5-2016

Building Voters: Exploring Interdependent Preferences in Binary Contexts

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Saint Paul, Minnesota

A thesis submitted for the degree of

Undergraduate Honors

May 2016

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Acknowledgements

First, I would like to thank Professor Andrew Beveridge for his constant support. Beginning as my first year advisor and now as my honors advisor, Andrew has always challenged me to approach problems in unique ways and encouraged me to trust my intuition. I owe my continued mathematical passion to him.

I would like to thank Professor Tom Halverson for his advice and contributions to this thesis. Professor Halverson's results about $\mathbb{C}\mathbb{Z}_2 \wr S_n$-modules (found in Appendix A) were the break-through that I needed to develop my vector space approach to constructing voter preferences.

I thank Professor Tom Halverson and Professor Lori Ziegelmeier for serving as readers of this project and providing invaluable feedback. I have great respect for them both: they are great mathematicians who have had faith in my mathematical pursuits.

Finally, I would like to thank my high school mentor Lynn Ibarra. She is the reason I came to Macalester and fell in love with academics.

Abstract

In this thesis we develop a new method for constructing binary preference orders for given interdependent structures, called characters. We introduce the preference space, which is a vector space of preference vectors. The preference vectors correspond to binary preference orders. We show that the hyperoctahedral group, $\mathbb{Z}_2 \wr S_n$, describes the symmetries of binary preferences orders and then define an action of $\mathbb{Z}_2 \wr S_n$ on our preference vectors. We find a natural basis for a preference space. These basis vectors are indexed by subsets of proposals. We show that when completely separable binary preference vectors are decomposed using this basis, basis vectors indexed by nontrivial, even sized subsets do not appear in the decomposition. We then use these basis vectors as building blocks for preference construction. In particular, we construct preference orders whose Hasse diagram of separable sets have a tree structure.

Contents

Chapter 1

Introduction

Underlying any choice we make is a preference. When shopping, a decision to purchase a computer indicates the computer is preferred to the money used to buy it. In a referendum election, a vote in favor of a proposal indicates that the voter would prefer the proposal to pass than not pass. Studying such preferences (have or have not, pass or pass not) at the single proposal/good level is trivial. Aggregating an individual's preferences on many goods or proposals produces preference orders on baskets of goods and election outcomes, respectively; these are immensely complex due to potential preference interdependence between subsets of goods or proposals. For example, a voter may only want one proposal to pass only if some other proposal also passes. Likewise, a consumer may only want good A in their basket of goods if good B is also present in their basket.

Studying the implications of such interdependences is critical in social choice theory and for understanding consumer choices. In referendum elections, the existence of preference interdependence among proposals is called the *separability problem* and can lead to election outcomes that are far from optimal [4]. Two examples which have been studied include the Los Angeles County in 1990 referendum election as well as the Colorado presidential election ballot in 2004 [4][12]. For vendors, understanding the interdependences among consumer preferences provides insight as to what goods are compliments or substitutes. Such information could help vendors significantly improve their marketing.

1.1 The Separability Problem

In a referendum election, voters communicate their preferences and opinions to policy makers directly. Based on the results of a referendum election policy makers institute changes which they believe are desired by voters. However, in the case of a multiple proposal referendum, researchers have found that an election can result in a winning outcome that is relatively undesirable or even the least desired outcome by all voters [9]. These democratic failures were first recognized in the 1990's. They can result when a voter or collection of voters perceive connections among a referendum's proposals. Lacy and Niou [13] defined a voter's preferences as nonseparable when these connections occur. They argue that multiple proposal referendums, "force people to separate their votes on issues that may be linked in their minds." Asking a voter to respond to a proposal or set of proposals without knowing the outcome of the others forces the voter to make predictions or simply ignore the interdependent structure of the ballot. This issue is called the *separability problem.*

One example of the separability problem occurring in the United States and causing an unsatisfactory election outcome is described in "Voting on Referenda" [4]. Brams et. al. argue that in 1990 the Los Angeles general election resulted in an undesirable outcome on three pro-environment proposals due to the nonseparability of voters' preferences on the proposals. They reason that, "Because all three propositions were pro-environment and involved the expenditure of substantial funds, there is good reason to believe that many voters saw them as related." They show that the winning outcome on the three proposals was only the fifth most popular outcome out of the eight possible outcomes. In fact, the winning outcome of YNY[1] appeared on only 99,176 ballots, while the outcomes of YYY and NNN appeared on 430,807 and 422,916 ballots respectfully [4].

Example 1.1.1 (Professional Soccer in Dubuque, Iowa[2])**.** *Due to the increasing popularity of soccer in Dubuque, the city has become interested in having their own professional soccer team and possibly a new stadium. They hold a referendum with the following three proposals:*

- *Proposal 1: Have a professional women's soccer team.*
- *Proposal 2: Have a professional men's soccer team.*
- *Proposal 3: Build a new stadium.*

The percent of citizens voting for a given outcome is shown in Table 1.1.

[1]In their paper, outcomes on referendums of n questions can be represented as n-tuples where "Y" in the nth position represents the passing of the nth proposal and "N" represents its failing.

[2]This example was created with Jon Hodge and Brea Beals at the 2014 Grand Valley State University REU [1].

Outcome	Percent of Vote
NNN	26%
NNY	0%
NYN	10%
YNN	10%
YYN	0%
YNY	15%
NYY	15%
YYY	24%

Table 1.1: The table indicates the percentage of voters voting for a specific outcome on the three proposals in the Dubuque referendum.

While the outcome with the most votes is "no" on all proposals with 26% of the votes, the winning outcome is determined by counting up votes on all of the proposals individually. Counting votes by proposal indicates that:

- *Proposal 1 fails 49% Yes to 51% No.*
- *Proposal 2 fails 49% Yes to 51% No.*
- *Proposal 3 passes 54% Yes to 46% No.*

The winning outcome is therefore NNY. This is an interesting result because, as we can observe in Table 1, this outcome received 0 votes. If policy makers were bound by the referendum in the previous example they would be forced to construct a new stadium despite not having a professional team, an expensive and useless result. The failure of this referendum to produce a desirable result is due to voters' preferences being interdependent. The voters in this election likely perceived connections among the proposals. We can see one connection in Table 1.1. Every voter that voted for the stadium also voted for one or both of the professional soccer teams; they only wanted a stadium if there was a team to play in it. This connection indicates that some of the voters preferences were likely interdependent. This is the source of the separability problem arising in this case.

1.2 Roadmap

In this thesis, we will explore the separability problem. That is, we will explore interdependent preferences, or nonseparable preferences. We will begin in Chapter 2 by comparing and contrasting the manners in which economists, mathematicians, and social choice theorists represent preference orders of individuals. This discussion provides a foundational understanding of preference orders and also justifies how we have chosen to represent individual preferences. After this discussion, we will mathematically define separability, preference independence and preference interdependence. We will then be able to describe the specific type of interdependence implied by a preference order. From here, we will be more prepared to discuss the ultimate goal of this thesis, preference construction.

The hyperoctahedral group and its relationship with separability of binary preference orderings will be outlined in Chapter 3. We will then find a basis, called the voter basis, for our preference spaces which is a transformation of the basis found by Tom Halverson in Appendix A. We will describe the separability of voter basis vectors then prove an interesting result describing the relationship between the voter basis and completely separable preference vectors.

In Chapter 4, we will present an algorithm for generating preference orderings with specific interdependence qualities using one of the bases from Chapter 3. The reliability of this algorithm will be rigorously proven and the significance of the algorithm will be compared with previous preference construction methods. Our main contribution is the ability to create preference orderings whose separable sets have a tree-like structure with respect to set inclusion.

We conclude this thesis by summarizing the important results and future directions. The results include Theorem 3.2.17 in and Theorem 4.2.6. Future directions focus on a counting the class of characters described in Chapter 4, finding other classes of characters for which preference orders can be created, and furthering results in constructing completely separable preference vectors.

Chapter 2

Binary Preference Orders

In this chapter, we define binary relations and the conditions necessary for a binary relation to be a preference relation. These preference relations serve to mathematically represent all the possible "rational" preferences an individual could exhibit. We then describe binary preference matrices, which were created specifically to demonstrate the preference orders of voters in referendum elections [8]. We show that vectors within vector spaces, called *preference spaces*, are also capable of representing the preference orders of voters in referendum elections and offer some significant advantages over preference matrices. After proving a few results related to preference spaces, we mathematically describe preference interdependence through the definition of *separability* and the *character of a preference order*.

2.1 Binary Relations and Preference Matrices

Preference orders are a special class of binary relations so we begin by defining general binary relations, then discuss the properties which make binary relation a preference relation. Finally, we show how to produce the preference order for a preference relation.

Definition 2.1.1. *A* ***binary relation*** *R over the set A and set B is any subset of the Cartesian product between A and B. That is $R \subseteq A \times B$.*

For a binary relation $R \subseteq A \times B$, if $(a, b) \in R$, we write aRb.

We focus on binary relations that describe individuals' preferences. Because of this, we make two assumptions based on rational choice theory: *completeness* and *transitivity*.

Definition 2.1.2. *If for all $(x, y) \in A \times B$ either xRy, yRx, or both, then R is said to be* ***complete****.*

Completeness guarantees that every pair of outcomes can be compared. With respect to preference, this is clearly a valid assumption. With perfect information about the outcomes, an individual is always able to look at two outcomes and describe whether they prefer one over the other or are indifferent between the two.

Definition 2.1.3. *If for all $(x, y), (y, z), (x, z) \in R$,*

$$xRy \text{ and } yRz \implies xRz$$

*then R is said to be **transitive**.*

If an individual's preferences were not transitive then they would have "cycles of unsatisfaction" which could be exploited. Imagine an individual who prefers good 1 to good 2, good 2 to good 3, and good 3 to good 1. If they started with good 1, they would be willing to pay some amount to swap good 1 for good 2, then pay some amount to swap good 2 for good 3, then pay some amount to swap good 3 for good 1 again. This would mean that the individual is indifferent to having good 1 and having good 1 with less money, which is economically irrational.

With these properties, we are now prepared to define our preference relation $\succeq$.

Definition 2.1.4. *Let X be a set of outcomes. The binary relation $R \subseteq X \times X$ is called a **preference relation** if it is both complete and transitive. We denote a preference relation as $\succeq$.*

We define two other relations in terms of $\succeq$. The first is *strict preference*, $\succ$. If $a \succeq b$ and $b \not\succeq a$, then $a \succ b$; that is a is strictly preferred to b. The second is *indifference*. If $a \succeq b$ and $b \succeq a$, then $a \sim b$; that is a is neither preferred to nor more preferable than b.

Now we can describe preference relations with $\succ$ and $\sim$. These are superior in describing the preference relationship between two goods. In English, $a \succeq b$, means that "a is at least as good as b," but it doesn't actually say whether or not a is more preferred than b. With $\succ$ and $\sim$ there is no ambiguity since $a \succ b$ means that a is more preferred than b and $a \sim b$ means that there is no difference between a and b preference-wise.

Preference relations allow us to make rankings or preference orders on all the possible outcomes.

Example 2.1.5 (Preference Relation to Preference Order)**.** *Consider $\{x_1 \sim x_2, x_3 \succ x_1, x_3 \succ x_2\} = R \subset \{x_1, x_2, x_3\} \times \{x_1, x_2, x_3\}$. From this we can write the corresponding ranking, or preference order on the outcomes as follows:*

$$x_3 \succ x_1 \sim x_2.$$

Preference orders on an entire set of outcomes are only guaranteed by the rationality conditions. Without completeness, we could not ensure that we could connect all the outcomes in a preference order. Without transitivity, we might end up with cycles as we try to rank outcomes in the preference order.

Mathematicians and social choice theorists have used *binary preference matrices* to visualize the preferences of an individual. This choice reflects their motivation, which is the separability problem in referendum elections [8].

In the previous chapter, we considered the outcome of a referendum as a series of yes/no choices. For example, YNY is an outcome for an election with three ordered proposals. From here forward, we use 1 to denote a "yes" and a 0 to denote a "no," so that this outcome would then be the bitstring 101. Of course, there is a natural bijection from outcomes on n proposals to subsets of $[n]$, so this outcome could also be expressed as $\{1, 3\}$. We will adhere to the binary notation for an election outcome. This emphasizes the context in which we are studying the outcome: we obtain a yes/no choice of every element in $[n]$.

Definition 2.1.6. *Let $[n]$ be a referendum election with n proposals. Let*

$$X = \{x_1 x_2 \cdots x_n \mid x_i \in \{0, 1\}\}$$

be the set of possible outcomes on $[n]$ with a 1 representing an outcome of "passes" and a 0 representing an outcome of "fails to pass" for a proposal. Let P be a $2^n \times n$ matrix whose rows are a permutations of the elements of X. P is defined as a ***binary preference matrix*** *for a voter where the k^{th} row of P corresponds to the k^{th} most preferred outcome by the voter.*

The binary preference order implied by a binary preference matrix is quite clear: the outcomes appear in the order that the voter prefers.

Example 2.1.7 (Binary Preference Matrix to Binary Preference Order)**.** *Let $[n]$ be a set of three proposals. The following 8×3 matrix, P, represents the preferences of an individual voter on the outcomes of the referendum election $[n]$. For example, the preference matrix*

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

gives the preference order

$$(111) \succ (010) \succ (011) \succ (110) \succ (000) \succ (001) \succ (101) \succ (100).$$

Binary preference matrices are appealing because they visually communicate preference orders. However, these matrices do not allow for preference orders with indifference between two outcomes; the order of the rows gives a complete ordering of the outcomes (no ties allowed). Additionally, these matrices cannot be used as meaningful linear transformations. In other words, while a binary preference matrix is a simple way to express a linear preference order, its use as a mathematical object is limited.

2.2 The Preference Space P^n

In economics, the most common way to represent preference orders is with utility functions. Given some utility function $f(\cdot)$, an outcome x is preferred to some other outcome y if and only if $f(x) > f(y)$. We combine this economic thinking with recent developments in the area of algebraic voting theory. Daughtry et al. define a set of vectors called a *profile space*, where the entries of vectors correspond to preference orders on election outcomes and the value of these entries correspond to the number of voters with the given preference order [5].[1] In short, the vectors in a profile space represent the aggregation of the preference orders of voters.

Consider a referendum election with n proposals. Let X be the set of all outcomes, so that $|X| = 2^n$. We introduce a 2^n-dimensional vector space P^n called the *preference space* for X. The entries of vectors in this space are indexed by the election outcomes, listed in reverse lexicographical order, for example: 11, 10, 01, 00. A *preference vector* $\vec{p} \in P^n$ corresponds to the preferences of a single voter. Suppose that this voter has utility function $f : X \to \mathbb{Q}$. For $x \in X$, the xth entry of $\vec{p}$ is $f(x)$.

Our preference space is very different from the profile space of Daughtry et al. They focus on an election as a whole, while we focus on the individual voter. Furthermore, our preference space is a vector space. For simplicity, we start off by considering preference vectors with nonnegative integer entries. Later on, we start using preference vectors with rational entries.

Definition 2.2.1. *Let $[n]$ be a referendum election with n proposals. The* ***preference space*** *P^n is the 2^n-dimensional vector space over $\mathbb{Q}$ where the entries of vectors are indexed by the outcomes of this referendum, listed in reverse lexicographical order.*

[1] Note that the vectors in the profile space only have non-negative entries.

Example 2.2.2. *We revisit Example 2.1.7, where we expressed our preferences using a binary preference matrix. The binary preference order in this example is*

$$(111) \succ (101) \succ (000) \succ (001) \succ (110) \succ (011) \succ (010) \succ (100).$$

We assign a unique integer weight $0 \leq w \leq 7$ *to each outcome, where a higher weight corresponds to stronger preference. The result is the following preference vector, with the reverse lexicographical indexing system displayed to the right, for convenience:*

$$\vec{p} = \begin{pmatrix} 7 \\ 3 \\ 6 \\ 0 \\ 2 \\ 1 \\ 4 \\ 5 \end{pmatrix} \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 100 \\ 011 \\ 010 \\ 001 \\ 000. \end{matrix}$$

Our most preferred outcome is 111*, so it received weight 7. Our second choice is* 101*, so that entry has weight 6. This ordinal ranking continues, until we reach our least preferred outcome of* 100*, and this receives weight 0.*

Preference vectors enjoy many advantages over binary preference matrices. For example, taking a linear combination of preference vectors has a natural interpretation, whereas a linear combination of binary preference matrices does not make sense. This vector space structure is the fundamental reason why we can use representation theory to understand voter preferences.

For a preference vector $\vec{p}$, we denote the value of the entry indexed by the outcome x as $[x]_{\vec{p}}$, rather than the more traditional notation $\vec{p}_x$. In Example 2.2.2, we have $[110]_{\vec{p}} = 3$. We opt for this notation to highlight the outcome x rather than the vector $\vec{p}$.This will be particularly useful when we start constructing a preference vector with a desired property.

We will consider two types of preference vectors: *ordinal* preference vectors and *cardinal* preference vectors. Ordinal preference vectors *only* consider the preference order of outcomes, while cardinal preference vectors consider the utilities associated with outcomes. Given a preference vector $\vec{p}$ there is always a corresponding preference order, which is denoted $\succeq_{\vec{p}}$. For the preference vector $\vec{p}$ and any two election outcomes x and y, we have

$$x \succeq_{\vec{p}} y \quad \text{if and only if} \quad [x]_{\vec{p}} \geq [y]_{\vec{p}}.$$

Definition 2.2.3. *An* ***ordinal preference space*** *in P^n is a vector for which the value of the entry indexed by the ith least preferred outcome is $i-1$.*

The preference order

$$(111) \succ (101) \succ (000) \succ (001) \succ (110) \succ (011) \succ (010) \succ (100)$$

corresponds to the ordinal preference vector in Example 2.2.2. This example does not have any indifference (ties between outcomes), but you can create an ordinal preference vector when there is indifference between outcomes. For example, the preference order

$$(11) \sim (00) \succ (01) \sim (10)$$

has ordinal preference vector

$$\vec{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \begin{array}{c} 11 \\ 10 \\ 01 \\ 00. \end{array}$$

Proposition 2.2.4. *There exists a bijection between binary preference orders on an n proposal referendum and ordinal preference vectors in P^n.*

Proof. The definition of a ordinal preference vector defines the bijective function between binary preference orders and ordinal preference vectors. □

We now turn to cardinal preference vectors, whose values do not have to be drawn from $\{0, 1, 2, \ldots, n-1\}$.

Definition 2.2.5. *Let $f : X \to \mathbb{Q}$ be a utility function, where X is the set of all outcomes on a referendum election with n proposals. The corresponding* ***cardinal preference vector*** *$\vec{p} \in P^n$ is the vector with xth entry $[x]_{\vec{p}} = f(x)$.*

Cardinal preferences are often used in economics because they allow individuals or groups to describe *how much* one outcome is preferred to another. This additional information can be just as significant as knowing the preference order itself.

Example 2.2.6 (Cardinal Preference Vector)**.** *In P^3, the utility function*

$$f(b) = \begin{cases} 35 & b \in \{000\} \\ 40 & b \in \{110, 101\} \\ 75 & b \in \{001, 111, 011\} \\ 100 & b \in \{100\} \\ 120 & b \in \{010\} \end{cases}$$

has corresponding cardinal preference vector

$$\vec{p} = \begin{array}{c|l} \left(\begin{array}{c} 75 \\ 40 \\ 40 \\ 100 \\ 75 \\ 120 \\ 75 \\ 35 \end{array} \right) & \begin{array}{l} 111 \\ 110 \\ 101 \\ 100 \\ 011 \\ 010 \\ 001 \\ 000 \end{array} \end{array}.$$

This corresponds to the preference order

$$(010) \succ (100) \succ (111) \sim (011) \sim (001) \succ (110) \sim (101) \succ (000).$$

Note that the difference in utility between first- and second-most preferred outcomes is four times as much as the difference in utility between the least and second least preferred outcomes.

