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Non-additive Lie centralizer of strictly upper triangular matrices

Driss Aiat Hadj Ahmed

Centre Régional des Metiers d'Education et de Formation (CRMEF)
Tangier, Morocco

ait_hadj@yahoo.com

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Abstract: Let $\mathcal{F}$ be a field of zero characteristic, let $N_n(\mathcal{F})$ denote the algebra of $n \times n$ strictly upper triangular matrices with entries in $\mathcal{F}$, and let $f : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ be a non-additive Lie centralizer of $N_n(\mathcal{F})$, that is, a map satisfying that $f([X, Y]) = [f(X), Y]$ for all $X, Y \in N_n(\mathcal{F})$. We prove that $f(X) = \lambda X + \eta(X)$ where $\lambda \in \mathcal{F}$ and η is a map from $N_n(\mathcal{F})$ into its center $\mathcal{Z}(N_n(\mathcal{F}))$ satisfying that $\eta([X, Y]) = 0$ for every X, Y in $N_n(\mathcal{F})$.

Key words: Lie centralizer, strictly upper triangular matrices, commuting map.

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1. Introduction

Consider a ring R. An additive mapping $T : R \to R$ is called a left (respectively right) centralizer if $T(ab) = T(a)b$ (respectively $T(ab) = aT(b)$) for all $a, b \in R$. The map T is called a centralizer if it is a left and a right centralizer. The characterization of centralizers on algebras or rings has been a widely discussed subject in various areas of mathematics.

In [13] Zalar proved the following interesting result: if R is a 2-torsion free semiprime ring and T is an additive mapping such that $T(a^2) = T(a)a$ (or $T(a^2) = aT(a)$), then T is a centralizer. Vukman [12] considered additive maps satisfying similar conditions, namely $2T(a^2) = T(a)a + aT(a)$ for any $a \in R$, and showed that if R is a 2-torsion free semiprime ring then T is also a centralizer. Since then, the centralizers have been intensively investigated by many mathematicians (see, e.g., [3, 4, 5, 6, 8]).

Let R be a ring. An additive map $f : R \to R$, is called a Lie centralizer of R if

$$f([x, y]) = [f(x), y] \qquad \text{for all } x, y \in R, \tag{1.1}$$

where $[x, y]$ is the Lie product of x and y.

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Recently, Ghomanjani and Bahmani [9] dealt with the structure of Lie centralizers of trivial extension algebras, whereas Fošner and Jing [7] studied Lie centralizers of triangular rings.

The inspiration of this paper comes from the articles [1, 5, 7] in which the authors deal with the Lie centralizer maps of triangular algebras and rings. In this note we will consider non-additive Lie centralizers on strictly upper triangular matrices over a field of zero characteristic.

Throughout this article, $\mathcal{F}$ is a field of zero characteristic. Let $M_n(\mathcal{F})$ and $N_n(\mathcal{F})$ denote the algebra of all $n \times n$ matrices and the algebra of all $n \times n$ strictly upper triangular matrices over $\mathcal{F}$, respectively. We use diag(a_1,a_2, . . . ,a_n) to represent a diagonal matrix with diagonal $(a_1, a_2, \ldots, a_n)$ where $a_i \in \mathcal{F}$. The set of all $n \times n$ diagonal matrices over $\mathcal{F}$ is denoted by $D_n(\mathcal{F})$. Let I_n be the identity in $M_n(\mathcal{F})$, $J = \sum\limits_{i=1}^{n-1} E_{i,i+1}$ and $\{E_{ij} : 1 \leq i, j \leq n\}$ the canonical basis of $M_n(\mathcal{F})$, where E_{ij} is the matrix with 1 in the (i, j) position and zeros elsewhere. By $C_{N_n(\mathcal{F})}(X)$ we will denote the centralizer of the element X in the ring $N_n(\mathcal{F})$.

The notation $f : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ means a non-additive map satisfying $f([X, Y]) = [f(X), Y]$ for all $X, Y \in N_n(\mathcal{F}$.

Notice that it is easy to check that $\mathcal{Z}\left(N_n(\mathcal{F})\right) = \mathcal{F}E_{1n}$.

