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ON ALMOST ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENCE OF SEQUENCES OF SETS

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Abstract. In this paper we study the concepts of Wijsman almost asymptotically statistical equivalent, Wijsman almost asymptotically lacunary statistical equivalent and Wijsman strongly almost asymptotically lacunary equivalent sequences of sets and investigate the relationship between them.

1. Introduction and Background

The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [6] and Schoenberg [16].

Definition 1.1. *(Fridy, [7]) The sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\varepsilon > 0$,*

$$\lim_n \frac{1}{n}|\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

(denoted by $st - \lim x_k = L$).

The concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence (see, [2, 4, 12, 17, 19]).

Let (X, ρ) be a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} \rho(x, A).$$

Definition 1.2. *(Baronti & Papini, [2]) Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman convergent to A if*

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim A_k = A$.

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Let (X, ρ) a metric space. For any non-empty closed subsets A_k of X, the sequence $\{A_k\}$ is said to be bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$.

Nuray and Rhoades [12] extended the notion of convergence of set sequences to statistical convergence, and gave some basic theorems. Also the concept of almost statistical convergence for sequences of sets was given by Nuray and Rhoades in [12].

Definition 1.3. *(Nuray & Rhoades,* [12]*) Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman statistical convergent to A if $\{d(x, A_k)\}$ is statistically convergent to $d(x, A)$; that is, for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

Definition 1.4. *(Nuray & Rhoades,* [12]*) Let (X, ρ) be a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman almost statistical convergent to A if for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |d(x, A_{k+i}) - d(x, A)| \geq \varepsilon\}| = 0,$$

uniformly in i.

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

Ulusu and Nuray [17] defined the Wijsman lacunary statistical convergence of sequences of sets, and considered its relation with Wijsman statistical convergence, which was defined by Nuray and Rhoades. Also, the concept of Wijsman lacunary almost statistical convergence and Wijsman lacunary strongly almost convergence were given by Ulusu and Nuray in [17].

Definition 1.5. *(Ulusu & Nuray,* [17]*) Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary statistical convergent to A if $\{d(x, A_k)\}$ is lacunary statistically convergent to $d(x, A)$; that is, for $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{r} \frac{1}{h_r} |\, \{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\} \,| = 0.$$

In this case we write $S_\theta - \lim_W = A$ or $A_k \to A(WS_\theta)$.

Definition 1.6. *(Ulusu and Nuray,* [17]*) Let (X, ρ) a metric space and $\theta = \{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary almost statistical convergent to A if for each $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_{r\to\infty} \frac{1}{h_r} |\, \{k \in I_r : |d(x, A_{k+i}) - d(x, A)| \geq \varepsilon\} \,| = 0,$$

uniformly in i.

Definition 1.7. *(Ulusu and Nuray, [17]) Let (X,ρ) a metric space and $\theta=\{k_r\}$ be a lacunary sequence. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman lacunary strongly almost convergent to A if for each $x \in X$,*

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|d(x,A_{k+i})-d(x,A)|=0,$$

uniformly in i.

Marouf [11] presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. Patterson [13] defined asymptotically statistical equivalent sequences by using the definition of statistical convergence.

Definition 1.8. *(Marouf, [11]) Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically equivalent if*

$$\lim_k \frac{x_k}{y_k}=1,$$

(denoted by $x\sim y$).

Definition 1.9. *(Patterson, [13]) Two nonnegative sequences $x=(x_k)$ and $y=(y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon>0$,*

$$\lim_n \frac{1}{n}\left\{\text{the number of } k<n : \left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}=0,$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent if $L=1$.

Patterson and Savaş [14] extended the definitions presented in [13] to lacunary sequences. In addition to these definitions, natural inclusion theorems were presented.

