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Zusammenfassung

In dieser Arbeit betrachten wir Stochastische Evolutionsgleichungen getrieben durch raue Pfade. Das erste Ziel ist die Existenz und Eindeutigkeit von milden Lösungen zu zeigen. Dazu werden wichtige Grundlagen zur Halbgruppentheorie sowie zur Theorie der rauen Pfade präsentiert. Anschließend entwickeln wir, basierend auf heuristischen Betrachtungen, eine Lösungstheorie und zeigen die Existenz einer eindeutigen globalen Lösung.
Als Anwendung für das treibende stochastische Rauschen betrachten wir eine fraktale Brownsche Bewegung, welche zu einem rauen Pfad geliftet wird und analysieren daran einfache dynamische Eigenschaften der milden Lösung. Wir zeigen, dass diese Lösung ein zufälliges dynamisches System generiert und untersuchen ihr Langzeitverhalten unter zusätzlichen Voraussetzungen an die nichtlinearen Koeffizienten, d.h. wir zeigen sowohl lokale als auch globale Stabilität der trivialen Lösung.

Abstract

In this thesis we consider Stochastic Evolution Equations driven by rough paths. The first aim is to show existence and uniqueness of mild solutions. Therefore, important basics on semigroup theory and on the theory of rough paths are introduced. Afterwards, we develop a solution theory based on heuristic considerations and use this to prove the existence of a global-in-time solution. Then, as leading example for the driving noise we consider a fractional Brownian motion which, can be lifted to a rough path, and analyze simple dynamic properties of the mild solution. We show that the solution generates a random dynamical system and investigate its long-time behavior under additional assumptions on the coefficients, i.e. we show local as well as global stability of the trivial solution.

Contents

1. Introduction

Starting with Lyons, [48] in the 1990's the rough path theory has become a widely used mathematical theory for the analysis of stochastic differential equations. Gubinelli extended Lyons first approach and introduced the concept of a controlled rough paths in [35]. Further, Gubinelli and Tindel expanded the rough path theory in order to analyze stochastic partial differential equations in [37]. Later on Hairer adopted the ideas of the rough paths theory and developed the theory of regularity structures in [38] for solving singular partial differential equations.

In this thesis we analyze existence and dynamics of solutions for rough stochastic partial differential equations (SPDEs)

$$\begin{cases} dy_t = (Ay_t + F(y_t))dt + G(y_t)d\omega_t, & t \in [0,T] \\ y_0 = \xi. \end{cases} \tag{1.1}$$

Here $T > 0$ is a fixed time-horizon, the linear part A is the generator of an analytic C_0-semigroup $(S(t))_{t\in[0,T]}$ on a separable Banach space W and $\xi \in W$ denotes the initial condition. Furthermore, F and G are the nonlinear coefficients. The precise assumptions on the coefficients will be stated in Section 2.1. Finally, ω denotes a rough random input which can be lifted to a geometric rough path, for instance a fractional Brownian motion with Hurst parameter $H \in (1/3, 1/2]$ introduced in Section 4.1. In order to solve (1.1) we rely on the pathwise construction of the rough integral

$$\int_0^t S(t-r)G(y_r)d\omega_r. \tag{1.2}$$

Results in this context are available in [51] and [27] via fractional calculus and in Gubinelli et al in [35], [10], [36], [37], [32] using rough paths techniques.
In this work, we combine Gubinelli's approach with the arguments employed by [27] to solve (1.1). The eventual aim of this thesis is to investigate the long-time behavior of solutions of (1.1). Thus for, we first establish the existence of a pathwise global solution. Consequently, we can show that the solution operator of (1.1) generates an infinite-dimensional random dynamical system, see [2]. The analysis of many dynamical aspects for (1.1) such as asymptotic stability, Lyapunov exponents, multiplicative ergodic theorems, random attractors, random invariant manifolds etc. relies on the random dynamical system approach, see e.g. [25], [3], [26], [41], [14] or [5].
The generation of a random dynamical system from an Itô-type SPDE is in general still an open question for multiplicative noise. Here, we benefit strongly from the pathwise construction of the solution since no exceptional sets occur. Recently, there has been a growing interest to give a pathwise meaning to the solutions of SPDEs by various techniques, see e.g. [37], [10], [27] or [38]. However, there are only few works that explore the pathwise character of the solutions to analyze random dynamical systems and their long-time behavior. Progress in this direction was made for instance in [30] and [27] where the authors use fractional calculus for dealing with random dynamical systems for SPDEs driven by a fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$ and $H \in (1/3, 1/2]$. Furthermore, in [16], [17] the random input is given by rough Gaussian noise and is handled by rough path techniques.

To the best of our knowledge there are only few works that connect the rough paths and random dynamical systems perspectives such as [19], [3], [34], [16] or [17]. Here we contribute to this aspect and provide a general framework of random dynamical systems for rough evolution equations under natural assumptions on the coefficients. The crucial result that opens the door for the random dynamical systems theory is the existence of a global pathwise solution for (1.1). It is known that global-in-time existence of solutions is a challenging question in the context of rough paths techniques, compare [27], [37], [32]. This is due to the fact that one obtains certain quadratic estimates on the norms of the solution of (1.1). Hence, it is not straightforward to extend the local solution to an arbitrary time horizon. Using additional restrictions on the coefficients or on the noisy input, see [27] shows global-in-time existence for (1.1) driven by a fractional Brownian motion with Hurst index $H \in (1/3, 1/2]$. However, in this thesis using regularizing properties of analytic C_0-semigroups, a-priori estimates on certain remainder terms and a standard concatenation procedure enables us to prove the global-in-time existence of solutions.

There are several techniques for analyzing the long-time behavior of a global-in-time solution of 1.1, see for instance [12], [23], [14] or [23]. In this work we use a truncation technique as used in [30] and a stopping time approach, compare [18], [16], [17], in order to show stability in zero.

This work is structured as follows. In Section 2.1 we collect well-known properties and estimates of analytic semigroups and introduce important notation, which are necessary in this framework. Further in Section 2.2 we give a short introduction into the rough path theory
In Chapter 3 we develop a solution theory in order to solve (1.1) in the rough case. For introducing the general ideas we first analyze a more regular case (Young case) in Section 3.1. Section 3.2 provides the general intuition of the required techniques in the rough case. It describes the story in a nutshell and provides an insight into the rough path theory pointing out the main obstacles which occur in the infinite-dimensional setting. We state basic concepts and indicate how an appropriate pathwise integral should be constructed and how a solution of a rough evolution equation should look like. The next sections rigorously justify the steps presented in Section 3.2. Section 3.3 is the core of Chapter 3. Here we introduce a modified version of the Sewing Lemma, compare [37], [36]. This is a very general fundamental result which entails the existence of a rough integral in a suitable analytic and algebraic framework. This is, of course, the first main ingredient for solving (1.1). In contrast to [37], we work with modified Hölder spaces and so the version of the Sewing Lemma precisely fits in this setting. Section 3.4 is devoted to the construction of supporting processes which are necessary to give an appropriate meaning of the rough integral. Inspired by [27], in order to define the supporting processes we first consider smooth approximations of the noise and thereafter pass to the limit. The existence of the corresponding processes is derived via classical tools, such as an integration by parts formula or using the Sewing Lemma introduced in Section 3.3. For a better comprehension, we point out an example in which one can construct a pathwise integral using the integration by parts formula as well as the Sewing Lemma. Section 3.5 shows the existence of a unique local solution of (1.1) by using a fixed point argument. Since certain a-priori estimates presented in Section 3.5 contain quadratic terms one cannot immediately conclude the existence of a global solution. Thus, in Section 3.6 we introduce a specific functional in order to derive a linear a-priori estimate for the solution of (1.1). Eventually, we show the existence of a global solution by concatenation techniques. We present an application of our theory in Section 3.7. Chapter 3 concludes with some remarks on the nonlinear coefficients F and G in Section 3.8.
In Chapter 4 we analyze the dynamics of solutions of (1.1) we derived in Chapter 3. We introduce the concept of a random dynamical system in Section 4.1. As leading example we present a Hilbert space valued fractional Brownian motion, lift it to a rough path and prove finally, that a solution of (1.1) driven by a fractional Brownian motion generates a random dynamical system.
Sections 4.2 and 4.3 deal with the long-time behavior of this solution, compare for instance [30], [16] or [12]. By assuming $F(0) = 0$ and $G(0) = 0$ we guarantee the existence of the trivial solution of (1.1). In Section 4.2 we use a truncation technique to show local exponential stability of the

trivial solution, compare [30]. In Section 4.3 we restrict ourselves to the simpler Young case and show global exponential stability by exploiting a stopping time argument, as introduced in [18]. In Chapter 5 we summarize the main results of this thesis and discuss possible extensions to our work. Finally, we collect some important results and computations in the Appendix A.

This work is based on the two papers [40] and [39]. More precisely, Sections 3.2–3.5 and 3.7 arise from [40]. Section 3.6 and 4.1 are a revised version of [39].

2. Preliminaries

Let $T > 0$. V stands for a separable Hilbert space and W denotes a separable Banach space. Furthermore, for any compact interval $J \subset \mathbb{R}$ we set $\Delta_J := \{(t,s) \in J^2 \colon t \geq s\}$ and $\Delta_T := \Delta_{[0,T]}$. For notational simplicity, if not further stated, we write $|\cdot|$ for the norm of an arbitrary Banach space. Furthermore, C denotes a universal constant which may vary from line to line. The explicit dependence of C on certain parameters will be precisely stated, whenever required. Finally, we fix $\alpha \in (\frac{1}{3}, \frac{1}{2})$. This parameter indicates the Hölder-regularity of the random input.

2.1 Notation and Deterministic Background

We start by introducing the assumptions on the linear part of (1.1) and on the coefficients F and G.

Since we are in the parabolic setting, i.e. A is a sectorial operator, we can introduce its fractional powers $(-A)^\gamma$ for $\gamma \geq 0$, see [55, Section 2.6] or [47]. We denote the domains of the fractional powers of $(-A)$ with D_γ, i.e. $D_\gamma := D((-A)^\gamma)$, and recall the following estimates.
For $\eta, \kappa \in \mathbb{R}$ we have

$$\|S(t)\|_{\mathcal{L}(D_\kappa, D_\eta)} = \|(-A)^\eta S(t)\|_{\mathcal{L}(D_\kappa, W)} \leq C t^{k-\eta}, \qquad \text{for } \eta \geq \kappa \tag{2.1}$$

$$\|S(t) - \mathrm{Id}\|_{\mathcal{L}(D_\kappa, D_\eta)} \leq C t^{\kappa-\eta}, \qquad \text{for } \kappa - \eta \in [0,1]. \tag{2.2}$$

Furthermore, one can show that the following assertions hold true, consult [55, Chapter 3].