Given a cardinal preference vector $\vec{p}$, we use $\succeq_{\vec{p}}$ to denote the *preference order induced by* $\vec{p}$. Next, we show that a cardinal preference vector induces an ordinal preference order. First, we create an equivalence relation on cardinal preference vectors.

Theorem 2.2.7. *Let $\vec{p}$ and $\vec{q}$ be cardinal preference vectors where $\vec{p} \sim \vec{q}$, if $\vec{p}$ and $\vec{q}$ have the same underlying preference order. This relation is an equivalence relation and the equivalence classes are represented by the set of ordinal preference vectors.*

Proof. We check the three conditions for an equivalence relation.

1. Reflexive: Consider an arbitrary cardinal preference vector $\vec{p}$. The vector $\vec{p}$ has a unique underlying preference order $\succeq_{\vec{p}}$, so $\vec{p} \sim \vec{p}$.

2. Symmetric: Consider two cardinal preference vectors $\vec{p}$ and $\vec{q}$ where $\vec{p} \sim \vec{q}$. Since $\vec{p} \sim \vec{q}$, both vectors must have the same underlying preference order, so $\vec{q} \sim \vec{p}$.

3. Transitive: Consider three cardinal preference vectors $\vec{p}$, $\vec{q}$, and $\vec{r}$ where $\vec{p} \sim \vec{q}$ and $\vec{q} \sim \vec{r}$. Since $\vec{p} \sim \vec{q}$, $\vec{p}$ and $\vec{q}$ must have the same unique underlying preference order, that is, $\succeq_{\vec{p}}$ is the same as $\succeq_{\vec{q}}$. Since $\vec{q} \sim \vec{r}$, $\vec{q}$ and $\vec{r}$ must have the same unique underlying preference order, that is, $\succeq_{\vec{q}}$ is the same as $\succ_{\vec{r}}$. Thus $\succeq_{\vec{p}}$ is the same as $\succ_{\vec{r}}$. Therefore, $\vec{p} \sim \vec{r}$.

Now for every ordinal preference vector, we could construct a function $u : X \to \mathbb{Q}$ so that the resulting cardinal preference vector $\vec{p}$ is identical to the original ordinal preference vector. There is a natural bijection between ordinal preference vectors and binary preference orders, so the equivalence classes of $\sim$ can be represented by the set of ordinal preference vectors. □

As long as we are concerned only with the underlying preference order of preference vectors, we do not need to worry about the differences between ordinal preferences or cardinal preferences. This allows us to speak generally about preference vectors during our analysis.

We end this section with two final observations about ordinal preference vectors and the equivalence classes of cardinal preference vectors. First, we observe that we can count the number of ordinal preference vectors.

An ordered set partition of $[N]$ of size k is an ordered list of disjoint subets $A_1, A_2, \ldots, A_k$ whose union is the entire set: $[N] = \cup_{i=1}^{k} A_k$. The number F_N of ordered set partitions of $[N]$ into any number of parts is called the Nth *Fubini number*, and is also known as the Nth *ordered Bell number*.

The set of Fubini numbers F_N satisfies the recurrence

$$F_0 = 1, \quad F_N = \sum_{k=0}^{N-1} \binom{N}{k} F_k \quad k \geq 1.$$

We also have the formula

$$F_N = \sum_{k=0}^{n} k!\, S(N, k) = \sum_{k=0}^{N} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n = \frac{1}{2} \sum_{m=0}^{\infty} \frac{m^N}{2^m}.$$

We now show that the set of ordinal preference vectors in P_n has size $F(2^n)$.

Corollary 2.2.8. *For P^n, the equivalence classes of $\sim$, the ordinal preference vectors, and the binary preference orders on the outcomes are counted by $F(2^n)$.*

Proof. There is natural bijection between the set of ordinal preference vectors for the set X of outcomes on an n proposal referendum and the set of ordered partitions of $[2^n]$. We view each outcome $x \in X$ as the (natural) binary representation of the number $b(x) \in [2^n]$. The ith set S_i in the ordered partition is

$$S_i = \{b(x) \mid u(x) = i\}.$$

□

Finally, we make the following algebraic observation.

Theorem 2.2.9. *Equivalence classes of the preference vector $\sim$ are closed under addition.*

Proof. Consider cardinal preference vectors $\vec{p}$ and $\vec{q}$, where $\vec{p} \sim \vec{q}$ and the underlying preference order is $\succeq_{\vec{p}}$. We show that given two outcomes x and y, the preference relation between them implied by $\succeq_{\vec{p}}$ is the same as that for $\vec{p} + \vec{q}$.

- **Case 1:** $x \succ y$. On $\succeq_{\vec{p}}$ let $x \succ y$. This implies $[x]_{\vec{p}} > [y]_{\vec{p}}$ and $[x]_{\vec{q}} > y_{\vec{q}}$. Thus, we know $[x]_{\vec{p}} + [x]_{\vec{q}} > [y]_{\vec{p}} + [y]_{\vec{q}}$ which guarantees $[x]_{\vec{p}} + [x]_{\vec{q}} > [y]_{\vec{p}+\vec{q}}$. Therefore, on $\succeq_{\vec{p}+\vec{q}}$ we have $x \succ y$.

- **Case 2:** $y \succ x$ Similar to the previous case.

- **Case 3:** $x \sim y$. On $\succeq_{\vec{p}}$ let $x \sim y$. This implies $[x]_{\vec{p}} = [y]_{\vec{p}}$ and $[x]_{\vec{q}} = y_{\vec{q}}$. Thus, we know $[x]_{\vec{p}} + [x]_{\vec{q}} = [y]_{\vec{p}} + [y]_{\vec{q}}$ which guarantees $[x]_{\vec{p}} + [x]_{\vec{q}} = [y]_{\vec{p}+\vec{q}}$. Therefore, on $\succeq_{\vec{p}+\vec{q}}$ we have $x \sim y$.

□

2.3 Separability

Much of the literature in the area of preference interdependence focuses on *separability* [3, 9, 11, 10, 13]. Before we introduce separability we define some additional notation.

So far we have represented outcomes as binary strings, but we can also represent an outcome as the set of passing proposals. For the outcome x, we represent the set of passing proposals as $\widehat{x}$. For example, in a referendum on five proposals, we have

$$\widehat{10110} = \{1, 3, 4\}.$$

Later we will combine partial election outcomes on disjoint proposal sets; with this new notation this becomes easy, $\widehat{x} \cup \widehat{y} = \widehat{xy}$. More specifically, suppose that we are considering a referendum election on 8 proposals. Let x be a partial outcome, where we know the outcomes on the first five proposals, which can denote

$$x = 01101 * * * .$$

We define $\widehat{x} = \{2, 3, 5\}$, though proposals are excluded for one of two reasons: either they fail (as is the case for 1 and 4), or the outcome on that election is not specified. Let y be a partial outcome on the last three proposals:

$$y = * * * * 101$$

so that $\widehat{y} = \{6, 8\}$. Note that the supporting sets of x and y are disjoint, so we can concatenate x and y to obtain

$$xy = 01101101$$

in the natural way. Then we have

$$\widehat{xy} = \widehat{01101101} = \{2, 3, 5, 6, 8\} = \{2, 3, 5\} \cup \{6, 8\} = \widehat{x} \cup \widehat{y}.$$

We are now ready to define separability. A set of proposals S is separable with respect to a preference order if the knowledge of whether proposals outside of S, that is $[n] - S$, pass or fail does not change the voter's preference on S. That is, an individual's preference for S is independent of the individual's preference outside of S. Here is the formal definition.

Definition 2.3.1. *Consider a referendum with a set of proposals $[n]$ and $\succeq_{[n]}$ be a preference order on $X_{[n]}$, the set of possible outcomes on $[n]$. The set $S \subset [n]$ is* **separable** *with respect to $\succeq_{[n]}$, provided that for all $x_S, y_S \in X_{[n]}$, we have*

$$(x_S u_{[n]-S}) \succeq (y_S u_{[n]-S}) \text{ for some } u_{[n]-S} \in X_{[n]-S}$$

implies

$$(x_S v_{[n]-S}) \succeq (y_S v_{[n]-S}) \text{ for all } v_{[n]-S} \in X_{[n]-S}$$

Recall the soccer stadium referendum example in Chapter 1 with the types of votes shown in Table 1.1. As we stated before, every voter that was in favor of the new stadium also voted for one or both of the professional soccer teams. Thinking of one of these individual voter's preference order on the outcomes, we would find that their preference on the stadium proposal was nonseparable. They would change their vote from "yes" for the stadium to "no" if both team proposals failed.

Definition 2.3.2. *A set S is called* **trivially separable** *on $\succeq_{[n]}$ if for all $x_S, y_S \in X_S$ and $u_{[n]-S} \in X_{[n]-S}$*

$$(x_S u_{[n]-S}) \sim (y_S u_{[n]-S})$$

When individuals have trivially separable preferences on a set S of proposals, they are entirely indifferent between the possible outcomes of the proposals in S. This means they are actually indifferent between voting and not voting at all on the proposals in S. Such indifference is the simplest preference order an individual can have on a set.

Lemma 2.3.3. *Given a referendum on the proposals $[n]$, the empty set $\emptyset$ is trivially separable.*

Proof. There are no outcomes on $\emptyset$, so $(x_\emptyset u_{[n]}) \sim (u_{[n]}) \sim (y_\emptyset u_{[n]})$ for all $u_{[n]}$, so the lemma is vacuously true. $\square$

The set $[n]$ is always separable, but typically $[n]$ is not trivially separable.

Lemma 2.3.4. *Given a referendum on the proposals $[n]$, the set $[n]$ is separable.*

Proof. This is vacuously true because there are no proposals outside of $[n]$ for there to be alternative outcomes on. $\square$

The separability or nonseparability of a given set on an individual's preference order provides us with valuable information about how the individual relates the set to other proposals. By aggregating all the separability and nonseparability information for a single voter, we have a comprehensive summary of how the voter understands the referendum in its entirety.

Definition 2.3.5. *The collection of all $S \subseteq [n]$ that are separable with respect to a preference order $\succeq_{[n]}$ is called the* ***character of*** $\succeq_{[n]}$ *and is denoted by* $\mathrm{char}(\succeq_{[n]})$.

Example 2.3.6 (Character of a Preference Vector)**.** *Consider the following preference vector in P^3:*

$$\vec{p} = \begin{pmatrix} 7 \\ 5 \\ 6 \\ 2 \\ 3 \\ 0 \\ 4 \\ 1 \end{pmatrix} \qquad \begin{matrix} 111 \\ 110 \\ 101 \\ 100 \\ 011 \\ 010 \\ 001 \\ 000. \end{matrix}$$

This preference vector has the preference order $\succeq_{\vec{p}}$ given by

$$111 \succ 101 \succ 110 \succ 001 \succ 011 \succ 100 \succ 000 \succ 010.$$

To check separability on $\{1\}$, we see if the preference order of outcomes on $\{1\}$ is consistent regardless of the outcome outside of $\{1\}$, that is the outcome on the set $\{2, 3\}$. We have:

Outcome on $\{2,3\}$	*Preference order on* $\{1\}$
-11	$1--\succ 0--$
-10	$1--\succ 0--$
-01	$1--\succ 0--$
-00	$1--\succ 0--$

The previous table shows that voters with this preference vector would want proposal 1 to pass no matter what the outcome on proposals 2 and 3. Their preference on $\{1\}$ *is separable. The same is true for* $\{3\}$.

Now let us check separability on $\{2\}$.

Outcome on $\{1,3\}$	*Preference order on* $\{2\}$
$1-1$	$-1-\succ -0-$
$1-0$	$-1-\succ -0-$
$0-1$	$-0-\succ -1-$
$0-0$	$-0-\succ -1-$

Looking at this table, we see that voters with this preference vector would want proposal 2 to pass if and only if proposal 1 also passes. Their preference on $\{2\}$ *is nonseparable.*

Let us look at the larger sets. First, for the $\{1,2\}$ *we have*

Outcome on $\{3\}$	*Preference order on* $\{1,2\}$
$--1$	$11-\succ 10-\succ 00-\succ 01-$
$--0$	$11-\succ 10-\succ 00-\succ 01-$

The outcome on proposal 3 does not influence the preference order on $\{1,2\}$.

Now consider the set $\{2,3\}$. *We have*

Outcome on $\{1\}$	*Preference order on* $\{2,3\}$
$1--$	$-11\succ -01\succ -10\succ -00$
$0--$	$-01\succ -11\succ -00\succ -10$

The preference order on $\{2,3\}$ *depends on the outcome on* $\{1\}$*, so the set* $\{2,3\}$ *is nonseparable. A similar result would be found for* $\{1,3\}$.

In conclusion, the set of separable sets for this $\vec{p}$*'s underlying preference order is*

$$\text{char}(\succeq_{\vec{p}}) = \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{1,2,3\}\}.$$

It is easy to see that for any given preference order we can always determine its character. But when given some character χ, are we always able to find a preference order where char($\succeq$)? This question was first posed by Hodge and TerHaar. They defined this quality as *admissibility* [11].

Definition 2.3.7. *For a referendum on the set of proposals* $[n]$*, the character* χ *is called* ***admissible*** *if and only if there exists some preference order,* $\succeq_{[n]}$*, on the outcomes with* $\text{char}(\succeq_{[n]}) = \chi$.

Hodge and HerHaar [11] prove that closure under intersections is a necessary condition for admissibility. Additionally, we note that the presence of $[n]$ and the $\emptyset$ are necessary conditions for admissibility. They show that these conditions are sufficient for $n \leq 3$, but not for larger n. When $n = 4$ there is exactly one character satisfying these conditions which is inadmissible:

$$\{\emptyset, \{1,2\}, \{2\}, \{2,3\}, \{3\}, \{3,4\}, \{1,2,3,4\}\}$$

Much of Hodge's work has focused on admissibility and building preference orders with specific characters. We add to this work by finding a basis for our preferences spaces that shows promise in allowing us to produce preference orders for given characters.

Chapter 3

Finding a Basis

In the last chapter, we defined the preference space P^n. In this chapter, we develop a vector encoding for binary preference orderings on referendum outcomes. This vector encoding will allow us to apply techniques from linear algebra and representation theory to construct preference orderings. The key will be to identify the symmetry group for referendum outcomes: the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$. This group preserves the interdependence structure of binary preference orderings on referendum outcomes.

In his honors thesis, Stephen Lee follows the algebraic voting theory approach of his adviser, Michael Orrison, in illuminating the problem of voting for committees [14]. He describes a specific case of voting for committees, but does not discuss the role of separability in voting for committees. He constructs a profile space, which structurally is similar to our preference space P^n. Lee creates and decomposes modules using his profile space and the hyperoctahedral group. We will follow and expand upon Lee's approach. Appendix A contains a proof of a conjecture posed in Lee's thesis (which was recently proven by Tom Halverson [7]). We connect Halveron's new representation theory result to the separability of preference orders.

3.1 The Hyperoctahedral Group $\mathbb{Z}_2 \wr S_n$

The hyperoctahedral group, $\mathbb{Z}_2 \wr S_n$, is the wreath product between the cyclic group $\mathbb{Z}_2$ and the symmetric group, S_n. It is best recognized as the group of symmetries of the n-dimensional hypercube. Elements of the group are commonly represented as signed permutations. We will define an action of $\pi \in \mathbb{Z}_2 \wr S_n$ on an outcome of an n proposal referendum. This allows us to define the action of π on $v \in P^n$.

3.1.1 Action of $\mathbb{Z}_2 \wr S_n$ on Referendum Outcomes

Let $\pi \in \mathbb{Z}_2 \wr S_n$. Similar to a (regular) permutation, we can represent the signed permutation π using two-line notation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$$

where $\pi_i \in \{\pm 1, \pm 2, \ldots, \pm n\}$. We can also write a signed permutation using cycle notation

$$(\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,k_1})(\alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,k_2}) \cdots (\alpha_{\ell,1}, \alpha_{\ell,2}, \ldots, \alpha_{\ell,k_\ell})$$

where $k_1 + k_2 + \cdots + k_\ell = n$ and each $\alpha_{i,j} \in \pi_i \in \{\pm 1, \pm 2, \ldots, \pm n\}$.

Element $i \in [n]$ corresponds to the ith proposal. The hyperoctahedral group has a natural action on an outcome. The signed permutation π permutes the indices of the proposal according to the action of the symmetric group. A negative sign changes a "yes" vote to a "no" vote and a "no" vote to a "yes vote".

Example 3.1.1 (Action on an Outcome, $\pi \cdot x$)**.** *Let x be an outcome for a referendum with 5 proposals represented by the binary string 10010 and $\pi = (+1,\ -2,\ -3)(+4,\ +5) \in \mathbb{Z}_2 \wr S_5$. To determine $\pi \cdot x$, we first "flip" the digits in the binary string according the signs of π. This yields the intermediate outcome* 11110*. Then we permute the order of the digits according to the permutation* $(1,\ 2,\ 3)(4,\ 5)$*, which yields 11101. Therefore,*

$$\pi \cdot x = (+1,\ -2,\ -3)(+4,\ +5) \cdot 10010 = 11101.$$

We can extend this action to vectors in P^n, as follows

$$\pi \cdot \vec{p} = \pi \cdot \begin{pmatrix} [1 \cdots 11]_{\vec{p}} \\ [1 \cdots 10]_{\vec{p}} \\ \cdot \\ \cdot \\ \cdot \\ [0 \cdots 01]_{\vec{p}} \\ [0 \cdots 00]_{\vec{p}} \end{pmatrix} = \begin{pmatrix} [\pi^{-1} \cdot (1 \cdots 11)]_{\vec{p}} \\ [\pi^{-1} \cdot (1 \cdots 10)]_{\vec{p}} \\ \cdot \\ \cdot \\ \cdot \\ [\pi^{-1} \cdot (0 \cdots 01)]_{\vec{p}} \\ [\pi^{-1} \cdot (0 \cdots 00)]_{\vec{p}} \end{pmatrix}.$$

This action can be also extended to binary preference orders corresponding to referendum outcomes. Consider a binary preference order over all the possible outcomes for a referendum $[n]$, $\succeq_P = x^{(1)} \succeq x^{(2)} \cdots \succeq x^{(2^n)}$, where $\{x^{(1)}, x^{(2)}, ..., x^{(2^n)}\}$ is the set of outcomes of an n proposal referendum. The action of π on $\succeq_P$ is as follows:

$$\pi \cdot \succeq_P = \pi \cdot (x^{(1)} \succeq x^{(2)} \succeq \cdots \succeq x^{(2^n)}) = \pi \cdot x^{(1)} \succeq \pi \cdot x^{(2)} \succeq \cdots \succeq \pi \cdot x^{(2^n)}.$$

Note that in a defined preference order we use only strict preference, $\succ$, or indifference, $\sim$, in a preference order. Here we must use $\succeq$ because we do not know whether a given relationship is strict preference or indifference.

