ARTICLE

A problem with viscoelastic mixtures: numerical analysis and computational experiments

J. R. Fernández* and M. Masid

Departamento de Matemática Aplicada I, Universidade de Vigo, ETSI Telecomunicación, Campus As Lagoas Marcosende s/n, 36310 Vigo, Spain

A. Magaña and R. Quintanilla

Departamento de Matemáticas, E.S.E.I.A.T.-U.P.C., Colom 11, 08222 Terrassa, Barcelona, Spain

(*April 2018*)

In this paper, we study, from the numerical point of view, a dynamic problem involving a mixture of two viscoelastic solids. The mechanical problem is written as a system of two coupled partial differential equations. Its variational formulation is derived and an existence and uniqueness result, and an energy decay property, are recalled. Then, fully discrete approximations are introduced by using the classical finite element method and the implicit Euler scheme. A discrete stability property and a priori error estimates are shown, from which we deduce the linear convergence of the algorithm. Finally, some numerical simulations, including examples in one and two dimensions, are presented to show the accuracy of the approximation and the behaviour of the solution.

Keywords: Mixtures; viscoelasticity; finite element approximations; error estimates; numerical simulations.

AMS Subject Classifications: 74E30; 74D05; 65M60; 65M15; 65M12.

1. Introduction

The interacting continua theory came from the decade of 1960's. The current formulation of this theory comes from the classical works of Truesdell and Toupin [1], Kelly [2], Eringen and Ingram [3], Green and Naghdi [4, 5], Müller [6] and Bowen and Wiese [7]. Some other presentations of this theory can be found in several review articles (see [8–12], among others). The first proposals considered an Eulerian description where the independent variables were the gradient of the displacement of each constituent of the mixture and the relative velocity. Bedford and Stern [13, 14] provided a Lagrangian description that has been widely accepted. In this case, the independent variables are the gradient of the displacement of each constituent and the relative displacement. In this paper, we consider this approach.

In fact, as the physical models to describe the behavior of the materials become more accurate, the corresponding mathematical problems become more complex. The systems of partial differential equations that arise for different kind of mixtures have been the aim of study of many authors. For example, Ieşan and Quintanilla [15] proved the existence and uniqueness of solutions in the isothermal case and, later,

*Corresponding author. Email: jose.fernandez@uvigo.es

Martínez and Quintanilla [16] extended some of these results for the non-isothermal problem.

During the last fifty years the literature concerning binary mixtures has increased drastically because their applications can be found in different fields. For instance, they can be used to model the well-known composites (see, e.g., [17]), used in the automotive or nautical industries, or even some type of cells in biology (see [18]).

There exists a large number of papers dealing with qualitative properties of the solutions of the mathematical problem associated with the mixtures (see, for instance, [19–28] and the numerous references cited therein). The asymptotic behavior of the solutions for the one-dimensional case when the only dissipation mechanism is the temperature was developed in [19, 26, 27]. The exponential stability of solutions has been proved [28] when the dissipation mechanisms are the gradient of the velocity of a constituent and the relative velocity. Moreover, the numerical analysis of this mixtures problem between a thermoviscoelastic constituent and an elastic one, assuming anti-plane shear deformations, was provided in [29].

Different dissipation mechanisms have been proposed for interacting continua. We should cite the work of Iesan and Quintanilla [25] where a very general kind of dissipation mechanisms was proposed and the work of Fernández *et al.* [30] where a qualitative study was developed.

In this paper, we continue the research developed in [30], where the mixture of two viscoelastic solids problem was described and the necessary and sufficient conditions to guarantee the positive definiteness of the internal energy and the dissipation were derived. The existence of a unique solution, analyticity of the associated semigroup and an energy decay property were also shown. We are interested in the problem determined by an isothermal viscoelastic mixture when the dissipation mechanisms are the gradient of the velocity of each constituent and the relative velocity. Now, our aim is to give a numerical counterpart of this problem, providing fully discrete approximations, proving a discrete energy decay property and a priori error estimates and performing some numerical simulations.

The outline of this paper is as follows. In Section 2, we describe the mathematical problem (following [30]) and we derive its variational formulation. Existence and uniqueness and an energy decay property are recalled. Then, fully discrete approximations are introduced in Section 3 by using the finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. A discrete stability property and an error estimate result are proved, from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 4 some one- and two-dimensional numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solutions.

2. The mechanical model and its variational formulation

In this section, we present a brief description of the model (details can be found in [30]). As we did there, we deal with a mixture of two interacting continua restricting ourselves, for the sake of simplicity, to the case of isotropic homogeneous materials with centre of symmetry.

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be the domain and denote by $[0, T]$, $T > 0$, the time interval of interest. The boundary of the body $\Gamma = \partial\Omega$ is assumed to be Lipschitz, with outward unit normal vector $\boldsymbol{\nu} = (\nu_i)_{i=1}^d$. Moreover, let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x} = (x_j)_{j=1}^d$ and t, and a subscript after a comma under a variable represents its spatial derivative with respect to the prescribed variable, i.e. $f_{i,j} = \dfrac{\partial f_i}{\partial x_j}$. The time derivatives are represented as a point for the first order, and two points for the second order, over each variable. Finally, as usual the repeated index notation is used for the summation.

Let us denote by $\mathbb{S}^d$ the space of second-order tensors on $\mathbb{R}^d$ and let $\boldsymbol{u} \in \mathbb{R}^d$ and $\boldsymbol{w} \in \mathbb{R}^d$ be the displacement field of the first constituent and the displacement field of the second constituent, respectively.

For the sake of simplicity in the writing of the problem and the analysis presented in the next section, we will assume homogeneous boundary conditions for both displacements and so,

$$\boldsymbol{u} = \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0, T).$$

Therefore, the mathematical problem of the interaction between two isotropic homogeneous viscoelastic mixtures with centre of symmetry is written as follows (see [30]).

Problem P. *Find the displacement of the first constituent* $\boldsymbol{u} = (u_i)_{i=1}^d : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ *and the displacement of the second constituent* $\boldsymbol{w} = (w_i)_{i=1}^d : \overline{\Omega} \times [0, T] \to \mathbb{R}^d$ *such that*

$$\begin{aligned}
&(\lambda + 2\nu + \mu + 2\zeta + \alpha + 2\gamma)u_{j,ji} + (\mu + 2\kappa + 2\zeta)u_{i,jj} + (\alpha + \nu + 2\kappa + \zeta)w_{j,ji} \\
&\quad +(2\gamma + \zeta)w_{i,jj} + (\lambda^* + \nu^* + \nu_1^* + \mu^* + \zeta^* + \zeta_1^* + \alpha^* + 2\gamma^*)\dot{u}_{j,ji} \\
&\quad +(\mu^* + 2\kappa^* + \zeta^* + \zeta_1^*)\dot{u}_{i,jj} + (\alpha^* + \nu^* + 2\kappa^* + \zeta^*)\dot{w}_{j,ji} + (2\gamma^* + \zeta^*)\dot{w}_{i,jj} \\
&\quad -\xi(u_i - w_i) - \xi^*(\dot{u}_i - \dot{w}_i) = \rho_1 \ddot{u}_i \quad \textit{in} \quad \Omega \times (0, T), \qquad (1) \\
&(\nu + \zeta + \alpha + 2\kappa)u_{j,ji} + (\zeta + 2\gamma)u_{i,jj} + (\alpha + 2\gamma)w_{j,ji} + 2\kappa w_{i,jj} \\
&\quad +(\nu_1^* + \zeta_1^* + \alpha^* + 2\kappa^*)\dot{u}_{j,ji} + (\zeta_1^* + 2\gamma^*)\dot{u}_{i,jj} + (\alpha^* + 2\gamma^*)\dot{w}_{j,ji} \\
&\quad +2\kappa^* \dot{w}_{i,jj} + \xi(u_i - w_i) + \xi^*(\dot{u}_i - \dot{w}_i) = \rho_2 \ddot{w}_i \quad \textit{in} \quad \Omega \times (0, T), \qquad (2) \\
&\boldsymbol{u} = \boldsymbol{w} = \boldsymbol{0} \quad \textit{on} \quad \Gamma \times (0, T), \qquad (3) \\
&\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \dot{\boldsymbol{u}}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Omega, \qquad (4) \\
&\boldsymbol{w}(\boldsymbol{x}, 0) = \boldsymbol{w}_0(\boldsymbol{x}), \quad \dot{\boldsymbol{w}}(\boldsymbol{x}, 0) = \boldsymbol{e}_0(\boldsymbol{x}) \quad \textit{for a.e.} \quad \boldsymbol{x} \in \Omega. \qquad (5)
\end{aligned}$$

Here, ρ_1 and ρ_2 denote the densities of the viscoelastic materials, assumed to be constant for the sake of simplicity, λ, ν, μ, ζ, α, γ, κ, ξ, λ^*, ν_1^*, μ^*, ζ_1^*, α^*, ν^*, γ^*, ζ^*, κ^* and ξ^* are the constitutive coefficients over which we will state the necessary conditions to guarantee the positivity of the internal energy and also of the dissipation, and $\boldsymbol{u}_0$, $\boldsymbol{v}_0$, $\boldsymbol{w}_0$ and $\boldsymbol{e}_0$ are given initial conditions.

In order to simplify the writing of the problem let us redefine the coefficients in the following form: $a_1 = \lambda + 2\nu + \mu + 2\zeta + \alpha + 2\gamma$, $a_2 = \mu + 2\kappa + 2\zeta$, $a_3 = \alpha + \nu + 2\kappa + \zeta$, $a_4 = 2\gamma + \zeta$, $a_5 = \lambda^* + \nu^* + \nu_1^* + \mu^* + \zeta^* + \zeta_1^* + \alpha^* + 2\gamma^*$, $a_6 = \mu^* + 2\kappa^* + \zeta^* + \zeta_1^*$, $a_7 = \alpha^* + \nu^* + 2\kappa^* + \zeta^*$, $a_8 = 2\gamma^* + \zeta^*$, $b_1 = \nu + \zeta + \alpha + 2\kappa$, $b_2 = \zeta + 2\gamma$, $b_3 = \alpha + 2\gamma$, $b_4 = 2\kappa$, $b_5 = \nu_1^* + \zeta_1^* + \alpha^* + 2\kappa^*$, $b_6 = \zeta_1^* + 2\gamma^*$, $b_7 = \alpha^* + 2\gamma^*$ and $b_8 = 2\kappa^*$.

Now, to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$, $H = [L^2(\Omega)]^d$ and $Q = [L^2(\Omega)]^{d \times d}$ and denote by $(\cdot, \cdot)_Y$, $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_Q$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$, $\|\cdot\|_H$ and $\|\cdot\|_Q$. Moreover, let us define the variational space V as follows,

$$V = \{\boldsymbol{z} \in [H^1(\Omega)]^d ; \ \boldsymbol{z} = \boldsymbol{0} \quad \text{on} \quad \Gamma\},$$

with respective scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$.

By using Green's formula and boundary conditions (3), we write the variational formulation of Problem P in terms of the velocity of the first constituent $\boldsymbol{v} = \dot{\boldsymbol{u}}$ and the velocity of the second constituent $\boldsymbol{e} = \dot{\boldsymbol{w}}$.