The main result in this paper is the following:

Theorem 1.1. *Let $\mathcal{F}$ be a field of zero characteristic. If $f : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ is a non-additive Lie centralizer then there exists $\lambda \in \mathcal{F}$ and a map $\eta : N_n(\mathcal{F}) \to \mathcal{Z}\left(N_n(\mathcal{F})\right)$ satisfying $\eta([X, Y]) = 0$ for every X, Y in $N_n(\mathcal{F})$ such that $f(X) = \lambda X + \eta\,(X)$ for all X in $N_n(\mathcal{F})$.*

Notice that the converse is trivially true: every map $f(X) = \lambda X + \eta\,(X)$ with η satisfying the condition in Theorem 1.1 is a (non-additive) Lie centralizer.

2. Proofs

Let's start with some basic properties of Lie centralizers.

Lemma 2.1. *Let f be a non-additive Lie centralizer of $N_n(\mathcal{F})$. Then:*

(1) $f(0) = 0$;

(2) *for every $X, Y \in N_n(\mathcal{F})$, we have $f([X, Y]) = [X, f(Y)]$;*

(3) *f is a commuting map, i.e., $f(X)X = Xf(X)$ for all $X \in N_n(\mathcal{F})$.*

Proof. To prove (1) it suffices to notice that

$$f(0) = f([0,0]) = [f(0), 0] = 0.$$

(2) Observe that if $f([X,Y]) = [f(X),Y]$, then we have

$$f(XY - YX) = f(X)Y - Yf(X).$$

Interchanging X and Y in the above identity, we have

$$f(YX - XY) = f(Y)X - Xf(Y).$$

Replacing X with $-X$ in the above relation, we arrive at $f(XY - YX) = Xf(Y) - f(Y)X$ which can be written as $f([X,Y]) = [X, f(Y)]$.

From (1) one also gets (3):

$$[f(X), X] = f([X,X]) = f(0) = 0.$$

∎

Remark 2.1. Let f be a non-additive Lie centralizer of $N_n(\mathcal{F})$ and $X \in C_{N_n(\mathcal{F})}(Y)$. Then $f(X) \in C_{N_n(\mathcal{F})}(Y)$. Indeed, if $X \in C_{N_n(\mathcal{F})}(Y)$, then $[X,Y] = 0$ and

$$0 = f(0) = f([X,Y]) = [f(X),Y].$$

LEMMA 2.2. *Let f be a non-additive Lie centralizer of $N_n(\mathcal{F})$. Then:*

(1) $f\left(\sum_{i=1}^{n-1} a_i E_{i,i+1}\right) = \sum_{i=1}^{n-1} b_i E_{i,i+1}$;

(2) *there exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$.*

Proof. Let $D_0 = \sum_{i=1}^{n} (n-i)\, E_{i,i}$.

(1) Consider $A \in M_n(\mathcal{F})$. It is well known that $[D_0, A] = A$ if and only if $A = \sum_{i=1}^{n-1} a_i E_{i,i+1}$.

Hence, if $A = \sum_{i=1}^{n-1} a_i E_{i,i+1}$, we have $[D_0, A] = A$. Thus $f([D_0, A]) = [D_0, f(A)] = f(A)$. Therefore $f(A) = \sum_{i=1}^{n-1} b_i E_{i,i+1}$.

(2) As in (1), consider $A = \sum_{i=1}^{n-1} a_i E_{i,i+1}$ for some $a_i \in \mathcal{F}$. Then $[J, A] = 0$ if and only if $A = aJ$ for some $a \in \mathcal{F}$.

Indeed, $f(J) = \sum_{i=1}^{n-1} a_i E_{i,i+1}$ by (1). Thus, $0 = f(0) = f([J, J]) = [J, f(J)]$. Hence, there exists $\lambda \in \mathcal{F}$ such that $f(J) = \lambda J$. ∎

We will need the following lemma.

Lemma 2.3. (Lemma 2.1, [14]) *Suppose that $\mathcal{F}$ is an arbitrary field. If $G, H \in UT_n(\mathcal{F})$ are such that $g_{i,i+1} = h_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, then G and H are conjugated in $UT_n(\mathcal{F})$.*

Here $UT_n(\mathcal{F})$ is the multiplicative group of $n \times n$ upper triangular matrices with only 1's in the main diagonal. From the lemma above we obtain the following corollary.