Lemma 2.1. *For any $\nu, \eta, \mu \in [0,1]$, $\kappa, \gamma, \varrho \geq 0$ such that $\kappa \leq \gamma + \mu$, there exists a constant $C > 0$ such that for $0 < q < r < s < t$ we have that*

$$\|S(t-r) - S(t-q)\|_{\mathcal{L}(D_\kappa, D_\gamma)} \leq C (r-q)^\mu (t-r)^{-\mu-\gamma+\kappa},$$

$$\|S(t-r) - S(s-r) - S(t-q) + S(s-q)\|_{\mathcal{L}(D_\varrho, D_\varrho)} \leq C (t-s)^\eta (r-q)^\nu (s-r)^{-(\nu+\eta)}.$$

On the coefficients we impose the following conditions:

- **(F)** $F \colon W \to W$ is Lipschitz continuous.
- **(G)** $G \colon W \to \mathcal{L}(V; D_\beta)$ is bounded and three times Frechét differentiable with bounded derivatives for some $\beta \in (0, \alpha)$ satisfying $\alpha + 2\beta > 1$.

Remark 2.2.

(i) For most parts of Chapter 3 we set $F \equiv 0$ for simplicity, since this term does not cause additional technical difficulties. However, we will discuss the differences in Section 3.8.

*(ii) The boundedness of G is only necessary for deriving a global solution. Clearly, in the case of linear noise **(G)** is not fulfilled. However, in Section 3.8 we will show that in this special case we will obtain a global solution, too.*

*(iii) The assumptions **(F)** and **(G)** are more general than the assumptions made on the coefficients of the SPDE* (1.1) *in many works, compare [27] and [37] and the references specified therein. Recently, in [32] the authors consider slightly more general assumptions but for the case of a finite-dimensional noisy input.*

For our aims we introduce the following function spaces. Let $\beta \in (0,1)$ be fixed and let $\overline{W}$ stand for a further separable Banach space. We recall that $C^{\beta}([0,T],W)$ represents the space of W-valued β-Hölder continuous functions on $[0,T]$ and denote by $C^{\alpha}(\Delta_T,\overline{W})$ the space of $\overline{W}$-valued functions on Δ_T with $z_{t,t}=0$ for all $t\in[0,T]$ and

$$\|z\|_{\alpha} := \sup_{0\leq t\leq T} |z_{t0}| + \sup_{0\leq s<t\leq T} \frac{|z_{ts}|}{(t-s)^{\alpha}} < \infty.$$

Furthermore, we define $C^{\beta,\beta}([0,T],W)$ as the space of W-valued continuous functions on $[0,T]$ endowed with the norm

$$\|y\|_{\beta,\beta} := \|y\|_{\infty} + \|\!|y|\!\|_{\beta,\beta} := \sup_{0\leq t\leq T} |y_t| + \sup_{0<s<t\leq T} s^{\beta}\frac{|y_t-y_s|}{(t-s)^{\beta}} < \infty.$$

Similarly we introduce $C^{\alpha+\beta,\beta}(\Delta_T,\overline{W})$ with the norm

$$\|z\|_{\alpha+\beta,\beta} := \sup_{0\leq t\leq T} |z_{t0}| + \sup_{0<s<t\leq T} s^{\beta}\frac{|z_{ts}|}{(t-s)^{\alpha+\beta}} < \infty.$$

Again $z_{t,t}=0$ for all $t\in[0,T]$.

These modified Hölder spaces are well-known in the theory of maximal regularity for parabolic evolution equations, see [47]. These were also used in [27].

Note that for notational simplicity we do not state the dependence of the (modified) Hölder spaces on the underlying time interval/area.
In case we want to point out this dependence, see Section 3.6 and Section 4.3 eventually, we denote $\|\cdot\|_{\alpha,T}$ respectively $\|\cdot\|_{\beta,\beta,T}$. In case we consider a restriction on a certain time interval $[\check{T},\hat{T}]$ and we set $\|\cdot\|_{\alpha,[\check{T},\hat{T}]}$ respectively $\|\cdot\|_{\beta,\beta,[\check{T},\hat{T}]}$.

It is well-known that analytic C_0-semigroups are not Hölder continuous in 0. However, the following lemma holds true.

Lemma 2.3. *Let $(S(t))_{t\geq 0}$ be an analytic C_0-semigroup on W. Then we have for all $x\in W$ and all $\beta\in[0,1]$ that*

$$\|S(\cdot)x\|_{\beta,\beta} \leq C\,|x|\,,$$

where C depends only on the semigroup and on β.

Proof. The definition of $\|\cdot\|_{\beta,\beta}$ and tjhe estimates recall (2.1) and (2.2) entail

$$\begin{aligned}\|S(\cdot)x\|_{\beta,\beta} &= \sup_{0\leq t\leq T}|S(t)x| + \sup_{0<s<t\leq T} s^{\beta}\frac{|(S(t)-S(s))x|}{(t-s)^{\beta}}\\ &\leq \sup_{0\leq t\leq T}|S(t)x| + \sup_{0<s<t\leq T} s^{\beta}\frac{|(S(t-s)-\mathrm{Id})S(s)x|}{(t-s)^{\beta}}\\ &\leq C|x|,\end{aligned}$$

□

This justifies our choice of working with the function space $C^{\beta,\beta}$. Note that if one lets $x\in D_{\beta}$ it suffices to consider only C^{β}. However, since we want to analyze random dynamical systems generated by (1.1) in W (compare Section 4.1), we need to take the initial condition $\xi\in W$ instead of D_{β}.

Furthermore, we fix some important notations. These are also used in [37] and [10].
Notations: For $y\in C([0,T],W)$ and $z\in C(\Delta_T,\overline{W})$ we set for $0\leq s\leq\tau\leq t$

$$(\delta y)_{ts} := y_t - y_s, \tag{2.3}$$

$$(\hat{\delta} y)_{ts} := y_t - S(t-s)y_s, \tag{2.4}$$

$$(\delta_2 z)_{t\tau s} := z_{ts} - z_{t\tau} - z_{\tau s}, \tag{2.5}$$

$$(\hat{\delta}_2 z)_{t\tau s} := z_{ts} - z_{t\tau} - S(t-\tau)z_{\tau s}. \tag{2.6}$$

At this moment these notations probably appear unreasonable. In Section 3.2 and 3.3 we will show the connection to the solution theory.

Let us state some important algebraic properties. For more details and a more general framework see [37].

Lemma 2.4. *The following statements hold true:*

(i) $\hat{\delta}_2 \circ \hat{\delta} \equiv 0$.

(ii) *Let* $N \in C(\Delta_T, W)$ *with* $\hat{\delta}_2 N \equiv 0$. *Then there exists* $y \in C([0,T], W)$ *with* $(\hat{\delta} y)_{ts} = N_{ts}$.

(iii) *Consider* $y^1, y^2 \in C([0,T], W)$ *with* $y^1_0 = y^2_0$ *and* $(\hat{\delta} y^1)_{ts} = (\hat{\delta} y^2)_{ts}$. *Then* $y^1 \equiv y^2$.

Proof.

(i) Take an arbitrary $y \in C([0,T], W)$.

$$\begin{aligned}(\hat{\delta}_2 \hat{\delta} y)_{t\tau s} &= (\hat{\delta} y)_{ts} - (\hat{\delta} y)_{t\tau} - S(t-\tau)(\hat{\delta} y)_{\tau s} \\ &= y_t - S(t-s)y_s - y_t + S(t-\tau)y_\tau - S(t-\tau)y_\tau + S(t-s)y_s = 0.\end{aligned}$$

(ii) Let $\hat{\delta}_2 N \equiv 0$. Set $y_t := N_{t0}$. Then, we have

$$(\hat{\delta} y)_{ts} = N_{t0} - S(t-s)N_{s0} = N_{ts}.$$

(iii) Consider

$$y^1_t = (\hat{\delta} y^1)_{t0} + S(t)y^1_0 = (\hat{\delta} y^2)_{t0} + S(t)y^2_0 = y^2_t.$$

□

The second assertion of the previous Lemma is extremely important for the deliberations made in Section 3.3, especially for Theorem 3.6, which ensures the existence of rough integrals.

Concluding this section we denote by $V \otimes V$ the usual tensor product of Hilbert spaces. In case of Banach spaces $W \otimes \overline{W}$ stands for the projective tensor product for given Banach spaces $W, \overline{W}$ which is an extension of the tensor product for Hilbert space. Then the property

$$\mathcal{L}(W; \mathcal{L}(\overline{W}; W)) \hookrightarrow \mathcal{L}(W \otimes \overline{W}; W) \tag{2.7}$$

holds true, consult [56, Theorem 2.9].

2.2 Basics of Rough Paths Theory

Although the techniques provided by the rough paths theory are fundamental for the whole thesis, in this section we only state the very basics. We will focus on further results of the general theory in Sections 3.2, 3.3 and 4.1. For a broad overview of the rough paths theory we recommend [20].

Definition 2.5. *(α-Hölder rough path)* Let $J \subset \mathbb{R}$ be a compact interval. The pair $\boldsymbol{\omega} := (\omega, \omega^{(2)})$ is called V-valued α-Hölder rough path if $\omega \in C^\alpha(J, V)$ and $\omega^{(2)} \in C^{2\alpha}(\Delta_J, V \otimes V)$. Furthermore, ω and $\omega^{(2)}$ are connected via Chen's relation, meaning that

$$\omega^{(2)}_{ts} - \omega^{(2)}_{us} - \omega^{(2)}_{tu} = (\omega_u - \omega_s) \otimes (\omega_t - \omega_u), \quad \text{for } s, u, t \in J, \quad s \leq u \leq t. \tag{2.8}$$

Using the notation introduced in (2.3) and (2.5) we can shorten this and obtain

$$(\delta_2 \, \omega^{(2)})_{tus} = (\delta\omega)_{us} \otimes (\delta\omega)_{tu}.$$

In the literature $\omega^{(2)}$ is referred to as Lévy-area or second order process. We denote $\mathscr{C}^\alpha = \mathscr{C}^\alpha(J, V)$ as the set of all V-valued α-Hölder rough path on J.