Problem VP. *Find the velocity of the first constituent $\boldsymbol{v} = (v_i)_{i=1}^d : [0,T] \to V$ and the velocity of the second constituent $\boldsymbol{e} = (e_i)_{i=1}^d : [0,T] \to V$ such that $\boldsymbol{v}(0) = \boldsymbol{v}_0$, $\boldsymbol{e}(0) = \boldsymbol{e}_0$ and, for a.e. $t \in (0,T)$,*

$$
\begin{aligned}
\rho_1(\dot{\boldsymbol{v}}(t), \boldsymbol{z})_H &+ a_1(\operatorname{div} \boldsymbol{u}(t), \operatorname{div} \boldsymbol{z})_Y + a_2(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{z})_Q + a_3(\operatorname{div} \boldsymbol{w}(t), \operatorname{div} \boldsymbol{z})_Y \\
&+a_4(\nabla \boldsymbol{w}(t), \nabla \boldsymbol{z})_Q + a_5(\operatorname{div} \boldsymbol{v}(t), \operatorname{div} \boldsymbol{z})_Y + a_6(\nabla \boldsymbol{v}(t), \nabla \boldsymbol{z})_Q \\
&+a_7(\operatorname{div} \boldsymbol{e}(t), \operatorname{div} \boldsymbol{z})_Y + a_8(\nabla \boldsymbol{e}(t), \nabla \boldsymbol{z})_Q + \xi(\boldsymbol{u}(t) - \boldsymbol{w}(t), \boldsymbol{z})_H \\
&+\xi^*(\boldsymbol{v}(t) - \boldsymbol{e}(t), \boldsymbol{z})_H = 0 \quad \forall \boldsymbol{z} \in V, \qquad (6)\\
\rho_2(\dot{\boldsymbol{e}}(t), \boldsymbol{r})_H &+ b_1(\operatorname{div} \boldsymbol{u}(t), \operatorname{div} \boldsymbol{r})_Y + b_2(\nabla \boldsymbol{u}(t), \nabla \boldsymbol{r})_Q + b_3(\operatorname{div} \boldsymbol{w}(t), \operatorname{div} \boldsymbol{r})_Y \\
&+b_4(\nabla \boldsymbol{w}(t), \nabla \boldsymbol{r})_Q + b_5(\operatorname{div} \boldsymbol{v}(t), \operatorname{div} \boldsymbol{r})_Y + b_6(\nabla \boldsymbol{v}(t), \nabla \boldsymbol{r})_Q \\
&+b_7(\operatorname{div} \boldsymbol{e}(t), \operatorname{div} \boldsymbol{r})_Y + b_8(\nabla \boldsymbol{e}(t), \nabla \boldsymbol{r})_Q - \xi(\boldsymbol{u}(t) - \boldsymbol{w}(t), \boldsymbol{r})_H \\
&-\xi^*(\boldsymbol{v}(t) - \boldsymbol{e}(t), \boldsymbol{r})_H = 0 \quad \forall \boldsymbol{r} \in V, \qquad (7)
\end{aligned}
$$

where the displacement fields $\boldsymbol{u}$ and $\boldsymbol{w}$ are then recovered from the relations:

$$
\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\, ds + \boldsymbol{u}_0, \quad \boldsymbol{w}(t) = \int_0^t \boldsymbol{e}(s)\, ds + \boldsymbol{w}_0. \qquad (8)
$$

In [30] the following theorem was proved.

THEOREM 2.1 *Let ρ_1, $\rho_2 > 0$ and assume that*

$$
\begin{aligned}
&\kappa > \gamma, \\
&\mu > 0, \\
&(\gamma + \kappa)\mu > \zeta^2, \\
&\lambda + \mu > 0, \\
&(\alpha + \gamma + \kappa)(\lambda + \mu) > (\zeta + \nu)^2, \\
&\xi > 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\kappa^* > \gamma^*, \\
&\mu^* > 0, \\
&4\mu^*(\gamma^* + \kappa^*) > (\zeta^* + \zeta_1^*)^2, \\
&\lambda^* + \mu^* > 0, \\
&4(\lambda^* + \mu^*)(\alpha^* + \gamma^* + \kappa^*) > (\zeta^* + \zeta_1^* + \nu^* + \nu_1^*)^2, \\
&\xi^* > 0
\end{aligned}
$$

for the two-dimensional setting (i.e. $d = 2$), and

$$\begin{aligned}
&\kappa > \gamma,\\
&\mu > 0,\\
&(\gamma + \kappa)\mu > \zeta^2,\\
&3\lambda + 2\mu > 0,\\
&(3\lambda + 2\mu)(3\alpha + 2\gamma + 2\kappa) > (2\zeta + 3\nu)^2,\\
&\xi > 0,
\end{aligned}$$

and

$$\begin{aligned}
&\kappa^* > \gamma^*,\\
&\mu^* > 0,\\
&4\mu^*(\gamma^* + \kappa^*) > (\zeta^* + \zeta_1^*)^2,\\
&3\lambda^* + 2\mu^* > 0,\\
&4(3\lambda^* + 2\mu^*)(3\alpha^* + 2\gamma^* + 2\kappa^*) > (2\zeta^* + 2\zeta_1^* + 3\nu^* + 3\nu_1^*)^2,\\
&\xi^* > 0
\end{aligned}$$

for the three-dimensional case ($d = 3$). Therefore, there exists a unique solution to Problem VP with the following regularity:

$$\begin{aligned}
&\boldsymbol{u} \in C^1([0,T];V) \cap C([0,T];[H^2(\Omega)]^d) \cap H^2(0,T;H),\\
&\boldsymbol{w} \in C^1([0,T];V) \cap C([0,T];[H^2(\Omega)]^d) \cap H^2(0,T;H).
\end{aligned}$$

Moreover, this solution is exponentially stable and analytic.

Remark 1 As it was commented in [30], a consequence of the analyticity of the solutions is the impossibility of localization, which means that the unique solution that can be identically zero after a finite period of time is the null solution.

3. Fully discrete approximations: a priori error estimates

In this section, we now consider a fully discrete approximation of Problem VP. This is done in two steps. First, we assume that the domain $\overline{\Omega}$ is polyhedral and we denote by $\mathcal{T}^h$ a triangulation. Thus, we construct the finite dimensional space $V^h \subset V$ given by

$$V^h = \{\boldsymbol{z}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{z}^h_{|_T} \in [P_1(T)]^d, \quad \boldsymbol{z}^h = \boldsymbol{0} \quad \text{on} \quad \Gamma\}, \tag{9}$$

where $P_1(T)$ represents the space of polynomials of degree less or equal to one in the element T; i.e. the finite element space V^h is composed of continuous and piecewise affine functions. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$, $\boldsymbol{w}_0^h$ and $\boldsymbol{e}_0^h$, are given by

$$\boldsymbol{u}_0^h = \mathcal{P}^h \boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \mathcal{P}^h \boldsymbol{v}_0, \quad \boldsymbol{w}_0^h = \mathcal{P}^h \boldsymbol{w}_0, \quad \boldsymbol{e}_0^h = \mathcal{P}^h \boldsymbol{e}_0. \tag{10}$$

where $\mathcal{P}^h$ is the classical finite element interpolation operator over V^h (see, e.g., [31]).

Secondly, we consider a partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

Problem VP^{hk}. *Find the discrete velocity of the first constituent* $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ *and the discrete velocity of the second constituent* $\boldsymbol{e}^{hk} = \{\boldsymbol{e}_n^{hk}\}_{n=0}^N \subset V^h$ *such that* $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$, $\boldsymbol{e}_0^{hk} = \boldsymbol{e}_0^h$, *and, for* $n = 1, \ldots, N$,

$$
\begin{aligned}
&\rho_1(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{z}^h)_H + a_1(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{z}^h)_Y + a_2(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{z}^h)_Q + a_3(\operatorname{div} \boldsymbol{w}_n^{hk}, \operatorname{div} \boldsymbol{z}^h)_Y \\
&\quad + a_4(\nabla \boldsymbol{w}_n^{hk}, \nabla \boldsymbol{z}^h)_Q + a_5(\operatorname{div} \boldsymbol{v}_n^{hk}, \operatorname{div} \boldsymbol{z}^h)_Y + a_6(\nabla \boldsymbol{v}_n^{hk}, \nabla \boldsymbol{z}^h)_Q \\
&\quad + a_7(\operatorname{div} \boldsymbol{e}_n^{hk}, \operatorname{div} \boldsymbol{z}^h)_Y + a_8(\nabla \boldsymbol{e}_n^{hk}, \nabla \boldsymbol{z}^h)_Q + \xi(\boldsymbol{u}_n^{hk} - \boldsymbol{w}_n^{hk}, \boldsymbol{z}^h)_H \\
&\quad + \xi^*(\boldsymbol{v}_n^{hk} - \boldsymbol{e}_n^{hk}, \boldsymbol{z}^h)_H = 0 \quad \forall \boldsymbol{z}^h \in V^h,
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
&\rho_2(\delta \boldsymbol{e}_n^{hk}, \boldsymbol{r}^h)_H + b_1(\operatorname{div} \boldsymbol{u}_n^{hk}, \operatorname{div} \boldsymbol{r}^h)_Y + b_2(\nabla \boldsymbol{u}_n^{hk}, \nabla \boldsymbol{r}^h)_Q + b_3(\operatorname{div} \boldsymbol{w}_n^{hk}, \operatorname{div} \boldsymbol{r}^h)_Y \\
&\quad + b_4(\nabla \boldsymbol{w}_n^{hk}, \nabla \boldsymbol{r}^h)_Q + b_5(\operatorname{div} \boldsymbol{v}_n^{hk}, \operatorname{div} \boldsymbol{r}^h)_Y + b_6(\nabla \boldsymbol{v}_n^{hk}, \nabla \boldsymbol{r}^h)_Q \\
&\quad + b_7(\operatorname{div} \boldsymbol{e}_n^{hk}, \operatorname{div} \boldsymbol{r}^h)_Y + b_8(\nabla \boldsymbol{e}_n^{hk}, \nabla \boldsymbol{r}^h)_Q - \xi(\boldsymbol{u}_n^{hk} - \boldsymbol{w}_n^{hk}, \boldsymbol{r}^h)_H \\
&\quad - \xi^*(\boldsymbol{v}_n^{hk} - \boldsymbol{e}_n^{hk}, \boldsymbol{r}^h)_H = 0 \quad \forall \boldsymbol{r}^h \in V^h,
\end{aligned} \tag{12}
$$

where the discrete displacements $\boldsymbol{u}_n^{hk}$ and $\boldsymbol{w}_n^{hk}$ are now recovered from the respective formulas:

$$
\boldsymbol{u}_n^{hk} = k \sum_{j=1}^n \boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad \boldsymbol{e}_n^{hk} = k \sum_{j=1}^n \boldsymbol{w}_j^{hk} + \boldsymbol{w}_0^h.
$$

In order to provide the numerical analysis of Problem VP^{hk}, we assume the following conditions on the constitutive coefficients:

$$
a_1,\, b_3,\, a_5,\, b_7 > 0, \tag{13}
$$

$$
a_1\, b_3 - a_3^2 > 0, \tag{14}
$$

$$
a_5\, b_7 - \left(\frac{a_7 + b_5}{2}\right)^2 > 0, \tag{15}
$$

$$
a_2,\, b_4,\, a_6,\, b_8 > 0, \tag{16}
$$

$$
a_2\, b_4 - a_4^2 > 0, \tag{17}
$$

$$
a_6\, b_8 - \left(\frac{a_8 + b_6}{2}\right)^2 > 0. \tag{18}
$$

Remark 2 We note that, from their definition, coefficients b_1 and b_2 equal a_3 and a_4, respectively. Moreover, after tedious algebraic manipulations, from the conditions of Theorem 2.1, for each dimension considered, we can derive conditions (16)-(18). However, even if we are able to simplify conditions (13)-(15), using the conditions of Theorem 2.1, we could not conclude them from there.

