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ASYMPTOTICALLY $\mathcal{I}_2$-LACUNARY STATISTICAL EQUIVALENCE OF DOUBLE SEQUENCES OF SETS

UĞUR ULUSU, ERDİNÇ DÜNDAR

Abstract. In this paper, we introduce the concepts of Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence, Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalence and Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of double sequences of sets and investigate the relationship between them.

1. introduction, definitions and notations

Throughout the paper $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [11] and Schoenberg [30]. This concept was extended to the double sequences by Mursaleen and Edely [19]. Çakan and Altay [6] presented multidimensional analogues of the results presented by Fridy and Orhan [13].

The idea of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [17] as a generalization of statistical convergence which is based on the structure of the idea $\mathcal{I}$ of subset of the set of natural numbers. Recently, Das et al. [7] introduced new notions, namely $\mathcal{I}$-statistical convergence and $\mathcal{I}$-lacunary statistical convergence by using ideal. Das, Kostyrko, Wilczyński and Malik [8] introduced the concept of $\mathcal{I}$-convergence of double sequences in a metric space and studied some properties of this convergence.

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [3–5, 34, 35]). Nuray and Rhoades [20] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [32] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Kişi and Nuray [15] introduced a new convergence notion, for sequences of sets, which is called Wijsman $\mathcal{I}$-convergence by using ideal. Recently, Ulusu and Dündar [31] studied the concepts of Wijsman $\mathcal{I}$-statistical convergence, Wijsman $\mathcal{I}$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}$-lacunary convergence of sequences of sets.

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Nuray et al. [22] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigate the relationship between them. Nuray et al. [21] studied the concepts of Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-convergence and Wijsman $\mathcal{I}_2$, $\mathcal{I}_2^*$-Cauchy double sequences of sets. Dündar et al. [10] introduced the concepts of the Wijsman $\mathcal{I}_2$-statistical convergence, Wijsman $\mathcal{I}_2$-lacunary statistical convergence and Wijsman strongly $\mathcal{I}_2$-lacunary convergence of double sequences of sets.

Marouf [18] peresented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [26] extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions. Patterson and Savaş [27] extend the definitions presented in [26] to lacunary sequences. In addition to these definitions, natural inclusion theorems were presented. Recently, Savaş [28] presented the concept of $\mathcal{I}$-asymptotically lacunary statistically equivalence which is a natural combination of the definitions for asymptotically equivalence and $\mathcal{I}$-lacunary statistical convergence.

The concept of asymptotically equivalence of sequences of real numbers which is defined by Marouf [18] has been extended by Ulusu and Nuray [33] to concept of Wijsman asymptotically equivalence of set sequences. In addition to these definitions, natural inclusion theorems are presented. Kişi et al. [16] introduced the concept of Wijsman $\mathcal{I}$-asymptotically equivalence of sequences of sets.

Now, we recall the basic definitions and concepts (See [1–3, 8–10, 12–14, 17, 18, 21–25, 29]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1.$$

It is denoted by $x \sim y$.

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

Let (X, ρ) be a metric space and A, A_k be any non-empty closed subsets of X. The sequence $\{A_k\}$ is Wijsman convergent to A if for each $x \in X$,

$$\lim_{k \to \infty} d(x, A_k) = d(x, A).$$

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if

(i) $\emptyset \in \mathcal{I}$, (ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$, (iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$.

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Throughout the paper we take $\mathcal{I}_2$ as an admissible ideal in $\mathbb{N} \times \mathbb{N}$.

A non-trivial ideal $\mathcal{I}_2$ of $\mathbb{N} \times \mathbb{N}$ is called strongly admissible if $\{i\} \times \mathbb{N}$ and $\mathbb{N} \times \{i\}$ belongs to $\mathcal{I}_2$ for each $i \in N$.

It is evident that a strongly admissible ideal is admissible also.