Lemma 3.1.2. *Let $\succeq_P$ be a binary preference order corresponding to a referendum election and x and y be two outcomes. Let $\pi \in \mathbb{Z}_2 \wr S_n$. If $x \succeq_P y$, then $\pi \cdot x \succeq_P \pi \cdot y$ for $\pi \cdot \succeq_P$.*

Proof. The result follows immediately from the action of the group.

$$\succeq_P = \cdots \succeq x \succeq \cdots \succeq y \cdots$$
$$\pi \cdot \succeq_P = \cdots \succeq \pi \cdot x \succeq \cdots \succeq \pi \cdot y \cdots$$

□

The action of this group corresponds to two intuitive alterations of a referendum. The first comes from the $\mathbb{Z}_2$ aspect of this group, or the sign aspect of the hyperoctahedral group. For proposal i in a referendum, we could negate the statement of the proposal. If this was done for every outcome where the voter originally preferred the proposal to pass, they would now rather it fail, and similarly, if they originally preferred the proposal to fail, they would now want it to pass. Clearly this single proposal negation does not change any interdependencies between proposals.

The second action comes from the symmetric group S_n, which permutes the order of the proposals. This reordering does not change how an individual voter ranks the outcomes of the collections of proposal (though it does change the indices of the proposals themselves).

We state this invariance of interdependence more formally in the following theorem.

Theorem 3.1.3. *Let $\succeq_P$ be a binary preference order corresponding to a referendum election and $\pi \in \mathbb{Z}_2 \wr S_n$. The characters* $\mathrm{char}(\succeq_P)$ *and* $\mathrm{char}(\pi \cdot \succeq_P)$ *are isomorphic up to the permutation $|\pi|$.*

Proof. We will show this result in two pieces. First, we will show that if a set S is separable on $\succeq_P$, then on $\pi \cdot \succeq_P$ the set W, where W is the collection of proposals mapped to from S by $|\pi|$, is separable. Then we will show that if a set S is non-separable on $\succeq_P$, then on $\pi \cdot \succeq_P$ the set W is separable.

Case 1: S is separable on $\succeq_P$

Because S is separable we know that given

$$(x_S u_{[n]-S}) \succeq (y_S u_{[n]-S})$$

for some $u_{[n]-S} \in X_{[n]-S}$. It must be true that

$$(x_S v_{[n]-S}) \succeq (y_S v_{[n]-S})$$

for all $v_{[n]-S} \in X_{[n]-S}$.

By Lemma 3.1.2, this implies that for $\pi \cdot \succeq_P$

$$\pi \cdot (x_S v_{[n]-S}) \succeq \pi \cdot (y_S v_{[n]-S})$$

for all $v_{[n]-S} \in X_{[n]-S}$. Let $\pi \cdot (x_S v_{[n]-S}) = (\alpha_W v_{[n]-W})$ and $\pi \cdot (y_S v_{[n]-S}) = (\beta_W v_{[n]-W})$ for all $v_{[n]-S} \in X_{[n]-S}, v_{[n]-W} \in X_{[n]-W}$. We then see the separability of W more clearly.

Given

$$\pi \cdot (x_S u_{[n]-S}) = (\alpha_W \omega_{[n]-W}) \succeq (\beta_W \omega_{[n]-W}) = \pi \cdot (y_S u_{[n]-S})$$

for some $\omega_{[n]-W} \in X_{[n]-W}$

$$(\alpha_W v_{[n]-W}) \succeq (\beta_W v_{[n]-W})$$

for all $v_{[n]-W} \in X_{[n]-W}$. Therefore, W is separable on $\pi \succeq_P$.

Case 2: S is nonseparable on $\succeq_P$

Because S is nonseparable, we know there must be two outcomes on $[n] - S$ and two outcomes on S such that

$$(x_S u_{[n]-S}) \succeq (y_S u_{[n]-S})$$

and

$$(x_S v_{[n]-S}) \prec (y_S v_{[n]-S}).$$

Using Lemma 3.1.2 yields

$$\pi \cdot (x_S u_{[n]-S}) = (\alpha_W \omega_{[n]-W}) \succeq (\beta_W \omega_{[n]-W}) = \pi \cdot (y_S u_{[n]-S})$$

and

$$\pi \cdot (x_S u_{[n]-S}) = (\alpha_W v_{[n]-W}) \prec (\beta_W v_{[n]-W}) = \pi \cdot (y_S u_{[n]-S})$$

These two cases are sufficient for proving that the characters char($\succeq_P$) and char($\pi \cdot \succeq_P$) are isomorphic up to a permutation. $\square$

This result carries through with preference vectors.

Corollary 3.1.4. *Let v be a preference vector in P^n and $\pi \in \mathbb{Z}_2 \wr S_n$. The characters of the underlying preference orderings for v and $\pi \cdot v$ are isomorphic up to a permutation, the S_n component of π.*

Proof. The result follows from Theorem 3.1.3. $\square$

While these results seem obvious from our intuition, they are immensely important as they demonstrate that the action of the hyperoctahedral group preserves the interdependent structure of a preference ordering corresponding to a referendum election.

3.2 A Tale of Two Bases

In Appendix A, it is shown that with the field $\mathbb{C}$, the action of $\mathbb{Z}_2 \wr S_n$ on our preferences vectors makes our preference spaces $\mathbb{C}\mathbb{Z}_2 \wr S_n$-modules. These modules capture the symmetries implied by our rational preference assumptions. That is, the interdependent structure of the preference order implied by a preference vector is invariant under the group action. In this section we will show that the basis vectors implied by the submodule decomposition in Appendix A are significant in the separability of their underlying preference orders. With minimal manipulation, we transform this basis into a new basis, the vectors of which serve as building blocks for building preference vectors with specific characters.

3.2.1 The Irreducible Basis

The irreducible submodule decomposition of our $\mathbb{C}\mathbb{Z}_2 \wr S_n$-module, P^n, in Appendix A provides us with the basis $\{\mathsf{u}_T \mid T \subseteq [n]\}$ for the space. Let us denote the basis vector indexed by set T as u_T.

Recall that the entries of vectors in P^n are indexed by outcomes on $[n]$. We describe the structure of u_T in terms of the subsets and outcomes of $[n]$. There is a natural bijection between subsets of $[n]$ and outcomes on $[n]$. We could index both the basis vectors and the entries of vectors by either subsets or outcomes. Our choice here reflects our desire to ease composition in later proofs.

Irreducible Basis Vector Rule: The entry of u_T indexed by the outcome x is $(-1)^{|\widehat{x} \cup T|}$.

Example 3.2.1 (The Irreducible Basis for P^3)**.** *The following table shows the basis vectors* u_T *for the preference space* P^3 *from Appendix A.*

	$\mathsf{u}_{\{1,2,3\}}$	$\mathsf{u}_{\{1,2\}}$	$\mathsf{u}_{\{1,3\}}$	$\mathsf{u}_{\{1\}}$	$\mathsf{u}_{\{2,3\}}$	$\mathsf{u}_{\{2\}}$	$\mathsf{u}_{\{3\}}$	$\mathsf{u}_{\emptyset}$
111	1	1	1	1	1	1	1	1
110	1	-1	1	-1	1	-1	1	-1
101	1	1	-1	-1	1	1	-1	-1
100	1	-1	-1	1	1	-1	-1	1
011	1	1	1	1	-1	-1	-1	-1
010	1	-1	1	-1	-1	1	-1	1
001	1	1	-1	-1	-1	-1	1	1
000	1	-1	-1	1	-1	1	1	-1

This basis has some important qualities. First, as we have already pointed out, the basis vectors are indexed by subsets of $[n]$. As we move forward in trying to construct preference orderings with specific separability structures, or characters, this indexing will be essential. Second, basis vectors indexed by sets that are the same order span a submodule of P^n. This is a direct result from Appendix A. That is, for the module P^n, the basis vectors indexed by sets of order k span an $\binom{n}{k}$ dimensional submodule. While we do not use this result directly in the rest of the this paper, we recognize it could be useful for future analysis. Finally, we can also completely characterize the separability of the preference orderings underlying the basis vectors. Before we do this, however, we are going to make a slight alteration to the basis to make it more manageable.

3.2.2 The Voter Basis

In this section, we define another basis $\{\mathsf{v}_T \mid T \in 2^{[n]}\}$ for the preference space P^n that we call the *voter basis*.

Irreducible Basis Vector Rule: In the **voter basis** the entry of v_A indexed by the outcome x is 1 if $\widehat{x} \cap T$ is even and 0 if it is odd.

We have the following bijection between the irreducible basis and the voting basis[1]: If T^c is odd.

$$\mathsf{v}_T = \frac{\mathsf{u}_{T^c} + \vec{1}}{2}$$

If T^c is even

$$\mathsf{v}_T = \frac{-\mathsf{u}_{T^c} + \vec{1}}{2}$$

Below, we see an example of this map.

[1]The vector $\vec{1}$ is the all ones vector.

Example 3.2.2 (Voter Basis from Irreducible Basis)**.**

$$\mathsf{v}_{100} = \frac{-\mathsf{u}_{011} + \vec{1}}{2} = -\frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{array}{c} 111 \\ 110 \\ 101 \\ 100 \\ 011 \\ 010 \\ 001 \\ 000. \end{array}$$

The structural similarities between both bases can be observed by comparing the following example with Example 3.2.1.

Example 3.2.3 (The Voter Basis for P^3)**.** *Here are the eight vectors that make up the voter basis for P^3 with basis vectors indexed by subset of* $\{1, 2, 3\}$.

	$\mathsf{v}_{\{1,2,3\}}$	$\mathsf{v}_{\{1,2\}}$	$\mathsf{v}_{\{1,3\}}$	$\mathsf{v}_{\{1\}}$	$\mathsf{v}_{\{2,3\}}$	$\mathsf{v}_{\{2\}}$	$\mathsf{v}_{\{3\}}$	$\mathsf{v}_{\emptyset}$
111	0	1	1	0	1	0	0	1
110	1	1	0	0	0	0	1	1
101	1	0	1	0	0	1	0	1
100	0	0	0	0	1	1	1	1
011	1	0	0	1	1	0	0	1
010	0	0	1	1	0	0	1	1
001	0	1	0	1	0	1	0	1
000	1	1	1	1	1	1	1	1

While we typically use the subset labeling scheme for voter basis vectors, we could also index the basis vectors using outcomes just as we do for the entries. This is shown below, which reveals the source of some of the symmetries.

	v_{111}	v_{110}	v_{101}	v_{100}	v_{011}	v_{010}	v_{001}	v_{000}
111	0	1	1	0	1	0	0	1
110	1	1	0	0	0	0	1	1
101	1	0	1	0	0	1	0	1
100	0	0	0	0	1	1	1	1
011	1	0	0	1	1	0	0	1
010	0	0	1	1	0	0	1	1
001	0	1	0	1	0	1	0	1
000	1	1	1	1	1	1	1	1

The voter basis has the same symmetries as the irreducible basis, while having many zero entries. We exploit this sparsity in the following chapter.

A more subtle difference between the irreducible basis and the voter basis is the switch from defining the basis vector rule in terms of *unions* of sets to *intersections* of sets. Using intersections rather than unions gives consistency in the behavior of the basis for P^n, regardless of the parity of n. It turns out that the "largest union" of sets is important in the irreducible basis, while the "smallest intersection" of sets is important in the voter basis. The largest union is $[n]$ while the the smallest intersection is $\emptyset$. Clearly, the parity of the former changes with n, while the parity of the latter is always even.

This voter basis is our foundation for the construction and analysis of preference vectors in P^n. Before we demonstrate its significance we add a small bit of terminology.

Definition 3.2.4. *We say that* **"the outcome x benefits from v_A"** *when x corresponds to a nonzero entry of v_A. We also say* **"the outcome x is even"** *if its binary representation has an even number of 1's and* **"the outcome x is odd"** *if its binary representation has an even number of 0's.*

In Example 3.2.2, only the outcomes 011, 010, 001, and 000 benefit from $\mathsf{v}_{\{1\}}$. This is because $\widehat{011} \cap \{1\}$, $\widehat{010} \cap \{1\}$, $\widehat{001} \cap \{1\}$, and $\widehat{000} \cap \{1\}$ are all even sets.

We are now prepared to discuss the separability of voter basis vectors.

Theorem 3.2.5. *The voter basis vector v_A is separable on S if and only if $A \subseteq S$ or $A \cap S = \emptyset$. That is,* $\mathrm{char}(\mathsf{v}_A) = \{X | A \subseteq X \text{ or } A \cap X = \emptyset\}$.

Proof. This proof is organized according to five types of sets:

1. S where $A \subset S$ is separable on v_A.
2. S where $S = A$ is separable on v_A.
3. S where $A \cap S = \emptyset$ is separable on v_A.
4. S where $\emptyset \subset S \subset A$ is nonseparable on v_A.
5. S where $S \cap A \neq \emptyset$ and $A - S \neq \emptyset$ is nonseparable on v_A.

Type 1: v_A is separable on S where $A \subset S$.

Let x and y be outcomes, represented as bitstrings of length n. Consider the entries $[x]_{\mathsf{v}_A}$ and $[y]_{\mathsf{v}_A}$ of v_A, which we decompose as

$$[x]_{\mathsf{v}_A} = [x_S u_{[n]-S}]_{\mathsf{v}_A} = [x_A x_{S-A} u_{[n]-S}]_{\mathsf{v}_A},$$
$$[y]_{\mathsf{v}_A} = [y_S u_{[n]-S}]_{\mathsf{v}_A} = [y_A y_{S-A} u_{[n]-S}]_{\mathsf{v}_A}.$$

We want to determine the preference relation between these outcomes and show that it is independent of the choice of outcomes on the shared binary digits $u_{[n]-S}$.

Case 1: x_A and y_A are the same parity
If x_A and y_A are the same parity, then both or neither the outcomes benefit from v_A. This guarantees that

$$[x_S u_{[n]-S}]_{\mathsf{v}_A} = [y_S u_{[n]-S}]_{\mathsf{v}_A}$$

for all $u_{[n]-S} \in X_{[n]-S}$.

Case 2: x_A and y_A are not the same parity
Assume without loss of generality that x_A is even y_A is odd. This implies that $x_S u_{[n]-S}$ benefits from v_A while $y_S u_{[n]-S}$ does not. This guarantees that

$$[x_S u_{[n]-S}]_{\mathsf{v}_A} > [y_S u_{[n]-S}]_{\mathsf{v}_A}$$

for all $u_{[n]-S} \in X_{[n]-S}$.

These cases together demonstrate that the preference between these outcomes depends only on the relative parities of x_A and y_A and not on $u_{[n]-S}$. Therefore, S where $A \subset S$ is separable on v_A.

Type 2: S where $S = A$ is separable on v_A.
This is a special case of the previous case where $S = A$, that is, where $S - A = \emptyset$. Thus, S is separable on v_A .

Type 3: S where $A \cap S = \emptyset$ is separable on v_A.
Consider the entries $[x]_{\mathsf{v}_A}$ and $[y]_{\mathsf{v}_A}$ of v_A, which we decompose as

$$[x]_{\mathsf{v}_A} = [x_S u_A u_{[n]-S-A}]_{\mathsf{v}_A},$$
$$[y]_{\mathsf{v}_A} = [y_S u_A u_{[n]-S-A}]_{\mathsf{v}_A}.$$

The outcomes are identical on A. This implies they both benefit from v_A or neither of them do. Therefore,

$$[y_S u_A u_{[n]-S-A}]_{\mathsf{v}_A} = [y_S u_A u_{[n]-S-A}]_{\mathsf{v}_A}$$

for all $u_{[n]-S} \in X_{[n]-S}$ which means that S is trivially separable on v_A.

Type 4: S where $\emptyset \subset S \subset A$ is nonseparable on v_A
Consider the entries $[x]_{\mathsf{v}_A}$, $[y]_{\mathsf{v}_A}$ of v_A, which we decompose as

$$[x]_{\mathsf{v}_A} = [x_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A},$$
$$[y]_{\mathsf{v}_A} = [y_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A}.$$

We focus on outcomes where x_S and y_S are different parities. Assume without loss of generality that x_S is even and y_S is odd.

Case 1: u_{A-S} is even. If u_{A-S} is even, then $x_S u_{A-S}$ is even and $y_S u_{A-S}$ is odd. Therefore, $x_S u_{A-S} u_{[n]-A}$ benefits from v_A and $y_S u_{A-S} u_{[n]-A}$ does not, guaranteeing that

$$[x_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A} > [y_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A}$$

Case 2: u_{A-S} is odd. If u_{A-S} is odd, then $x_S u_{A-S}$ is odd and $y_S u_{A-S}$ is even. Therefore, $x_S u_{A-S} u_{[n]-A}$ does not benefit from v_A and $y_S u_{A-S} u_{[n]-A}$ does, guaranteeing that

$$[x_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A} < [y_S u_{A-S} u_{[n]-A}]_{\mathsf{v}_A}$$

Type 5: S where $S \cap A \neq \emptyset$ and $A - S \neq \emptyset$ is nonseparable on v_A. Consider the entries $[x]_{\mathsf{v}_A}$, $[y]_{\mathsf{v}_A}$ of v_A, which we decompose as

$$\begin{aligned} [x]_{\mathsf{v}_A} &= [x_{S-A} x_{A\cap S} u_{A-S} u_{[n]-A-S}]_{\mathsf{v}_A}, \\ [y]_{\mathsf{v}_A} &= [y_{S-A} y_{A\cap S} u_{A-S} u_{[n]-A-S}]_{\mathsf{v}_A}. \end{aligned}$$

We focus on outcomes where $x_{A\cap S}$ and $y_{A\cap S}$ are different parities. Assume without loss of generality that $x_{A\cap S}$ is even and $y_{A\cap S}$ is odd.

Case 1: u_{A-S} is even. If u_{A-S} is even, then $x_{A\cap S} u_{A-S}$ is even and $y_{A\cap S} u_{A-S}$ is odd. Therefore, $x_{S-A} x_{A\cap S} u_{A-S} u_{[n]-A-S}$ benefits from v_A and $y_{S-A} y_{A\cap S} u_{A-S} u_{[n]-A-S}$ does not, guaranteeing that

$$[x]_{\mathsf{v}_A} > [y]_{\mathsf{v}_A}$$

Case 2: u_{A-S} is odd. If u_{A-S} is odd, then $x_{A\cap S} u_{A-S}$ is odd and $y_{A\cap S} u_{A-S}$ is even. Therefore, $x_{S-A} x_{A\cap S} u_{A-S} u_{[n]-A-S}$ does not benefit from v_A and $y_{S-A} y_{A\cap S} u_{A-S} u_{[n]-A-S}$ does benefit from v_A, guaranteeing that

$$[x]_{\mathsf{v}_A} < [y]_{\mathsf{v}_A}$$

Together these cases demonstrate that the preference between these outcomes depends on u_{A-S}, which means that S where $\emptyset \subset S \subset A$ is nonseparable on v_A. $\square$

Theorem 5 justifies our choice in basis as well as overall our approach. It is important to fully appreciate the manner in which these basis vectors capture separability with respect to their indexing sets.

