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Chapter 7
Radial and Angular Derivatives of Distributions

Fred Brackx

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract When expressing a distribution in Euclidean space in spherical co-ordinates, derivation with respect to the radial and angular co-ordinates is far from trivial. Exploring the possibilities of defining a radial derivative of the delta distribution $\delta(\underline{x})$ (the angular derivatives of $\delta(\underline{x})$ being zero since the delta distribution is itself radial) led to the introduction of a new kind of distributions, the so-called *signumdistributions*, as continuous linear functionals on a space of test functions showing a singularity at the origin. In this paper we search for a definition of the radial and angular derivatives of a general standard distribution and again, as expected, we are inevitably led to consider signumdistributions. Although these signumdistributions provide an adequate framework for the actions on distributions aimed at, it turns out that the derivation with respect to the radial distance of a general (signum)distribution is still not yet unambiguous.

Keywords Distribution · Radial derivative · Angular derivative · Signumdistribution

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7.1 Introduction

Let us consider a scalar-valued distribution $T(\underline{x}) \in \mathcal{D}'(\mathbb{R}^m)$ expressed in terms of spherical co-ordinates: $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} = \sum_{j=1}^m e_j \omega_j \in \mathbb{S}^{m-1}$, $(e_1, e_2, \dots, e_m)$ being an orthonormal basis of $\mathbb{R}^m$ and $\mathbb{S}^{m-1}$ being the unit sphere in $\mathbb{R}^m$. The aim of this paper is to search for an adequate definition of the radial and angular derivatives $\partial_r T$ and $\partial_{\omega_j} T$, $j = 1, \dots, m$. This problem was treated in [2] for the special and interesting case of the delta distribution $\delta(\underline{x})$, the following spherical co-ordinates expression of which is often encountered in physics texts:

$$\delta(\underline{x}) = \frac{1}{a_m} \frac{\delta(r)}{r^{m-1}} \tag{7.1.1}$$

where $a_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the area of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$. Apparently expression (7.1.1) can mathematically be explained in the following way. Write the action of the delta distribution as an integral:

$$\begin{aligned}\varphi(0) = \langle\, \delta(\underline{x}), \varphi(\underline{x}) \,\rangle &= \int_{\mathbb{R}^m} \delta(\underline{x})\, \varphi(\underline{x})\, dV(\underline{x}) \\ &= \int_0^\infty r^{m-1} \delta(\underline{x})\, dr \int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\, dS_{\underline{\omega}} \\ &= a_m \int_0^\infty r^{m-1}\, \delta(\underline{x})\, \Sigma^0[\varphi](r)\, dr\end{aligned}$$

introducing the so-called *spherical mean* of the test function φ given by

$$\Sigma^0[\varphi](r) = \frac{1}{a_m} \int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\, dS_{\underline{\omega}}.$$

As it is easily seen that $\Sigma^0[\varphi](0) = \varphi(0)$, it follows that

$$a_m \int_0^\infty r^{m-1}\, \delta(\underline{x})\, \Sigma^0[\varphi](r)\, dr = \int_0^\infty \delta(r)\, \Sigma^0[\varphi](r)\, dr = \langle\, \delta(r), \Sigma^0[\varphi](r) \,\rangle$$

which explains (7.1.1). However we prefer to interpret this expression as

$$\varphi(0) = \langle\, \delta(\underline{x}), \varphi(\underline{x}) \,\rangle = \langle\, \delta(r), \Sigma^0[\varphi](r) \,\rangle = \Sigma^0[\varphi](0). \tag{7.1.2}$$

Straightforward successive derivation with respect to r of (7.1.1) leads to

$$\partial_r^{2\ell}\, \delta(\underline{x}) = \frac{1}{(2\ell)!}(m)(m+1)\cdots(m+2\ell-1)\, \frac{1}{a_m}\, \frac{\delta^{(2\ell)}(r)}{r^{m-1}}, \tag{7.1.3}$$

$$\partial_r^{2\ell+1}\, \delta(\underline{x}) = \frac{1}{(2\ell+1)!}(m)(m+1)\cdots(m+2\ell)\, \frac{1}{a_m}\, \frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}. \tag{7.1.4}$$

Expression (7.1.3) then is interpreted as

$$\langle\, \partial_r^{2\ell}\, \delta(\underline{x}), \varphi(\underline{x}) \,\rangle = \frac{1}{(2\ell)!}(m)(m+1)\cdots(m+2\ell-1)\, \langle\, \delta^{(2\ell)}(r), \Sigma^0[\varphi](r) \,\rangle$$

which is meaningful and which can serve as the definition of the even order derivatives with respect to r of the delta distribution. However expression (7.1.4) makes no sense since the spherical mean $\Sigma^0[\varphi](r)$ is an even function of r, whence its odd order derivatives vanish at the origin:

$$\langle\, -\partial_r^{2\ell+1}\, \delta(r), \Sigma^0[\varphi](r) \,\rangle = \{\partial_r^{2\ell+1}\, \Sigma^0[\varphi](r)\}|_{r=0} = 0.$$

How to explain this fact that, proceeding stepwise by derivation with respect to r, the even order derivatives of $\delta(\underline{x})$ apparently make sense, while its odd order derivatives are zero distributions, in this way violating the basic requirement of any derivation procedure that $\partial_r\, \partial_r$ should equal ∂_r^2? Let us to that end have a quick look at the functional analytic background of this phenomenon; for a more systematic treatment we refer to [2].

When expressing a scalar-valued test function $\varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)$ in spherical co-ordinates, one obtains a function $\widetilde{\varphi}(r, \underline{\omega}) = \varphi(r\underline{\omega}) \in \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$, but it is evident that not all functions $\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$ stem from a test function in $\mathcal{D}(\mathbb{R}^m)$. However a one-to-one correspondence may be established between the usual space of test functions $\mathcal{D}(\mathbb{R}^m)$ and a specific subspace of $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$.

Lemma 7.1.1 (See [5]) *There is a one-to-one correspondence* $\varphi(\underline{x}) \leftrightarrow \widetilde{\varphi}(r, \underline{\omega}) = \varphi(r\underline{\omega})$ *between the spaces* $\mathcal{D}(\mathbb{R}^m)$ *and* $\mathcal{V} = \{\phi(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1}) : \phi$ *is even, i.e.* $\phi(-r, -\underline{\omega}) = \phi(r, \underline{\omega})$, *and* $\{\partial_r^n\, \phi(r, \underline{\omega})\}|_{r=0}$ *is a homogeneous polynomial of degree* n *in* $(\omega_1, \dots, \omega_m)$, $\forall n \in \mathbb{N}\}$.

Clearly $\mathcal{V}$ is a closed (but not dense) subspace of $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$ and even of $\mathcal{D}_E(\mathbb{R} \times \mathbb{S}^{m-1})$, where the subscript E refers to the even character of the test functions in that space; this space $\mathcal{V}$ is endowed with the induced topology of $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$. The one-to-one correspondence between the spaces of test functions $\mathcal{D}(\mathbb{R}^m)$ and $\mathcal{V}$ translates into a one-to-one correspondence between the standard distributions $T \in \mathcal{D}'(\mathbb{R}^m)$ and the bounded linear functionals in $\mathcal{V}'$, this correspondence being given by

$$\langle\, T(\underline{x}), \varphi(\underline{x}) \,\rangle = \langle\, \widetilde{T}(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega}) \,\rangle.$$

By Hahn-Banach's theorem the bounded linear functional $\widetilde{T}(r, \underline{\omega}) \in \mathcal{V}'$ may be extended to the distribution $\mathbb{T}(r, \underline{\omega}) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1})$; such an extension is called a *spherical representation* of the distribution T (see e.g. [9]). However as the subspace $\mathcal{V}$ is not dense in $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$, the spherical representation of a distribution is *not* *unique*, but if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of the same distribution T, their restrictions to $\mathcal{V}$ coincide:

$$\langle\, \mathbb{T}_1(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega}) \,\rangle = \langle\, \mathbb{T}_2(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega}) \,\rangle = \langle\, \widetilde{T}(r, \underline{\omega}), \varphi(r\underline{\omega}) \,\rangle = \langle\, T(\underline{x}), \varphi(\underline{x}) \,\rangle.$$

For test functions in $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$ the spherical variables r and $\underline{\omega}$ are ordinary variables, and thus smooth functions. It follows that for distributions in $\mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1})$ multiplication by r and ω_j, $j = 1, \ldots, m$ and differentiation with respect to r and ω_j, $j = 1, \ldots, m$ are well-defined standard operations, whence

$$\langle\, \partial_r \mathbb{T}(r, \underline{\omega}), \Xi(r, \underline{\omega}) \,\rangle = -\langle\, \mathbb{T}(r, \underline{\omega}), \partial_r \Xi(r, \underline{\omega}) \,\rangle \tag{7.1.5}$$

for all test functions $\Xi(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$, and similar expressions for $\partial_{\omega_j} \mathbb{T}$, $r\,\mathbb{T}$ and $\underline{\omega}\,\mathbb{T}$. However if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of the same distribution $T \in \mathcal{D}'(\mathbb{R}^m)$, then, upon restriction to test functions $\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{V}$, we are stuck with

$$-\langle\, \mathbb{T}_1(r, \underline{\omega}), \partial_r \widetilde{\varphi}(r, \underline{\omega}) \,\rangle \neq -\langle\, \mathbb{T}_2(r, \underline{\omega}), \partial_r \widetilde{\varphi}(r, \underline{\omega}) \,\rangle$$

because $\partial_r \widetilde{\varphi}(r, \underline{\omega})$ does no longer belong to $\mathcal{V}$ (and neither do $\partial_{\omega_j} \widetilde{\varphi}(r, \underline{\omega})$, $r\, \widetilde{\varphi}(r, \underline{\omega})$ and $\underline{\omega}\, \widetilde{\varphi}(r, \underline{\omega})$) since it is an odd function in the variables $(r, \underline{\omega})$. And it is also clear that the action (7.1.5) might be unambiguously restricted to testfunctions in $\mathcal{V}$ if the test function Ξ were in a subspace of $\mathcal{D}(\mathbb{R} \times S^{m-1})$ consisting of odd functions. The conclusion is that the concept of spherical representation of a distribution does not allow for an unambiguous definition of the actions proposed. What is more, it becomes apparent that there is a need for a subspace of odd test functions. And at the same time it becomes clear why even order derivatives with respect to r of the delta distribution and of a standard distribution in general, are well-defined instead. Indeed, we have e.g.

$$\langle\, \partial_r^{2\ell}\, \mathbb{T}(r, \underline{\omega}), \Xi(r, \underline{\omega}) \,\rangle = \langle\, \mathbb{T}(r, \underline{\omega}), \partial_r^{2\ell}\, \Xi(r, \underline{\omega}) \,\rangle$$

where now $\partial_r^{2\ell}\, \Xi(r, \underline{\omega})$ belongs to $\mathcal{D}_E(\mathbb{R} \times \mathbb{S}^{m-1})$ which enables restriction to test functions in $\mathcal{V}$ in an unambiguous way.