Definition 1.10. *(Patterson and Savaş, [14]) Let $\theta=\{k_r\}$ be a lacunary sequence, two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon>0$,*

$$\lim_r \frac{1}{h_r}\left|\left\{k\in I_r : \left|\frac{x_k}{y_k}-L\right|\geq\varepsilon\right\}\right|=0,$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L=1$. Furthermore, let S_θ^L denote the set of x and y such that $x \overset{S_\theta^L}{\sim} y$.

Definition 1.11. *(Patterson and Savaş, [14]) Let $\theta=\{k_r\}$ be a lacunary sequence, two number sequences $x=(x_k)$ and $y=(y_k)$ are said to be strong asymptotically lacunary equivalent of multiple L provided that,*

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k}-L\right|=0,$$

(denoted by $x \overset{N_\theta^L}{\sim} y$) and strong simply asymptotically lacunary equivalent if $L=1$. In addition, let N_θ^L denote the set of x and y such that $x \overset{N_\theta^L}{\sim} y$.

Ulusu and Nuray [18] extended the definitions presented in [14] to sequences of sets in the Wijsman sense. In addition to these definitions, natural inclusion theorems are presented.

Definition 1.12. *(Ulusu & Nuray,* [18]*) Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$are Wijsman asymptotically statistical equivalent of multiple L if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_n \frac{1}{n}\left|\left\{k\leq n : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|=0,$$

(denoted by $\{A_k\}\overset{WS_L}{\sim}\{B_k\}$) and simply Wijsman asymptotically statistical equivalent if $L=1$.

Definition 1.13. *(Ulusu & Nuray,*[18]*) Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k\subseteq X$ such that $d(x,A_k)>0$ and $d(x,B_k)>0$ for each $x\in X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman asymptotically lacunary statistical equivalent of multiple L if for every $\varepsilon>0$ and each $x\in X$,*

$$\lim_r \frac{1}{h_r}\left|\left\{k\in I_r : \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|=0,$$

(denoted by $\{A_k\}\overset{WS_\theta^L}{\sim}\{B_k\}$) and simply Wijsman asymptotically lacunary statistical equivalent if $L=1$.

2. Main Results

In this section we shall give some new definitions and new theorems.

Definition 2.1. *Let (X,ρ) be a metric space. For any non-empty closed subsets A_k, $B_k\subseteq X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman almost asymptotically statistical equivalent of multiple L if for every $\varepsilon>0$ and for each $x\in X$,*

$$\lim_n \frac{1}{n}\left|\{k\leq n : |d(x;A_{k+i},B_{k+i})-L|\geq\varepsilon\}\right|=0,$$

uniformly in i where

$$d(x;A_k,B_k)=\begin{cases}\dfrac{d(x,A_k)}{d(x,B_k)} & , \quad x\notin A_k\cup B_k\\[2ex] L & , \quad x\in A_k\cup B_k.\end{cases}$$

In this case we write $\{A_k\}\overset{(\widehat{WS})_L}{\sim}\{B_k\}$ and simply Wijsman almost asymptotically statistical equivalent if $L=1$. Furthermore, let $\left(\widehat{WS}\right)_L$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\}\overset{(\widehat{WS})_L}{\sim}\{B_k\}$.

Example 2.2. *Consider the following sequences;*

$$A_k=\begin{cases}\{(x,y):x^2+y^2-2kx=0\} & , \quad \textit{if } k \textit{ is a square integer,}\\ \{(1,1)\} & , \quad \textit{otherwise.}\end{cases}$$

and

$$B_k = \begin{cases} \{(x,y) : x^2+y^2+2kx=0\} & , \text{ if } k \text{ is a square integer,} \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

Since

$$\lim_n \frac{1}{n} |\{k \leq n : |d(x; A_{k+i}, B_{k+i}) - 1| \geq \varepsilon\}| = 0,$$

uniformly in i, the sequences $\{A_k\}$ and $\{B_k\}$ is Wijsman almost asymptotically statistical equivalent. That is $\{A_k\} \overset{(\widehat{WS})_1}{\sim} \{B_k\}$.

