible for all $X \in L_3(\mathcal{A})$. This is a contradiction to Theorem 2.6. □

To give a complete list of all possible Coxeter graphs of irreducible supersolvable simplicial 4-arrangements we need the explicit description of the change of Coxeter graphs for adjacent chambers in the three possible irreducible rank 3 localizations given by the next lemma.

Lemma 6.2.5. *Let $\mathcal{A}$ be one of the arrangements $\mathcal{A}(A_3)$, $\mathcal{A}(B_3)$ or $\mathcal{A}_3^2$. Then Figure 6.10 gives a complete description of the change of the Coxeter graphs for adjacent chambers where an arrow labeled with σ_i means crossing the i-th wall corresponding to the i-th vertex of the Coxeter graph.*

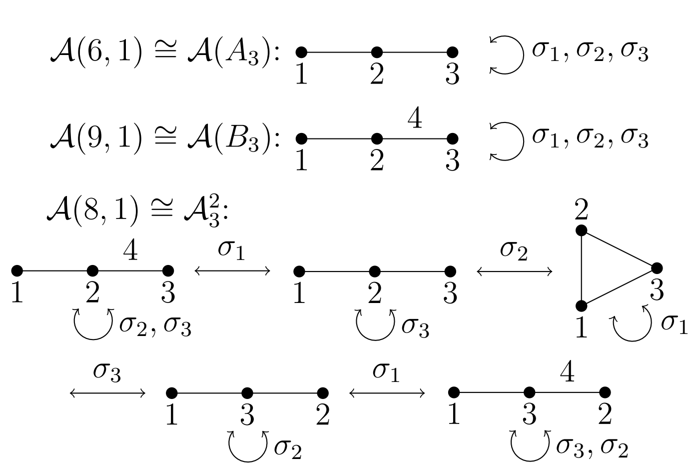


Figure 6.10: Diagrams for the change of Coxeter graphs of adjacent chambers in $\mathcal{A}(A_3), \mathcal{A}(B_3)$, and $\mathcal{A}_3^2$ respectively.

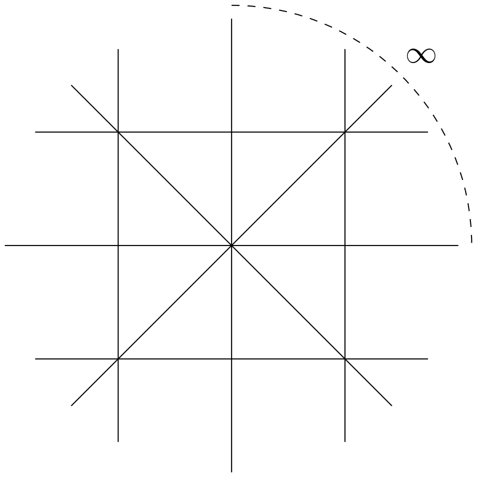


Figure 6.11: The arrangements $\mathcal{A}(8,1)$ and $\mathcal{A}(9,1)$

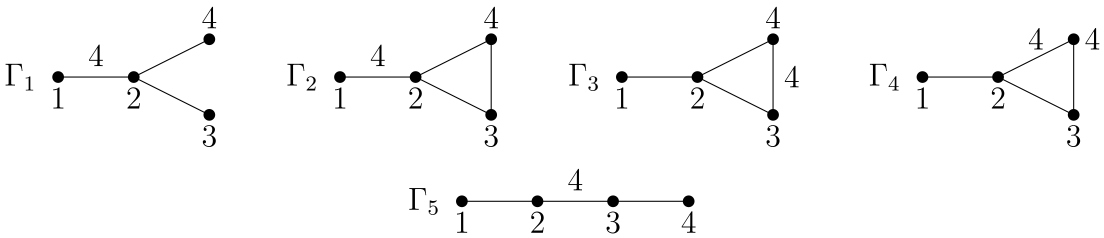


Figure 6.12: Impossible Coxeter graphs for an irreducible supersolvable simplicial 4-arrangement.

Proof. The diagrams for $\mathcal{A}(A_3)$ and $\mathcal{A}(B_3)$ are obvious since they are reflection arrangements and hence for all chambers the Coxeter graph is the same.

The third diagram can be seen by looking at a projective picture of the arrangement (see Figure 6.11) or as a special case of [CH15a, Prop. 3.8]. $\square$

Lemma 6.2.6. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ is not one of the graphs of Figure 6.12.*

Proof. Let $B^K = \{\alpha_1, \dots, \alpha_4\}$ be a basis for K.

