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ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENCE OF DOUBLE SEQUENCES OF SETS

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Abstract. The concepts of Wijsman asymptotically equivalence, Wijsman asymptotically statistically equivalence, Wijsman asymptotically lacunary equivalence and Wijsman asymptotically lacunary statistical equivalence for sequences of sets were studied by Ulusu and Nuray [24]. In this paper, we get analogous results for double sequences of sets.

1. Introduction

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [7] and Schoenberg [20]. This concept was extended to the double sequences by Mursaleen and Edely [11]. Çakan and Altay [6] presented multidimensional analogues of the results presented by Fridy and Orhan [8].

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets (see, [3, 4, 5, 12, 25, 26]). Nuray and Rhoades [12] extended the notion of convergence of set sequences to statistical convergence and gave some basic theorems. Ulusu and Nuray [23] defined the Wijsman lacunary statistical convergence of sequence of sets and considered its relation with Wiijsman statistical convergence, which was defined by Nuray and Rhoades. Nuray et. al. [13] studied Wijsman statistical convergence, Hausdorff statistical convergence and Wijsman statistical Cauchy double sequences of sets and investigated the relationship between them.

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Marouf [10] presented definitions for asymptotically equivalent and asymptotic regular matrices. Patterson [16] extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. Patterson and Savaş [17] extended the definitions presented in [16] to lacunary sequences.

The concepts of Wijsman asymptotically equivalence, Wijsman asymptotically statistically equivalence, Wijsman asymptotically lacunary equivalence and Wijsman asymptotically lacunary statistical equivalence for sequences of sets were studied by Ulusu and Nuray [24]. In this paper, we get analogous results for double sequences of sets.

2. Definitions and notations

Now, we recall the basic definitions and concepts (See [1, 2, 3, 4, 5, 9, 10, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 26]).

Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, a).$$

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper, the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Throughout the paper, we let $\theta = \{k_r\}$ be a lacunary sequence and A, A_k be any non-empty closed subsets of X.

We say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A),$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

We say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if, for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} \frac{1}{n} |\{k \le n : |d(x, A_k) - d(x, A)| \ge \varepsilon\}| = 0.$$

We say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; i.e., for $\varepsilon > 0$

and for each $x \in X$,

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

Let us consider non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$. Then, we remember following definitions:

We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense) if for each $x \in X$,

$$\lim_k \frac{d(x, A_k)}{d(x, B_k)} = 1$$

(denoted by $A_k \sim B_k$).

We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistical equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_n \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $\{A_k\} \overset{WS_L}{\sim} \{B_k\}$) and simply asymptotically statistical equivalent (Wijsman sense) if $L = 1$.

We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunary equivalent (Wijsman sense) of multiple L if for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \frac{d(x, A_k)}{d(x, B_k)} = L$$

(denoted by $\{A_k\} \overset{WN_\theta^L}{\sim} \{B_k\}$) and simply asymptotically lacunary equivalent (Wijsman sense) if $L = 1$.

We say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly asymptotically lacunary equivalent (Wijsman sense) of multiple L if for each $x \in X$,

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| = 0$$

(denoted by $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$) and simply strongly asymptotically lacunary equivalent (Wijsman sense) if $L = 1$.

We say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically lacunary statistical equivalent (Wijsman sense) of multiple L if for every $\varepsilon > 0$ and each $x \in X$,

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| = 0$$

(denoted by $\{A_k\} \overset{WS_\theta^L}{\sim} \{B_k\}$) and simply asymptotically lacunary statistical equivalent (Wijsman sense) if $L = 1$.

A double sequence $x = (x_{kj})_{k,j \in \mathbb{N}}$ of real numbers is said to be convergent to $L \in \mathbb{R}$ in Pringsheim's sense if for any $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $|x_{kj} - L| < \varepsilon$, whenever $k, j > N_\varepsilon$. In this case we write

$$P - \lim_{k,j \to \infty} x_{kj} = L.$$

The double sequence $\{A_{kj}\}$ is Wijsman convergent to A if

$$P - \lim_{k,j \to \infty} d(x, A_{kj}) = d(x, A)$$

for each $x \in X$.

