Balanced Model Order Reduction Method for Systems Depending on a Parameter [⋆]

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Abstract: We provide an analytical framework for balanced realization model order reduction of linear control systems which depend on an unknown parameter. Besides recovering known results for the first order corrections, we obtain explicit novel expressions for the form of second order corrections for singular values and singular vectors. The final result of our procedure is an order reduced model which incorporates the uncertain parameter. We apply our algorithm to a system of masses and springs with parameter dependent coefficients.

Keywords: model reduction, singular value decomposition, uncertain linear systems

1. INTRODUCTION

Order reduced models (Schilders et al. (2008)) are useful to simulate very large models using less computational resources, allowing, for instance, the exploration of parameter regions. The lower order model should have some desirable properties, such as being easily computable, preserving some of the structural properties of the full model and, more importantly, yielding an error with respect to the original model that can be bounded in terms of the complexity of the approximating model. In particular, for linear time-invariant MIMO systems, model order reduction (MOR) based on the truncation of balanced realizations preserves the stability, controllability and observability of the full model, and furthermore provides bounds for the norm of the error system (Antoulas (2005)).

The computation of a balanced realization for a linear system relies on numerical linear algebra algorithms, and does not allow for the presence of symbolic parameters in the model. Hence, if a system contains an uncertain parameter, appearing, for instance, due to a physical coefficient which is only known to belong to a given interval, or due to the specification of a working point in a nonlinear system, the balancing procedure must be carried out for each numerical value of the parameter. This results in a set of reduced order models, which are difficult to work with if they are to be used to design a controller and, in any case, the explicit dependence on the original parameter is lost in the reduced system.

In this paper we work out an algorithm to obtain a reduced order model which incorporates the original, symbolical parameter through a polynomial of arbitrary degree. To this end, we solve each step of the balanced realization procedure in powers of the symbolical parameter, although for the last step, which involves a singular value decomposition (SVD), we only provide explicit expressions up to second order corrections. Up to our knowledge, the second order correction to the singular subspaces that we obtain has not been reported in the literature, and it may be useful in other applications of SVD.

The paper is organized as follows. Section 2 reviews the steps of the computation of the balanced realization for linear systems, and how a reduced order model can be constructed from it. Section 3 develops a power series expansion for each of the above steps. We give explicit algorithms for each step, except for the singular value decomposition, which we develop only to second order. Section 4 applies the procedure to a system of masses and springs with parameter dependent coefficients, and, finally, we discuss our results and point to possible improvements in Section 5.

An extended version of this work, with some detailed developments that are not presented here, has been posted on `arXiv[cs.SY]` (Batlle and Roqueiro (2016)).

2. BALANCED REALIZATION PROCEDURE

Consider the nonlinear control system

$$\dot{x} = f(x) + g(x)u, \qquad (1)$$
$$y = h(x), \qquad (2)$$

with $x \in \mathbb{R}^N$, $u \in \mathbb{R}^M$, $y \in \mathbb{R}^P$ and $f(0) = 0$.

The controllability function $L_c(x)$ is the solution of the optimal control problem

$$L_c(x) = \inf_{u \in L^2((-\infty,0),\mathbb{R}^M)} \frac{1}{2} \int_{-\infty}^{0} ||u(t)||^2 \mathrm{d}t \qquad (3)$$

subject to the boundary conditions $x(-\infty) = 0$, $x(0) = x$ and the system (1). Roughly speaking, $L_c(x)$ measures the

minimum 2-norm of the input signal necessary to bring the system to the state x from the origin.

The observability function $L_o(x)$ is the 2-norm of the output signal obtained when the system is relaxed from the state x

$$L_o(x) = \frac{1}{2} \int_0^\infty ||y(t)||^2 \mathrm{d}t = \frac{1}{2} \int_0^\infty ||h(x(t))||^2 \mathrm{d}t, \quad (4)$$

with $x(0) = x$ and subjected to (1) with $u = 0$.