First suppose that there is a $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K) = \Gamma_1$. By Lemma 3.4.5(1) the arrangements $\mathcal{A}^K_{123} := \mathcal{A}_X/X$ with $X = \alpha_1^\perp \cap \alpha_2^\perp \cap \alpha_3^\perp$ and $\mathcal{A}^K_{124} := \mathcal{A}_{X'}/X'$ with $X' = \alpha_1^\perp \cap \alpha_2^\perp \cap \alpha_4^\perp$ are irreducible supersolvable simplicial arrangements. By Lemma 5.2.5, Lemma 5.2.3 and Corollary 6.1.11 the two arrangements are either $\mathcal{A}(8,1)$ or $\mathcal{A}(9,1)$. Since $|\mathcal{A}^K_{23}| = |\mathcal{A}^K_{24}| = 3$ by Lemma 6.1.4 the intersection $Y = \alpha_1^\perp \cap \alpha_2^\perp$ is modular in $\mathcal{A}_X$ and $\mathcal{A}_{X'}$. Let $H = \sigma_2^K(\alpha_1) \in \mathcal{A}_Y$ then by Lemma 5.2.8 the Coxeter graph of $K_2^H \in \mathcal{K}(\mathcal{A}^H)$ contains a subgraph of the form shown in Figure 6.13.

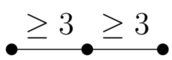


Figure 6.13: Subgraph of a chamber in $\mathcal{A}^H$.

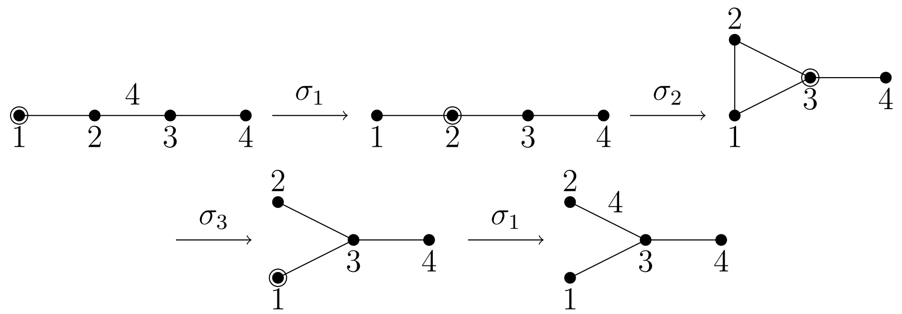


Figure 6.14: A sequence of graphs of chambers in $\mathcal{A}$ starting at K and leading to a contradiction.

For the arrangements $\mathcal{A}(8,1)$ and $\mathcal{A}(9,1)$ in both cases we have $|\mathcal{A}_X^H| = |\mathcal{A}_{X'}^H| = 4$. So actually both labels of the Coxeter subgraph are equal to 4 and $\Gamma(K_2^H)$ contains a subgraph as in Figure 6.8. This is a contradiction to Corollary 6.1.11 and we can exclude the graph Γ_1 from the list of possible Coxeter graphs of irreducible supersolvable simplicial 4-arrangements.

Secondly suppose that $\Gamma(K) = \Gamma_2$. Then by Lemma 5.2.8 there is a hyperplane $H \in \mathcal{A}_{23}^K$ and a chamber $K^H \in \mathcal{K}(\mathcal{A}^H)$ such that the graph shown in figure 6.8 is a subgraph of $\Gamma(K^H)$ contradicting Corollary 6.1.11 again.

For the graphs Γ_3 and Γ_4 the localization $\mathcal{A}_{234}^K$ is an irreducible supersolvable simplicial 3-arrangement. By Theorem 6.2.2 it hast rank 2 localizations of size at most 4. By Lemma 5.2.5 there is a chamber in $\mathcal{A}_{234}^K$ with Coxeter graph the induced subgraph on the vertices $\{2,3,4\}$. But this a contradiction to Corollary 6.1.11.

Finally, suppose that there is a $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K) = \Gamma_5$ and let $B^K = \{\alpha_1, \dots, \alpha_4\}$. Let $X = \alpha_1 \cap \alpha_2 \cap \alpha_3$, $X' = \alpha_2 \cap \alpha_3 \cap \alpha_4$, $\mathcal{A}_{123} = \mathcal{A}_X/X$ and $\mathcal{A}_{234} = \mathcal{A}_{X'}/X'$. By Lemma 3.4.5(1) these arrangements are supersolvable and simplicial, and by Lemma 5.2.5, Lemma 5.2.3 and Corollary 6.1.11 the two arrangements are either $\mathcal{A}(8,1)$ or $\mathcal{A}(9,1)$. If both arrangements are $\mathcal{A}(9,1)$ then for all $H \in \mathcal{A}_Y$ with $Y = \alpha_2 \cap \alpha_3$ we have $|\mathcal{A}_X^H| = |\mathcal{A}_{X'}^H| = 4$ (see Figure 6.11) and similarly to the first part of this proof we can find an H' and a $K'^{H'} \in \mathcal{K}(\mathcal{A}^{H'})$ which contains the forbidden Coxeter subgraph of Figure 6.8. So assume without loss of generality that $\mathcal{A}_{123}$ is equal to $\mathcal{A}(8,1)$. We use Lemma 5.2.5, Lemma 5.2.2 and Lemma 6.2.5 to get for example the following sequence of Coxeter graphs for the corresponding sequence of chambers of Figure 6.14. But the last graph in this sequence is (after renumbering the vertices) the graph Γ_1 which we already excluded. Similarly there are a few other possible sequences of graphs which we omit here all ending in a Coxeter graph already excluded. Hence Γ_5 is not the Coxeter graph of a chamber of an irreducible supersolvable simplicial 4-arrangement. □