Corollary 2.1. *Let $\mathcal{F}$ be a field. For every $A = \sum_{1 \leq i < j \leq n} a_{ij} E_{ij}$, where $a_{i,i+1} \neq 0$ for all $1 \leq i \leq n-1$, there exists $B \in T_n(\mathcal{F})$ such that $B^{-1}AB = J$ and $T_n(\mathcal{F})$ is the ring of upper triangular matrices.*

Proof. Let A be a matrix in $N_n(\mathcal{F})$ of the mentioned form. Then $I_n + A$ is a unitriangular matrix. Let's notice first that there exists $B_1 \in D_n(\mathcal{F})$ such that $(B_1^{-1}AB_1)_{i,i+1} = 1$ for all $i \in \mathbb{N}$. We can construct $B_1 \in D_n(\mathcal{F})$ recursively by:

$$(B_1)_{11} = 1, \qquad (B_1)_{i+1,i+1} = (B_1)_{ii} \cdot (A_{i,i+1})^{-1} \quad \text{for } i \geq 1.$$

Consider the matrix $I_n + B_1^{-1}AB \ \in UT_n(\mathcal{F})$. The unitriangular matrices $I_n + J$ and $I_n + B_1^{-1}AB$ fulfill the condition in Lemma 2.3. Hence, there exists $B_2 \in UT_n(\mathcal{F})$ such that

$$I_n + J = B_2^{-1}(I_n + B_1^{-1}AB_1)B_2.$$

Then $J = B_2^{-1}(B_1^{-1}AB_1)B_2$. Taking $B = B_1B_2 \in T_n(\mathcal{F})$, we get $J = B^{-1}AB$ as wanted. ∎

Lemma 2.4. *Let $A = \sum_{i<j} a_{ij} E_{ij}$ be a matrix in $N_n(\mathcal{F})$ with $a_{i,i+1} \neq 0$ for every $i = 1, \ldots, n-1$. Then there exists $\lambda_A \in \mathcal{F}$ such that $f(A) = \lambda_A A$.*

Proof. Since $A = \sum_{1 \le i < j \le n} a_{ij} E_{ij}$, where $a_{i,i+1} \ne 0$, there exists $T \in T_n(\mathcal{F})$ such that $TAT^{-1} = J$ by the previous corollary. Define $h : N_n(\mathcal{F}) \to N_n(\mathcal{F})$ by $h(X) = Tf(T^{-1}XT)T^{-1}$. Then h is a non-additive Lie centralizer. Indeed, for all $A, B \in N_n(\mathcal{F})$ we have:

$$\begin{aligned}
h([A,B]) &= Tf\left(T^{-1}[A,B]T\right)T^{-1} \\
&= Tf\left(T^{-1}(AB - BA)T\right)T^{-1} \\
&= Tf\left(T^{-1}ATT^{-1}BT - T^{-1}BTT^{-1}AT\right)T^{-1} \\
&= Tf\left(\left[T^{-1}AT, T^{-1}BT\right]\right)T^{-1} \\
&= T\left[f\left(T^{-1}AT\right), T^{-1}BT\right]T^{-1} \\
&= T\left(f\left(T^{-1}AT\right)T^{-1}BT - T^{-1}BTf\left(T^{-1}AT\right)\right)T^{-1} \\
&= Tf\left(T^{-1}AT\right)T^{-1}B - BTf\left(T^{-1}AT\right)T^{-1} \\
&= \left[Tf\left(T^{-1}AT\right)T^{-1}, B\right] \\
&= [h(A), B].
\end{aligned}$$

Hence, $h(J) = \lambda_A J$ by Lemma 2.2. Then

$$Tf(A)T^{-1} = Tf(T^{-1}(TAT^{-1})T)T^{-1} = h(J) = \lambda_A J = \lambda_A TAT^{-1}.$$

Multiplying the left and right sides by T^{-1} and T respectively yields $f(A) = \lambda_A A$. ∎

Now we wish to extend Lemma 2.4 to all elements of $N_n(\mathcal{F})$. In order to do this, let's introduce the following set:

$$\mathcal{S} = \{B = (b_{ij}) \in N_n(\mathcal{F}) \,:\, b_{i,i+1} \ne 0 \quad \forall\, i = 1, \dots, n-1\}.$$

This set has an important property that is established below.