Definition 2.3. *Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman almost asymptotically lacunary statistical equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,*

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}| = 0,$$

uniformly in i. In this case we write $\{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\}$ and simply Wijsman almost asymptotically lacunary statistical equivalent if $L = 1$. In addition, let $\left(\widehat{WS_\theta}\right)_L$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\}$.

Example 2.4. *Consider the following sequences;*

$$A_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 = \dfrac{1}{k}\right\} & , \begin{array}{l} \text{if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}] \text{ and} \\ k \text{ is a square integer,} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

and

$$B_k := \begin{cases} \left\{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 = \dfrac{1}{k}\right\} & , \begin{array}{l} \text{if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}] \text{ and} \\ k \text{ is a square integer,} \end{array} \\ \{(0,0)\} & , \text{ otherwise.} \end{cases}$$

Since

$$\lim_r \frac{1}{h_r} |\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - 1| \geq \varepsilon\}| = 0,$$

uniformly in i, the sequences $\{A_k\}$ and $\{B_k\}$ is Wijsman almost asymptotically lacunary statistical equivalent. That is $\{A_k\} \overset{(\widehat{WS_\theta})_1}{\sim} \{B_k\}$.

Definition 2.5. *Let (X,ρ) be a metric space and θ be a lacunary sequence. For any non-empty closed subsets A_k, $B_k \subseteq X$. We say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman strongly almost asymptotically lacunary equivalent of multiple L if for each $x \in X$,*

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |d(x; A_{k+i}, B_{k+i}) - L| = 0,$$

uniformly in i. In this case we write $\{A_k\} \overset{[\widehat{WN_\theta}]_L}{\sim} \{B_k\}$ and simply Wijsman strongly almost asymptotically lacunary equivalent if $L = 1$. In addition, let $\left(\widehat{WN_\theta}\right)_L$ denote the set of $\{A_k\}$ and $\{B_k\}$ such that $\{A_k\} \overset{(\widehat{WN_\theta})_L}{\sim} \{B_k\}$.

Example 2.6. *Consider the following sequences;*

$$A_k := \begin{cases} \left\{ (x,y) \in \mathbb{R}^2 : \dfrac{(x+\sqrt{k})^2}{k} + \dfrac{y^2}{2k} = 1 \right\} & , \text{ if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}] \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

and

$$B_k := \begin{cases} \left\{ (x,y) \in \mathbb{R}^2 : \dfrac{(x-\sqrt{k})^2}{k} + \dfrac{y^2}{2k} = 1 \right\} & , \text{ if } k_{r-1} < k < k_{r-1} + [\sqrt{h_r}] \\ \{(1,1)\} & , \text{ otherwise.} \end{cases}$$

Since

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} |d(x; A_{k+i}, B_{k+i}) - 1| = 0,$$

uniformly in i, the sequences $\{A_k\}$ and $\{B_k\}$ is Wijsman strongly almost asymptotically lacunary equivalent. That is $\{A_k\} \overset{[\widehat{WN}_\theta]_1}{\sim} \{B_k\}$.

Theorem 2.7. *Let (X,ρ) be a metric space, $\theta = \{k_r\}$ be a lacunary sequence and A_k, B_k be non-empty closed subsets of X;*

(i) (a) $\{A_k\} \overset{[\widehat{WN}_\theta]_L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{(\widehat{WS}_\theta)_L}{\sim} \{B_k\}$,

(b) $\left[\widehat{WN}_\theta\right]_L$ *is a proper subset of* $\left(\widehat{WS}_\theta\right)_L$;

(ii) $d(x, A_k) = O(d(x, B_k))$ *and* $\{A_k\} \overset{(\widehat{WS}_\theta)_L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{[\widehat{WN}_\theta]_L}{\sim} \{B_k\}$.