We say that the double sequence $\{A_{kj}\}$ is Wijsman statistically convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,

$$P - \lim_{m,n \to \infty} \frac{1}{mn} |\{k \leq m, j \leq n : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

The double sequence $\theta = \{(k_r, j_s)\}$ is called double lacunary sequence if there exist two increasing sequences of integers such that

$$k_0 = 0, \quad h_r = k_r - k_{r-1} \to \infty \quad \text{as} \quad r \to \infty \quad \text{and}$$
$$j_0 = 0, \quad \bar{h}_u = j_u - j_{u-1} \to \infty \quad \text{as} \quad u \to \infty.$$

We use following notations in the sequel:

$$k_{ru} = k_r j_u, \ \ h_{ru} = h_r \bar{h}_u, \ \ I_{ru} = \{(k, j) : k_{r-1} < k \leq k_r \ \text{ and } \ j_{u-1} < j \leq j_u\},$$
$$q_r = \frac{k_r}{k_{r-1}} \ \text{ and } \ q_u = \frac{j_u}{j_{u-1}}.$$

We say that the double sequence $\{A_{kj}\}$ is Wijsman lacunary statistically convergent to A, if for each $\varepsilon > 0$ and for each $x \in X$,

$$P - \lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} |\{(k, j) \in I_{ru} : |d(x, A_{kj}) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case, we write $st_2 - \lim_{W_\theta} A_{kj} = A$.

Let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence. The double sequence $\{A_{kj}\}$ is Wijsman strongly lacunary convergent to A if for each $x \in X$,

$$P - \lim_{r,u \to \infty} \frac{1}{h_r \bar{h}_u} \sum_{k=k_{r-1}+1}^{k_r} \sum_{j=j_{u-1}+1}^{j_u} |d(x, A_{kj}) - d(x, A)| = 0.$$

3. Main results

Throughout the paper, we let $\theta = \{(k_r, j_s)\}$ be a double lacunary sequence and A, A_{kj}, B_{kj} be any non-empty closed subsets of X. We define

$d(x; A_{kj}, B_{kj})$ as follows:

$$d(x; A_{kj}, B_{kj}) = \begin{cases} \dfrac{d(x, A_{kj})}{d(x, B_{kj})}, & x \notin A_{kj} \cup B_{kj}, \\ L, & x \in A_{kj} \cup B_{kj}. \end{cases}$$

Definition 3.1. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically equivalent of multiple L if for each $x \in X$

$$P - \lim_{k,j\to\infty} d(x; A_{kj}, B_{kj}) = L,$$

in this case we write $\{A_{kj}\} \overset{W_2^L}{\sim} \{B_{kj}\}$, and simply Wijsman asymptotically equivalent if $L = 1$.

As an example, consider the following double sequences of circles in the (x, y)-plane:

$$A_{kj} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - 2kx - 2jy = 0\}$$

and

$$B_{kj} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 + 2kx + 2jy = 0\}.$$

Since

$$P - \lim_{k,j\to\infty} d(x; A_{kj}, B_{kj}) = 1,$$

the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically equivalent. Thus, $A_{kj} \overset{W_2^1}{\sim} B_{kj}$.

Definition 3.2. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically C-equivalent of multiple L if for each $x \in X$

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} d(x; A_{kj}, B_{kj}) = L,$$

in this case we write $\{A_{kj}\} \overset{W_2C^L}{\sim} \{B_{kj}\}$, and simply Wijsman asymptotically C-equivalent if $L = 1$.

Definition 3.3. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically C-equivalent of multiple L if for each $x \in X$

$$P - \lim_{m,n\to\infty} \frac{1}{mn} \sum_{k,j=1,1}^{m,n} |d(x; A_{kj}, B_{kj} - L| = 0,$$

in this case we write $\{A_{kj}\} \overset{[W_2C^L]}{\sim} \{B_{kj}\}$, and simply Wijsman strongly asymptotically C-equivalent if $L = 1$.