Proposition 6.2.7. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and $K \in \mathcal{K}(\mathcal{A})$. Then $\Gamma(K)$ is one of the Coxeter graphs displayed in Figure 6.15.*

Proof. By Lemma 6.2.4 no big circles are possible and by Proposition 6.2.1 all labels are at most 4. Furthermore, with Lemma 5.2.5, Lemma 5.2.8 and Corollary 6.1.11 we see that the graph cannot contain two edges labeled with 4 by looking at the

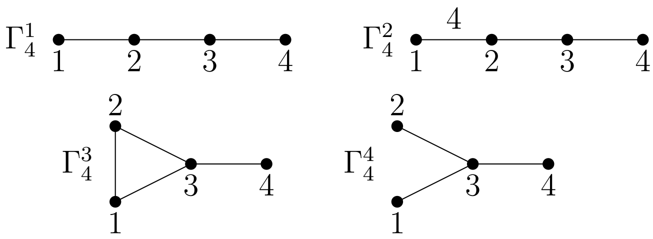


Figure 6.15: Possible Coxeter graphs for an irreducible supersolvable simplicial 4-arrangement.

appropriate restriction respectively localization not fitting into the classification of rank 3 arrangements and their Coxeter graphs (see Theorem 6.1.8 and Corollary 6.1.11). Now by Lemma 6.2.6 the only possible Coxeter graphs left are the ones of Figure 6.15. □

Proposition 6.2.8. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and $K \in \mathcal{K}(\mathcal{A})$.*

(1) There exists a locally crystallographic basis B^K for K such that the Cartan matrix C^K with respect to B^K is of type A, C, D' or D.

(2) If B^K is a locally crystallographic basis for K with C^K of type A, C, D' or D, then for $1 \leq i \leq 4$ the basis $B^{K_i} = \sigma_i^K(B^K) = \{\alpha_j + c_{ij}^K \alpha_i \mid 1 \leq j \leq 4\}$ is a locally crystallographic basis with Cartan matrix C^{K_i} of type A, C, D' or D.

Proof. Part (1). By Proposition 6.2.7 the Coxeter graph $\Gamma(K)$ is one of the graphs of Figure 6.15. Let $W^K = \{H_1, \ldots, H_4\}$, and $\Gamma(K) = (\mathcal{V}, \mathcal{E})$ with numbering of the walls corresponding to the numbering of the vertices of the graphs in Figure 6.15.

Firstly suppose that $\Gamma(K) = \Gamma_4^1$. By Lemma 3.4.5 and Lemma 5.2.5 the localization $\mathcal{A}_{123}^K$ adjacent to K is an irreducible supersolvable simplicial 3-arrangement with a modular rank 2 intersection of size at most 4 by Theorem 6.2.2. Hence by Theorem 6.1.8 and Corollary 6.1.11 it is the arrangement $\mathcal{A}(6,1)$ or $\mathcal{A}(8,1)$ and in particular crystallographic (see Example 3.3.24). By choosing a crystallographic root system for $\mathcal{A}_{123}^K$ and taking the corresponding basis for the chamber in the localization by Example 3.3.24 there are $\alpha_1, \alpha_2, \alpha_3 \in (\mathbb{R}^4)^*$ such that $\alpha_i^\perp = H_i$, $K \subseteq \alpha_i^+$, $(\alpha_1 + \alpha_2)^\perp \in W^{K_1}, W^{K_2}$, and $(\alpha_2 + \alpha_3)^\perp \in W^{K_2}, W^{K_3}$. Let $\tilde{\alpha_4} \in (\mathbb{R}^4)^*$ such that $\tilde{\alpha_4}^\perp = H_4$ and $\tilde{\alpha_4}^+ \supseteq K$. Since $\{3,4\} \in \mathcal{E}$ with label $m^K(3,4) = 3$ there is a unique $\lambda \in \mathbb{R}_{>0}$ such that $(\alpha_3 + \lambda\tilde{\alpha_4})^\perp \in W^{K_3}, W^{K_4}$. But then with $\alpha_4 := \lambda\tilde{\alpha_4}$ we have $(\alpha_3 + \alpha_4)^\perp \in W^{K_3}, W^{K_4}$. Hence $B^K := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a locally crystallographic basis for K with Cartan matrix $C^K = (c_{ij}^K)$ of type A.