$\mathcal{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : (\exists m(A) \in \mathbb{N})(i, j \geq m(A) \Rightarrow (i, j) \notin A)\}$. Then $\mathcal{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathcal{I}_2$ is strongly admissible if and only if $\mathcal{I}_2^0 \subset \mathcal{I}_2$.

The class of all $A \subset \mathbb{N} \times \mathbb{N}$ which has natural density zero denoted by $\mathcal{I}_2^f$. Then $\mathcal{I}_2^f$ is strongly admissible ideal.

A family of sets $F \subseteq 2^{\mathbb{N}}$ is called a filter if and only if

(i) $\emptyset \notin F$, (ii) For each $A, B \in F$ we have $A \cap B \in F$, (iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$.

$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if $F(\mathcal{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathcal{I})(M = \mathbb{N} \backslash A)\}$ is a filter in $\mathbb{N}$.

An admissible ideal $\mathcal{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{A_1, A_2, ...\}$ belonging to $\mathcal{I}_2$, there exists a countable family of sets $\{B_1, B_2, ...\}$ such that $A_j \Delta B_j \in \mathcal{I}_2^0$, i.e., $A_j \Delta B_j$ is included in the finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}_2$ (hence $B_j \in \mathcal{I}_2$ for each $j \in \mathbb{N}$).

A double sequence $x = (x_{kj})_{k,j\in\mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case, we write

$$P - \lim_{k,j\to\infty} x_{kj} = L \quad \text{or} \quad \lim_{k,j\to\infty} x_{kj} = L.$$

Throughout the paper, we let (X, ρ) be a separable metric space, $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal and A, A_{kj} be any non-empty closed subsets of X.

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if for each $x \in X$,

$$P - \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A) \quad or \quad \lim_{k,j\to\infty} d(x, A_{kj}) = d(x, A).$$

The double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0,$$

that is, $|d(x, A_{kj}) - d(x, A)| < \varepsilon$ for almost every (k, j).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad as \quad r \to \infty$$

and

$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad as \quad u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u, \;\; h_{ru} = h_r \bar{h}_u, \;\; I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \;\; and \;\; j_{u-1} < j \leq j_u\},$$

$$q_r = \frac{k_r}{k_{r-1}} \;\; and \;\; q_u = \frac{j_u}{j_{u-1}}.$$

Let θ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

Let θ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} |\{(k,j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

The double sequence of sets $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-convergent to A, if for every $\varepsilon > 0$ and for each $x \in X$,

$$\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case we write $\mathcal{I}_{W_2} - \lim\limits_{k,j\to\infty} A_{kj} = A$.

We define $d(x; A_{kj}, B_{kj})$ as follows:

$$d(x; A_{kj}, B_{kj}) = \begin{cases} \dfrac{d(x, A_{kj})}{d(x, B_{kj})} & , \quad x \notin A_{kj} \cup B_{kj} \\ \\ L & , \quad x \in A_{kj} \cup B_{kj}. \end{cases}$$

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically equivalent of multiple L if for each $x \in X$, $\lim\limits_{k,j\to\infty} d(x; A_{kj}, B_{kj}) = L$.

The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically statistical equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{m,n\to\infty} \frac{1}{mn} \left| \left\{ k \leq m, j \leq n : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon \right\} \right| = 0.$$

Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically lacunary equivalent of multiple L if for each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| = 0.$$

Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically lacunary statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \left| \left\{ (k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon \right\} \right| = 0.$$

The sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-statistical convergent to A or $S(\mathcal{I}_{W_2})$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{ (m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(S(\mathcal{I}_{W_2}))$.

Let θ be a double lacunary sequence. The sequence $\{A_{kj}\}$ is said to be Wijsman strongly $\mathcal{I}_2$-lacunary convergent to A or $N_\theta[\mathcal{I}_{W_2}]$-convergent to A if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ (r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r\bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x, A_{kj}) - d(x, A)| \geq \varepsilon \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\,(N_\theta[\mathcal{I}_{W_2}])$.