7.2 Preliminaries

In this paper vectors in $\mathbb{R}^m$ will be interpreted as Clifford 1-vectors in the Clifford algebra $\mathbb{R}_{0,m}$, where the basis vectors $(e_j, e_2, \ldots, e_m)$ of $\mathbb{R}^m$, satisfy the relations $e_j^2 = -1$, $e_i \wedge e_j = e_i e_j = -e_j e_i = -e_j \wedge e_i$, $e_i \cdot e_j = 0, i \neq j = 1, \ldots, m$. This allows for the use of the very efficient *geometric* or *Clifford product* of Clifford vectors:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

for which, in particular,

$$\underline{x}\,\underline{x} = \underline{x} \cdot \underline{x} = -\,|\underline{x}|^2$$

$\underline{x}$ being the Clifford 1-vector $\underline{x} = \sum_{j=1}^m e_j\, x_j$, whence also

$$\underline{\omega}\,\underline{\omega} = \underline{\omega} \cdot \underline{\omega} = -\,|\underline{\omega}|^2 = -1.$$

For more on Clifford algebras we refer to e.g. [6].

The Dirac operator $\underline{\partial} = \sum_{j=1}^m e_j\, \partial_{x_j}$, which may be seen as a Stein-Weiss projection of the gradient operator (see e.g. [8]) and which underlies the higher dimensional theory of monogenic functions (see e.g. [3, 4]), linearizes the Laplace operator: $\underline{\partial}^2 = -\,\Delta$. Its action on a scalar-valued standard distribution $T(\underline{x})$ results into the vector-valued distribution $\underline{\partial}\, T(\underline{x})$ given for all $\varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)$ by

$$\begin{aligned} \langle\, \underline{\partial}\, T(\underline{x})\,, \varphi(\underline{x}) \,\rangle &= \sum_{j=1}^m e_j \,\langle\, \partial_{x_j} T(\underline{x})\,, \varphi(\underline{x}) \,\rangle = -\sum_{j=1}^m e_j \,\langle\, T(\underline{x})\,, \partial_{x_j} \varphi(\underline{x}) \,\rangle \\ &= -\,\langle\, T(\underline{x})\,, \underline{\partial}\, \varphi(\underline{x}) \,\rangle \end{aligned}$$

which is a meaningful operation since only derivatives with respect to the cartesian co-ordinates are involved.

Two fundamental formulae in monogenic function theory are

$$\{\underline{x}, \underline{\partial}\} = \underline{x}\,\underline{\partial} + \underline{\partial}\,\underline{x} = -\,2\mathbb{E} - m \qquad \text{and} \qquad [\underline{x}, \underline{\partial}] = \underline{x}\,\underline{\partial} - \underline{\partial}\,\underline{x} = m - 2\Gamma$$

where

$$\mathbb{E} = \sum_{j=1}^m x_j\, \partial_{x_j}$$

is the scalar Euler operator, and

$$\Gamma = \sum_{j<k} e_j e_k\, L_{jk} = \sum_{j<k} e_j e_k (x_j \partial x_k - x_k \partial_{x_j})$$

is the bivector angular momentum operator. It follows that

$$\underline{x}\,\underline{\partial} = -\mathbb{E} - \Gamma$$

or more precisely

$$\underline{x} \cdot \underline{\partial} = -\mathbb{E} \qquad \text{and} \qquad \underline{x} \wedge \underline{\partial} = -\Gamma.$$

7.3 Signumdistributions

As already observed in the introduction, $\underline{\omega}$ is an ordinary (vector) variable in $\mathbb{R} \times \mathbb{S}^{m-1}$, whence it makes sense to consider the following subspace of vector-valued test functions in $\mathbb{R} \times \mathbb{S}^{m-1}$:

$$\mathcal{W} = \underline{\omega}\mathcal{V} \subset \mathcal{D}_O(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m) \subset \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$$

where now the subscript O refers to the odd character of the test functions under consideration, i.e. $\psi(-r, -\underline{\omega}) = -\psi(r, \underline{\omega}), \forall \psi \in \mathcal{D}_O(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$. This space $\mathcal{W}$ is endowed with the induced topology of $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$. By definition there is a one-to-one correspondence between the spaces $\mathcal{V}$ and $\mathcal{W}$.

For each $\mathbb{U}(r, \underline{\omega}) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$ we define $\widetilde{U}(r, \underline{\omega}) \in \mathcal{W}'$ by the restriction

$$\langle \widetilde{U}(r, \underline{\omega}), \underline{\omega}\widetilde{\varphi}(r, \underline{\omega}) \rangle = \langle \mathbb{U}(r, \underline{\omega}), \underline{\omega}\widetilde{\varphi}(r, \underline{\omega}) \rangle, \quad \forall\, \underline{\omega}\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{W}.$$

In $\mathbb{R}^m$ we consider the space $\Omega(\mathbb{R}^m; \mathbb{R}^m) = \{\underline{\omega}\varphi(\underline{x}) : \varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)\}$. Clearly the functions in $\Omega(\mathbb{R}^m; \mathbb{R}^m)$ are no longer differentiable in the whole of $\mathbb{R}^m$, since they are not defined at the origin due to the function $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$. By definition there is a one-to-one correspondence between the spaces $\mathcal{D}(\mathbb{R}^m)$ and $\Omega(\mathbb{R}^m; \mathbb{R}^m)$.

For each $\widetilde{U}(r, \underline{\omega}) \in \mathcal{W}'$ we define ${}^sU(\underline{x})$ by

$$\langle {}^sU(\underline{x}), \underline{\omega}\varphi(\underline{x}) \rangle = \langle \widetilde{U}(r, \underline{\omega}), \underline{\omega}\widetilde{\varphi}(r, \underline{\omega})) \rangle, \quad \forall\, \underline{\omega}\varphi(\underline{x}) \in \Omega(\mathbb{R}^m; \mathbb{R}^m).$$

Clearly ${}^sU(\underline{x})$ is a bounded linear functional on $\Omega(\mathbb{R}^m; \mathbb{R}^m)$, for which, in [2], we coined the term *signumdistribution*.

Now start with a standard distribution $T(\underline{x}) \in \mathcal{D}'(\mathbb{R}^m)$ and let $\mathbb{T}(r, \underline{\omega}) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1})$ be one of its spherical representations. Put $\mathbb{S}(r, \underline{\omega}) = \underline{\omega}\,\mathbb{T}(r, \underline{\omega})$ which in its turn leads to the signumdistribution ${}^sS(\underline{x}) \in \Omega'(\mathbb{R}^m; \mathbb{R}^m)$. Then we consecutively have

$$\begin{aligned}\langle {}^sS(\underline{x}), \underline{\omega}\varphi(\underline{x}) \rangle &= \langle \mathbb{S}(r, \underline{\omega}), \underline{\omega}\widetilde{\varphi}(r, \underline{\omega}) \rangle = \langle \underline{\omega}\,\mathbb{T}(r, \underline{\omega}), \underline{\omega}\widetilde{\varphi}(r, \underline{\omega}) \rangle \\ &= -\langle \mathbb{T}(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega}) \rangle = -\langle T(\underline{x}), \varphi(\underline{x}) \rangle\end{aligned}$$

since $\underline{\omega}^2 = -1$. We call ${}^sS(\underline{x})$ a signumdistribution associated to the distribution $T(\underline{x})$ and denote it by $T^{\vee}(\underline{x})$. It thus holds that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^m)$

$$\langle T^{\vee}(\underline{x}), \underline{\omega}\varphi(\underline{x}) \rangle = -\langle T(\underline{x}), \varphi(\underline{x}) \rangle. \tag{7.3.1}$$

At the same time we call $T(\underline{x})$ the distribution associated to the signumdistribution ${}^sS(\underline{x})$ and we denote this distribution by ${}^sS^{\wedge}(\underline{x})$. Formula (7.3.1) then also reads

$$\langle {}^sS(\underline{x}), \underline{\omega}\varphi(\underline{x}) \rangle = -\langle {}^sS^{\wedge}(\underline{x}), \varphi(\underline{x}) \rangle \tag{7.3.2}$$

and it is clear that

$$T^{\vee\wedge} = T \qquad \text{and} \qquad {}^sS^{\wedge\vee} = {}^sS.$$

At first sight for a given distribution $T(\underline{x})$ the associated signumdistribution $T^{\vee}(\underline{x})$ is not uniquely defined since its construction involves the not uniquely defined spherical representation $\mathbb{T}$ of $T(\underline{x})$. Nevertheless it follows from (7.3.1) that for a given distribution $T(\underline{x})$ its associated signumdistribution $T^{\vee}(\underline{x})$ is unique, what can also be proven directly as follows.