Remark 3 In the particular case where we consider anti-plane shear deformations; that is, if $\boldsymbol{u}(x_1, x_2, x_3, t) = (0, 0, \tilde{u}(x_1, x_2))$ and $\boldsymbol{w}(x_1, x_2, x_3, t) = (0, 0, \tilde{w}(x_1, x_2))$, where $\tilde{u}$ and $\tilde{w}$ are now scalar functions, then the terms involving divergence-like functions disappear and the necessary conditions imposed in (13)-(18) are obtained straightforwardly from Theorem 2.1.

First, we will prove a stability result.

THEOREM 3.1 *Under the assumptions of Theorem 2.1 and conditions (13)-(18), it follows that the sequences* $\{\boldsymbol{u}^{hk}, \boldsymbol{v}^{hk}, \boldsymbol{w}^{hk}, \boldsymbol{e}^{hk}\}$*, generated by Problem* VP^{hk}*, satisfy the stability estimate:*

$$\begin{aligned}&\|\boldsymbol{v}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{u}_n^{hk}\|_Q^2 + \|\mathrm{div}\, \boldsymbol{u}_n^{hk}\|_Y^2 + \|\boldsymbol{e}_n^{hk}\|_H^2 + \|\nabla \boldsymbol{w}_n^{hk}\|_Q^2 + \|\mathrm{div}\, \boldsymbol{w}_n^{hk}\|_Y^2 \\ &\quad + \xi \|\boldsymbol{u}_n^{hk} - \boldsymbol{w}_n^{hk}\|_H^2 \le C,\end{aligned}$$

where C *is a positive constant assumed to be independent of the discretization parameters* h *and* k*.*

Proof. In order to simplify the writing, we remove the superscripts h and k in all the variables.

Taking $\boldsymbol{z}^h = \boldsymbol{v}_n$ and $\boldsymbol{r}^h = \boldsymbol{e}_n$ in equations (11) and (12), respectively, we find that

$$\begin{aligned}&\rho_1(\delta \boldsymbol{v}_n, \boldsymbol{v}_n)_H + a_1(\mathrm{div}\, \boldsymbol{u}_n, \mathrm{div}\, \boldsymbol{v}_n)_Y + a_2(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{v}_n)_Q + a_3(\mathrm{div}\, \boldsymbol{w}_n, \mathrm{div}\, \boldsymbol{v}_n)_Y \\ &\quad + a_4(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{v}_n)_Q + a_5(\mathrm{div}\, \boldsymbol{v}_n, \mathrm{div}\, \boldsymbol{v}_n)_Y + a_6(\nabla \boldsymbol{v}_n, \nabla \boldsymbol{v}_n)_Q \\ &\quad + a_7(\mathrm{div}\, \boldsymbol{e}_n, \mathrm{div}\, \boldsymbol{v}_n)_Y + a_8(\nabla \boldsymbol{e}_n, \nabla \boldsymbol{v}_n)_Q + \xi(\boldsymbol{u}_n - \boldsymbol{w}_n, \boldsymbol{v}_n)_H \\ &\quad + \xi^*(\boldsymbol{v}_n - \boldsymbol{e}_n, \boldsymbol{v}_n)_H = 0, \qquad (19) \\ &\rho_2(\delta \boldsymbol{e}_n, \boldsymbol{e}_n)_H + b_1(\mathrm{div}\, \boldsymbol{u}_n, \mathrm{div}\, \boldsymbol{e}_n)_Y + b_2(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{e}_n)_Q + b_3(\mathrm{div}\, \boldsymbol{w}_n, \mathrm{div}\, \boldsymbol{e}_n)_Y \\ &\quad + b_4(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{e}_n)_Q + b_5(\mathrm{div}\, \boldsymbol{v}_n, \mathrm{div}\, \boldsymbol{e}_n)_Y + b_6(\nabla \boldsymbol{v}_n, \nabla \boldsymbol{e}_n)_Q \\ &\quad + b_7(\mathrm{div}\, \boldsymbol{e}_n, \mathrm{div}\, \boldsymbol{e}_n)_Y + b_8(\nabla \boldsymbol{e}_n, \nabla \boldsymbol{e}_n)_Q - \xi(\boldsymbol{u}_n - \boldsymbol{w}_n, \boldsymbol{e}_n)_H \\ &\quad - \xi^*(\boldsymbol{v}_n - \boldsymbol{e}_n, \boldsymbol{e}_n)_H = 0. \qquad (20)\end{aligned}$$

It is straightforward to obtain that

$$\begin{aligned}&a_1(\mathrm{div}\, \boldsymbol{u}_n, \mathrm{div}\, \boldsymbol{v}_n)_Y \ge \frac{a_1}{2k}\Big[\|\mathrm{div}\, \boldsymbol{u}_n\|_Y^2 - \|\mathrm{div}\, \boldsymbol{u}_{n-1}\|_Y^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_{n-1})\|_Y^2\Big], \\ &a_2(\nabla \boldsymbol{u}_n, \nabla \boldsymbol{v}_n)_Q \ge \frac{a_2}{2k}\Big[\|\nabla \boldsymbol{u}_n\|_Q^2 - \|\nabla \boldsymbol{u}_{n-1}\|_Q^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_{n-1})\|_Q^2\Big], \\ &b_3(\mathrm{div}\, \boldsymbol{w}_n, \mathrm{div}\, \boldsymbol{e}_n)_Y \ge \frac{b_3}{2k}\Big[\|\mathrm{div}\, \boldsymbol{w}_n\|_Y^2 - \|\mathrm{div}\, \boldsymbol{w}_{n-1}\|_Y^2 + \|\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_{n-1})\|_Y^2\Big], \\ &b_4(\nabla \boldsymbol{w}_n, \nabla \boldsymbol{e}_n)_Q \ge \frac{b_4}{2k}\Big[\|\nabla \boldsymbol{w}_n\|_Q^2 - \|\nabla \boldsymbol{w}_{n-1}\|_Q^2 + \|\nabla(\boldsymbol{w}_n - \boldsymbol{w}_{n-1})\|_Q^2\Big], \\ &\xi(\boldsymbol{u}_n - \boldsymbol{w}_n, \boldsymbol{v}_n - \boldsymbol{e}_n) \ge \frac{\xi}{2k}\Big[\|\boldsymbol{u}_n - \boldsymbol{w}_n\|_H^2 - \|\boldsymbol{u}_{n-1} - \boldsymbol{w}_{n-1}\|_H^2\Big], \\ &\rho_1(\delta \boldsymbol{v}_n, \boldsymbol{v}_n)_H \ge \frac{\rho_1}{2k}\Big[\|\boldsymbol{v}_n\|_H^2 - \|\boldsymbol{v}_{n-1}\|_H^2\Big], \\ &\rho_2(\delta \boldsymbol{e}_n, \boldsymbol{e}_n)_H \ge \frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n\|_H^2 - \|\boldsymbol{e}_{n-1}\|_H^2\Big].\end{aligned}$$

Using the fact that $a_3 = b_1$ and $a_4 = b_2$, from Theorem 2.1 and conditions (13), (14), (16) and (17) we easily find that

$$\begin{aligned}
&a_2\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_{n-1})\|_Q^2+b_4\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_{n-1})\|_Q^2\\
&\qquad\qquad+2a_4(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_{n-1}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_{n-1}))_Q\geq 0,\\
&a_1\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_{n-1})\|_Y^2+b_3\|\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_{n-1})\|_Y^2\\
&\qquad\qquad+2a_3(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_{n-1}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_{n-1}))_Y\geq 0.
\end{aligned}$$

Keeping in mind that

$$\begin{aligned}
&a_3(\mathrm{div}\,\boldsymbol{w}_n,\mathrm{div}\,\boldsymbol{v}_n)_Y+b_1(\mathrm{div}\,\boldsymbol{u}_n,\mathrm{div}\,\boldsymbol{e}_n)_Y\\
&\qquad=\frac{a_3}{k}\Big[(\mathrm{div}\,\boldsymbol{u}_n,\mathrm{div}\,\boldsymbol{w}_n)_Y-(\mathrm{div}\,\boldsymbol{u}_{n-1},\mathrm{div}\,\boldsymbol{w}_{n-1})_Y\\
&\qquad\qquad+(\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_{n-1}),\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_{n-1}))_Y\Big],\\
&a_4(\nabla\boldsymbol{w}_n,\nabla\boldsymbol{v}_n)_Q+b_2(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{e}_n)_Q\\
&\qquad=\frac{a_4}{k}\Big[(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{w}_n)_Q-(\nabla\boldsymbol{u}_{n-1},\nabla\boldsymbol{w}_{n-1})_Q\\
&\qquad\qquad+(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_{n-1}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_{n-1}))_Q\Big]
\end{aligned}$$

and combining (19) and (20), we have

$$\begin{aligned}
&\frac{\rho_1}{2k}\Big[\|\boldsymbol{v}_n\|_H^2-\|\boldsymbol{v}_{n-1}\|_H^2\Big]+\frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n\|_H^2-\|\boldsymbol{e}_{n-1}\|_H^2\Big]\\
&\quad+\frac{a_1}{2k}\Big[\|\mathrm{div}\,\boldsymbol{u}_n\|_Y^2-\|\mathrm{div}\,\boldsymbol{u}_{n-1}\|_Y^2\Big]+\frac{a_2}{2k}\Big[\|\nabla\boldsymbol{u}_n\|_Q^2-\|\nabla\boldsymbol{u}_{n-1}\|_Q^2\Big]\\
&\quad+\frac{\xi}{2k}\Big[\|\boldsymbol{u}_n-\boldsymbol{w}_n\|_H^2-\|\boldsymbol{u}_{n-1}-\boldsymbol{w}_{n-1}\|_H^2\Big]\\
&\quad+\frac{b_3}{2k}\Big[\|\mathrm{div}\,\boldsymbol{w}_n\|_Y^2-\|\mathrm{div}\,\boldsymbol{w}_{n-1}\|_Y^2\Big]+\frac{b_4}{2k}\Big[\|\nabla\boldsymbol{w}_n\|_Q^2-\|\nabla\boldsymbol{w}_{n-1}\|_Q^2\Big]\\
&\quad+\frac{a_3}{k}\Big[(\mathrm{div}\,\boldsymbol{u}_n,\mathrm{div}\,\boldsymbol{w}_n)_Y-(\mathrm{div}\,\boldsymbol{u}_{n-1},\mathrm{div}\,\boldsymbol{w}_{n-1})_Y\Big]\\
&\quad+\frac{a_4}{k}\Big[(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{w}_n)_Q-(\nabla\boldsymbol{u}_{n-1},\nabla\boldsymbol{w}_{n-1})_Q\Big]\\
&\quad+a_5\|\mathrm{div}\,\boldsymbol{v}_n\|_Y^2+a_6\|\nabla\boldsymbol{v}_n\|_Q^2+(a_7+b_5)(\mathrm{div}\,\boldsymbol{e}_n,\mathrm{div}\,\boldsymbol{v}_n)_Y\\
&\quad+b_7\|\mathrm{div}\,\boldsymbol{e}_n\|_Y^2+b_8\|\nabla\boldsymbol{e}_n\|_Q^2+(a_8+b_6)(\nabla\boldsymbol{e}_n,\nabla\boldsymbol{v}_n)_Q\\
&\leq C\Big(\|\boldsymbol{u}_n\|_H^2+\|\boldsymbol{v}_n\|_H^2+\|\boldsymbol{w}_n\|_H^2+\|\boldsymbol{e}_n\|_H^2\Big).
\end{aligned}$$

