WITNESSING INCOMPATIBILITY OF QUANTUM CHANNELS

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ABSTRACT. We introduce the notion of incompatibility witness for quantum channels, defined as an affine functional that is non-negative on all pairs of compatible channels and strictly negative on some incompatible pair. This notion extends the recent definition of incompatibility witnesses for quantum measurements. We utilize the general framework of channels acting on arbitrary finite-dimensional von Neumann algebras, thus allowing us to investigate incompatibility witnesses on measurement-measurement, measurement-channel and channel-channel pairs. We prove that any incompatibility witness can be implemented as a state discrimination task in which some intermediate classical information is obtained before completing the task. This implies that any incompatible pair of channels gives an advantage over compatible pairs in some such state discrimination task.

1. INTRODUCTION

Two input-output devices, such as measurements, channels or instruments, are called incompatible if they are not parts of a common third device [1]. The concept of incompatibility, taken at this level of generality, gives a common ground for several important notions and statements of quantum information. For instance, the 'no cloning' theorem declares that two identity channels are incompatible, and statements about optimal quantum cloning devices are then statements about compatibility of some channels, such as depolarizing channels [2, 3, 4, 5, 6, 7]. As another example, antidegradable channels are exactly those channels that are compatible with themselves, whereas entanglement breaking channels are those channels that are compatible with arbitrary many copies of themselves [8].

The traditional and most extensively studied topic in the area of incompatibility is the incompatibility of pairs of measurement devices. It has been recently shown that two quantum measurements are incompatible if and only if they give an advantage in some state discrimination task [9, 10, 11]. Physically speaking, the connection with state discrimination tasks and incompatibility of measurement devices can be understood by comparing two state discrimination scenarios, where partial information is given either before or after measurements are to be performed [12]. Indeed, only for compatible pairs of measurements the state discrimination capability is unaffected by the stage when partial information is given, as pairs of this kind can be postprocessed from a single measurement device performed with no reference to partial information. In the present paper, we show that this physical interpretation, with a slight modification, carries also to the incompatibility of quantum channels.

In Ref. [9], the above result was obtained by introducing the concept of incompatibility witnesses and then proving that, up to detection equivalence, every incompatibility witness is associated to some state discrimination task with partial intermediate information. We generalize this approach and prove that a similar statement holds for all incompatible pairs of quantum channels. To do it, we first define the concept of channel incompatibility witness and then prove that all such witnesses can be brought into a standard form related to a variation of the state discrimination task described in Ref. [9]. The state discrimination task we will consider does not require to couple the measured system with any ancillary system. In particular, the advantage of our approach is that it does not rely on entanglement.

In the formulation used in the current work, channels are completely positive linear maps between finite-dimensional von Neumann algebras. This framework (or something closely related to it) has been used in several earlier studies, e.g. Refs. [13, 14]. A measurement can be seen as a channel from an abelian von Neumann algebra, and our formalism therefore covers the main theorem of Ref. [9] as a special case. It also allows to treat the incompatibility between channels and measurements that is the underlying source for fundamental noise-disturbance trade-off in quantum measurements [15, 16, 17, 18].

Our investigation is organized as follows. After having recalled some elementary facts about channels and von Neumann algebras in Sec. 2 and described the convex compact set of all compatible pairs of channels in Sec. 3, in Sec. 4 we provide the definition and main properties of channel incompatibility witnesses. Section 5 then describes the particular state discrimination task we will be concerned with, and contains the proof that any incompatibility witness is associated to a task of this kind for some choice of the state ensemble to be detected. Finally, Secs. 6 and 7 contain some examples of channel incompatibility witnesses. The examples of Sec. 6 are derived from the measurement incompatibility witnesses constructed in Ref. [9] by means of two mutually unbiased bases, while the example of Sec. 7 is related to the optimal approximate cloning method of Refs. [3, 4].

2. Preliminaries

We consider systems described by finite-dimensional von Neumann algebras, that is, complex $*$-algebras that are isomorphic to block matrix algebras endowed with the uniform matrix norm $\|\cdot\|$. If $\mathcal{A}$ is such an algebra, its *predual* $\mathcal{A}_*$ coincides with the linear dual $\mathcal{A}^*$. We denote by $\langle\, a\,,\, A\,\rangle$ the canonical pairing between elements $a \in \mathcal{A}_*$ and $A \in \mathcal{A}$. The notations $\mathcal{A}^{\mathrm{sa}}$ and $\mathcal{A}^+$ are used for the set of all selfadjoint and all positive elements of $\mathcal{A}$, respectively. The analogous subsets of $\mathcal{A}_*$ are

$$\mathcal{A}_*^{\mathrm{sa}} = \{a \in \mathcal{A}_* \mid \langle\, a\,,\, A\,\rangle \in \mathbb{R} \ \ \forall A \in \mathcal{A}^{\mathrm{sa}}\}\,,$$
$$\mathcal{A}_*^+ = \{a \in \mathcal{A}_* \mid \langle\, a\,,\, A\,\rangle \geq 0 \ \ \forall A \in \mathcal{A}^+\}\,.$$

The *states* of $\mathcal{A}$ constitute the convex set $\mathcal{S}(\mathcal{A}) = \{a \in \mathcal{A}_*^+ \mid \langle\, a\,,\, \mathbb{1}_{\mathcal{A}}\,\rangle = 1\}$, where $\mathbb{1}_{\mathcal{A}}$ is the identity element of $\mathcal{A}$. A *measurement* with a finite outcome set X is described by a map $\mathsf{M} : X \to \mathcal{A}$ such that $\mathsf{M}(x) \in \mathcal{A}^+$ for all $x \in X$ and $\sum_{x \in X} \mathsf{M}(x) = \mathbb{1}_{\mathcal{A}}$. The probability of obtaining an outcome x by performing the measurement M in the state a is then $\langle\, a\,,\, \mathsf{M}(x)\,\rangle$. A measurement M is called *informationally complete* if the associated probability distributions are different for all states, i.e., for any two states $a \neq a'$ there is an outcome x such that $\langle\, a\,,\, \mathsf{M}(x)\,\rangle \neq \langle\, a'\,,\, \mathsf{M}(x)\,\rangle$. The informational completeness of M is equivalent to

the condition that the real linear span of the set $\{\mathsf{M}(x) \mid x \in X\}$ coincides with the real vector space $\mathcal{A}^{\mathrm{sa}}$ (cf. Ref. [19]).

A finite-dimensional quantum system is associated with the von Neumann algebra $\mathcal{L}(\mathcal{H})$ of all linear maps on a finite-dimensional complex Hilbert space $\mathcal{H}$, whereas a finite classical system is described by the von Neumann algebra $\ell^\infty(X)$ of all complex functions on a finite set X. The respective norms are the uniform operator norm $\|A\| = \max\{\|Au\| / \|u\| \mid u \in \mathcal{H} \setminus \{0\}\}$ and the sup norm $\|F\| = \max\{|F(x)| \mid x \in X\}$. In these two extreme cases, the states of the system are described by positive trace-one operators and classical probability distributions, respectively. The framework of general von Neumann algebras allows us to consider also hybrid systems, like e.g. the classical-quantum output of a quantum measuring process, or quantum systems subject to superselection rules.

Let $\mathcal{B}$ be another finite-dimensional von Neumann algebra. A *channel* connecting the system $\mathcal{A}$ with the system $\mathcal{B}$ is a linear map $\Phi : \mathcal{A}_* \to \mathcal{B}_*$ such that its adjoint Φ^* is completely positive and unital. The adjoint of Φ is the linear map $\Phi^* : \mathcal{B} \to \mathcal{A}$ defined by

$$\langle\, a \,,\, \Phi^*(B) \,\rangle = \langle\, \Phi(a) \,,\, B \,\rangle$$

for all $a \in \mathcal{A}_*$ and $B \in \mathcal{B}$.

A measurement $\mathsf{M} : X \to \mathcal{A}$ can be regarded as a channel $\widehat{\mathsf{M}} : \mathcal{A}_* \to \ell^1(X)$, where $\ell^1(X) = \ell^\infty(X)_*$ is the ℓ^1-space of all complex functions on X. This identification is obtained by setting $\widehat{\mathsf{M}}(a) = \langle\, a \,,\, \mathsf{M}(\cdot) \,\rangle$, or, equivalently, $\widehat{\mathsf{M}}^*(\delta_x) = \mathsf{M}(x)$, where $\delta_x \in \ell^\infty(X)$ denotes the Kronecker delta function at x.