For linear control systems,

$$\dot{x} = Ax + Bu, \quad (5)$$
$$y = Cx, \quad (6)$$

assumed to be observable, controllable and Hurwitz, both $L_c(x)$ and $L_o(x)$ are quadratic functions

$$L_c(x) = \frac{1}{2} x^T W_c^{-1} x, \quad (7)$$
$$L_0(x) = \frac{1}{2} x^T W_o x, \quad (8)$$

where $W_c > 0$ and $W_o > 0$, the controllability and observability Gramians, are the solutions to the matrix Lyapunov equations

$$AW_c + W_c A^T + BB^T = 0, \quad (9)$$
$$A^T W_o + W_o A + C^T C = 0. \quad (10)$$

As shown by Moore (Moore (1981); see also Laub et al. (1987) and Verriest and Kailath (1983)), the matrix W_c provides information about the states that are easy to control (in the sense that signals u of small norm can be used to reach them), while W_o allows to find the states that are easily observable (in the sense that they produce outputs of large norm). From the point of view of the input-output map given by (5) (6), one would like to select the states that score well on both counts, and this leads to the concept of *balanced realization*, for which $W_c = W_o$.

The balanced realization is obtained by means of a linear transformation $x = Tz$, with T computed as follows:

(1) Solve the Lyapunov equations (9) and (10).
(2) Perform Cholesky factorizations of the Gramians:
$$W_c = XX^T, \quad W_o = YY^T. \quad (11)$$
(3) Compute the SVD of $Y^T X$:
$$Y^T X = U\Sigma V^T, \quad (12)$$
with U and V orthogonal and
$$\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N), \quad \sigma_1 \geq \cdots \geq \sigma_N > 0. \quad (13)$$
(4) The balancing transformation is given then by
$$T = XV\Sigma^{-1/2}, \quad \text{with } T^{-1} = \Sigma^{-1/2} U^T Y^T. \quad (14)$$
(5) The balanced realization is given by the linear system
$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad (15)$$
and, in the new coordinates,
$$\tilde{W}_c = T^{-1} W_c T^{-T} = \Sigma, \quad \tilde{W}_o = T^T W_o T = \Sigma. \quad (16)$$

Notice that, in the balanced realization,

$$\tilde{L}_c(z) = \frac{1}{2} \sum_{i=1}^N \frac{z_i^2}{\sigma_i} = \frac{1}{2} z^T \Sigma^{-1} z, \quad (17)$$

$$\tilde{L}_o(z) = \frac{1}{2} \sum_{i=1}^N \sigma_i z_i^2 = \frac{1}{2} z^T \Sigma z, \quad (18)$$

so that the state with only nonzero coordinate z_i is both easier to control and easier to observe than the state corresponding to z_{i+1}, for $i = 1, 2, \ldots, N-1$. If, for a given r, $1 \leq r < N$, one has $\sigma_r \gg \sigma_{r+1}$, it may be sensible, from the point of view of the map between u and y, to keep just the states corresponding to the coordinates $z_1, z_2, \ldots, z_r$, and this is what is known as balanced realization model order reduction.

$\mathcal{H}_\infty$-norm lower and upper error bounds of the balanced truncation method are given by

$$\sigma_{r+1} \leq ||G(s) - G_r(s)||_{\mathcal{H}_\infty} \leq 2 \sum_{i=r+1}^N \sigma_i, \quad (19)$$

where $\sigma_i, i = 1, \ldots, N$, are the Hankel singular values of the system (Glover (1984)Enns (1984)). From these inequalities it follows that, in order to get the smallest error for the truncated system, one should disregard the states associated with the smallest Hankel singular values.

If we denote by $\tilde{A}_r$ the upper-left square block of $\tilde{A}$ formed by the first r rows and columns, and by $\tilde{B}_r$ and $\tilde{C}_r$ the matrices obtained from the first r rows or columns of $\tilde{B}$ or $\tilde{C}$, respectively, the reduced system of order r obtained by balanced truncation is given by
$$\dot{Z}_r = \tilde{A}_r Z_r + \tilde{B}_r u, \quad y = \tilde{C}_r Z_r, \quad (20)$$
with $Z_r = (z_1, \ldots, z_r)$.

One of the problems of the above procedure is that it does not allow for the presence of symbolic parameters in the problem, since the solution of the matrix equations involved relies on numerical methods. In this paper we address this issue, assuming that the linear system is given by matrices $A(m)$, $B(m)$ and $C(m)$ which depend analytically on the parameter m. This may represent an uncertain physical coefficient (this is the case of the example in Section 4), or it may appear by considering an unspecified working point in the linearization of a nonlinear system.