Proof. $(i)-(a)$. Let $\varepsilon > 0$ and $\{A_k\} \overset{[WN]_\theta^L}{\sim} \{B_k\}$. For each $x \in X$, we can write

$$\begin{aligned} \sum_{k \in I_r} |d(x; A_{k+i}, B_{k+i}) - L| &\geq \sum_{\substack{k \in I_r \\ |d(x;A_{k+i},B_{k+i})-L| \geq \varepsilon}} |d(x; A_{k+i}, B_{k+i}) - L| \\ &\geq \varepsilon \cdot |\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}| \end{aligned}$$

uniformly in i which yields the result.

$(i)-(b)$. Suppose that $\left[\widehat{WN}_\theta\right]_L \subset \left(\widehat{WS}_\theta\right)_L$. Let $\{A_k\}$ and $\{B_k\}$ be following sequences;

$$A_k = \begin{cases} \{k\} & , \text{ if } \ k_{r-1} < k \leq k_{r-1} + \left[\sqrt{h_r}\right] \qquad r = 1, 2, \cdots \\ \{0\} & , \text{ otherwise.} \end{cases}$$

and

$$B_k = \{0\} \quad \text{for all } k.$$

Note that $\{A_k\}$ is not bounded. For every $\varepsilon > 0$ and for each $x \in X$, we have

$$\frac{1}{h_r} |\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - 1| \geq \varepsilon\}| \leq \frac{\left[\sqrt{h_r}\right]}{h_r} \to 0, \qquad \text{as } r \to \infty,$$

uniformly in i, that is, $\{A_k\} \overset{(\widehat{WS}_\theta)_1}{\sim} \{B_k\}$. On the other hand, there exists x in X

$$\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_{k+i},B_{k+i})-L| \not\to 0, \qquad \text{as } r\to\infty,$$

uniformly in i. Hence $\{A_k\} \overset{[\widehat{WN}_\theta]_L}{\not\sim} \{B_k\}$.

(ii) Suppose that $d(x,A_k)=O(d(x,B_k))$ and $\{A_k\} \overset{(\widehat{WS}_\theta)_L}{\sim} \{B_k\}$. Then we can assume that

$$|d(x;A_{k+i},B_{k+i})-L| \le M$$

for each $x\in X$, for all k and uniformly in i.

Given $\varepsilon>0$ and for each $x\in X$, we get

$$\begin{aligned}
\frac{1}{h_r}\sum_{k\in I_r}|d(x;A_{k+i},B_{k+i})-L| &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon}}|d(x;A_{k+i},B_{k+i})-L| \\
&+ \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |d(x;A_{k+i},B_{k+i})-L|<\varepsilon}}|d(x;A_{k+i},B_{k+i})-L| \\
&\le \frac{M}{h_r}\left|\{k\in I_r : |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon\}\right| + \varepsilon,
\end{aligned}$$

uniformly in i. Therefore $\{A_k\} \overset{[\widehat{WN}_\theta]_L}{\sim} \{B_k\}$. □

Lemma 2.8. *Suppose that for given $\varepsilon_1>0$ and every $\varepsilon>0$ there exist i_0 and j_0 such that*

$$\frac{1}{j}\left|\{0\le k\le j-1 : |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon\}\right| < \varepsilon_1,$$

for each $x\in X$, for all $j\ge j_0$ and $i\ge i_0$, then $\{A_k\} \overset{(\widehat{WS})_L}{\sim} \{B_k\}$.

Proof. Let $\varepsilon_1>0$ be given. For every $\varepsilon>0$ and for each $x\in X$, choose j_0' and i_0 such that

$$\frac{1}{j}\left|\{0\le k\le j-1 : |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon\}\right| < \frac{\varepsilon_1}{2},$$

for all $j\ge j_0'$ and $i\ge i_0$. It is enough to prove that there exists j_0'' such that

$$\frac{1}{j}\left|\{0\le k\le j-1 : |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon\}\right| < \varepsilon_1, \tag{2.1}$$

