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Jordan Triple Elementary Maps on Alternative Rings

J.C.M. Ferreira, B.L.M. Ferreira, H. Guzzo Jr*

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC,
Av. dos Estados 5001, 09210–580, *Santo André, Brazil*

Coordenação de Engenharia Mecânica, Universidade Tecnológica Federal do Paraná,
Av. Professora Laura Pacheco Bastos 800, 85053–510, *Guarapuava, Brazil*

Instituto de Matemática e Estatística, Universidade de São Paulo,
Rua do Matão 1010, 05508–090, *São Paulo, Brazil*

joao.cmferreira@ufabc.edu.br , brunoferreira@utfpr.edu.br , guzzo@ime.usp.br

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Abstract: Let $\mathfrak{R}$ and $\mathfrak{R}'$ be alternative rings. We study the additivity of surjective Jordan triple elementary maps of $\mathfrak{R} \times \mathfrak{R}'$. We prove that if $\mathfrak{R}$ contains a non-trivial idempotent satisfying some conditions, then they are additives.

Key words: Jordan triple elementary maps, alternative rings, additivity.

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1. Alternative rings and Jordan triple elementary maps

Let $\mathfrak{R}$ be a ring not necessarily associative or commutative and consider the following convention for its multiplication operation: that $xy \cdot z = (xy)z$ and $x \cdot yz = x(yz)$ for $x, y, z \in \mathfrak{R}$, to the reduction in the number of necessary parentheses. We denote the *associator* of $\mathfrak{R}$ by $(x, y, z) = xy \cdot z - x \cdot yz$ for $x, y, z \in \mathfrak{R}$.

Let $\mathfrak{X} = \{x_i\}_{i\in\mathbb{N}}$ be an arbitrary set of variables. A *non-associative monomial of degree* 1 is any element of $\mathfrak{X}$. Given a natural number $n > 1$, a *non-associative monomial of degree* n is an expression of the form $(u)(v)$, where u is a non-associative monomial of some degree i and v a non-associative monomial of degree $n - i$. A *non-associative polynomial* f over a ring $\mathfrak{R}$ is any formal linear combination of non-associative monomials with coefficients in $\mathfrak{R}$. If f includes no variables except $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ is a set of elements of $\mathfrak{R}$, then $f(a_1, a_2, \ldots, a_n)$ is an element of $\mathfrak{R}$ which results

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by applying the sequence of operations forming f to $a_1, a_2, \ldots, a_n$ in place of $x_1, x_2, \ldots, x_n$.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two rings and let $M : \mathfrak{R} \to \mathfrak{R}'$ and $M^* : \mathfrak{R}' \to \mathfrak{R}$ be two maps. We call the ordered pair (M, M^*) a *Jordan triple elementary map of* $\mathfrak{R} \times \mathfrak{R}'$ if for all non-associative monomial $f = f(x_1, x_2, x_3)$ of degree 3

$$\begin{aligned} &M\big(f\big(a, M^*(x), b\big) + f\big(b, M^*(x), a\big)\big) \\ &\qquad\qquad = f\big(M(a), x, M(b)\big) + f\big(M(b), x, M(a)\big)\,, \\ &M^*\big(f\big(x, M(a), y\big) + f\big(y, M(a), x\big)\big) \\ &\qquad\qquad = f\big(M^*(x), a, M^*(y)\big) + f\big(M^*(y), a, M^*(x)\big) \end{aligned}$$

for all $a, b \in \mathfrak{R}$ and $x, y \in \mathfrak{R}'$.

We say that a Jordan triple elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is *additive* (resp., *injective, surjective, bijective*) if both maps M and M^* are additive (resp., injective, surjective, bijective).

A ring $\mathfrak{R}$ is said to be *alternative* if $(x, x, y) = 0 = (y, x, x)$ for all $x, y \in \mathfrak{R}$. One easily sees that any associative ring is an alternative ring.

An alternative ring $\mathfrak{R}$ is called *k-torsion free* if $k\,x = 0$ implies $x = 0$, for any $x \in \mathfrak{R}$, where $k \in \mathbb{Z}$, $k > 0$, and *prime* if $\mathfrak{A}\mathfrak{B} \neq 0$ for any two nonzero ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{R}$.

Let us consider $\mathfrak{R}$ an alternative ring and fix a nontrivial idempotent $e_1 \in \mathfrak{R}$, i.e., $e_1^2 = e_1$, $e_1 \neq 0$ and e_1 is not a unity element. Let $e_2 \colon \mathfrak{R} \to \mathfrak{R}$ and $e_2' \colon \mathfrak{R} \to \mathfrak{R}$ be linear operators given by $e_2(a) = a - e_1 a$ and $e_2'(a) = a - a e_1$. Clearly $e_2^2 = e_2$, $(e_2')^2 = e_2'$ and we denote $e_2(a)$ by $e_2 a$ and $e_2'(a)$ by $a e_2$. Let us note that if $\mathfrak{R}$ has a unity, then we can consider $e_2 (= e_2') = 1 - e_1 \in \mathfrak{R}$. It is easy to see that $e_i a \cdot e_j = e_i \cdot a e_j$ $(i, j = 1, 2)$ for all $a \in \mathfrak{R}$. Then $\mathfrak{R}$ has a Peirce decomposition $\mathfrak{R} = \mathfrak{R}_{11} \oplus \mathfrak{R}_{12} \oplus \mathfrak{R}_{21} \oplus \mathfrak{R}_{22}$, where $\mathfrak{R}_{ij} = e_i \mathfrak{R} e_j$ $(i, j = 1, 2)$, (see [3]) satisfying the multiplicative relations:

(i) $\mathfrak{R}_{ij}\mathfrak{R}_{jl} \subseteq \mathfrak{R}_{il}$ $\;(i, j, l = 1, 2)$;

(ii) $\mathfrak{R}_{ij}\mathfrak{R}_{ij} \subseteq \mathfrak{R}_{ji}$ $\;(i, j = 1, 2)$;

(iii) $\mathfrak{R}_{ij}\mathfrak{R}_{kl} = 0$ if $j \neq k$ and $(i, j) \neq (k, l)$ $\;(i, j, k, l = 1, 2)$;

(iv) $x_{ij}^2 = 0$ for all $x_{ij} \in \mathfrak{R}_{ij}$ $\;(i, j = 1, 2;\; i \neq j)$.

For the case of alternative rings the notion of Jordan triple elementary map takes the following equivalent form:

PROPOSITION 1.1. *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two alternative rings and let $M : \mathfrak{R} \to \mathfrak{R}'$ and $M^* : \mathfrak{R}' \to \mathfrak{R}$ be two maps. The following assertions are equivalent:*

(i) *the ordered pair (M, M^*) is a Jordan triple elementary map of $\mathfrak{R} \times \mathfrak{R}'$;*

(ii) *there is a non-associative monomial $f = f(x_1, x_2, x_3)$ of degree 3 such that*

$$\begin{aligned} &M\big(f\big(a, M^*(x), b\big) + f\big(b, M^*(x), a\big)\big) \\ &\qquad\qquad = f\big(M(a), x, M(b)\big) + f\big(M(b), x, M(a)\big), \\ &M^*\big(f\big(x, M(a), y\big) + f\big(y, M(a), x\big)\big) \\ &\qquad\qquad = f\big(M^*(x), a, M^*(y)\big) + f\big(M^*(y), a, M^*(x)\big) \end{aligned}$$

for all $a, b \in \mathfrak{R}$ and $x, y \in \mathfrak{R}'$.