Example 3.2.6 (Character of the voting basis vector $\mathsf{v}_{\{1,2\}} \in P^4$.). *We consider the basis element $\mathsf{v}_{\{1,2\}} = \mathsf{v}_{1100}$ in P^4, its underlying preference ordering, and corresponding character.*

$$\mathsf{v}_{\{1,2\}} = \mathsf{v}_{1100} = \begin{array}{c} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \end{array} \begin{array}{c} 1111 \\ 1110 \\ 1101 \\ 1100 \\ 1011 \\ 1010 \\ 1001 \\ 1000 \\ 0111 \\ 0110 \\ 0101 \\ 0100 \\ 0011 \\ 0010 \\ 0001 \\ 0000. \end{array}$$

The underlying preference order of $\mathsf{v}_{\{1,2\}} = \mathsf{v}_{1100}$ *is*

$$\begin{aligned} &1111 \sim 1110 \sim 1101 \sim 1100 \sim 0011 \sim 0010 \sim 0001 \sim 0000 \\ &\quad \succ 1011 \sim 1010 \sim 1001 \sim 1000 \sim 0111 \sim 0110 \sim 0101 \sim 0100) \end{aligned}$$

and one can check that its character is

$$\mathrm{char}(\mathsf{v}_{1100}) = \{\{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}\}.$$

Reflecting on the previous example and Theorem 3.2.5, we recognize the potential for these basis vectors to serve as discrete bricks for building specific types of separability.

3.2.3 Voter Basis Symmetries

Reflecting on Example 3.2.3, we observe that vectors in our voter basis have some fascinating symmetries. Interestingly, the symmetry that a basis vector possesses depends on the parity of the indexing set. The preference vector symmetry associated with basis vectors indexed by odd sets is a *folding symmetry*[2]. The preference vector symmetry associated with basis vectors indexed by even sets is a *reflecting symmetry*.

Before we define these symmetries, we define the **reversing function**. The reversing function takes in a preference vector and outputs a new preference vector where the values of entries indexed by bitwise complements are swapped.

[2] A preference vector having folding symmetry is analogous to a binary preference matrix being bitwise symmetric.

Definition 3.2.7. *The* ***reversing function*** $\{\mathrm{rev} : P^n \to P^n\}$ *for* P^n *is the function given by applying the element* $\sigma = (-1)(-2)\cdots(-n) \in \mathbb{Z}_2 \wr S_n$ *to a preference vector. In other words,*

$$\mathrm{rev}(\vec{p}) = (-1)(-2)\cdots(-n) \cdot \vec{p}.$$

In terms of a referendum election, the effect of the reversal function on a preference vector is to negate every question in the referendum.[3] With this function in hand, we can now describe the folding and reflecting symmetry.

Definition 3.2.8. *A preference vector* $\vec{p}$ *has* ***folding symmetry*** *if*

$$\vec{p} + \mathrm{rev}(\vec{p}) = a\mathsf{v}_\emptyset = a\vec{1}$$

for some $a \in \mathbb{C}$.

Equivalently, a preference vector $\vec{p}$ has folding symmetry if there exists some $a \in \mathbb{C}$ such that for any outcome x, $[x]_{\vec{p}} + [x^c]_{\vec{p}} = a$ where x^c denotes the bitwise complement of x.

Definition 3.2.9. *A preference vector* $\vec{p}$ *has* ***reflecting symmetry*** *if*

$$\vec{p} = \mathrm{rev}(\vec{p}).$$

Equivalently, a preference vector $\vec{p}$ has reflecting symmetry if for any outcome x, $[x]_{\vec{p}} = [x^c]_{\vec{p}}$ where x^c denotes the bitwise complement of x.

The following proposition describes part of the relationship between folding and reflecting symmetry.

Proposition 3.2.10. *A preference vector* $\vec{p}$ *has both folding and reflecting symmetry if and only if there is some* $c \in \mathbb{C}$ *for which* $c\vec{p} = \mathsf{v}_\emptyset = \vec{1}$.

Proof. Let $\vec{p}$ be a vector with both folding and reflecting symmetry. This implies

$$\vec{p} = \mathrm{rev}(\vec{p})$$

and

$$\vec{p} + \mathrm{rev}(\vec{p}) = c\vec{1}$$

[3] Informally, this function simply flips a vector, i.e. the value of the first entry becomes the value of the last, while the value of the last entry becomes the value of the first.

for some $c \in \mathbb{C}$. We combine these to conclude that

$$\begin{aligned}\vec{p} + \text{rev}(\vec{p}) &= c\vec{1} \\ \vec{p} + \vec{p} &= c\vec{1} \\ 2\vec{p} &= c\vec{1} \\ \frac{2}{c}\vec{p} &= \vec{1}.\end{aligned}$$

□

The next two propositions tie these symmetries back in with our voter basis.

Proposition 3.2.11. *When T is an odd sized set, the vector v_T has folding symmetry.*

Proof. Consider v_T where $T \subset [n]$ is odd. Consider some outcome arbitrary outcome x such that $\widehat{x} \subset [n]$. If $|\widehat{x} \cap T|$ is even, then $|\widehat{x}^c \cap T|$ is odd because $|T|$ is odd. Similarly, if $|\widehat{x} \cap T|$ is odd, then $|\widehat{x}^c \cap T|$ is even. This means that for all x, only one of the outcomes x and x^c will benefit from v_T. Thus, $[x]_{\vec{p}} + [x^c]_{\vec{p}} = 1$ for all x. Therefore, v_T has folding symmetry.

□

Proposition 3.2.12. *When T is a set of even size, the vector v_T has reflecting symmetry.*

Proof. Consider v_T where $T \subset [n]$ is even. Consider an outcome x such that $\widehat{x} \subset [n]$. If $|\widehat{x} \cap T|$ is even, then $|\widehat{x}^c \cap T|$ is even because $|T|$ is even. Similarly, if $|\widehat{x} \cap T|$ is odd, then $|\widehat{x}^c \cap T|$ is odd. This means that for all outcomes x, $[x]_{\mathsf{v}_T} = [x^c]_{\mathsf{v}_T}$. Therefore, v_T has reflecting symmetry. □

The next three lemmas describe the effect of combining vectors with these symmetries.

Lemma 3.2.13. *Let $\vec{p}$ and $\vec{q}$ be preference vectors with folding symmetry. The sum $\vec{p} + \vec{q}$ has folding symmetry.*

Proof. Let $\vec{p}$ and $\vec{q}$ be preference vectors with folding symmetry. We have

$$\begin{aligned}\vec{p} + \vec{q} &= a_p\vec{1} - \text{rev}(\vec{p}) + a_q\vec{1} - \text{rev}(\vec{q}) \\ \vec{p} + \vec{q} + \text{rev}(\vec{p}) + \text{rev}(\vec{q}) &= (a_p + a_q)\vec{1} \\ (\vec{p} + \vec{q}) + (\text{rev}(\vec{p}) + \text{rev}(\vec{q})) &= (a_p + a_q)\vec{1}.\end{aligned}$$

Therefore, the sum $(\vec{p} + \vec{q})$ has folding symmetry. □

Lemma 3.2.14. *Let $\vec{p}$ and $\vec{q}$ be preference vectors with reflecting symmetry. The sum $\vec{p} + \vec{q}$ has reflecting symmetry.*

Proof. Let $\vec{p}$ and $\vec{q}$ be preference vectors with reflecting symmetry. For any outcome x, we have

$$[x]_{\vec{p}} = [x^c]_{\vec{p}} \quad \text{and} \quad [x]_{\vec{q}} = [x^c]_{\vec{q}}.$$

This implies

$$[x]_{(\vec{p}+\vec{q})} = [x^c]_{(\vec{p}+\vec{q})}.$$

Therefore, the sum $\vec{p} + \vec{q}$ has reflecting symmetry. □

Lemma 3.2.15. *Let $\vec{p}$ be a preference vector with folding symmetry and $\vec{q}$ be a preference vector without folding symmetry. The sum $\vec{p} + \vec{q}$ does not have folding symmetry.*

Proof. Let $\vec{p}$ be a preference vector with folding symmetry and $\vec{q}$ be a preference vector without folding symmetry. Now AFTOC $\vec{p} + \vec{q}$ has folding symmetry. We have

$$\begin{aligned}
(\vec{p} + \vec{q}) + \operatorname{rev}(\vec{p} + \vec{q}) &= a\vec{1} \\
\vec{p} + \vec{q} + \operatorname{rev}(\vec{p}) + \operatorname{rev}(\vec{q}) &= a\vec{1} \\
(\vec{p} + \operatorname{rev}(\vec{p})) + \vec{q} + \operatorname{rev}(\vec{q}) &= a\vec{1} \\
a_p\vec{1} + \vec{q} + \operatorname{rev}(\vec{q}) &= a\vec{1} \\
\vec{q} + \operatorname{rev}(\vec{q}) = (a - a_p)\vec{1}.
\end{aligned}$$

This shows that $\vec{q}$ has folding symmetry, but that contradicts one of our original conditions so our assumption, that $\vec{p} + \vec{q}$ has folding symmetry, is incorrect. Thus, $v + w$ does not have folding symmetry. □

The previous lemmas allow us to make the following conclusion.

Theorem 3.2.16. *When a preference vector with folding symmetry is written as a linear combination of voter basis vectors, vectors corresponding to nonempty subsets of even size have a 0 coefficient.*

Proof. Consider a preference vector $\vec{p}$ with folding symmetry. We may write $\vec{p}$ as

$$\vec{p} = \vec{o} + \vec{e}$$

where $\vec{o}$ is the projection of $\vec{p}$ onto the space spanned by basis vectors corresponding to odd sized sets and $\vec{e}$ is the projection of $\vec{p}$ onto the space spanned by basis vectors corresponding to even sized sets. We know that $\vec{o}$ has folding symmetry because it is the sum of vectors with folding symmetry. Since $\vec{p}$ has folding symmetry and $\vec{o}$ has folding symmetry, it must be true that $\vec{e}$ also has folding symmetry. However, $\vec{e}$ is the sum of vectors with reflecting symmetry, none of which can have folding symmetry except $\mathsf{v}_\emptyset = \vec{1}$. Now assume that when $\vec{e}$ is written as linear combination of voter basis vectors corresponding to subsets of even size, there is at least one nonzero coefficient on a vector other than $\mathsf{v}_\emptyset = \vec{1}$. This means that $\vec{e}$ has reflecting symmetry and does not have folding symmetry. Thus, the sum $\vec{p} = \vec{o} + \vec{e}$ cannot have folding symmetry. This, however, contradicts $\vec{p}$ possessing folding symmetry, so our assumption is incorrect. Therefore, when $\vec{p}$ is written as a linear combination of voter basis vectors, vectors corresponding to nonempty subsets of even size must have a coefficient of 0.

□

A result from previous research is that every completely separable preference matrix is bitwise symmetric which is the preference matrix equivalent to folding symmetry for preference vectors [9].[4] This leads us to the following corollary.

Corollary 3.2.17. *When a completely separable preference vector with distinct entries (no indifference) is written as a linear combination of voter basis vectors, vectors corresponding to nonempty subsets of even size have a coefficient of 0.*

Proof. A completely separable preference vector with distinct entries has a binary preference matrix equivalent. All completely separable preference matrices must be bitwise symmetric. Bitwise symmetry is equivalent to folding symmetry, so all completely separable preference vectors must have folding symmetry. By Theorem 3.2.16, this guarantees that when a completely separable preference vector with distinct entries (no indifference) is written as a linear combination of voter basis vectors, vectors corresponding to nonempty subsets of even size have a coefficient of 0. □

In the case of $n = 4$, there are 14 distinct preference orders that are completely separable.[11] We transform these into 14 distinct ordinal preference vectors and scale them by 4 to avoid decimal coefficients. These coefficients are shown in Table 3.1. The patterns of zeros in these coefficients reveals the significance of Corollary 3.2.17. Currently, the number of completely separable preference orders for arbitrary n is not known, but we believe this result is a step towards solving this problem.

[4] Recall that preference matrices do not allow for indifference.

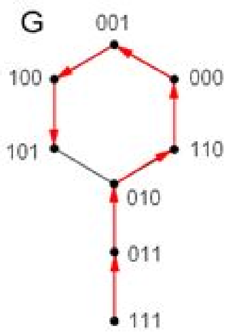
	$\vec{p_1}$	$\vec{p_2}$	$\vec{p_3}$	$\vec{p_4}$	$\vec{p_5}$	$\vec{p_6}$	$\vec{p_7}$	$\vec{p_8}$	$\vec{p_9}$	$\vec{p_{10}}$	$\vec{p_{11}}$	$\vec{p_{12}}$	$\vec{p_{13}}$	$\vec{p_{14}}$
$\mathsf{v}_{\emptyset}$	64	64	64	64	64	64	64	64	64	64	64	64	64	64
$\mathsf{v}_{\{1\}}$	-4	-1	-5	-4	-10	-7	-6	-3	-7	-6	-9	-12	1	-2
$\mathsf{v}_{\{2\}}$	-8	-9	-9	-12	-6	-3	-14	-11	-11	-10	-5	-8	-5	-8
$\mathsf{v}_{\{3\}}$	-16	-17	-17	-20	-20	-17	-18	-15	-15	-14	-15	-18	-15	-18
$\mathsf{v}_{\{4\}}$	-32	-31	-31	-28	-28	-31	-28	-31	-31	-32	-31	-28	-31	-28
$\mathsf{v}_{\{1,2\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{1,3\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{1,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{2,3\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{2,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{3,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathsf{v}_{\{1,2,3\}}$	0	3	-1	0	0	3	2	5	1	2	5	2	-5	-8
$\mathsf{v}_{\{1,2,4\}}$	0	-3	1	0	0	-3	0	-3	1	0	-3	0	7	10
$\mathsf{v}_{\{1,3,4\}}$	0	-3	1	0	6	3	0	-3	1	0	3	6	-7	-4
$\mathsf{v}_{\{2,3,4\}}$	0	1	1	4	-2	-5	4	1	1	0	-5	-2	-5	-2
$\mathsf{v}_{\{1,2,3,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Table 3.1: The coefficients with respect to the voting basis for the list of the 14 completely separable preferences when $n = 4$.

Chapter 4

Character Construction

For a referendum of n questions there are $2^n!$ preference orders without indifference, meaning that for even moderately large n, random generation is not an effective technique for finding a preference order for a given character. Adding to this difficulty is the fact that for a referendum $[n]$, the probability of a randomly selected preference order having the character $\chi = \{\emptyset, [n]\}$ approaches 1 as n increases [11]. Methods have been outlined for systematically creating preference orders for characters possessing certain properties. One such method is called a preseparable extension. Using this method, preference orders can be created for admissible characters which include a set and its compliment. Most recent methods have used hamiltonian paths on labeled graphs to generate orders on outcomes. These methods include exploring hypercube graphs and bead graphs[1, 2].[1]



$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Figure 4.1: G is a graph labeled with the binary outcomes of a three proposal referendum. The hamiltonian path shown in G generates the preference matrix P. From P, char(P)= $\{\emptyset, \{2\}, \{2,3\}, \{1,2,3\}\}$.

In this section we demonstrate the potential for our linear algebraic approach

[1]Interestingly, the symmetry group of the hypercube is the hyperoctahedral group, or $\mathbb{Z}_2 \wr S_n$.

to add to these techniques. Most notably we prove that any element of a class of character, called tree characters, can always be created by combining vectors from the voter basis systematically. We begin by discussing some general results about vectors in preference spaces. This is followed by defining a class of characters called tree characters and describing an algorithm for generating a preference vector for any given tree character. The final section proves our algorithm is correct.

4.1 Construction Techniques with Voter Basis

Here we prove some general results regarding the separability exhibited by preference vectors. Regardless of what class of characters we are trying to create preference vectors for, these results are essential.

Proposition 4.1.1 (Disjoint Set Proposition)**.** *Consider two entries of a preference vector $\vec{p}$ in P^n where the outcomes corresponding to these entries are the same on $([n]-S)$, namely*

$$(x_S u_{([n]-S)}) \text{ and } (y_S u_{([n]-S)})$$

Let $T \subset [n]$. Then the outcome $(x_S u_{([n]-S)})$ benefits from v_T, where $S \cap T = \emptyset$, if and only if $(y_S u_{([n]-S)})$ also benefits from v_T.

Proof. Assume $(x_S u_{([n]-S)})$ benefits from v_T. We have

$$\begin{aligned}
(x_S u_{([n]-S)}) &\text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow |\widehat{x_S u_{([n]-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{x_S u_T u_{[n]-S-T}} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{u_T} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{y_S u_T u_{[n]-S-T}} \cap T| \text{ is even.}\\
&\Leftrightarrow (y_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T.
\end{aligned}$$

□

This result highlights the following rule: if outcomes agree on some subset U, then vectors indexed by subsets of U will not contribute to preference differences between the two outcomes. The lemma above is proven with $U = [n] - S$.

Lemma 4.1.2. *Consider the entries of a preference vector $\vec{p}$ where the outcomes corresponding to these entries are the same on $([n]-S)$. That is, entries of the form*

$$(x_S u_{([n]-S)}) \text{ and } (y_S u_{([n]-S)}).$$

Suppose that $|\widehat{x_S} \cap S|$ *and* $|\widehat{y_S} \cap S|$ *are the same parity. If* $S \subset T$ *then the outcome* $(x_S u_{([n]-S)})$ *benefits from* v_T *if and only if* $(y_S u_{([n]-S)})$ *also benefits from* v_T.

Proof. **Case 1:** $|\widehat{x_S} \cap S|$ **and** $|\widehat{y_S} \cap S|$ **are even.**

Assume $(x_S u_{([n]-S)})$ benefits from v_T.

$$\begin{aligned}
&(x_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T\\
&\Leftrightarrow (x_S u_{(T-S)} u_{([n]-T)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow |\widehat{x_S u_{(T-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{x_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{y_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{y_S u_{(T-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow (y_S u_{(T-S)} u_{([n]-T)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow (y_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T.
\end{aligned}$$

Case 2: $|x_S \cap S|$ **and** $|y_S \cap S|$ **are odd.**

Assume $(x_S u_{([n]-S)})$ benefits from v_T. We have

$$\begin{aligned}
&(x_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T\\
&\Leftrightarrow (x_S u_{(T-S)} u_{([n]-T)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow |\widehat{x_S u_{(T-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{x_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is odd.}\\
&\Leftrightarrow |\widehat{y_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{y_S u_{(T-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow (y_S u_{(T-S)} u_{([n]-T)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow (y_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T.
\end{aligned}$$

□

Lemma 4.1.3. *Consider two entries of a preference vector* v *where the outcomes corresponding to these entries are the same on* $([n]-S)$*, namely*

$$(x_S u_{([n]-S)}) \text{ and } (y_S u_{([n]-S)}).$$

Suppose that $|x_S \cap S|$ *and* $|y_S \cap S|$ *are not the same parity. Then the outcome* $(x_S u_{([n]-S)})$ *receives the benefit of* v_T*, where* $S \subset T$*, if and only if* $(y_S u_{([n]-S)})$ *does not receive the benefit of* v_T*.*

Proof. Without loss of generality, assume $|x_S \cap S|$ is even and $|y_S \cap S|$ is odd. $(x_S u_{([n]-S)})$ benefits from v_T.

$$\begin{aligned}
&(x_S u_{([n]-S)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow (x_S u_{(T-S)} u_{([n]-T)}) \text{ benefits from } \mathsf{v}_T.\\
&\Leftrightarrow |\widehat{x_S u_{(T-S)}} \cap T| \text{ is even.}\\
&\Leftrightarrow |\widehat{x_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is even.}\\
&\Leftrightarrow |\widehat{y_S} \cap S| + |\widehat{u_{(T-S)}} \cap (T-S)| \text{ is odd.}\\
&\Leftrightarrow |\widehat{y_S u_{(T-S)}} \cap T| \text{ is odd.}\\
&\Leftrightarrow (y_S u_{(T-S)} u_{([n]-T)}) \text{ does not benefit from } \mathsf{v}_T.\\
&\Leftrightarrow (y_S u_{([n]-S)}) \text{ does not benefit from } \mathsf{v}_T.
\end{aligned}$$

□

The previous two lemmas can be combined to form the following proposition.