Using conditions (13), (15), (16) and (18) it follows that there exist ζ_1, $\zeta_2>0$ such that

$$\frac{\frac{a_7+b_5}{2}}{a_5}<\zeta_1<\frac{b_7}{\frac{a_7+b_5}{2}},\qquad\frac{\frac{a_8+b_6}{2}}{a_6}<\zeta_2<\frac{b_8}{\frac{a_8+b_6}{2}},$$

which lead to the following estimates

$$\begin{aligned}
&a_5\|\mathrm{div}\,\boldsymbol{v}_n\|_Y^2+b_7\|\mathrm{div}\,\boldsymbol{e}_n\|_Y^2+(a_7+b_5)(\mathrm{div}\,\boldsymbol{e}_n,\mathrm{div}\,\boldsymbol{v}_n)_Y\\
&\qquad\geq\left(a_5-\frac{\frac{a_7+b_5}{2}}{\zeta_1}\right)\|\mathrm{div}\,\boldsymbol{v}_n\|_Y^2+(b_7-\zeta_1\tfrac{a_7+b_5}{2})\|\mathrm{div}\,\boldsymbol{e}_n\|_Y^2,\\
&a_6\|\nabla\boldsymbol{v}_n\|_Q^2+b_8\|\nabla\boldsymbol{e}_n\|_Q^2+\frac{b_6+a_8}{2}(\nabla\boldsymbol{v}_n,\nabla\boldsymbol{e}_n)_Q\\
&\qquad\geq\left(a_6-\frac{\frac{b_6+a_8}{2}}{\zeta_2}\right)\|\nabla\boldsymbol{v}_n\|_Q^2+(b_8-\zeta_2\tfrac{b_6+a_8}{2})\|\nabla\boldsymbol{e}_n\|_Q^2.
\end{aligned}$$

Therefore, by induction we obtain

$$\begin{aligned}
&\rho_1\|\boldsymbol{v}_n\|_H^2+\rho_2\|\boldsymbol{e}_n\|_H^2+a_1\|\mathrm{div}\,\boldsymbol{u}_n\|_Y^2+a_2\|\nabla\boldsymbol{u}_n\|_Q^2+b_3\|\mathrm{div}\,\boldsymbol{w}_n\|_Y^2+b_4\|\nabla\boldsymbol{w}_n\|_Q^2\\
&\qquad+\xi\|\boldsymbol{u}_n-\boldsymbol{w}_n\|_H^2+2a_3(\mathrm{div}\,\boldsymbol{u}_n,\mathrm{div}\,\boldsymbol{w}_n)_Y+2a_4(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{w}_n)_Q\\
&\quad\le Ck\sum_{j=1}^n\Big(\|\boldsymbol{u}_j\|_H^2+\|\boldsymbol{v}_j\|_H^2+\|\boldsymbol{w}_j\|_H^2+\|\boldsymbol{e}_j\|_H^2\Big)\\
&\qquad+C\Big(\|\boldsymbol{u}_0\|_V^2+\|\boldsymbol{v}_0\|_H^2+\|\boldsymbol{w}_0\|_V^2+\|\boldsymbol{e}_0\|_H^2\Big).
\end{aligned}$$

Now, using again conditions (13), (14), (16) and (17) it follows that there exist ζ_3, $\zeta_4>0$ such that

$$\frac{a_3}{a_1}<\zeta_3<\frac{b_3}{a_3},\quad \frac{a_4}{a_2}<\zeta_4<\frac{b_4}{a_4},$$

which imply that

$$\begin{aligned}
&a_1\|\mathrm{div}\,\boldsymbol{u}_n\|_Y^2+b_3\|\mathrm{div}\,\boldsymbol{w}_n\|_Y^2+2a_3(\mathrm{div}\,\boldsymbol{u}_n,\mathrm{div}\,\boldsymbol{w}_n)_Y\\
&\quad\ge\left(a_1-\tfrac{a_3}{\zeta_3}\right)\|\mathrm{div}\,\boldsymbol{u}_n\|_Y^2+(b_3-\zeta_3a_3)\|\mathrm{div}\,\boldsymbol{w}_n\|_Y^2,\\
&a_2\|\nabla\boldsymbol{u}_n\|_Q^2+b_4\|\nabla\boldsymbol{w}_n\|_Q^2+2a_4(\nabla\boldsymbol{u}_n,\nabla\boldsymbol{w}_n)_Q\\
&\quad\ge\left(a_2-\tfrac{a_4}{\zeta_4}\right)\|\nabla\boldsymbol{u}_n\|_Q^2+(b_4-\zeta_4a_4)\|\nabla\boldsymbol{w}_n\|_Q^2.
\end{aligned}$$

Finally, using a discrete version of Gronwall's inequality (see, for instance, [32]), we conclude the stability property. ■

From Theorem 3.1, we obtain the following discrete version of the energy decay property.

Corollary 3.2 *If we define the discrete energy at time $t=t_n$, E_n^{hk}, as follows*

$$\begin{aligned}
E_n^{hk}&=\rho_1\|\boldsymbol{v}_n^{hk}\|_H^2+\rho_2\|\boldsymbol{e}_n^{hk}\|_H^2+a_1\|\mathrm{div}\,\boldsymbol{u}_n^{hk}\|_Y^2+a_2\|\nabla\boldsymbol{u}_n^{hk}\|_Q^2+b_3\|\mathrm{div}\,\boldsymbol{w}_n^{hk}\|_Y^2\\
&\quad+\xi\|\boldsymbol{u}_n^{hk}-\boldsymbol{w}_n^{hk}\|_H^2+b_4\|\nabla\boldsymbol{w}_n^{hk}\|_Q^2+2a_3(\mathrm{div}\,\boldsymbol{u}_n^{hk},\mathrm{div}\,\boldsymbol{w}_n^{hk})_Y\\
&\quad+2a_4(\nabla\boldsymbol{u}_n^{hk},\nabla\boldsymbol{w}_n^{hk})_Q, \qquad (21)
\end{aligned}$$

then we have

$$\frac{E_n^{hk}-E_{n-1}^{hk}}{k}\le 0.$$

Next, we obtain some a priori error estimates on the numerical errors $\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}$, $\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}$, $\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}$ and $\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}$. Therefore, assume the following additional regularity on the continuous solution:

$$\boldsymbol{u},\boldsymbol{w}\in C^2([0,T];H). \qquad (22)$$

We have the following.

Theorem 3.3 *Under the assumptions of Theorem 3.1 and the additional regularity conditions (22), if we denote by $(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w},\boldsymbol{e})$ and $(\boldsymbol{u}^{hk},\boldsymbol{v}^{hk},\boldsymbol{w}^{hk},\boldsymbol{e}^{hk})$ the solutions* *to problems VP and VP^{hk}, respectively, then we have the following a priori error estimates, for all $\boldsymbol{z}^h = \{\boldsymbol{z}_j^h\}_{j=0}^N$, $\boldsymbol{r}^h = \{\boldsymbol{r}_j^h\}_{j=0}^N \subset V^h$,*

$$\begin{aligned}
&\max_{0\le n\le N} \Big\{ \|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\|_H^2 + \|\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}\|_H^2 + \|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk})\|_Q^2 \\
&\quad + \|\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk})\|_Y^2 + \|\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk})\|_Q^2 + \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk})\|_H^2 \Big\} \\
&\le C\Big(k\sum_{j=1}^N \|\dot{\boldsymbol{v}}_j - \delta\boldsymbol{v}_j\|_H^2 + \|\dot{\boldsymbol{e}}_j - \delta\boldsymbol{e}_j\|_H^2 + \|\dot{\boldsymbol{u}}_j - \delta\boldsymbol{u}_j\|_V^2 + \|\dot{\boldsymbol{w}}_j - \delta\boldsymbol{w}_j\|_V^2 \\
&\quad + \|\boldsymbol{v}_j - \boldsymbol{z}_j^h\|_V^2 + \|\boldsymbol{e}_j - \boldsymbol{r}_j^h\|_V^2 \Big) + C \max_{0\le n\le N} \|\boldsymbol{v}_n - \boldsymbol{z}_n^h\|_H^2 + C \max_{0\le n\le N} \|\boldsymbol{e}_n - \boldsymbol{r}_n^h\|_H^2 \\
&\quad + \frac{C}{k}\sum_{j=1}^{N-1} \Big[\|\boldsymbol{v}_j - \boldsymbol{z}_j^h - (\boldsymbol{v}_{j+1} - \boldsymbol{z}_{j+1}^h)\|_H^2 + \|\boldsymbol{w}_j - \boldsymbol{r}_j^h - (\boldsymbol{w}_{j+1} - \boldsymbol{r}_{j+1}^h)\|_H^2 \Big] \\
&\quad + C\Big(\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V^2 + \|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\|_H^2 + \|\boldsymbol{w}_0 - \boldsymbol{w}_0^h\|_V^2 + \|\boldsymbol{e}_0 - \boldsymbol{e}_0^h\|_H^2 \Big),
\end{aligned} \tag{23}$$

where $\delta\boldsymbol{u}_n = (\boldsymbol{u}_n - \boldsymbol{u}_{n-1})/k$ and $\delta\boldsymbol{w}_n = (\boldsymbol{w}_n - \boldsymbol{w}_{n-1})/k$, and C is a positive constant assumed to be independent of the discretization parameters h and k but depending on the continuous solution.

Proof. First, we subtract variational equation (6) at time $t = t_n$ for a test function $\boldsymbol{z} = \boldsymbol{z}^h \in V^h \subset V$ and discrete variational equation (11) to obtain, for all $\boldsymbol{z}^h \in V^h$,

$$\begin{aligned}
&a_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,\boldsymbol{z}^h)_Y + a_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla\boldsymbol{z}^h)_Q \\
&\quad + a_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,\boldsymbol{z}^h)_Y + a_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla\boldsymbol{z}^h)_Q \\
&\quad + a_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,\boldsymbol{z}^h)_Y + a_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla\boldsymbol{z}^h)_Q \\
&\quad + a_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,\boldsymbol{z}^h)_Y + a_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla\boldsymbol{z}^h)_Q \\
&\quad + \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{z}^h)_H + \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{z}^h)_H \\
&\quad + \rho_1(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{z}^h)_H = 0,
\end{aligned}$$

and so, for all $\boldsymbol{z}^h \in V^h$ we find that

$$\begin{aligned}
&a_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y + a_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\quad + a_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y + a_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\quad + a_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y + a_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\quad + a_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Y + a_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}))_Q \\
&\quad + \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H + \rho_1(\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H \\
&\quad + \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{v}_n^{hk})_H
\end{aligned}$$

$$
\begin{aligned}
&= a_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{z}^h))_Y + a_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{z}^h))_Q \\
&\quad + a_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{z}^h))_Y + a_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{z}^h))_Q \\
&\quad + a_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{z}^h))_Y + a_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{z}^h))_Q \\
&\quad + a_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{z}^h))_Y + a_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla(\boldsymbol{v}_n - \boldsymbol{z}^h))_Q \\
&\quad + \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{z}^h)_H + \rho_1(\dot{\boldsymbol{v}}_n - \delta \boldsymbol{v}_n^{hk}, \boldsymbol{v}_n - \boldsymbol{z}^h)_H \\
&\quad + \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{v}_n - \boldsymbol{z}^h)_H. \qquad (24)
\end{aligned}
$$