Definition 3.4. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically lacunary equivalent of multiple L if for each $x \in X$

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} d(x; A_{kj}, B_{kj}) = L,$$

in this case we write $\{A_{kj}\} \overset{W_2N_\theta^L}{\sim} \{B_{kj}\}$, and Wijsman asymptotically lacunary equivalent if $L = 1$.

Definition 3.5. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically lacunary equivalent of multiple L if for each $x \in X$

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} \left| d(x; A_{kj}, B_{kj}) - L \right| = 0,$$

in this case we write $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$, and Wijsman strongly asymptotically lacunary equivalent if $L = 1$.

As an example, consider the following double sequences;

$$A_{kj} := \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : \dfrac{(x-\sqrt{kj})^2}{kj} + \dfrac{y^2}{2kj} = 1\right\}, & \text{if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}], \end{array} \\ \{(1,1)\}, & \text{otherwise.} \end{cases}$$

and

$$B_{kj} := \begin{cases} \left\{(x,y)\in\mathbb{R}^2 : \dfrac{(x+\sqrt{kj})^2}{kj} + \dfrac{y^2}{2kj} = 1\right\}, & \text{if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}], \end{array} \\ \{(1,1)\}, & \text{otherwise.} \end{cases}$$

Since

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u} \sum_{k,j\in I_{ru}} \left| d(x; A_{kj}, B_{kj}) - 1 \right| = 0,$$

the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman strongly asymptotically lacunary equivalent. Thus, $\{A_{kj}\} \overset{W_2N_\theta^1}{\sim} \{B_{kj}\}$.

Theorem 3.1. *For any double lacunary sequence θ, if*

$$1 < \liminf_r q_r \le \limsup_r q_r < \infty \quad \text{and} \quad 1 < \liminf_u q_u \le \limsup_u q_u < \infty,$$

then $\{A_{kj}\} \overset{[W_2C^L]}{\sim} \{B_{kj}\}$ *if and only if* $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$.

Proof. Firstly, we assume that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there exist $\lambda, \mu > 0$ such that

$q_r \geq 1+\lambda$ and $q_u \geq 1+\mu$ for all $r, u \geq 1$, which implies that

$$\frac{k_r j_u}{h_r \bar{h}_u} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.$$

Let $A_{kj} \overset{[W_2 C^L]}{\sim} B_{kj}$. We can write

$$\begin{aligned}
\frac{1}{h_r \bar{h}_u} \sum_{k,j \in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| &= \frac{1}{h_r \bar{h}_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| \\
&\quad - \frac{1}{h_r \bar{h}_u} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x; A_{is}, B_{is}) - L| \\
&= \frac{k_r j_u}{h_r \bar{h}_u} \left(\frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| \right) \\
&\quad - \frac{k_{r-1} j_{u-1}}{h_r \bar{h}_u} \left(\frac{1}{k_{r-1} j_{u-1}} \sum_{i=1}^{k_{r-1}, j_{u-1}} |d(x; A_{is}, B_{is}) - L| \right).
\end{aligned}$$

Since $A_{kj} \overset{[W_2 C^L]}{\sim} B_{kj}$, the terms

$$\frac{1}{k_r j_u} \sum_{i,s=1,1}^{k_r, j_u} |d(x; A_{is}, B_{is}) - L| \quad \text{and} \quad \frac{1}{k_{r-1} j_{u-1}} \sum_{i,s=1,1}^{k_{r-1}, j_{u-1}} |d(x; A_{is}, B_{is}) - L|$$

both convergent to 0, and it follows that

$$\frac{1}{h_r \bar{h}_u} \sum_{k,j \in I_{ru}} |d(x; A_{kj}, B_{kj}) - L| \to 0,$$

that is, $A_{kj} \overset{[W_2 N_\theta^L]}{\sim} B_{kj}$. Secondly, we assume that $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there exist $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r, u. Let $\{A_{kj}\} \overset{[W_2 N_\theta^L]}{\sim} \{B_{kj}\}$ and $\varepsilon > 0$. Then we can find $R, U > 0$ and $K > 0$ such that

$$\sup_{i \geq R, s \geq U} \tau_{is} < \varepsilon \qquad \text{and} \qquad \tau_{is} < K \qquad \text{for all } i, s = 1, 2, \cdots,$$

where

$$\tau_{ru} = \frac{1}{h_r \bar{h}_u} \sum_{I_{ru}} |d(x; A_{kj}, B_{kj}) - L|.$$