Our goal is to develop a power series expansion in m of the balanced model order reduction algorithm for the linear input/output system given by $A(m)$, $B(m)$, $C(m)$. This will facilitate the analysis of how much the important degrees of freedom vary when m is changed and, more importantly, will yield a reduced order model, suitable for control design, which incorporates the dependence on m in an explicit way. A survey of other approaches to this problem is presented in Benner et al. (2015).

3. POWER SERIES EXPANSION FOR THE BALANCED REALIZATION

Following the previous discussion, consider the control system

$$\dot{x} = A(m)x + B(m)u, \quad (21)$$
$$y = C(m)x, \quad (22)$$

with m a symbolic parameter. The controllability Gramian $W^c(m)$ will be the solution to the Lyapunov equation

$$A(m)W^c(m) + W^c(m)A^T(m) + B(m)B^T(m) = 0. \quad (23)$$

Assume that $A(m)$, $B(m)$ and $C(m)$ are analytic in m,

$$A(m) = \sum_{k=0}^{\infty} A_k m^k, \quad (24)$$

$$B(m) = \sum_{k=0}^{\infty} B_k m^k, \quad (25)$$

$$C(m) = \sum_{k=0}^{\infty} C_k m^k, \quad (26)$$

and let us look for likewise solutions of the form

$$W^c(m) = \sum_{k=0}^{\infty} W_k^c m^k. \quad (27)$$

Putting the above expansions into (23) one obtains

$$\sum_{s=0}^{r} \left(A_{r-s} W_s^c + W_s^c A_{r-s}^T + B_{r-s} B_s^T \right) = 0, \ r = 0, 1, 2, \ldots. \quad (28)$$

These are equivalent to the set of Lyapunov equations

$$A_0 W_0^c + W_0^c A_0^T + B_0 B_0^T = 0, \quad (29)$$

$$A_0 W_r^c + W_r^c A_0^T + P_r = 0, \ r = 1, 2, \ldots, \quad (30)$$

with

$$P_r = B_0 B_r^T + \sum_{s=0}^{r-1} \left(A_{r-s} W_s^c + W_s^c A_{r-s}^T + B_{r-s} B_s^T \right). \quad (31)$$

These equations can be solved recursively to the desired order, starting with the zeroth order Lyapunov equation (29). Observe that the internal dynamics is always given by A_0, and that it is only the effective control term P_r the one that changes with the order.

Similarly, the observability Gramian $W^o(m)$ satisfies

$$A^T(m)W^o(m) + W^o(m)A(m) + C^T(m)C(m) = 0, \quad (32)$$

and its power series solution

$$W^o(m) = \sum_{k=0}^{\infty} W_k^o m^k, \quad (33)$$

can be obtained recursively from

$$A_0^T W_0^o + W_0^o A_0 + C_0^T C_0 = 0, \quad (34)$$

$$A_0^T W_r^o + W_r^o A_0 + Q_r = 0, \ r = 1, 2, \ldots, \quad (35)$$

with

$$Q_r = C_0^T C_r + \sum_{s=0}^{r-1} \left(A_{r-s}^T W_s^o + W_s^o A_{r-s} + C_{r-s}^T C_s \right). \quad (36)$$

After computing $W^c(m)$ and $W^o(m)$ at the desired order, the next step in the balancing transformation procedure is to compute $X(m)$ and $Y(m)$ such that

$$W^c(m) = X(m)X^T(m), \quad (37)$$

$$W^o(m) = Y(m)Y^T(m). \quad (38)$$

If

$$X(m) = \sum_{k=0}^{\infty} X_k m^k, \quad (39)$$

one gets

$$W_k^c = \sum_{s=0}^{k} X_{k-s} X_s^T, \quad (40)$$

which, again, are solved recursively as

$$X_0 X_0^T = W_0^c, \quad (41)$$

$$X_k X_0^T + X_0 X_k^T = W_k^c - \sum_{s=1}^{k-1} X_{k-s} X_s^T. \quad (42)$$

Similarly, for

$$Y(m) = \sum_{k=0}^{\infty} Y_k m^k, \quad (43)$$

one arrives at

$$Y_0 Y_0^T = W_0^o, \quad (44)$$

$$Y_k Y_0^T + Y_0 Y_k^T = W_k^o - \sum_{s=1}^{k-1} Y_{k-s} Y_s^T. \quad (45)$$

Equations (41) and (44) are standard Cholesky equations, but (42) and (45) are not Lyapunov (or Sylvester) equations for X_k or Y_k because of the presence of X_k^T and Y_k^T, respectively.