According to [5], *"The first result about the additivity of maps on rings was given by Martindale III in an excellent paper [6]. He established a condition on an associative ring $\mathfrak{R}$ such that every multiplicative bijective map on $\mathfrak{R}$ is additive"*. Jing [5] considered also the investigation of the additivity of maps for the case of Jordan triple elementary maps on associative rings. He proved the following theorem.

THEOREM 1.1. (JING [5]) *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two associative rings. Suppose that $\mathfrak{R}$ is a 2-torsion free unital ring containing a nontrivial idempotent e_1 and satisfies*

$$e_i a e_j \mathfrak{R} e_k = 0 \;\text{ or }\; e_k \mathfrak{R} e_i a e_j = 0 \quad \Rightarrow \quad e_i a e_j = 0 \qquad (1 \le i, j, k \le 2)\,.$$

Then every surjective Jordan triple elementary map (M, M^*) of $\mathcal{R} \times \mathcal{R}'$ is additive.

The hypotheses of the Jing's Theorem [5] allowed the author to make its proof based on calculus using the Peirce decomposition notion for associative rings.

The notion of Peirce decomposition for the alternative rings is similar to the notion of Peirce decomposition for the associative rings. However, the similarity of this notion is only in its written form, but not in its theoretical structure because the Peirce decomposition for alternative rings is the generalization of the Peirce decomposition for associative rings. Taking this fact into account, in the present paper we generalize the main Jing's Theorem [5] to the class of alternative rings. For this, we adopt and follow the same structure of the demonstration presented in [5], in order: to preserve the author's ideas and to highlight the generalization of the associative results to the alternative results. Therefore, our lemmas and the theorem that seem to be equal in written form with the lemmas and the theorem proposed in Jing [5], are distinguished by a fundamental item: the use of the non-associative multiplications. The symbol "$\cdot$", as defined in the introduction section of our article, is essential to elucidate how the non-associative multiplication should be done, and also the symbol "$\cdot$" is used to simplify the notation. Therefore, the symbol "$\cdot$" is crucial to the logic, characterization and generalization of associative results to the alternative results.

2. The main result

Let's state the main result of this paper.

Theorem 2.1. *Let $\mathfrak{R}$ and $\mathfrak{R}'$ be two alternative rings. Suppose that $\mathfrak{R}$ is a 2-torsion free unital ring containing a nontrivial idempotent e_1 and satisfies:*

(i) $(e_iae_j)\cdot\mathfrak{R}e_k = 0$ *or* $(e_iae_j)\mathfrak{R}\cdot e_k = 0 \;\Rightarrow\; e_iae_j = 0 \;\; (1 \leq i,j,k \leq 2)$;

(ii) $e_k\mathfrak{R}\cdot(e_iae_j) = 0$ *or* $e_k\cdot\mathfrak{R}(e_iae_j) = 0 \;\Rightarrow\; e_iae_j = 0 \;\; (1 \leq i,j,k \leq 2)$.

Then every surjective Jordan elementary map (M, M^*) of $\mathfrak{R}\times\mathfrak{R}'$ is additive.

To prove Theorem 2.1 we introduced a set of lemmas, similar to introduced by Jing [5]. The first one is:

Lemma 2.1. $M(0) = 0$ *and* $M^*(0) = 0$.

Proof. $M(0) = M\big(0M^*(0)\cdot 0 + 0M^*(0)\cdot 0\big) = M(0)0\cdot M(0) + M(0)0\cdot M(0) = 0$. Similarly, we prove $M^*(0) = 0$. ∎

The following lemma is verified by direct calculations, from the conditions (i) and (ii) of Theorem 2.1.

Lemma 2.2. *Let $a = a_{11} + a_{12} + a_{21} + a_{22} \in \mathfrak{R}$.*

(i) *If $a_{ij}t_{jk} = 0$ for each $t_{jk} \in \mathfrak{R}_{jk}$ $(1 \leq i,j,k \leq 2)$, then $a_{ij} = 0$. Dually, if $t_{ki}a_{ij} = 0$ for each $t_{ki} \in \mathfrak{R}_{ki}$ $(1 \leq i,j,k \leq 2)$, then $a_{ij} = 0$.*

(ii) *If $a_{ij}t + ta_{ij} \in \mathfrak{R}_{ij}$ for every $a_{ij} \in \mathfrak{R}_{ij}$ $(1 \leq i,j \leq 2)$, then $t_{ji} = 0$.*

(iii) *If* $a_{ii}t + ta_{ii} = 0$ *for every* $a_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$*, then* $t_{ii} = 0$.

(iv) *If* $a_{jj}t + ta_{jj} \in \mathfrak{R}_{ij}$ *for every* $a_{jj} \in \mathfrak{R}_{jj}$ $(1 \le i \ne j \le 2)$*, then* $t_{ji} = 0$ *and* $t_{jj} = 0$*. Dually, if* $a_{jj}t + ta_{jj} \in \mathfrak{R}_{ji}$ *for every* $a_{jj} \in \mathfrak{R}_{jj}$ $(1 \le i \ne j \le 2)$*, then* $t_{ij} = 0$ *and* $t_{jj} = 0$*.*

LEMMA 2.3. *M and M^* are bijective.*

Proof. Suppose that $M(a) = M(b)$ for arbitrary elements $a, b \in \mathfrak{R}$ and let us write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. For arbitraries $t_{ij} \in \mathfrak{R}_{ij}$ and $s_{kl} \in \mathfrak{R}_{kl}$ $(1 \le i, j, k, l \le 2)$, there are $x(i, j) \in \mathfrak{R}'$ and $y(k, l) \in \mathfrak{R}'$ such that $M^*\big(x(i, j)\big) = t_{ij}$ and $M^*\big(y(k, l)\big) = s_{kl}$ since M^* is surjective. It follows that

$$\begin{aligned}
s_{kl}a \cdot t_{ij} + t_{ij}a \cdot s_{kl} &= M^*\big(y(k,l)\big)a \cdot M^*\big(x(i,j)\big) + M^*\big(x(i,j)\big)a \cdot M^*\big(y(k,l)\big) \\
&= M^*\big(y(k,l)M(a) \cdot x(i,j) + x(i,j)M(a) \cdot y(k,l)\big) \\
&= M^*\big(y(k,l)M(b) \cdot x(i,j) + x(i,j)M(b) \cdot y(k,l)\big) \\
&= M^*\big(y(k,l)\big)b \cdot M^*\big(x(i,j)\big) + M^*\big(x(i,j)\big)b \cdot M^*\big(y(k,l)\big) \\
&= s_{kl}b \cdot t_{ij} + t_{ij}b \cdot s_{kl}\,.
\end{aligned}$$