Proposition 7.3.1 *Given the distribution $T(\underline{x})$ its associated signumdistribution $T^{\vee}(\underline{x})$ is uniquely determined.*

Proof Assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of T, i.e. for all test functions $\Xi(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times S^{m-1}; \mathbb{R}^m)$ it holds that

$$\langle \mathbb{T}_1, \Xi(r, \underline{\omega}) \rangle \neq \langle \mathbb{T}_2, \Xi(r, \underline{\omega}) \rangle$$

while for all test functions $\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{V}$ it holds that

$$\langle \mathbb{T}_1, \widetilde{\varphi}(r, \underline{\omega}) \rangle = \langle \mathbb{T}_2, \widetilde{\varphi}(r, \underline{\omega}) \rangle = \langle \widetilde{T}, \widetilde{\varphi}(r, \underline{\omega}) \rangle.$$

Let $T_1^{\vee}$ and $T_2^{\vee}$ be the associated signumdistributions to T through the spherical representations $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. Then for $j = 1, 2$ it holds that

$$\langle T_j^{\vee}, \underline{\omega}\varphi(\underline{x}) \rangle = \langle \mathbb{T}_j, \widetilde{\varphi}(r, \underline{\omega}) \rangle$$

whence $T_1^{\vee} = T_2^{\vee}$ on $\Omega(\mathbb{R}^m; \mathbb{R}^m)$. □

Conversely, for a given signumdistribution ${}^sU \in \Omega'(\mathbb{R}^m; \mathbb{R}^m)$ we define the associated distribution ${}^sU^{\wedge}$ by

$$\langle {}^sU^{\wedge}(\underline{x}), \varphi(\underline{x}) \rangle = -\langle {}^sU(\underline{x}), \underline{\omega}\varphi(\underline{x}) \rangle \qquad \forall \varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m).$$

Clearly it holds that

$$T^{\vee\wedge} = T \qquad \text{and} \qquad {}^sU^{\wedge\vee} = {}^sU.$$

Example As an example consider the distribution $T(\underline{x}) = \delta(\underline{x})$. Our aim is to define the signumdistribution $\delta^{\vee}(\underline{x})$. A spherical representation of the delta distribution is given by

$$\langle \mathbb{T}(r, \omega), \Xi(r, \underline{\omega}) \rangle = \Sigma^0[\Xi(r, \underline{\omega})]\}|_{r=0}.$$

Indeed, when restricting to the space $\mathcal{V}$ and taking into account property (7.1.2), we obtain

$$\langle \mathbb{T}(r,\omega), \widetilde{\varphi}(r,\underline{\omega}) \rangle = \Sigma^0[\varphi(r\,\underline{\omega})]\}|_{r=0} = \langle \delta(\underline{x}), \varphi(\underline{x}) \rangle.$$

This particular spherical representation of $T(\underline{x})$ induces the signumdistribution associated to $\delta(\underline{x})$, which we define to be $\delta^{\vee}(\underline{x})$. It thus holds that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^m)$

$$\langle \delta^{\vee}(\underline{x}), \underline{\omega}\,\varphi(\underline{x}) \rangle = -\langle \delta(\underline{x}), \varphi(\underline{x}) \rangle. \qquad (7.3.3)$$

For further examples we refer to [2].

7.4 The Dirac Operator in Spherical Co-ordinates

Passing to spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} = \sum_{j=1}^{m} e_j\,\omega_j \in \mathbb{S}^{m-1}$, the Dirac operator takes the form

$$\underline{\partial} = \underline{\partial}_{rad} + \underline{\partial}_{ang}$$

with

$$\underline{\partial}_{rad} = \underline{\omega}\,\partial_r \qquad \text{and} \qquad \underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}.$$

To give an idea what the angular differential operator $\partial_{\underline{\omega}} = \sum_{j=1}^{m} e_j\,\partial_{\omega_j}$ looks like, let us mention its explicit form in dimension $m = 2$:

$$\partial_{\underline{\omega}} = e_\theta\,\partial_\theta$$

and in dimension $m = 3$:

$$\partial_{\underline{\omega}} = e_\theta\,\partial_\theta + e_\varphi\,\frac{1}{\sin\theta}\,\partial_\varphi,$$

the meaning of the polar co-ordinates θ and φ being straightforward. The operator $\partial_{\underline{\omega}}$ is sometimes called the *spherical Dirac operator.*

Taking into account that $\partial_{\underline{\omega}}$ is orthogonal to $\underline{\omega}$, the Euler operator in spherical co-ordinates then reads:

$$\mathbb{E} = -\underline{x}\cdot\underline{\partial} = -r\underline{\omega}\cdot\underline{\partial}_{rad} = -r\underline{\omega}\cdot\underline{\omega}\,\partial_r = r\,\partial_r$$

while the angular momentum operator Γ takes the form

$$\Gamma = -\underline{x}\wedge\underline{\partial} = -r\underline{\omega}\wedge\underline{\partial}_{ang} = -r\underline{\omega}\wedge\frac{1}{r}\,\partial_{\underline{\omega}} = -\underline{\omega}\wedge\partial_{\underline{\omega}} = -\underline{\omega}\,\partial_{\underline{\omega}}.$$

The question now is how to define, if possible, the action of the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a standard distribution. To that end both operators should be expressed in terms of cartesian derivatives. This is achieved as follows.

Definition 7.4.1 The actions of the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a distribution T are given by

$$\underline{\partial}_{rad}\,T = \underline{\omega}\,\partial_r\,T = -\frac{1}{\underline{x}}\,\mathbb{E}\,T$$

and

$$\underline{\partial}_{ang}\,T = \frac{1}{r}\,\partial_{\underline{\omega}}\,T = -\frac{1}{\underline{x}}\,\Gamma\,T.$$

It becomes clear at once that, in this way, the actions of $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a standard distribution $T(\underline{x})$ are well-defined but not uniquely defined. Indeed, due to the division by the analytic function $\underline{x}$, both expressions

$$\underline{\partial}_{rad}\,T(\underline{x}) = \underline{\omega}\,\partial_r\,T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T(\underline{x})\right] \qquad (7.4.1)$$

and

$$\underline{\partial}_{ang}\,T(\underline{x}) = \frac{1}{r}\,\partial_{\underline{\omega}}\,T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\Gamma\,T(\underline{x})\right] \qquad (7.4.2)$$

represent equivalent classes of distributions each two of which differ by a vector multiple of the delta distribution $\delta(\underline{x})$. However if $S_1 = \underline{\partial}_{rad}\,T(\underline{x})$ and $S_2 = \underline{\partial}_{ang}\,T(\underline{x})$ are distributions arbitrarily chosen in the equivalent classes (7.4.1) and (7.4.2) respectively, i.e.

$$\underline{x}\,S_1 = -\mathbb{E}\,T(\underline{x}) \qquad \text{and} \qquad \underline{x}\,S_2 = -\Gamma T(\underline{x})$$

this choice is not completely arbitrary since S_1 and S_2 always must satisfy the relation

$$S_1 + S_2 = \underline{\partial}_{rad}\,T(\underline{x}) + \underline{\partial}_{ang}\,T(\underline{x}) = \underline{\partial}\,T(\underline{x}) \qquad (7.4.3)$$

where the right-hand side, quite naturally, is a known distribution once the distribution T has been given. One could say that the differential operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$

are *entangled* in the sense that the results of their actions on a distribution are subject to (7.4.3).

Example Let us give a simple example to illustrate the above phenomenon. Consider the regular distribution $T(\underline{x}) = \underline{x}$. Then $\underline{\partial}\, \underline{x} = -m$, $\mathbb{E}\, \underline{x} = \underline{x}$ and $\Gamma\, \underline{x} = (m-1)\, \underline{x}$, whence

$$(\underline{\omega}\, \partial_r)\, \underline{x} = -1 + \underline{c}_1\, \delta(\underline{x}) \qquad \text{and} \qquad \left(\frac{1}{r}\, \partial_{\underline{\omega}}\right) \underline{x} = 1 - m + \underline{c}_2\, \delta(\underline{x})$$

with the restriction that the vector constants $\underline{c}_1$ and $\underline{c}_2$ always must satisfy the entanglement condition $\underline{c}_1 + \underline{c}_2 = 0$.

Apparently there seems to be no possibility to uniquely define the actions of the $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ operators on a standard distribution by singling out specific distributions in the equivalent classes (7.4.1) and (7.4.2), except for the following two special cases.

(i) If the distribution $T(\underline{x})$ is *radial*, i.e. only depends on $r = |\underline{x}|$, then we put $\frac{1}{r}\, \partial_{\underline{\omega}}\, T = 0$ and $\underline{\omega}\, \partial_r\, T = \underline{\partial}\, T$. This first special case is illustrated by the delta distribution (see also [2]): $\frac{1}{r}\, \partial_{\underline{\omega}}\, \delta(\underline{x}) = 0$ and $\underline{\omega}\, \partial_r\, \delta(\underline{x}) = \underline{\partial}\, \delta(\underline{x})$.
(ii) If the distribution $T(\underline{x})$ is *angular*, i.e. only depends on $\underline{\omega} = \frac{\underline{\omega}}{|\underline{x}|}$, then we put $\underline{\omega}\, \partial_r\, T = 0$ and $\frac{1}{r}\, \partial_{\underline{\omega}}\, T = \underline{\partial}\, T$. This second special case is illustrated by the regular distribution $\underline{\omega}$ for which $\underline{\omega}\, \partial_r\, \underline{\omega} = 0$ and $\frac{1}{r}\, \partial_{\underline{\omega}}\, \underline{\omega} = \underline{\partial}\, \underline{\omega} = -(m-1)\frac{1}{r}$.

In Sect. 7.6 we will expose two other cases where the actions of the $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ operators are uniquely defined.

7.5 The Laplace Operator in Spherical Co-ordinates

As was already observed in Sect. 7.2, the Dirac operator factorizes the Laplace operator: $-\Delta = \underline{\partial}^2$. As the Laplace operator is a scalar operator it holds that

$$\Delta = -\underline{\partial} \cdot \underline{\partial} = |\underline{\partial}|^2.$$

Passing to spherical co-ordinates we obtain, in view of

$$\begin{aligned}
\underline{\partial}_{rad}\, \underline{\partial}_{rad} &= -\, \partial_r^2 \\
\underline{\partial}_{rad}\, \underline{\partial}_{ang} &= -\frac{1}{r^2}\, \underline{\omega}\, \partial_{\underline{\omega}} + \frac{1}{r}\, \underline{\omega}\, \partial_{\underline{\omega}}\, \partial_r \\
\underline{\partial}_{ang}\, \underline{\partial}_{rad} &= -\,(m-1)\frac{1}{r}\, \partial_r - \frac{1}{r}\, \partial_r\, \underline{\omega}\, \partial_{\underline{\omega}} \\
\underline{\partial}_{ang}\, \underline{\partial}_{ang} &= \frac{1}{r^2}\, \partial_{\underline{\omega}}^2
\end{aligned}$$

the following expression for the Laplace operator:

$$\begin{aligned}
\Delta &= -\,(\underline{\partial}_{rad} + \underline{\partial}_{ang})^2 \\
&= \partial_r^2 + (m-1)\frac{1}{r}\, \partial_r + \frac{1}{r^2}\, (\underline{\omega}\, \partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2) \\
&= \partial_r^2 + (m-1)\frac{1}{r}\, \partial_r + \frac{1}{r^2}\, \Delta^*
\end{aligned}$$

where

$$\Delta^* = \underline{\omega}\, \partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2$$

is the Laplace-Beltrami operator, sometimes denoted by Δ_0. The Laplace-Beltrami operator is a purely angular scalar operator; as $\underline{\omega}\, \partial_{\underline{\omega}} = -\,\Gamma$ is a bivector operator, it follows that

$$\Delta^* = -\, \partial_{\underline{\omega}} \cdot \partial_{\underline{\omega}} = |\partial_{\underline{\omega}}|^2 \qquad \text{and} \qquad \underline{\omega}\, \partial_{\underline{\omega}} = \partial_{\underline{\omega}} \wedge \partial_{\underline{\omega}} = -\,\Gamma.$$

It is a nice observation that while the Laplace operator Δ is the normsquared of the Dirac operator, the spherical Laplace or Laplace-Beltrami operator is the normsquared of the spherical Dirac operator.