The same arguments work for the Coxeter graphs Γ_4^3 and Γ_4^4 since the vertex denoted as 4 is only connected with the vertex 3 and the localization $\mathcal{A}_{123}^K$ is $\mathcal{A}(6,1)$ or $\mathcal{A}(8,1)$ by Theorem 6.1.8. So similarly there is a locally crystallographic basis B^K for K such that the Cartan matrix is of type D' if $\Gamma(K) = \Gamma_4^3$, or of type D if $\Gamma(K) = \Gamma_4^4$.

Now assume that $\Gamma(K) = \Gamma_4^2$. Then $\mathcal{A}_{123}^K$ is $\mathcal{A}(8,1) = \mathcal{A}_3^2$ or $\mathcal{A}(9,1) = \mathcal{A}(B_3)$. Then there are $\alpha_1, \alpha_2, \alpha_3 \in (\mathbb{R}^4)^*$ such that $\alpha_i^\perp = H_i$, $K \subseteq \alpha_i^+$, $(2\alpha_1 + \alpha_2)^\perp \in W^{K_2}$, $(\alpha_1 + \alpha_2)^\perp \in W^{K_1}$, and $(\alpha_2 + \alpha_3)^\perp \in W^{K_2}, W^{K_3}$ (by choosing a proper crystallographic root system for the localization and taking the corresponding basis for the chamber in the localization). Again it is clear that we can find an $\alpha_4 \in (\mathbb{R}^4)^*$, $K \subseteq \alpha_4^+$ such that $(\alpha_3+\alpha_4)^\perp \in W^{K_3}, W^{K_4}$ and hence $B^K := \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a locally crystallographic basis for K with Cartan matrix $C^K = (c_{ij}^K)$ of type C.

Part (2). For the second part we use Proposition 6.2.7, Lemma 3.3.26, Lemma 3.3.27, Lemma 5.2.1, Lemma 5.2.4, and Lemma 6.2.5 to obtain the Coxeter graphs for the adjacent chamber K_i and deduce the claimed property of the induced basis B^{K_i} and the coefficients c_{ij}^K:

We check the cases in turn. First assume that $\Gamma(K) = \Gamma_4^1$, B^K is locally crystallographic and C^K is of type A. As we have seen in the proof of Part (1), the localization $\mathcal{A}_{123}^K$ is the arrangement $\mathcal{A}(6,1)$ or $\mathcal{A}(8,1)$.

Let $i=1$. By Proposition 6.2.7 The Coxeter graph $\Gamma(K_1)$ is one of the four graphs of Figure 6.15 and by Lemma 3.3.27 and Lemma 5.2.4 only Γ_4^1 is possible. Thus $\Gamma(K_1) = \Gamma(K)$ and by Lemma 3.3.26, Lemma 3.3.27, Lemma 5.2.1, and Lemma 5.2.4 the basis B^{K_1} induced by C^K and B^K is locally crystallographic with Cartan matrix $C^{K_1} = C^K$ of type A.

Next let $i=2$. If the localizations $\mathcal{A}_{123}^K$ and $\mathcal{A}_{234}^K$ are both isomorphic to $\mathcal{A}(6,1)$ then using the same lemmas from Subsection 3.3 as above, the basis B^{K_2} defined by C^K is locally crystallographic with Cartan matrix $C^{K_2} = C^K$ of type A. If $\mathcal{A}_{123}^K$ is the arrangement $\mathcal{A}(8,1)$ then $\mathcal{A}_{234}^K$ has to be the arrangement $\mathcal{A}(6,1)$ and $\Gamma(K_2) = \Gamma_4^3$. Otherwise by Lemma 6.2.5 we would get a forbidden Coxeter graph of Figure 6.12 for K_2. With the lemmas from Subsection 3.3 and Section 5 we again obtain all coefficients $c_{ij}^{K_2}$ except the ones with $\{i,j\} = \{1,3\}$. But $\mathcal{A}_{123}^K = \mathcal{A}_{123}^{K_2}$ is the arrangement $\mathcal{A}(8,1)$ for which we know that with respect to the basis $B_{123}^K = \{\alpha_1,\alpha_2,\alpha_3\} \subseteq B^K$ we have $(\alpha_1+2\alpha_2+\alpha_3)^\perp \in \mathcal{A}_{123}^K$ and in particular $(\alpha_1+2\alpha_2+\alpha_3)^\perp = (\sigma_2^K(\alpha_1)+\sigma_2^K(\alpha_3))^\perp \in W^{(K_2)_1}, W^{(K_2)_3}$. So $c_{13}^{K_2} = c_{31}^{K_2} = -1$ and B^{K_2} is locally crystallographic with Cartan matrix C^{K_2} of type D'.