If t, v are any integers with $k_{r-1} < t \leq k_r$ and $j_{u-1} < v \leq j_u$, where $r > R$ and $u > U$, then we can write

$$\begin{aligned}
\frac{1}{tv}\sum_{i,s=1,1}^{t,v}|d(x;A_{is},B_{is})-L| &\le \frac{1}{k_{r-1}j_{u-1}}\sum_{i,s=1,1}^{k_r,j_u}|d(x;A_{is},B_{is})-L|\\
&= \frac{1}{k_{r-1}j_{u-1}}\Bigg(\sum_{I_{11}}|d(x;A_{is},B_{is})-L|+\sum_{I_{12}}|d(x;A_{is},B_{is})-L|\\
&\quad+\sum_{I_{21}}|d(x;A_{is},B_{is})-L|+\sum_{I_{22}}|d(x;A_{is},B_{is})-L|+\cdots\\
&\quad+\sum_{I_{ru}}|d(x;A_{is},B_{is})-L|\Bigg)\\
&\le \frac{k_1j_1}{k_{r-1}j_{u-1}}.\tau_{11}+\frac{k_1(j_2-j_1)}{k_{r-1}j_{u-1}}.\tau_{12}+\frac{(k_2-k_1)j_1}{k_{r-1}j_{u-1}}.\tau_{21}\\
&\quad+\frac{(k_2-k_1)(j_2-j_1)}{k_{r-1}j_{u-1}}.\tau_{22}+\cdots+\frac{(k_R-k_{R-1})(j_U-j_{U-1})}{k_{r-1}j_{u-1}}\tau_{RU}\\
&\quad+\cdots+\frac{(k_r-k_{r-1})(j_u-j_{u-1})}{k_{r-1}j_{u-1}}\tau_{ru}\\
&\le \Big(\sup_{i,s\ge 1,1}\tau_{is}\Big)\frac{k_Rj_U}{k_{r-1}j_{u-1}}+\Big(\sup_{i\ge R,s\ge U}\tau_{is}\Big)\frac{(k_r-k_R)(j_u-j_U)}{k_{r-1}j_{u-1}}\\
&\le K\frac{k_Rj_U}{k_{r-1}j_{u-1}}+\varepsilon MN.
\end{aligned}$$

Since $k_{r-1}, j_{u-1}\to\infty$ as $t,v\to\infty$, it follows that

$$\frac{1}{tv}\sum_{i,s=1,1}^{t,v}|d(x;A_{is},B_{is})-L|\to 0$$

and consequently $\{A_{kj}\}\overset{[W_2C^1]}{\sim}\{B_{kj}\}$. Hence we obtain the desired result. ∎

Definition 3.6. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$are Wijsman asymptotically statistical equivalent of multiple L if for each $\varepsilon>0$ and for each $x\in X$

$$P-\lim_{m,n\to\infty}\frac{1}{mn}\left|\left\{k\le m, j\le n: \left|d(x;A_{kj},B_{kj})-L\right|\ge\varepsilon\right\}\right|=0,$$

in this case we write $\{A_k\}\overset{W_2S^L}{\sim}\{B_k\}$, and simply Wijsman asymptotically statistical equivalent if $L=1$.

As an example, consider the following double sequences of circles in the (x,y)-plane:

$$A_{kj}=\begin{cases}\{(x,y)\in\mathbb{R}^2: x^2+y^2+kjy=0\}, & \text{if } k \text{ and } j \text{ are a square integer},\\ \{(1,1)\}, & \text{otherwise.}\end{cases}$$

and

$$B_{kj} = \begin{cases} \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 - kjy = 0\}, & \text{if } k \text{ and } j \text{ are a square integer,} \\ \{(1,1)\}, & \text{otherwise.} \end{cases}$$

Since

$$P - \lim_{m,n\to\infty} \frac{1}{mn}\left|\left\{k \le m, j \le n : \left|d(x; A_{kj}, B_{kj}) - 1\right| \ge \varepsilon\right\}\right| = 0,$$

the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically statistical equivalent. Thus, $\{A_k\} \overset{W_2S^1}{\sim} \{B_k\}$.