We further describe an appropriate distance between two α-Hölder rough paths.

Definition 2.6. Let $\boldsymbol{\omega}$ and $\widetilde{\boldsymbol{\omega}}$ be two α-Hölder rough paths. We introduce the α-Hölder rough path (inhomogeneous) metric

$$d_{\alpha,J}(\boldsymbol{\omega},\widetilde{\boldsymbol{\omega}}) := \sup_{(t,s)\in\Delta_J} \frac{|\omega_t-\omega_s-\widetilde{\omega}_t+\widetilde{\omega}_s|}{|t-s|^{\alpha}} + \sup_{(t,s)\in\Delta_J} \frac{|\,\omega^{(2)}_{ts}-\widetilde{\omega}^{(2)}_{ts}|}{|t-s|^{2\alpha}}. \tag{2.9}$$

We set $d_{\alpha,T} := d_{\alpha,[0,T]}$.

For more details on this topic consult [20, Chapter 2]. We stress that in our situation we always have that $\omega(0)=0$ and therefore, (2.9) is a metric.

Remark 2.7. *In general, the main issue is to find an adequate second order process $\omega^{(2)}$ for a given path ω. We call this* to lift ω *to a rough path* $\boldsymbol{\omega}=(\omega,\omega^{(2)})$.
Note that $\omega^{(2)}$ is uniquely determined up to an increment of some path $\psi\in C^{2\alpha}(V\otimes V)$, see [20, Section 2.1], i.e. if $(\omega,\omega^{(2)})$ is an α-Hölder rough path then $(\omega,\omega^{(2)}+\delta\varphi)\in\mathscr{C}^{\alpha}$ for all $\varphi\in C^{2\alpha}(V\otimes V)$. Moreover, every lifted rough path has this form.
However, under the assumption that ω is a smooth path, e.g. continuously differentiable, we can define $\omega^{(2)}$ by

$$\omega^{(2)}_{ts} = \int_s^t (\delta\omega)_{rs}\otimes d\omega_r. \tag{2.10}$$

We call this the canonical lift.

Furthermore, θ denotes the *Wiener shift* (this represents an appropriate shift with respect to the noise), more precisely $\theta=(\theta_\tau)_{\tau\in\mathbb{R}}$ with

$$\theta_\tau\omega_t := \omega_{t+\tau}-\omega_\tau. \tag{2.11}$$

The Wiener shift is explained in detail in Section 4.1 and is mainly required in the theory of random dynamical systems.

Moreover, we use the notation $\widetilde{\theta}$ in order to indicate the usual shift, namely $\widetilde{\theta}=(\widetilde{\theta}_\tau)_{\tau\in\mathbb{R}}$ with

$$\widetilde{\theta}_\tau y_t := y_{t+\tau}, \tag{2.12}$$

$$\widetilde{\theta}_\tau z_{ts} := z_{t+\tau,s+\tau}. \tag{2.13}$$

Note that, at this moment y and z are meant to be arbitrary first order respectively second order processes. In particular, the definition of $\widetilde{\theta}$ is also applicable to the noise, e.g. $\widetilde{\theta}_\tau\,\omega^{(2)}_{ts} := \omega^{(2)}_{t+\tau,s+\tau}$.

We indicate the following result regarding the shift-property of an α-Hölder rough path. Eventually, it will be crucial in Section 3.6 as well as in Section (4.1).

Lemma 2.8. *For an α-Hölder rough path $(\omega,\omega^{(2)})$ on the time interval J and for $\tau\in\mathbb{R}$, the time-shift $(\theta_\tau\omega,\widetilde{\theta}_\tau\,\omega^{(2)})$ is an α-Hölder rough path on the time interval $J-\tau$.*

Proof. The time-regularity is straightforward and one can easily verify Chen's relation (2.8).

$$\begin{aligned}\widetilde{\theta}_\tau\,\omega^{(2)}_{ts}-\widetilde{\theta}_\tau\,\omega^{(2)}_{us}-\widetilde{\theta}_\tau\,\omega^{(2)}_{tu} &= \omega^{(2)}_{t+\tau,s+\tau}-\omega^{(2)}_{u+\tau,s+\tau}-\omega^{(2)}_{t+\tau,u+\tau}\\ &=\omega_{u+\tau,s+\tau}\otimes\omega_{t+\tau,u+\tau}, \qquad (2.14)\\ &=(\omega_{u+\tau}-\omega_\tau-\omega_{s+\tau}+\omega_\tau)\otimes(\omega_{t+\tau}-\omega_\tau-\omega_{u+\tau}+\omega_\tau)\\ &=(\delta\theta_\tau\omega)_{us}\otimes(\delta\theta_\tau\omega)_{tu}.\end{aligned}$$

where in (2.14) we use Chen's relation (2.8). □

To close this chapter let us introduce the concept of a *geometric rough path*, see [20, Section 2.2].

Definition 2.9. Let $\boldsymbol{\omega}$ be a V-valued α-Hölder rough path and $(e_i)_{i\in\mathbb{N}}$ be an orthonormal basis of V. For $i,j\in\mathbb{N}$ define $\omega^i = (\omega, e_i)_V$ and $\omega^{(2),ij} = (\omega^{(2)}, e_i\otimes e_j)_{V\otimes V}$. We call $\boldsymbol{\omega}$ a geometric V-valued α-Hölder rough path if

$$\omega^{(2),ij}_{ts} + \omega^{(2),ji}_{ts} = (\delta\omega^i)_{ts}(\delta\omega^j)_{ts}.$$

We denote $\mathscr{C}^\alpha_g$ as the set of all geometric α-Hölder rough paths.

Note that we do not need this property for deriving a solution of (1.1) in Chapter 3. However, in many applications, e.g. [20, Chapter 10], [11] or [25], the rough noise ω is approximated by a sequence of (piecewise) smooth paths ω^n. These are lifted to rough paths $\boldsymbol{\omega^n}$ according to (2.10) and one proves the convergence of these sequence in $d_{\alpha,J}$. We will present this procedure for our example of a fractional Brownian motion in Section 4.1. The next lemma shows that in this case we always end up with a geometric rough path, see [20, Section 2.2].

Lemma 2.10. *Let $\mathscr{C}^{0,\alpha}_g$ be the closure of canonical lifts of piecewise smooth paths w.r.t. $d_{\alpha,J}$. Then*

$$\mathscr{C}^{0,\alpha}_g \subseteq \mathscr{C}^\alpha_g.$$

In particular, every canonical lift of a piecewise smooth path is a geometric rough path.

3. Solution Theory for Rough Evolution Equations

In this chapter we rigorously define (1.2) in the rough case and eventually show the existence of a global-in-time solution of (1.1). Recall from Chapter 2 that C denotes a generic constant which may vary from line to line. This constant can depend on the semigroup S, the nonlinear coefficients F and G, the parameter $\alpha \in (\frac{1}{3}, \frac{1}{2})$, the rough path $\boldsymbol{\omega}$ and the eventual solution space.
For notational simplicity we omit these dependencies. Furthermore, in the following we drop the tensor symbol for the same reason. Finally, for simplicity we set $F \equiv 0$.
In Section 3.8 we will have a closer look on the case where the drift term does not vanish and further give a comment about the generic constant C.

3.1 The Young Case

Before considering the rough case we want to give a concise motivational overview of a more regular case. In this section we assume $\alpha > \frac{1}{2}$.
In 1936 Young proved the existence of a deterministic integral as limit of Riemann-Stieltjes sums, see [61]. In [62] the Young integral was first applied for defining a pathwise stochastic integral with respect to a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.
For defining the Young integral one exploits the regularity of the integrand and the integrator, more precisely for an integrand with finite p-variation and an integrator with finite q-variation one demands $\frac{1}{p} + \frac{1}{q} > 1$. Clearly every α-Hölder function has finite p-variation for all $p < \frac{1}{\alpha}$. Hence, for simplicity we state Young's result for Hölder continuous functions.

Theorem 3.1. *Let $\alpha, \beta > 0$ with $\alpha + \beta > 1$ and $\omega \in C^{\alpha}([0,T], V)$ $y \in C^{\beta}([0,T], \mathcal{L}(V;W))$. Then, for all $0 \leq s \leq t \leq T$ there exists*

$$\int_s^t y_r \, d\omega_r := \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} y_u (\delta\omega)_{vu}, \tag{3.1}$$

where $\mathcal{P} = \mathcal{P}(s,t)$ is an arbitrary partition of $[s,t]$ and $|\mathcal{P}|$ indicates the mesh of this partition. Furthermore, we obtain $\int_0^{\cdot} y_r \, d\omega_r \in C^{\alpha}([0,T], W)$ and the Hölder norm can be estimated by

$$\left\| \int_0^{\cdot} y_r \, d\omega_r \right\|_{\alpha} \leq C \left\| y \right\|_{\beta} \left\| \omega \right\|_{\alpha}. \tag{3.2}$$

It is important to mention that the limit in (3.1) does not depend on the choice of the partition neither the choice of the node.

Consider the simple Young equation of the form

$$y_t = \xi + \int_0^t G(y_r) \, d\omega_r, \qquad 0 \leq t \leq T$$

with $\omega \in C^\alpha$. Let C^β be the space where we want to find the solution in, e.g. by a fixed point argument. So, let $y \in C^\beta$ then, even for smooth G, Theorem 3.1 demands $\alpha+\beta>1$ and grants $y \in C^\alpha$. Consequently, if we want to iterate the Young integral, we need $\alpha \geq \beta$. This makes clear why we have to assume $\alpha > \frac{1}{2}$.

Indeed, if $\alpha > \frac{1}{2}$ there are several possibilities for defining (1.2) as a Young integral, see [36] or [50]. Note that we can not apply Theorem 3.1 for defining (1.2) since the semigroup S is not Hölder continuous, see Section 2.1. However, Lemma 2.3 guarantees $S(\cdot)x \in C^{\beta,\beta}$ for all $x \in W$. Consequently, we can not expect a solution y of (1.1) to be Hölder regular but to fulfill $y \in C^{\beta,\beta}$. The next Lemma follows by a more general abstract result we are going to prove in Section 3.3. In order to motivate such a result, we first give a sketch for a more regular $\omega \in C^\alpha$, $\alpha > \frac{1}{2}$ here.