Now let $i=3$. If the localizations $\mathcal{A}_{123}^K$ and $\mathcal{A}_{234}^K$ are both isomorphic to $\mathcal{A}(6,1)$ or if $\mathcal{A}_{234}^K$ is the arrangement $\mathcal{A}(8,1)$ then by symmetry we already handled these cases. So suppose that $\mathcal{A}_{123}^K$ is the arrangement $\mathcal{A}(8,1)$. Then by Lemma 6.2.5 and the lemmas from Subsection 3.3 and Section 5 we have $\Gamma(K_3) = \Gamma_4^2$ of Figure 6.15 and we also obtain all $c_{ij}^{K_3}$ except $c_{21}^{K_3}$. But with respect to the basis $B_{123}^K = \{\alpha_1,\alpha_2,\alpha_3\} \subseteq B^K$ we have $(2\alpha_1+\alpha_2+\alpha_3)^\perp = (2\sigma_3^K(\alpha_1)+\sigma_3^K(\alpha_2))^\perp \in \mathcal{A}_{123}^K$ so $c_{21}^{K_3} = -2$ and B^{K_3} is locally crystallographic with Cartan matrix C^{K_3} of type C.

By symmetry we have also shown the claim for $i=4$, for $\Gamma(K) = \Gamma_4^2$ and $i=3$, $\Gamma(K) = \Gamma_3^4$ and $i \in \{2,3\}$. All the other remaining cases can be handled completely analogously. □

Proposition 6.2.8 immediately tells us that an irreducible supersolvable simplicial 4-arrangement defines a Weyl groupoid and thus a crystallographic arrangement (cf. [Cun11a, Thm. 1.1]). Since we did not introduce the notion of a Weyl groupoid, we repeat the argument without this terminology:

Proposition 6.2.9. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement, and fix a chamber $K^0 \in \mathcal{K}(\mathcal{A})$. Then there exists a basis B^{K^0} for K^0 such that*

$$R := \bigcup_{G \in \mathcal{G}(K^0, \mathcal{A})} B_G$$

is a crystallographic root system for $\mathcal{A}$.

Proof. By Proposition 6.2.8(1) for K^0 there exists a locally crystallographic basis B^{K^0} with Cartan matrix of type A, C, D' or D. Such a basis will have the desired property and from now on we fix it.

First we show that for $K \in \mathcal{K}(\mathcal{A})$ the basis B_G^K does not depend on the chosen $G \in \mathcal{G}(K^0, \mathcal{A})$ with $e(G) = K$. Let $G, \tilde{G} \in \mathcal{G}(K^0, \mathcal{A})$ with $e(G) = e(\tilde{G}) = K$, say

$$G = (K^0, K^1, \ldots, K^{n-1}, K^n = K),$$

and

$$\tilde{G} = (K^0, \tilde{K}^1, \ldots, \tilde{K}^{m-1}, \tilde{K}^m = K).$$

Then

$$B_G = (\sigma_{\mu_{n-1}}^{K^{n-1}} \circ \ldots \circ \sigma_{\mu_0}^{K^0})(B^{K^0}),$$

where the linear map $\sigma_{\mu_{n-1}}^{K^{n-1}} \circ \ldots \circ \sigma_{\mu_0}^{K^0}$ is represented with respect to B^{K^0} by a product of reflection matrices

$$S_{\mu_{n-1}}^{K^{n-1}} \cdots S_{\mu_0}^{K^0} =: S.$$

By Proposition 6.2.8(2) and an easy induction over the length n of G all reflection matrices $S_{\mu_i}^{K^i}$ are integral matrices with determinant ± 1. Hence the product S has only entries in $\mathbb{Z}$ and has determinant ± 1. Similarly for $\tilde{G}$ we have

$$B_{\tilde{G}} = (\sigma_{\tilde{\mu}_{m-1}}^{\tilde{K}^{m-1}} \circ \ldots \circ \sigma_{\tilde{\mu}_0}^{K^0})(B^{K^0}),$$

where the linear map is represented with respect to B^{K^0} by a product of integral reflection matrices

$$S_{\tilde{\mu}_{m-1}}^{\tilde{K}^{n-1}} \cdots S_{\tilde{\mu}_0}^{K^0} =: \tilde{S},$$

and $\tilde{S}$ also has only entries in $\mathbb{Z}$ and determinant equal to ± 1. Now $S\tilde{S}^{-1} = \operatorname{diag}(\lambda_1, \ldots, \lambda_4)$ with $\lambda_i \in \mathbb{R}_{>0}$ but also has determinant 1 and all entries are in $\mathbb{Z}$. Thus $S\tilde{S}^{-1}$ is the identity matrix and $B_G = B_{\tilde{G}}$.