Taking $i = j = k = 1$ and $l = 2$ we have $s_{12}a \cdot t_{11} + t_{11}a \cdot s_{12} = s_{12}b \cdot t_{11} + t_{11}b \cdot s_{12}$, which implies that $a_{11} = b_{11}$ and $a_{21} = b_{21}$, by directness of the Peirce decomposition and Lemma 2.2-(i). Now, if we take $i = k = l = 2$ and $j = 1$, we obtain $a_{12} = b_{12}$ and $a_{22} = b_{22}$. This implies that $a = b$ and hence M is injective. Next, let $x, y \in \mathfrak{R}'$ such that $M^*(x) = M^*(y)$. For arbitrary elements $t_{ij} \in \mathfrak{R}_{ij}$ and $s_{kl} \in \mathfrak{R}_{kl}$, there are $c(i, j) \in \mathfrak{R}$ and $d(k, l) \in \mathfrak{R}$ such that $M^*M\big(c(i, j)\big) = t_{ij}$ and $M^*M\big(d(k, l)\big) = s_{kl}$, by the surjectivity of M^*M. It follows that

$$\begin{aligned}
&t_{ij}M^{-1}(x) \cdot s_{kl} \;+\; s_{kl}M^{-1}(x) \cdot t_{ij} \\
&\qquad = M^*M\big(c(i,j)\big)M^{-1}(x) \cdot M^*M\big(d(k,l)\big) \\
&\qquad\qquad + M^*M\big(d(k,l)\big)M^{-1}(x) \cdot M^*M\big(c(i,j)\big) \\
&\qquad = M^*\Big(M\big(c(i,j)\big)x \cdot M\big(d(k,l)\big) + M\big(d(k,l)\big)x \cdot M\big(c(i,j)\big)\Big) \\
&\qquad = M^*M\big(c(i,j)M^*(x) \cdot d(k,l) + d(k,l)M^*(x) \cdot c(i,j)\big) \\
&\qquad = M^*M\big(c(i,j)M^*(y) \cdot d(k,l) + d(k,l)M^*(y) \cdot c(i,j)\big)
\end{aligned}$$

$$
\begin{aligned}
&= M^*\Big(M\big(c(i,j)\big)y \cdot M\big(d(k,l)\big) + M\big(d(k,l)\big)y \cdot M\big(c(i,j)\big)\Big)\\
&= M^*M\big(c(i,j)\big)M^{-1}(y) \cdot M^*M\big(d(k,l)\big)\\
&\quad\; + M^*M\big(d(k,l)\big)M^{-1}(y) \cdot M^*M\big(c(i,j)\big)\\
&= \; t_{ij}M^{-1}(y) \cdot s_{kl} + s_{kl}M^{-1}(y) \cdot t_{ij}\,.
\end{aligned}
$$

Taking a same argument as in the first case, we can conclude that $M^{-1}(x) = M^{-1}(y)$. It follows that $x = y$ and hence M^* is injective.

Since M and M^* are also surjective, then both are bijective. ∎

Lemma 2.4. *The pair* $(M^{*^{-1}}, M^{-1})$ *is a Jordan triple elementary map on* $\mathfrak{R} \times \mathfrak{R}'$.

Proof. For arbitrary elements $a, b \in \mathfrak{R}$ and $x, y \in \mathfrak{R}'$, we have

$$
\begin{aligned}
M^*\big(&M^{*^{-1}}(a)x \cdot M^{*^{-1}}(b) + M^{*^{-1}}(b)x \cdot M^{*^{-1}}(a)\big)\\
&= M^*\big(M^{*^{-1}}(a)MM^{-1}(x) \cdot M^{*^{-1}}(b)\\
&\quad\; + M^{*^{-1}}(b)MM^{-1}(x) \cdot M^{*^{-1}}(a)\big)\\
&= M^*M^{*^{-1}}(a)M^{-1}(x) \cdot M^*M^{*^{-1}}(b)\\
&\quad\; + M^*M^{*^{-1}}(b)M^{-1}(x) \cdot M^*M^{*^{-1}}(a)\\
&= aM^{-1}(x) \cdot b + bM^{-1}(x) \cdot a\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
M^{*^{-1}}&\big(aM^{-1}(x) \cdot b + bM^{-1}(x) \cdot a\big)\\
&= M^{*^{-1}}(a)x \cdot M^{*^{-1}}(b) + M^{*^{-1}}(b)x \cdot M^{*^{-1}}(a)\,,
\end{aligned}
$$

by Lemma 2.3. Similarly, we prove that

$$
\begin{aligned}
M^{-1}&\big(xM^{*^{-1}}(a) \cdot y + yM^{*^{-1}}(a) \cdot x\big)\\
&= M^{-1}(x)a \cdot M^{-1}(y) + M^{-1}(y)a \cdot M^{-1}(x)\,.
\end{aligned}
$$

By Proposition 1.1, we infer that the pair $(M^{*^{-1}}, M^{-1})$ is a Jordan triple elementary map on $\mathfrak{R} \times \mathfrak{R}'$. ∎

LEMMA 2.5. *Let $a, b, c \in \mathfrak{R}$ such that $M(c) = M(a) + M(b)$. Then*

$$M^{*^{-1}}(tc \cdot s + sc \cdot t) = M^{*^{-1}}(ta \cdot s + sa \cdot t) + M^{*^{-1}}(tb \cdot s + sb \cdot t)$$

for all $t, s \in \mathfrak{R}$.

Proof. For arbitrary elements $t, s \in \mathfrak{R}$, by Lemma 2.4, we have

$$\begin{aligned}
M^{*^{-1}}(tc \cdot s + sc \cdot t) &= M^{*^{-1}}\big(tM^{-1}M(c) \cdot s + sM^{-1}M(c) \cdot t\big) \\
&= M^{*^{-1}}(t)M(c) \cdot M^{*^{-1}}(s) + M^{*^{-1}}(s)M(c) \cdot M^{*^{-1}}(t) \\
&= M^{*^{-1}}(t)\big(M(a) + M(b)\big) \cdot M^{*^{-1}}(s) \\
&\quad + M^{*^{-1}}(s)\big(M(a) + M(b)\big) \cdot M^{*^{-1}}(t) \\
&= M^{*^{-1}}(t)M(a) \cdot M^{*^{-1}}(s) + M^{*^{-1}}(s)M(a) \cdot M^{*^{-1}}(t) \\
&\quad + M^{*^{-1}}(t)M(b) \cdot M^{*^{-1}}(s) + M^{*^{-1}}(s)M(b) \cdot M^{*^{-1}}(t) \\
&= M^{*^{-1}}(ta \cdot s + sa \cdot t) + M^{*^{-1}}(tb \cdot s + sb \cdot t)\,,
\end{aligned}$$

and the lemma is proved. ∎

LEMMA 2.6. *For arbitraries $a_{11} \in \mathfrak{R}_{11}$ and $b_{22} \in \mathfrak{R}_{22}$, we have:*

(i) $M(a_{11} + b_{22}) = M(a_{11}) + M(b_{22})$;
(ii) $M^{*^{-1}}(a_{11} + b_{22}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{22})\,.$

Proof. Suppose that $M(c) = M(a_{11}) + M(a_{22})$ for some $c \in \mathfrak{R}$ and let us write $c = c_{11} + c_{12} + c_{21} + c_{22}$. For arbitrary elements $t_{21} \in \mathfrak{R}_{21}$ and $s_{11} \in \mathfrak{R}_{11}$, by Lemma 2.5 we have

$$\begin{aligned}
M^{*^{-1}}(t_{21}c \cdot s_{11} + s_{11}c \cdot t_{21}) &= M^{*^{-1}}(t_{21}a_{11} \cdot s_{11} + s_{11}a_{11} \cdot t_{21}) \\
&\quad + M^{*^{-1}}(t_{21}b_{22} \cdot s_{11} + s_{11}b_{22} \cdot t_{21}) \\
&= M^{*^{-1}}(t_{21}a_{11} \cdot s_{11})\,.
\end{aligned}$$

This implies that $t_{21}c \cdot s_{11} + s_{11}c \cdot t_{21} = t_{21}a_{11} \cdot s_{11}$. It follows that $t_{21}c_{11} \cdot s_{11} + s_{11}c_{12} \cdot t_{21} = t_{21}a_{11} \cdot s_{11}$, and so $c_{11} = a_{11}$ and $c_{12} = 0$, by directness of the Peirce decomposition and Lemma 2.2-(i).