Definition 3.7. We say that the double sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ are Wijsman asymptotically lacunary statistical equivalent of multiple L if for every $\varepsilon > 0$ and each $x \in X$

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u}\left|\left\{(k,j) \in I_{ru} : \left|d(x; A_{kj}, B_{kj}) - L\right| \ge \varepsilon\right\}\right| = 0,$$

in this case we write $\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}$, and simply Wijsman asymptotically lacunary statistical equivalent if $L = 1$.

As an example, consider the following double sequences;

$$A_{kj} := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \dfrac{1}{kj}\right\}, & \text{if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}] \\ \text{and } k \text{ is a square integer,} \end{array} \\ \{(0,0)\}, & \text{otherwise,} \end{cases}$$

and

$$B_{kj} := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : x^2 + (y+1)^2 = \dfrac{1}{kj}\right\}, & \text{if } \begin{array}{l} k_{r-1} < k < k_{r-1} + [\sqrt{h_r}], \\ j_{u-1} < j < j_{u-1} + [\sqrt{\bar{h}_u}] \\ \text{and } k \text{ is a square integer,} \end{array} \\ \{(0,0)\}, & \text{otherwise.} \end{cases}$$

Since

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\bar{h}_u}\left|\left\{(k,j) \in I_{ru} : \left|d(x; A_{kj}, B_{kj}) - 1\right| \ge \varepsilon\right\}\right| = 0,$$

the sequences $\{A_{kj}\}$ and $\{B_{kj}\}$ is Wijsman asymptotically lacunary statistical equivalent. Thus, $\{A_{kj}\} \overset{W_2S_\theta^1}{\sim} \{B_k\}$.

Theorem 3.2. *(i)* $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$ *implies* $\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}$,
(ii) $\left[W_2N_\theta^L\right]$ *is a proper subset of* $W_2S_\theta^L$.

Proof. (i) Let $\varepsilon > 0$ and $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$. Then we can write

$$\sum_{k,j\in I_{ru}} |d(x;A_{kj},B_{kj}) - L| = \sum_{\substack{k,j\in I_{ru}\\ |d(x;A_{kj},B_{kj})-L|\ge\varepsilon}} |d(x;A_{kj},B_{kj}) - L| + \sum_{\substack{k,j\in I_{ru}\\ |d(x;A_{kj},B_{kj})-L|<\varepsilon}} |d(x;A_{kj},B_{kj}) - L|$$
$$\ge \varepsilon . |\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj}) - L| \ge \varepsilon\}|$$

which yields the result.

(ii) Suppose that $[W_2N_\theta^L] \subset W_2S_\theta^L$. Let $\{A_{kj}\}$ and $\{B_{kj}\}$ be following sequences;

$$A_{kj} = \begin{cases} \{kj\}, & \text{if } k_{r-1} < k \le k_{r-1} + \left[\sqrt{h_r}\right],\ j_{u-1} < j \le j_{u-1} + \left[\sqrt{\overline{h}_u}\right], \\ \{0\}, & \text{otherwise.} \end{cases}$$

$B_{kj} = \{0\}$ for all k and j. Note that $\{A_{kj}\}$ is not bounded. We have, for every $\varepsilon > 0$ and for each $x \in X$,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u} |\{(k,j)\in I_{ru} : |d(x;A_{kj},B_{kj}) - 1| \ge \varepsilon\}|$$
$$= P - \lim_{r,u\to\infty} \frac{\left[\sqrt{h_r}\right]\left[\sqrt{\overline{h}_u}\right]}{h_r\overline{h}_u} = 0.$$

Thus, $\{A_{kj}\} \overset{W_2S_\theta^1}{\sim} \{B_{kj}\}$. On the other hand,

$$P - \lim_{r,u\to\infty} \frac{1}{h_r\overline{h}_u} \sum_{k,j\in I_{ru}} |d(x;A_{kj},B_{kj}) - L| \neq 0.$$