Proposition 4.1.4 (Superset Proposition)**.** *Consider two entries of a preference vector* v *where the outcomes corresponding to these entries are the same on* $([n]-S)$*, namely*

$$(x_S u_{([n]-S)}) \text{ and } (y_S u_{([n]-S)}).$$

If $S \subset T$ *then exactly one of the two outcomes benefits from* v_T*, where* $S \subset T$*, if and only if* $|x_S \cap S|$ *and* $|y_S \cap S|$ *are not the same parity*

Proof. The result follows immediately the previous two lemmas □

The Disjoint Set Proposition and Superset Proposition are reminiscent of the separability implied by a single basis vector. Recall that the basis vector $\vec{A}$ is trivially separable on sets disjoint from A and separable on supersets of A. These results will be used frequently in the following proofs.

4.2 Tree Characters

Given a collection of sets $\chi = \{\emptyset = A_0, A_1, A_2, \ldots, A_m = [n]\}$, we use set inclusion to obtain a natural partially ordered set (poset). That is, we have $A_i < A_j$ when $A_i \subset A_j$. We say that A_j **covers** A_i when there is no A_k such that $A_i < A_k < A_j$. The **Hasse diagram** of a poset is a graph that describes its structure. Each set in χ is represented by a vertex. We add an edge between A_i and A_j whenever A_j covers A_i. Furthermore, the vertex for set A_j is located above all of the sets that it covers. So $A_m = [n]$ is the highest vertex and $A_0 = \emptyset$ is the lowest vertex.

In this section, we will focus on characters χ such that the Hasse diagram of $\chi - \{\emptyset\}$ is a tree. We describe how to use our voting basis to create a preference vector for such a tree character χ.

Definition 4.2.1. *A* ***tree character*** χ *is a character such that the Hasse diagram of* $\chi - \{\emptyset\}$ *is a tree. In other words, for any A,B* $\in \chi$*, one of the following is true:*

$$A = B, A \subsetneq B, A \supsetneq B, \text{ or } A \cap B = \emptyset.$$

Example 4.2.2 (Examples and Nonexamples of Tree Characters.). *Here are some examples of tree characters:*

$$\{\emptyset, \{1\}, \{3,4\}, \{1,2,3,4\}, \{5\}, \{5,6\}, \{1,2,3,4,5,6\}\},$$
$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{5,6\}, \{1,2,3,4,5,6\}\},$$
$$\{\emptyset, \{1,2,3\}\}.$$

Here is a simple example of a set that is not a tree character:

$$\{\emptyset, \{1\}, \{1,2\}\{1,3\}, \{1,2,3\}\}.$$

We visualize these tree characters as rooted trees by omitting the trivially separable $\emptyset$. We visualize the first character from Example 4.2.2 in Figure 4.2.

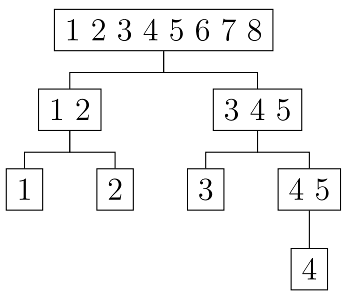


Figure 4.2: The Hasse diagram of a tree character (omitting the $\emptyset$).

As the name suggests, we think of a tree character as a tree structure that is rooted at $[n]$. We will take advantage of some terminology from rooted trees.

Definition 4.2.3. *Let χ be a tree character. If A_j covers A_i, then A_j is the* **parent** *of A_j and A_i is the* **child** *of A_j. The children of A_j are called* **siblings**. *We can also speak of the k***th generation** *of sets, which consists of all sets that are at distance k from the root. We also refer to* **ancestors** *and* **descendants** *in the natural way.*

As we construct our preference vector for χ, we will also need to keep track of the elements of $[n]$ that appear in generation k, but do not appear in generation $k+1$. For convenience, we collect these elements into naturally defined sets that we call *ghost children*. Figure 4.3 shows the Hasse diagram of the same character as Figure 4.2, but includes the ghost children as well. Note that when we include the ghost children, every element in the set $[n]$ appears in a leaf of the tree. We refer to a Hasse diagram that includes sets of ghost children as a *haunted Hasse diagram*.[2]

Definition 4.2.4. *Let $A_j \in \chi$ with children $A_{j_1}, \dots A_{j_k}$. If $\cup_i A_{j_i} \neq A_j$, then the* **ghost child** *of A_j is $A_j - (\cup_i A_{j_i})$. In other words, the ghost child of A_j is the relative complement of the union of its children. The Hasse diagram that also shows the ghost children is called a* **haunted Hasse diagram**.

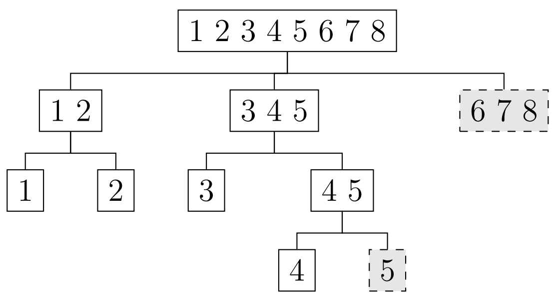


Figure 4.3: The haunted Hasse diagram for a tree character. The ghost sets are shown with grey background and dashed outlines.

The ghost children come out to play when we want to ensure that certain unions of siblings are not separable. For example, in Figure 4.3, the children of the set $[8]$ are $\{1,2\}$ and $\{3,4,5\}$. In order to prevent the set $\{1,2,3,4,5\}$ from also being separable, the ghost child $\{6,7,8\}$ will be used during our vector construction. More precisely, to break unwanted separability on unions of siblings, we will add a small weight to each set in a *sibling chain*, as defined below.

[2]Be sure to say "haunted Hasse diagram" out loud.

Definition 4.2.5. *Let $A \in \chi$ with its children listed in some order $\psi = (S_1, S_2, ..., S_k)$. If $\cup_i S_k = A$ (so there is no ghost child of A), then the* **sibling chain** *ρ of these children is*

$$\rho = \{S_1 \cup S_2, S_2 \cup S_3, ..., S_{k-1} \cup S_k\}.$$

If A has a nonempty ghost child $G = A - (\cup_i S_i)$ then the sibling chain ρ also includes the set $S_k \cup G$. That is,

$$\rho = \{S_1 \cup S_2, S_2 \cup S_3, ..., S_{k-1} \cup S_k, S_k \cup G\}.$$

Elements of a sibling chain ρ are called **siblinks***. A siblink $S_i \cup S_{i+1}$ is an* **odd link** *if i is odd and an* **even link** *if i is even. Likewise the siblink $S_i \cup G$ is odd or even depending on the parity of i.*

Siblinks are important for insuring the union of siblings in χ is not separable unless it is also in χ.

Consider the tree character $\chi = \{\emptyset, A_1, A_2, ..., [n]\}$. We construct a vector v_χ such that $\mathrm{char}(v_\chi) = \chi$ as follows. Define $a_i = 2^{r_i}$ where r_i is the generation of A_i in χ. Let $\omega = \{C_1, C_2, ..., C_h\}$ be the set of all siblinks from the sibling chains that can be made from χ. Define $c_i = \frac{1}{2^n}$ if C_i is an odd link and $c_i = -\frac{1}{2^n}$ if C_i is an even link. Our preference vector v_χ is given by

$$v_\chi = a_1 \mathsf{v}_{A_1} + a_2 \mathsf{v}_{A_2} + \cdots + a_{[n]} \mathsf{v}_{[n]} + c_1 \mathsf{v}_{C_1} + c_2 \mathsf{v}_{C_2} + \cdots + c_h \mathsf{v}_{C_h}. \tag{4.1}$$

Theorem 4.2.6. *The preference vector v_χ has character χ. That is, $char(v) = \chi$.*

The proof of Theorem 4.2.6 is quite technical, so we prove it via a series of lemmas. Before providing those details, we give the proof of the theorem itself, referring to the lemmas that we will prove thereafter. This will help to motivate those lemmas, since it will be clear how they come together into the main argument.

Proof. Lemma 4.3.4 below shows that the vector v is separable on every element of χ. Remaining subsets of $[n]$ fall into three categories

- S, where S does not have a fine χ-decomposition. Lemma 4.3.8 below proves nonseparability for this category
- S, where S has a fine χ-decomposition, but at least two elements of the decomposition do not have the same parent in χ. Lemma 4.3.11 below proves nonseparability for this category.

- S, where S has a fine χ-decomposition, and every element of the decomposition has the same parent in χ. Lemma 4.3.12 below proves nonseparability for this category.

In summary, the only elements of χ are separable. Therefore, $char(v) = \chi$. $\square$

We take up the proof of the four lemmas in the next subsection. But before that, we give an example of the preference vector construction described in the theorem.

Example 4.2.7 (Tree character χ to $\vec{p}_\chi$.)**.** *Consider the tree character*

$$\chi = \{\{1,2,3,4,5,6\},\{7,8,9\},\{1\},\{2\},\{3,4\},\{6\},\{7,8\},\{3\},\{7\}\,\emptyset\}$$

with haunted Hasse diagram.

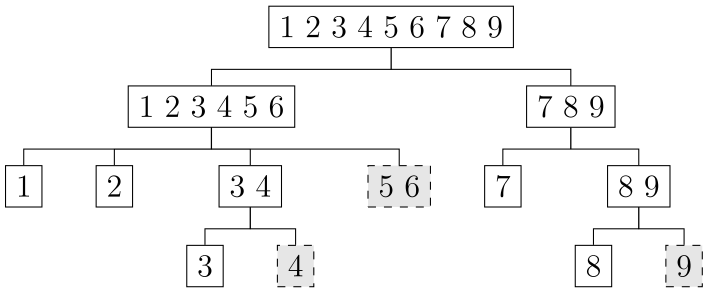


Figure 4.4: The tree character with ghost children shown.

First we add basis vectors indexed by elements of χ. The coefficient of v_S is determined by the generation of S in the haunted Hasse diagram. This gives us the first part of our vector $\vec{p}_\chi$:

$$1\mathsf{v}_{[9]} + 2\mathsf{v}_{[6]} + 2\mathsf{v}_{\{7,8,9\}} + 4\mathsf{v}_{\{1\}} + 4\mathsf{v}_{\{2\}} + 4\mathsf{v}_{\{3,4\}} + 4\mathsf{v}_{\{7\}} + 4\mathsf{v}_{\{8,9\}} + 8\mathsf{v}_{\{3\}} + 8\mathsf{v}_{\{8\}}.$$

Next, we construct the sibling chains. In this case we have five chains:

$$\begin{aligned}
\rho_1 =& \{\{1,2,3,4,5,6\}\cup\{7,8,9\}\} = \{\{1,2,3,4,5,6,7,8,9\}\},\\
\rho_2 =& \{\{1\}\cup\{2\},\{2\}\cup\{3,4\},\{3,4\}\cup\{5,6\}\} = \{\{1,2\},\{2,3,4\},\{3,4,5,6\}\},\\
\rho_3 =& \{\{7\}\cup\{8,9\}\} = \{\{7,8,9\}\},\\
\rho_4 =& \{\{3\}\cup\{4\}\} = \{\{3,4\}\},\\
\rho_5 =& \{\{8\}\cup\{9\}\} = \{\{8,9\}\}.
\end{aligned}$$

So the set of all siblinks is

$$\omega = \{\{1,2,3,4,5,6,7,8,9\},\{1,2\},\{2,3,4\},\{3,4,5,6\},\{7,8,9\},\{3,4\},\{8,9\}\}.$$

Finally we add basis vectors indexed by elements of ω. The coefficients of these basis vectors is based on whether or not the indexing element is an odd or even siblink. Note that a sets may appear in both χ and ω. For example, $\{3,4\} \in \chi$ and this set also appears as the siblink $\{3\} \cup \{4\}$. We have chosen our coefficients to that we still get our desired outcome: that is, $\mathrm{char}(v_\chi) = \chi$. The final result is our desired preference vector:

$$\begin{aligned}\vec{p}_\chi =& 1\mathsf{v}_{[9]} + 2\mathsf{v}_{[6]} + 2\mathsf{v}_{\{7,8,9\}} + 4\mathsf{v}_{\{1\}} + 4\mathsf{v}_{\{2\}} + 4\mathsf{v}_{\{3,4\}} + 4\mathsf{v}_{\{7\}} + 4\mathsf{v}_{\{8,9\}} \\ & + 8\mathsf{v}_{\{3\}} + 8\mathsf{v}_{\{8\}} + \frac{1}{2^6}\mathsf{v}_{[9]} + \frac{1}{2^6}\mathsf{v}_{\{1,2\}} - \frac{1}{2^6}\mathsf{v}_{\{2,3,4\}} + \frac{1}{2^6}\mathsf{v}_{[6]-[2]} + \frac{1}{2^6}\mathsf{v}_{\{7,8,9\}} \\ & + \frac{1}{2^6}\mathsf{v}_{\{3,4\}} + \frac{1}{2^6}\mathsf{v}_{\{8,9\}}\end{aligned}$$

One can check that this vector (which lies in $\mathbb{R}^{2^9}$) satisfies $\mathrm{char}(\succ_{\vec{p}_\chi}) = \chi$.

4.3 Proof of Correctness for Constructing Tree Characters

The remainder of this paper is devoted to proving the four lemmas required for the proof of Theorem 4.2.6. We start with some intermediary lemmas to handle the various cases.
Before we prove Theorem 4.2.6, we establish a few helpful lemmas.

Lemma 4.3.1. *Consider the coefficient a_l corresponding to the set $A_l \in \chi$. The coefficient a_l is one greater than the sum of all the coefficients corresponding to supersets of $A_l \in \chi$. That is, $a_l + 1 = \sum_{\{s|A_l \subset A_s\}} a_s$.*

Proof. Recall that the structure of our character ensures that when $i < j$ we either have $A_j \subset A_i$ or $A_j \cap A_i = \emptyset$. Let $a_l = 2^{r_l}$. Consider two arbitrary supersets A_i, A_j of A_l, where $i < j$. Since their intersection is nonempty, we have $A_j \subset A_i$. More generally, all the supersets of A_l are nested. Therefore, for every natural number up to r_l, there is exactly one superset of A_l in generation r_l. Thus, the sum of the coefficients of the supersets of A_l is $2^0 + 2^1 + 2^2 + \cdots + 2^{r_l - l} = 2^{r_l} - 1 = a_l - 1$. □

Lemma 4.3.2. *Let $c_1, c_2, \ldots, c_h$ be all of the coefficients associated with elements of the siblinks. Then $\left|\sum_1^h c_i\right| < 1$.*

Proof. The siblinks of are made from the union of two distinct sets, so singletons cannot be elements of a sibling chain. Thus, h must be less than than the total number of possible subsets of $[n]$, that is $h < 2^n$, so that

$$\left|\sum_1^h c_i\right| \leq \sum_1^h |c_i| = h \cdot \frac{1}{2^n} < 1.$$

$\square$

Relative parity of sets plays an important role in our character construction, so we introduce the following notation. Let $A \subset B \subset [n]$. We define

$$\mathcal{E}(A, B) = \{S \subset B : |S \cap A| \text{ is even}\}$$

and

$$\mathcal{O}(A, B) = \{S \subset B : |S \cap A| \text{ is odd}\}.$$

We also define

$$S(x, A) = \sum_{\{s: A_s \in \mathcal{E}(\widehat{x_A}, A)\}} a_s.$$

This brings us to our next lemma.

Lemma 4.3.3. *Consider two distinct entries of v whose outcomes are the same on $([n] - A_l)$, namely $(x_{A_l} u_{([n]-A_l)})$ and $(y_{A_l} u_{([n]-A_l)})$ where $x_{A_l} \neq y_{A_l}$. Then*

$$|S(x, A_l) - S(y, A_l)| \geq a_l.$$

Proof. Note that $|S(x, A_l) - S(y, A_l)| > 0$ is an integer. If $A_s \in \mathcal{E}(\widehat{x_{A_l}}, A_l)$ then $A_s \subset A_l$, so that $a_s = 2^r_s \geq 2^{r_l} = a_l$. Since a_l divides every term in both $S(x, A_l)$ and $S(y, A_l)$, it also divides $|S(x, A_l) - S(y, A_l)|$. $\square$

4.3.1 Separability on χ

Lemma 4.3.4. *The preference vector v_χ is separable on every element in χ.*

Proof. Consider a set $A_l \in \{A_1, A_2, ..., [n]\}$ and the outcomes

$$(x_{A_l} u_{([n]-A_l)}) \text{ and } (y_{A_l} u_{([n]-A_l)}).$$

We claim that:

- If $S(x, A_l) > S(y, A_l)$ then $(x_{A_l} u_{([n]-A_l)}) > (y_{A_l} u_{([n]-A_l)})$.
- If $S(x, A_l) = S(y, A_l)$ then $(x_{A_l} u_{([n]-A_l)}) = (y_{A_l} u_{([n]-A_l)})$.
- If $S(x, A_l) < S(y, A_l)$ then $(x_{A_l} u_{([n]-A_l)}) < (y_{A_l} u_{([n]-A_l)})$.

Of course, the third case follows from the first by interchanging the roles of x and y.

Case 1: $S(x, A_l) > S(y, A_l)$.

First, we recognize by Proposition 4.1.1 that $(x_{A_l}u_{([n]-A_l)})$ benefits from a_s, where $A_s \cap A_l = \emptyset$, if and only if $(y_{A_l}u_{([n]-A_l)})$ also does. Let us denote the sum of all such a_s that both outcomes benefit from as $\mathbf{F}$.

Now it must be true that

$$[x_{A_l}u_{([n]-A_l)}]_v \geq S(x, A_l) + \mathbf{F}$$

and that

$$S(y, A_l) + \mathbf{N_{A_l}} + \mathbf{F} + \mathbf{C} \geq [y_{A_l}u_{([n]-A_l)}]_v,$$

where $\mathbf{N_{A_l}} = \sum_{\{a_s | A_l \subset A_s\}} a_s$ is the sum of all the coefficients corresponding to supersets of A_l in χ, and $\mathbf{C} = \sum_1^h c_i$, the sum of the coefficients from the sibling chains.

Next, we subtract the upperbound of $[y_{A_l}u_{([n]-A_l)}]_v$ from the lowerbound of $[x_{A_l}u_{([n]-A_l)}]_v$.

$$\begin{aligned} S(x, A_l) + \mathbf{F} - (S(x, A_l) + \mathbf{N_{A_l}} + \mathbf{F} + \mathbf{C}) &= (S(x, A_l) - S(y, A_l)) \\ &\qquad - \mathbf{N_{A_l}} - \mathbf{C} \\ &\geq a_l - \mathbf{N_{A_l}} - \mathbf{C} && \text{by } \textbf{Lemma 4.3.3} \\ &\geq 1 - \mathbf{C} && \text{by } \textbf{Lemma 4.3.1} \\ &> 0 && \text{by } \textbf{Lemma 4.3.2} \end{aligned}$$

The lowerbound of $[x_{A_l}u_{([n]-A_l)}]_v$ is greater than the upperbound of $[y_{A_l}u_{([n]-A_l)}]_v$, thus $[x_{A_l}u_{([n]-A_l)}]_v > [y_{A_l}u_{([n]-A_l)}]_v$ for an arbitrary $u_{([n]-A_l)}$. Therefore, for all $w_{([n]-A_l)}$ we have $(x_{A_l}w_{([n]-A_l)}) > (y_{A_l}w_{([n]-A_l)})$. This concludes Case 1.