Secondly, subtracting variational equation (7), at time $t = t_n$ and for a test function $\boldsymbol{r} = \boldsymbol{r}^h \in V^h \subset V$, and discrete variational equation (12), we get, for all $\boldsymbol{r}^h \in V^h$,

$$
\begin{aligned}
& b_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,\boldsymbol{r}^h)_Y + b_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla \boldsymbol{r}^h)_Q \\
&\quad + b_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,\boldsymbol{r}^h)_Y + b_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla \boldsymbol{r}^h)_Q \\
&\quad + b_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,\boldsymbol{r}^h)_Y + b_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla \boldsymbol{r}^h)_Q \\
&\quad + b_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,\boldsymbol{r}^h)_Y + a_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla \boldsymbol{r}^h)_Q \\
&\quad - \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{r}^h)_H - \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{r}^h)_H \\
&\quad + \rho_2(\dot{\boldsymbol{e}}_n - \delta \boldsymbol{e}_n^{hk}, \boldsymbol{r}^h)_H = 0,
\end{aligned}
$$

and so, we have,

$$
\begin{aligned}
& b_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Y + b_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Q \\
&\quad + b_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Y + b_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Q \\
&\quad + b_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Y + b_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Q \\
&\quad + b_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Y + b_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}))_Q \\
&\quad - \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{e}_n - \boldsymbol{e}_n^{hk})_H + \rho_2(\dot{\boldsymbol{e}}_n - \delta \boldsymbol{e}_n^{hk}, \boldsymbol{e}_n - \boldsymbol{e}_n^{hk})_H \\
&\quad - \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{e}_n - \boldsymbol{e}_n^{hk})_H \\
&= b_1(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{r}^h))_Y + b_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{r}^h))_Q \\
&\quad + b_3(\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{r}^h))_Y + b_4(\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{r}^h))_Q \\
&\quad + b_5(\mathrm{div}\,(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{r}^h))_Y + b_6(\nabla(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{r}^h))_Q \\
&\quad + b_7(\mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \mathrm{div}\,(\boldsymbol{e}_n - \boldsymbol{r}^h))_Y + b_8(\nabla(\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \nabla(\boldsymbol{e}_n - \boldsymbol{r}^h))_Q \\
&\quad - \xi^*(\boldsymbol{v}_n - \boldsymbol{v}_n^{hk} - (\boldsymbol{e}_n - \boldsymbol{e}_n^{hk}), \boldsymbol{e}_n - \boldsymbol{r}^h)_H + \rho_2(\dot{\boldsymbol{e}}_n - \delta \boldsymbol{e}_n^{hk}, \boldsymbol{e}_n - \boldsymbol{r}^h)_H \\
&\quad - \xi(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{w}_n - \boldsymbol{w}_n^{hk}), \boldsymbol{e}_n - \boldsymbol{r}^h)_H \quad \forall \boldsymbol{r}^h \in V^h. \qquad (25)
\end{aligned}
$$

Using the fact that $a_3 = b_1$ and $a_4 = b_2$ from Theorem 2.1 and conditions (13)-(18), we easily find that

$$
\begin{aligned}
& a_2\|\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}))\|_Q^2 + b_4\|\nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}))\|_Q^2 \\
&\quad + 2b_2(\nabla(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})), \nabla(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk})))_Q \geq 0, \\
& a_1\|\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk}))\|_Y^2 + b_3\|\mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk}))\|_Y^2 \\
&\quad + 2a_3(\mathrm{div}\,(\boldsymbol{u}_n - \boldsymbol{u}_n^{hk} - (\boldsymbol{u}_{n-1} - \boldsymbol{u}_{n-1}^{hk})), \mathrm{div}\,(\boldsymbol{w}_n - \boldsymbol{w}_n^{hk} - (\boldsymbol{w}_{n-1} - \boldsymbol{w}_{n-1}^{hk})))_Y \geq 0.
\end{aligned}
$$

Taking into account that

$$
\begin{aligned}
&a_1(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y \geq a_1(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Y\\
&\qquad+\frac{a_1}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\\
&\qquad+\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}))\|_Y^2\Big],\\
&a_2(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q \geq a_2(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n))_Q\\
&\qquad+\frac{a_2}{2k}\Big[\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{v}_{n-1}^{hk})\|_Q^2\\
&\qquad+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}))\|_Q^2\Big],\\
&b_3(\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Y \geq b_3(\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\operatorname{div}(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n))_Y\\
&\qquad+\frac{b_3}{2k}\Big[\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_Y^2\\
&\qquad+\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))\|_Y^2\Big],\\
&b_4(\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Q \geq b_4(\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\nabla(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n))_Q\\
&\qquad+\frac{b_4}{2k}\Big[\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_Q^2\\
&\qquad+\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))\|_Q^2\Big],\\
&\rho_1(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})_H \geq \frac{\rho_1}{2k}\Big[\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big],\\
&\rho_2(\delta\boldsymbol{e}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})_H \geq \frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big],\\
&a_3(\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y+b_1(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Y\\
&\quad=\frac{a_3}{k}\Big[(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk},\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Y-(\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}),\operatorname{div}(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))_Y\\
&\qquad+(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})),\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})))_Y\Big],\\
&a_4(\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q+b_2(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Q\\
&\quad=\frac{a_4}{k}\Big[(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Q-(\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}),\nabla(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))_Q\\
&\qquad+(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})))_Q\Big],\\
&\xi(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}-(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_H\\
&\quad=\xi(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}),\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n-(\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n))_H\\
&\qquad+\frac{\xi}{2k}\Big[\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_H^2-\|\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_H^2\Big],
\end{aligned}
$$

where we recall that $\delta\boldsymbol{u}_n^{nk}=\boldsymbol{v}_n^{hk}$ and $\delta\boldsymbol{w}_n^{nk}=\boldsymbol{e}_n^{hk}$, combining equations (24) and (25) and using several times Cauchy-Schwarz inequality and Young's inequality $ab\leq\epsilon a^2+\frac{1}{4\epsilon}b^2$, a, b, $\epsilon\in\mathbb{R}$, $\epsilon>0$, we find that

$$
\begin{aligned}
&\frac{\rho_1}{2k}\Big[\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2-\|\boldsymbol{v}_{n-1}-\boldsymbol{v}_{n-1}^{hk}\|_H^2\Big]+\frac{\rho_2}{2k}\Big[\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2-\|\boldsymbol{e}_{n-1}-\boldsymbol{e}_{n-1}^{hk}\|_H^2\Big]\\
&\quad+\frac{a_1}{2k}\Big[\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Y^2\Big]\\
&\quad+\frac{a_2}{2k}\Big[\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk})\|_Q^2\Big]\\
&\quad+\frac{b_3}{2k}\Big[\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2-\|\operatorname{div}(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_Y^2\Big]\\
&\quad+\frac{b_4}{2k}\Big[\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2-\|\nabla(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_Q^2\Big]\\
&\quad+\frac{\xi}{2k}\Big[\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_H^2-\|\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}-(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk})\|_H^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{a_3}{k}\Big[(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Y-(\operatorname{div}(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}),\operatorname{div}(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))_Y\Big]\\
&+\frac{a_4}{k}\Big[(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Q-(\nabla(\boldsymbol{u}_{n-1}-\boldsymbol{u}_{n-1}^{hk}),\nabla(\boldsymbol{w}_{n-1}-\boldsymbol{w}_{n-1}^{hk}))_Q\Big]\\
&+a_6\|\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Q^2+(a_7+b_5)(\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&+b_8\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2+(a_8+b_6)(\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Q\\
&+a_5\|\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Y^2+b_7\|\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2\\
&\le C\Big(\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_H^2+\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\|\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}\|_H^2+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2\\
&\quad+\|\boldsymbol{v}_n-\boldsymbol{z}^h\|_V^2+\|\boldsymbol{e}_n-\boldsymbol{r}^h\|_V^2+\|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H^2+\|\dot{\boldsymbol{e}}_n-\delta\boldsymbol{e}_n\|_H^2\\
&\quad+\|\dot{\boldsymbol{u}}_n-\delta\boldsymbol{u}_n\|_V^2+\|\dot{\boldsymbol{w}}_n-\delta\boldsymbol{w}_n\|_V^2+(\delta\boldsymbol{v}_n-\delta\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{z}^h)_H\\
&\quad+(\delta\boldsymbol{e}_n-\delta\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{r}^h)_H+\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2\\
&\quad+\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2+\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2\Big)+\epsilon\|\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Y^2\\
&\quad+\epsilon\|\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Q^2+\epsilon\|\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2+\epsilon\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2,
\end{aligned}
$$

where $\epsilon>0$ is assumed small enough.

Keeping in mind the definitions of the positive parameters ζ_1 and ζ_2 introduced in the proof of Theorem 3.1, we obtain

$$
\begin{aligned}
&a_5\|\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Y^2+b_7\|\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2+(a_7+b_5)(\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}),\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}))_Y\\
&\quad\ge\left(a_5-\frac{\frac{a_7+b_5}{2}}{\zeta_1}\right)\|\operatorname{div}(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Y^2+(b_7-\zeta_1\tfrac{a_7+b_5}{2})\|\operatorname{div}(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Y^2,\\
&a_6\|\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Q^2+b_8\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2+\frac{b_6+a_8}{2}(\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}),\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}))_Q\\
&\quad\ge\left(a_6-\frac{\frac{b_6+a_8}{2}}{\zeta_2}\right)\|\nabla(\boldsymbol{v}_n-\boldsymbol{v}_n^{hk})\|_Q^2+(b_8-\zeta_2\tfrac{b_6+a_8}{2})\|\nabla(\boldsymbol{e}_n-\boldsymbol{e}_n^{hk})\|_Q^2.
\end{aligned}
$$

By induction it follows that

$$
\begin{aligned}
&\rho_1\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H^2+\rho_2\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H^2+a_1\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+a_2\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2\\
&\quad+b_3\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2+2a_3(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Y\\
&\quad+b_4\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2+2a_4(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Q\\
&\quad+\xi\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}-(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_H^2\\
&\le Ck\sum_{j=1}^n\Big(\|\boldsymbol{u}_j-\boldsymbol{u}_j^{hk}\|_H^2+\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\|_H^2+\|\boldsymbol{w}_j-\boldsymbol{w}_j^{hk}\|_H^2+\|\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}\|_H^2\\
&\quad+\|\boldsymbol{v}_j-\boldsymbol{z}_j^h\|_V^2+\|\boldsymbol{e}_j-\boldsymbol{r}_j^h\|_V^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{e}}_j-\delta\boldsymbol{e}_j\|_H^2\\
&\quad+\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_V^2+(\delta\boldsymbol{v}_j-\delta\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{z}_j^h)_H\\
&\quad+(\delta\boldsymbol{e}_j-\delta\boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{r}_j^h)_H+\|\operatorname{div}(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Y^2+\|\nabla(\boldsymbol{u}_j-\boldsymbol{u}_j^{hk})\|_Q^2\\
&\quad+\|\operatorname{div}(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk})\|_Y^2+\|\nabla(\boldsymbol{w}_j-\boldsymbol{w}_j^{hk})\|_Q^2\Big)+C\Big(\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2\\
&\quad+\|\boldsymbol{w}_0-\boldsymbol{w}_0^h\|_V^2+\|\boldsymbol{e}_0-\boldsymbol{e}_0^h\|_H^2\Big)\quad\forall\{\boldsymbol{z}_j^h\}_{j=0}^n,\ \{\boldsymbol{r}_j^h\}_{j=0}^n\subset V^h.
\end{aligned}
$$

Using now the positive parameters ζ_3 and $\zeta_4>0$ defined in the proof of Theorem 3.1, we have

$$
\begin{aligned}
&a_1\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+b_3\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2+2a_3(\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Y\\
&\quad\ge\left(a_1-\frac{a_3}{\zeta_3}\right)\|\operatorname{div}(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y^2+(b_3-\zeta_3a_3)\|\operatorname{div}(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y^2,\\
&a_2\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+b_4\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2+2a_4(\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}),\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk}))_Q\\
&\quad\ge\left(a_2-\frac{a_4}{\zeta_4}\right)\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q^2+(b_4-\zeta_4a_4)\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q^2.
\end{aligned}
$$