for each $x\in X$, for $j\ge j_0''$ and $0\le i\le i_0$. If we let $j_0=\max\{j_0',j_0''\}$, (2.1) will be true for $j>j_0$ and for all i. Once i_0 has been chosen, i_0 is fixed, so

$$\left|\{0\le k\le i_0-1 : |d(x;A_{k+i},B_{k+i})-L|\ge\varepsilon\}\right| = M,$$

for each $x \in X$. Now, taking $0 \leq i \leq i_0$, and $j > i_0$, for each $x \in X$ we have

$$\begin{aligned}
&\frac{1}{j}\left|\{0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad \leq \frac{1}{j}\left|\{0 \leq k \leq i_0 - 1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad\quad + \left|\{i_0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad \leq \frac{M}{j} + \frac{1}{j}\left|\{i_0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad \leq \frac{M}{j} + \frac{\varepsilon_1}{2}.
\end{aligned}$$

Thus, for j sufficiently large and for each $x \in X$

$$\frac{1}{j}\left|\{0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \leq \frac{M}{j} + \frac{\varepsilon_1}{2} < \varepsilon_1,$$

uniformly in i which gives (2.1) and this step concludes the proof. □

Theorem 2.9. *For every lacunary sequence $\theta = \{k_r\}$, $\left(\widehat{WS_\theta}\right)_L = \left(\widehat{WS}\right)_L$.*

Proof. Let $\{A_k\} \in \left(\widehat{WS_\theta}\right)_L$, then from Definition (2.3) assures us that, given $\varepsilon_1 > 0$ there exist $\varepsilon > 0$, $\exists r_0$ and L such that

$$\frac{1}{h_r}\left|\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| < \varepsilon_1,$$

for each $x \in X$, for $r \geq r_0$ and $i = k_{r-1} + 1 + v$ where $v \geq 0$.

Let $j \geq h_r$ and write $j = m \cdot h_r + t$ where $0 \leq t \leq h_r$ and m is an integer. Since $j \geq h_r$ and $m \geq 0$, we obtain the following, for each $x \in X$,

$$\begin{aligned}
&\frac{1}{j}\left|\{0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad \leq \frac{1}{j}\left|\{0 \leq k \leq (m+1)h_r - 1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad = \frac{1}{j}\sum_{n=0}^{m}\left|\{n \cdot h_r \leq k \leq (n+1)h_r - 1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \\
&\qquad \leq \frac{(m+1)}{j} h_r \cdot \varepsilon_1 \\
&\qquad \leq \frac{2m \cdot h_r \cdot \varepsilon_1}{j} \quad \text{for } m \geq 1,
\end{aligned}$$

uniformly in i.

For $\frac{h_r}{j} \leq 1$, since $\frac{m \cdot h_r}{j} \leq 1$ we have, for each $x \in X$,

$$\frac{1}{j}\left|\{0 \leq k \leq j-1 : |d(x; A_{k+i}, B_{k+i}) - L| \geq \varepsilon\}\right| \leq 2\varepsilon_1,$$

uniformly in i. Then, Lemma 2.8 implies

$$\left(\widehat{WS_\theta}\right)_L \subseteq \left(\widehat{WS}\right)_L .$$

It is also clear that

$$\left(\widehat{WS}\right)_L \subseteq \left(\widehat{WS_\theta}\right)_L$$

for every lacunay sequence θ. Hence, we have the result. $\square$

Theorem 2.10. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ be a lacunary sequence with* $\liminf_r q_r > 1$, *then*

$$\{A_k\} \overset{(\widehat{WS})_L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\}.$$