Case 2: $S(x, A_l) = S(y, A_l)$.

First, we note that either both outcomes receive the benefit from a_l or neither do, or equivalently, $|x_{A_l} \cap A_l|$ and $|y_{A_l} \cap A_l|$ are the same parity. If this were not true, then exactly one of $S(x, A_l)$ and $S(y, A_l)$ would benefit from a_l, which would guarantee $S(x, A_l) \neq S(y, A_l)$. Indeed, a_l is the smallest summand in each of these sums, so no combination of other summands could properly compensate for the small difference.

Let

$$M(x, A_l) = \sum_{\{a_s | A_l \subset A_s \text{ and } |x_{A_l} \cap A_s| \text{ is even}\}} a_s + \sum_{\{c_s | A_l \subset C_s \text{ and } |x_{A_l} \cap A_s| \text{is even}\}} a_s$$

denote the sum of the coefficients corresponding to supersets of A_l that $(x_{A_l}v_{([n]-A_l)})$ benefits from and let

$$M(y, A_l) = \sum_{\{a_s | A_l \subset A_s \text{ and } |y_{A_l} \cap A_s| \text{ is even}\}} a_s + \sum_{\{c_s | A_l \subset C_s \text{ and } |y_{A_l} \cap A_s| \text{ is even}\}} a_s$$

denote the sum of the coefficients corresponding to supersets of A_l that $(y_{A_l}v_{([n]-A_l)})$ benefits from. Because $|x_{A_l} \cap A_l|$ and $|y_{A_l} \cap A_l|$ are the same parity, we may use **Proposition 4.1.4** to guarantee $M(x, A_l) = M(y, A_l)$. Considering the same $\mathbf{F}$ from the previous case, the result follows.

$$\begin{aligned}
[x_{A_l}u_{[n]-A_l}]_v &= S(x, A_l) + M(x, A_l) + \mathbf{F} \\
&= S(y, A_l) + M(y, A_l) + \mathbf{F} \\
&= [y_{A_l}u_{[n]-A_l}]_v
\end{aligned}$$

We have shown for an arbitrary $u_{([n]-A_l)}$, that $[x_{A_l}u_{([n]-A_l)}]_v = [y_{A_l}u_{([n]-A_l)}]_v$. Therefore, for all $w_{([n]-A_l)}$ we have $(x_{A_l}w_{([n]-A_l)}) = (y_{A_l}w_{([n]-A_l)})$. This concludes Case 2.

The combination of Case 1 and Case 2 demonstrate that the relation R does not depend on $u_{([n]-A_l)}$. That is, if for some $u_{([n]-A_l)}$, $(x_{A_l}u_{([n]-A_l)})R(y_{A_l}u_{([n]-A_l)})$ then for all $w_{([n]-A_l)}$ $(x_{A_l}w_{([n]-A_l)})R(y_{A_l}w_{([n]-A_l)})$. Therefore, our arbitrarily chosen A_l is separable implying that every set in $\{A_1, A_2, ..., [n]\}$ is separable. □

4.3.2 Nonseparability Outside of χ

Definition 4.3.5. *Consider a set B and a set of sets χ. The set B is* χ**-constructable** *when there exists $\phi \subset \chi$, such that $B = \cup_{A\in\phi}A$. If such a set ϕ exists, then it called a* χ**-construction** *of B.*

Note that any $A \in \chi$ is χ-constructable, since we can take the trivial construction set $\phi = \{A\}$. Any other χ-construction is called *nontrivial*.

Definition 4.3.6. *The subset $\phi \subset \chi$ is a* **fine** *χ-construction of B when no element in ϕ has a nontrivial χ-construction.*

Example 4.3.7 (χ-constructable sets: examples and non-examples)**.** *Consider the tree character*

$$\chi = \{\{1, 2, 3, 4, 5, 6\}, \{7, 8, 9\}, \{1\}, \{2\}, \{3, 4\}, \{6\}, \{7, 8\}, \{3\}, \{7\}\, \emptyset\}$$

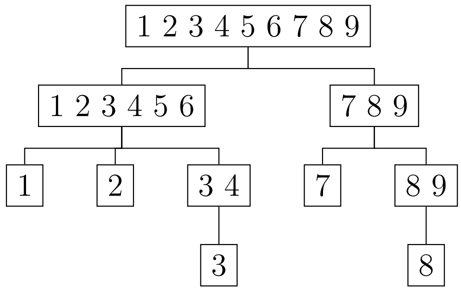


Figure 4.5: The tree character χ

with the following Hasse diagram.

Here are four examples of χ-constructable sets:

$$\begin{aligned}
\{1,2\} &= \{1\} \cup \{2\}, \\
\{1,2,3,4\} &= \{1\} \cup \{2\} \cup \{3,4\}, \\
\{2,8,9\} &= \{2\} \cup \{8,9\}, \\
\{1,3,8,9\} &= \{1\} \cup \{3\} \cup \{7,8,9\}.
\end{aligned}$$

Note that the first two sets $\{1,2\} = \{1\} \cup \{2\}$ and $\{1,2,3,4\} = \{1\} \cup \{2\} \cup \{3,4\}$ are χ-constructed using only siblings in χ. Meanwhile, the sets $\{2,8,9\} = \{2\} \cup \cup \{8,9\}$ and $\{1,2,8,9\} = \{1\} \cup \{3\} \cup \{7,8,9\}$ are χ-constructed using elements that are not siblings. The differences between these construction methods will come into play later.

It is also important to recognize that the first three examples are fine χ-constructions. The fourth example is not a fine χ-construction because the set $\{7,8,9\}$ is χ-constructable, namely $\{7,8,9\} = \{7\} \cup \{8,9\}$.

The following four sets are not χ-constructable

$$\{9\},\ \{1,2,4\},\ \{2,6\},\ \{1,2,3,7,9\}.$$

We are now ready for our next lemma.

Lemma 4.3.8. *Let B be a set which is not χ-constructable. Then the set B is not separable on $\vec{p_\chi}$.*

Proof. We will construct two outcomes $x_B \neq y_B$ on B so that the relation between $x = x_B u_{([n]-B)}$ and $y = y_B u_{([n]-B)}$ depends on $u_{([n]-B)}$.

Let

$$D = \cup_{A_k \subset B} A_k.$$

This set $D \subset B$ is the largest χ-constructable subset of B, so $B - D \neq \emptyset$. (Note that we might have $D = \emptyset$. Indeed, the set $B = \{9\}$ in the Hasse diagram above is one such B.) Consider the set $\mathcal{F}$ of fundamental sets that intersect $B - D$. Pick a minimal set $A_i \in \mathcal{F}$, meaning that A_i does not contain any other member of $\mathcal{F}$. Observe that $A_i - B \neq \emptyset$ by our choice of D. Let $g \in A_i \cap (B - D)$ and let $h \in A_i - B$. By the structure of our tree character, if $g \in A_j$ for some $A_j \in \chi$, then $A_i \subset A_j$.

Now we rewrite our as outcomes

$$x = x_g x_{B-g} u_h u_{([n]-B-h)},$$
$$y = y_g y_{B-g} u_h u_{([n]-B-h)}.$$

We have the freedom to construct x and y. For simplicity, we will take both to be all-zero on the sets $B - g$ and $[n] - B - h$, and denote these all-zero outcomes as $z_{(B-g)}$ and $z_{([n]-B-h)}$. So our outcomes are

$$x = x_g z_{B-g} u_h z_{([n]-B-h)},$$
$$y = y_g z_{B-g} u_h z_{([n]-B-h)}.$$

We take $x_g = 0$ and $y_g = 1$. By Proposition 4.1.1, the outcome x will benefit from a_s or c_s, where $g \notin S_s$ or $g \notin C_s$ if and only if the outcome y also benefits from a_s or c_s. This implies that any difference between $[x]_v$ and $[y]_v$ must be caused by coefficients associated with sets in χ and ω that contain g. Such sets must contain A_i, and therefore they also contain h.

Since χ is a tree character, the χ-supersets of A_i are nested. Let $N(A_i) = \sum_{\{s:A_i \subsetneq A_s\}} a_s$ be the sum of the coefficients of proper supersets of A_i in χ. Let $\mathbf{C} = \sum_1^h c_i$ be the sum of the coefficients from the sibling chains that contain A_i. We show that R depends on the parity of u_h.

Case 1: u_h is 0. We have

$x = x_g z_{B-g} u_h z_{([n]-B-h)}$ is even and $x = y_g z_{B-g} u_h z_{([n]-B-h)}$ is odd

$\Rightarrow x$ benefits from a_{A_i} and y does not benefit from a_{A_i}

$\Rightarrow [x]_v - [y]_v$

$\geq a_{A_i} - N(A_i) - \mathbf{C}$

$\geq 1 - \mathbf{C}$ **Lemma 4.3.1**

> 0 **Lemma 4.3.2**

$\Rightarrow x > y$.

Case 2: u_h **is 1**. We have

$x = x_g z_{B-g} u_h z_{([n]-B-h)}$ is odd and $x = y_g z_{B-g} u_h z_{([n]-B-h)}$ is even

$\Rightarrow x$ does not benefit from a_{A_i} and y benefits from a_{A_i}

$\Rightarrow [x]_v - [y]_v$

$\leq \mathbf{N_S} + \mathbf{C} - a_g$

$\leq \mathbf{C} - 1$ **Lemma 4.3.1**

< 0 **Lemma 4.3.2**

$\Rightarrow x < y$.

We have shown that the relation between x and y depends on u_{A_h}, so the set B is non-separable on v. □

We can now construct an outcome that only benefits from a specified family of nested subsets.

Theorem 4.3.9. *Consider a nesting $A_{l_1} \subsetneq A_{l_2} \subsetneq \cdots \subsetneq A_{l_r}$. For any subset of coefficients $\{a_{l_1}, a_{l_2}, \ldots, a_{l_r}\}$, an outcome on A_{l_r} can be constructed that only benefits from the coefficients in that subset.*

Proof. We construct the outcome

$$x = x_{A_{l_1}} x_{A_{l_2}-A_{l_1}} x_{A_{l_3}-A_{l_2}} \cdots x_{A_{l_r}-A_{l_{r-1}}} z_{([n]-A_{l_r})}$$

where $z_{([n]-A_{l_r})}$ is all-zero and the rest is determined recursively as follows. Begin with A_{l_1}. If $a_{l_1} > 0$, then take $x_{A_{l_1}}$ to be odd; otherwise take $x_{A_{l_1}}$ to be even. Next, choose $x_{A_{l_2}-A_{l_1}}$ to be even or odd, depending on both the parity of $x_{A_{l_1}}$ and whether or not $a_2 > 0$. Continue in this way to create the entire outcome. This construction works because the sets are nested and $A_{l_i} - A_{l_{i-1}} \neq \emptyset$. □

Example 4.3.10 (Building an outcome on nested sets.)**.** *Consider the sets where* $[n] = [10]$

$$\{1,2\}, \{1,2,3\}, \{1,2,3,4,5,6\}, \{1,2,3,4,5,6,7,8\}.$$

We have an outcome on $[10]$ *which we can write as*

$$x = x_{\{1,2\}} x_{\{3\}} x_{\{4,5,6\}} x_{\{7,8\}} z_{\{9,10\}}$$

If we want the outcome to benefits only from $\mathsf{v}_{\{1,2\}}$ *and* $\mathsf{v}_{[8]}$*, we start by making the outcome* $x_{\{1,2\}}$ *even, which means* x *benefits from* $v_{\{1,2\}}$*. From here we make the* *outcome* $x_{\{3\}}$ *odd, guaranteeing* $x_{\{1,2\}}x_{\{3\}}$ *is odd, so* x *does not benefit from* $v_{\{1,2,3\}}$*. Now we make the outcome* $x_{\{4,5,6\}}$ *even so* $x_{\{1,2\}}x_{\{3\}}x_{\{4,5,6\}}$ *is odd, so* x *does not benefit from* $v_{\{1,2,3,4,5,6\}}$*. Finally, make* $x_{\{7,8\}}$ *odd so* $x_{\{1,2\}}x_{\{3\}}x_{\{4,5,6\}}x_{\{7,8\}}$*, implying* x *benefits from* $v_{\{1,2,3,4,5,6,7,8\}}$*. One such outcome is*

$$x = 1110001100100$$

Lemma 4.3.11. *Consider a set* $B \subset [n]$ *where* $\phi = \{A_0 = \emptyset, A_1, A_2, \cdots, A_k = [n]\}$ *is a fine set decomposition of* B *with* χ*, but at least two elements of* ϕ *do not have the same parent in* χ*. The set* B *is not separable on* $\vec{p_\chi}$*.*

Proof. Denote the set of all children of A_i in χ as $C(A_i)$. Let A_l and A_r be two sets in ϕ that are not siblings. First, we note that we can ignore the effect of the siblinks. In total, these change the value by less than one. We will see below that the sets in χ will create a difference of at least one, so the siblinks do not have a large enough effect to change the relative values.

Now consider the following two outcomes

$$x = (e_{(A_l - C(A_l))} o_{(A_r - C(A_r))} z_{C(A_l)} z_{C(A_r)} z_{(B - A_l - A_r)} u_{([n]-B)}),$$
$$y = (o_{(A_l - C(A_l))} e_{(A_r - C(A_r))} z_{C(A_l)} z_{C(A_r)} z_{(B - A_l - A_r)} u_{([n]-B)})$$

where for any set S, e_S is even, o_S is odd and z_S is all-zero. (Take a moment to recognize that two outcomes with these conditions must exist.) We determine the $u_{([n]-B)}$ below.

These outcomes can receive the benefit of coefficients associated with elements of $\phi = \{\emptyset, A_1, A_2, ..., [n]\}$. We place these elements in six categories

- Proper subsets of A_l. $S \subset A_l$
- Proper subsets of A_r. $S \subset A_r$
- Sets disjoint from both A_l and A_r. $S \cap (A_l \cup A_r) = \emptyset$
- Supersets of $A_l \cup A_r$. $(A_l \cup A_r) \subseteq S$
- Supersets of A_l, but not A_r. $A_l \subseteq S$, $A_r \not\subseteq S$
- Supersets of A_r, but not A_l. $A_r \subseteq S$, $A_l \not\subseteq S$

Both x and y benefit from the same descendants of A_l and descedants of A_r because these outcomes agree on $C(A_l)$ and $C(A_r)$.

By Proposition 4.1.1, both x and y benefit from the same sets that are disjoint from $(A_l \cup A_r)$.

It is also true that x has the same parity as y, so by Proposition 4.1.4, x and y benefit from the same ancestors of $(A_l \cup A_r)$.

So, if there is a preference difference between these outcomes, then it must come from coefficients associated with sets from the last two categories. We have constructed our outcomes so that only x benefits from the coefficient of A_l, while only y benefits from the coefficient of A_r.

From here we must make a few notes. First, by Lemma 4.3.1, we recognize that

$$\begin{aligned} a_r &= 1 + \sum_{(a_i | A_r \subset A_i)} a_i \\ &= 1 + \sum_{(a_i | A_r \subset A_i \text{ and } A_l \not\subset A_i)} a_i + \sum_{(a_i | A_r \cup A_l \subset A_i)} a_i, \end{aligned}$$

and likewise

$$\begin{aligned} a_l &= 1 + \sum_{(a_i | A_l \subset A_i)} a_i \\ &= 1 + \sum_{(a_i | A_l \subset A_i \text{ and } A_r \not\subset A_i)} a_i + \sum_{(a_i | A_r \cup A_l \subset A_i)} a_i. \end{aligned}$$

We also note that either $\sum_{(a_i | A_l \subset A_i \text{ and } A_r \not\subset A_i)} a_i$ or $\sum_{(a_i | A_r \subset A_i \text{ and } A_l \not\subset A_i)} a_i$ could be 0, but both at least one of them must be nonzero; if both were 0 then A_l and A_r would have to have the same parent, a contradiction. So let us assume that $\sum_{(a_i | A_l \subset A_i \text{ and } A_l \not\subset A_i)} a_i > 0$. We have

$$\sum_{(a_i | A_r \subset A_i \text{ and } A_l \not\subset A_i)} a_i > \sum_{(a_i | A_r \cup A_l \subset A_i)} a_i$$

and

$$\sum_{(a_i | A_l \subset A_i \text{ and } A_r \not\subset A_i)} a_i > \sum_{(a_i | A_r \cup A_l \subset A_i)} a_i$$

or

$$\sum_{(a_i | A_l \subset A_i \text{ and } A_r \not\subset A_i)} a_i = 0.$$

Now, use Lemma 4.3.9 to construct two different values for $u_{([n]-B)}$ that change the preference between the outcomes x and y.

Case 1: Choose $u_{([n]-B)}$ so that x benefits from all supersets of A_l, but not A_r ($\{S|A_l \subset S$ and $A_r \not\subset S\}$) and all supersets of A_r, but not A_l ($\{S|A_r \subset S$ and $A_l \not\subset S\}$

Because x benefits from these supersets, y does not as by Proposition 4.1.4. Therefore, we get the following difference

$$\begin{aligned}
&[x_{(A_l-C(A_l))}y_{(A_r-C(A_r))}z_{C(A_l)}z_{C(A_r)}z_{(B-A_l-A_r)}u_{([n]-B)}]_v \\
&- [y_{(A_l-C(A_l))}x_{(A_r-C(A_r))}z_{C(A_l)}z_{C(A_r)}z_{(B-A_l-A_r)}u_{([n]-B)}]_v \\
&= a_l + \sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i + \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i - a_r \\
&= a_l + \sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i + \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i \\
&\qquad - \left(1 + \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i + \sum_{(a_i|A_r\cup A_l\subset A_i)} a_i\right) \\
&= a_l + \sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i - \sum_{(a_i|A_r\cup A_l\subset A_i)} a_i - 1 \\
&\geq 0
\end{aligned}$$

This implies either $x > y$ when $\sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i \neq 0$, or R is $x = y$ when $\sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i = 0$.

Case 2: Choose $u_{([n]-B)}$ so that x benefits from no supersets of A_l, but not A_r ($\{S|A_l \subset S$ and $A_r \not\subset S\}$) and no supersets of A_r, but not A_l ($\{S|A_r \subset S$ and $A_l \not\subset S\}$

Because x does not benefit from any of these superset, y must benefit from all of them by Proposition 4.1.4. Therefore, we get the following difference

$$\begin{aligned}
&[x_{(A_l-C(A_l))}y_{(A_r-C(A_r))}z_{C(A_l)}z_{C(A_r)}z_{(B-A_l-A_r)}u_{([n]-B)}]_v \\
&-[y_{(A_l-C(A_l))}x_{(A_r-C(A_r))}z_{C(A_l)}z_{C(A_r)}z_{(B-A_l-A_r)}u_{([n]-B)}]_v \\
&= a_l - \left(\sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i + \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i + a_r\right) \\
&= \left(1 + \sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i + \sum_{(a_i|A_r\cup A_l\subset A_i)} a_i\right) \\
&\quad - \left(\sum_{(a_i|A_l\subset A_i \text{ and } A_r\not\subset A_i)} a_i + \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i + a_r\right) \\
&= 1 + \sum_{(a_i|A_r\cup A_l\subset A_i)} a_i - \sum_{(a_i|A_r\subset A_i \text{ and } A_l\not\subset A_i)} a_i - a_r \\
&< 0
\end{aligned}$$

This implies that $x < y$.