When $\mathcal{A} = \mathcal{B} = \mathcal{L}(\mathcal{H})$, any channel connecting the system $\mathcal{A}$ with the system $\mathcal{B}$ is a quantum channel in the usual sense. Moreover, any measurement $\mathsf{M} : X \to \mathcal{L}(\mathcal{H})$ is a quantum measurement in the usual sense and can be identified with a positive operator valued measure [20]. In this case, the predual $\mathcal{L}^1(\mathcal{H}) = \mathcal{L}(\mathcal{H})_*$ is the normed space of all linear operators on $\mathcal{H}$ endowed with the trace-class norm. When instead $\mathcal{A} = \mathcal{B} = \ell^\infty(X)$, channels connecting $\mathcal{A}$ with $\mathcal{B}$ constitute classical data processings and just coincide with measurements $\mathsf{M} : X \to \ell^1(X)$ (cf. Ref. [21]).

3. Incompatibility of channels

The incompatibility of quantum channels has been defined and studied in Refs. [8, 22, 23]. That definition has been generalized in Ref. [24] for different types of devices in general probabilistic theories, while in Ref. [25] it has been extended to cover the case of two channels with arbitrary outcome algebras. In the following we state the definition of (in)compatible channels explicitly in our current framework.

If $\mathcal{B}_1$ and $\mathcal{B}_2$ are two von Neumann algebras, we denote by $\mathcal{B}_1 \bar\otimes \mathcal{B}_2$ their algebraic tensor product canonically regarded as a von Neumann algebra; see e.g. Ref. [26, Sec. IV, Def. 1.3]. The projection onto the ith algebra is the channel $\Pi_i : (\mathcal{B}_1 \bar\otimes \mathcal{B}_2)_* \to \mathcal{B}_{i*}$ with $\Pi_1^*(B_1) = B_1 \otimes \mathbb{1}_{\mathcal{B}_2}$ and $\Pi_2^*(B_2) = \mathbb{1}_{\mathcal{B}_1} \otimes B_2$ for all $B_i \in \mathcal{B}_i$. The ith *margin* of a channel $\Phi : \mathcal{A}_* \to (\mathcal{B}_1 \bar\otimes \mathcal{B}_2)_*$ is then defined as the composition channel $\Pi_i \circ \Phi$. Two channels $\Phi_1 : \mathcal{A}_* \to \mathcal{B}_{1*}$ and $\Phi_2 : \mathcal{A}_* \to \mathcal{B}_{2*}$ are *compatible* if there exists a channel Φ such that $\Phi_1 = \Pi_1 \circ \Phi$ and $\Phi_2 = \Pi_2 \circ \Phi$. In this case, we say that Φ is a *joint channel* of Φ_1 and Φ_2. Otherwise, Φ_1 and Φ_2 are called *incompatible*.

The compatibility of Φ_1 and Φ_2 is preserved if they are concatenated with other channels $\Psi_i : \mathcal{B}_{i*} \to \mathcal{C}_{i*}$. Indeed, if Φ is a joint channel of Φ_1 and Φ_2, then the

composition $(\Psi_1 \otimes \Psi_2) \circ \Phi$ is a joint channel of $\Psi_1 \circ \Phi_1$ and $\Psi_2 \circ \Phi_2$. The tensor product of two channels is defined by the relation $(\Psi_1 \otimes \Psi_2)^* = \Psi_1^* \otimes \Psi_2^*$.

In the particular case $\mathcal{B}_1 = \ell^\infty(X_1)$ and $\mathcal{B}_2 = \ell^\infty(X_2)$, the compatibility of channels coincides with the usual notion of compatibility for measurements due to the aforementioned identification $\mathsf{M} \simeq \widehat{\mathsf{M}}$; see Ref. [8, Prop. 5]. Indeed, let π_i be the projection onto the ith factor of the Cartesian product $X_1 \times X_2$, and recall that two measurements $\mathsf{M}_1 : X_1 \to \mathcal{A}$ and $\mathsf{M}_2 : X_2 \to \mathcal{A}$ are called *compatible* if there exists a third measurement $\mathsf{M} : X_1 \times X_2 \to \mathcal{A}$ such that its *margins* $\pi_i \mathsf{M}(x_i) = \sum_{(y_1, y_2) \in \pi_i^{-1}(x_i)} \mathsf{M}(y_1, y_2)$ coincide with M_i. The equivalence of the two notions of compatibility then directly follows from the equality $\Pi_i \circ \widehat{\mathsf{M}} = \widehat{\pi_i \mathsf{M}}$.

Similarly, when $\mathcal{B}_1 = \ell^\infty(X)$ and $\mathcal{B}_2 = \mathcal{L}(\mathcal{H})$, any channel connecting the system $\mathcal{A}$ with the system $\mathcal{B}_1 \bar\otimes \mathcal{B}_2$ can be identified with an *instrument* [20]. In this case, compatibility of two channels $\Phi_1 : \mathcal{A}_* \to \ell^1(X)$ and $\Phi_2 : \mathcal{A}_* \to \mathcal{L}^1(\mathcal{H})$ amounts to measurement-channel compatibility in the sense of Ref. [15].

We denote by $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ the convex compact set of all pairs of channels (Φ_1, Φ_2), where $\Phi_i : \mathcal{A}_* \to \mathcal{B}_{i*}$. Convex combinations in $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ are defined componentwise. We let $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ be the subset of all compatible pairs of channels. This subset is itself convex and compact, since it is the image of the convex compact set of channels $\Phi : \mathcal{A}_* \to (\mathcal{B}_1 \bar\otimes \mathcal{B}_2)_*$ under the affine mapping $\Phi \mapsto (\Pi_1 \circ \Phi, \Pi_2 \circ \Phi)$. In the following, we show that the inclusion $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) \subseteq \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is strict unless $\mathcal{A}$ is a commutative algebra or either $\mathcal{B}_1$ or $\mathcal{B}_2$ is trivial. A different but related result has been proven in Ref. [27].

Proposition 1. $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) = \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ *if and only if $\mathcal{A}$ is an abelian von Neumann algebra or $\mathcal{B}_i = \mathbb{C}$ for $i \in \{1, 2\}$.*

Proof. If $\mathcal{B}_i = \mathbb{C}$ for – say – $i = 1$, then the trivial channel $\Phi_1 = \langle \cdot, \mathbb{1}_\mathcal{A} \rangle$ is the unique channel connecting the system $\mathcal{A}$ with the system $\mathcal{B}_1$. This channel is compatible with any channel $\Phi_2 : \mathcal{A}_* \to \mathcal{B}_{2*}$. Indeed, Φ_2 is itself a joint channel of Φ_1 and Φ_2 since $\mathcal{B}_1 \bar\otimes \mathcal{B}_2 = \mathcal{B}_2$. Thus, $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) = \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ in this case.

If $\mathcal{A}$ is abelian, there exists a finite set X such that $\mathcal{A}$ is isomorphic to the von Neumann algebra $\ell^\infty(X)$. Since $\ell^\infty(X)_* = \ell^1(X)$ and $(\ell^\infty(X) \bar\otimes \ell^\infty(X))_* = \ell^1(X \times X)$, we can define a broadcasting map $\Gamma : \mathcal{A}_* \to (\mathcal{A} \bar\otimes \mathcal{A})_*$ as

$$[\Gamma(f)](x, y) = \begin{cases} f(x) & \text{if } x = y, \\ 0 & \text{if } x \neq y. \end{cases}$$

The adjoint Γ^* is positive and unital, hence Γ is a channel. Indeed, for any linear map having an abelian von Neumann algebra as its domain or image, positivity implies complete positivity by Ref. [28, Thms. 3.9 and 3.11]. The two margins of Γ are the identity channel $\mathrm{id} : \mathcal{A}_* \to \mathcal{A}_*$. Therefore, the fact that compatibility is preserved in concatenation implies that any two channels $\Phi_1 = \Phi_1 \circ \mathrm{id}$ and $\Phi_2 = \Phi_2 \circ \mathrm{id}$ are compatible. We conclude that $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) = \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ also in this case.

Finally, if $\mathcal{C}_c(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) = \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, then either $\mathcal{B}_i = \mathbb{C}$ for some $i \in \{1, 2\}$, or for all $i \in \{1, 2\}$ there exist two disjointly supported states $b_{i,1}, b_{i,2} \in \mathcal{S}(\mathcal{B}_i)$. In the latter case, either $\mathcal{A} = \mathbb{C}$ and thus $\mathcal{A}$ is abelian, or, for any two fixed projections $P_{1,1}, P_{2,1} \in \mathcal{A} \setminus \{0, \mathbb{1}_\mathcal{A}\}$, let $P_{1,2}, P_{2,2} \in \mathcal{A}$ be such that $P_{i,1} + P_{i,2} = \mathbb{1}_\mathcal{A}$ for $i = 1, 2$.