Lemma 3.2. *Let $\alpha, \beta > 0$ with $\alpha + \beta > 1$ and $\omega \in C^\alpha([0,T],V)$, $y \in C^{\beta,\beta}([0,T],\mathcal{L}(V;W))$. Furthermore, let $(S_t)_{t\geq 0}$ be an analytic semigroup on W and $G\colon W \to \mathcal{L}(V;W)$ be Lipschitz continuous. Then, for all $0 \leq s \leq t \leq T$ there exists*

$$\int_s^t S(t-r)G(y_r)\,d\omega_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} S(t-v)\,\omega^S_{vu}(G(y_u)), \tag{3.3}$$

where for a placeholder $K \in \mathcal{L}(V;W)$ and $0 \leq s \leq t \leq T$ we define

$$\omega^S_{ts}(K) := \int_s^t S(t-r)\,K\;d\omega_r. \tag{3.4}$$

Furthermore, for all $0 < s \leq t \leq T$ we obtain

$$\left|\int_s^t S(t-r)G(y_r)\,d\omega_r\right| \leq C\,\|y\|_{\beta,\beta}\,\|\omega\|_\alpha\, s^{-\beta}(t-s)^\alpha. \tag{3.5}$$

Sketch of Proof. One can define (3.4) using integration by parts see (3.51) and derive the estimate

$$\left|\omega^S_{ts}(K)\right| \leq C\,\|\omega\|_\alpha\,|K|\,(t-s)^\alpha, \qquad \text{compare (3.47).}$$

For simplicity let $\mathcal{P}_n = \mathcal{P}_n(s,t)$ by the n-th dyadic partition of $[s,t]$ with $0 < s \leq t \leq T$ and define

$$N^n_{ts} := \sum_{[u,v]\in\mathcal{P}_n} S(t-v)\,\omega^S_{vu}(G(y_u)), \qquad \text{for } n \in \mathbb{N}_0. \tag{3.6}$$

For a rigorous proof we had to consider an arbitrary sequence of partitions whose meshes converge to zero and also handle the case $s=0$.

We consider the difference of two adjacent partitions

$$\begin{aligned} N^n_{ts} - N^{n+1}_{ts} &= \sum_{[u,v]\in\mathcal{P}_n} S(t-v)\,\omega^S_{vu}(G(y_u)) - \sum_{[u,v]\in\mathcal{P}_{n+1}} S(t-v)\,\omega^S_{vu}(G(y_u)) \\ &= \sum_{[u,v]\in\mathcal{P}_n} \left[S(t-v)\,\omega^S_{vu}(G(y_u)) - S(t-v)\,\omega^S_{vm}(G(y_m)) - S(t-m)\,\omega^S_{mu}(G(y_u))\right] \end{aligned}$$

where $m = \frac{u+v}{2}$. Furthermore, we have

$$\begin{aligned} &S(t-v)\,\omega^S_{vu}(G(y_u)) - S(t-v)\,\omega^S_{vm}(G(y_m)) - S(t-m)\,\omega^S_{mu}(G(y_u)) \\ &= S(t-v)\,(\hat{\delta}_2\omega^S)_{vmu}(G(y_u)) + S(t-v)\,\omega^S_{vm}(G(y_u)-G(y_m)). \end{aligned}$$

We will show in Lemma 3.17 that $(\hat{\delta}_2\omega^S) \equiv 0$. So, since for $[u,v] \in \mathcal{P}_n$ we have $v - u = \frac{t-s}{2^n}$, we obtain the estimate

$$\begin{aligned}\left|N^n_t s - N^{n+1}_{ts}\right| \leq & C\,\|\omega\|_\alpha \sum_{[u,v]\in\mathcal{P}_n} |G(y_u) - G(y_m)|\,(v-m)^\alpha \\ \leq & C\,\|\omega\|_\alpha\,\|y\|_{\beta,\beta}\; s^{-\beta}\frac{(t-s)^{\alpha+\beta}}{2^{n(\alpha+\beta-1)}},\end{aligned}$$

since G is Lipschitz continuous and $y \in C^{\beta,\beta}$. The sequence $(N^n_{ts} - N^{n+1}_{ts})_{n\in\mathbb{N}_0}$ is absolutely summable As $\alpha + \beta > 1$. Thus, we can define the limit of N^n_{ts} by the limit of the telescopic sums over the differences of two adjacent partitions. Hence, we derive the estimate

$$\Big|\lim_{n\to\infty}\sum_{[u,v]\in\mathcal{P}_n} S(t-v)\,\omega^S_{vu}(G(y_u)) - \omega^S_{ts}(G(y_s))\Big| \leq C\,\|\omega\|_\alpha\,\|y\|_{\beta,\beta}\,s^{-\beta}(t-s)^{\alpha+\beta}. \qquad \square$$

In Section 3.3 we will give a rigorous proof in a more general setting and obtain further estimates for the Young integral.

3.2 Heuristic Considerations

The goal is again to obtain a criterion that enables us to obtain the integral as a limit of Riemann-Stieltjes sums as we did above in the Young case. This abstract result is called Sewing Lemma and is stated in the following Section 3.3. To this aim we need certain algebraic and analytic conditions.

For a better comprehension and in order to point out the difficulties that arise in the infinite-dimensional setting, we shortly sketch the well-known results in the finite dimensional case. In finite dimensions, the solution theory for (1.1) is well-established and one needs very few ingredients to define (1.2) by rough paths techniques. This immediately entails a suitable solution concept for (1.1). Regard that we use the notation introduced in Chapter 2.

Since the trajectories of the noise are irregular, i.e. Hölder continuous with exponent $\alpha < 1/2$, the Young integral defined as (3.1) can no longer be used. Therefore, Gubinelli [35] introduced the concept of a controlled rough integral, which extends the Young case. Regarding (3.3), it turns out that we have to consider additional terms satisfying certain algebraic and analytic properties. This reads as follows

$$\int_s^t y_r d\omega_r = \lim_{|\mathcal{P}|\to 0}\Big(\sum_{[u,v]\in\mathcal{P}} y_u(\delta\omega)_{vu} + y'_u\,\omega^{(2)}_{vu}\Big). \tag{3.7}$$

Here the pair (y, y') stands for a controlled rough path. This can be interpreted as an abstract Taylor series, namely one assumes that there exists $y' \in C^\alpha$, which is called Gubinelli's derivative, such that for all $0 \leq s \leq t \leq T$ we have

$$y_t = y_s + y'_s(\delta\omega)_{ts} + R^y_{ts}, \tag{3.8}$$

where the remainder R^y is 2α-Hölder regular. Here $\omega \in C^\alpha$ and $\omega^{(2)} \in C^{2\alpha}$ are connected via Chen's relation (recall Definition 2.5), meaning that for $0 \leq s \leq u \leq t$

$$(\delta_2\,\omega^{(2)})_{tus} = (\delta\omega)_{us} \otimes (\delta\omega)_{tu}.$$

Consequently, $\omega^{(2)}$ can be thought of as the iterated integral

$$\omega^{(2)}_{ts} = \int_s^t (\delta\omega)_{rs} \otimes d\omega_r. \tag{3.9}$$

We emphasize that in order to construct (3.7) and thereafter the solution of (1.1), one needs an appropriate algebraic and analytic setting which will be in detail analyzed in this work for stochastic evolution equation. The rigorous existence proof of (3.7) is based on a Sewing Lemma, see Lemma 4.2 in [20]. For more details on this topic consult [35] and [20, Chapter 4].

This opens the door for the theory of rough SDEs using a completely pathwise approach. The only part where stochastic analysis plays a role is hidden in (3.9). Keeping this in mind, one can solve (1.1) by a fixed-point argument in the space of controlled rough paths. Regarding this, one can easily show that the solution of (1.1) with $F \equiv 0$ is given by the pair

$$(y, y') = \Big(S(\cdot)\xi + \int\limits_0^{\cdot} S(\cdot - r)G(y_r)d\omega_r, G(y_\cdot) \Big). \tag{3.10}$$

The essential tool in defining (3.7) and proving that (3.10) is the right object to solve (1.1) in the finite-dimensional case is the regularity of $(S(t))_{t\geq 0}$. Note that a (semi)group generated by a linear bounded operator is Lipschitz continuous therefore, the required Hölder regularity of the terms appearing in (3.8) and (3.7) cannot be influenced. More precisely, one can easily show that for a controlled rough path (y, y') as specified in (3.8), the convolution with $(S(t))_{t\geq 0}$, i.e. $(S(t-\cdot)y, S(t-\cdot)y')$, is again a controlled rough path. Due to this fact one can define $\int\limits_s^t S(t-r)y_r d\omega_r$ by (3.7) and show that the mapping

$$(y, y') \mapsto \Big(\int\limits_0^{\cdot} S(\cdot - r)y_r d\omega_r, y_\cdot \Big)$$

is linear and continuous on the space of controlled rough paths. Moreover, the composition of a controlled rough path with a regular function is a well-defined operation according to Lemma 7.3 in [20]. Consequently, $\int\limits_s^t S(t-r)G(y_r)d\omega_r$ fits perfectly in the framework of (3.7). Regarding the notation introduced above one observes that

$$\int\limits_s^t S(t-r)G(y_r)d\omega_r$$

corresponds to

$$(\hat{\delta} y)_{ts} = (\delta y)_{ts} - (S(t-s) - \mathrm{Id})y_s.$$

Finally, by an appropriate fixed-point argument one establishes that (3.10) solves (1.1). This would be the story in a nutshell in the finite-dimensional setting. For further information and applications see [35], [20], [22].

However, in the infinite-dimensional case, since the analytic C_0-semigroup $(S(t))_{t\geq 0}$ is not Lipschitz continuous (not even Hölder continuous in 0, recall Lemma 2.3) it is no longer straightforward what are the appropriate objects required in order to obtain something similar to (3.10). It turns out that one has to construct additional supporting processes. Consult also [10] in order to find the right way to define (1.2) together with the corresponding pair (y, y') that solves (1.1). This is the main topic of our work and for the beginning we illustrate heuristically the main ideas, which will be justified by the computations in the next sections. Furthermore, we stress that the noisy input ω is infinite-dimensional in contrast to [10]. Therefore, one needs to make sure that the Lévy-area $\omega^{(2)}$ exists in this case, see [24] and the references specified therein.