Conclusion We have shown that there will always exists a pair of outcomes on B for which the preference between them will change depending on the outcome on $u_{[n]-B}$. Thus, B is nonseparable. □

Lemma 4.3.12. *Consider a set $B \notin \chi$ where ϕ is a fine set decomposition of B with χ and all the elements of ϕ are siblings in χ. The set B is not separable on the order implied by v.*

Proof. We will construct two outcomes $x_B \neq y_B$ on B so that the relation between $x = x_B u_{([n]-B)}$ and $y = y_B u_{([n]-B)}$ depends on $u_{([n]-B)}$.

We begin by selecting two elements in ϕ, the set of siblinks, to focus on. All the elements of ϕ are siblings so they must have the same parent A_p. We have $B \subsetneq A_p$ because $B \notin \chi$. Let $Q = A_p - B$, which includes elements of A_p found in children of A_p outside of B, including any ghost children of A_p.

Recall that the sibling chain ω is the set of siblinks for the children of A_p. This sibling chain "connects" all the children and the ghost child of A_p. Therefore, there must be a siblink made up from the union of two children of A_p such that one child is in B and one child (or ghost child) is in Q. Let $A_l \subset B$ and $A_s \subset Q$. Let A_r be any other element in ϕ. This A_r must exist: there is at least one more set from χ contained in B because $B \notin \chi$. We denote the set of all children (including ghost children) of set A_i in χ as $C(A_i)$.

We construct the following two outcomes:

$$x = (e_{(A_l - C(A_l))} o_{(A_r - C(A_r))} z_{C(A_l)} z_{C(A_r)} z_{(B - A_l - A_r)} u_{A_s} u_{([n] - B - A_s)})$$
$$y = (o_{(A_l - C(A_l))} e_{(A_r - C(A_r))} z_{C(A_l)} z_{C(A_r)} z_{(B - A_l - A_r)} u_{A_s} u_{([n] - B - A_s)})$$

where $e_{(A_l - C(A_l))}$ and $e_{(A_r - C(A_r))}$ are even, $o_{(A_l - C(A_l))}$ and $o_{(A_r - C(A_r))}$ are odd, and $z_{C(A_l)}$ and $z_{C(A_r)}$ are outcomes represented by a string of zeros. Note that $A_l - C(A_l) \neq \emptyset$ and that $A_r - C(A_r) \neq \emptyset$ because they are part of a fine χ-construction of B Take a moment to recognize that two outcomes with these conditions must exist.

Just as in the proof of the previous lemma, these outcomes benefit from the same coefficients associated with elements of $\{\emptyset, A_1, A_2, ..., [n]\}$ that fall into the following categories:

- Proper subsets of A_l. $S \subset A_l$
- Proper subsets of A_r. $S \subset A_r$
- Sets disjoint from both A_l and A_r. $S \cap (A_l \cup A_r) = \emptyset$
- Supersets of $A_l \cup A_r$. $(A_l \cup A_r) \subseteq S$

Now because A_l and A_r have the same parent, there are no elements of χ that fall into the categories

- Supersets of A_l, but not A_r. $A_l \subseteq S$, $A_r \not\subseteq S$
- Supersets of A_r, but not A_l. $A_r \subseteq S$, $A_l \not\subseteq S$

except for A_l and A_r.

From our choice of outcomes, we know x will benefit from a_l, but not benefit from a_r. Similarly, y will benefit from a_r, but not benefit from a_l. Now because A_l and A_r have the same parent, they are in the same generation. Therefore, $a_l = a_r$. This implies that the sums of the coefficients associated with elements of χ that x and y benefit from are equal. So any preference difference between these outcomes must come from coefficients associated with elements of ω.

Due to the structure for sibling chains, there are at most four elements of ω, the union of sibling chains, which could be supersets of either A_l or A_r, but not both.[3]

$$(A_\alpha \cup A_l), \quad (A_l \cup A_\beta), \quad (A_\gamma \cup A_r), \quad (A_r \cup A_\delta)$$

[3]The number of such links varies between two and four depending on whether A_s, A_l, and A_r are the outside components of the outside siblinks and where A_l and A_r are in relation to one another in the sibling chain. Note that the links $(A_l \cup A_\beta)$ and must exist $(A_\gamma \cup A_r)$; the other two do not depending on if A_l or A_r are outside components of the outside siblinks.

The general situation for the parent A_p and its various children is depicted in Figure 4.6. In the cases we handle below, some of these child sets coincide, or are absent.

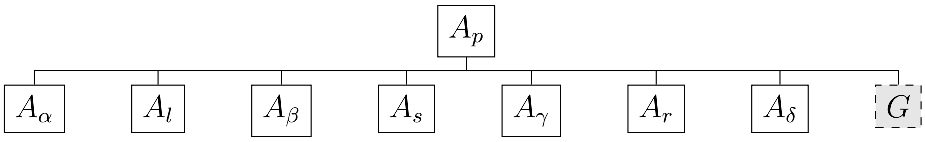


Figure 4.6: A parent A_p and its children. The sets A_l and A_r are subsets of the χ-constructable set B, while A_s is not.

We chose our A_l intentionally to guarantee that either $A_\alpha = A_s$ or $A_\beta = A_s$. We use this to split into two cases. Before tackling these two cases, we define some supporting notation and clarify some assumptions. First, we assume that A_l appears before A_r in the sibling chain; this can be done because if A_r appeared before A_l in the sibling chain then we could simply "flip" the sibling chain by reversing the order we used to construct it making A_l come before A_r without changing structure.[4] Now we also denote the coefficient corresponding to the link $(A_a \cup A_b)$ as c_{ab}. Finally, reacall that all links have the same magnitude. Furthermore, all even links have a positive coefficient and all odd links have a negative coefficient. Compactly, we have $c_{ab} = \pm c$.

Case 1: $A_\alpha = A_s$

When $A_\alpha = A_s$ there are six distinct sibling chain structures describing the relationship between the potential links.

1. $\{\cdots (A_s \cup A_l), (A_l \cup A_r)\}$
2. $\{\cdots (A_s \cup A_l), (A_l \cup A_r), (A_r \cup A_\delta) \cdots\}$
3. $\{\cdots (A_s \cup A_l), (A_l \cup A_\beta), (A_\beta \cup A_r)\}$
4. $\{\cdots (A_s \cup A_l), (A_l \cup A_\beta), (A_\beta \cup A_r), (A_r \cup A_\delta) \cdots\}$
5. $\{\cdots (A_s \cup A_l), (A_l \cup A_\beta) \cdots (A_\gamma \cup A_r), (A_r \cup A_\delta) \cdots\}$
6. $\{\cdots (A_s \cup A_l), (A_l \cup A_\beta) \cdots (A_\gamma \cup A_r)\}$

In the following subcases, we show that regardless of the sibling chain structure, we can always construct outcomes which contradict separability on B.

Subcase 1.1: $\{\cdots (A_s \cup A_l), (A_l \cup A_r)\}$

[4] The only change "flipping" the chain could make is making even links into odd links and making odd links into even links which would change the signs of coefficient. The change of sign does not matter, however, because if two links had the same sign before the flip they will have the same sign after. Similarly, if two links had the different signs before the flip they will have different signs after.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{sl}	Only y benefits from c_{sl}
Neither x nor y benefit form c_{lr}	Neither x nor y benefit form c_{lr}
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} = c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = -c_{sl} = -c$

Subcase 1.2: $\{\cdots(A_s \cup A_l), (A_l \cup A_r), (A_r \cup A_\delta)\cdots\}$ In both our outcomes x and y we make the outcome on A_δ even, while maintaining agreement between x and y on this set. [5]

u_{A_s} is even, $u_{[n]-B-A_s}$ is all-zero	u_{A_s} is odd, $u_{[n]-B-A_s}$ is all-zero
Only x benefits from c_{sl}	Only y benefits from c_{sl}
Neither x nor y benefit form c_{lr}	Neither x nor y benefit form c_{lr}
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} - c_{r\delta} = 0$	$[x]_{\vec{p}} - [y]_{\vec{p}} = -c_{sl} - c_{r\delta} = -2c$

Subcase 1.3: $\{\cdots(A_s \cup A_l), (A_l \cup A_\beta), (A_\beta \cup A_r)\}$
Subcase 1.3.1: $A_\beta \in B$
If $A_\beta \in B$, then we can swap the roles of A_β and A_r to get

$$x = (e_{(A_l - C(A_l))} o_{(A_\beta - C(A_\beta))} z_{C(A_l)} z_{C(A_\beta)} z_{(B-A_l-A_\beta)} u_{A_s} u_{([n]-B-A_s)})$$
$$y = (o_{(A_l - C(A_l))} e_{(A_\beta - C(A_\beta))} z_{C(A_l)} z_{C(A_\beta)} z_{(B-A_l-A_\beta)} u_{A_s} u_{([n]-B-A_s)})$$

For which the sibling chain structure is covered by Subcase 1.2.
Subcase 1.3.2: $A_\beta \notin B$

u_{A_s} is even, u_{A_β} is odd	u_{A_s} is odd, u_{A_β} is even
Only x benefits from c_{sl}	Only y benefits from c_{sl}
Only y benefits from $c_{l\beta}$	Only x benefits from $c_{l\beta}$
Only x benefits from $c_{\beta r}$	Only y benefits from $c_{\beta r}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{\beta r} - c_{l\beta} = 3c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{l\beta} - c_{sl} - c_{\beta r} = -3c$

Subcase 1.4: $\{\cdots(A_s \cup A_l), (A_l \cup A_\beta), (A_\beta \cup A_r), (A_r \cup A_\delta)\cdots\}$

In both our outcomes x and y we make the outcome on A_β odd and the outcome on A_δ even, while maintaining agreement between x and y on these sets.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{sl}	Only y benefits from c_{sl}
Only y benefits from $c_{l\beta}$	Only y benefits from $c_{l\beta}$
Only x benefits from $c_{\beta r}$	Only x benefits from $c_{\beta r}$
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{\beta r} - c_{l\beta} - c_{r\delta} = 0$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\beta r} - c_{sl} - c_{l\beta} - c_{r\delta} = -2c$

[5] The outcomes still agree on everything except $A_l - C(A_l)$ and $A_r - C(A_r)$, so our previous analysis still holds regardless of whether these sets are in B or $[n] - B$, which is why we do not specify where they are unless necessary.

Subcase 1.5: $\{\cdots(A_s \cup A_l),(A_l \cup A_\beta)\cdots(A_\gamma \cup A_r),(A_r \cup A_\delta)\cdots\}$

In both our outcomes x and y we make the outcome on A_β even and the outcome on A_δ even, while maintaining agreement between x and y on these sets.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{sl}	Only y benefits from c_{sl}
Only x benefits from $c_{l\beta}$	Only x benefits from $c_{l\beta}$
Only y benefits from $c_{\beta r}$	Only y benefits from $c_{\beta r}$
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{l\beta} - c_{\beta r} - c_{r\delta} = 0$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{l\beta} - c_{sl} - c_{\beta r} - c_{r\delta} = -2c$

Subcase 1.6: $\{\cdots(A_s \cup A_l),(A_l \cup A_\beta)\cdots(A_\gamma \cup A_r)\}$
Subcase 1.6.1: $A_\gamma \in B$
If $A_\gamma \in B$, then we can swap our outcomes x and y for

$$x = (e_{(A_l - C(A_l))} o_{(A_\gamma - C(A_\gamma))} z_{C(A_l)} z_{C(A_\gamma)} z_{(B - A_l - A_\gamma)} u_{A_s} u_{([n]-B-A_s)})$$
$$y = (o_{(A_l - C(A_l))} e_{(A_\gamma - C(A_\gamma))} z_{C(A_l)} z_{C(A_\gamma)} z_{(B - A_l - A_\gamma)} u_{A_s} u_{([n]-B-A_s)})$$

For which the sibling chain structure will be covered by a previous case, either 1.2, 1.4, or 1.5. The particular case depends on whether $A_\gamma \subset B$
Subcase 1.6.2: $A_\gamma \notin B$

In both our outcomes x and y we make the outcome on A_β even, while maintaining agreement between x and y on this set.

u_{A_s} is even, u_{A_γ} is odd	u_{A_s} is even, u_{A_γ} is even
Only x benefits from c_{sl}	Only x benefits from c_{sl}
Only x benefits from $c_{l\beta}$	Only x benefits from $c_{l\beta}$
Only x benefits from $c_{\gamma r}$	Only y benefits from $c_{\gamma r}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{l\beta} + c_{\gamma r} = c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{l\beta} + c_{\gamma r} = -c$

Case 2: $A_\beta = A_s$
When $A_\beta = A_s$ there are eight distinct sibling chain structures describing the relationship between the potential links.

1. $\{(A_l \cup A_s),(A_s \cup A_r)\}$
2. $\{\cdots(A_\alpha \cup A_l),(A_l \cup A_s),(A_s \cup A_r)\}$
3. $\{(A_l \cup A_s),(A_s \cup A_r),(A_r \cup A_\delta)\cdots\}$
4. $\{\cdots(A_\alpha \cup A_l),(A_l \cup A_s),(A_s \cup A_r),(A_r \cup A_\delta)\cdots\}$
5. $\{(A_l \cup A_s)\cdots(A_\gamma \cup A_r)\}$

6. $\{(A_l \cup A_s) \cdots (A_\gamma \cup A_r), (A_r \cup A_\delta) \cdots\}$

7. $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s) \cdots (A_\gamma \cup A_r), (A_r \cup A_\delta) \cdots\}$

8. $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s) \cdots (A_\gamma \cup A_r)\}$

In the following subcases, we show that regardless of the sibling chain structure, we can always construct outcomes which contradict separability on B.

Subcase 2.1: $\{(A_l \cup A_s), (A_s \cup A_r)\}$

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from c_{sr}	Only y benefits from c_{sr}
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{ls} - c_{sr} = 2c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sr} - c_{ls} = -2c$

Subcase 2.2: $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s), (A_s \cup A_r)\}$

In both our outcomes x and y we make the outcome on A_α even, while maintaining agreement between x and y on this set.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from $c_{\alpha l}$	Only x benefits from $c_{\alpha l}$
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from c_{sr}	Only x benefits from c_{sr}
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} + c_{ls} - c_{sr} = c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} + c_{sr} - c_{ls} = -3c$

Subcase 2.3: $\{(A_l \cup A_s), (A_s \cup A_r), (A_r \cup A_\delta) \cdots\}$

In both our outcomes x and y we make the outcome on A_δ odd, while maintaining agreement between x and y on this set.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from c_{sr}	Only x benefits from c_{sr}
Only x benefits from $c_{r\delta}$	Only x benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{ls} + c_{r\delta} - c_{sr} = 3c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sr} + c_{r\delta} - c_{ls} = -c$

Subcase 2.4: $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s), (A_s \cup A_r), (A_r \cup A_\delta) \cdots\}$

In both our outcomes x and y we make the outcome on A_α even and the outcome on A_δ even, while maintaining agreement between x and y on these set.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from $c_{\alpha l}$	Only x benefits from $c_{\alpha l}$
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from c_{sr}	Only x benefits from c_{sr}
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} + c_{ls} - c_{sr} - c_{r\delta} = 0$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} + c_{sr} - c_{ls} - c_{r\delta} = -4c$

Subcase 2.5: $\{(A_l \cup A_s) \cdots (A_\gamma \cup A_r)\}$

In both our outcomes x and y we make the outcome on A_γ even, while maintaining agreement between x and y on this set.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from $c_{\gamma r}$	Only y benefits from $c_{\gamma r}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{ls} - c_{\gamma r}$	$[x]_{\vec{p}} - [y]_{\vec{p}} = -c_{ls} - c_{\gamma r}$

If $c_{ls} = c_{\gamma r}$ then in the first column the difference is 0 and in the second column the difference is $-2c_{ls}$. If $c_{ls} = -c_{\gamma r}$ then in the first column the difference is $2c_{ls}$ and in the second column the difference is 0.

Subcase 2.6: $\{(A_l \cup A_s) \cdots (A_\gamma \cup A_r), (A_r \cup A_\delta) \cdots\}$

In both our outcomes x and y we make the outcome on A_α even and the outcome on A_γ even, while maintaining agreement between x and y on these sets.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from $c_{\gamma r}$	Only y benefits from $c_{\gamma r}$
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{ls} - c_{\gamma r} - c_{r\delta} = c$	$[x]_{\vec{p}} - [y]_{\vec{p}} = -c_{ls} - c_{\gamma r} - c_{r\delta} = -c$

Subcase 2.7: $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s) \cdots (A_\gamma \cup A_r), (A_r \cup A_\delta) \cdots\}$

In both our outcomes x and y we make the outcome on A_α even, the outcome on A_γ even, and the outcome on A_δ even while maintaining agreement between x and y on these sets.

u_{A_s} is even	u_{A_s} is odd
Only x benefits from $c_{\alpha l}$	Only x benefits from $c_{\alpha l}$
Only x benefits from c_{ls}	Only y benefits from c_{ls}
Only y benefits from $c_{\gamma r}$	Only y benefits from $c_{\gamma r}$
Only y benefits from $c_{r\delta}$	Only y benefits from $c_{r\delta}$
$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} + c_{ls} - c_{\gamma r} - c_{r\delta} = 0$	$[x]_{\vec{p}} - [y]_{\vec{p}} = c_{\alpha l} - c_{ls} - c_{\gamma r} - c_{r\delta} = -2c$

Subcase 2.8: $\{\cdots (A_\alpha \cup A_l), (A_l \cup A_s) \cdots (A_\gamma \cup A_r)\}$

Subcase 2.8.1: $A_\gamma \in B$

If $A_\gamma \in B$, then we can swap our outcomes x and y for

$$x = (e_{(A_l - C(A_l))} o_{(A_\gamma - C(A_\gamma))} z_{C(A_l)} z_{C(A_\gamma)} z_{(B - A_l - A_\gamma)} u_{A_s} u_{([n]-B-A_s)})$$
$$y = (o_{(A_l - C(A_l))} e_{(A_\gamma - C(A_\gamma))} z_{C(A_l)} z_{C(A_\gamma)} z_{(B - A_l - A_\gamma)} u_{A_s} u_{([n]-B-A_s)})$$

For which the sibling chain structure will be covered by a previous case.

Subcase 2.8.2: $A_\gamma \notin B$

In both our outcomes x and y we make the outcome on A_α even, while maintaining agreement between x and y on this set.

u_{A_s} is even, u_{A_γ} is odd	u_{A_s} is even, u_{A_γ} is even
Only x benefits from $c_{\alpha l}$ Only x benefits from c_{ls} Only x benefits from $c_{\gamma r}$ $[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{l\beta} + c_{\gamma r} = c$	Only x benefits from $c_{\alpha l}$ Only x benefits from c_{ls} Only y benefits from $c_{\gamma r}$ $[x]_{\vec{p}} - [y]_{\vec{p}} = c_{sl} + c_{l\beta} + c_{\gamma r} = -c$

Conclusion

We have shown that we can always construct two outcomes x and y that agree on B for which the relation between them is depends the outcome on $u_{([n]-B)}$.

□

Chapter 5

Conclusion

In this thesis we presented a new method for representing, creating, and understanding preferences in binary contexts. Central to this method is the voter basis. Our primary results include Theorem 3.2.17 and Theorem 4.2.6.