Finally, keeping in mind that

$$
\begin{aligned}
k\sum_{j=1}^{n}(\delta \boldsymbol{v}_j-\delta \boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{z}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}-(\boldsymbol{v}_{j-1}-\boldsymbol{v}_{j-1}^{hk}),\boldsymbol{v}_j-\boldsymbol{z}_j^h)_H\\
&= (\boldsymbol{v}_n-\boldsymbol{v}_n^{hk},\boldsymbol{v}_n-\boldsymbol{z}_n^h)_H+(\boldsymbol{v}_0^h-\boldsymbol{v}_0,\boldsymbol{v}_1-\boldsymbol{z}_1^h)_H\\
&\quad +\sum_{j=1}^{n-1}(\boldsymbol{v}_j-\boldsymbol{v}_j^{hk},\boldsymbol{v}_j-\boldsymbol{z}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{z}_{j+1}^h))_H,\\
k\sum_{j=1}^{n}(\delta \boldsymbol{e}_j-\delta \boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{r}_j^h)_H &= \sum_{j=1}^{n}(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk}-(\boldsymbol{e}_{j-1}-\boldsymbol{e}_{j-1}^{hk}),\boldsymbol{e}_j-\boldsymbol{r}_j^h)_H\\
&= (\boldsymbol{e}_n-\boldsymbol{e}_n^{hk},\boldsymbol{e}_n-\boldsymbol{r}_n^h)_H+(\boldsymbol{e}_0^h-\boldsymbol{e}_0,\boldsymbol{e}_1-\boldsymbol{r}_1^h)_H\\
&\quad +\sum_{j=1}^{n-1}(\boldsymbol{e}_j-\boldsymbol{e}_j^{hk},\boldsymbol{e}_j-\boldsymbol{r}_j^h-(\boldsymbol{e}_{j+1}-\boldsymbol{r}_{j+1}^h))_H,
\end{aligned}
$$

applying again a discrete version of Gronwall's inequality (see, for instance, [32]), we obtain a priori error estimates (23). ■

We point out that estimates (23) can be used to obtain a convergence analysis. For example, assume the following additional regularity conditions on the solution to the continuous problem:

$$\boldsymbol{u},\,\boldsymbol{w}\in H^2(0,T;V)\cap H^3(0,T;H)\cap C^1([0,T];[H^2(\Omega)]^d). \tag{26}$$

We recall that the finite element space V^h was given in (9) and we use the discrete initial conditions $\boldsymbol{u}_0^h$, $\boldsymbol{v}_0^h$, $\boldsymbol{w}_0^h$ and $\boldsymbol{e}_0^h$ defined in (10). Thus, we have the following.

Corollary 3.4 *Let the assumptions of Theorem 3.3 still hold. Under the additional regularity conditions (26) the linear convergence of the algorithm is deduced; i.e. there exists a positive constant $C>0$, independent of the discretization parameters h and k, such that*

$$
\begin{aligned}
\max_{0\le n\le N}\Big\{&\|\boldsymbol{v}_n-\boldsymbol{v}_n^{hk}\|_H+\|\boldsymbol{e}_n-\boldsymbol{e}_n^{hk}\|_H+\|\mathrm{div}\,(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Y+\|\nabla(\boldsymbol{u}_n-\boldsymbol{u}_n^{hk})\|_Q\\
&+\|\mathrm{div}\,(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Y+\|\nabla(\boldsymbol{w}_n-\boldsymbol{w}_n^{hk})\|_Q\Big\}\le C(h+k).
\end{aligned}
$$

Corollary 3.4 is proved using classical arguments. Therefore, we have the following property of approximation by finite elements (see, e.g., [31]):

$$
\begin{aligned}
k\sum_{j=1}^{N}\Big(&\inf_{\boldsymbol{z}_j^h\in V^h}\|\boldsymbol{v}_j-\boldsymbol{z}_j^h\|_V^2+\inf_{\boldsymbol{r}_j^h\in V^h}\|\boldsymbol{e}_j-\boldsymbol{r}_j^h\|_V^2\\
&+\max_{0\le n\le N}\inf_{\boldsymbol{z}_n^h\in V^h}\|\boldsymbol{v}_n-\boldsymbol{z}_n^h\|_H^2+\max_{0\le n\le N}\inf_{\boldsymbol{r}_n^h\in V^h}\|\boldsymbol{e}_n-\boldsymbol{r}_n^h\|_H^2\Big)\\
&\le Ch^2\left(\|\boldsymbol{u}\|^2_{C^1([0,T];[H^2(\Omega)]^d)}+\|\boldsymbol{w}\|^2_{C^1([0,T];[H^2(\Omega)]^d)}\right).
\end{aligned}
$$

By using the additional regularity conditions (26), we find that the initial conditions have the regularity

$$\boldsymbol{u}_0,\,\boldsymbol{v}_0,\,\boldsymbol{w}_0,\,\boldsymbol{e}_0\in[H^2(\Omega)]^d,$$

and therefore,

$$\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{e}_0-\boldsymbol{e}_0^h\|_H^2+\|\boldsymbol{w}_0-\boldsymbol{w}_0^h\|_V^2$$
$$\leq Ch^2\left(\|\boldsymbol{u}_0\|^2_{[H^2(\Omega)]^d}+\|\boldsymbol{v}_0\|^2_{[H^2(\Omega)]^d}+\|\boldsymbol{w}_0\|^2_{[H^2(\Omega)]^d}+\|\boldsymbol{e}_0\|^2_{[H^2(\Omega)]^d}\right).$$

Moreover, using again regularity conditions (26), we have

$$k\sum_{j=1}^N\left[\|\dot{\boldsymbol{u}}_j-\delta\boldsymbol{u}_j\|_V^2+\|\dot{\boldsymbol{w}}_j-\delta\boldsymbol{w}_j\|_V^2+\|\dot{\boldsymbol{v}}_j-\delta\boldsymbol{v}_j\|_H^2+\|\dot{\boldsymbol{e}}_j-\delta\boldsymbol{e}_j\|_H^2\right]$$
$$\leq ck^2\big(\|\boldsymbol{u}\|^2_{H^2(0,T;V)}+\|\boldsymbol{u}\|^2_{H^3(0,T;H)}+\|\boldsymbol{w}\|^2_{H^2(0,T;V)}+\|\boldsymbol{w}\|^2_{H^3(0,T;H)}\big).$$

Finally, the remaining terms in estimates (23) can be bounded as follows (see [32, 33] for details),

$$\frac{1}{k}\sum_{j=1}^{N-1}\left[\|\boldsymbol{v}_j-\boldsymbol{z}_j^h-(\boldsymbol{v}_{j+1}-\boldsymbol{z}_{j+1}^h)\|_H^2+\|\boldsymbol{e}_j-\boldsymbol{r}_j^h-(\boldsymbol{e}_{j+1}-\boldsymbol{r}_{j+1}^h)\|_H^2\right]$$
$$\leq Ch^2(\|\boldsymbol{u}\|^2_{H^2(0,T;V)}+\|\boldsymbol{w}\|^2_{H^2(0,T;V)}).$$

Combining all these estimates, the linear convergence is deduced.

4. Numerical results

In this final section, we describe the numerical scheme implemented in MATLAB for solving Problem VP^{hk}, and we show some numerical examples to demonstrate the accuracy of the approximation and the behaviour of the solution.

Let the finite element space be defined in (9). For $n=1,2,\ldots,N$ and given $\boldsymbol{u}_{n-1}^{hk}$, $\boldsymbol{v}_{n-1}^{hk}$, $\boldsymbol{w}_{n-1}^{hk}$, $\boldsymbol{e}_{n-1}^{hk}\in V^h$ the discrete velocity of the first constituent $\boldsymbol{v}_n^{hk}$ and the discrete velocity of the second constituent $\boldsymbol{e}_n^{hk}$, at time $t=t_n$, are then obtained from equations (11) and (12). That is, we solve the following problem, for all $\boldsymbol{z}^h$, $\boldsymbol{r}^h\in V^h$,

$$\begin{aligned}
&\rho_1(\boldsymbol{v}_n^{hk},\boldsymbol{z}^h)_H+a_1k^2(\operatorname{div}\boldsymbol{v}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y+a_2k^2(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{z}^h)_Q\\
&\quad+a_3k^2(\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y+a_4k^2(\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{z}^h)_Q+a_5k(\operatorname{div}\boldsymbol{v}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y\\
&\quad+a_6k(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{z}^h)_Q+a_7k(\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y+a_8k(\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{z}^h)_Q\\
&\quad+\xi k^2(\boldsymbol{v}_n^{hk}-\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H+\xi^*k(\boldsymbol{v}_n^{hk}-\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H=\rho_1(\boldsymbol{v}_{n-1}^{hk},\boldsymbol{z}^h)_H\\
&\quad-a_1k(\operatorname{div}\boldsymbol{u}_{n-1}^{hk},\operatorname{div}\boldsymbol{z}^h)_Y-a_2k(\nabla\boldsymbol{u}_{n-1}^{hk},\nabla\boldsymbol{z}^h)_Q-a_3k(\operatorname{div}\boldsymbol{w}_{n-1}^{hk},\operatorname{div}\boldsymbol{z}^h)_Y\\
&\quad-a_4k(\nabla\boldsymbol{w}_{n-1}^{hk},\nabla\boldsymbol{z}^h)_Q-\xi k(\boldsymbol{u}_{n-1}^{hk}-\boldsymbol{w}_{n-1}^{hk},\boldsymbol{z}^h)_H,
\end{aligned}$$

$$
\begin{aligned}
&\rho_2(\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H + b_1k^2(\operatorname{div}\boldsymbol{v}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y + b_2k^2(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{z}^h)_Q\\
&\quad + b_3k^2(\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y + b_4k^2(\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{z}^h)_Q + b_5k(\operatorname{div}\boldsymbol{v}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y\\
&\quad + b_6k(\nabla\boldsymbol{v}_n^{hk},\nabla\boldsymbol{z}^h)_Q + b_7k(\operatorname{div}\boldsymbol{e}_n^{hk},\operatorname{div}\boldsymbol{z}^h)_Y + b_8k(\nabla\boldsymbol{e}_n^{hk},\nabla\boldsymbol{z}^h)_Q\\
&\quad - \xi k^2(\boldsymbol{v}_n^{hk}-\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H - \xi^* k(\boldsymbol{v}_n^{hk}-\boldsymbol{e}_n^{hk},\boldsymbol{z}^h)_H = \rho_2(\boldsymbol{e}_{n-1}^{hk},\boldsymbol{z}^h)_H\\
&\quad - b_1k(\operatorname{div}\boldsymbol{u}_{n-1}^{hk},\operatorname{div}\boldsymbol{z}^h)_Y - b_2k(\nabla\boldsymbol{u}_{n-1}^{hk},\nabla\boldsymbol{z}^h)_Q - b_3k(\operatorname{div}\boldsymbol{w}_{n-1}^{hk},\operatorname{div}\boldsymbol{z}^h)_Y\\
&\quad - b_4k(\nabla\boldsymbol{w}_{n-1}^{hk},\nabla\boldsymbol{z}^h)_Q + \xi k(\boldsymbol{u}_{n-1}^{hk}+\boldsymbol{w}_{n-1}^{hk},\boldsymbol{z}^h)_H,
\end{aligned}
$$

where the discrete displacements of the first and second constituents, $\boldsymbol{u}_n^{hk}$ and $\boldsymbol{w}_n^{hk}$ respectively, are now recovered from the relations:

$$
\boldsymbol{u}_n^{hk} = k\sum_{j=1}^{n}\boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h, \quad \boldsymbol{w}_n^{hk} = k\sum_{j=1}^{n}\boldsymbol{e}_j^{hk} + \boldsymbol{w}_0^h.
$$

Here, the discrete initial conditions are defined by (10).

This numerical scheme was implemented on a 2.8 GHz PC using MATLAB, and a typical 1D run ($h = k = 0.01$) took about 0.146 seconds of CPU time, meanwhile a 2D one (with the finite element shown later and $k = 0.001$) took about 19.062 seconds of CPU time.