As is the case for the Laplace operator $\Delta = \sum_{j=1}^m \partial_{x_j}^2$, also the Laplace-Beltrami operator may be expressed in terms of derivatives with respect to the cartesian co-ordinates.

Proposition 7.5.1 *The angular differential operators $\partial_{\underline{\omega}}^2$ and Δ^* may be written in terms of cartesian co-ordinates as*

$$\partial_{\underline{\omega}}^2 = \Gamma^2 - (m-1)\, \Gamma$$

and

$$\Delta^* = (m-2)\, \Gamma - \Gamma^2.$$

Proof One has

$$\begin{aligned}
\Gamma^2 &= (-\, \underline{\omega}\, \partial_{\underline{\omega}})^2 = \underline{\omega}\, \partial_{\underline{\omega}}\, \underline{\omega}\, \partial_{\underline{\omega}} \\
&= \underline{\omega}\, ((1-m) - \underline{\omega}\, \partial_{\underline{\omega}})\partial_{\underline{\omega}} \\
&= (1-m)\, \underline{\omega}\, \partial_{\underline{\omega}} + \partial_{\underline{\omega}}^2 \\
&= (m-1)\, \Gamma + \partial_{\underline{\omega}}^2
\end{aligned}$$

and

$$\begin{aligned}\Delta^* &= \underline{\omega}\,\partial_{\underline{\omega}} - \partial^2_{\underline{\omega}}\\ &= -\,\Gamma - \Gamma^2 + (m-1)\,\Gamma\\ &= (m-2)\,\Gamma - \Gamma^2.\end{aligned}$$

□

There is a second, and, quite naturally, equivalent, way to write the Laplace-Beltrami operator by means of cartesian derivatives. It only needs a straightforward calculation to prove the following result.

Proposition 7.5.2 *The Laplace-Beltrami operator may be written as*

$$\Delta^* = \sum_{j<k} L^2_{jk} = \sum_{j<k} (x_j\,\partial_{x_k} - x_k\,\partial_{x_j})^2.$$

The actions of the Laplace operator and the Laplace-Beltrami operator on a distribution being uniquely well-defined, the question arises how to define the actions on a distribution of the three parts of the Laplace operator expressed in spherical co-ordinates. It turns out that these actions are well-defined, though not uniquely, through equivalent classes of distributions.

Proposition 7.5.3 *Let T be a scalar distribution. One has*

(i) $\partial_r^2\,T = S_2 + \delta(\underline{x})\,c_2 - \sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x})$
for arbitrary constants c_2 and $c_{1,j}$, $j = 1,\dots,m$ and any distribution S_2 such that $\underline{x}\,S_2 = \mathbb{E}\,\underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$

(ii) $\frac{1}{r}\,\partial_r\,T = S_3 + \frac{1}{m}\sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$
for arbitrarily constant c_3 and any distribution S_3 such that $\underline{x}\,S_3 = \underline{S}_1$

(iii) $\frac{1}{r^2}\,\Delta^*\,T = S_4 + c_4\,\delta(\underline{x}) + \sum_{j=1}^m c_{5,j}\,\partial_{x_j}\delta(\underline{x})$
for arbitrary constants c_4 and $c_{5,j}$, $j = 1,\dots,m$ and any distribution S_4 such that $r^2\,S_4 = \Delta^*\,T$

Proof

(i) From Sect. 7.4 we know that

$$(\underline{\omega}\,\partial_r)\,T = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] = \underline{S}_1 + \delta(\underline{x})\,\underline{c}_1$$

with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$. It follows that

$$\begin{aligned}\partial_r^2\,T &= -\,(\underline{\omega}\,\partial_r)^2\,T\\ &= -\,(\underline{\omega}\,\partial_r)\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1)\\ &= \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{S}_1\right] - \underline{\partial}\delta(\underline{x})\,\underline{c}_1\\ &= S_2 + \delta(\underline{x})\,c_2 - \underline{\partial}\delta(\underline{x})\,\underline{c}_1\end{aligned}$$

with $\underline{x}\,S_2 = \mathbb{E}\,\underline{S}_1$.

(ii) We have consecutively

$$\begin{aligned}\frac{1}{r}\,\partial_r\,T &= \frac{1}{\underline{x}}\,(\underline{\omega}\,\partial_r)\,T\\ &= \frac{1}{\underline{x}}\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1)\\ &= S_3 + \frac{1}{\underline{x}}\,\delta(\underline{x})\,\underline{c}_1\\ &= S_3 + \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x})\,\underline{c}_1 + \delta(\underline{x})\,c_3\end{aligned}$$

with $\underline{x}\,S_3 = \underline{S}_1$.

(iii) The distribution $\Delta^*\,T$ is uniquely defined and r^2 is an analytic function with a second order zero at the origin. The result follows immediately.

□

Remark 7.5.4 The operators ∂_r^2, $\frac{1}{r}\,\partial_r$ and $\frac{1}{r^2}\,\Delta^*$ are *entangled* in the sense that, given a distribution T and having chosen appropriately the distributions $\underline{S}_1$, S_2, S_3 and S_4, all arbitrary constants appearing in the expressions of Proposition 7.5.3 should satisfy the entanglement condition

$$\partial_r^2\,T + (m-1)\frac{1}{r}\,\partial_r\,T + \frac{1}{r^2}\,\Delta^*\,T = \Delta\,T$$

the distribution at the right-hand side being uniquely determined.

Example Proposition 7.5.3 may be generalised to distributions which are e.g. vector valued. Let us illustrate this by considering the distribution $T = \underline{x}^3 = -\,r^3\,\underline{\omega}$, for which, by a direct computation, $\Delta\,T = \Delta(\underline{x}^3) = -\,2\,(m+2)\,\underline{x}$, and $\Delta^*\,T = \Delta^*\,(\underline{x}^3) = (m-1)\,r^2\,\underline{x} = -\,(m-1)\,\underline{x}^3$.

As $\mathbb{E}\,T = \mathbb{E}\,(\underline{x}^3) = 3\,\underline{x}^3$, we chose $S_1 = -\,3\,\underline{x}^2 = 3\,r^2$ satisfying $\underline{x}\,S_1 = -\,3\,\underline{x}^3$. As $\mathbb{E}\,S_1 = \mathbb{E}\,(-3\underline{x}^2) = -\,6\,\underline{x}^2 = 6\,r^2$, we chose $\underline{S}_2 = -\,6\,\underline{x}$ satisfying $\underline{x}\,\underline{S}_2 =$

$-6\,\underline{x}^2$, and $\underline{S}_3 = -3\,\underline{x}$ satisfying $\underline{x}\,\underline{S}_3 = -3\,\underline{x}^2$. Finally we chose $\underline{S}_4 = (m-1)\,\underline{x}$, satisfying $r^2\,\underline{S}_4 = \Delta^*\,T = (m-1)\,r^2\,\underline{x}$. This leads to:

(i) $\partial_r^2\,T = \partial_r^2\,(\underline{x}^3) = -6\,\underline{x} + \delta(\underline{x})\,c_2 - \sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x})$

(ii) $\frac{1}{r}\,\partial_r\,T = \frac{1}{r}\,\partial_r\,(\underline{x}^3) = -3\,\underline{x} + \frac{1}{m}\sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$

(iii) $\frac{1}{r^2}\,\Delta^*\,T = \frac{1}{r^2}\,\Delta^*\,(\underline{x}^3) = (m-1)\,\underline{x} + c_4\,\delta(\underline{x}) + \sum_{j=1}^{m} c_{5,j}\,\partial_{x_j}\delta(\underline{x})$

provided that the arbitrary constants should satisfy the entanglement conditions

$$\begin{cases} c_2 + (m-1)\,c_3 + c_4 = 0 \\ -\frac{1}{m}\,c_{1,j} + c_{5,j} = 0, \quad j = 1, \ldots, m. \end{cases}$$

7.6 Radial and Angular Derivatives of Distributions

In Sect. 7.1 we explained why it is impossible to define the radial derivative $\partial_r\,T$ and the vector angular derivative $\partial_{\underline{\omega}}\,T$ of a distribution T within the class of distributions. Neither is it possible to multiply a distribution by the non-analytic functions r and $\underline{\omega}$. For *legitimizing* those *forbidden actions* we have to take the signumdistributions into consideration instead.

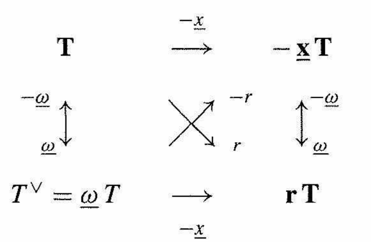
Definition 7.6.1 The product of a scalar-valued distribution T by the function $\underline{\omega}$ is the signumdistribution $T^\vee$ associated to T, and it holds that

$$\langle\,\underline{\omega}\,T\,,\,\underline{\omega}\,\varphi\,\rangle = \langle\,T^\vee\,,\,\underline{\omega}\,\varphi\,\rangle = -\langle\,T\,,\varphi\,\rangle.$$

Definition 7.6.2 The product of a scalar-valued distribution T by the function r is the signumdistribution $r\,T = (-\underline{x}\,T)^\vee$ given by

$$\langle\,r\,T\,,\,\underline{\omega}\,\varphi\,\rangle = \langle\,\underline{x}\,T\,,\varphi\,\rangle = \langle\,T\,,\,\underline{x}\,\varphi\,\rangle$$

according to (the boldface part of) the commutative diagram

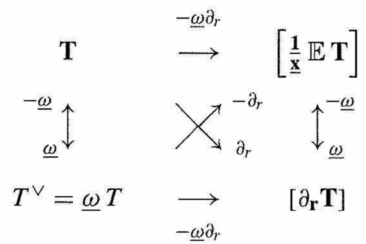


Remark 7.6.3 In the above commutative diagram, and in all the commutative diagrams in the sequel of this paper as well, the row above is situated in the world of distributions, while the objects in the row below are signumdistributions. Vertical transition from the distributions to the signumdistributions and vice versa is executed by the multiplication operators $\underline{\omega}$ and $-\underline{\omega}$ respectively. Each of the horizontally acting operators between distributions, has its counterpart in the world of signumdistributions, and vice versa; e.g. in the above commutative diagram the multiplication operator $-\underline{x}$ between the distributions T and $-\underline{x}\,T$ corresponds with the multiplication operator $-\underline{x}$ between the signumdistributions $T^\vee$ and $(-\underline{x}\,T)^\vee = r\,T$. In fact this implies the definition of the multiplication of the signumdistribution $T^\vee = \underline{\omega}\,T$ by the function $\underline{x}$ resulting in the signumdistribution $-r\,T$.