Theorem 3.2.17 described how completely separable preference vectors are written in terms of the voter basis. The finding, that basis vectors indexed by nontrivial, even subsets have a 0 coefficient is significant and provides insight into the properties of completely separable preference orders. We hope this will aid researches in determining the number of such orders for arbitrary n.

Theorem 4.2.6 proves, by construction, that for any tree character, we can always building a corresponding preference vector. We note that the number of tree characters grows very quickly with n. Proving that this such a large class of characters is admissible and that we can systematically create preference vectors for them greatly justifies our new approach to character construction.

We see this new approach to understanding preference separability as opening many doors for further analysis. We are eager to pursue many questions. When decomposed by the voter basis, what other patterns can we find for completely separable preference vectors? Can we count the number of tree characters for arbitrary n? Can we develop other algorithms using the voter basis to construct families other than tree characters? What insight does are approach have on the question on the existence of inadmissible characters? Additionally, we acknowledge a connection to boolean term orders, what implications does such a connection imply?

Glossary

binary preference matrix A matrix describing an individual's strict preferences on referendum election outcomes. 7

cardinal preference vector A preference vector where entries are based on utility of indexing outcomes. 10

character The set of all separable sets for a given preference order. 15

election outcome An outcome corresponding to a referendum with n proposals. Can be understood as either a binary string or as the set of passing proposals. Denoted by lowercase letters . 7

Hasse diagram A graph describing the structure of a poset. 38

ordinal preference space A preference vector where only the order of preference is considered. 10

partial election outcome An outcome corresponding to a subset of proposals from an n proposal referendum. Can be understood as either a binary string or as the set of passing proposals. Denoted by lowercase letters with a subscript corresponding to relevant subset. Disjoint partial election outcomes can be concatenated to produce larger partial election outcomes, $x_S u_{[n]-S}$. 13

preference order A ranking over a set based on a preference relation. Denoted as $\succeq_P$ or $\succeq_[n]$, where P is the preference relation and $[n]$ is the set of proposals for a referendum. 6

preference relation A binary relation that is both complete and transitive. 6

preference space A 2^n dimensional vector space corresponding to a referendum with n proposals where entries of vectors are indexed by election outcomes. Denoted at P^n. 8

separability problem The existence of preference interdependence among proposals for an individual voter in a referendum election. 1

separable The quality of preference independence between a set and its complement. 14

tree character A character where the corresponding Hass diagram is a tree. 38

trivially separable The quality of preference independence between a set and its complement where all outcomes on the set are equally preferred. 14

Appendix A

$\mathbb{C}\mathbb{Z}_2 \wr S_n$-Modules

Note: The results of this section we proven by Tom Halverson [7].

The action defined in the previous section makes our preference space P^n a $\mathbb{C}\mathbb{Z}_2 \wr S_n$-module.

A.1 Set notation

Throughout this section, let $n \in \mathbb{Z}_{\geq 0}$ and define

$$[n] = \{1, 2, \ldots, n\} \tag{A.1}$$

$$2^{[n]} = \text{the set of all subsets of } [n] \tag{A.2}$$

$$\binom{[n]}{k} = \text{the set of all subsets of } [n] \text{ of order } k \tag{A.3}$$

so that $\left|2^{[n]}\right| = 2^n$ and $\left|\binom{[n]}{k}\right| = \binom{n}{k}$.

A.2 The hyperoctahedral group $\mathsf{G}_{2,n} = \mathbb{Z}_2 \wr \mathsf{S}_n$

Let $\mathsf{G}_{2,n} = \mathbb{Z}_2 \wr \mathsf{S}_n = \mathbb{Z}_2^n \rtimes \mathsf{S}_n$ be the hyperoctahdral group. It is the wreath product of $\mathbb{Z}_2$ with S_n and, equivalently, the semidirect product of $\mathbb{Z}_2^n$ with S_n. It is generated by the symmetric group S_n and the elements $\mathsf{t}_1, \ldots, \mathsf{t}_n$ which generate $\mathbb{Z}_2^n$. These can be represented as all monomial matrices with nonzero entries chosen from $\{1, -1\}$. Then S_n is the subgroup of all matrices whose nonzero entries are 1, and $\mathbb{Z}_2^n$ is the subgroup of diagonal matrices, and t_i is the diagonal matrix with -1 in position (i, i) and 1's in the other diagonal positions.

Generators and relations $\mathsf{G}_{2,n}$ is generated by $\mathsf{t}_1, \mathsf{s}_1, \ldots, \mathsf{s}_{n-1}$, where $\mathsf{s}_i = (i\, i_1)$, subject to the relations

$$\mathsf{s}_i^2 = 1, \qquad \mathsf{s}_i\mathsf{s}_j = \mathsf{s}_j\mathsf{s}_i, \qquad \mathsf{s}_i\mathsf{s}_{i+1}\mathsf{s}_i = \mathsf{s}_{i+1}\mathsf{s}_i\mathsf{s}_{i+1} \qquad (t_1 s_1)^4 = 1.$$

(need to put in appropriate bounds on subscripts)

Conjugacy classes

A.3 Irreducible $\mathbb{C}\mathsf{G}_{2,n}$ modules

The irreducible representations of $\mathsf{G}_{2,n}$ are indexed by pairs of partitions (λ, μ) with $|\lambda|+|\mu|= n$. We let $\mathsf{V}^{(\lambda,\mu)}$ denote the irreducible $\mathbb{C}\mathsf{G}_{2,n}$ module indexed by (λ, μ). A. Young has given an explicit description of the action of the generators of $\mathsf{G}_{2,n}$ on this module. This representation is known as Young's *seminormal* representation (see for example [15] or [6]). Let $\mathcal{S}_{\lambda,\mu}$ be the set of standard tableaux of shape (λ, μ). More needs to be said here. The basis of $\mathsf{V}^{(\lambda,\mu)}$ is indexed by

$$\mathsf{V}^{(\lambda,\mu)} = \mathbb{C}\text{-span}\{\ \mathsf{y}_t \mid t \in \mathcal{S}_{\lambda,\mu}\}$$

Action of generators: it would be nice to spell it out here.

A.4 One-line $\mathbb{C}\mathsf{G}_{2,n}$ modules

In the case where the two partitions λ and μ consist of a single part this representation is especially simple and elegant. Let $(\lambda, \mu) = ((k), (n-k))$, which we will simply denote as $(k, n-k)$. In this case, there is a natural bijection between: (i) standard tableaux of shape $(k, n-k)$; (ii) binary strings of length n with k ones; and (iii) subsets $\binom{[n]}{k}$ as illustrated here:

$(\boxed{1\,\vert\,2\,\vert\,3}, \boxed{4\,\vert\,5})$	$(\boxed{1\,\vert\,2\,\vert\,4}, \boxed{3\,\vert\,5})$	$(\boxed{1\,\vert\,2\,\vert\,5}, \boxed{3\,\vert\,4})$	$(\boxed{1\,\vert\,3\,\vert\,4}, \boxed{2\,\vert\,5})$	$(\boxed{1\,\vert\,3\,\vert\,5}, \boxed{2\,\vert\,4})$
11100	11010	11001	10110	10101
$\{1,2,3\}$	$\{1,2,4\}$	$\{1,2,5\}$	$\{1,3,4\}$	$\{1,3,5\}$
$(\boxed{1\,\vert\,4\,\vert\,5}, \boxed{2\,\vert\,3})$	$(\boxed{2\,\vert\,3\,\vert\,4}, \boxed{1\,\vert\,5})$	$(\boxed{2\,\vert\,3\,\vert\,5}, \boxed{1\,\vert\,4})$	$(\boxed{2\,\vert\,4\,\vert\,5}, \boxed{1\,\vert\,3})$	$(\boxed{3\,\vert\,4\,\vert\,5}, \boxed{1\,\vert\,2})$
10011	01110	01101	01011	00111
$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$	$\{2,4,5\}$	$\{3,4,5\}$

The symmetric group S_n acts naturally on subsets in $\binom{[n]}{k}$ by permutation of the elements of the subset. That is, if $A = \{a_1, a_2, \ldots, a_k\} \subseteq [n]$ and $\sigma \in \mathsf{S}_n$, then $\sigma(A) = \{\sigma(a_1), \sigma(a_2), \ldots, \sigma(a_k)\}$. For example if $\sigma = (1\,2\,5)(3\,4)$ in cycle notation, then in our three equivalent notations

$$\sigma((\boxed{1\,\vert\,2\,\vert\,3}, \boxed{4\,\vert\,5})) = (\boxed{2\,\vert\,4\,\vert\,5}, \boxed{1\,\vert\,3})$$

$$\sigma(11100) = 01011$$

$$\sigma(\{1,2,3\}) = \{2,4,5\}$$

For $0 \le k \le n$, let

$$\mathsf{V}^{(k,n-k)} = \mathbb{C}\text{-span}\left\{ \mathsf{y}_T \mid T \in \binom{[n]}{k} \right\}.$$

The seminormal action of $\mathsf{G}_{2,n}$ on this basis is given as follows. For any basis element y_t of $\mathsf{V}^{(k,n-k)}$ with $T \in \binom{[n]}{k}$, define

$$\begin{aligned} \sigma \cdot \mathsf{y}_T &= \mathsf{y}_{\sigma(T)}, && \text{for } \sigma \in \mathsf{S}_n, \\ \mathsf{t}_i \cdot \mathsf{y}_T &= \begin{cases} \mathsf{y}_T, & \text{if } i \in T, \\ -\mathsf{y}_T, & \text{if } i \notin T, \end{cases} && \text{for } 1 \le i \le n. \end{aligned} \tag{A.4}$$

Extend these actions linearly to all of $\mathsf{V}^{(k,n-k)}$ and extend them to all of $\mathsf{G}_{2,n}$ using the fact that $\sigma \in \mathsf{S}_n$ and $\mathsf{t}_i \in \mathbb{Z}_k^n$ generate $\mathsf{G}_{2,n}$. Observe that in both cases of (A.4), the cardinality of the subset is preserved and so $\mathsf{V}^{(k,n-k)}$ is $\mathsf{G}_{2,n}$-invariant. A. Young proves that this action makes $\mathsf{V}^{(k,n-k)}$ a $\mathsf{G}_{2,n}$ module and that it is irreducible. That it is a representation is easy enough; check the relations: $s_i^2 = 1, s_i s_j = s_j s_i, s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and $(t_1 s_1)^4 = 1$. Irreducibility is done by induction on n and the restriction from $\mathsf{G}_{2,n}$ to $\mathsf{G}_{2,n-1}$.

Note: This can all be done over the field $\mathbb{Q}$ for the hyperoctahedral group. But if we generalize to $\mathsf{G}_{r,n}$ then we will need to use $\mathbb{C}$ because the eigenvalues of the t_i are complex.

A.5 The Binomial Representation

Let W^n be the 2^n dimensional $\mathbb{C}$-vector space

$$\mathsf{W}^n = \mathbb{C}\text{-span}\left\{ \mathsf{w}_T \mid T \in 2^{[n]} \right\} \tag{A.5}$$

Define an action of $\mathsf{G}_{2,n}$ on W^n as follows. For any basis element w_T of W^n, with $T \in 2^{[n]}$, define

$$\begin{aligned} \sigma \cdot \mathsf{w}_T &= \mathsf{w}_{\sigma(t)}, && \text{for } \sigma \in \mathsf{S}_n, \\ \mathsf{t}_i \cdot \mathsf{w}_T &= \mathsf{w}_{\mathsf{t}_i(T)}, && \text{for } 1 \le i \le n, \end{aligned} \tag{A.6}$$

where

$$\mathsf{t}_i(T) = \begin{cases} T \cup \{i\}, & \text{if } i \notin T, \\ T \setminus \{i\}, & \text{if } i \in T. \end{cases} \tag{A.7}$$

The action in (A.6) is extended linearly to all of $\mathsf{V}^{(k,n-k)}$ and extended to all of $\mathsf{G}_{2,n}$ using the fact that $\sigma \in \mathsf{S}_n$ and $\mathsf{t}_i \in \mathbb{Z}_k^n$ generate $\mathsf{G}_{2,n}$.

Remark A.5.1. *A few observations.*

1. *If T is given in binary string notation then t_i "flips" the ith bit, and if T is given in tableau notation then t_i moves i from one tableau to the other.*

2. *The action of a permutation σ preserves the cardinality of the indexing subset T but the action of t_i either increases or decreases the cardinality of T by 1.*

Proposition A.5.2. *The action defined in* (A.6) *makes W^k a $\mathbb{C}\mathsf{G}_{2,n}$ module.*

Proof. We must check the defining relations. □

Change basis

Inside of W^k, for each $S \in 2^{[n]}$, we define,

$$\mathsf{u}_T = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, S)\, \mathsf{w_S} \tag{A.8}$$

where

$$\mathsf{sgn}(T, S) = (-1)^{|T \cup S|}. \tag{A.9}$$

For example, here is a table of the values of the coefficients $\mathsf{sgn}(T, S) = (-1)^{|T \cup S|}$ for $n = 3$ and using binary string notation for the subsets.

	111	110	101	011	100	010	001	000
111	1	1	1	1	1	1	1	1
110	1	−1	1	1	−1	−1	1	−1
101	1	1	−1	1	−1	1	−1	−1
011	1	1	1	−1	1	−1	−1	−1
100	1	−1	−1	1	1	−1	−1	1
010	1	−1	1	−1	−1	1	−1	1
001	1	1	−1	−1	−1	−1	1	1
000	1	−1	−1	−1	1	1	1	−1

Proposition A.5.3. *For $0 \le k \le n$ and $T \in \binom{[n]}{k}$, we have*

$$\begin{aligned} \sigma \cdot \mathsf{u}_T &= \mathsf{u}_{\sigma(T)}, && \text{for } \sigma \in \mathsf{S}_n, \\ \mathsf{t}_i \cdot \mathsf{u}_T &= \begin{cases} \mathsf{u}_T, & \text{if } i \in T, \\ -\mathsf{u}_T, & \text{if } i \notin T, \end{cases} && \text{for } 1 \le i \le n. \end{aligned} \tag{A.10}$$

Proof. First, observe that for any $\sigma \in \mathsf{S}_n$ and any $S, T \in 2^{[n]}$ we have $\mathsf{sgn}(\sigma(T), \sigma(S)) = \mathsf{sgn}(T, S)$. Thus

$$\sigma \cdot \mathsf{u}_T = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, S)\, \sigma \cdot \mathsf{w}_\mathsf{S} = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, S)\, \mathsf{w}_{\sigma(\mathsf{S})} = \sum_{S \in 2^{[n]}} \mathsf{sgn}(\sigma(T), \sigma(S))\, \mathsf{w}_{\sigma(\mathsf{S})} = \mathsf{u}_{\sigma(T)},$$

where the last equality comes from the fact that summing over S is equivalent to summing over $\sigma(S)$. Second, observe that if $i \in T$ then $\mathsf{sgn}(T, S) = \mathsf{sgn}(T, \mathsf{t}_i(S))$ and if $i \notin T$ then $\mathsf{sgn}(T, S) = -\mathsf{sgn}(T, \mathsf{t}_i(S))$, and these statements hold for all $S, T \in 2^{[n]}$. Thus,

$$t_i \cdot \mathsf{u}_T = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, S)\, t_i \cdot \mathsf{w}_\mathsf{S} = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, S)\, \mathsf{w}_{\mathsf{t}_i(\mathsf{S})} = \sum_{S \in 2^{[n]}} \mathsf{sgn}(T, \mathsf{t}_i(S))\, \mathsf{w}_{\mathsf{t}_i(\mathsf{S})}.$$

Now if $i \in T$ then this sum equals u_T and if $i \notin T$ then this sum equals $-\mathsf{u}_T$. This uses the fact that summing over S is equivalent to summing over $\mathsf{t}_i(S)$. (maybe we should write out both cases). □

Now, for each $0 \leq k \leq n$, define the subspace

$$\mathsf{W}^{(k,n-k)} = \mathbb{C}\text{-span}\left\{ \mathsf{u}_T \mid T \in \binom{[n]}{k} \right\} \subseteq \mathsf{W}^n. \tag{A.11}$$

Corollary A.5.4. *For each $0 \leq k \leq n$ the subspace $\mathsf{W}^{(k,n-k)}$ is an irreducible $\mathsf{G}_{2,n}$-module that is isomorphic to $\mathsf{V}^{(k,n-k)}$, the action of $\mathsf{G}_{2,n}$ on the basis $\{\mathsf{u}_T \mid T \in \binom{[n]}{k}\}$ is exactly Young's seminormal action* (A.10)*, and W^k decomposes into irreducible $\mathsf{G}_{2,n}$-modules as*

$$\mathsf{W}^k \cong \bigoplus_{k=0}^{n} \mathsf{W}^{(k,n-k)}. \tag{A.12}$$

In particular, for each $0 \leq k \leq n$ the module $\mathsf{W}^{(k,n-k)}$ appears as a submodule of W^k with multiplicity exactly 1.

Bibliography

[1] Brea Beals, Ian Calaway, and Jonathan Hodge. A graph theoretical approach to the admissibility problem, bead graphs and character construction. 2014.

[2] Beth Bjorkman, Sean Gravelle, and Jonathan Hodge. Graph theoretic models of interdependent prefences in referendum elections. 2013.

[3] W. James Bradley, Jonathan K. Hodge, and D. Marc Kilgour. Separable discrete preferences. *Mathematical Social Sciences*, 49(1):335–353, 2005.

[4] Steven J. Brams, D. Marc Kilgour, and William S. Zwicker. Voting on referenda: The separability problem and possible solutions. *Electoral Studies*, 16:359–377, 1997.

[5] Zajj Daugherty, Alexander K. Eustis, Gregory Minton, and Michael E. Orrison. Voting, the symmetric group, and representation theory. *The American Mathematical Monthly*, 116(8):667–687, 2009.

[6] T. Halverson and A. Ram. Murnaghan-Nakayama rules for characters of Iwahori-Hecke algebras of classical type. *Trans. Am. Math. Soc.*, 10:3967–3995, 1996.

[7] Thomas Halverson. personal communication. 2016.

[8] J. K. Hodge and R. E. Klima. *The Mathematics of Voting and Elections: A Hands-On Approach*. Volume 22 of Mathematical World. American Mathematical Society, Providence, 2005.

[9] Jonathan K. Hodge. The mathematics of referendums and separable preferences. *Mathematics Magazine*, 84(4):268–277, 2011.

[10] Jonathan K. Hodge, Mark Krines, and Jennifer Lahr. Preseparable extensions of multidimmensional preferences. *Order*, 26(2):125–147, 2009.

[11] Jonathan K. Hodge and Micah TerHaar. Classifying interdependence in multidimensional binary preferences. *Mathematical Social Sciences*, 55(2):190–204, 2008.

[12] Jeffrey A. Karp and Susan A. Banducci. Reforming the electoral college by initiative: Assessing the relative importance of strategic choice and proportional outcomes. *Party Politics*, 13(2):217–234, 2007.

[13] Dean Lacy and Emerson M.S. Niou. A problem with referendums. *Journal of Theoretical Politics*, 12(1):5–31, 2000.

[14] Stephen Lee. Understanding voting for committees using wreath products. Honors thesis, Harvey Mudd College, May 2010.

[15] A. Ram. Seminormal representations of Weyl groups and Iwahori-Hecke algebras. *Proc. London Math. Soc.*, 75(3):99–133, 1997.