Equations of the form $AX + X^T B = C$ for X have been studied in Vorontsov and Ikramov (2011), where the problem is reduced to a sequence of low-order linear systems for the entries of X. However, the conditions for the uniqueness of the solution stated in Vorontsov and Ikramov (2011) are not satisfied by equations of the form of (42). Indeed, in order to solve (42) one has to consider $\det(X_0 + \lambda(X_0^T)^T) = (1 + \lambda)^N \det X_0$, which vanishes for $\lambda = -1$ and thus violates condition (2) of Theorem 3 in Vorontsov and Ikramov (2011). Notice, however, that the right-hand side of (42) is a symmetric matrix. If one splits X_k into symmetric, S_k, and skew-symmetric, T_k, parts, one gets, after some calculations, that they obey

$$S_k X_0^T + X_0 S_k = W_k^c - \sum_{s=1}^{k-1} X_{k-s} X_s^T, \quad (46)$$

$$T_k X_0^T + X_0 T_k = 0. \quad (47)$$

Equations (46) and (47) are Lyapunov equations, and in fact the generic solution to (47) is $T_k = 0$. Hence, we have that the solution to (42) is given by $X_k = S_k$, with S_k the solution to the Lyapunov equation (46), and an analogous reasoning applies to the solution of (45).

The last nontrivial step in the balancing algorithm is the singular value decomposition (SVD) of the product $Y^T(m)X(m)$,

$$Y^T(m)X(m) = U(m)\Sigma(m)V^T(m), \quad (48)$$

where

$$\Sigma(m) = \text{diag}(\sigma_1(m) \geq \sigma_2(m) \geq \ldots \geq \sigma_n(m) > 0), \quad (49)$$

and $U(m)$ and $V(m)$ are $N \times N$ orthogonal matrices.

Let us denote by R_k the coefficients of the power series of $Y^T(m)X(m)$,

$$Y^T(m)X(m) = \sum_{k=0}^{\infty} R_k m^k, \tag{50}$$

with

$$R_k = \sum_{s=0}^{k} Y_{k-s}^T X_s = \sum_{s=0}^{k} Y_s^T X_{k-s}. \tag{51}$$

We also denote by U_k, V_k, and Σ_k the coefficients of the power series of $U(m)$, $V(m)$ and $\Sigma(m)$, respectively.

From now on we will consider approximations only up to second order. We will write

$$R(m) = R_0 + mR_1 + m^2 R_2, \tag{52}$$
$$U(m) = U_0 + mU_1 + m^2 U_2, \tag{53}$$
$$V(m) = V_0 + mV_1 + m^2 V_2, \tag{54}$$
$$\Sigma(m) = \Sigma_0 + m\Sigma_1 + m^2 \Sigma_2, \tag{55}$$

with the understanding that any higher order contribution is neglected. From $R = U\Sigma V^T$ one gets the identities

$$RV = U\Sigma, \quad R^T U = V\Sigma, \tag{56}$$

which in turn inply

$$R^T R V = V\Sigma^2, \tag{57}$$
$$RR^T U = U\Sigma^2. \tag{58}$$

If we denote by $u_j^{(k)}$ the jth column vector of U_k, and by $v_j^{(k)}$ the one of V_k, equations (56) imply, to zeroth, first and second order in m,

$$R_0 v_j^{(0)} = \sigma_j^{(0)} u_j^{(0)}, \tag{59}$$
$$R_0^T u_j^{(0)} = \sigma_j^{(0)} v_j^{(0)}, \tag{60}$$
$$R_1 v_j^{(0)} + R_0 v_j^{(1)} = \sigma_j^{(0)} u_j^{(1)} + \sigma_j^{(1)} u_j^{(0)}, \tag{61}$$
$$R_1^T u_j^{(0)} + R_0^T u_j^{(1)} = \sigma_j^{(0)} v_j^{(1)} + \sigma_j^{(1)} v_j^{(0)}, \tag{62}$$
$$R_0 v_j^{(2)} + R_1 v_j^{(1)} + R_2 v_j^{(0)} = \sigma_j^{(0)} u_j^{(2)} + \sigma_j^{(1)} u_j^{(1)} + \sigma_j^{(2)} u_j^{(0)}, \tag{63}$$
$$R_0^T u_j^{(2)} + R_1^T u_j^{(1)} + R_2^T u_j^{(0)} = \sigma_j^{(0)} v_j^{(2)} + \sigma_j^{(1)} v_j^{(1)} + \sigma_j^{(2)} v_j^{(0)}. \tag{64}$$