Let θ be a double lacunary sequence. The sequence $\{A_{kj}\}$ is Wijsman $\mathcal{I}_2$-lacunary statistical convergent to A or $S_\theta\left(\mathcal{I}_{W_2}\right)$-convergent to A if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u} \left| \left\{(k,j) \in I_{ru} : |d(x,A_{kj}) - d(x,A)| \geq \varepsilon\right\} \right| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \to A\left(S_\theta\left(\mathcal{I}_{W_2}\right)\right)$.

$X \subset \mathbb{R}$, $f,g : X \to \mathbb{R}$ functions and a point $a \in X'$ are given. If $f(x) = \alpha(x)g(x)$ for $\forall x \in \overset{o}{U}_\delta(a) \cap X$, then for $x \in X$ we write $f = \mathcal{O}(g)$ as $x \to a$, where for any $\delta > 0$, $\alpha : X \to \mathbb{R}$ is bounded function on $\overset{o}{U}_\delta(a) \cap X$. In this case, if there exists a $c \geq 0$ such that $|f(x)| \leq c|g(x)|$ for $\forall x \in \overset{o}{U}_\delta(a) \cap X$, then for $x \in X$, $f = \mathcal{O}(g)$ as $x \to a$.

2. MAIN RESULTS

In this section, we define the concepts of Wijsman asymptotically $\mathcal{I}_2$-statistical equivalence, Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalence and Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalence of double sequences of sets and investigate the relationship between them.

Definition 2.1. *The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-equivalent of multiple L if for every $\varepsilon > 0$ and each $x \in X$*

$$\{(k,j) \in \mathbb{N}\times\mathbb{N} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{\mathcal{I}^L_{W_2}}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-equivalent if $L = 1$.

Definition 2.2. *The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,*

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn} |\{k \leq m, j \leq n : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}| \geq \delta \right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{S(\mathcal{I}^L_{W_2})}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent if $L = 1$. The set of Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent double sequences will be denoted by $\left\{S\left(\mathcal{I}^L_{W_2}\right)\right\}$.

For $\mathcal{I}_2 = \mathcal{I}_2^f$, Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L coincides with Wijsman asymptotically statistical equivalent of multiple L which is defined in [23].

As an example, consider the following double sequences;

$$A_{kj} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 + kjy = 0\} & , \text{ if } k \text{ and } j \text{ are a square integer,} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

and

$$B_{kj} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - kjy = 0\} & , \text{ if } k \text{ and } j \text{ are a square integer,} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, since

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}|\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{I}_2^f,$$

then the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent.

Definition 2.3. *Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-lacunary equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \Big(\frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} d(x; A_{kj}, B_{kj}) - L\Big) \ge \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{N_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj}$ and simply Wijsman asymptotically $\mathcal{I}_2$-lacunary equivalent if $L = 1$.

Definition 2.4. *Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are said to be Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{N_\theta\left[\mathcal{I}_{W_2}^L\right]}{\sim} B_{kj}$ and simply Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent if $L = 1$. The set of Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent double sequences will be denoted by $\left\{N_\theta\left[\mathcal{I}_{W_2}^L\right]\right\}$.

As an example, consider the following double sequences;

$$A_{kj} := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : \dfrac{(x-\sqrt{kj})^2}{kj} + \dfrac{y^2}{2kj} = 1\right\} & , \text{ if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}]. \end{array} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

and

$$B_{kj} := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : \dfrac{(x+\sqrt{kj})^2}{kj} + \dfrac{y^2}{2kj} = 1\right\} & , \text{ if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}]. \end{array} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, since

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\right\} \in \mathcal{I}_2^f,$$

then the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent.