Lemma 2.5. *Let $\mathcal{F}$ be a field. Every element of $N_n(\mathcal{F})$ can be written as a sum of at most two elements of $\mathcal{S}$.*

Proof. If $a_{i,i+1} \ne 0$ for all $i = 1, \dots, n-1$, then A belongs to $\mathcal{S}$, so there is nothing to prove. If A is not in $\mathcal{S}$, then we can define B_1 and B_2 as follows:

$$(B_1)_{ij} = \begin{cases} a_{i,i+1} - b_i & \text{if } j = i+1, \\ a_{ij} & \text{if } j > i+1, \end{cases} \qquad (B_2)_{ij} = \begin{cases} b_i & \text{if } j = i+1, \\ 0 & \text{otherwise}, \end{cases}$$

where b_i is an element in $\mathcal{F}$ different from $a_{i,i+1}$. It is easy to see that B_1, B_2 are in $\mathcal{S}$, and $A = B_1 + B_2$, so we wanted. ∎

Lemma 2.6. *Let $\mathcal{F}$ be a field. For arbitrary elements A, B of $N_n(\mathcal{F})$, there exists $\lambda_{A,B} \in F$ such that*

$$f(A+B) = f(A) + f(B) + \lambda_{A,B} E_{1n}.$$

Proof. For any A, B, X of $N_n(\mathcal{F})$, we have

$$\begin{aligned} [f(A+B), X] &= f([A+B, X]) \\ &= [A+B, f(X)] \\ &= [A, f(X)] + [B, f(X)] \\ &= [f(A), X] + [f(B), X] \\ &= [f(A)+f(B), X], \end{aligned}$$

which implies that $f(A+B) - f(A) - f(B) \in \mathcal{Z}(N_n(\mathcal{F}))$. Thus, there exists $\lambda_{A,B} \in \mathcal{F}$ such that $f(A+B) = f(A) + f(B) + \lambda_{A,B} E_{1n}$. ∎

Now we can prove the main theorem.

Proof of Theorem 1.1. Let A, $B \in \mathcal{S}$ be two non-commuting elements. By Lemma 2.4, $f(A) = \lambda_A A$, $f(B) = \lambda_B B$, $\lambda_A, \lambda_B \in \mathcal{F}$.

Since f is a non-additive Lie centralizer, we get,

$$\begin{aligned} f([A, B]) = [f(A), B] &= \lambda_A [A, B] \\ = [A, f(B)] &= \lambda_B [A, B]. \end{aligned}$$

Then, $[A, B] \neq 0$ implies that $\lambda_A = \lambda_B$.

If A, $B \in \mathcal{S}$ commute, then we take $C \in \mathcal{S}$ that does not commute neither with A nor with B. As we have just seen, $\lambda_A = \lambda_C$ and $\lambda_B = \lambda_C$. So $\lambda_A = \lambda_B = \lambda$ for arbitrary elements $A, B \in \mathcal{S}$. Given $X \in N_n(\mathcal{F})$ we know, by Lemma 2.5, that there exists $A, B \in \mathcal{S}$ such that $X = A + B$ (we can assume that $X \notin S$). Then $f(X) - f(A) - f(B) \in \mathcal{Z}(N_n(\mathcal{F}))$ by Lemma 2.6.

That is $f(X) - \lambda_A A - \lambda_B B = f(X) - \lambda X \in \mathcal{Z}(N_n(\mathcal{F}))$ for $\lambda \in \mathcal{F}$ such that $f(A) = \lambda A$ for each $A \in S$.

We can define $\eta : N_n(\mathcal{F}) \to \mathcal{Z}(N_n(\mathcal{F}))$ such that $\eta(X) = f(X) - \lambda X$, that is, $f(X) = \lambda X + \eta(X)$.

Notice that $\eta(A) = 0$ for each $A \in S$. Furthermore, if $X, Y \in N_n(\mathcal{F})$, then

$$\begin{aligned} f([X, Y]) &= \lambda [X, Y] + \eta([X, Y]) = [f(X), Y] \\ &= [\lambda X + \eta(X), Y] = \lambda [X, Y], \end{aligned}$$

since $\eta(X) \in \mathcal{Z}(N_n(\mathcal{F}))$.

Consequently, $\eta([X, Y]) = 0$ and Theorem 1.1 is proved. ∎

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