Further, let $\Phi_i : \mathcal{A}_* \to \mathcal{B}_{i*}$ be the linear map defined as

$$\Phi_i(a) = \sum_{k=1,2} \langle\, a\,,\, P_{i,k}\,\rangle\, b_{i,k}\,.$$

The unitality of Φ_i^* is clear. Moreover, since Φ_i^* is positive and its image $\Phi_i^*(\mathcal{B}_i) = \mathrm{span}\,\{P_{i,1}, P_{i,2}\}$ is a commutative algebra, it follows that Φ_i^* is completely positive by Ref. [28, Thm. 3.9]. Thus, Φ_i is a channel, and by the assumed hypothesis we can pick a joint channel Φ of Φ_1 and Φ_2. If $Q_{i,1}$ and $Q_{i,2}$ are the support projections of the states $b_{i,1}$ and $b_{i,2}$, respectively, then

$$0 \leq \Phi^*(Q_{1,h} \otimes Q_{2,k}) \leq \begin{cases} \Phi^*(Q_{1,h} \otimes \mathbb{1}_{\mathcal{B}_2}) = \Phi_1^*(Q_{1,h}) = P_{1,h}\,, \\ \Phi^*(\mathbb{1}_{\mathcal{B}_1} \otimes Q_{2,k}) = \Phi_2^*(Q_{2,k}) = P_{2,k} \end{cases}$$

for all $h, k = 1, 2$. It follows that $\Phi^*(Q_{1,h} \otimes Q_{2,k})\Phi^*(Q_{1,h'} \otimes Q_{2,k'}) = 0$ whenever $(h, k) \neq (h', k')$, and hence the projections

$$P_{1,1} = \Phi^*(Q_{1,1} \otimes Q_{2,1}) + \Phi^*(Q_{1,1} \otimes Q_{2,2})\,, \qquad P_{2,1} = \Phi^*(Q_{1,1} \otimes Q_{2,1}) + \Phi^*(Q_{1,2} \otimes Q_{2,1})$$

commute. Since the choice of $P_{1,1}$ and $P_{2,1}$ was arbitrary, this proves that all projections commute in $\mathcal{A}$, which again implies that $\mathcal{A}$ is abelian. □

The next corollary is a restatement of Ref. [29, Thm. 3] within the framework of von Neumann algebras. Interestingly, the assumption that $\mathcal{A}$ is finite-dimensional is essential for its validity; see Ref. [30, Thm. 3.10].

Corollary 1. *The identity channel* $\mathrm{id} : \mathcal{A}_* \to \mathcal{A}_*$ *is compatible with itself if and only if* $\mathcal{A}$ *is abelian.*

Proof. Since any channel $\Phi : \mathcal{A}_* \to \mathcal{A}_*$ is the composition $\Phi = \Phi \circ \mathrm{id}$, the inclusion $(\mathrm{id}, \mathrm{id}) \in \mathcal{C}_{\mathrm{c}}(\mathcal{A}; \mathcal{A}, \mathcal{A})$ is equivalent to the equality $\mathcal{C}_{\mathrm{c}}(\mathcal{A}; \mathcal{A}, \mathcal{A}) = \mathcal{C}(\mathcal{A}; \mathcal{A}, \mathcal{A})$, and then to $\mathcal{A}$ being abelian by Proposition 1. □

4. CHANNEL INCOMPATIBILITY WITNESSES

From now on, we will always assume that the inclusion $\mathcal{C}_{\mathrm{c}}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) \subseteq \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is strict. In view of Proposition 1, this amounts to require that $\mathcal{A}$ is not abelian and $\dim \mathcal{B}_i \geq 2$ for all $i = 1, 2$.

For convenience, we denote $\vec{\Phi} = (\Phi_1, \Phi_2)$. A *(channel) incompatibility witness* (CIW) is a map $\xi : \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) \to \mathbb{R}$ having the following three properties:

(W1) $\xi(\vec{\Phi}) \geq 0$ for all $\vec{\Phi} \in \mathcal{C}_{\mathrm{c}}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$;
(W2) $\xi(\vec{\Phi}) < 0$ at least for some incompatible pair $\vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$;
(W3) $\xi(t\vec{\Phi} + (1-t)\vec{\Psi})) = t\xi(\vec{\Phi}) + (1-t)\xi(\vec{\Psi})$ for all $\vec{\Phi}, \vec{\Psi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ and $t \in (0, 1)$.

We denote by $\mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ the set of all such maps ξ.

If $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ and $\xi(\vec{\Phi}) < 0$, we say that ξ *detects* the incompatible pair of channels $\vec{\Phi}$; the set of all detected pairs is denoted by $\mathcal{D}(\xi)$. The larger the set $\mathcal{D}(\xi)$, the more efficient is the CIW ξ in detecting incompatibility. Given another $\xi' \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, we say that ξ' is *finer* than ξ whenever $\mathcal{D}(\xi) \subseteq \mathcal{D}(\xi')$. Further, two witnesses ξ and ξ' are called *detection equivalent* if $\mathcal{D}(\xi) = \mathcal{D}(\xi')$. For any choice of ξ, we can always construct another CIW $\bar{\xi}$ which is finer than ξ by setting

$$\bar{\xi}(\vec{\Phi}) = \xi(\vec{\Phi}) - \min\{\xi(\vec{\Psi}) \mid \vec{\Psi} \in \mathcal{C}_{\mathrm{c}}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)\}\,. \tag{1}$$
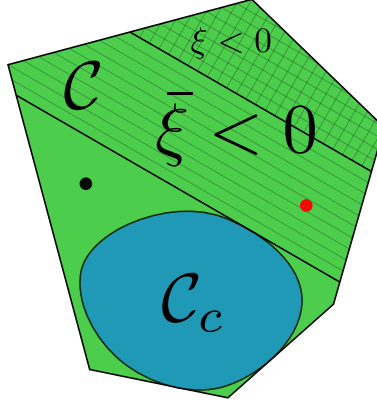
FIGURE 1. The set of incompatible pairs of channels which are detected by an incompatibility witness ξ (cross-hatched region) and those which are detected by the tight version $\bar{\xi}$ of ξ (hatched region). The red point is detected by $\bar{\xi}$ but not by ξ, while the black point is not detected by either witness.

In the case $\xi = \bar{\xi}$, we say that ξ is *tight* (see Fig. 1).

We observe that, when restricting to the particular case in which $\mathcal{A}$ is a full matrix algebra and the algebras $\mathcal{B}_i$ are abelian, the above definition of CIW coincides with the definition of incompatibility witnesses for quantum measurements introduced in Ref. [9]. Indeed, as we have already seen, measurements $\mathsf{M}_i : X_i \to \mathcal{A}$ and channels $\Phi_i : \mathcal{A}_* \to \mathcal{B}_{i*}$ are naturally identified if $\mathcal{B}_i = \ell^\infty(X_i)$, and the two notions of compatibility for measurements and channels are the same under this identification. Properties (W1)-(W3) are then a rewriting of the similar ones stated in Ref. [9]. Related investigations on incompatibility witnesses have been reported in Refs. [31, 32].

By standard separation results for convex compact sets, witnesses are enough to detect all incompatible pairs of channels.

Proposition 2. *For any incompatible pair of channels $\Phi_1 : \mathcal{A}_* \to \mathcal{B}_{1*}$ and $\Phi_2 : \mathcal{A}_* \to \mathcal{B}_{2*}$, there exist a channel incompatibility witness $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ detecting the pair (Φ_1, Φ_2).*

Proof. Denote by $\mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})$ the real vector space of all complex linear maps $\Phi_i : \mathcal{A}_* \to \mathcal{B}_{i*}$ satisfying $\Phi_{i*}(\mathcal{A}_*^{\mathrm{sa}}) \subseteq \mathcal{B}_{i*}^{\mathrm{sa}}$. Then, the sets $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ and $\mathcal{C}_\mathrm{c}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ are convex compact subsets of the Cartesian product $\mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{1*}) \times \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{2*})$. If $\vec{\Phi} \notin \mathcal{C}_\mathrm{c}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, by Ref. [33, Cor. 11.4.2] there exist elements $\phi_i \in \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})^*$ and $\delta \in \mathbb{R}$ such that $\sum_{i=1,2} \langle \phi_i, \Phi_i \rangle > \delta$ and $\sum_{i=1,2} \langle \phi_i, \Psi_i \rangle \leq \delta$ for all $\vec{\Psi} \in \mathcal{C}_\mathrm{c}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. Here, $\mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})^*$ denotes the linear dual of $\mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})$, and $\langle \phi_i, \Phi_i \rangle$ is the canonical pairing between elements $\phi_i \in \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})^*$ and $\Phi_i \in \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})$. Setting $\xi(\vec{\Psi}) = \delta - \sum_{i=1,2} \langle \phi_i, \Psi_i \rangle$ for all $\vec{\Psi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, we thus obtain a CIW for which $\vec{\Phi} \in \mathcal{D}(\xi)$. $\square$

5. Channel incompatibility witnesses as a state discrimination task

We consider the following state discrimination task, in which Bob is asked to retrieve a string of classical information which Alice sends to him through some communication channel which can be classical, quantum or semi-quantum.