Now, for arbitrary elements $t_{12} \in \mathfrak{R}_{12}$ and $s_{22} \in \mathfrak{R}_{22}$, by Lemma 2.5, we obtain

$$\begin{aligned} M^{*^{-1}}(t_{12}c \cdot s_{22} + s_{22}c \cdot t_{12}) &= M^{*^{-1}}(t_{12}a_{11} \cdot s_{22} + s_{22}a_{11} \cdot t_{12}) \\ &\quad + M^{*^{-1}}(t_{12}b_{22} \cdot s_{22} + s_{22}b_{22} \cdot t_{12}) \\ &= M^{*^{-1}}(t_{12}b_{22} \cdot s_{22}) . \end{aligned}$$

It follows that $t_{12}c \cdot s_{22} + s_{22}c \cdot t_{12} = t_{12}b_{22} \cdot s_{22}$, which implies $t_{12}c_{22} \cdot s_{22} + s_{22}c_{21} \cdot t_{12} = t_{12}b_{22} \cdot s_{22}$, and so $c_{21} = 0$ and $c_{22} = b_{22}$. Therefore, $c = a_{11} + b_{22}$.

By Lemma 2.4 we can infer that (ii) holds. ∎

Lemma 2.7. *For arbitrary elements $a_{12} \in \mathfrak{R}_{12}$ and $b_{21} \in \mathfrak{R}_{21}$, we have:*

(i) $M(a_{12} + b_{21}) = M(a_{12}) + M(b_{21})$;

(ii) $M^{*^{-1}}(a_{12} + b_{21}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{21})$.

Proof. Suppose that $M(c) = M(a_{12}) + M(b_{21})$ for some $c \in \mathfrak{R}$ and let us write $c = c_{11} + c_{12} + c_{21} + c_{22}$. For arbitrary elements $t_{21} \in \mathfrak{R}_{21}$ and $s_{11} \in \mathfrak{R}_{11}$, by Lemma 2.5 we have

$$\begin{aligned} M^{*^{-1}}(t_{21}c \cdot s_{11} + s_{11}c \cdot t_{21}) &= M^{*^{-1}}(t_{21}a_{12} \cdot s_{11} + s_{11}a_{12} \cdot t_{21}) \\ &\quad + M^{*^{-1}}(t_{21}b_{21} \cdot s_{11} + s_{11}b_{21} \cdot t_{21}) \\ &= M^{*^{-1}}(s_{11}a_{12} \cdot t_{21}) , \end{aligned}$$

which implies that $t_{21}c \cdot s_{11} + s_{11}c \cdot t_{21} = s_{11}a_{12} \cdot t_{21}$ resulting in $t_{21}c_{11} \cdot s_{11} + s_{11}c_{12} \cdot t_{21} = s_{11}a_{12} \cdot t_{21}$. It follows that $c_{11} = 0$ and $c_{12} = a_{12}$, by directness of the Peirce decomposition and Lemma 2.2-(i).

Now, for arbitrary elements $t_{12} \in \mathfrak{R}_{12}$ and $s_{22} \in \mathfrak{R}_{22}$, by Lemma 2.5, we obtain

$$\begin{aligned} M^{*^{-1}}(t_{12}c \cdot s_{22} + s_{22}c \cdot t_{12}) &= M^{*^{-1}}(t_{12}a_{12} \cdot s_{22} + s_{22}a_{12} \cdot t_{12}) \\ &\quad + M^{*^{-1}}(t_{12}b_{21} \cdot s_{22} + s_{22}b_{21} \cdot t_{12}) \\ &= M^{*^{-1}}(s_{22}b_{21} \cdot t_{12}) , \end{aligned}$$

which implies $t_{12}c \cdot s_{22} + s_{22}c \cdot t_{12} = s_{22}b_{21} \cdot t_{12}$ resulting in $t_{12}c_{22} \cdot s_{22} + s_{22}c_{21} \cdot t_{12} = s_{22}b_{21} \cdot t_{12}$. It follows that $c_{21} = b_{21}$ and $c_{22} = 0$.

By Lemma 2.4 we can infer that (ii) holds. ∎

LEMMA 2.8. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ *and* $b_{ij} \in \mathfrak{R}_{ij}$ $(1 \leq i \neq j \leq 2)$. *Then*

(i) $M(a_{ii} + b_{ij}) = M(a_{ii}) + M(b_{ij})$;

(ii) $M^{*^{-1}}(a_{ii} + b_{ij}) = M^{*^{-1}}(a_{ii}) + M^{*^{-1}}(b_{ij})$.

Proof. Let us consider $c \in \mathfrak{R}$ such that $M(c) = M(a_{ii}) + M(b_{ij})$ and let us write $c = c_{11} + c_{12} + c_{21} + c_{22}$. For arbitrary elements $t_{ii} \in \mathfrak{R}_{ii}$ and $d_{ij} \in \mathfrak{R}_{ij}$, by Lemma 2.5 and Lemma 2.7-(ii) we have

$$\begin{aligned} M^{*^{-1}}(t_{ii}c \cdot d_{ij} + d_{ij}c \cdot t_{ii}) &= M^{*^{-1}}(t_{ii}a_{ii} \cdot d_{ij} + d_{ij}a_{ii} \cdot t_{ii}) \\ &\quad + M^{*^{-1}}(t_{ii}b_{ij} \cdot d_{ij} + d_{ij}b_{ij} \cdot t_{ii}) \\ &= M^{*^{-1}}(t_{ii}a_{ii} \cdot d_{ij} + t_{ii}b_{ij} \cdot d_{ij} + d_{ij}b_{ij} \cdot t_{ii}) . \end{aligned}$$

It follows that $t_{ii}c \cdot d_{ij} + d_{ij}c \cdot t_{ii} = t_{ii}a_{ii} \cdot d_{ij} + t_{ii}b_{ij} \cdot d_{ij} + d_{ij}b_{ij} \cdot t_{ii}$ which yields

$$t_{ii}c_{ii} \cdot d_{ij} + t_{ii}c_{ij} \cdot d_{ij} + d_{ij}c_{ij} \cdot t_{ii} + d_{ij}c_{ji} \cdot t_{ii} = t_{ii}a_{ii} \cdot d_{ij} + t_{ii}b_{ij} \cdot d_{ij} + d_{ij}b_{ij} \cdot t_{ii} .$$

By directness of the Peirce decomposition and Lemma 2.2-(i), we have $c_{ii} = a_{ii}$ and $c_{ji} = 0$.