Definition 2.5. *Let θ be a double lacunary sequence. The double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L if for every $\varepsilon > 0$, $\delta > 0$ and for each $x \in X$,*

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u} |\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon\}| \ge \delta\right\} \in \mathcal{I}_2.$$

In this case, we write $A_{kj} \overset{S_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj}$ *and simply Wijsman asymptotically* $\mathcal{I}_2$*-lacunary statistical equivalent if* $L=1$*. The set of Wijsman asymptotically* $\mathcal{I}_2$*-lacunary statistical equivalent double sequences will be denoted by* $\left\{S_\theta\left(\mathcal{I}_{W_2}^L\right)\right\}$*.*

For $\mathcal{I}_2 = \mathcal{I}_2^f$, Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L coincides with Wijsman asymptotically lacunary statistical equivalent of multiple L which is defined in [23].

As an example, consider the following double sequences;

$$A_{kj} := \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : x^2+(y-1)^2 = \dfrac{1}{kj}\right\} & , \text{ if } \begin{array}{c} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\overline{h}_u}] \\ \text{and } k \text{ is a square integer,} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

and

$$B_{kj} := \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : x^2+(y+1)^2 = \dfrac{1}{kj}\right\} & , \text{ if } \begin{array}{c} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\overline{h}_u}] \\ \text{and } k \text{ is a square integer,} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

If we take $\mathcal{I}_2 = \mathcal{I}_2^f$, since

$$\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u} \left|\left\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj})-L| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}_2^f,$$

then the sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent.

Theorem 2.6. *Let* θ *be a double lacunary sequence. Then,*

$$A_{kj} \overset{N_\theta\left[\mathcal{I}_{W_2}^L\right]}{\sim} B_{kj} \Rightarrow A_{kj} \overset{S_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj}.$$

Proof. Suppose that $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman strongly asymptotically $\mathcal{I}_2$-lacunary equivalent of multiple L. Given $\varepsilon > 0$ and for each $x \in X$ we can write

$$\begin{aligned} \sum_{(k,j)\in I_{ru}} |d(x;A_{kj},B_{kj})-L| &\geq \sum_{\substack{(k,j)\in I_{ru} \\ |d(x;A_{kj},B_{kj})-L|\geq\varepsilon}} |d(x;A_{kj},B_{kj})-L| \\ &\geq \varepsilon.\left|\left\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj})-L|\geq\varepsilon\right\}\right| \end{aligned}$$

and so we get

$$\frac{1}{\varepsilon\cdot h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x;A_{kj},B_{kj})-L| \geq \frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj})-L|\geq\varepsilon\right\}\right|.$$

Hence, for each $x \in X$ and for any $\delta > 0$, we have

$$\begin{aligned} &\left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\left|\left\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj})-L|\geq\varepsilon\right\}\right| \geq \delta\right\} \\ &\qquad \subseteq \left\{(r,u)\in\mathbb{N}\times\mathbb{N} : \frac{1}{h_r\overline{h}_u}\sum_{(k,j)\in I_{ru}} |d(x;A_{kj},B_{kj})-L| \geq \varepsilon\cdot\delta\right\} \in \mathcal{I}_2 \end{aligned}$$

and so $A_{kj} \overset{S_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj}$. □

Theorem 2.7. *Let θ be a double lacunary sequence and $d(x, A_{kj})\mathcal{O}\big(d(x, B_{kj})\big)$. Then,*

$$A_{kj} \overset{S_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj} \Rightarrow A_{kj} \overset{N_\theta\left[\mathcal{I}_{W_2}^L\right]}{\sim} B_{kj}.$$

Proof. Suppose that $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L and $d(x, A_{kj})\mathcal{O}\big(d(x, B_{kj})\big)$. Then, there exists an $M > 0$ such that

$$|d(x; A_{kj}, B_{kj}) - L| \le M,$$

for each $x \in X$ and all $k, j \in \mathbb{N}$. Given $\varepsilon > 0$, for each $x \in X$ we get

$$\frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L|$$

$$= \frac{1}{h_r \bar{h}_u} \sum_{\substack{(k,j)\in I_{ru} \\ |d(x;A_{kj},B_{kj})-L| \ge \frac{\varepsilon}{2}}} |d(x; A_{kj}, B_{kj}) - L|$$

$$+ \frac{1}{h_r \bar{h}_u} \sum_{\substack{(k,j)\in I_{ru} \\ |d(x;A_{kj},B_{kj})-L| < \frac{\varepsilon}{2}}} |d(x; A_{kj}, B_{kj}) - L|$$

$$\le \frac{M}{h_r \bar{h}_u} \Big| \big\{ (k, j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \ge \tfrac{\varepsilon}{2} \big\} \Big| + \frac{\varepsilon}{2}.$$