(i) Alice randomly picks a label z with probability $p(z)$ and she encodes it into a state a_z. The label z is chosen within either one of two finite disjoint sets X_1 and X_2. The state a_z belongs to the predual of the von Neumann agebra $\mathcal{A}$ which describes Alice's system.

(ii) Alice then sends the state a_z to Bob. At a later and still unspecified time, she also communicates him the set X_i from which she picked the label z.

(iii) Bob processes the received state a_z by converting it into a bipartite system $\mathcal{B} = \mathcal{B}_1 \bar{\otimes} \mathcal{B}_2$. This amounts to applying a channel $\Phi : \mathcal{A}_* \to (\mathcal{B}_1 \bar{\otimes} \mathcal{B}_2)_*$, thus obtaining the bipartite state $\Phi(a_z)$ on Bob's side.

(iv) In order to retrieve the label z, Bob performs two local measurements M_1 on the subsystem $\mathcal{B}_1$ and M_2 on the subsystem $\mathcal{B}_2$. Each measurement M_j has outcomes in the corresponding label set X_j. The probability that Bob jointly obtains the outcomes x_1 and x_2 from the respective measurements M_1 and M_2 is thus $\langle \Phi(a_z), \mathsf{M}_1(x_1) \otimes \mathsf{M}_2(x_2) \rangle$.

(v) Finally, according to the set X_i communicated by Alice, Bob's guess for the label z is the outcome x_i.

The disjoint sets X_1 and X_2, the probability p on the union $X_1 \cup X_2$ and the states $\{a_z \mid z \in X_1 \cup X_2\}$ used by Alice in her encoding are fixed and known by both parties. Also the two measurements M_1 and M_2 used by Bob are fixed. Only the channel $\Phi : \mathcal{A}_* \to (\mathcal{B}_1 \bar{\otimes} \mathcal{B}_2)_*$ can be freely chosen by Bob.

According to the time when Alice communicates to Bob the chosen set X_i, two scenarios then arise.

(a) *Preprocessing information scenario*: Alice communicates the value of i to Bob *before* he processes the received state a_z. Bob can then optimize the choice of Φ according to Alice's information. If $\Phi_{(i)}$ is the channel he uses when Alice communicates him the set X_i, his probability of guessing the correct label is

$$P_{\text{guess}}^{\text{prior}} = \sum_{i=1,2} \sum_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} p(x_i) \langle \Phi_{(i)}(a_{x_i}), \mathsf{M}_1(x_1) \otimes \mathsf{M}_2(x_2) \rangle$$
$$= \sum_{i=1,2} \sum_{z \in X_i} p(z) \langle \Pi_i \circ \Phi_{(i)}(a_z), \mathsf{M}_i(z) \rangle.$$

This quantity depends only on the two margin channels $\Phi_1 = \Pi_1 \circ \Phi_{(1)}$ and $\Phi_2 = \Pi_2 \circ \Phi_{(2)}$. Since $\Phi_{(1)}$ and $\Phi_{(2)}$ are arbitrary, the pair (Φ_1, Φ_2) can be any element of $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$.

(b) *Postprocessing information scenario*: Alice communicates the value of i to Bob *after* he processes the received state a_z. Bob is then forced to choose Φ without knowing the set X_i chosen by Alice. His channel Φ is thus the same regardless of the value of i. In this scenario, Bob's probability of guessing the correct label
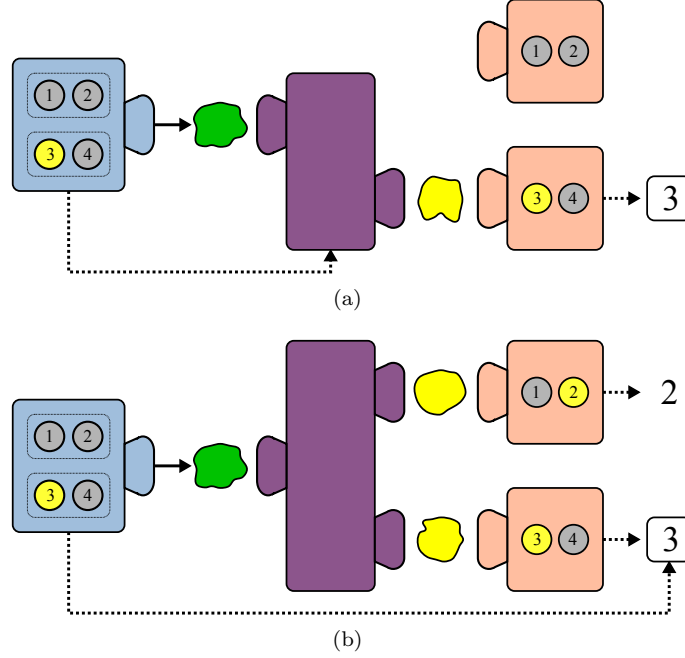
FIGURE 2. Alice encodes a label $z \in X_i$ into a state a_z, which then she sends to Bob. Bob uses a channel Φ to convert a_z into a bipartite system, and subsequently he performs a local measurement on each subsystem. According to the value of i communicated by Alice, Bob's guess for z is the result obtained in the i-th subsystem. In the *preprocessing information scenario* (a), Alice announces the value of i before Bob arranges the channel Φ, thus allowing him to optimize his choice for the i-th measurement. In the *postprocessing information scenario* (b), Alice announces i only after Bob has performed both measurements. In all scenarios, the channel Φ can be modified by Bob, while his measurements remain fixed.

is

$$
\begin{aligned}
P_{\text{guess}}^{\text{post}} &= \sum_{i=1,2} \sum_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} p(x_i) \langle\, \Phi(a_{x_i})\,,\, \mathsf{M}_1(x_1) \otimes \mathsf{M}_2(x_2)\,\rangle \\
&= \sum_{i=1,2} \sum_{z \in X_i} p(z) \langle\, \Pi_i \circ \Phi(a_z)\,,\, \mathsf{M}_i(z)\,\rangle\,.
\end{aligned}
$$

The latter quantity depends on the two margins $\Phi_1 = \Pi_1 \circ \Phi$ and $\Phi_2 = \Pi_2 \circ \Phi$ of a single channel Φ. These need to be a pair of compatible channels $(\Phi_1, \Phi_2) \in \mathcal{C}_{\text{c}}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$.

The two scenarios are depicted in Fig. 2.

It is useful to merge the probability distribution p on $X_1 \cup X_2$ and the states $\{a_z \mid z \in X_1 \cup X_2\}$ into a single map $\mathcal{E} : X_1 \cup X_2 \to \mathcal{A}_*$, defined as $\mathcal{E}(z) = p(z)\, a_z$. We call this map a *state ensemble* with label set $X_1 \cup X_2$. Its defining properties

are that $\mathcal{E}(z) \in \mathcal{A}_*^+$ for all z and $\sum_{z \in X_1 \cup X_2} \mathcal{E}(z) \in \mathcal{S}(\mathcal{A})$. We further denote by $\mathscr{P}$ the pair of disjoint sets (X_1, X_2), and we collect the two measurements M_1 and M_2 within a single vector $\vec{\mathsf{M}} = (\mathsf{M}_1, \mathsf{M}_2)$. The procedure described in steps (i)-(v) is thus completely determined by the triple $(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$, together with the choice between scenarios (a) and (b).

In the two guessing probabilities described above, the pair of sets $\mathscr{P}$, the state ensemble $\mathcal{E}$ and the measurement vector $\vec{\mathsf{M}}$ are fixed objects, while the channels $\Phi_i = \Pi_i \circ \Phi_{(i)}$ and Φ are variable quantities. To stress it, we rewrite

$$P_{\text{guess}}^{\text{prior}}(\Phi_1, \Phi_2 \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) = \sum_{i=1,2} \sum_{z \in X_i} \langle \Phi_i(\mathcal{E}(z)), \mathsf{M}_i(z) \rangle, \tag{2}$$

$$P_{\text{guess}}^{\text{post}}(\Phi \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) = P_{\text{guess}}^{\text{prior}}(\Pi_1 \circ \Phi, \Pi_2 \circ \Phi \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}). \tag{3}$$

Optimizing these probabilities over the respective sets of channels, we obtain Bob's maximal guessing probabilities in the two scenarios:

$$P_{\text{guess}}^{\text{prior}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) = \max\{P_{\text{guess}}^{\text{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) \mid \vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)\}, \tag{4}$$

$$P_{\text{guess}}^{\text{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) = \max\{P_{\text{guess}}^{\text{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) \mid \vec{\Phi} \in \mathcal{C}_{\text{c}}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)\}. \tag{5}$$

Clearly, $P_{\text{guess}}^{\text{prior}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) \geq P_{\text{guess}}^{\text{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$. Whenever the inequality is strict, the expression

$$\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}(\vec{\Phi}) = P_{\text{guess}}^{\text{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) - P_{\text{guess}}^{\text{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) \qquad \forall \vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) \tag{6}$$

defines a tight CIW $\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}} \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. We call it the CIW *associated with the state discrimination task* $(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$. Remarkably, no generality is lost in considering only CIWs of this form, as it is shown in the following main result.