Definition 7.6.4 The derivative with respect to the radial distance r of a scalar-valued distribution T is the equivalent class of signumdistributions

$$[\partial_r\,T] = [-\underline{\omega}\,\partial_r\,T]^\vee = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right]^\vee = (\underline{S} + \underline{c}\,\delta(\underline{x}))^\vee = \underline{\omega}\,\underline{S} + \underline{\omega}\,\delta(\underline{x})\,\underline{c}$$

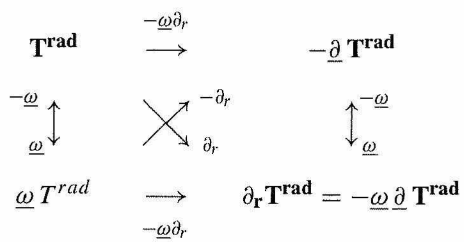
for any vector distribution $\underline{S}$ satisfying $\underline{x}\,\underline{S} = \mathbb{E}\,T$, according to (the boldface part of) the commutative diagram



Remark 7.6.5 In the special case of a scalar-valued radial distribution T^{rad}, its radial derivative $\partial_r\,T^{rad}$ is uniquely determined as the signumdistribution $\partial_r\,T^{rad} = (-\underline{\partial}\,T^{rad})^\vee$ given by

$$\langle\,\partial_r\,T^{rad}\,,\,\underline{\omega}\,\varphi\,\rangle = \langle\,\underline{\omega}\partial_r\,T^{rad}\,,\,\varphi\,\rangle = \langle\,\underline{\partial}\,T^{rad}\,,\,\varphi\,\rangle$$

according to (the boldface part of) the commutative diagram



Remark 7.6.6 The commutative diagram of Definition 7.6.4 implies the definition of the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\, \partial_r$ on the signumdistribution $T^{\vee} = \underline{\omega}\, T$ resulting in the (equivalence class of) signumdistributions $-[\partial_r T]$. In the special case where the distribution T is radial: $T = T^{rad}$, the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\, \partial_r$ on $\underline{\omega}\, T^{rad}$ is the uniquely determined signumdistribution

$$(\underline{\omega}\, \partial_r)\underline{\omega}\, T^{rad} = -\, \partial_r\, T^{rad} = \underline{\omega}(\underline{\omega}\, \partial_r)T = \underline{\omega}\, \underline{\partial}\, T = (\underline{\partial}\, T)^{\vee}$$

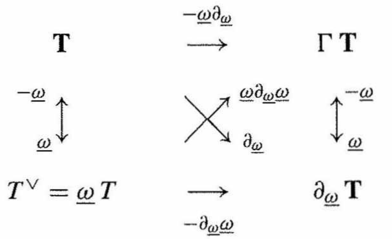
and for all test functions $\underline{\omega}\, \varphi$ it holds that

$$\begin{aligned}\langle -\underline{\omega}\, \partial_r\, T^{\vee} , \underline{\omega}\varphi \rangle &= \langle (\underline{\omega}\, \partial_r\, T^{\vee})^{\wedge} , \varphi \rangle = \langle \partial_r\, T^{\vee} , \varphi \rangle \\ &= \langle -(\partial_r\, T)^{\wedge} , \varphi \rangle = \langle \partial_r\, T , \underline{\omega}\varphi \rangle .\end{aligned}$$

Definition 7.6.7 The angular $\partial_{\underline{\omega}}$-derivative of a scalar-valued distribution T is the signumdistribution $\partial_{\underline{\omega}}\, T = (\Gamma\, T)^{\vee}$ given by

$$\langle \underline{\omega}\varphi , \partial_{\underline{\omega}}\, T \rangle = \langle \varphi , \underline{\omega}\, \partial_{\underline{\omega}}\, T \rangle = \langle \varphi , -\Gamma\, T \rangle$$

according to (the boldface part of) the commutative diagram



Remark 7.6.8 The commutative diagram of Definition 7.6.7 implies the definition of the action of the operator $\partial_{\underline{\omega}}\, \underline{\omega}$ on the signumdistribution $T^{\vee} = \underline{\omega}\, T$ resulting in the signumdistribution $-\, \partial_{\underline{\omega}}\, T$, which in its turn implies the definition of the action of the Γ-operator on the signumdistribution $T^{\vee} = \underline{\omega}\, T$ resulting in the signumdistribution

$$\Gamma(\underline{\omega}\, T) = (m-1)\, \underline{\omega}\, T - \partial_{\underline{\omega}}\, T$$

since

$$\partial_{\underline{\omega}}\, \underline{\omega} = (1-m)\mathbf{1} - \underline{\omega}\, \partial_{\underline{\omega}} = (1-m)\mathbf{1} + \Gamma .$$

7.7 Actions on Signumdistributions

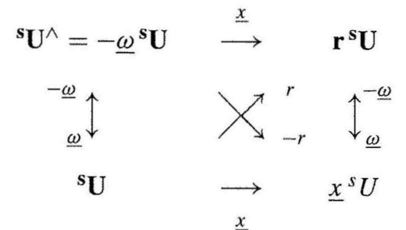
Definition 7.7.1 The product of a scalar-valued signumdistribution ${}^sU$ by the function $\underline{\omega}$ is the distribution $-\, {}^sU^{\wedge}$ associated to $-\, {}^sU$, and it holds that

$$\langle \underline{\omega}\, {}^sU , \varphi \rangle = \langle -\, {}^sU^{\wedge} , \varphi \rangle = \langle {}^sU , \underline{\omega}\varphi \rangle .$$

Definition 7.7.2 The product of a scalar-valued signumdistribution ${}^sU$ by the function r is the distribution $r\, {}^sU = \underline{x}\, ({}^sU)^{\wedge}$ given by

$$\langle r\, {}^sU , \varphi \rangle = \langle \underline{x}\, (-\underline{\omega}\, {}^sU) , \varphi \rangle = \langle -\underline{\omega}\, {}^sU , \underline{x}\, \varphi \rangle = \langle {}^sU , -\underline{\omega}\, (\underline{x}\, \varphi) \rangle .$$

according to (the boldface part of) the commutative diagram



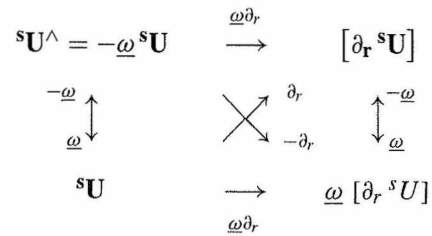
Remark 7.7.3 The commutative diagram of Definition 7.7.2 implies the definition of the multiplication of the signumdistribution ${}^sU$ by the function $\underline{x}$ resulting in the signumdistribution $\underline{x}\, {}^sU$ given by

$$\underline{x}\, {}^sU = (r\, {}^sU)^{\vee} = \underline{\omega}\, (\underline{x}\, {}^sU^{\wedge}) = \underline{\omega}\, \big(\underline{x}\, (-\underline{\omega})\, {}^sU\big) .$$

Definition 7.7.4 The derivative with respect to the radial distance r of a scalar-valued signumdistribution ${}^sU$ is the equivalent class of distributions

$$[\partial_r\, {}^sU] = [\underline{\omega}\, \partial_r\, {}^sU^{\wedge}] = \left[-\frac{1}{\underline{x}}\, \mathbb{E}\, {}^sU^{\wedge} \right] = \left[\frac{1}{\underline{x}}\, \mathbb{E}\, \underline{\omega}\, {}^sU \right] = T + c\, \delta(\underline{x})$$

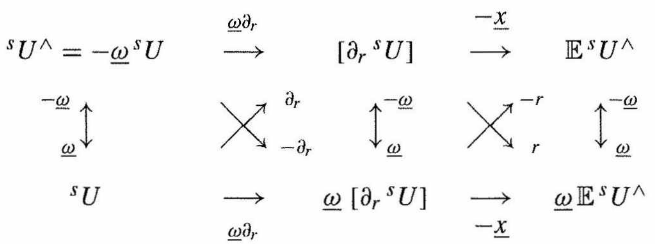
for any scalar distribution T satisfying $\underline{x}\, T = -\, \mathbb{E}\, {}^sU^{\wedge} = \mathbb{E}\, \underline{\omega}\, {}^sU$, according to (the bold face part of) the commutative diagram



Remark 7.7.5 As we have now at our disposal the definitions of the multiplication by r (Definition 7.7.2) and of the radial derivative ∂_r (Definition 7.7.4) of a signumdistribution, we are able to define the action of the Euler operator $\mathbb{E} = r\,\partial_r$ on the signumdistribution ${}^sU$, resulting into the unique signumdistribution $\mathbb{E}\,{}^sU$ given by

$$\mathbb{E}\,{}^sU = (r\,\partial_r)\,{}^sU = r\,(\partial_r\,{}^sU) = \underline{\omega}\,(-\underline{x}\,[\partial_r\,{}^sU]) = \underline{\omega}\,(\mathbb{E}\,{}^sU^\wedge) = \underline{\omega}\,(-\underline{x}\,T) = r\,T$$

for any distribution T satisfying $\underline{x}\,T = -\mathbb{E}\,{}^sU^\wedge$, according to the commutative diagram



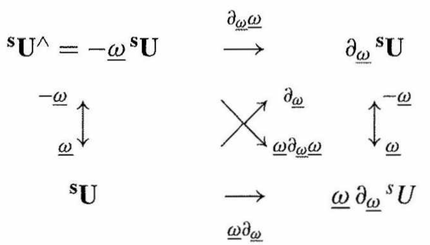
Remark 7.7.6 The commutative diagram of Definition 7.7.4 implies the definition of the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\,\partial_r$ on the signumdistribution ${}^sU$ resulting in the signumdistribution $\underline{\omega}\,\partial_r\,{}^sU$ given by the equivalence class

$$[\underline{\omega}\,\partial_r\,{}^sU] = \underline{\omega}\,\left[\partial_r\,{}^sU\right] = \underline{\omega}\left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU^\wedge\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{\omega}\,{}^sU\right] = \left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU\right].$$