We make preliminary deliberations which will lead us to the right definition of (1.2).

To this aim, similar to [27], we firstly assume that ω is smooth and consider the following approximation of the integral:

$$\begin{aligned}\int_s^t S(t-r)G(y_r)d\omega_r &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)G(y_r)d\omega_r \\ &\approx \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)\left[G(y_u)+DG(y_u)(\delta y)_{ru}\right]d\omega_r \\ &=: \sum_{[u,v]\in\mathcal{P}} S(t-v)\Big[\omega^S_{vu}(G(y_u)) + \int_u^v S(v-r)DG(y_u)(\delta y)_{ru}d\omega_r\Big].\end{aligned}$$

In the first step we just plugged in the definition of the integral using Riemann-Stieltjes sums and in the second step we employed a Taylor expansion for G. Furthermore, we introduced the notation

$$\omega^S_{vu}(G(y_u)) := \int_u^v S(v-r)G(y_u)d\omega_r, \tag{3.11}$$

respectively

$$z_{vu}(DG(y_u)) := \int_u^v S(v-r)DG(y_u)(\delta y)_{ru}d\omega_r. \tag{3.12}$$

Since ω is smooth, all the expressions above are well-defined. We argue in Section 3.4 how to define the first integral (3.11) for a rough input ω and derive important properties of ω^S, using an integration by parts formula and regularizing properties of analytic semigroups. Unfortunately, it is not at all clear how to define z if ω is not smooth. Therefore, we have to continue our considerations.

The strategy is to construct the integral using an appropriate Sewing Lemma as derived in Section 3.3. To this aim we need to introduce several processes satisfying appropriate analytic and algebraic conditions. We describe the general intuition of this approach which will allow us to define the integral $\mathcal{I}$ of

$$\Xi^{(y)}_{vu} := \Xi^{(y)}_{vu}(y,z) := \omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u)).$$

In order to employ the Sewing Lemma (Theorem 3.6) to obtain the existence together with suitable estimates of $\mathcal{I}\Xi^{(y)}$ we firstly have to compute (as rigorously justified in Section 3.3)

$$(\hat{\delta}_2\Xi^{(y)})_{vmu} = \Xi^{(y)}_{vu} - \Xi^{(y)}_{vm} - S(v-m)\Xi^{(y)}_{mu}.$$

We can easily check

$$\begin{aligned}(\hat{\delta}_2\Xi^{(y)})_{vmu} &= (\hat{\delta}_2\omega^S)_{vmu}(G(y_u)) + \omega^S_{vm}(G(y_u)-G(y_m)) \\ &\quad + (\hat{\delta}_2 z)_{vmu}(DG(y_u)) + z_{vm}(DG(y_u)-DG(y_m)).\end{aligned}$$

The first term obviously results in

$$\begin{aligned}(\hat{\delta}_2\omega^S)_{vmu}(G(y_u)) &= \omega^S_{vu}(G(y_u)) - \omega^S_{vm}(G(y_u)) - S(v-m)\omega^S_{mu}(G(y_u)) \\ &= \int_u^v S(v-r)G(y_u)d\omega_r - \int_m^v S(v-r)G(y_u)d\omega_r - S(v-m)\int_u^m S(m-r)G(y_u)d\omega_r \\ &= 0.\end{aligned}$$

Consequently,

$$(\hat{\delta}_2\Xi^{(y)})_{vmu} = \omega^S_{vm}(G(y_u) - G(y_m)) + (\hat{\delta}_2 z)_{vmu}(DG(y_u)) + z_{vm}(DG(y_u) - DG(y_m)). \tag{3.13}$$

Hence, it remains to investigate $\hat{\delta}_2 z(E)$, where $E \in \mathcal{L}(W \otimes V; W)$ denotes a placeholder. For smooth paths ω we have z canonically given by (3.12) and compute

$$\begin{aligned}(\hat{\delta}_2 z)_{vmu}(E) &= z_{vu}(E) - z_{vm}(E) - S(v-m)z_{mu}(E)\\ &= \int_u^v S(v-r)E(\delta y)_{ru}d\omega_r - \int_m^v S(v-r)E(\delta y)_{rm}d\omega_r - \int_u^m S(v-r)E(\delta y)_{ru}d\omega_r\\ &= \int_m^v S(v-r)E(\delta y)_{mu}d\omega_r = \omega^S_{vm}(E(\delta y)_{mu}).\end{aligned}$$

As already mentioned, this term indeed exists even for a rough trajectory $\omega \in C^\alpha$ with $\alpha \in (\frac{1}{3}, \frac{1}{2}]$. If we assume that the algebraic relation

$$(\hat{\delta}_2 z)_{vmu}(E) = \omega^S_{vm}(E(\delta y)_{mu}) \tag{3.14}$$

holds true for any $E \in \mathcal{L}(W \otimes V; W)$ we obtain

$$(\hat{\delta}_2\Xi^{(y)})_{vmu} = \omega^S_{vm}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}) + z_{vm}(DG(y_u) - DG(y_m)).$$

Having this structure for $(\hat{\delta}_2\Xi^{(y)})_{vmu}$, under suitable regularity assumptions on y and z specified in Section 3.3, we are able to define

$$\overline{y}_t := \mathcal{I}\Xi^{(y)}_t \quad \text{and} \quad \widetilde{y}_t := S(t)\xi + \overline{y}_t. \tag{3.15}$$

Note that $\mathcal{I}\Xi^{(y)}_t$ corresponds to (1.2) and

$$(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} = \int_s^t S(t-r)G(y_r)d\omega_r.$$

Remark 3.3.

1. *Since we demand the existence of a suitable z in order to construct the rough integral, it is necessary to define $\widetilde{z}$ fulfilling $(\hat{\delta}_2\widetilde{z})_{vmu}(E) = \omega^S_{vm}(E(\delta\widetilde{y})_{mu})$. Only if this is valid we are able to iterate the solution mapping.*
2. *Note that if $S(\cdot) = Id$, then the algebraic relation (3.14) reduces to*

$$(\delta_2 z)_{vmu}(E) = E(\delta y)_{mu} \otimes (\delta\omega)_{vm}, \qquad \textit{compare } (2.8).$$

Again, for smooth ω, $\widetilde{z}$ is canonically given by

$$\begin{aligned}\widetilde{z}_{ts}(E) &= \int_s^t S(t-r)E(\delta\widetilde{y})_{rs}d\omega_r = \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E(\delta\widetilde{y})_{rs}d\omega_r\\ &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E(\hat{\delta}\widetilde{y})_{ru}d\omega_r\\ &\quad + \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)ES(r-u)\widetilde{y}_u d\omega_r - \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\widetilde{y}_s d\omega_r.\end{aligned}$$

Since $(\hat{\delta}\widetilde{y})_{ru} = \int_u^r S(r-q)G(y_q)d\omega_q$ we have

$$\begin{aligned}
\widetilde{z}_{ts}(E) &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_u^r S(r-q)G(y_q)d\omega_q d\omega_r \\
&\quad + \sum_{[u,v]\in\mathcal{P}} S(t-v)a_{vu}(E,\widetilde{y}_u) - \omega^S_{ts}(E\widetilde{y}_s) \\
&\approx \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_u^r S(r-q)G(y_u)d\omega_q d\omega_r \\
&\quad + \sum_{[u,v]\in\mathcal{P}} S(t-v)a_{vu}(E,\widetilde{y}_u) - \omega^S_{ts}(E\widetilde{y}_s) \\
&=: \sum_{[u,v]\in\mathcal{P}} S(t-v)\left[b_{vu}(E,G(y_u)) + a_{vu}(E,\widetilde{y}_u)\right] - \omega^S_{ts}(E\widetilde{y}_s).
\end{aligned}$$

Here we introduced the notation

$$\begin{aligned}
a_{vu}(E,\widetilde{y}_u) &:= \int_u^v S(v-r)ES(r-u)\widetilde{y}_u d\omega_r, \qquad \text{and} \\
b_{vu}(E,G(y_u)) &:= \int_u^v S(v-r)E\int_u^r S(r-q)G(y_u)d\omega_q d\omega_r.
\end{aligned}$$

Hence, we set

$$\begin{aligned}
&\Xi^{(z)}(y,\widetilde{y})_{vu}(E) := b_{vu}(E,G(y_u)) + a_{vu}(E,\widetilde{y}_u), \\
&\widetilde{z}_{ts}(E) := (\hat{\delta}\mathcal{I}\Xi^{(z)}(y,\widetilde{y}))_{ts}(E) - \omega^S_{ts}(E\widetilde{y}_s).
\end{aligned}$$

This means that we have to define a, b and ω^S in order to describe $\widetilde{z}$. At the very first sight, it is not straightforward under which assumptions b is well-defined, compare Remark 4.3 in [10]. This problem will be addressed in Section 3.4. For the sake of completeness we provide here a possible heuristic definition of b which will be shown to be the right one in Section 3.4. For a smooth path ω and a placeholder K which stands for $G(y_\cdot)$ we have

$$\begin{aligned}
\int_s^t S(t-r)E\int_s^r S(r-q)Kd\omega_q d\omega_r &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_s^r S(r-q)Kd\omega_q d\omega_r \\
&= \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_s^u S(r-q)Kd\omega_q d\omega_r \\
&\quad + \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_u^r S(r-q)Kd\omega_q d\omega_r \\
&\approx \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_s^u S(u-q)Kd\omega_q d\omega_r \\
&\quad + \sum_{[u,v]\in\mathcal{P}} S(t-v)\int_u^v S(v-r)E\int_u^r Kd\omega_q d\omega_r \\
&=: \sum_{[u,v]\in\mathcal{P}} S(t-v)\left[\omega^S_{vu}(E\omega^S_{us}(K)) + c_{vu}(E,K)\right], \qquad (3.16)
\end{aligned}$$

where

$$c_{ts}(E,K) := \int_s^t S(t-r)EK(\delta\omega)_{rs}d\omega_r.$$

The existence of a, b, c and of all other auxiliary processes required to give meaning to (1.2) will be justified in Section 3.4. Motivated by this heuristic computations we first define similar to [27] these processes for smooth paths ω^n approximating ω. Thereafter passing to the limit entails a suitable construction/interpretation of all these expressions.