Hence $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\nsim} \{B_k\}$. ∎

Theorem 3.3. *$d(x,A_{kj}) = O\big(d(x,B_{kj})\big)$ and $\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}$ then $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$.*

Proof. Suppose that $d(x,A_{kj}) = O\big(d(x,B_{kj})\big)$ and $\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}$. Then, we can assume that

$$|d(x;A_{kj},B_{kj}) - L| \le M$$

for each $x \in X$ and all k and all j. Given $\varepsilon > 0$, we get

$$\frac{1}{h_r\bar{h}_u}\sum_{k,j\in I_{ru}}|d(x;A_{kj},B_{kj})-L| = \frac{1}{h_r\bar{h}_u}\sum_{\substack{k,j\in I_{ru}\\ \left|d(x;A_{kj},B_{kj})-L\right|\geq\varepsilon}}|d(x;A_{kj},B_{kj})-L|$$

$$+\frac{1}{h_r\bar{h}_u}\sum_{\substack{k,j\in I_{ru}\\ \left|d(x;A_{kj},B_{kj})-L\right|<\varepsilon}}|d(x;A_{kj},B_{kj})-L|$$

$$\leq \frac{M}{h_r\bar{h}_u}\left|\{(k,j)\in I_{ru}:|d(x;A_{kj},B_{kj})-L|\geq\varepsilon\}\right|+\varepsilon.$$

Therefore $\{A_{kj}\} \overset{[W_2N_\theta^L]}{\sim} \{B_{kj}\}$. ■

Theorem 3.4. *If* $\theta=\{(k_r,j_s)\}$ *is a double lacunary sequence with* $\liminf_r q_r > 1$, $\liminf_u q_u > 1$, *then*

$$\{A_{kj}\} \overset{W_2S^L}{\sim} \{B_{kj}\} \quad \textit{implies} \quad \{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}.$$

Proof. Suppose first that $\liminf_r q_r > 1$ and $\liminf_u q_u > 1$, then there exist $\lambda, \mu > 0$ such that $q_r \geq 1+\lambda$ and $q_u \geq 1+\mu$ for all $r,u \geq 1$, which implies that

$$\frac{k_r j_u}{h_r\bar{h}_u} \leq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.$$

If $\{A_{kj}\} \overset{W_2S^L}{\sim} \{B_{kj}\}$, then for every $\varepsilon > 0$, for sufficiently large r,u and for each $x \in X$, we have

$$\frac{1}{k_rj_u}\left|\{k\leq k_r,\ j\leq j_u:|d(x;A_{kj},B_{kj})-L|\geq\varepsilon\}\right|$$

$$\geq \frac{1}{k_rj_u}\left|\{(k,j)\in I_{ru}:|d(x;A_{kj},B_{kj})-L|\geq\varepsilon\}\right|$$

$$\geq \frac{(1+\lambda)(1+\mu)}{\lambda\mu}.\left(\frac{1}{h_r\bar{h}_u}\left|\{(k,j)\in I_{ru}:|d(x;A_{kj},B_{kj})-L|\geq\varepsilon\}\right|\right),$$

this completes the proof. ■

Theorem 3.5. *If* $\theta=\{(k_r,j_s)\}$ *is a double lacunary sequence with* $\limsup_r q_r < \infty$, $\limsup_u q_{ru} < \infty$ *then*

$$\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\} \quad \textit{implies} \quad \{A_{kj}\} \overset{W_2S^L}{\sim} \{B_{kj}\}.$$

Proof. Assume that $\limsup_r q_r < \infty$ and $\limsup_u q_u < \infty$, then there exist $M, N > 0$ such that $q_r < M$ and $q_u < N$, for all r,u. Let $\{A_{kj}\} \overset{W_2S_\theta^L}{\sim} \{B_{kj}\}$

and $\varepsilon > 0$. There exists $R > 0$ such that for every $r, s \geq R$

$$A_{rs} = \frac{1}{h_r \overline{h}_s} \left|\{(k,j) \in I_{rs} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| < \varepsilon.$$