Proof. Suppose first that $\liminf_r q_r > 1$, then there exists a $\lambda > 0$ such that $q_r \geq 1+\lambda$ for sufficiently large r, which implies that

$$\frac{\lambda}{1+\lambda} \leq \frac{h_r}{k_r}.$$

If $\{A_k\} \overset{(\widehat{WS})_L}{\sim} \{B_k\}$, then for every $\varepsilon > 0$, for each $x \in X$ and for sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r}\left|\{k \leq k_r : |d(x;A_{k+i},B_{k+i}) - L| \geq \varepsilon\}\right| &\geq \frac{1}{k_r}\left|\{k \in I_r : |d(x;A_{k+i},B_{k+i}) - L| \geq \varepsilon\}\right| \\ &\geq \frac{\lambda}{1+\lambda}.\left(\frac{1}{h_r}\left|\{k \in I_r : |d(x;A_{k+i},B_{k+i}) - L| \geq \varepsilon\}\right|\right),\end{aligned}$$

uniformly in i. This completes the proof. $\square$

Theorem 2.11. *Let (X,ρ) be a metric space and A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ be a lacunary sequence with* $\limsup_r q_r < \infty$, *then*

$$\{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\} \Rightarrow \{A_k\} \overset{(\widehat{WS})_L}{\sim} \{B_k\}.$$

Proof. Let $\theta = \{k_r\}$ be a lacunary sequence with $\limsup_r q_r < \infty$. Then there is an $M > 0$ such that $q_r < M$ for all $r \geq 1$. Let $\{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\}$, and $\varepsilon_1 > 0$. There exists $R > 0$ and $\varepsilon > 0$ such that for every $j \geq R$ and for each $x \in X$,

$$A_j = \frac{1}{h_j}\left|\{k \in I_j : |d(x;A_{k+i},B_{k+i}) - L| \geq \varepsilon\}\right| < \varepsilon_1,$$

uniformly in i. We can also find $H > 0$ such that $A_j < H$ for all $j = 1, 2, \ldots$. Now let t be any integer with satisfying $k_{r-1} < t \leq k_r$, where $r > R$. Then we can write, for each $x \in X$,

$$\frac{1}{t}\left|\{k \le t : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right| \le \frac{1}{k_{r-1}}\left|\{k \le k_r : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|$$

$$= \frac{1}{k_{r-1}}\left\{\left|\{k \in I_1 : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|\right\}$$

$$+ \frac{1}{k_{r-1}}\left\{\left|\{k \in I_2 : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|\right\}$$

$$+ \cdots + \frac{1}{k_{r-1}}\left\{\left|\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|\right\}$$

$$= \frac{k_1}{k_{r-1} \cdot k_1}\left|\{k \in I_1 : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|$$

$$+ \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\{k \in I_2 : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|$$

$$+ \cdots + \frac{k_R - k_{R-1}}{k_{r-1}(k_R - k_{R-1})}\left|\{k \in I_R : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|$$

$$+ \cdots + \frac{k_r - k_{r-1}}{k_{r-1}(k_r - k_{r-1})}\left|\{k \in I_r : |d(x; A_{k+i}, B_{k+i}) - L| \ge \varepsilon\}\right|$$

$$= \frac{k_1}{k_{r-1}}A_1 + \frac{k_2 - k_1}{k_{r-1}}A_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}}A_R$$

$$+ \frac{k_{R+1} - k_R}{k_{r-1}}A_{R+1} + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}}A_r$$

$$\le \left\{\sup_{j \ge 1} A_j\right\}\frac{k_R}{k_{r-1}} + \left\{\sup_{j \ge R} A_j\right\}\frac{k_r - k_R}{k_{r-1}}$$

$$\le H\frac{k_R}{k_{r-1}} + \varepsilon M,$$

uniformly in i. This completes the proof. □

Combining Theorem (2.10) and (2.11) we have following Theorem.

Theorem 2.12. *Let (X, ρ) be a metric space and A_k, B_k be non-empty closed subsets of X. If $\theta = \{k_r\}$ be a lacunary sequence with $1 < \liminf_r q_r \le \limsup_r q_r < \infty$, then*

$$\{A_k\} \overset{(\widehat{WS_\theta})_L}{\sim} \{B_k\} \Leftrightarrow \{A_k\} \overset{(\widehat{WS})_L}{\sim} \{B_k\}.$$

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