with $\sigma_j^{(k)}$ the jth element of the diagonal matrix Σ_k. From the orthogonality condition $U^T(m)U(m) = \mathbb{I}$ one gets

$$\left\langle u_i^{(1)}, u_i^{(0)} \right\rangle = 0, \tag{65}$$
$$\left\langle u_i^{(2)}, u_i^{(0)} \right\rangle = -\frac{1}{2} ||u_j^{(1)}||^2, \tag{66}$$

and similarly, from $V^T(m)V(m) = \mathbb{I}$, one has

$$\left\langle v_i^{(1)}, v_i^{(0)} \right\rangle = 0, \tag{67}$$
$$\left\langle v_i^{(2)}, v_i^{(0)} \right\rangle = -\frac{1}{2} ||v_j^{(1)}||^2. \tag{68}$$

The inner product of $u_i^{(0)}$ with (61) yields (Stewart (1991))

$$\sigma_i^{(1)} = \left\langle u_i^{(0)}, R_1 v_i^{(0)} \right\rangle = \left\langle v_i^{(0)}, R_1^T u_i^{(0)} \right\rangle, \tag{69}$$

where the second form can also be obtained operating from (62). In order to complete the first order correction one

needs to compute the corrections to the singular subspaces, *i.e.* the vectors $u_i^{(1)}$ and $v_i^{(1)}$. To compute $u_i^{(1)}$, we act on (62) with R_0 and then use (61) to get rid of $v_i^{(1)}$. One obtains then the system

$$\left(R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \right) u_i^{(1)}$$
$$= -R_0 R_1^T u_i^{(0)} + \sigma_i^{(1)} R_0 v_i^{(0)} - \sigma_i^{(0)} R_1 v_i^{(0)} + \sigma_i^{(0)} \sigma_i^{(1)} u_i^{(0)}$$
$$= 2\sigma_i^{(0)} \sigma_i^{(1)} u_i^{(0)} - R_0 R_1^T u_i^{(0)} - \sigma_i^{(0)} R_1 v_i^{(0)}. \tag{70}$$

This is a system of N equations for the N components of $u_i^{(1)}$, but the equations are not independent. Indeed, from (58) one has, to zeroth order,

$$R_0 R_0^T u_i^{(0)} = (\sigma_i^{(0)})^2 u_i^{(0)}, \tag{71}$$

so that $(\sigma_i^{(0)})^2$ is an eigenvalue of $R_0 R_0^T$ and $R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I}$ is not invertible. Assuming that the eigenvalues are simple, one must find an extra equation in order to be able to obtain $u_i^{(1)}$, and this is provided by (65). Denoting by $Q_i^{(1)}$ the vector in the right-hand side of (70),

$$Q_i^{(1)} = 2\sigma_i^{(0)} \sigma_i^{(1)} u_i^{(0)} - R_0 R_1^T u_i^{(0)} - \sigma_i^{(0)} R_1 v_i^{(0)}, \tag{72}$$

it turns out that each $u_i^{(1)}$ can be uniquely computed as the solution to the system

$$\left(\begin{array}{c} R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \\ (u_i^{(0)})^T \end{array} \right) u_i^{(1)} = \left(\begin{array}{c} Q_i^{(1)} \\ 0 \end{array} \right). \tag{73}$$

An explicit form of the solution to (73) for the more general case of non-square matrices is given in Liu et al. (2008). Similarly, for $v_i^{(1)}$ one has

$$\left(\begin{array}{c} R_0^T R_0 - (\sigma_i^{(0)})^2 \mathbb{I} \\ (v_i^{(0)})^T \end{array} \right) v_i^{(1)} = \left(\begin{array}{c} P_i^{(1)} \\ 0 \end{array} \right), \tag{74}$$

with

$$P_i^{(1)} = 2\sigma_i^{(0)} \sigma_i^{(1)} v_i^{(0)} - R_0^T R_1 v_i^{(0)} - \sigma_i^{(0)} R_1^T u_i^{(0)}. \tag{75}$$