To conclude this heuristic computations, we introduce the following definition of a solution for (1.1) (compare (3.10)). This is the counterpart of the solution concepts investigated in [42] and [25]. Recalling that ω is still assumed to be a smooth path we have.

Definition 3.4. We call a pair (y,z) a mild solution for (1.1) if

$$\begin{aligned} y_t &= S(t)\xi + \mathcal{I}\Xi^{(y)}(y,z)_t \\ &= S(t)\xi + \lim_{|\mathcal{P}(0,t)|\to 0} \sum_{[u,v]\in\mathcal{P}(0,t)} S(t-v)[\omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u))] \end{aligned} \tag{3.17}$$

$$\begin{aligned} z_{ts}(E) &= (\hat{\delta}\mathcal{I}\Xi^{(z)}(y,y))_{ts}(E) - \omega^S_{ts}(Ey_s) \\ &= \lim_{|\mathcal{P}(s,t)|\to 0} \sum_{[u,v]\in\mathcal{P}(s,t)} S(t-v)[b_{vu}(E,G(y_u)) + a_{vu}(E,y_u)] - \omega^S_{ts}(Ey_s). \end{aligned} \tag{3.18}$$

Our aim is to rigorously justify this solution theory for a rough path $\boldsymbol{\omega} = (\omega, \omega^{(2)})$ with $1/3 < \alpha \leq 1/2$. This will be carried out in Section 3.5 by means of a fixed-point argument in a suitable function space which is specially designed to incorporate the analytic and algebraic properties of the solution pair (y,z).

Finally, we introduce further notation which will turn out to be useful for the computations in Section 3.5.

Remark 3.5. *Going back to the definition of $\widetilde{z}$, recalling (3.15), applying the bilinearity of a and the linearity of ω^S entails*

$$\begin{aligned} \widetilde{z}_{ts}(E) \approx \sum_{[u,v]\in\mathcal{P}} & S(t-v)\left[b_{vu}(E,G(y_u)) + a_{vu}(E,\overline{y}_u) + a_{vu}(E,S(u)\xi)\right] \\ & - \omega^S_{ts}(E\overline{y}_s) - \omega^S_{ts}(ES(s)\xi). \end{aligned}$$

As justified in Section 3.4 (Corollary 3.22) we get

$$\begin{aligned} \widetilde{z}_{ts}(E) \approx \sum_{[u,v]\in\mathcal{P}} & S(t-v)\left[b_{vu}(E,G(y_u)) + a_{vu}(E,\overline{y}_u)\right] - \omega^S_{ts}(E\overline{y}_s) \\ & + a_{ts}(E,S(s)\xi) - \omega^S_{ts}(ES(s)\xi). \end{aligned}$$

Therefore, we can define

$$\begin{aligned} &\Xi^{(z)}(y,\overline{y})_{vu}(E) := b_{vu}(E,G(y_u)) + a_{vu}(E,\overline{y}_u), \\ &\overline{z}_{ts}(E) := (\hat{\delta}\mathcal{I}\Xi^{(z)}(y,\overline{y}))_{ts}(E) - \omega^S_{ts}(E\overline{y}_s), \quad \textit{which yields} \\ &\widetilde{z}_{ts}(E) = \overline{z}_{ts}(E) + a_{ts}(E,S(s)\xi) - \omega^S_{ts}(ES(s)\xi). \end{aligned}$$

In Section 3.5 we will see that it is more convenient to estimate $\overline{z}$ than $\widetilde{z}$.

3.3 The Sewing Lemma

The following result is crucial for our work, since it gives us the existence of the rough integral together with all the necessary properties required to solve (1.1). Since we work with the weighted Hölder spaces introduced in Section 2.1 we have to extend the results obtained in [37, Section 3] using similar techniques. The next statement is the analogue of Theorem 3.5 in [37] in our framework.

Theorem 3.6 (Sewing Lemma). *Let W be a separable Banach space and $(S(t))_{t\geq 0}$ be an analytic C_0-semigroup on W with $\|S(t)\|_{\mathcal{L}(W)} \leq c_S$ for all $t \leq T$. Furthermore, let $\Xi \in C(\Delta_T, W)$ be an approximation term satisfying the following properties for all $0 \leq u \leq m \leq v \leq T$:*

$$|\Xi_{vu}| \leq c_1 (v-u)^{\alpha}, \tag{3.19}$$

$$\left|\left(\hat{\delta}_2 \Xi\right)_{vmu}\right| \leq c_2 u^{-\beta} (v-u)^{\nu}, \quad \text{for } u \neq 0. \tag{3.20}$$

Here we impose $0 < \alpha, \beta \leq 1$, $\nu > 1$ and $\alpha + \beta \leq \nu$.

Then there exists a unique $\mathcal{I}\Xi \in C([0,T], W)$, such that

$$\mathcal{I}\Xi_0 = 0, \tag{3.21}$$

$$\left|\left(\hat{\delta}\mathcal{I}\Xi\right)_{ts}\right| \leq C (c_1 + c_2) (t-s)^{\alpha} \tag{3.22}$$

$$\left|(\hat{\delta}\mathcal{I}\Xi)_{ts} - \Xi_{ts}\right| \leq C c_2 s^{-\beta} (t-s)^{\nu}, \quad \text{for } s \neq 0. \tag{3.23}$$

Proof. Firstly, note that the uniqueness of $\mathcal{I}\Xi$ immediately follows from Lemma A.2. Assuming by contradiction that there are two candidates $\mathcal{I}^1$ and $\mathcal{I}^2$ for a given Ξ, we have

$$\begin{aligned}
\mathcal{I}^1_0 - \mathcal{I}^2_0 &= 0, \\
\left|(\hat{\delta}(\mathcal{I}^1 - \mathcal{I}^2))_{ts}\right| &\leq C (c_1 + c_2) (t-s)^{\alpha}, \\
\left|(\hat{\delta}(\mathcal{I}^1 - \mathcal{I}^2))_{ts}\right| &\leq C c_2 s^{-\beta} (t-s)^{\nu}, \quad \text{for } s \neq 0.
\end{aligned}$$

Hence, Lemma A.2 implies that $\mathcal{I}^1 \equiv \mathcal{I}^2$.

The following deliberations are conducted in order to prove the existence of $\mathcal{I}\Xi$. To this aim, given $0 \leq s < t \leq T$, we let $\mathcal{P}_n = \mathcal{P}_n(s,t)$ be the n-th dyadic partition of $[s,t]$ for $n \in \mathbb{N}_0$ and define

$$\begin{aligned}
N^n_{ts} &:= \sum_{[u,v]\in\mathcal{P}_n} S(t-v)\Xi_{vu}, \\
M^n_{ts} &:= \Xi_{ts} - N^n_{ts}.
\end{aligned}$$

Note that $N^0_{ts} = \Xi_{ts}$ which implies that $M^0_{ts} = 0$.

Furthermore, setting $m := \frac{u+v}{2}$, we derive

$$\begin{aligned}
N^n_{ts} - N^{n+1}_{ts} = M^{n+1}_{ts} - M^n_{ts} &= \sum_{[u,v]\in\mathcal{P}_n} \left(S(t-v)\Xi_{vu} - S(t-v)\Xi_{vm} - S(t-m)\Xi_{mu}\right) \\
&= \sum_{[u,v]\in\mathcal{P}_n} S(t-v)(\hat{\delta}_2\Xi)_{vmu}.
\end{aligned}$$

Hence, we obtain

$$\left|M^n_{ts} - M^{n+1}_{ts}\right| \leq C \sum_{[u,v]\in\mathcal{P}_n} \left|(\hat{\delta}_2\Xi)_{vmu}\right|. \tag{3.24}$$

Since we also have to deal with the case $s = 0$, we apply (3.19) to the first term and use (3.20) to estimate the other terms in (3.24). This further entails

$$\begin{aligned}\left|M^n_{ts} - M^{n+1}_{ts}\right| \leq & C c_1 (t-s)^{\alpha} 2^{-n\alpha} + C \sum_{\substack{[u,v]\in\mathcal{P}_n \\ u\neq s}} c_2 u^{-\beta} (t-s)^{\nu} 2^{-n\nu} \\ \leq & C c_1 (t-s)^{\alpha} 2^{-n\alpha} + C c_2 (t-s)^{\nu-1} 2^{-n(\nu-1)} \sum_{\substack{[u,v]\in\mathcal{P}_n \\ u\neq s}} u^{-\beta}(t-s)2^{-n} \\ \leq & C c_1 (t-s)^{\alpha} 2^{-n\alpha} + C c_2 (t-s)^{\nu-1} 2^{-n(\nu-1)} \int_s^t q^{-\beta} dq \\ \leq & C c_1 (t-s)^{\alpha} 2^{-n\alpha} + C c_2 (t-s)^{\nu-\beta} 2^{-n(\nu-1)}.\end{aligned}$$

Since this expression is summable, we conclude that $M^n_{ts} \to M_{ts}$ as $n \to \infty$ for all $0 \leq s \leq t \leq T$. The previous computations give us the estimate

$$|M_{ts}| \leq C (c_1 + c_2) (t-s)^{\alpha}. \tag{3.25}$$

Note that this is valid due to the fact that $\alpha + \beta \leq \nu$.

Setting $N_{ts} := \Xi_{ts} - M_{ts}$ immediately entails $N^n_{ts} \to N_{ts}$ as $n \to \infty$ for all $0 \leq s \leq t \leq T$ and

$$|N_{ts}| \leq C (c_1 + c_2) (t-s)^{\alpha}. \tag{3.26}$$

Furthermore, this also yields that $(\hat{\delta}_2 N) \equiv 0$.

To prove this statement, note that it is equivalent to show that

$$(\hat{\delta}_2 N^n)_{t\tau s} \to 0, \quad \text{as } n \to 0,$$

for all $0 \leq s \leq \tau \leq t \leq T$.