We can also find $H > 0$ such that $A_{rs} < H$ for all $r, s = 1, 2, \ldots$. Now let m, n be any integers satisfying $k_{r-1} < m \leq k_r$ and $j_{u-1} < n \leq j_u$, where $r, s > R$. Then we can write

$$\begin{aligned}
&\frac{1}{mn} \left|\{k \leq m, j \leq n : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\quad \leq \frac{1}{k_{r-1} j_{u-1}} \left|\{k \leq k_r, j \leq j_u : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\quad = \frac{1}{k_{r-1} j_{u-1}} \left|\{(k,j) \in I_{11} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{1}{k_{r-1} j_{u-1}} \left|\{(k,j) \in I_{21} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{1}{k_{r-1} j_{u-1}} \left|\{(k,j) \in I_{12} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{1}{k_{r-1} j_{u-1}} \left|\{(k,j) \in I_{22} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad \vdots \\
&\qquad + \frac{1}{k_{r-1} j_{u-1}} \left|\{(k,j) \in I_{ru} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\quad = \frac{k_1 j_1}{k_{r-1} j_{u-1} k_1 j_1} \left|\{(k,j) \in I_{11} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1} (k_2 - k_1) j_1} \left|\{(k,j) \in I_{21} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{k_1 (j_2 - J_1)}{k_{r-1} j_{u-1} k_1 (j_2 - j_1)} \left|\{(k,j) \in I_{12} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad + \frac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1} (k_2 - k_1)(j_2 - j_1)} \left|\{(k,j) \in I_{22} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad \vdots \\
&\qquad + \frac{(k_R - k_{R-1})(j_R - j_{R-1})}{k_{r-1} j_{u-1} (k_R - k_{R-1})(j_R - j_{R-1})} \left|\{(k,j) \in I_{RR} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right| \\
&\qquad \vdots \\
&\qquad + \frac{(k_r - k_{r-1})(j_r - j_{r-1})}{k_{r-1} j_{u-1} (k_r - k_{r-1})(j_r - j_{r-1})} \left|\{(k,j) \in I_{rr} : |d(x; A_{kj}, B_{kj}) - L| \geq \varepsilon\}\right|
\end{aligned}$$

$$= \frac{k_1 j_1}{k_{r-1} j_{u-1}} A_{11} + \frac{(k_2 - k_1) j_1}{k_{r-1} j_{u-1}} A_{21} + \frac{k_1 (j_2 - j_1)}{k_{r-1} j_{u-1}} A_{12} + \frac{(k_2 - k_1)(j_2 - j_1)}{k_{r-1} j_{u-1}} A_{22}$$
$$\vdots$$
$$+ \frac{(k_R - k_{R-1})(j_R - j_{R-1})}{k_{r-1} j_{u-1}} A_{RR} + \ldots + \frac{(k_r - k_{r-1})(j_r - j_{r-1})}{k_{r-1} j_{u-1}} A_{rr}$$
$$\leq \left\{ \sup_{r,s \geq 1} A_{rs} \right\} \frac{k_R J_R}{k_{r-1} j_{u-1}} + \left\{ \sup_{r,s \geq R} A_{rs} \right\} \frac{(k_r - k_R)(j_r - j_R)}{k_{r-1} j_{u-1}}$$
$$\leq H \cdot \frac{k_R j_R}{k_{r-1} j_{u-1}} + \varepsilon \cdot M \cdot N.$$

This completes the proof. ■

Combining Theorem 3.4 and Theorem 3.5, we have

Theorem 3.6. *If $\theta = \{(k_r, j_s)\}$ is a double lacunary sequence with*

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty \quad \text{and} \quad 1 < \liminf_u q_u \leq \limsup_u q_u < \infty$$

then

$$\{A_{kj}\} \overset{W_2 S_\theta^L}{\sim} \{B_{kj}\} \quad \text{if and only if} \quad \{A_{kj}\} \overset{W_2 S^L}{\sim} \{B_{kj}\}.$$

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