Now for arbitrary elements $t_{ji} \in \mathfrak{R}_{ji}$ and $d_{jj} \in \mathfrak{R}_{jj}$, by Lemma 2.5 and Lemma 2.7 we have

$$\begin{aligned} M^{*^{-1}}(t_{ji}c \cdot d_{jj} + d_{jj}c \cdot t_{ji}) &= M^{*^{-1}}(t_{ji}a_{ii} \cdot d_{jj} + d_{jj}a_{ii} \cdot t_{ji}) \\ &\quad + M^{*^{-1}}(t_{ji}b_{ij} \cdot d_{jj} + d_{jj}b_{ij} \cdot t_{ji}) \\ &= M^{*^{-1}}(t_{ji}b_{ij} \cdot d_{jj}) , \end{aligned}$$

which implies $t_{ji}c \cdot d_{jj} + d_{jj}c \cdot t_{ji} = t_{ji}b_{ij} \cdot d_{jj}$. It follows that $t_{ji}c_{ij} \cdot d_{jj} + d_{jj}c_{jj} \cdot t_{ji} = t_{ji}b_{ij} \cdot d_{jj}$ which results $c_{ij} = b_{ij}$ and $c_{jj} = 0$. Consequently, $c = a_{ii} + b_{ij}$.

By Lemma 2.4 we can infer that (ii) holds. ∎

The following lemma can be proved similarly as in the Lemma 2.8. Therefore we omit its proof.

LEMMA 2.9. *Let* $a_{ii} \in \mathfrak{R}_{ii}$ *and* $b_{ji} \in \mathfrak{R}_{ji}$ $(1 \leq i \neq j \leq 2)$. *Then*

(i) $M(a_{ii} + b_{ji}) = M(a_{ii}) + M(b_{ji})$;

(ii) $M^{*^{-1}}(a_{ii} + b_{ji}) = M^{*^{-1}}(a_{ii}) + M^{*^{-1}}(b_{ji})$.

Lemma 2.10. *The following hold:*

(i) $M(a_{12}s_{22} + b_{12}c_{22} \cdot s_{22}) = M(a_{12}s_{22}) + M(b_{12}c_{22} \cdot s_{22})$;

(ii) $M^{*^{-1}}(a_{12}s_{22} + b_{12}c_{22} \cdot s_{22}) = M^{*^{-1}}(a_{12}s_{22}) + M^{*^{-1}}(b_{12}c_{22} \cdot s_{22})$;

(iii) $M(a_{21}s_{11} + c_{22}b_{21} \cdot s_{11}) = M(a_{21}s_{11}) + M(c_{22}b_{21} \cdot s_{11})$;

(iv) $M^{*^{-1}}(a_{21}s_{11} + c_{22}b_{21} \cdot s_{11}) = M^{*^{-1}}(a_{21}s_{11}) + M^{*^{-1}}(c_{22}b_{21} \cdot s_{11})$.

Proof. First at all, let us note that

$$a_{12}s_{22} + b_{12}c_{22} \cdot s_{22} = (e_1 + b_{12})(a_{12} + c_{22}) \cdot s_{22} + s_{22}(a_{12} + c_{22}) \cdot (e_1 + b_{12}).$$

Hence

$$\begin{aligned}
M(a_{12}s_{22} + b_{12}c_{22} \cdot s_{22}) &= M\big((e_1 + b_{12})(a_{12} + c_{22}) \cdot s_{22} \\
&\qquad + s_{22}(a_{12} + c_{22}) \cdot (e_1 + b_{12})\big) \\
&= M\big((e_1 + b_{12})M^*M^{*^{-1}}(a_{12} + c_{22}) \cdot s_{22} \\
&\qquad + s_{22}M^*M^{*^{-1}}(a_{12} + c_{22}) \cdot (e_1 + b_{12})\big) \\
&= M(e_1 + b_{12})M^{*^{-1}}(a_{12}) \cdot M(s_{22}) \\
&\qquad + M(e_1 + b_{12})M^{*^{-1}}(c_{22}) \cdot M(s_{22}) \\
&\qquad + M(s_{22})M^{*^{-1}}(a_{12}) \cdot M(e_1 + b_{12}) \\
&\qquad + M(s_{22})M^{*^{-1}}(c_{22}) \cdot M(e_1 + b_{12}) \\
&= M\big((e_1 + b_{12})a_{12} \cdot s_{22} + s_{22}a_{12} \cdot (e_1 + b_{12})\big) \\
&\qquad + M\big((e_1 + b_{12})c_{22} \cdot s_{22} + s_{22}c_{22} \cdot (e_1 + b_{12})\big) \\
&= M(a_{12}s_{22}) + M(b_{12}c_{22} \cdot s_{22}).
\end{aligned}$$

Similarly, we obtain $M(a_{21}s_{11} + c_{22}b_{21} \cdot s_{11}) = M(a_{21}s_{11}) + M(b_{22}c_{21} \cdot s_{11})$ from the identity

$$a_{21}s_{11} + c_{22}b_{21} \cdot s_{11} = (a_{21} + c_{22})(e_1 + b_{21}) \cdot s_{11} + s_{11}(e_1 + b_{21}) \cdot (a_{21} + c_{22}).$$

By Lemma 2.4, we can conclude that (ii) and (iv) follow from (i) and (iii), respectively. ∎

Lemma 2.11. (i) $M(a_{12} + b_{12}c_{22}) = M(a_{12}) + M(b_{12}c_{22})$;

(ii) $M^{*^{-1}}(a_{12}+b_{12}c_{22}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{12}c_{22})$;

(iii) $M(a_{21}+c_{22}b_{21}) = M(a_{21}) + M(c_{22}b_{21})$;

(iv) $M^{*^{-1}}(a_{21}+c_{22}b_{21}) = M^{*^{-1}}(a_{21}) + M^{*^{-1}}(c_{22}b_{21})$.

Proof. Suppose that $M(d) = M(a_{12}) + M(b_{12}c_{22})$ for some $d \in \mathfrak{R}$ and let us write $d = d_{11} + d_{12} + d_{21} + d_{22}$. For arbitrary elements $t_{11} \in \mathfrak{R}_{11}$ and $s_{22} \in \mathfrak{R}_{22}$, by Lemma 2.5 and Lemma 2.10-(ii), we have

$$\begin{aligned} M^{*^{-1}}(t_{11}d \cdot s_{22} + s_{22}d \cdot t_{11}) &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22} + s_{22}a_{12} \cdot t_{11}) \\ &\quad + M^{*^{-1}}\big(t_{11}(b_{12}c_{22}) \cdot s_{22} + s_{22}(b_{12}c_{22}) \cdot t_{11}\big) \\ &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22}) + M^{*^{-1}}\big(t_{11}(b_{12}c_{22}) \cdot s_{22}\big) \\ &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22}) + M^{*^{-1}}\big((t_{11}b_{12})c_{22} \cdot s_{22}\big) \\ &= M^{*^{-1}}\big(t_{11}a_{12} \cdot s_{22} + t_{11}(b_{12}c_{22}) \cdot s_{22}\big)\,. \end{aligned}$$

Therefore $t_{11}d \cdot s_{22} + s_{22}d \cdot t_{11} = t_{11}a_{12} \cdot s_{22} + t_{11}(b_{12}c_{22}) \cdot s_{22}$ which implies $t_{11}d_{12} \cdot s_{22} + s_{22}d_{21} \cdot t_{11} = t_{11}a_{12} \cdot s_{22} + t_{11}(b_{12}c_{22}) \cdot s_{22}$. It follows that $d_{12} = a_{12}+b_{12}c_{22}$ and $d_{21} = 0$, by directness of the Peirce decomposition and Lemma 2.2-(i).