Hence, for each $x \in X$ we have

$$\left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \sum_{(k,j)\in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \ge \varepsilon \right\}$$

$$\subseteq \left\{ (r, u) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_r \bar{h}_u} \Big| \big\{ (k, j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \ge \tfrac{\varepsilon}{2} \big\} \Big| \ge \tfrac{\varepsilon}{2M} \right\} \in \mathcal{I}_2$$

and so $A_{kj} \overset{N_\theta\left[\mathcal{I}_{W_2}^L\right]}{\sim} B_{kj}$. □

We have the following Theorem by Theorem 2.6 and Theorem 2.7.

Theorem 2.8. *Let θ be a double lacunary sequence. If $d(x, A_{kj})\mathcal{O}\big(d(x, B_{kj})\big)$, then*

$$\left\{ S_\theta\left(\mathcal{I}_{W_2}^L\right) \right\} = \left\{ N_\theta\left[\mathcal{I}_{W_2}^L\right] \right\}.$$

Theorem 2.9. *Let θ be a double lacunary sequence. If $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$ then,*

$$A_{kj} \overset{S\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj} \Rightarrow A_{kj} \overset{S_\theta\left(\mathcal{I}_{W_2}^L\right)}{\sim} B_{kj}.$$

Proof. Assume that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there exist $\lambda, \mu > 0$ such that

$$q_r \ge 1 + \lambda \ \ \textit{and} \ \ q_u \ge 1 + \mu$$

for sufficiently large r, u which implies that

$$\frac{h_r \bar{h}_u}{k_{ru}} \ge \frac{\lambda\mu}{(1 + \lambda)(1 + \mu)}.$$

If $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, u, we get

$$\frac{1}{k_r j_u}\Big|\big\{k \leq k_r, j \leq j_u : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big|$$

$$\geq \frac{1}{k_r j_u}\Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big|$$

$$\geq \frac{\lambda\mu}{(1+\lambda)(1+\mu)}.\left(\frac{1}{h_r \bar{h}_u}\Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big|\right).$$

Hence, for each $x \in X$ and for any $\delta > 0$ we have

$$\left\{(r,u) : \frac{1}{h_r \bar{h}_u}\Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}$$

$$\subseteq \left\{(r,u) : \frac{1}{k_r j_u}\Big|\big\{k \leq k_r, j \leq j_u : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big| \geq \tfrac{\delta\lambda\mu}{(1+\lambda)(1+\mu)}\right\} \in \mathcal{I}_2$$

and so $A_{kj} \overset{S_\theta^L(\mathcal{I}_{W_2})}{\sim} B_{kj}$. □

Theorem 2.10. *Let θ be a double lacunary sequence. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then*

$$A_{kj} \overset{S_\theta(\mathcal{I}_{W_2}^L)}{\sim} B_{kj} \Rightarrow A_{kj} \overset{S(\mathcal{I}_{W_2}^L)}{\sim} B_{kj}.$$

Proof. If $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there is an $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r, u. Suppose that $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L and let

$$U_{ru} = U(r,u,x) := \Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big|.$$

Since $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L, it follows that for every $\varepsilon > 0$ and $\delta > 0$, for each $x \in X$,

$$\left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{1}{h_r \bar{h}_u}\Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\big\}\Big| \geq \delta\right\}$$

$$= \left\{(r,u) \in \mathbb{N}\times\mathbb{N} : \frac{U_{ru}}{h_r \bar{h}_u} \geq \delta\right\} \in \mathcal{I}_2.$$

Hence, we can choose a positive integers $r_0, u_0 \in \mathbb{N}$ such that

$$\frac{U_{ru}}{h_r \bar{h}_u} < \delta, \text{ for all } r > r_0, u > u_0.$$

Now let

$$K := \max\big\{U_{ru} : 1 \leq r \leq r_0, 1 \leq u \leq u_0\big\}$$

and let t and v be any integers satisfying $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$.