From the above consideration we obtain

$$B_G \subseteq \sum_{\alpha \in B_{G'}} \mathbb{Z}\alpha,$$

for $G, G' \in \mathcal{G}(K^0, \mathcal{A})$. Hence for R we have

$$R \subseteq \sum_{\alpha \in B_R^K} \mathbb{Z}\alpha,$$

for all $K \in \mathcal{K}(\mathcal{A})$ since $B_R^K = B_G$ for some $G \in \mathcal{G}(K^0, \mathcal{A})$ with $e(G) = K$ and each $\beta \in R$ is contained in some $B_{G'}$, $G' \in \mathcal{G}(K^0, \mathcal{A})$.

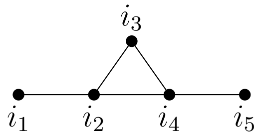


Figure 6.16: Forbidden subgraph

It remains to show that R is reduced, i.e. that for $\beta \in R$ we have $R \cap \mathbb{R}\beta = \{\pm\beta\}$. So suppose that $\beta \in R$ and $\lambda\beta \in R$ for some $\lambda \in \mathbb{R}\setminus\{0\}$. Then there are $G, G' \in \mathcal{G}(K^0, \mathcal{A})$ such that $\beta \in B_G$ and $\lambda\beta \in B_{G'}$. But as above $\lambda\beta \in \mathbb{Z}\beta$ and $\beta \in \mathbb{Z}(\lambda\beta)$. Hence $\lambda \in \{\pm 1\}$. □

Theorem 6.2.10. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement. Then $\mathcal{A}$ is isomorphic to either one of the reflection arrangements $\mathcal{A}(A_4)$, $\mathcal{A}(B_4)$, or isomorphic to $\mathcal{A}_4^3 = \mathcal{A}(B_4) \setminus \{\{x_1 = 0\}\}$. In particular $\mathcal{A}$ is crystallographic.*

Proof. By Proposition 6.2.9 there exists a crystallographic root system for $\mathcal{A}$, so the arrangement $\mathcal{A}$ is crystallographic. By Theorem 3.4.11 the only irreducible crystallographic 4-arrangements which are supersolvable are the three arrangements $\mathcal{A}(A_4)$, $\mathcal{A}(B_4)$, and $\mathcal{A}_4^3 = \mathcal{A}(B_4) \setminus \{\{x_1 = 0\}\}$. □

Corollary 6.2.11. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial 4-arrangement and $K \in \mathcal{K}(\mathcal{A})$. Then $\Gamma(K)$ is not the Coxeter graph Γ_4^4 of Figure 6.15.*

6.3 The rank ≥ 5 case

Lemma 6.3.1. *Let $\mathcal{A}$ be an irreducible simplicial supersolvable ℓ-arrangement and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ has no circles with more than 3 vertices.*

Proof. Suppose there is a chamber $K \in \mathcal{K}(\mathcal{A})$ such that $\Gamma(K)$ has a circle with more than three vertices. Then we localize at the intersection of the walls corresponding to these vertices and use Lemma 5.2.5 and Lemma 5.2.8 (possibly several times) to arrive at an 4-arrangement which is irreducible by Lemma 5.2.9, simplicial and supersolvable by Lemma 3.4.5, and contains a chamber K' such that the Coxeter graph $\Gamma(K')$ contains a subgraph of the form displayed in Figure 6.9. This is a contradiction to Lemma 6.2.4. □

Lemma 6.3.2. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement, $\ell \geq 5$ and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then the Coxeter graph $\Gamma(K)$ does not contain a subgraph of the form shown in Figure 6.16.*

Proof. Let $B^K = \{\alpha_1, \ldots, \alpha_\ell\}$, suppose that $\Gamma(K)$ has a subgraph of this form containing the vertices $\{i_1, \ldots, i_5\} \subseteq \{1, \ldots, \ell\}$. By Lemma 5.2.5 and Lemma 5.2.8 localizing $\mathcal{A}^K_{i_1 \cdots i_5}$ and restricting to $H = \sigma^K_{i_2}(\alpha_{i_3})^\perp$ gives the irreducible supersolvable simplicial 4-arrangement $(\mathcal{A}^K_{i_1 \cdots i_5})^H$ which contains a chamber with a Coxeter graph not included in the list of Proposition 6.2.7. Hence $\Gamma(K)$ could not have such a subgraph in the first place. □