Theorem 1. *Suppose that X_1 and X_2 are two finite disjoint sets, that $\mathsf{M}_1 : X_1 \to \mathcal{B}_1$ and $\mathsf{M}_2 : X_2 \to \mathcal{B}_2$ are two informationally complete measurements, and let $\mathscr{P} = (X_1, X_2)$ and $\vec{\mathsf{M}} = (\mathsf{M}_1, \mathsf{M}_2)$. Then, for any channel incompatibility witness $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, there exists a state ensemble $\mathcal{E} : X_1 \cup X_2 \to \mathcal{A}_*$ and real constants $\alpha > 0$ and $P_{\text{guess}}^{\text{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) \leq \delta < P_{\text{guess}}^{\text{prior}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$ such that*

$$\xi(\vec{\Phi}) = \alpha[\delta - P_{\text{guess}}^{\text{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})] \qquad \forall \vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2). \tag{7}$$

In particular, $\bar{\xi} = \alpha \xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}$, and thus the channel incompatibility witness $\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}$ is finer than ξ.

We emphasize that in Theorem 1 the sets X_1, X_2 and the measurements M_1, M_2 are fixed quantities, while the state ensemble $\mathcal{E}$ and the real constants α, δ depend upon the CIW at hand. Thus, the only free parameters which effectively enter the description of an arbitrary CIW are just the quantities $\mathcal{E}$, α and δ. For a tight CIW, the free parameters actually reduce to only $\mathcal{E}$ and α. We further note that by Ref. [34, Prop. 1], there exist informationally complete measurements $\mathsf{M}_1 : X_1 \to \mathcal{B}_1$ and $\mathsf{M}_2 : X_2 \to \mathcal{B}_2$ such that the cardinalities of the respective outcome sets are $|X_i| = \dim \mathcal{B}_i$. As a consequence of this fact, one can always choose X_1, X_2 with cardinalities $|X_i| = \dim \mathcal{B}_i$.

Proof of Theorem 1. As already seen in the proof of Proposition 2, the set $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ is a convex subset of the Cartesian product $\mathcal{L}^{\text{sa}}(\mathcal{A}_*; \mathcal{B}_{1*}) \times \mathcal{L}^{\text{sa}}(\mathcal{A}_*; \mathcal{B}_{2*})$, where we denote by $\mathcal{L}^{\text{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})$ the real vector space of all complex linear maps $\Phi_i :$

$\mathcal{A}_* \to \mathcal{B}_{i*}$ satisfying $\Phi_{i*}(\mathcal{A}_*^{\mathrm{sa}}) \subseteq \mathcal{B}_{i*}^{\mathrm{sa}}$. Then, for any CIW $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, by Ref. [9, Prop. S2 of the Supplementary Material], there exist a dual element $(\phi_1, \phi_2) \in \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{1*})^* \times \mathcal{L}(\mathcal{A}_*; \mathcal{B}_{2*})^*$ and $\delta_0 \in \mathbb{R}$ such that

$$\xi(\vec{\Phi}) = \delta_0 - \sum_{i=1,2} \langle\, \phi_i \,,\, \Phi_i \,\rangle \qquad \forall \vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)\,.$$

The dual space $\mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})^*$ is identified with the real algebraic tensor product $\mathcal{A}_*^{\mathrm{sa}} \otimes \mathcal{B}_i^{\mathrm{sa}}$ by setting

$$\langle\, a \otimes B_i \,,\, \Phi_i \,\rangle = \langle\, \Phi_i(a) \,,\, B_i \,\rangle \qquad \forall a \in \mathcal{A}_*^{\mathrm{sa}}, B_i \in \mathcal{B}_i^{\mathrm{sa}}, \Phi_i \in \mathcal{L}^{\mathrm{sa}}(\mathcal{A}_*; \mathcal{B}_{i*})\,.$$

Then, since the set $\{\mathsf{M}_i(z) \mid z \in X_i\}$ spans $\mathcal{B}_i^{\mathrm{sa}}$, we have

$$\phi_i = \sum_{z \in X_i} a_i(z) \otimes \mathsf{M}_i(z)$$

for some choice of elements $\{a_i(z) \mid z \in X_i\}$. Now, fix any faithful state of $\mathcal{A}$, that is, any $a_0 \in \mathcal{S}(\mathcal{A})$ such that $\langle\, a_0 \,,\, A \,\rangle > 0$ for all $A \in \mathcal{A}^+$ with $A \neq 0$. Such a state exists by standard arguments; see Ref. [26, Sec. I.9, Exercise 3.(b)]. Then, if $\beta \in \mathbb{R}$ is such that

$$\beta > \frac{\max\{\|a_i(z)\| \mid z \in X_i\,, i = 1,2\}}{\min\{\langle\, a_0 \,,\, A \,\rangle \mid A \in \mathcal{A}^+, \|A\| = 1\}}\,,$$

we have $\langle\, \beta a_0 + a_i(z) \,,\, A \,\rangle > 0$ for all $A \in \mathcal{A}^+$ with $A \neq 0$ and $z \in X_i$, $i = 1, 2$. Therefore, we can define the state ensemble $\mathcal{E} : X_1 \cup X_2 \to \mathcal{A}_*$ given by

$$\mathcal{E}(z) = \frac{1}{\alpha}\left(\beta a_0 + a_i(z)\right) \qquad \forall z \in X_i, i = 1, 2\,,$$

where the normalization constant $\alpha > 0$ is

$$\alpha = \sum_{i=1,2} \sum_{z \in X_i} \langle\, \beta a_0 + a_i(z) \,,\, \mathbb{1}_{\mathcal{A}} \,\rangle\,.$$

For the state ensemble $\mathcal{E}$, we have

$$\xi(\vec{\Phi}) = \delta_0 - \sum_{i=1,2} \sum_{z \in X_i} \langle\, a_i(z) \otimes \mathsf{M}_i(z) \,,\, \Phi_i \,\rangle = \delta_0 - \sum_{i=1,2} \sum_{z \in X_i} \langle\, \Phi_i(a_i(z)) \,,\, \mathsf{M}_i(z) \,\rangle$$

$$= \delta_0 + 2\beta - \alpha \sum_{i=1,2} \sum_{z \in X_i} \langle\, \Phi_i(\mathcal{E}(z)) \,,\, \mathsf{M}_i(z) \,\rangle = \alpha\big[\delta - P_{\mathrm{guess}}^{\mathrm{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})\big]\,,$$

in which we set $\delta = (\delta_0 + 2\beta)/\alpha$. Since ξ is a CIW, property (W1) and (5) imply the inequality $\delta \geq P_{\mathrm{guess}}^{\mathrm{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$, while on the other hand property (W2) and (4) require that $\delta < P_{\mathrm{guess}}^{\mathrm{prior}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$. By inserting (7) into (1) and using again (5), we immediately obtain the equality $\bar{\xi} = \alpha \xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}$, and hence the CIW $\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}$ is finer than ξ. $\qquad\square$

As a consequence of Theorem 1, for any pair of incompatible channels $(\Phi_1, \Phi_2) \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$, there exists some state discrimination task in which Bob can improve his guessing probability by choosing among Φ_1 and Φ_2 according to the preprocessing information. From an equivalent point of view, whenever Bob's strategy is to arrange his channel Φ_i after he knows the value of i, one can find a triple $(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$ that reveals Bob's use of preprocessing information. This is the content of the next corollary.

Corollary 2. *Let $\mathscr{P} = (X_1, X_2)$ and $\vec{\mathsf{M}} = (\mathsf{M}_1, \mathsf{M}_2)$, with X_i and M_i as in Theorem 1. Two channels $\Phi_1 : \mathcal{A}_* \to \mathcal{B}_{1*}$ and $\Phi_2 : \mathcal{A}_* \to \mathcal{B}_{2*}$ are incompatible if and only if there exists some state ensemble $\mathcal{E} : X_1 \cup X_2 \to \mathcal{A}_*$ such that*

$$P_{\mathrm{guess}}^{\mathrm{prior}}(\Phi_1, \Phi_2 \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) > P_{\mathrm{guess}}^{\mathrm{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}). \tag{8}$$

As in the statement of Theorem 1, also in the above corollary the sets X_1, X_2 and the measurements M_1, M_2 are independent of the incompatible channels Φ_1 and Φ_2. Indeed, only the state ensemble $\mathcal{E}$ needs to be arranged to detect incompatibility.