In particular, when ${}^sU$ is a radial signumdistribution: ${}^sU = {}^sU^{rad}$, we define the action of the Dirac operator $\underline{\partial}$ on ${}^sU^{rad}$ to be

$$\underline{\partial}\,{}^sU^{rad} = \left[\underline{\omega}\,\partial_r\,{}^sU^{rad}\right] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{\omega}\,{}^sU^{rad}\right].$$

Definition 7.7.7 The angular $\partial_{\underline{\omega}}$-derivative of a scalar-valued signumdistribution ${}^sU$ is the distribution $\partial_{\underline{\omega}}\,{}^sU = \partial_{\underline{\omega}}\,\underline{\omega}\,{}^sU^\wedge$ according to (the boldface part of) the commutative diagram



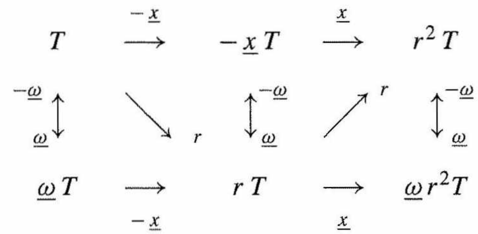
Remark 7.7.8 The commutative diagram of Definition 7.7.7 implies the definition of the action of the Γ-operator on the signumdistribution ${}^sU$ resulting in the signumdistribution $\Gamma\,{}^sU$ given by

$$-\,\Gamma\,{}^sU = \underline{\omega}\,\partial_{\underline{\omega}}\,{}^sU = \underline{\omega}\,(\partial_{\underline{\omega}}\,{}^sU) = \underline{\omega}\,(\partial_{\underline{\omega}}\,\underline{\omega}\,{}^sU^\wedge) = (\partial_{\underline{\omega}}\,\underline{\omega}\,{}^sU^\wedge)^\vee.$$

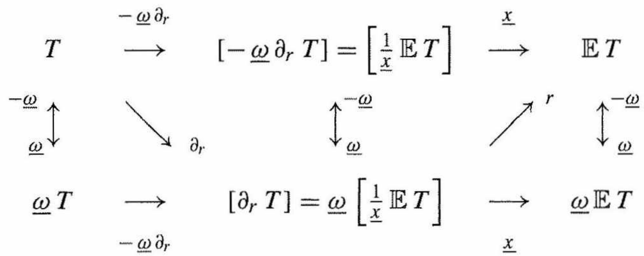
7.8 Composite Actions of Two Operators

In the preceding sections we were able to define the actions on (signum-) distributions of the operators r, $\underline{\omega}$, ∂_r, and $\partial_{\underline{\omega}}$. In Sect. 7.1 it was argued that the composite action by any two of those operators should lead to a *legal* action on distributions. Let us find out now if this is indeed the case.

1. Multiplication of a distribution T by the analytic function $r^2 = -\underline{x}^2 = \sum_{j=1}^{m} x_j^2$ is well defined. Through the following commutative diagram it is shown that $r(rT) = r^2\,T$:



2. Multiplication of a distribution T by the analytic function $\underline{x} = r\,\underline{\omega}$ is well defined. Through the commutative diagram of Definition 7.6.2 it is shown that $r(\underline{\omega}\,T) = \underline{x}\,T$.
3. The action of the Euler operator $\mathbb{E} = \sum_{j=1}^{m} x_j\,\partial_{x_j}$ on a distribution is well defined. Through the following commutative diagram it is shown that $r(\partial_r\,T) = \mathbb{E}\,T$:



4. The action of the operator $\underline{x}\,\Gamma = \underline{x}\left(\sum_{j<k} e_j e_k (x_j \partial x_k - x_k \partial_{x_j})\right)$ on a distribution is well defined. Through the following commutative diagram it is shown that $r(\partial_{\underline{\omega}}\, T) = \underline{x}\,\Gamma\, T$:

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\,\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{\underline{x}} & \underline{x}\,\Gamma\, T \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow {\scriptstyle \partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow {\scriptstyle r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\, T & \underset{\underline{x}}{\longrightarrow} & \underline{x}\,\partial_{\underline{\omega}}\, T
\end{array}$$

5. It is clear that $\underline{\omega}\,(\underline{\omega}\, T) = -T$.
6. The action of the operator $\underline{\omega}\,\partial_r$ on a distribution is well defined, albeit not uniquely but through an equivalence class instead, see (7.4.1). Definition 7.6.4 implies that $\underline{\omega}\,[\partial_r\, T] = [(\underline{\omega}\,\partial_r)\, T]$.
7. The action of the operator $\underline{\omega}\,\partial_{\underline{\omega}} = -\,\Gamma$ on a distribution is well defined. Definition 7.6.7 implies that $\underline{\omega}\,(\partial_{\underline{\omega}}\, T) = -\,\Gamma\, T$.
8. The action of the operator ∂_r^2 on a distribution was defined in Sect. 7.5 by the equivalence class

$$\partial_r^2\, T = \left[-\,(\underline{\omega}\,\partial_r)^2\, T\right] = S_2 + \delta(\underline{x})\, c_2 - \sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x})$$

for arbitrary constants c_2 and $c_{1,j}$, $j = 1, \ldots, m$ and any distribution S_2 such that $\underline{x}\, S_2 = \mathbb{E}\,\underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\, T$, which is in complete agreement with the commutative diagram

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\,\partial_r} & [-\,\underline{\omega}\,\partial_r\, T] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\, T\right] & \xrightarrow{\underline{\omega}\,\partial_r} & \left[-\,(\underline{\omega}\,\partial_r)^2\, T\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow {\scriptstyle \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow {\scriptstyle \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\underline{\omega}\,\partial_r}{\longrightarrow} & [\partial_r\, T] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\, T\right] & \underset{\underline{\omega}\,\partial_r}{\longrightarrow} & \underline{\omega}\left[-\,(\underline{\omega}\,\partial_r)^2\, T\right]
\end{array}$$

9. Start with the observation that for a distribution T,

$$\partial_r\,\partial_{\underline{\omega}}\, T = \underline{\omega}\,\partial_r\,(-\,\underline{\omega}\,\partial_{\underline{\omega}})\, T = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,\Gamma\, T\right]$$

to see that the action of the operator $\partial_r\,\partial_{\underline{\omega}}$ on a distribution is well-defined, though not uniquely. Then the commutative diagram

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\,\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{\underline{\omega}\,\partial_r} & \partial_r\,\partial_{\underline{\omega}}\, T \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow {\scriptstyle \partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow {\scriptstyle \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\, T & \underset{\underline{\omega}\,\partial_r}{\longrightarrow} & -\,\partial_r\,\Gamma\, T
\end{array}$$

shows that indeed $\partial_r\,(\partial_{\underline{\omega}}\, T) = \partial_r\,\partial_{\underline{\omega}}\, T$.

10. Start with the observation that for a distribution T,

$$\partial_{\underline{\omega}}^2\, T = \partial_{\underline{\omega}}\,\underline{\omega}\,(-\,\underline{\omega}\,\partial_{\underline{\omega}})\, T = (1-m)\,\Gamma\, T + \Gamma^2\, T$$

to see that the action of the operator $\partial_{\underline{\omega}}$ on a distribution is well-defined. Applying twice the commutative diagram of Definition 7.6.7 we obtain

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\,\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{\partial_{\underline{\omega}}\,\underline{\omega}} & \partial_{\underline{\omega}}^2\, T \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow {\scriptstyle \partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow {\scriptstyle \partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\, T & \underset{\underline{\omega}\,\partial_{\underline{\omega}}}{\longrightarrow} & \underline{\omega}\,\partial_{\underline{\omega}}^2\, T
\end{array}$$

showing that indeed $\partial_{\underline{\omega}}\,(\partial_{\underline{\omega}}\, T) = \partial_{\underline{\omega}}^2\, T$.

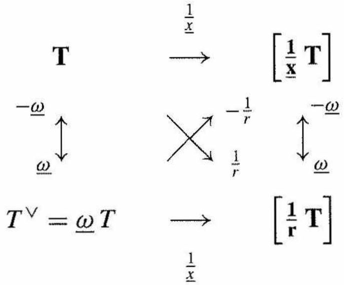
7.9 Division of (Signum)Distributions by r

Division of a standard distribution T by an analytic function $\alpha(\underline{x})$ resulting in an equivalent class of distributions S such that $\alpha(\underline{x})\, S = T$, we expect the division of a standard distribution by the non-analytic function r to lead to an equivalence class of signumdistributions. Let us make this precise.

Definition 7.9.1 The quotient of a scalar distribution T by the radial distance r is the equivalence class of signumdistributions

$$\left[\frac{1}{r}\, T\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\, T\right] = \underline{\omega}\,(\underline{S} + \delta(\underline{x})\,\underline{c}) = \underline{\omega}\,\underline{S} + \underline{\omega}\,\delta(\underline{x})\,\underline{c} = \underline{S}^{\vee} + \delta(\underline{x})^{\vee}\,\underline{c}$$

for any vector-valued distribution $\underline{S}$ for which $\underline{x}\,\underline{S} = T$, according to (the boldface part of) the commutative diagram



Example Let us illustrate Definition 7.9.1 by the case of the delta-distribution: $T = \delta(\underline{x})$. As $\underline{x}\,\underline{\partial}\,\delta(\underline{x}) = m\,\delta(\underline{x})$ and $\underline{x}\,\delta(\underline{x}) = 0$ we have

$$\frac{1}{\underline{x}}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) + \delta(\underline{x})\,\underline{c}_0$$

with $\underline{c}_0$ an arbitrary constant vector. It then follows that

$$\left[\frac{1}{r}\,\delta(\underline{x})\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\delta(\underline{x})\right] = \underline{\omega}\left[\frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) + \delta(\underline{x})\,\underline{c}_0\right]$$

$$= \frac{1}{m}\,\underline{\omega}\,\underline{\partial}\,\delta(\underline{x}) + \underline{\omega}\,\delta(\underline{x})\,\underline{c}_0 = \frac{1}{m}\,(\underline{\partial}\,\delta(\underline{x}))^{\vee} + \delta(\underline{x})^{\vee}\,\underline{c}_0$$

or, in view of the definition of $\partial_r\,\delta(\underline{x})$,

$$\left[\frac{1}{r}\,\delta(\underline{x})\right] = -\frac{1}{m}\,\partial_r\,\delta(\underline{x}) + \underline{\omega}\,\delta(\underline{x})\,\underline{c}_0.$$

However in this particular case of the delta-distribution it turns out that $\frac{1}{r}\,\delta(\underline{x})$ is uniquely determined. Indeed, as $\partial_r\,\delta(\underline{x})$ is a radial signumdistribution and as we expect the signumdistribution $\frac{1}{r}\,\delta(\underline{x})$ to be SO(m)-invariant as well, the arbitrary vector constant $\underline{c}_0$ should be zero, eventually leading to

$$\frac{1}{r}\,\delta(\underline{x}) = -\frac{1}{m}\,\partial_r\,\delta(\underline{x}).$$

For the general case of the division of the delta-distribution by natural powers of r we refer to [2].