Now, for arbitrary elements $t_{ii}, s_{ii} \in \mathfrak{R}_{ii}$ $(i = 1, 2)$, by Lemma 2.5 again, we obtain

$$\begin{aligned} M^{*^{-1}}(t_{ii}d \cdot s_{ii} + s_{ii}d \cdot t_{ii}) &= M^{*^{-1}}(t_{ii}a_{12} \cdot s_{ii} + s_{ii}a_{12} \cdot t_{ii}) \\ &\quad + M^{*^{-1}}\big(t_{ii}(b_{12}c_{22}) \cdot s_{ii} + s_{ii}(b_{12}c_{22}) \cdot t_{ii}\big) = 0\,. \end{aligned}$$

It follows that $t_{ii}d \cdot s_{ii} + s_{ii}d \cdot t_{ii} = 0$ which implies $2d_{ii} = 0$ resulting in $d_{ii} = 0$. Therefore, $d = a_{12} + b_{12}c_{22}$.

Similarly, we prove (iii).

By Lemma 2.4, (ii) and (iv) follow from (i) and (iii), respectively. ∎

Lemma 2.12. *For any $a_{12}, b_{12} \in \mathfrak{R}_{12}$, we have:*

(i) $M(a_{12}+b_{12}) = M(a_{12}) + M(b_{12})$;

(ii) $M^{*^{-1}}(a_{12}+b_{12}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{12})$.

Proof. Suppose that $M(c) = M(a_{12}) + M(b_{12})$, for some $c \in \mathfrak{R}$, and let us write $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathfrak{R}$. For arbitrary elements $t_{11} \in \mathfrak{R}_{11}$ and

$s_{22} \in \mathfrak{R}_{22}$, by Lemma 2.5 and Lemma 2.11-(ii) we have

$$\begin{aligned} M^{*^{-1}}(t_{11}c \cdot s_{22} + s_{22}c \cdot t_{11}) &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22} + s_{22}a_{12} \cdot t_{11}) \\ &\quad + M^{*^{-1}}(t_{11}b_{12} \cdot s_{22} + s_{22}b_{12} \cdot t_{11}) \\ &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22}) + M^{*^{-1}}(t_{11}b_{12} \cdot s_{22}) \\ &= M^{*^{-1}}(t_{11}a_{12} \cdot s_{22} + t_{11}b_{12} \cdot s_{22}) . \end{aligned}$$

It follows that $t_{11}c \cdot s_{22} + s_{22}c \cdot t_{11} = t_{11}a_{12} \cdot s_{22} + t_{11}b_{12} \cdot s_{22}$ which implies $t_{11}c_{12} \cdot s_{22} + s_{22}c_{21} \cdot t_{11} = t_{11}a_{12} \cdot s_{22} + t_{11}b_{12} \cdot s_{22}$ resulting in $c_{12} = a_{12} + b_{12}$ and $c_{21} = 0$.

Now, taking a similar argument in the demonstration of the previous lemma, we can show that $c_{ii} = 0$ $(i = 1, 2)$. Therefore, $c = a_{12} + b_{12}$.

By Lemma 2.4, we can infer that (ii) holds. ∎

Similarly, we have the following result.

LEMMA 2.13. *For any $a_{21}, b_{21} \in \mathfrak{R}_{21}$, we have:*

(i) $M(a_{21} + b_{21}) = M(a_{21}) + M(b_{21})$;
(ii) $M^{*^{-1}}(a_{21} + b_{21}) = M^{*^{-1}}(a_{21}) + M^{*^{-1}}(b_{21})$.

LEMMA 2.14. *For any $a_{11}, b_{11} \in \mathfrak{R}_{11}$, we have:*

(i) $M(a_{11} + b_{11}) = M(a_{11}) + M(b_{11})$;
(ii) $M^{*^{-1}}(a_{11} + b_{11}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{11})$.

Proof. Suppose that $M(c) = M(a_{11}) + M(b_{11})$, for some $c \in \mathfrak{R}$, and let us write $c = c_{11} + c_{12} + c_{21} + c_{22} \in \mathfrak{R}$. For arbitrary elements $t_{22} \in \mathfrak{R}_{22}$ and $s_{ij} \in \mathfrak{R}_{ij}$, we have

$$\begin{aligned} M^{*^{-1}}(t_{22}c \cdot s_{ij} + s_{ij}c \cdot t_{22}) &= M^{*^{-1}}(t_{22}a_{11} \cdot s_{ij} + s_{ij}a_{11} \cdot t_{22}) \\ &\quad + M^{*^{-1}}(t_{22}b_{11} \cdot s_{ij} + s_{ij}b_{11} \cdot t_{22}) = 0 , \end{aligned}$$

by Lemma 2.5, which implies that $t_{22}c \cdot s_{ij} + s_{ij}c \cdot t_{22} = 0$. Taking $i = j = 1$ in the last identity, we have $t_{22}c_{21} \cdot s_{11} + s_{11}c_{12} \cdot t_{22} = 0$ resulting in $c_{12} = c_{21} = 0$.

If we take $i = 2$ and $j = 1$, then we have $t_{22}c_{22} \cdot s_{21} = 0$ which implies $c_{22} = 0$. Now, for arbitrary elements $t_{12} \in \mathfrak{R}_{12}$ and $s_{11} \in \mathfrak{R}_{11}$, we have

$$\begin{aligned} M^{*^{-1}}(t_{12}c \cdot s_{11} + s_{11}c \cdot t_{12}) &= M^{*^{-1}}(t_{12}a_{11} \cdot s_{11} + s_{11}a_{11} \cdot t_{12}) \\ &\quad + M^{*^{-1}}(t_{12}b_{11} \cdot s_{11} + s_{11}b_{11} \cdot t_{12}) \\ &= M^{*^{-1}}(s_{11}a_{11} \cdot t_{12}) + M^{*^{-1}}(s_{11}b_{11} \cdot t_{12}) \\ &= M^{*^{-1}}(s_{11}a_{11} \cdot t_{12} + s_{11}b_{11} \cdot t_{12}), \end{aligned}$$

by Lemma 2.12-(ii). It follows that $t_{12}c \cdot s_{11} + s_{11}c \cdot t_{12} = s_{11}a_{11} \cdot t_{12} + s_{11}b_{11} \cdot t_{12}$ which yields $s_{11}c_{11} \cdot t_{12} = s_{11}a_{11} \cdot t_{12} + s_{11}b_{11} \cdot t_{12}$. By Lemma 2.2-(i), we have $c_{11} = a_{11} + b_{11}$.

By Lemma 2.4, we can conclude that (ii) holds. ∎

Similarly, we have

LEMMA 2.15. *For arbitrary $a_{22}, b_{22} \in \mathfrak{R}_{22}$, we have:*

(i) $M(a_{22} + b_{22}) = M(a_{22}) + M(b_{22})$;

(ii) $M^{*^{-1}}(a_{22} + b_{22}) = M^{*^{-1}}(a_{22}) + M^{*^{-1}}(b_{22})$.

LEMMA 2.16. *For any $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, and $c_{21} \in \mathfrak{R}_{21}$, we have:*

(i) $M(a_{11} + b_{12} + c_{21}) = M(a_{11}) + M(b_{12}) + M(c_{21})$;

(ii) $M^{*^{-1}}(a_{11} + b_{12} + c_{21}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12}) + M^{*^{-1}}(c_{21})$.