To this aim we consider a fixed interval $[u, v] \in \mathcal{P}_n(s, t)$ and a finite number of nodes $u = r_0 < r_1 < r_2 < \ldots < r_k < r_{k+1} = v$, for $k \in \mathbb{N}$. Then,

$$S(t-v)\Xi_{vu} - \sum_{j=0}^{k} S(t - r_{j+1})\Xi_{r_{j+1}r_j} = \sum_{j=0}^{k-1} S(t-v)(\hat{\delta}_2\Xi)_{v r_{j+1} r_j}.$$

Hence, we estimate

$$\left| S(t-v)\Xi_{vu} - \sum_{j=0}^{k} S(t - r_{j+1})\Xi_{r_{j+1}r_j} \right| \leq \begin{cases} C c_2 k u^{-\beta}(v-u)^{\nu}, & u > s \\ C c_1 k (v-u)^{\alpha}, & u = s. \end{cases} \tag{3.27}$$

For fixed $0 \leq s \leq \tau \leq t \leq T$ we define $\pi_n(s,t) := \mathcal{P}_n(s,t) \cup \mathcal{P}_n(s,\tau) \cup \mathcal{P}_n(\tau,t)$ and $\pi_n(s,\tau) := \pi_n(s,t) \cap [s,\tau]$, $\pi_n(\tau,t) := \pi_n(s,t) \cap [\tau,t]$.

We define $N^{\pi_n}_{ts}$, $N^{\pi_n}_{t\tau}$ and $N^{\pi_n}_{\tau s}$ analogously to N^n, i.e.

$$N^{\pi_n}_{ts} := \sum_{[u,v]\in\pi_n(s,t)} S(t-v)\Xi_{vu}.$$

Since $\pi_n(s,t) = \pi_n(s,\tau) \cup \pi_n(\tau,t)$ one obtains that $(\hat{\delta}_2 N^{\pi_n})_{t\tau s} = 0$. Consequently, we estimate

$$\left|(\hat{\delta}_2 N^n)_{t\tau s}\right| \leq |N^n_{ts} - N^{\pi_n}_{ts}| + |N^n_{t\tau} - N^{\pi_n}_{t\tau}| + |S(t-\tau)\,(N^n_{\tau s} - N^{\pi_n}_{\tau s})|.$$

Therefore, it is left to show that all three summands tend to zero as $n \to \infty$. To this aim consider an arbitrary interval $[u,v] \in \mathcal{P}_n(s,t)$ and let $k := \left\lfloor \frac{t-s}{\tau-s} \right\rfloor \vee \left\lfloor \frac{t-s}{t-\tau} \right\rfloor$. Then there are at most $k+1$ many nodes of π_n between u and v. Thus, by (3.27) we have

$$|N^n_{ts} - N^{\pi_n}_{ts}| \leq C(k+1)(c_1+c_2)\left(\left(\frac{t-s}{2^n}\right)^\alpha + \sum_{\substack{[u,v]\in\mathcal{P}_n(s,t)\\ u\neq s}} u^{-\beta}\left(\frac{t-s}{2^n}\right)^\nu\right) \to 0, \text{ as } n\to\infty.$$

The other summands tend to zero analogously.

Since $(\hat{\delta}_2 N) \equiv 0$ we can apply Lemma 2.4. This ensures the unique existence of $\mathcal{I}\Xi \in C([0,T],W)$ such that

$$\begin{aligned} \mathcal{I}\Xi_0 &= 0, \\ (\hat{\delta}\mathcal{I}\Xi)_{ts} &= N_{ts} = \Xi_{ts} - M_{ts}, \text{ for all } 0 \leq s < t \leq T. \end{aligned} \tag{3.28}$$

Hence, (3.26) implies

$$\left|(\hat{\delta}\mathcal{I}\Xi)_{ts}\right| \leq C\,(c_1+c_2)\,(t-s)^\alpha .$$

We now show (3.23). To this aim, if $s>0$ we apply (3.20) to all summands in (3.24) and obtain

$$\begin{aligned} \left|M^n_{ts} - M^{n+1}_{ts}\right| \leq& C \sum_{[u,v]\in\mathcal{P}_n} c_2 u^{-\beta}\,(t-s)^\nu\, 2^{-n\nu} \\ \leq& C c_2 s^{-\beta}\,(t-s)^\nu\, 2^{-n(\nu-1)}. \end{aligned}$$

Consequently,

$$|M_{ts}| \leq C c_2 s^{-\beta}\,(t-s)^\nu\,, \tag{3.29}$$

which yields (3.23). ☐

The following result gives us an additional estimate necessary for the fixed-point argument.

Corollary 3.7. *Additionally to the assumptions of Theorem 3.6, we let*

$$\left|\left(\hat{\delta}_2\Xi\right)_{vmu}\right| \leq c_3 u^{-\beta'}\,(v-u)^{\nu'}\,, \tag{3.30}$$

with $0 \leq \beta,\ \beta' \leq 1$ *and* $\nu' - \beta' \leq \nu - \beta$.

Then it holds

$$\left|(\hat{\delta}\mathcal{I}\Xi)_{ts} - \Xi_{ts}\right| \leq C(c_2+c_3)s^{-\beta'}\,(t-s)^{\nu'} . \tag{3.31}$$

Proof. The proof is analogous to the previous one. Recalling that

$$\left|M^n_{ts} - M^{n+1}_{ts}\right| \leq C \sum_{[u,v]\in\mathcal{P}_n} \left|(\hat{\delta}_2\Xi)_{vmu}\right|,$$

we apply (3.30) to the first summand and again (3.20) to the other terms. This leads to

$$\begin{aligned} \left|M^n_{ts} - M^{n+1}_{ts}\right| \leq& C c_3 s^{-\beta'}\,(t-s)^{\nu'}\, 2^{-n\nu'} + C \sum_{\substack{[u,v]\in\mathcal{P}_n\\ u\neq s}} c_2 u^{-\beta}\,(t-s)^\nu\, 2^{-n\nu} \\ \leq& C c_3 s^{-\beta'}\,(t-s)^{\nu'}\, 2^{-n\nu'} + C c_2 s^{-\beta'}\,(t-s)^{\nu-1}\, 2^{-n(\nu-1)} \sum_{\substack{[u,v]\in\mathcal{P}_n\\ u\neq s}} u^{-(\beta-\beta')}(t-s)2^{-n} \\ \leq& C c_3 s^{-\beta'}\,(t-s)^{\nu'}\, 2^{-n\nu'} + C c_2 s^{-\beta'}\,(t-s)^{\nu-1}\, 2^{-n(\nu-1)} \int_s^t q^{-(\beta-\beta')}dq \\ \leq& C c_3 s^{-\beta'}\,(t-s)^{\nu'}\, 2^{-n\nu'} + C c_2 s^{-\beta'}\,(t-s)^{\nu-\beta+\beta'}\, 2^{-n(\nu-1)}. \end{aligned}$$

Since $\nu - \beta + \beta' \geq \nu'$ we have

$$|M_{ts}| \leq C(c_2 + c_3)s^{-\beta'}(t-s)^{\nu'}.$$

□

In order to give a meaning to $\mathcal{I}\Xi$ as a rough integral we firstly describe it as a limit of finite sums, compare Corollary 3.6 in [35]. Note that in our case technical difficulties occur in the proof due to (3.29).

Corollary 3.8. *Under the assumptions of Theorem 3.6 it holds that*

$$\left(\hat{\delta}\mathcal{I}\Xi\right)_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} S(t-v)\Xi_{vu}, \tag{3.32}$$

where $|\mathcal{P}|$ stands for the mesh of the given partition $\mathcal{P} = \mathcal{P}(s,t)$.

Proof. Consider an arbitrary partition $\mathcal{P}$ of $[s,t]$. Then we have by (3.28) that

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} S(t-v)\Xi_{vu} &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\left((\hat{\delta}\mathcal{I}\Xi)_{vu} + M_{vu}\right) \\ &= (\hat{\delta}\mathcal{I}\Xi)_{ts} + \sum_{[u,v]\in\mathcal{P}} S(t-v)M_{vu}.\end{aligned}$$

Therefore, it is left to show that

$$\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} S(t-v)M_{vu} = 0.$$

To this aim, we prove the sufficient statement

$$\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} |M_{vu}| = 0.$$

As concluded within the proof of Theorem 3.6 we have two estimates for M, recall (3.25) and (3.29). Namely we obtained that

$$\begin{aligned}|M_{vu}| &\leq C(c_1 + c_2)(v-u)^{\alpha}, \\ |M_{vu}| &\leq Cc_2 u^{-\beta}(v-u)^{\nu}, \quad \text{for } u \neq 0.\end{aligned}$$

Clearly, which one of them is more restrictive depends on the relation between u and $v-u$.

Hence, we introduce $\widetilde{\mathcal{P}} := \{[u,v] \in \mathcal{P} \colon u < v - u\}$. We order the intervals of $\widetilde{\mathcal{P}}$ by their starting point and write $\widetilde{\mathcal{P}} = \{[\widetilde{u}_k, \widetilde{v}_k] : k = 1, \ldots, m\}$, where $s \leq \widetilde{u}_1 < \widetilde{v}_1 \leq \widetilde{u}_2 < \widetilde{v}_2 \leq \ldots \leq \widetilde{u}_m < \widetilde{v}_m \leq t$.