Under the assumption that the singular values $\sigma_i^{(0)}$ are non-degenerate, *i.e.* the solution spaces of equations (57) and (58) are one-dimensional, the above systems have unique solutions that can be numerically computed. Indeed, if there is a vector $u \neq 0$ such that

$$\left(\begin{array}{c} R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \\ (u_i^{(0)})^T \end{array} \right) u = 0,$$

then $(R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I})u = 0$, and hence, due to the non-degeneracy, $u = \lambda u_i^{(0)}$ for some λ, which contradicts the last relation $(u_i^{(0)})^T u = 0$.

In order to obtain the second order corrections one has to work with (63), (64) and (66). For instance, multiplying (63) with $u_i^{(0)}$, and using (66) and (65), one gets the second order correction to the singular values of R

$$\sigma_i^{(2)} = \frac{1}{2} \sigma_i^{(0)} \left(||u_i^{(1)}||^2 - ||v_i^{(1)}||^2 \right) + \left\langle u_i^{(0)}, R_1 v_i^{(1)} + R_2 v_i^{(0)} \right\rangle. \tag{76}$$

Notice that the right-hand side depends only on data from the zeroth and first order approximations, plus the second order perturbation R_2.

To compute the second order correction to the singular subspaces one must solve (63) and (64) for $u_i^{(2)}$ and $v_i^{(2)}$. One obtains that the $u_i^{(2)}$ are the unique solution to

$$\begin{pmatrix} R_0 R_0^T - (\sigma_i^{(0)})^2 \mathbb{I} \\ (u_i^{(0)})^T \end{pmatrix} u_i^{(2)} = \begin{pmatrix} Q_i^{(2)} \\ -\dfrac{1}{2}||u_i^{(1)}||^2 \end{pmatrix}, \qquad (77)$$

with

$$Q_i^{(2)} = -R_0 R_1^T u_i^{(1)} - R_0 R_2^T u_i^{(0)} + \sigma_i^{(0)} \sigma_i^{(1)} u_i^{(1)} + \sigma_i^{(1)} R_0 v_i^{(1)}$$
$$+ 2\sigma_i^{(0)} \sigma_i^{(2)} u_i^{(0)} - \sigma_i^{(0)} R_1 v_i^{(1)} - \sigma_i^{(0)} R_2 v_i^{(0)}. \qquad (78)$$

Similarly, the $v_i^{(2)}$ are given by the solution to

$$\begin{pmatrix} R_0^T R_0 - (\sigma_i^{(0)})^2 \mathbb{I} \\ (v_i^{(0)})^T \end{pmatrix} v_i^{(2)} = \begin{pmatrix} P_i^{(2)} \\ -\dfrac{1}{2}||v_i^{(1)}||^2 \end{pmatrix}, \qquad (79)$$

with

$$P_i^{(2)} = -R_0^T R_1 v_i^{(1)} - R_0^T R_2 v_i^{(0)} + \sigma_i^{(0)} \sigma_i^{(1)} v_i^{(1)} + \sigma_i^{(1)} R_0^T u_i^{(1)}$$
$$+ 2\sigma_i^{(0)} \sigma_i^{(2)} v_i^{(0)} - \sigma_i^{(0)} R_1^T u_i^{(1)} - \sigma_i^{(0)} R_2^T u_i^{(0)}. \qquad (80)$$

Notice that the matrices appearing on the left hand-sides of (77) and (79) are the same than the ones in (73) and (74), respectively, and hence the solutions are unique.

This procedure can be repeated to obtain higher order corrections in m. At order m, one obtains first an explicit expression for the corrections $\sigma_i^{(m)}$ to the singular values, and then one can write systems of equations for the corrections $u_i^{(m)}$ and $v_i^{(m)}$ to the singular vectors, with the same matrices appearing in previous orders but with different right-hand sides.