Remark 7.9.2 The commutative diagram of Definition 7.9.1 implies the definition of the quotient of the signumdistribution $T^{\vee} = \underline{\omega}\,T$ by r, viz. the equivalence class of distributions

$$\left[\frac{1}{r}\,(\underline{\omega}\,T)\right] = \left[-\frac{1}{\underline{x}}\,T\right]$$

as well as the quotient of the same signumdistribution by $\underline{x}$, viz. the equivalence class of signumdistributions

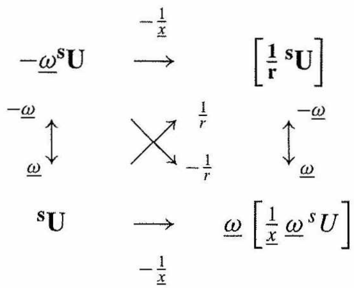
$$\left[\frac{1}{\underline{x}}\,(\underline{\omega}\,T)\right] = \left[\frac{1}{r}\,T\right].$$

It is also interesting and useful to define the division by r of a signumdistribution, because it will lead to the definition of the action of the angular part $\underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}$ of the Dirac operator on a signumdistribution, leading in its turn to the definition of the action of the Dirac operator on a signumdistribution.

Definition 7.9.3 The quotient of a scalar-valued signumdistribution ${}^sU$ by the radial distance r is the equivalence class of distributions

$$\left[\frac{1}{r}\,{}^sU\right] = \left[\frac{1}{\underline{x}}\,\underline{\omega}\,{}^sU\right] = S + \delta(\underline{x})\,c$$

for any scalar-valued distribution S for which $\underline{x}\,S = \underline{\omega}\,{}^sU$, according to (the boldface part of) the commutative diagram



Remark 7.9.4 The commutative diagram of Definition 7.9.3 implies the definition of the quotient of the signumdistribution ${}^sU$ by $\underline{x}$, viz. the equivalence class of signumdistributions

$$\left[\frac{1}{\underline{x}}\,{}^sU\right] = -\underline{\omega}\left[\frac{1}{\underline{x}}\,\underline{\omega}^s U\right]$$

as well as the quotient of the distribution $\underline{\omega}\, {}^sU$ by r, viz. the equivalence class of signumdistributions

$$\left[\frac{1}{r}\,(\underline{\omega}\, {}^sU)\right] = \left[-\frac{1}{\underline{x}}\, {}^sU\right].$$

Now as we know how to act with the operator $\partial_{\underline{\omega}}$ on a distribution (see Definition 7.6.7) and how to act with the operator $\frac{1}{r}$ on a signumdistribution (see Definition 7.9.3) we are now in the position to check the action on a distribution of the composition of both operators, viz. the angular part $\underline{\partial}_{ang}$ of the Dirac operator. The outcome should match expression (7.4.2); that this is indeed the case is shown by the following commutative diagram:

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{-\frac{1}{\underline{x}}} & \left[\frac{1}{r}\,\partial_{\underline{\omega}}\, T\right] = \left[-\frac{1}{\underline{x}}\,\Gamma\, T\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow^{\partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{\frac{1}{r}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\partial_{\underline{\omega}}\underline{\omega}]{} & \partial_{\underline{\omega}}\, T & \xrightarrow[-\frac{1}{\underline{x}}]{} & \underline{\omega}\left[-\frac{1}{\underline{x}}\,\Gamma\, T\right]
\end{array}$$

In the same order of ideas we can define the action of $\underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}$ on a signumdistribution through the commutative diagram

$$\begin{array}{ccccc}
{}^sU^\wedge = -\underline{\omega}\, {}^sU & \xrightarrow{\partial_{\underline{\omega}}\,\underline{\omega}} & \partial_{\underline{\omega}}\, {}^sU & \xrightarrow{\frac{1}{\underline{x}}} & \left[\frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\, {}^sU\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow^{\partial_{\underline{\omega}}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow^{\frac{1}{r}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
{}^sU & \xrightarrow[\underline{\omega}\partial_{\underline{\omega}}]{} & \underline{\omega}\,\partial_{\underline{\omega}}\, {}^sU & \xrightarrow[\frac{1}{\underline{x}}]{} & \left[\frac{1}{r}\,\partial_{\underline{\omega}}\, {}^sU\right]
\end{array}$$

in other words

$$\left[\underline{\partial}_{ang}\, {}^sU\right] = \left[\frac{1}{r}\,\partial_{\underline{\omega}}\, {}^sU\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\, {}^sU\right] = \left[-\frac{1}{\underline{x}}\,\Gamma\, {}^sU\right].$$

Combining the actions on a signumdistribution of the radial and angular parts of the Dirac operator, we are able to define the action of the Dirac operator itself on a signumdistribution.

Definition 7.9.5 The action of the Dirac operator $\underline{\partial}$ on the signumdistribution ${}^sU$ is given by the equivalence class of signumdistributions

$$\begin{aligned}
\left[\underline{\partial}\, {}^sU\right] &= \left[(\underline{\omega}\,\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}})\, {}^sU\right] \\
&= \left[-\frac{1}{\underline{x}}\,\mathbb{E}\, {}^sU\right] + \left[-\frac{1}{\underline{x}}\,\Gamma\, {}^sU\right] \\
&= \left[-\frac{1}{\underline{x}}\,(\mathbb{E}+\Gamma)\, {}^sU\right] \\
&= \left[\frac{1}{\underline{x}}\,(\underline{x}\,\underline{\partial})\, {}^sU\right]
\end{aligned}$$

according to the commutative diagram

$$\begin{array}{ccc}
{}^sU^\wedge = -\underline{\omega}\, {}^sU & \xrightarrow{D} & \left[\partial_r\, {}^sU + \frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\, {}^sU\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
{}^sU & \xrightarrow[\underline{\partial}]{} & \left[\underline{\partial}\, {}^sU\right]
\end{array}$$

where D stands for the operator

$$\begin{aligned}
D &= \underline{\omega}\,\partial_r + \frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\,\underline{\omega} \\
&= \underline{\omega}\,\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega} \\
&= \underline{\omega}\,\partial_r - \frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\,\underline{\omega}.
\end{aligned}$$

Example Let us illustrate Definition 7.9.5 with the following simple example; Sect. 7.10 will offer more elaborated ones. Consider the signumdistribution $\underline{x}$ defined by

$$\langle\, \underline{x}\,,\ \underline{\omega}\varphi\,\rangle = \langle\, \underline{x}\,\underline{\omega}\,,\ \varphi\,\rangle = \langle\, -r\,,\ \varphi\,\rangle = \int_{\mathbb{R}^m} r\,\tilde{\varphi}(r,\underline{\omega})\, d\underline{x}$$

for which

$$\mathbb{E}\,\underline{x} = \underline{x}$$

and

$$\Gamma\, \underline{x} = (m-1)\underline{x}$$

whence

$$[\underline{\partial}\, \underline{x}] = \left[-\frac{1}{\underline{x}}\, m\, \underline{x} \right] = [-m] = -m + \delta(\underline{x})\, c$$

As $\underline{x}^\wedge = r$ and $D\,r = [m\,\underline{\omega}]$, this result fits into the following commutative diagram:

$$\begin{array}{ccc} r & \xrightarrow{D} & [m\,\underline{\omega}] \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\ \underline{x} & \xrightarrow[\underline{\partial}]{} & [-m] \end{array}$$

Remark 7.9.6 The commutative diagram of Definition 7.9.5 shows that the Dirac operator acting on signumdistributions, corresponds with the operator D acting on distributions. We can wonder which operator acting on signumdistributions corresponds to the Dirac operator $\underline{\partial} = \underline{\partial}_{rad} + \underline{\partial}_{ang} = \underline{\omega}\,\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}}$ acting on distributions. From the commutative diagram in Definition 7.6.4 we learn that $\underline{\omega}\,\partial_r$ corresponds with $\underline{\omega}\,\partial_r$, while we saw above that $\frac{1}{r}\,\partial_{\underline{\omega}}$ corresponds with $-\frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega}$. It follows that the Dirac operator acting on distributions corresponds with the operator $\underline{\omega}\,\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega}$, which is precisely the operator D, acting on signumdistributions.

Finally, as we know how to act with the multiplication operator $\frac{1}{r}$ on a signumdistribution, we can check the action on a distribution T of the composite operator $(\frac{1}{r} \circ \partial_r)\,T = \frac{1}{r}\,(\partial_r\, T)$ which should coincide with the action $(\frac{1}{r}\,\partial_r)\,T$, defined, though not uniquely, in Proposition 7.5.3 by $\frac{1}{r}\,\partial_r\, T = S_3 + \frac{1}{m}\sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$ for arbitrarily constant c_3 and any distribution S_3 such that $\underline{x}\,S_3 = \underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\mathbb{E}\,T$. That this is indeed the case is shown by the following commutative diagram:

$$\begin{array}{ccccc} T & \xrightarrow{-\underline{\omega}\,\partial_r} & [-\underline{\omega}\,\partial_r\, T] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] & \xrightarrow{-\frac{1}{\underline{x}}} & \left[\frac{1}{r^2}\,\mathbb{E}\,T\right] \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow{\scriptstyle \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \nearrow{\scriptstyle \frac{1}{r}} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\ \underline{\omega}\,T & \xrightarrow[-\underline{\omega}\,\partial_r]{} & [\partial_r\, T] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] & \xrightarrow[-\frac{1}{\underline{x}}]{} & \underline{\omega}\left[\frac{1}{r^2}\,\mathbb{E}\,T\right] \end{array}$$

7.10 Two Families of Specific (Signum)Distributions

In the context of Clifford analysis a number of families of distributions were thoroughly studied, see e.g. [1]. Of particular importance are the families T_λ and U_λ, λ being a complex parameter. They are defined as follows.

$$\langle\, T_\lambda, \varphi(\underline{x})\,\rangle := a_m\,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^0[\varphi](r)\,\rangle_r$$

$$\langle\, U_\lambda, \varphi(\underline{x})\,\rangle := a_m\,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^1[\varphi](r)\,\rangle_r$$

where the so-called spherical means Σ^0 and Σ^1 are given by

$$\Sigma^0[\varphi](r) = \frac{1}{a_m}\int_{S^{m-1}} \varphi(r\underline{\omega})\, dS(\underline{\omega})$$

$$\Sigma^1[\varphi](r) = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\,\varphi(r\underline{\omega})\, dS(\underline{\omega})$$

and $\mathrm{Fp}\, r_+^\mu$ stands for the *finite part* distribution on the one-dimensional r-axis.