Proof of Corollary 2. The 'if' statement trivially follows from the definition (5) of the guessing probability $P_{\mathrm{guess}}^{\mathrm{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$, so we prove the 'only if' part. By Proposition 2, there exists a witness $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ such that $\vec{\Phi} \in \mathcal{D}(\xi)$. On the other hand, by Theorem 1, we can construct a state ensemble $\mathcal{E} : X_1 \cup X_2 \to \mathcal{A}_*$ such that the CIW $\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}$ is finer than ξ. This means that in (6) we have $\xi_{\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}}(\vec{\Phi}) < 0$, that is, $P_{\mathrm{guess}}^{\mathrm{prior}}(\vec{\Phi} \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}}) > P_{\mathrm{guess}}^{\mathrm{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$. □

In Corollary 2, the probability $P_{\mathrm{guess}}^{\mathrm{post}}(\mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$ can be calculated analytically or numerically, or at least upper bounded tightly enough, by solving a convex optimization problem. On the other hand, the probability $P_{\mathrm{guess}}^{\mathrm{prior}}(\Phi_1, \Phi_2 \parallel \mathscr{P}, \mathcal{E}, \vec{\mathsf{M}})$ is assessable by using Alice's classical information, and then performing quantum measurements only on Bob's side. Since no entangled state is shared in the state discrimination protocol, Corollary 2 provides a more practical way to detect incompatibility than schemes based on Bell experiments or steering. In particular, as a fundamental fact, entanglement is not needed to detect incompatibility.

A particular instance of the scheme introduced in this section is the discrimination task with pre- and postmeasurement information described and studied in Refs. [12, 35, 36, 37]. In the latter task, Bob is asked to retrieve Alice's label $z \in X_1 \cup X_2$ by simply performing a measurement N on the received state a_z, without making any processing of a_z before that. The outcome set of N is assumed to be the Cartesian product $X_1 \times X_2$. When Bob obtains the outcome (x_1, x_2) and Alice communicates him that $z \in X_i$, his guess for z is the value x_i. According to the time when Bob is informed about i – either before or after he performs the measurement – the choice of N optimizing the correct guessing probability may be different. Consequently, also in this task Bob's maximal guessing probability may vary according to the pre- or postmeasurement information scenario.

The state discrimination task with pre- or postmeasurement information can be recast into the general scheme described at the beginning of this section by fixing the commutative algebras $\mathcal{B}_i = \ell^\infty(X_i)$ as Bob's subsystems, identifying the measurement N with the channel $\widehat{\mathsf{N}} : \mathcal{A}_* \to \ell^1(X_1 \times X_2)$ and letting $\mathsf{M}_i : X \to \mathcal{B}_i$ be the projective measurements corresponding to simply reading off the outcome of N. In this way, Ref. [9, Thms. 1 and 2] are particular instances of the above Theorem 1 and Corollary 2.

6. FROM MEASUREMENT TO CHANNEL INCOMPATIBILITY WITNESSES

In this section, we provide examples of a tight channel incompatibility witness $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ for each of the three cases $\mathcal{B}_1 = \mathcal{B}_2 = \ell^\infty(X)$ (incompatibility of two measurements), $\mathcal{B}_1 = \ell^\infty(X)$ and $\mathcal{B}_2 = \mathcal{L}(\mathcal{H})$ (incompatibility of a measurement and a channel) and $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{L}(\mathcal{H})$ (incompatibility of two channels). We always

consider the standard quantum input $\mathcal{A} = \mathcal{L}(\mathcal{H})$. Moreover, we assume that the cardinality of the outcome set X equals the dimension d of the Hilbert space $\mathcal{H}$. Our examples are based on the fact that, by using the next simple observation, the results of Ref. [9] immediately yield instances of CIWs also for $\mathcal{B}_1$ and $\mathcal{B}_2$ being non-abelian.

Proposition 3. *Suppose $\xi \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \ell^\infty(X))$ and let $\mathsf{P} : X \to \mathcal{B}_2$ be a measurement such that $\mathsf{P}(x)$ is a non-zero projection of $\mathcal{B}_2$ for all $x \in X$. Define the map $\xi_\mathsf{P} : \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2) \to \mathbb{R}$ as*

$$\xi_\mathsf{P}(\vec{\Phi}) = \xi(\Phi_1, \widehat{\mathsf{P}} \circ \Phi_2) \qquad \forall \vec{\Phi} \in \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2). \tag{9}$$

Then $\xi_\mathsf{P} \in \mathcal{W}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$. Moreover, ξ_P is tight if ξ is such.

Proof. Properties (W1) and (W3) for ξ_P follow from the analogous properties for ξ and from the fact that $\Phi_1 = \mathrm{id} \circ \Phi_1$ and $\widehat{\mathsf{P}} \circ \Phi_2$ are compatible if Φ_1 and Φ_2 are. In order to prove property (W2), fix any faithful state $b_0 \in \mathcal{S}(\mathcal{B}_2)$. For all $x \in X$, let $b_{0,x} \in \mathcal{S}(\mathcal{B}_2)$ be given by $\langle\, b_{0,x}\,,\, B\,\rangle = \langle\, b_0\,,\, \mathsf{P}(x)B\mathsf{P}(x)\,\rangle / \langle\, b_0\,,\, \mathsf{P}(x)\,\rangle$ for all $B \in \mathcal{B}_2$. Further, define the linear map $\Psi : \ell^1(X) \to \mathcal{B}_{2*}$ with $\Psi(f) = \sum_{x \in X} f(x)\, b_{0,x}$. Such a map is a channel, since its adjoint $\Psi^* : \mathcal{B}_2 \to \ell^\infty(X)$ is unital and $\ell^\infty(X)$ is abelian. Then, it is easy to check that the composition channel $\widehat{\mathsf{P}} \circ \Psi$ is the identity map of $\ell^1(X)$, from which it follows that, for any measurement $\mathsf{M} : X \to \mathcal{A}$, we have $\widehat{\mathsf{P}} \circ \Psi \circ \widehat{\mathsf{M}} = \widehat{\mathsf{M}}$. In particular, for $\Phi_2 = \Psi \circ \widehat{\mathsf{M}}$ we have $\xi_\mathsf{P}(\Phi_1, \Phi_2) < 0$ if $(\Phi_1, \widehat{\mathsf{M}}) \in \mathcal{D}(\xi)$, thus showing property (W2) for ξ_P. If instead Φ_1 and $\widehat{\mathsf{M}}$ are compatible and $\xi(\Phi_1, \widehat{\mathsf{M}}) = 0$, then also $\Phi_1 = \mathrm{id} \circ \Phi_1$ and $\Phi_2 = \Psi \circ \widehat{\mathsf{M}}$ are compatible and $\xi_\mathsf{P}(\Phi_1, \Phi_2) = 0$, thus implying that ξ_P is tight whenever ξ is such. $\qquad \square$

The composition channel $\Psi \circ \widehat{\mathsf{M}}$ introduced in the previous proof is the *measure-and-prepare channel* associated with the measurement M and the family of states $\{b_{0,x} \mid x \in X\} \subset \mathcal{S}(\mathcal{B}_2)$. Explicitly,

$$(\Psi \circ \widehat{\mathsf{M}})(a) = \sum_{x \in X} \langle\, a\,,\, \mathsf{M}(x)\,\rangle\, b_{0,x} \qquad \forall a \in \mathcal{A}_*.$$

Note that $\Psi \circ \widehat{\mathsf{M}}$ is a channel even if the supports of the states $\{b_{0,x} \mid x \in X\}$ are not orthogonal.