Proof. Suppose that $M(d) = M(a_{11}) + M(b_{12}) + M(c_{21})$ for some $c \in \mathfrak{R}$ and let us write $d = d_{11} + d_{12} + d_{21} + d_{22}$. By Lemma 2.8 and Lemma 2.9, $M(d)$ can be represented in the following two forms:

$$M(d) = M(a_{11} + b_{12}) + M(c_{21}) \tag{1}$$

and

$$M(d) = M(a_{11} + c_{21}) + M(b_{12}). \tag{2}$$

Hence, for arbitrary elements $t_{22} \in \mathfrak{R}_{22}$ and $s_{12} \in \mathfrak{R}_{12}$, by Lemma 2.5 and identity (1) we have

$$\begin{aligned} M^{*^{-1}}(t_{22}d \cdot s_{12} + s_{12}d \cdot t_{22}) &= M^{*^{-1}}\big((t_{22}(a_{11} + b_{12}) \cdot s_{12} + s_{12}(a_{11} + b_{12}) \cdot t_{22}\big) \\ &\quad + M^{*^{-1}}\big(t_{22}c_{21} \cdot s_{12} + s_{12}c_{21} \cdot t_{22}\big) \\ &= M^{*^{-1}}(t_{22}c_{21} \cdot s_{12}), \end{aligned}$$

which implies that $t_{22}d \cdot s_{12} + s_{12}d \cdot t_{22} = t_{22}c_{21} \cdot s_{12}$. It follows that $t_{22}d_{21} \cdot s_{12} + s_{12}d_{22} \cdot t_{22} = t_{22}c_{21} \cdot s_{12}$ which yields $d_{21} = c_{21}$ and $d_{22} = 0$. Now, for arbitrary elements $t_{11} \in \mathfrak{R}_{11}$ and $s_{21} \in \mathfrak{R}_{21}$, using Lemma 2.5 and Lemma 2.9-(ii) and identity (2) we have

$$\begin{aligned} M^{*^{-1}}(t_{11}d \cdot s_{21} + s_{21}d \cdot t_{11}) &= M^{*^{-1}}\big(t_{11}(a_{11} + c_{21}) \cdot s_{21} + s_{21}(a_{11} + c_{21}) \cdot t_{11}\big) \\ &\quad + M^{*^{-1}}\big(t_{11}b_{12} \cdot s_{21} + s_{21}b_{12} \cdot t_{11}\big) \\ &= M^{*^{-1}}(s_{21}a_{11} \cdot t_{11} + t_{11}b_{12} \cdot s_{21}), \end{aligned}$$

which implies that $t_{11}d \cdot s_{21} + s_{21}d \cdot t_{11} = s_{21}a_{11} \cdot t_{11} + t_{11}b_{12} \cdot s_{21}$. It follows that $t_{11}d_{12} \cdot s_{21} + s_{21}d_{11} \cdot t_{11} = s_{21}a_{11} \cdot t_{11} + t_{11}b_{12} \cdot s_{21}$ which yields $d_{11} = a_{11}$ and $d_{12} = b_{12}$, by directness of the Peirce decomposition and Lemma 2.2-(i). Therefore, $d = a_{11} + b_{12} + c_{21}$.

By Lemma 2.4, we can conclude that (ii) holds. ∎

Similarly, we can prove the following result.

Lemma 2.17. *For any $a_{12} \in \mathfrak{R}_{12}$, $b_{21} \in \mathfrak{R}_{21}$, and $c_{22} \in \mathfrak{R}_{22}$, we have:*

(i) $M(a_{12} + b_{21} + c_{22}) = M(a_{12}) + M(b_{21}) + M(c_{22})$;

(ii) $M^{*^{-1}}(a_{12} + b_{21} + c_{22}) = M^{*^{-1}}(a_{12}) + M^{*^{-1}}(b_{21}) + M^{*^{-1}}(c_{22})$.

Lemma 2.18. *For any $a_{11} \in \mathfrak{R}_{11}$, $b_{12} \in \mathfrak{R}_{12}$, $c_{21} \in \mathfrak{R}_{21}$, and $d_{22} \in \mathfrak{R}_{22}$, we have:*

(i) $M(a_{11} + b_{12} + c_{21} + d_{22}) = M(a_{11}) + M(b_{12}) + M(c_{21}) + M(d_{22})$;

(ii) $M^{*^{-1}}(a_{11} + b_{12} + c_{21} + d_{22}) = M^{*^{-1}}(a_{11}) + M^{*^{-1}}(b_{12}) + M^{*^{-1}}(c_{21}) + M^{*^{-1}}(d_{22})$.

Proof. Suppose that

$$\begin{aligned} M(f) &= M(a_{11}) + M(b_{12}) + M(c_{21}) + M(d_{22}) \\ &= M(a_{11} + d_{22}) + M(b_{12} + c_{21}) \end{aligned}$$

for some $f \in \mathfrak{R}$ and let us write $f = f_{11} + f_{12} + f_{21} + f_{22} \in \mathfrak{R}$. For arbitrary elements $t_{11} \in \mathfrak{R}_{11}$ and $s_{12} \in \mathfrak{R}_{12}$, we have

$$\begin{aligned}
M^{*^{-1}}&(t_{11}f \cdot s_{12} + s_{12}f \cdot t_{11}) \\
&= M^{*^{-1}}\big(t_{11}(a_{11} + d_{22}) \cdot s_{12} + s_{12}(a_{11} + d_{22}) \cdot t_{11}\big) \\
&\quad + M^{*^{-1}}\big(t_{11}(b_{12} + c_{21}) \cdot s_{12} + s_{12}(b_{12} + c_{21}) \cdot t_{11}\big) \\
&= M^{*^{-1}}(t_{11}a_{11} \cdot s_{12}) \\
&\quad + M^{*^{-1}}(t_{11}b_{12} \cdot s_{12} + s_{12}b_{12} \cdot t_{11} + s_{12}c_{21} \cdot t_{11}) \\
&= M^{*^{-1}}(t_{11}a_{11} \cdot s_{12} + t_{11}b_{12} \cdot s_{12} + s_{12}b_{12} \cdot t_{11} + s_{12}c_{21} \cdot t_{11}),
\end{aligned}$$

by Lemma 2.16-(ii). It follows that

$$t_{11}f \cdot s_{12} + s_{12}f \cdot t_{11} = t_{11}a_{11} \cdot s_{12} + t_{11}b_{12} \cdot s_{12} + s_{12}b_{12} \cdot t_{11} + s_{12}c_{21} \cdot t_{11},$$

which implies

$$\begin{aligned}
t_{11}f_{11} \cdot s_{12} &+ t_{11}f_{12} \cdot s_{12} + s_{12}f_{12} \cdot t_{11} + s_{12}f_{21} \cdot t_{11} \\
&= t_{11}a_{11} \cdot s_{12} + t_{11}b_{12} \cdot s_{12} + s_{12}b_{12} \cdot t_{11} + s_{12}c_{21} \cdot t_{11}.
\end{aligned}$$