The final step of the procedure for the construction of the balanced realization is to use (14) with (52)—(55), keeping terms up to order m^2. Since the matrix $\Sigma(m)$ is diagonal, $\Sigma(m)^{-1/2}$ is defined diagonal-wise, and for each entry $\sigma_i(m)$ we have, up to order m^2,

$$(\sigma_i(m))^{-1/2} = \frac{1}{(\sigma_i^{(0)})^{1/2}} - m\frac{\sigma_i^{(1)}}{2(\sigma_i^{(0)})^{3/2}}$$
$$+ m^2 \left(-\frac{\sigma_i^{(2)}}{2(\sigma_i^{(0)})^{3/2}} + \frac{3(\sigma_i^{(1)})^2}{8(\sigma_i^{(0)})^{5/2}} \right) \quad (81)$$
$$\equiv s_i^{(0)} + m s_i^{(1)} + m^2 s_i^{(2)}. \qquad (82)$$

Hence, if $S_a = \mathrm{diag}(s_i^{(a)})$, $a = 0, 1, 2$,

$$\Sigma(m)^{-1/2} = S_0 + m S_1 + m^2 S_2. \qquad (83)$$

Up to order m^2, the matrix $T(m)$ for the transformation from the original x coordinates to the balanced ones z, $x = Tz$, and its inverse $T^{-1}(m)$, are given by $T(m) = T_2(m) + O(m^3)$ and $T^{-1}(m) = T_2^{-1}(m) + O(m^3)$, with

$$T_2(m) = X_0 V_0 S_0 + m(X_0 V_0 S_1 + X_0 V_1 S_0 + X_1 V_0 S_0)$$
$$+ m^2(X_0 V_0 S_2 + X_2 V_0 S_0 + X_0 V_2 S_0$$
$$+ X_0 V_1 S_1 + X_1 V_0 S_1 + X_1 V_1 S_0)$$
$$\equiv T_0 + m T_1 + m^2 T_2,$$
$$T_2^{-1}(m) = S_0 U_0^T Y_0^T + m(S_0 U_1^T Y_0^T + S_0 U_0^T Y_1^T + S_1 U_0^T Y_0^T)$$
$$+ m^2(S_0 U_0^T Y_2^T + S_0 U_2^T Y_0^T + S_2 U_0^T Y_0^T$$
$$+ S_1 U_1^T Y_0^T + S_1 U_0^T Y_1^T + S_0 U_1^T Y_1^T)$$
$$\equiv \hat{T}_0 + m \hat{T}_1 + m^2 \hat{T}_2.$$

From these, the approximation of the balanced realization, up to the second order in m, is given (see (15)) by

$$\tilde{A}_2(m) = \hat{T}_0 A_0 T_0 + m(\hat{T}_0 A_1 T_0 + \hat{T}_0 A_0 T_1 + \hat{T}_1 A_0 T_0)$$
$$+ m^2(\hat{T}_0 A_0 T_2 + \hat{T}_0 A_2 T_0 + \hat{T}_2 A_0 T_0$$
$$+ \hat{T}_0 A_1 T_1 + \hat{T}_1 A_0 T_1 + \hat{T}_1 A_1 T_0), \qquad (84)$$
$$\tilde{B}_2(m) = \hat{T}_0 B_0 + m(\hat{T}_0 B_1 + \hat{T}_1 B_0)$$
$$+ m^2(\hat{T}_0 B_2 + \hat{T}_2 B_0 + \hat{T}_1 B_1), \qquad (85)$$
$$\tilde{C}_2(m) = C_0 T_0 + m(C_0 T_1 + C_1 T_0)$$
$$+ m^2(C_0 T_2 + C_2 T_0 + C_1 T_1). \qquad (86)$$

Matrices (84)—(86) define a balanced realization of the original system which is exact for $m = 0$ and approximate to $O(m^3)$ for $m \neq 0$. A reduced system of order r is obtained by truncating this realization to the first r states.

4. A SYSTEM OF MASSES AND SPRINGS

We consider a system of N masses m_i and (linear)springs with constants k_i and natural lengths d_i, so that the ith spring lies between masses m_i and m_{i+1}, $i = 1, \ldots, N-1$, and the last spring connects mass m_N to a fixed wall. We also add a linear damper to each mass, with coefficients γ_i and, furthermore, act with an external force F on the first mass. After redefining the coordinates to absorb the lengths d_i and introducing the canonical momenta $p_i = \dot{x}_i/m_i$, the system can be put in the first order form

$$\dot{X} = \begin{pmatrix} 0_{N \times N} & \mathrm{diag}(1/m_1, \ldots, 1/m_N) \\ K_{N \times N} & -\mathrm{diag}(\gamma_1/m_1, \ldots, \gamma_N/m_N) \end{pmatrix} X + BF, \qquad (87)$$

where $X = (x_1, \ldots, x_N, p_1, \ldots, p_N)^T$,
$$B = (\underbrace{0, \ldots, 0}_{N}, 1, 0, \ldots, 0)^T,$$

and K is the corresponding stiffness matrix. If we measure the velocity of the first mass, we have also the output $y = CX$ with $C = (\, 1 \; 0 \; \cdots \; 0 \; 0 \; 0 \; \cdots \; 0\,)$.