The distributions T_λ are standard distributions in harmonic analysis; as functions of $\lambda \in \mathbb{C}$ they show simple poles at $\lambda = -m, -m-2, -m-4, \ldots$. The most important distribution is this family is $T_{-m+2} = \frac{1}{r^{m-2}}$, which is, up to a constant, the fundamental solution of the Laplace operator Δ.

The distributions U_λ form a typical Clifford analysis construct; they show simple poles at $\lambda = -m-1, -m-3, -m-5, \ldots$. The most important distribution in this family is $U_{-m+1} = \frac{\underline{\omega}}{r^{m-1}}$ which is, up to a constant, the fundamental solution of the Dirac operator $\underline{\partial}$ (see Sect. 7.4).

Both families of distributions are intertwined by the action of the Dirac operator $\underline{\partial}$, viz.

$$\underline{\partial}\, T_\lambda = \lambda\, U_{\lambda-1} \qquad \lambda \neq -m, -m-2, -m-4, \ldots$$

and

$$\underline{\partial}\, U_\lambda = -(\lambda+m-1)\, T_{\lambda-1} \qquad \lambda \neq -m+1, -m-1, -m-3, \ldots.$$

In the setting of spherical co-ordinates these formulae take the form:

$$\underline{\omega}\,\partial_r\, T_\lambda = \lambda\, U_{\lambda-1} \qquad \frac{1}{r}\,\partial_{\underline{\omega}}\, T_\lambda = 0 \qquad \lambda \neq -m, -m-2, -m-4, \ldots \tag{7.10.1}$$

and

$$\underline{\omega}\,\partial_r\, U_\lambda = -\lambda\, T_{\lambda-1} \qquad \frac{1}{r}\,\partial_{\underline{\omega}}\, U_\lambda = -(m-1)\, T_{\lambda-1} \qquad \lambda \neq -m+1, -m-1, \ldots. \tag{7.10.2}$$

When restricted to the half-plane $\mathbb{R}e\,\lambda > -m$ the distributions T_λ and U_λ are regular, i.e. locally integrable functions. We know from [2] that a locally integrable function can also be seen as a signumdistribution. Whence the definition of the following two families of signumdistributions:

$$\langle\, {}^sT_\lambda, \underline{\omega}\,\varphi(\underline{x})\,\rangle := \quad a_m\,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\, \Sigma^1[\varphi](r)\,\rangle_r$$

$$\langle\, {}^sU_\lambda, \underline{\omega}\,\varphi(\underline{x})\,\rangle := -\,a_m\,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1},\, \Sigma^0[\varphi](r)\,\rangle_r$$

It becomes clear at once that:

$$T_\lambda^\vee = \ {}^sU_\lambda$$

and

$$U_\lambda^\vee = -\,{}^sT_\lambda$$

Moreover ${}^sT_\lambda$ inherits the simple poles of U_λ, viz. $\lambda = -m-1, -m-3, \ldots$, while ${}^sU_\lambda$ inherits the simple poles of T_λ, viz. $\lambda = -m, -m-2, \ldots$.

Invoking the commutative diagrams of Sect. 7.6 we are now able to compute the radial derivative of the distributions T_λ and U_λ, which at the time [1] and related papers were written, we were not yet able to achieve. We obtain:

$$\begin{array}{ccccc}
-\frac{1}{\lambda}T_\lambda & \xrightarrow{-\underline{\omega}\partial_r} & U_{\lambda-1} & \xrightarrow{-\underline{\omega}\partial_r} & (\lambda-1)T_{\lambda-2} \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -\partial_r,\ \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -\partial_r,\ \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
-\frac{1}{\lambda}{}^sU_\lambda & \xrightarrow[-\underline{\omega}\partial_r]{} & -{}^sT_{\lambda-1} & \xrightarrow[-\underline{\omega}\partial_r]{} & (\lambda-1){}^sU_{\lambda-2}
\end{array}$$

whence, for general λ, i.e. λ not in the simple poles mentioned above:

$$\partial_r\, T_\lambda = \lambda\, {}^sT_{\lambda-1} \qquad \partial_r\, U_\lambda = \lambda\, {}^sU_{\lambda-1}$$

formulae one should expect right from the start where it not that the results are no longer distributions but signumdistributions instead.

For the exceptional values of the parameter λ, in particular for those values which give rise to the fundamental solutions of the Dirac and Laplace operators, we obtain, in a similar manner, the following commutative diagram:

$$\begin{array}{ccccc}
\frac{T_{-m+2}}{m-2} & \xrightarrow{-\underline{\omega}\partial_r} & U_{-m+1} & \xrightarrow{-\underline{\omega}\partial_r} & -(m-1)T_{-m} + a_m\delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -\partial_r,\ \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -\partial_r,\ \partial_r} & {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} \\
\frac{{}^sU_{-m+2}}{m-2} & \xrightarrow[-\underline{\omega}\partial_r]{} & -{}^sT_{-m+1} & \xrightarrow[-\underline{\omega}\partial_r]{} & -(m-1){}^sU_{-m} + a_m\underline{\omega}\delta\underline{x}
\end{array}$$

Notice that at in this way we proved the formula

$$\underline{\omega}\partial_r\, U_{-m+1} = (m-1)T_{-m} - a_m\delta(\underline{x})$$

which is additional to (7.10.2) and refines the traditional formula

$$\underline{\partial}\, U_{-m+1} = -a_m\,\delta(\underline{x}).$$

We also proved

$$(\underline{\omega}\,\partial_r)\,{}^sT_{-m+1} = -\,(m-1)\,{}^sU_{-m} + a_m\,\underline{\omega}\,\delta(\underline{x}).$$

Compared with the similar formula at the distribution level, viz.

$$(\underline{\omega}\,\partial_r)\,T_{-m+1} = -\,(m-1)\,U_{-m},$$

the appearance of the signumdeltadistribution $\underline{\omega}\,\delta(\underline{x})$ might be surprising, but is well understood when realizing that $\lambda = -m$ is a regular point for U_{-m} while it is a simple pole for ${}^sU_{-m}$.

7.11 Conclusion

In his famous and seminal book [7] Laurent Schwartz writes on page 51: *Using co-ordinate systems other than the cartesian ones should be done with the utmost care* [our translation]. And right he is! Indeed, just consider the delta distribution $\delta(\underline{x})$: it is pointly supported at the origin, it is rotation invariant: $\delta(A\,\underline{x}) = \delta(\underline{x}),\ \forall A \in \mathrm{SO}(m)$, it is even: $\delta(-\underline{x}) = \delta(\underline{x})$ and it is homogeneous of order $(-m)$: $\delta(a\underline{x}) = \frac{1}{|a|^m}\,\delta(\underline{x})$. So in a first, naive, approach, one could think of its radial derivative $\partial_r\,\delta(\underline{x})$ as a distribution which remains pointly supported at the origin, rotation invariant, even and homogeneous of degree $(-m-1)$. Temporarily leaving aside the even character, on the basis of the other cited characteristics the distribution $\partial_r\, \delta(\underline{x})$ should take the following form:

$$\partial_r\, \delta(\underline{x}) = c_0\, \partial_{x_1} \delta(\underline{x}) + \cdots + c_m\, \partial_{x_m} \delta(\underline{x})$$

and it becomes immediately clear that this approach to the radial derivation of the delta distribution is impossible since all distributions appearing in the sum at the right-hand side are odd and not rotation invariant, whereas $\partial_r\, \delta(\underline{x})$ is assumed to be even and rotation invariant. It could be that $\partial_r\, \delta(\underline{x})$ is either the zero distribution or is no longer pointly supported at the origin, but both those possibilities are unacceptable. So from the start we are warned by this example that introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in \mathbb{S}^{m-1}$ makes derivation of distributions in $\mathbb{R}^m$ a far from trivial action, as are, in principle "forbidden", actions such as multiplication by the non-analytic functions r and ω_j, $j = 1, \ldots, m$. But there is more: functional analytic considerations on the space $\mathcal{D}(\mathbb{R}^m)$ of compactly supported smooth test functions expressed in spherical co-ordinates, forced us to introduce a new space of continuous linear functionals on a auxiliary space of test functions showing a singularity at the origin, for which, in [2], we coined the term *signumdistributions*, bearing in mind that $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$ may be interpreted as the higher dimensional counterpart to the *signum* function on the real line. It turns out that the actions by r, $\underline{\omega}$, ∂_r and $\partial_{\underline{\omega}}$ map a distribution to a signumdistribution and vice versa. The basic idea behind the definition of these actions on a distribution $T \in \mathcal{D}'(\mathbb{R}^m)$, is to express the resulting signumdistributions as appropriate and "legal" actions on T. So, for example, we put $\langle rT, \underline{\omega}\varphi \rangle = \langle r\underline{\omega}T, \varphi \rangle = \langle \underline{x}T, \varphi \rangle, \forall \varphi \in \mathcal{D}(\mathbb{R}^m)$. This idea may seem to be rather simple, but it is backed up by the functional analytic considerations of Sect. 7.1, and it paves the way for easy to handle calculus rules as established in [2].

Of the four aforementioned actions only the radial derivative $\partial_r T$ escapes, in general, from an unambiguous definition, but leads to an equivalent class of signumdistributions instead. Still we are able to define unambiguously $\partial_r T$ in two particular cases: (i) when the given distribution T is radial, i.e. rotation invariant, and (ii) when $T = U^\wedge$ is the associated distribution to a given radial signumdistribution U, these two particular cases being quite interesting since they correspond to two families of frequently used distributions such as the fundamental solutions of the Laplace and the Dirac operator, in Clifford analysis.

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