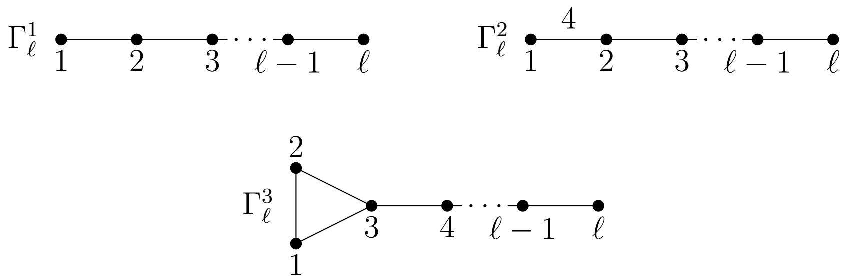


Figure 6.17: Possible Coxeter graphs for an irreducible supersolvable simplicial ℓ-arrangement ($\ell \geq 4$).

Proposition 6.3.3. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement, $\ell \geq 4$ and let $K \in \mathcal{K}(\mathcal{A})$ be a chamber. Then $\Gamma(K)$ is one of the Coxeter graphs of Figure 6.17.*

Proof. By Lemma 6.3.2 the Coxeter graph $\Gamma(K)$ cannot have a triangle somewhere in the middle.

The statement then follows by induction on ℓ, Lemma 5.2.5, Proposition 6.2.7, Corollary 6.2.11, and Lemma 6.3.1. □

Proposition 6.3.4. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement, $\ell \geq 4$ and $K \in \mathcal{K}(\mathcal{A})$.*

(1) There exists a locally crystallographic basis B^K for K such that the Cartan matrix C^K is of type A, C or D'.

(2) If B^K is a locally crystallographic basis for K with C^K of type A, C or D', then for $1 \leq i \leq \ell$ the basis $B^{K_i} = \sigma_i^K(B^K) = \{\alpha_j + c_{ij}^K \alpha_i \mid 1 \leq j \leq \ell\}$ is a locally crystallographic basis with Cartan matrix C^{K_i} of type A, C or D'.

Proof. We argue by induction on $\ell \geq 4$. For $\ell = 4$ this is Proposition 6.2.8. Let $\ell \geq 5$ and assume both statements are true for $\ell - 1$. By Proposition 6.3.3 the Coxeter graph $\Gamma(K)$ is one of the graphs of Figure 6.17. Let $W^K = \{H_1, \dots, H_\ell\}$ where the numbering of the walls corresponds to the numbering of the vertices in $\Gamma_\ell^1, \Gamma_\ell^2, \Gamma_\ell^3$ respectively.

By the induction hypothesis for the localization $\mathcal{A}_{12\cdots(\ell-1)}^K$ there are $\{\alpha_1, \dots, \alpha_{\ell-1}\} \subseteq (\mathbb{R}^\ell)^*$ which form a locally crystallographic basis for the corresponding chamber in $\mathcal{A}_{12\cdots(\ell-1)}^K$. Furthermore, $\alpha_i^\perp = H_i$ for $1 \leq i \leq \ell - 1$, there are $c_{ij}^K \in \mathbb{Z}, 1 \leq i, j \leq \ell - 1$ such that $(\alpha_j - c_{ij}^K \alpha_i)^\perp \in W^{K_i}$, and the matrix $C'^K = (c_{ij}^K)_{1 \leq i,j \leq \ell-1}$ is a Cartan matrix of type A, C, or D'. But in $\Gamma(K)$ the vertex ℓ is only connected to $\ell - 1$ by an edge with label 3. Hence there is an $\alpha_\ell \in (\mathbb{R}^\ell)^*$ such that $\alpha_\ell^\perp = H_\ell$, $K \subseteq \alpha_\ell^+$, $(\alpha_{\ell-1} + \alpha_\ell)^\perp \in W^{K_{\ell-1}}, W^{K_\ell}$. This is to say for $B^K := \{\alpha_1, \dots, \alpha_\ell\}$ we have $c_{(\ell-1)\ell}^K = c_{\ell(\ell-1)}^K = -1$, $c_{\ell j}^K = c_{j\ell}^K = 0$ for $j \notin \{\ell - 1, \ell\}$, the other c_{ij}^K are given by the localization $\mathcal{A}_{12\cdots(\ell-1)}^K$, and hence B^K is a locally crystallographic basis for K with Cartan matrix of type A, C, or D' if $\Gamma(K)$ is $\Gamma_\ell^1, \Gamma_\ell^2$, or Γ_ℓ^3 respectively.

Now let B^K be a locally crystallographic basis with Cartan matrix of type A, C, or D' and B^{K_i} the induced basis for K_i.