We start constructing our examples by recalling a family of inequivalent tight witnesses $\xi_{\boldsymbol{\mu}} \in \mathcal{W}(\mathcal{L}(\mathcal{H}); \ell^\infty(X), \ell^\infty(X))$ which was described in Ref. [9, Thm. 3]. This family is constructed by fixing two mutually unbiased bases $\{e_x \mid x \in X\}$ and $\{f_x \mid x \in X\}$ of $\mathcal{H}$, and it depends on the direction of a two-dimensional vector $\boldsymbol{\mu} \in \mathbb{R}^2$. Here we recall only the following example, which corresponds to the equally weighted choice $\boldsymbol{\mu} = (1,1)$ for the parameter $\boldsymbol{\mu}$:

$$\xi_{\mathrm{mm}}(\widehat{\mathsf{M}}, \widehat{\mathsf{N}}) = \frac{1}{2d} \left\{ \sqrt{d}(\sqrt{d} + 1) - \sum_{x \in X} \left[\langle\, e_x \mid \mathsf{M}(x)e_x\,\rangle + \langle\, f_x \mid \mathsf{N}(x)f_x\,\rangle \right] \right\}. \tag{10}$$

In the previous formula, $\langle\, \cdot \mid \cdot\,\rangle$ is the inner product of the Hilbert space $\mathcal{H}$. The measurement-measurement incompatibility witness (10) vanishes when evaluated on the compatible pair of quantum measurements

$$\mathsf{M}_0(x) = \gamma(d)|e_x\rangle\langle e_x| + (1 - \gamma(d))\frac{\mathbb{1}}{d}, \qquad \mathsf{N}_0(x) = \gamma(d)|f_x\rangle\langle f_x| + (1 - \gamma(d))\frac{\mathbb{1}}{d}, \tag{11}$$

where $\mathbb{1}$ is the identity operator on $\mathcal{H}$ and $\gamma(d)$ is the real constant

$$\gamma(d) = \frac{\sqrt{d}+2}{2(\sqrt{d}+1)} \ . \tag{12}$$

This implies that the measurements

$$\mathsf{M}(x) = \gamma|e_x\rangle\langle e_x| + (1-\gamma)\,\frac{\mathbb{1}}{d}\,, \qquad \mathsf{N}(x) = \gamma|f_x\rangle\langle f_x| + (1-\gamma)\,\frac{\mathbb{1}}{d}\,,$$

are incompatible if and only if $\gamma(d) < \gamma \leq 1$, a result that was earlier obtained in Refs. [38, 39] by using different methods.

The previous measurement-measurement witness can be immediately turned into a tight witness $\xi_{\mathrm{mc}} \in \mathcal{W}(\mathcal{L}(\mathcal{H}); \ell^\infty(X), \mathcal{L}(\mathcal{H}))$ by means of Proposition 3. Indeed, it is enough to fix another orthonormal basis $\{h_x \mid x \in X\}$, set $\mathsf{P}(x) = |h_x\rangle\langle h_x|$ and define

$$\begin{aligned}
\xi_{\mathrm{mc}}(\widehat{\mathsf{M}}, \Lambda) &= (\xi_{\mathrm{mm}})_{\mathsf{P}}(\widehat{\mathsf{M}}, \Lambda) = \xi_{\mathrm{mm}}(\widehat{\mathsf{M}}, \widehat{\mathsf{P}} \circ \Lambda) \\
&= \frac{1}{2d}\left\{ \sqrt{d}(\sqrt{d}+1) - \sum_{x \in X} \left[\langle\, e_x \mid \mathsf{M}(x)e_x \,\rangle + \langle\, h_x \mid \Lambda(|f_x\rangle\langle f_x|)\, h_x \,\rangle \right] \right\}.
\end{aligned} \tag{13}$$

We have $\xi_{\mathrm{mc}}(\widehat{\mathsf{M}}_0, \Lambda_{\mathsf{N}_0}) = 0$ for the compatible pair $(\widehat{\mathsf{M}}_0, \Lambda_{\mathsf{N}_0})$, in which $\widehat{\mathsf{M}}_0$ is given by (11) and Λ_{N_0} is the measure-and-prepare quantum channel

$$\Lambda_{\mathsf{N}_0}(a) = \sum_{x \in X} \mathrm{tr}\,[a\mathsf{N}_0(x)]\,|h_x\rangle\langle h_x| \qquad \forall a \in \mathcal{L}^1(\mathcal{H}) \tag{14}$$

with N_0 still given by (11). Here, tr denotes the trace of $\mathcal{H}$.

In order to find an example of a tight witness $\xi_{\mathrm{cc}} \in \mathcal{W}(\mathcal{L}(\mathcal{H}); \mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$, we can still proceed along the same lines as above. Specifically, we can use the witness (11) and any two bases $\{g_x \mid x \in X\}$ and $\{h_x \mid x \in X\}$ of $\mathcal{H}$ in order to construct ξ_{cc} by means of Proposition 3. In this way, dropping the irrelevant factor $1/(2d)$, the resulting witness is

$$\xi_{\mathrm{cc}}(\Theta, \Lambda) = \sqrt{d}(\sqrt{d}+1) - \sum_{x \in X} \left[\langle\, g_x \mid \Theta(|e_x\rangle\langle e_x|)\, g_x \,\rangle + \langle\, h_x \mid \Lambda(|f_x\rangle\langle f_x|)\, h_x \,\rangle \right] \tag{15}$$

for all $(\Theta, \Lambda) \in \mathcal{C}(\mathcal{L}(\mathcal{H}); \mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$.

7. Incompatibility witness related to approximate cloning

As we have seen, the measurement-channel and the channel-channel incompatibility witnesses ξ_{mc} and ξ_{cc} derived in the previous section are adaptations of the measurement-measurement witness ξ_{mm} found in Ref. [9] and constructed by means of two mutually unbiased bases. Here we show that, by using a different method, another tight witness $\zeta_{\mathrm{cc}} \in \mathcal{W}(\mathcal{L}(\mathcal{H}); \mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$ can also be derived by fixing only one arbitrary orthonormal basis $\{e_x \mid x \in X\}$ of $\mathcal{H}$ and setting

$$\zeta_{\mathrm{cc}}(\Theta, \Lambda) = d(d+1) - \sum_{x,y \in X} \langle\, e_x \mid (\Theta + \Lambda)(|e_x\rangle\langle e_y|)\, e_y \,\rangle \ . \tag{16a}$$

Actually, the dependence of ζ_{cc} on the choice of the basis of $\mathcal{H}$ is not relevant. Indeed, (16a) can be rewritten in a basis independent form by using the trace Tr of the linear space $\mathcal{L}(\mathcal{H})$, so that

$$\zeta_{\mathrm{cc}}(\Theta, \Lambda) = d(d+1) - \mathrm{Tr}[\Theta + \Lambda] \ . \tag{16b}$$

For the witness ζ_{cc}, we have $\zeta_{cc}(\Theta_0, \Lambda_0) = 0$ when Θ_0 and Λ_0 are the two margins of the optimal approximate cloning channel found in Refs. [3, 4], i.e., the depolarizing quantum channels

$$\Theta_0(a) = \Lambda_0(a) = \gamma(d^2)a + (1 - \gamma(d^2))\mathrm{tr}\,[a]\,\frac{\mathbb{1}}{d} \qquad (17)$$

with $\gamma(d^2)$ defined by (12).

One can show with some calculation that the witnesses ξ_{cc} and ζ_{cc} are detection inequivalent, since inserting Θ_0 and Λ_0 into (15) yields

$$\xi_{cc}(\Theta_0, \Lambda_0) = (\sqrt{d} + 2)(\sqrt{d} - 1) + \gamma(d^2)\left[2 - \sum_{x \in X}(|\langle\, e_x\,|\,g_x\,\rangle|^2 + |\langle\, f_x\,|\,h_x\,\rangle|^2)\right],$$

which is strictly positive for all $d \geq 2$ and any choice of the bases $\{g_x \mid x \in X\}$ and $\{h_x \mid x \in X\}$. Thus, for suitably small $\varepsilon > 0$, the CIW ξ_{cc} does not detect the incompatible channels $\Theta = (1 + \varepsilon)\Theta_0 - \varepsilon\mathrm{tr}\,[\cdot]\,\mathbb{1}/d$ and $\Lambda = (1 + \varepsilon)\Lambda_0 - \varepsilon\mathrm{tr}\,[\cdot]\,\mathbb{1}/d$, which instead are detected by ζ_{cc}.

In a similar way, if we insert the compatible measure-and-prepare channels

$$\Theta_{\mathsf{M}_0}(a) = \sum_{x \in X}\mathrm{tr}\,[a\mathsf{M}_0(x)]\,|g_x\rangle\langle g_x|, \qquad \Lambda_{\mathsf{N}_0}(a) = \sum_{x \in X}\mathrm{tr}\,[a\mathsf{N}_0(x)]\,|h_x\rangle\langle h_x| \qquad (18)$$

into (16), where M_0 and N_0 are the compatible measurements (11), we obtain

$$\zeta_{cc}(\Theta_{\mathsf{M}_0}, \Lambda_{\mathsf{N}_0}) = (d + 2)(d - 1) + \gamma(d)\left[2 - \sum_{x \in X}(|\langle\, e_x\,|\,g_x\,\rangle|^2 + |\langle\, f_x\,|\,h_x\,\rangle|^2)\right],$$

which is strictly positive for all $d \geq 2$ and any bases $\{g_x \mid x \in X\}$ and $\{h_x \mid x \in X\}$. Since on the other hand $\xi_{cc}(\Theta_{\mathsf{M}_0}, \Lambda_{\mathsf{N}_0}) = 0$, a similar reasoning as in the previous paragraph yields that $\mathcal{D}(\xi_{cc}) \not\subseteq \mathcal{D}(\zeta_{cc})$. Thus, neither ξ_{cc} is finer than ζ_{cc}, nor ζ_{cc} is finer than ξ_{cc}, thus proving that the two witnesses ξ_{cc} and ζ_{cc} are genuinely unrelated.