4.1. *First example: numerical convergence in a 1D example*

As an academical 1D example, in order to show the accuracy of the approximations we consider the following simpler problem.

Problem P^{ex}. *Find the displacement of the first constituent* $u : [0,1]\times[0,1] \to \mathbb{R}$ *and the displacement of the second constituent* $w : [0,1]\times[0,1] \to \mathbb{R}$ *such that*

$$
\begin{aligned}
&\ddot u = 9u_{xx} + 23u_{xx} + 23w_{xx} + 3w_{xx} + 9\dot u_{xx} + 23\dot u_{xx} + 23\dot w_{xx}\\
&\qquad + 3\dot w_{xx} - (u-w) - (\dot u - \dot w) + F_1 \quad \text{in} \quad (0,1)\times(0,1),\\
&\ddot w = 23u_{xx} + 3u_{xx} + 3w_{xx} + 20w_{xx} + 23\dot u_{xx} + 3\dot u_{xx} + 3\dot w_{xx}\\
&\qquad + 20\dot w_{xx} + (u-w) + (\dot u - \dot w) + F_2 \quad \text{in} \quad (0,1)\times(0,1),\\
&u(0,t) = w(0,t) = u(1,t) = w(1,t) = 0 \qquad \text{for a.e. } t \in (0,1),\\
&u(x,0) = x(x-1), \quad \dot u(x,0) = x(x-1) \quad \text{for a.e.} \quad x\in(0,1),\\
&w(x,0) = x(x-1), \quad \dot w(x,0) = x(x-1) \quad \text{for a.e.} \quad x\in(0,1),
\end{aligned}
$$

where the artificial volume forces F_1, F_2 are given by

$$
F_1(x,t) = e^t(x^2 - x - 232), \quad F_2(x,t) = e^t(x^2 - x - 196).
$$

We note that Problem P^{ex} corresponds to Problem P with the following data:

$$
\begin{aligned}
&\Omega = (0,1), \quad T = 1, \quad \rho_1 = 1, \quad \rho_2 = 1, \quad \lambda = 1, \quad \nu = 1, \quad \mu = 1, \quad \zeta = 1,\\
&\alpha = 1, \quad \gamma = 1, \quad \kappa = 10, \quad \xi = 1, \quad \lambda^* = 1, \quad \nu^* = 1, \quad \nu_1^* = 1, \quad \mu^* = 1,\\
&\zeta^* = 1, \quad \zeta_1^* = 1, \quad \alpha^* = 1, \quad \gamma^* = 1, \quad \kappa^* = 10, \quad \xi^* = 1,
\end{aligned}
$$

the following initial conditions, for $x \in (0,1)$,

$$u_0(x) = \dot{u}_0(x) = w_0(x) = \dot{w}_0(x) = x(x-1),$$

and homogeneous Dirichlet boundary conditions on $x = 0,\ 1$.

The exact solution to Problem P^{ex} can be easily calculated and it has the form, for $(x,t) \in (0,1) \times (0,1)$,

$$u(x,t) = e^t x(x-1), \quad w(x,t) = e^t x(x-1).$$

To show the numerical convergence and the asymptotic behaviour of the algorithm, the numerical errors given by

$$\max_{0 \le n \le N} \left\{ \|v_n - v_n^{hk}\|_Y + \|e_n - e_n^{hk}\|_Y + \|(u_n - u_n^{hk})_x\|_Y + \|(w_n - w_n^{hk})_x\|_Y \right\},$$

are calculated and presented (multiplied by 10^2) in Table 1 for several values of the discretization parameters h and k. Finally, the evolution of the error depending on the parameter $h+k$ is plotted in Figure 1. We notice that the convergence of the algorithm is clearly observed, and the linear convergence, stated in Corollary 3.4, is achieved.

$h \downarrow k \rightarrow$	0.01	0.005	0.001	0.0005	0.0002	0.0001	0.00005
0.01	8.619512	7.181511	6.389765	6.328891	6.297641	6.289279	6.287465
0.005	6.191994	4.380770	3.271737	3.195133	3.158866	3.148491	3.143712
0.001	4.751606	2.622495	0.889081	0.723513	0.654604	0.639072	0.632879
0.0005	4.662346	2.495919	0.652821	0.445403	0.348697	0.327318	0.319539
0.0002	4.634653	2.452818	0.544245	0.307391	0.178374	0.144805	0.132795
0.0001	4.630571	2.446212	0.520938	0.273441	0.131244	0.089220	0.072398
0.00005	4.629584	2.444574	0.514157	0.261904	0.112746	0.065676	0.044617

Table 1. Example 1: Numerical errors (x10^2) for some h and k.

If we assume now that there are not volume forces and we use the final time $T = 20$, being the remaining data the same than in the previous simulation and taking the discretization parameters $k = h = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk}, defined in (21), is plotted in Figure 2 in natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

4.2. *Second example: a 2D surface force in one component*

As a first two-dimensional example we study the effect of the application of a surface force in the first constituent. Thus, we solve Problem P with the following data:

$$\begin{gathered}\Omega = (0,5)\times(0,1), \quad \Gamma_D = \{0\}\times(0,1), \quad \Gamma_N = \Gamma - \Gamma_D, \quad T = 1, \quad \rho_1 = 1, \\ \rho_2 = 1, \quad \lambda = 4442.5, \quad \nu = 1, \quad \mu = 975.18, \quad \zeta = 1, \quad \alpha = 862.96, \quad \gamma = 37, \\ \kappa = 2013.6, \quad \xi = 130, \quad \lambda^* = 326.84, \quad \nu^* = 1, \quad \nu_1^* = 1, \quad \mu^* = 415.98, \\ \zeta^* = 1, \quad \zeta_1^* = 1, \quad \alpha^* = 390.31, \quad \gamma^* = 22, \quad \kappa^* = 539, \quad \xi^* = 98,\end{gathered}$$

and the initial conditions $\boldsymbol{u}_0 = \dot{\boldsymbol{u}}_0 = \boldsymbol{w}_0 = \dot{\boldsymbol{w}}_0 = \boldsymbol{0}$.

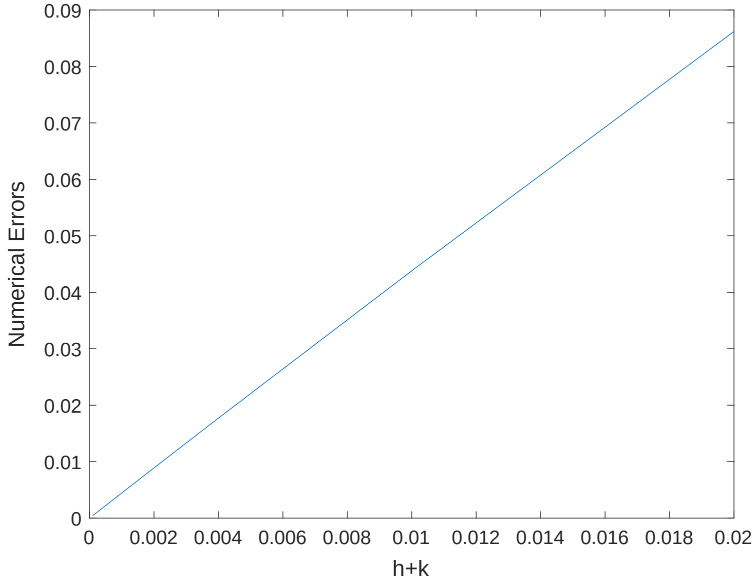

Figure 1. Example 1: Asymptotic constant error.

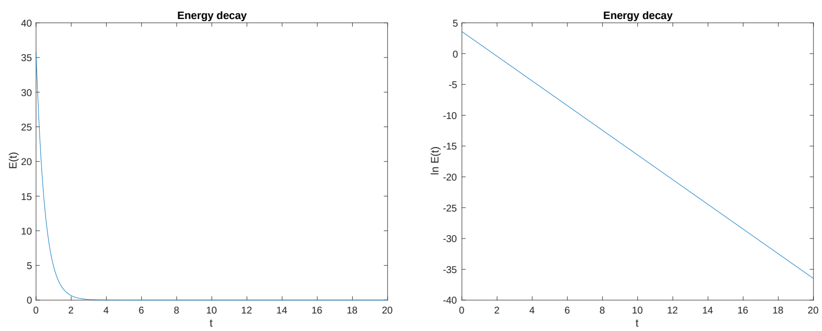

Figure 2. Example 1: Evolution of the discrete energy in the natural (left) and the semi-log scales (right).

We consider homogeneous Dirichlet boundary conditions on Γ_D for both components, that is,

$$\boldsymbol{u}(0,y,t)=\boldsymbol{w}(0,y,t)=\boldsymbol{0} \quad \text{for all } t\in(0,1),\ y\in(0,1).$$

Furthermore, we consider homogeneous Neumann boundary conditions on Γ_N for the displacements of the second constituent, $\boldsymbol{w}$, and non-homogeneous Neumann boundary conditions for the displacements of the first constituent, $\boldsymbol{u}$, in the following form:

$$\text{Force}=\begin{cases}(0,0) & \text{if } \ x\in(0,5), \quad y=0,\\ (573t,0) & \text{if } \ x=5, \quad y\in(0,1),\\ (0,0) & \text{if } \ x\in(0,5), \quad y=1.\end{cases}$$

Using the time discretization parameter $k=10^{-3}$ and the finite element mesh shown in Figure 3, the von Mises stress norm over the deformed mesh of the solid material, for the first constituent, is plotted at final time in Figure 4. As expected, there is an extension of the body in the X-direction and a compression in the

Y-direction.

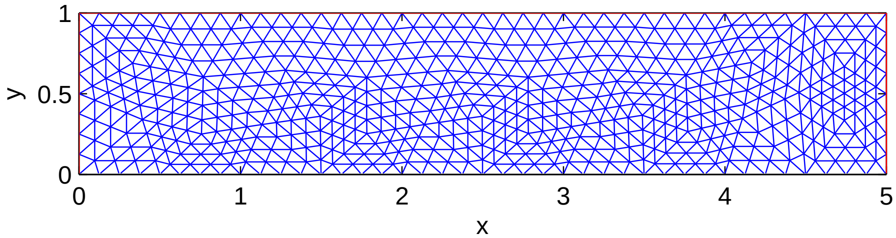

Figure 3. Example 2: Finite element mesh.

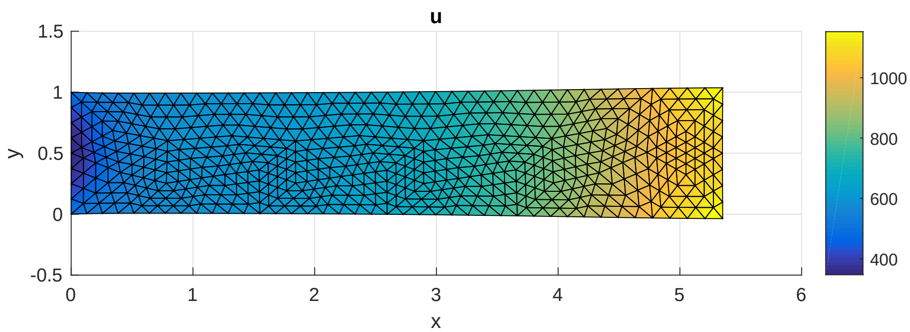

Figure 4. Example 2: von Mises stress norm for the displacements of the first constituent at final time over the deformed mesh.

The von Mises stress norm is shown in Figure 5 over the deformed mesh of the solid material for the second constituent and at final time. Now, there is a compression of the body in the X-direction. This is possible produced as a reaction to the effect due to the first constituent.