In order to obtain a test of our algorithm, we consider $k_i = 100(i+1)$, $m_i = i(1+m)$, $\gamma_i = 1$, $i = 1, \ldots, N$, with m the parameter of the Taylor expansion. We set $N = 10$, which yields a system with 20 states, and consider reduced systems with four states. Our procedure, which we have implemented entirely in `Matlab`, yields a linear reduced system (Batlle and Roqueiro (2016))

$$\dot{q} = A_4(m)q + B_4(m)u, \quad y = C_4(m)q \qquad (88)$$

with $q \in \mathbb{R}^4$, and matrices $A_4(m)$, $B_4(m)$ and $C_4(m)$ whose elements are polynomials of degree 2 im m.
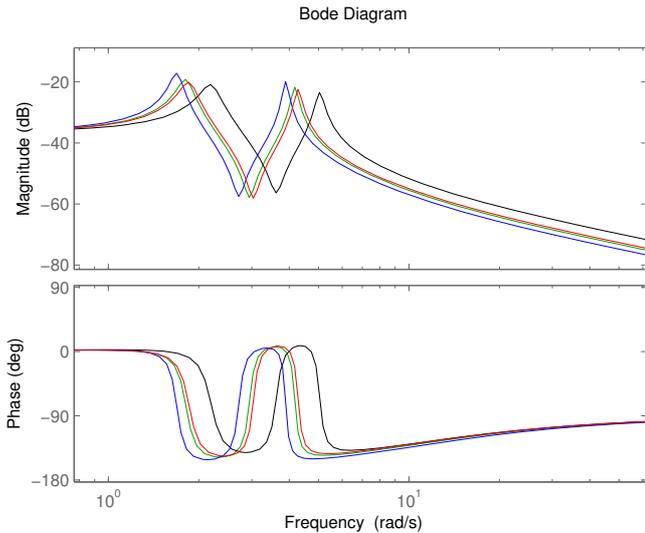
Fig. 1. Comparison of Bode plots for zeroth (black), first (blue) and second (red) order approximations for $m = 0.5$, together with the *exact* reduction of the system (green).

Figure 1 shows a detail of the Bode diagrams for $m = 0.5$ computed using the polynomial approximations of degree zero (black), one (blue) and two (red), together with the exact reduced system (green). It is clearly seen that the results improve as the order of the polynomial approximation is increased. Notice that the zeroth order polynomial approximation is equivalent to considering $m = 0$.

5. CONCLUSIONS

We have developed a parameter dependent model order reduction algorithm based on the balanced realization approximation. The algorithm yields a reduced order model which can be used to design a controller valid for a range of values of the parameter. As a by-product, we have obtained an expression for the second order perturbation of the singular subspaces (see equations (77) or (79)).

We should point out that, from the point of view of simulating a large system, it may be better to compute the *exact* reduced system for a given value of the parameter, since the truncation error of our second order polynomial approximation may become quite large for large m (or even yield unstable reduced systems). Our procedure is thus more relevant for control design than for simulation.

Some trivial extensions of our work, which we have not reported here for the sake of simplicity, include considering several parameters instead of one or computing some further higher order corrections of the parametrized SVD.

We have not addressed the issue of the estimation of the error of the reduced model. Notice that this error involves both the truncation errors of the different steps of the algorithm and the error which comes from the truncation of the balanced realization. The latter is the only present for $m = 0$, and is the one for which bounds are well known. We currently do not know how to deal with the former, and how it could be integrated with the latter. However, the simulations of the system that we have presented, together with some simulations of the individual steps (not reported here) seem to indicate that the errors due to the different polynomial truncations go down when higher order approximations are used. We plan to address this issue by relating our construction to the general framework of Beck et al. (1996), and by comparing it to the approaches in Benner et al. (2015).

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