The rest of this section is devoted to the proof that the map ζ_{cc} defined in (16) is a tight CIW, and that $\zeta_{cc}(\Theta_0, \Lambda_0) = 0$ when Θ_0 and Λ_0 are the compatible channels defined in (17).

For any pair of channels $(\Theta, \Lambda) \in \mathcal{C}(\mathcal{L}(\mathcal{H}); \mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$, let

$$\zeta_0(\Theta, \Lambda) = \sum_{i,j=1}^{d}\langle\, e_i\,|\,(\Theta + \Lambda)\,(|e_i\rangle\langle e_j|)e_j\,\rangle$$

be the linear part of the witness (16). By denoting

$$\omega = \frac{1}{\sqrt{d}}\sum_{i=1}^{d}e_i \otimes e_i$$

the maximally entangled state associated with the given basis, the linear functional ζ_0 can be rewritten as

$$\zeta_0(\Theta, \Lambda) = d^2\left\{\mathrm{tr}\,[|\omega\rangle\langle\omega|\,(\Theta^* \otimes \mathrm{id}^*)\,(|\omega\rangle\langle\omega|)] + \mathrm{tr}\,[|\omega\rangle\langle\omega|\,(\Lambda^* \otimes \mathrm{id}^*)\,(|\omega\rangle\langle\omega|)]\right\},$$

where $\mathrm{id} : \mathcal{L}^1(\mathcal{H}) \to \mathcal{L}^1(\mathcal{H})$ is the identity channel.

Now, suppose Θ and Λ are compatible, and let Φ be a joint channel for them. Moreover, denote by $F : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ the flip operator $F(u \otimes v) = v \otimes u$. Then,

using the marginality conditions $\Phi^*(A \otimes \mathbb{1}) = \Theta^*$ and $\Phi^*(\mathbb{1} \otimes B) = \Lambda^*$ together with the relation $A \otimes \mathbb{1} = F(\mathbb{1} \otimes A)F$, we have

$$
\begin{aligned}
\zeta_0(\Theta, \Lambda) &= d^2 \big\{ \mathrm{tr} \left[|\omega\rangle\langle\omega| \, (\Phi^* \otimes \mathrm{id}^*) \left((F \otimes \mathbb{1})(\mathbb{1} \otimes |\omega\rangle\langle\omega|)(F \otimes \mathbb{1}) \right) \right] \\
&\quad + \mathrm{tr} \left[|\omega\rangle\langle\omega| \, (\Phi^* \otimes \mathrm{id}^*) \left(\mathbb{1} \otimes |\omega\rangle\langle\omega| \right) \right] \big\} \\
&= d^2 \mathrm{tr} \left[(\Phi \otimes \mathrm{id})(|\omega\rangle\langle\omega|) \, E \right] ,
\end{aligned}
\tag{19}
$$

where E is the selfadjoint positive operator

$$
E = (F \otimes \mathbb{1})(\mathbb{1} \otimes |\omega\rangle\langle\omega|)(F \otimes \mathbb{1}) + \mathbb{1} \otimes |\omega\rangle\langle\omega| .
$$

Hence, for any compatible pair (Θ, Λ), we have the following upper bound for (19)

$$
\zeta_0(\Theta, \Lambda) \leq d^2 \lambda_{\max}(E) ,
\tag{20}
$$

where $\lambda_{\max}(E)$ is the maximal eigenvalue of E.

We now evaluate $\lambda_{\max}(E)$ by finding the eigenspace decomposition of E. To this aim, we introduce the two operators $S_\pm = \frac{1}{2}(\mathbb{1} \pm F)$, which are the orthogonal projections onto the symmetric and antisymmetric subspaces of $\mathcal{H} \otimes \mathcal{H}$, respectively. Since

$$
E\left((e_l \otimes e_m \pm e_m \otimes e_l) \otimes u \right) =
$$

$$
= \frac{1}{d} \left[\sum_{i=1}^{d} \langle e_m \,|\, u \rangle \, (e_l \otimes e_i \pm e_i \otimes e_l) \otimes e_i \pm \sum_{i=1}^{d} \langle e_l \,|\, u \rangle \, (e_m \otimes e_i \pm e_i \otimes e_m) \otimes e_i \right] ,
\tag{21}
$$

we conclude that

$$
E\left(S_\pm(\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H} \right) = \left\{ \sum_{i=1}^{d} (v \otimes e_i \pm e_i \otimes v) \otimes e_i \ \mid \ v \in \mathcal{H} \right\} .
$$

Since E commutes with both projections $S_+ \otimes \mathbb{1}$ and $S_- \otimes \mathbb{1}$, and moreover $S_+ + S_- = \mathbb{1}$, we have the orthogonal decomposition

$$
E(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) = E\left(S_+(\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H} \right) \oplus E\left(S_-(\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H} \right) .
$$

Now we show that $E\left(S_+(\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H} \right)$ and $E\left(S_-(\mathcal{H} \otimes \mathcal{H}) \otimes \mathcal{H} \right)$ are the eigenspaces corresponding to the only two non-zero eigenvalues $\lambda_\pm(E) = (d \pm 1)/d$ of E. Indeed, another application of (21) yields

$$
E\left(\sum_{i=1}^{d} (v \otimes e_i \pm e_i \otimes v) \otimes e_i \right) = \frac{d \pm 1}{d} \sum_{i=1}^{d} (v \otimes e_i \pm e_i \otimes v) \otimes e_i .
$$

We thus conclude that $\lambda_{\max}(E) = (d+1)/d$, hence, for any compatible pair (Θ, Λ), by (20) we have

$$
\zeta_0(\Theta, \Lambda) \leq d(d+1) .
\tag{22}
$$

On the other hand, since for the identity channel we have

$$
\zeta_0(\mathrm{id}, \mathrm{id}) = 2d^2 \, |\langle \omega \,|\, \omega \rangle|^2 = 2d^2 > d(d+1) ,
$$

it follows that (16) defines a CIW.

Finally, as we already noticed, by Refs. [3, 4] the two depolarizing channels Θ_0 and Λ_0 defined in (17) are compatible, and an easy calculation yields $\zeta_0(\Theta_0, \Lambda_0) = d(d+1)$. The bound (22) is thus attained on $\mathcal{C}_c(\mathcal{L}(\mathcal{H}); \mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$, hence the witness (16) is tight.

8. Discussion

We have proved that incompatibility can always be detected by means of a state discrimination protocol. We have done it for systems described by arbitrary finite-dimensional von Neumann algebras, thus encompassing all possible hybrid quantum-classical cases. Our approach was based on the notion of channel incompatibility witness and its connection with a state discrimination task with intermediate partial information. Once we established this connection in Theorem 1, the main result in Corollary 2 easily followed from standard separation results for convex compact sets. We pointed out that all incompatible pairs of channels can be detected by tuning only the state ensemble on Alice's side, while Bob can keep his measurements fixed to this purpose.

The essential point in the presented formalism is that the set of all compatible channels is a convex compact subset of all pairs of channels. In fact, a similar mathematical technique works for any binary relation $\mathcal{R} \subset \mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ that is convex and compact. The state discrimination protocol is hence useful to detect also other resources, mathematically described as subsets of $\mathcal{C}(\mathcal{A}; \mathcal{B}_1, \mathcal{B}_2)$ with convex compact complements.

We finally provided four examples of channel incompatibility witnesses with standard quantum input $\mathcal{A} = \mathcal{L}(\mathcal{H})$. The first example (10) applies to measurement-measurement incompatibility and was taken from Ref. [9], while the second (13) and the third (15) are adaptations of the former one. The last example of channel-channel incompatibility witness (16) is unrelated to the measurement-measurement case. It would be interesting to develop it into a whole family of inequivalent witnesses in analogy with the results of Ref. [9]. We also point out that the measurement-channel case deserves further study, as the only presented example relies upon the measurement-measurement case. We defer more detailed investigations on these topics to future work.

During the preparation of the manuscript we became aware of recent related works by Uola, Kraft and Abbott [40] and by Mori [41].

Acknowledgements

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