Then, we have

$$\frac{1}{tv}\Big|\big\{k \le t, j \le v : |d(x;A_{kj},B_{kj}) - L| \ge \varepsilon\big\}\Big|$$

$$\le \frac{1}{k_{r-1}j_{u-1}}\Big|\big\{k \le k_r, j \le j_u : |d(x;A_{kj},B_{kj}) - L| \ge \varepsilon\big\}\Big|$$

$$= \frac{1}{k_{r-1}j_{u-1}}\big(U_{11} + U_{12} + U_{21} + U_{22} + \cdots + U_{r_0u_0} + \cdots + U_{ru}\big)$$

$$\le \frac{K}{k_{r-1}j_{u-1}} \cdot r_0u_0$$

$$+\frac{1}{k_{r-1}j_{u-1}}\left(h_{r_0}\overline{h}_{u_0+1}\frac{U_{r_0,u_0+1}}{h_{r_0}\overline{h}_{u_0+1}} + h_{r_0+1}\overline{h}_{u_0}\frac{U_{r_0+1,u_0}}{h_{r_0+1}\overline{h}_{u_0}} + \cdots + h_r\overline{h}_u\frac{U_{ru}}{h_r\overline{h}_u}\right)$$

$$\le \frac{r_0u_0 \cdot K}{k_{r-1}j_{u-1}} + \frac{1}{k_{r-1}j_{u-1}}\left(\sup_{\substack{r>r_0\\u>u_0}}\frac{U_{ru}}{h_r\overline{h}_u}\right)\big(h_{r_0}\overline{h}_{u_0+1} + h_{r_0+1}\overline{h}_{u_0} + \cdots + h_r\overline{h}_u\big)$$

$$\le \frac{r_0u_0 \cdot K}{k_{r-1}j_{u-1}} + \varepsilon \cdot \frac{(k_r - k_{r_0})(j_u - j_{u_0})}{k_{r-1}j_{u-1}}$$

$$\le \frac{r_0u_0 \cdot K}{k_{r-1}j_{u-1}} + \varepsilon \cdot q_r \cdot q_u \le \frac{r_0u_0 \cdot K}{k_{r-1}j_{u-1}} + \varepsilon \cdot M \cdot N.$$

Since $k_{r-1}j_{u-1} \to \infty$ as $t, v \to \infty$, it follows that

$$\frac{1}{tv}\Big|\big\{k \le t, j \le v : |d(x;A_{kj},B_{kj}) - L| \ge \varepsilon\big\}\Big| \to 0$$

and consequently, for any $\delta_1 > 0$ the set

$$\left\{(t,v) \in \mathbb{N}\times\mathbb{N} : \frac{1}{tv}\Big|\big\{k \le t, j \le v : |d(x;A_{kj},B_{kj}) - L| \ge \varepsilon\big\}\Big| \ge \delta_1\right\} \in \mathcal{I}_2.$$

This shows that $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-statistical equivalent of multiple L. □

We have the following Theorem by Theorem 2.9 and Theorem 2.10.