If $i = \ell$ then in each case $\Gamma(K_i) = \Gamma(K)$ by Lemma 5.2.2, Lemma 5.2.4 and Proposition 6.3.3 where the vertex k in $\Gamma(K_i)$ corresponds to the root $\sigma_\ell^K(\alpha_k)$. In all graphs Γ_ℓ^k the vertex ℓ is not connected with the vertex j for $j \le \ell - 2$, and $m^K(i,j) = 3$ for all $i, j \in \{1, \dots, \ell\}$ except possibly for $\{i,j\} = \{1,2\}$. So by Lemma 3.3.26, and Lemma 3.3.27 the induced basis B^{K_ℓ} is locally crystallographic with Cartan matrix $C^{K_i} = C^K$ of type A, C, or D'.

For $i \in \{1, \dots, \ell-1\}$ we have $\mathcal{A}^K_{1\cdots(\ell-1)} = \mathcal{A}^{K_i}_{1\cdots(\ell-1)}$. So at least $C'^{K_i} = (c^{K_i})_{1 \le i,j \le \ell-1}$ is a Cartan matrix of type A, C, or D' by the induction hypothesis. If C'^{K_i} is of type C, or D then by Proposition 6.3.3 the Coxeter graph $\Gamma(K_i)$ is Γ_ℓ^2, or Γ_ℓ^3 respectively with the numbering of the vertices corresponding to $B^{K_i} = \{\sigma_i^K(\alpha_1), \dots, \sigma_i^K(\alpha_\ell)\}$. If C'^{K_i} is of type A we may also assume that $\Gamma(K_i)$ is the Coxeter graph Γ_ℓ^1 since otherwise we can renumber the bases B^K and B^{K_i} respectively such that C'^{K_i} is of type C, or D' and we actually are in one of the above cases. We observe next that in $\Gamma(K_i)$ the vertex ℓ is not connected with the vertex j for $j \le \ell - 2$. So $c^{K_i}_{\ell j} = c^{K_i}_{j\ell} = 0$ for $1 \le j \le \ell - 2$.

If $i \in \{1, \dots, \ell-2\}$ we have $c^K_{i\ell} = 0$ and then by Lemma 3.3.27 we get $c^{K_i}_{\ell(\ell-1)} = c^K_{\ell(\ell-1)}$. But $m^{K_i}(\ell-1, \ell) = 3$ and by Lemma 5.2.1 for the last remaining coefficient we obtain $c^{K_i}_{(\ell-1)\ell} = -1$ and B^{K_i} is a locally crystallographic basis with Cartan matrix of type A, C, or D'.

Finally, for $i = \ell - 1$ by Lemma 3.3.26 we also have $c^{K_{\ell-1}}_{(\ell-1)\ell} = c^K_{(\ell-1)\ell} = -1$. Again since $m^{K_{\ell-1}}(\ell-1, \ell) = 3$, by Lemma 5.2.1 for the remaining coefficient we get $c^{K_{\ell-1}}_{(\ell-1)\ell} = -1$ and $B^{K_{\ell-1}}$ is a locally crystallographic basis with Cartan matrix $C^{K_{\ell-1}}$ of type A, C, or D'. This finishes the proof. □

Proposition 6.3.5. *Let $\mathcal{A}$ be an irreducible supersolvable simplicial ℓ-arrangement, $\ell \ge 4$, and fix a chamber $K^0 \in \mathcal{K}(\mathcal{A})$. Then there exists a basis B^{K^0} for K^0 such that*

$$R := \bigcup_{G \in \mathcal{G}(K^0, \mathcal{A})} B_G$$

is a crystallographic root system for $\mathcal{A}$.

Proof. This is exactly the same as the proof of Proposition 6.2.9 using Proposition 6.3.4 instead of Proposition 6.2.8. □

Theorem 6.3.6. *Let $\mathcal{A}$ be an irreducible simplicial supersolvable ℓ-arrangement, $\ell \ge 4$. Then $\mathcal{A}$ is isomorphic to either one of the reflection arrangements $\mathcal{A}(A_\ell)$, $\mathcal{A}(B_\ell)$, or isomorphic to $\mathcal{A}_\ell^{\ell-1} = \mathcal{A}(B_\ell) \setminus \{\{x_1 = 0\}\}$. In particular $\mathcal{A}$ is crystallographic.*

Proof. By Proposition 6.3.5 there exists a crystallographic root system for $\mathcal{A}$, so the arrangement $\mathcal{A}$ is crystallographic. By Theorem 3.4.11 the only irreducible crystallographic ℓ-arrangements, $\ell \ge 4$ which are supersolvable are the arrangements $\mathcal{A}(A_\ell)$, $\mathcal{A}(B_\ell)$, and $\mathcal{A}_\ell^{\ell-1} = \mathcal{A}(B_\ell) \setminus \{\{x_1 = 0\}\}$. □

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