This results in $f_{11} = a_{11}$ and $f_{21} = c_{21}$, by directness of the Peirce decomposition and Lemma 2.2-(i). Now, for arbitrary elements $t_{22} \in \mathfrak{R}_{22}$ and $s_{21} \in \mathfrak{R}_{21}$, we have

$$\begin{aligned}
M^{*^{-1}}&(t_{22}f \cdot s_{21} + s_{21}f \cdot t_{22}) \\
&= M^{*^{-1}}\big(t_{22}(a_{11} + d_{22}) \cdot s_{21} + s_{21}(a_{11} + d_{22}) \cdot t_{22}\big) \\
&\quad + M^{*^{-1}}\big(t_{22}(b_{12} + c_{21}) \cdot s_{21} + s_{21}(b_{12} + c_{21}) \cdot t_{22}\big) \\
&= M^{*^{-1}}(t_{22}d_{22} \cdot s_{21}) \\
&\quad + M^{*^{-1}}(t_{22}c_{21} \cdot s_{21} + s_{21}b_{12} \cdot t_{22} + s_{21}c_{21} \cdot t_{22}) \\
&= M^{*^{-1}}(t_{22}d_{22} \cdot s_{21} + t_{22}c_{21} \cdot s_{21} + s_{21}b_{12} \cdot t_{22} + s_{21}c_{21} \cdot t_{22}).
\end{aligned}$$

which implies that

$$t_{22}f \cdot s_{21} + s_{21}f \cdot t_{22} = t_{22}d_{22} \cdot s_{21} + t_{22}c_{21} \cdot s_{21} + s_{21}b_{12} \cdot t_{22} + s_{21}c_{21} \cdot t_{22}$$

resulting in

$$\begin{aligned}
t_{22}f_{21} \cdot s_{21} &+ t_{22}f_{22} \cdot s_{21} + s_{21}f_{12} \cdot t_{22} + s_{21}f_{21} \cdot t_{22} \\
&= t_{22}d_{22} \cdot s_{21} + t_{22}c_{21} \cdot s_{21} + s_{21}b_{12} \cdot t_{22} + s_{21}c_{21} \cdot t_{22}.
\end{aligned}$$

Thus $f_{12} = b_{12}$ and $f_{22} = d_{22}$. Thus, we have $f = a_{11} + b_{12} + c_{21} + d_{22}$. ∎

Proof of Theorem 2.1. Let us consider arbitrary elements $a, b \in \mathfrak{R}$ and let us write $a = a_{11} + a_{12} + a_{21} + a_{22}$ and $b = b_{11} + b_{12} + b_{21} + b_{22}$. Then

$$\begin{aligned}
M(a+b) &= M\big((a_{11}+b_{11}) + (a_{12}+b_{12}) + (a_{21}+b_{21}) + (a_{22}+b_{22})\big) \\
&= M(a_{11}+b_{11}) + M(a_{12}+b_{12}) + M(a_{21}+b_{21}) + M(a_{22}+b_{22}) \\
&= M(a_{11}) + M(b_{11}) + M(a_{12}) + M(b_{12}) \\
&\quad + M(a_{21}) + M(b_{21}) + M(a_{22}) + M(b_{22}) \\
&= M(a_{11}+a_{12}+a_{21}+a_{22}) + M(b_{11}+b_{12}+b_{21}+b_{22}) \\
&= M(a) + M(b).
\end{aligned}$$

Thus, M is an additive map. Now, for arbitrary elements $x, y \in \mathfrak{R}'$, there are elements $c = c_{11} + c_{12} + c_{21} + c_{22}$ and $d = d_{11} + d_{12} + d_{21} + d_{22}$ in $\mathfrak{R}$ such that $c = M^*(x) + M^*(y)$ and $d = M^*(x+y)$, by Lema 2.3. It follows that for arbitrary elements $t_{ij} \in \mathfrak{R}_{ij}$ and $s_{kl} \in \mathfrak{R}_{kl}$ $(1 \leq i,j,k,l \leq 2)$, we have

$$\begin{aligned}
M(t_{ij}c \cdot s_{kl} + s_{kl}c \cdot t_{ij}) &= M\Big(t_{ij}\big(M^*(x) + M^*(y)\big) \cdot s_{kl} \\
&\qquad + s_{kl}\big(M^*(x) + M^*(y)\big) \cdot t_{ij}\Big) \\
&= M\big(t_{ij}M^*(x) \cdot s_{kl}\big) + M\big(t_{ij}M^*(y) \cdot s_{kl}\big) \\
&\qquad + M\big(s_{kl}M^*(x) \cdot t_{ij}\big) + M\big(s_{kl}M^*(y) \cdot t_{ij}\big) \\
&= M\big(t_{ij}M^*(x) \cdot s_{kl} + s_{kl}M^*(x) \cdot t_{ij}\big) \\
&\qquad + M\big(t_{ij}M^*(y) \cdot s_{kl} + s_{kl}M^*(y) \cdot t_{ij}\big) \\
&= M(t_{ij})x \cdot M(s_{kl}) + M(s_{kl})x \cdot M(t_{ij}) \\
&\qquad + M(t_{ij})y \cdot M(s_{kl}) + M(s_{kl})y \cdot M(t_{ij}) \\
&= M(t_{ij})(x+y) \cdot M(s_{kl}) + M(s_{kl})(x+y) \cdot M(t_{ij}) \\
&= M\big(t_{ij}M^*(x+y) \cdot s_{kl} + s_{kl}M^*(x+y) \cdot t_{ij}\big) \\
&= M(t_{ij}d \cdot s_{kl} + s_{kl}d \cdot t_{ij}),
\end{aligned}$$

which implies that

$$t_{ij}c \cdot s_{kl} + s_{kl}c \cdot t_{ij} = t_{ij}d \cdot s_{kl} + s_{kl}d \cdot t_{ij}\,. \tag{3}$$

Taking $i = j = k = 1$ and $l = 2$ in the last equality, we obtain

$$\begin{aligned}
&t_{11}c_{11} \cdot s_{12} + t_{11}c_{12} \cdot s_{12} + s_{12}c_{12} \cdot t_{11} + s_{12}c_{21} \cdot t_{11} \\
&\qquad = t_{11}d_{11} \cdot s_{12} + t_{11}d_{12} \cdot s_{12} + s_{12}d_{12} \cdot t_{11} + s_{12}d_{21} \cdot t_{11}\,,
\end{aligned}$$

which implies that $c_{11} = d_{11}$ and $c_{21} = d_{21}$. Now, if we take $i = j = k = 2$ and $l = 1$ in the identity (3), then we obtain

$$t_{22}c_{21} \cdot s_{21} + t_{22}c_{22} \cdot s_{21} + s_{21}c_{12} \cdot t_{22} + s_{21}c_{21} \cdot t_{22}$$
$$= t_{22}d_{21} \cdot s_{21} + t_{22}d_{22} \cdot s_{21} + s_{21}d_{12} \cdot t_{22} + s_{21}d_{21} \cdot t_{22}\,.$$

By directness of the Peirce decomposition and Lemma 2.2-(i), we have $c_{12} = d_{12}$ and $c_{22} = d_{22}$. It follows that $c = d$ and so $M^*(x + y) = M^*(x) + M^*(y)$.

The prove is complete. ∎

For the case of Jordan triple elementary maps on prime alternative rings we have the following result.

Corollary 2.1. *Let $\mathfrak{R}$ be a 2 and 3-torsion free unital prime alternative ring containing a nontrivial idempotent and let $\mathfrak{R}'$ be an arbitrary alternative ring. Then every surjective Jordan triple elementary map (M, M^*) of $\mathfrak{R} \times \mathfrak{R}'$ is additive.*

Proof. Since $\mathfrak{R}$ is prime, it is easy to check that the conditions (i) and (ii) of Theorem 2.1 hold true, by [2, Theorem 2.2]. Now the proof goes directly. ∎

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