Theorem 2.11. *Let θ be a double lacunary sequence. If*

$$1 < \liminf_r q_r \le \limsup_r q_r < \infty \text{ and } 1 < \liminf_u q_u \le \limsup_u q_u < \infty,$$

then

$$\big\{S_\theta(\mathcal{I}^L_{W_2})\big\} = \big\{S(\mathcal{I}^L_{W_2})\big\}.$$

Theorem 2.12. *Let $\mathcal{I}_2 \subseteq 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal satisfying property $(AP2)$ and $\theta \in \mathcal{F}(\mathcal{I}_2)$. If $\{A_{kj}\}, \{B_{kj}\} \in \big\{S(\mathcal{I}^{L_1}_{W_2})\big\} \cap \big\{S_\theta(\mathcal{I}^{L_2}_{W_2})\big\}$, then $L_1 = L_2$.*

Proof. Assume that $A_{kj} \overset{S^{L_1}(\mathcal{I}_{W_2})}{\sim} B_{kj}$, $A_{kj} \overset{S^{L_2}_\theta(\mathcal{I}_{W_2})}{\sim} B_{kj}$ and $L_1 \ne L_2$. Let

$$0 < \varepsilon < \frac{1}{2}|L_1 - L_2|.$$

Since $\mathcal{I}_2$ satisfies the property $(AP2)$, there exists $M \in \mathcal{F}(\mathcal{I}_2)$ $(i.e., \mathbb{N}\times\mathbb{N}\backslash M \in \mathcal{I}_2)$ such that for each $x \in X$ and for $(m,n) \in M$,

$$\lim_{m,n\to\infty} \frac{1}{mn}\Big|\big\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_1| \ge \varepsilon\big\}\Big| = 0.$$

Let the sets

$$P = \{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_1|)| \ge \varepsilon|\}$$

and

$$R = \{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\}.$$

Then, $mn = |P \cup R| \le |P| + |R|$. This implies that

$$1 \le \frac{|P|}{mn} + \frac{|R|}{mn}.$$

Since

$$\frac{|R|}{mn} \le 1 \quad and \quad \lim_{m,n\to\infty} \frac{|P|}{mn} = 0,$$

so we must have

$$\lim_{m,n\to\infty} \frac{|R|}{mn} = 1.$$

Let $M^* = M \cap \theta \in \mathcal{F}(\mathcal{I}_2)$. Then, for $(k_l, j_t) \in M^*$ the $k_l j_t$th term of the statistical limit expression

$$\frac{1}{mn}\Big|\big\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\big\}\Big|$$

is

$$\frac{1}{k_l j_t}\left|\left\{(k,j) \in \bigcup_{r,u=1,1}^{l,t} I_{ru} : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\right\}\right| = \frac{1}{\sum\limits_{r,u=1,1}^{l,t} h_r \overline{h}_u} \sum_{r,u=1,1}^{l,t} v_{ru} h_r \overline{h}_u, \tag{2.1}$$

where

$$v_{ru} = \frac{1}{h_r \overline{h}_u}\Big|\big\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\big\}\Big| \xrightarrow{\mathcal{I}_2} 0$$

because $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically $\mathcal{I}_2$-lacunary statistical equivalent of multiple L_2. Since θ is a double lacunary sequence, (2.1) is a regular weighted mean transform of v_{ru}'s and therefore it is also $\mathcal{I}_2$-convergent to 0 as $l, t \to \infty$, and so it has a subsequence which is convergent to 0 since $\mathcal{I}_2$ satisfies property $(AP2)$. But since this is a subsequence of

$$\left\{\frac{1}{mn}\Big|\big\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\big\}\Big|\right\}_{(m,n)\in M},$$

we infer that

$$\left\{\frac{1}{mn}\Big|\big\{k \le m, j \le n : |d(x; A_{kj}, B_{kj}) - L_2| \ge \varepsilon\big\}\Big|\right\}_{(m,n)\in M}$$

is not convergent to 1. This is a contradiction. Hence, the proof is completed. □

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Uğur Ulusu

Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, Afyonkarahisar, Turkey

E-mail address: ulusu@aku.edu.tr

Erdinç Dündar

Department of Mathematics, Faculty of Science and Literature, Afyon Kocatepe University, Afyonkarahisar, Turkey

E-mail address: edundar@aku.edu.tr