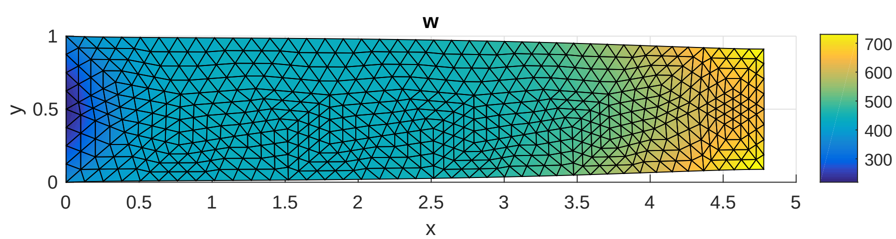

Figure 5. Example 2: von Mises stress norm for the displacements of the second constituent at final time over the deformed mesh.

As we did in the one-dimensional example, if we assume now that there are not applied forces and we use the final time $T = 5$ and the initial conditions:

$$u_0 = v_0 = e_0 = w_0 = (x^2 - 5x)(y^2 - y),$$

being the remaining data the same used in the previous simulation and taking the discretization parameter $k = 10^{-3}$, the evolution in time of the discrete energy E_n^{hk}, defined in (21), is plotted in Figure 6 in natural (left) and semi-log (right) scales. As can be seen, it converges to zero and an exponential decay seems to be achieved.

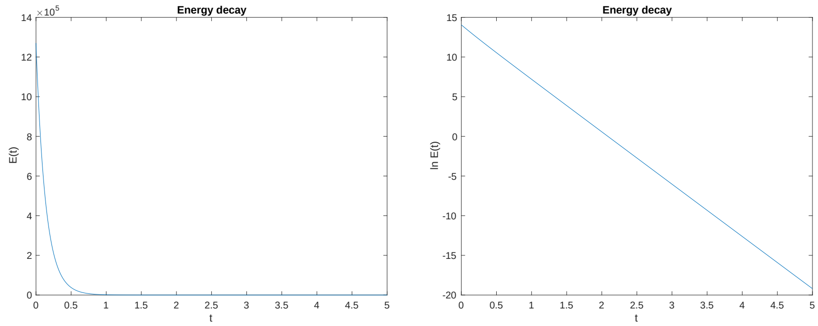

Figure 6. Example 2: Evolution of the discrete energy in the natural (left) and the semi-log scales (right).

4.3. *Third example: a 2D surface force in two components*

As a second two-dimensional example, we study the effect of the application of a surface force on both constituents. Hence, we solve Problem P considering the same data as in the previous example.

We also assume homogeneous Dirichlet boundary conditions on Γ_D for both components, that is,

$$\boldsymbol{u}(0, y, t) = \boldsymbol{w}(0, y, t) = 0 \quad \text{for all } t \in (0, 1),\ y \in (0, 1).$$

In this example, we consider non-homogeneous Neumann boundary conditions on Γ_N for the displacements of both constituents as follows,

$$\text{Force} = \begin{cases} (0,0) & \text{if } x \in (0,5), \quad y = 0, \\ (573t, 0) & \text{if } x = 5, \quad y \in (0,1), \\ (0,0) & \text{if } x \in (0,5), \quad y = 1. \end{cases}$$

Using again the time discretization parameter $k = 10^{-3}$ and the finite element mesh shown in Figure 3, the von Mises stress norm over the deformed mesh of the solid material, for the first constituent, is plotted at final time in Figure 7. As expected, there is an extension of the body in the X-direction and a compression in the Y-direction.

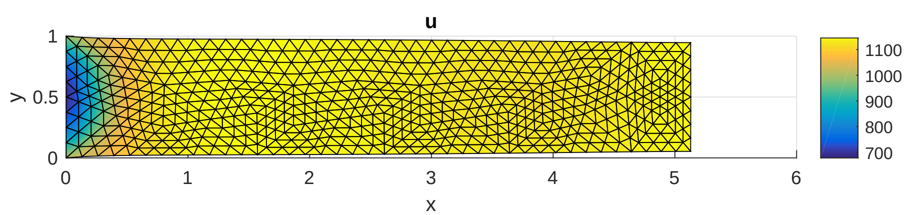

Figure 7. Example 3: von Mises stress norm for the displacements of the first constituent at final time over the deformed mesh.

Finally, the von Mises stress norm is shown in Figure 8 over the deformed mesh of the solid material for the second constituent and at final time. Now, the results are rather similar for both constituents maybe because the forces are applied at the same time.

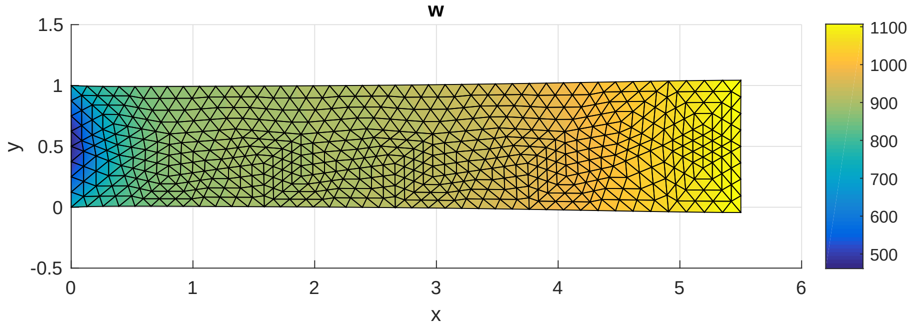

Figure 8. Example 3: von Mises stress norm for the displacements of the second constituent at final time over the deformed mesh.

5. Acknowledgements

The work of J.R. Fernández has been supported by the Ministerio de Economía y Competitividad under the research project MTM2015-66640-P (with the participation of FEDER).

The work of A. Magaña and R. Quintanilla has been supported by the Ministerio de Economía y Competitividad under the research projects "Análisis Matemático de las Ecuaciones en Derivadas Parciales de la Termomecánica" (MTM2013-42004-P) and "Análisis Matemático de Problemas de la Termomecánica" (MTM2016-74934-P), (AEI/FEDER, UE).

References

[1] C. Truesdell, R. A. Toupin, The classical field theories, Springer-Verlag, Berlin, 1960, handbuch der physik, vol. II/3.

[2] P. Kelly, Reacting continuum, Internat. J. Engng. Sci. 2 (1964) 129–153.

[3] A. Eringen, J. Ingram, A continuum theory of chemically reacting media-i, Internat. J. Engng. Sci. 3 (1965) 197–212.

[4] A. E. Green, P. M. Naghdi, A dynamical theory of interacting continua, Internat. J. Engng. Sci. 3 (1965) 231–241.

[5] A. E. Green, P. M. Naghdi, A note on mixtures, Internat. J. Engng. Sci. 6 (1968) 631–635.

[6] I. Müller, A thermodynamic theory of mixtures of fluids, Archive for Rational Mechanics and Analysis 28 (1968) 1–39.

[7] R. M. Bowen, J. C. Wiese, Diffusion in mixtures of elastic material, Internat. J. Engng. Sci. 7 (1969) 689–722.

[8] R. M. Bowen, Theory of mixtures, in: Continuum Physics. Vol. III, Academic Press, New York, 1976, pp. 3–120.

[9] R. J. Atkin, R. E. Craine, Continuum theories of mixtures. basic theory and historial development, Quart. J. of Mechanics and Applied Mathematics 29 (1972) 209–245.

[10] A. Bedford, D. S. Drumhelles, Theory of immiscible and structured mixtures, Int. J. of Engineering Science 21 (1983) 153–207.

[11] K. R. Rajagopal, A. S. Wineman, Developments in the mechanics of interactions between a fluid and a highly elastic solid, Pitmant Research Notes in Mathematics Series, Essex, 1990, vol. II.

[12] K. R. Rajagopal, L. Tao, Mechanics of Mixtures, World Scientific, Singapore, 1995.

[13] A. Bedford, M. Stern, A multi-continuum theory for composite elastic materials, Acta Mechanica 14 (1972) 85–102.

[14] A. Bedford, M. Stern, Towards a diffusing continuum theory of composite elastic materials, J. Appl. Mech. 38 (1972) 8–14.

[15] D. Ieşan, R. Quintanilla, Existence and continuous dependence results in the theory of interacting continua, J. Elasticity 36 (1994) 85–98.

[16] F. Martinez, R. Quintanilla, Some qualitative results for the linear theory of binary mixtures of thermoelastic solids, Collectanea Mathematica 46 (1995) 263–277.

[17] J. R. Vinson, R. L. Sierakowski, The behaviour of structures composed of composite materials, Kluwer Academics Publishing, Dordrecht, 2002.

[18] S. Elangovan, B. S. Altan, G. M. Odegard, An elastic micropolar mixture theory for predicting elastic properties of cellular materials, Mech. Mater. 40 (2008) 602–614.

[19] M. S. Alves, J. E. Muñoz Rivera, R. Quintanilla, Exponential decay in a thermoelastic mixture of solids, Internat. J. Solids Structures 46 (2009) 1659–1666.

[20] M. S. Alves, J. E. Muñoz Rivera, M. Sepúlveda, O. V. Villagrán, Exponential stability in thermoviscoelastic mixtures of solids, Internat. J. Solids Structures 46 (2009) 4151–4162.

[21] M. S. Alves, J. E. Muñoz Rivera, M. Sepúlveda, O. V. Villagrán, Stabilitization in three-dimensional porous thermoviscoelastic mixtures, Quart. J. Math. 64 (2013) 37–57.

[22] H. Fernández-Sare, J. Muñoz Rivera, R. Quintanilla, On the rate of decay in interacting continua with memory, J. Differential Equations 251 (2011) 3583–3605.

[23] D. Ieşan, A theory of mixtures of nonsimple elastic solids, Internat. J. Engrg. Sci. 30 (1992) 317–328.

[24] D. Ieşan, Continuous dependence in a nonlinear theory of viscoelastic porous mixtures, Internat. J. Engrg. Sci. 44 (2006) 1127–1145.

[25] D. Ieşan, R. Quintanilla, A theory of porous thermoviscoelastic mixtures, J. Thermal Stresses 30 (2007) 693–714.

[26] J. Muñoz Rivera, M. G. Naso, R. Quintanilla, Decay of solutions for a mixture of thermoelastic one-dimensional solids, Comput. Math. Appl. 66 (2013) 263–277.

[27] J. Muñoz Rivera, M. G. Naso, R. Quintanilla, Decay of solutions for a mixture of thermoelastic solids with different temperatures, Comput. Math. Appl. 71 (2016) 991–1009.

[28] R. Quintanilla, Existence and exponential decay in the linear theory of viscoelastic mixtures, Eur. J. Mech. A. Solids 24 (2005) 311–324.

[29] M. I. M. Copetti, J. R. Fernández, M. Masid, Numerical analysis of a viscoelastic mixture problem, Internat. J. Solids Structures 80 (2016) 393–404.

[30] J. R. Fernández, A. Magaña, M. Masid, R. Quintanilla, On the viscoelastic mixtures of solids, Appl. Math. Optim. in press.

[31] P. G. Ciarlet, The finite element method for elliptic problems, in: Handbook of Numerical Analysis. Vol. II, Handb. Numer. Anal., North-Holland, 1991, pp. 17–352.

[32] M. Campo, J. R. Fernández, K. L. Kuttler, M. Shillor, J. M. Viaño, Numerical analysis and simulations of a dynamic frictionless contact problem with damage, Comput. Methods Appl. Mech. Engrg. 196 (2006) 476–488.

[33] M. Barboteu, J. R. Fernández, T.-V. Hoarau-Mantel, A class of evolutionary variational inequalities with applications in viscoelasticity, Math. Models Methods Appl. Sci. 15 (10) (2005) 1595–1617.