For $k = 1, \ldots m-1$ we get

$$\widetilde{u}_k < \frac{\widetilde{v}_k}{2} \leq \frac{\widetilde{u}_{k+1}}{2},$$

which yields

$$\widetilde{u}_k < \widetilde{u}_m\, 2^{-(m-l)} < (\widetilde{v}_m - \widetilde{u}_m)\, 2^{-(m-l)} \leq |\mathcal{P}|\, 2^{-(m-l)}.$$

All in all this means that

$$\widetilde{v}_k - \widetilde{u}_k \leq \widetilde{v}_k \leq \widetilde{u}_{k+1} \leq |\mathcal{P}|\, 2^{-(m-l-1)}.$$

For $k=m$ we trivially have $\widetilde{v}_m - \widetilde{u}_m \le |\mathcal{P}| \le 2\,|\mathcal{P}|$. Hence, by using (3.25) we derive

$$\sum_{[u,v]\in\widetilde{\mathcal{P}}} |M_{vu}| \le C(c_1+c_2) \sum_{k=1}^{m} (\widetilde{v}_k - \widetilde{u}_k)^{\alpha} \le C(c_1+c_2)\,|\mathcal{P}|^{\alpha} \sum_{k=1}^{m} 2^{-(m-l-1)\alpha} \le C(c_1+c_2)\,|\mathcal{P}|^{\alpha}.$$

If $[u,v]\in\mathcal{P}\backslash\widetilde{\mathcal{P}}$ we have

$$v-u\le u, \text{ so } v\le 2u, \text{ therefore, } u^{-\beta}\le 2^{\beta}v^{-\beta}.$$

So, by applying (3.29) we infer

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}\backslash\widetilde{\mathcal{P}}} |M_{vu}| &\le Cc_2 \sum_{[u,v]\in\mathcal{P}\backslash\tilde{\mathcal{P}}} u^{-\beta}\,(v-u)^{\nu}\\ &\le Cc_2\,|\mathcal{P}|^{\nu-1} \sum_{[u,v]\in\mathcal{P}\backslash\tilde{\mathcal{P}}} v^{-\beta}\,(v-u)\\ &\le Cc_2\,|\mathcal{P}|^{\nu-1} \int_s^t q^{-\beta}dq\\ &\le Cc_2\,(t-s)^{1-\beta}\,|\mathcal{P}|^{\nu-1}.\end{aligned}$$

Consequently, putting both estimates together we have

$$\begin{aligned}\sum_{[u,v]\in\mathcal{P}} |M_{vu}| &\le \sum_{[u,v]\in\widetilde{\mathcal{P}}} |M_{vu}| + \sum_{[u,v]\in\mathcal{P}\backslash\widetilde{\mathcal{P}}} |M_{vu}|\\ &\le C(c_1+c_2)\,|\mathcal{P}|^{\alpha} + Cc_2\,(t-s)^{1-\beta}\,|\mathcal{P}|^{\nu-1},\end{aligned}$$

which tends to 0 as $|\mathcal{P}|\to 0$. This proves the statement. □

Remark 3.9. *Note that the above limit is independent of the approximating sequence of partitions. Hence, Corollary 3.8 implies the additivity of the rough integral.*

In order to introduce the shift property of the rough integral $\mathcal{I}\Xi$ we recall that for $\tau>0$

$$\widetilde{\theta}_{\tau}\Xi_{vu} = \Xi_{v+\tau,u+\tau},$$

see Section 2.2. Considering this, one can easily verify the shift property of $\mathcal{I}\Xi$.

Lemma 3.10. *Under the assumptions of Theorem 3.6 we have*

$$(\hat{\delta}\mathcal{I}\Xi)_{ts} = (\hat{\delta}\mathcal{I}\widetilde{\theta}_{\tau}\Xi)_{t-\tau,s-\tau}, \quad \textit{for } \tau\le s\le t.$$

Proof. The proof is a direct consequence of Corollary 3.8.

$$\begin{aligned}(\hat{\delta}\mathcal{I}\Xi)_{ts} &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(s,t)} S(t-v)\Xi_{vu}\\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(s-\tau,t-\tau)} S(t-\tau-v)\Xi_{v+\tau,u+\tau}\\ &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(s-\tau,t-\tau)} S(t-\tau-v)\widetilde{\theta}_{\tau}\Xi_{vu}\\ &= (\hat{\delta}\mathcal{I}\widetilde{\theta}_{\tau}\Xi)_{t-\tau,s-\tau}.\end{aligned}$$

□

The next result contains necessary estimates for $\hat{\delta}\mathcal{I}\Xi$ in a suitable fractional domain. These will be required later on in Corollary 3.12 to estimate $\delta\mathcal{I}\Xi$ and eventually in Section 3.6 for deriving a global solution.

Corollary 3.11. *Let the assumptions of Theorem 3.6 hold true and further assume that*

$$|S(v-u)\Xi_{vu}|_{D_\varepsilon} \leq c_1' \, (v-u)^{\alpha'} \,, \tag{3.33}$$

where $0 < \alpha' \leq 1$ *and* $0 \leq \varepsilon < (\nu - \beta) \wedge 1$*. Then we have*

$$\left|(\hat{\delta}\mathcal{I}\Xi)_{ts}\right|_{D_\varepsilon} \leq C \left(c_1'(t-s)^{\alpha'} + c_2(t-s)^{\nu-\beta-\varepsilon} \right). \tag{3.34}$$

In particular, under the assumptions of Theorem 3.6 we have

$$\left|\left(\hat{\delta}\mathcal{I}\Xi\right)_{ts}\right|_{D_\varepsilon} \leq C\,(c_1+c_2)\,(t-s)^{\alpha-\varepsilon}\,, \qquad \textit{for all } 0 \leq \varepsilon < \alpha. \tag{3.35}$$

Proof. Analogously to the proof of Theorem 3.6 we introduce

$$\widetilde{N}^n_{ts} := \sum_{\substack{[u,v]\in\mathcal{P}_n \\ v\neq t, u\neq s}} S(t-v)\Xi_{vu},$$
$$\widetilde{M}^n_{ts} := \Xi_{ts} - \widetilde{N}^n_{ts},$$

where $\mathcal{P}_n$ is the n-th dyadic partition of $[s,t]$. Similarly to the proof of Theorem 3.6 we have $\widetilde{N}^0_{ts} = \widetilde{N}^1_{ts} = 0$ which means that $\widetilde{M}^0_{ts} = \widetilde{M}^1_{ts} = \Xi_{ts}$.

We further set $\overline{v}_n := \max\{v < t \colon [u,v] \in \mathcal{P}_n\}$ and $\overline{u}_n := \min\{u > s \colon [u,v] \in \mathcal{P}_n\}$. Then we derive for $n \geq 1$

$$\begin{aligned}
\widetilde{N}^n_{ts} - \widetilde{N}^{n+1}_{ts} &= \widetilde{M}^n_{ts} - \widetilde{M}^{n+1}_{ts} \\
&= N^n_{ts} - N^{n+1}_{ts} - \Xi_{t\overline{v}_n} - S(t-\overline{u}_n)\Xi_{\overline{u}_n s} + \Xi_{t\overline{v}_{n+1}} + S(t-\overline{u}_n n+1)\Xi_{\overline{u}_n n+1 s} \\
&= \sum_{\substack{[u,v]\in\mathcal{P}_n \\ v\neq t, u\neq s}} S(t-v)(\hat{\delta}_2\Xi)_{vmu} + (\hat{\delta}\Xi)_{t\overline{v}_{n+1}\overline{v}_n} + S(t-\overline{u}_n)(\hat{\delta}\Xi)_{\overline{u}_n\overline{u}_{n+1}s} \\
&\quad - \Xi_{t\overline{v}_n} + \Xi_{t\overline{v}_{n+1}} - S(t-\overline{u}_n)\Xi_{\overline{u}_n s} + S(t-\overline{u}_{n+1})\Xi_{\overline{u}_{n+1}s} \\
&= \sum_{\substack{[u,v]\in\mathcal{P}_n \\ v\neq t, u\neq s}} S(t-v)(\hat{\delta}_2\Xi)_{vmu} - S(t-\overline{v}_{n+1})\Xi_{\overline{v}_{n+1}\overline{v}_n} - S(t-\overline{u}_n)\Xi_{\overline{u}_n\overline{u}_{n+1}}.
\end{aligned}$$

This yields

$$\left|\widetilde{N}^n_{ts} - \widetilde{N}^{n+1}_{ts}\right|_{D_\varepsilon} \leq \left|S(t-\overline{u}_n)\Xi_{\overline{u}_n\overline{u}_{n+1}}\right|_{D_\varepsilon} + \left|S(t-\overline{v}_{n+1})\Xi_{\overline{v}_{n+1}\overline{v}_n}\right|_{D_\varepsilon} + C \sum_{\substack{[u,v]\in\mathcal{P}_n \\ v\neq t, u\neq s}} (t-v)^{-\varepsilon}\left|(\hat{\delta}_2\Xi)_{vmu}\right|.$$

Note that $t - \overline{u}_n \geq t - \overline{v}_{n+1} = \overline{v}_{n+1} - \overline{v}_n = \overline{u}_n - \overline{u}_{n+1}$ and apply (3.33) to the first two summands to obtain.

$$\left|S(t-\overline{u}_n)\Xi_{\overline{u}_n\overline{u}_{n+1}}\right|_{D_\varepsilon} + \left|S(t-\overline{v}_{n+1})\Xi_{\overline{v}_{n+1}\overline{v}_n}\right|_{D_\varepsilon} \leq 2\,c_1'(t-s)^{\alpha'}2^{-n\alpha'}.$$

For last summand we apply (3.20) which entails

$$\begin{aligned}
\sum_{\substack{[u,v]\in\mathcal{P}_n \\ v\neq t, u\neq s}} (t-v)^{-\varepsilon}\left|(\hat{\delta}_2\Xi)_{vmu}\right| &\leq c_2 \sum_{\substack{[u,v]\in\mathcal{P}_n \\ u\neq s, v\neq t}} (t-v)^{-\varepsilon}\, u^{-\beta}(t-s)^{\nu}2^{-n\nu} \\
&= c_2(t-s)^{\nu}2^{-n\nu} \sum_{\substack{[u,v]\in\mathcal{P}_n \\ u\neq s, v\neq t}} (t-v)^{-\varepsilon}\, u^{-\beta}.
\end{aligned}$$

We now have to estimate the term

$$
\begin{aligned}
J_{ts} &:= \sum_{\substack{[u,v]\in\mathcal{P}_n \\ u\neq s, v\neq t}} (t-v)^{-\varepsilon}\, u^{-\beta} \\
&= \sum_{k=1}^{2^n-2} \left(s+\frac{k(t-s)}{2^n}\right)^{-\beta} \left(t-s-\frac{(k+1)(t-s)}{2^n}\right)^{-\varepsilon} \\
&\leq (t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{k=1}^{2^n-2} k^{-\beta}\,(2^n-1-k)^{-\varepsilon}.
\end{aligned}
$$

By Lemma A.3 we obtain

$$
\begin{aligned}
J_{ts} &\leq (t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{k=1}^{2^n-2} k^{-\beta}\,(2^n-1-k)^{-\varepsilon} \\
&\leq C\,(t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{k=0}^{2^n-2} (k+1)^{-\beta}\,(2^n-1-k)^{-\varepsilon} \\
&= C\,(t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{j=1}^{2^n-1} j^{-\varepsilon}\,(2^n-j)^{-\beta}.
\end{aligned}
$$

Using again Lemma A.3 entails

$$
\begin{aligned}
J_{ts} &\leq C\,(t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{j=1}^{2^n-1} j^{-\varepsilon}\,(2^n-j)^{-\beta} \\
&\leq C\,(t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \sum_{j=0}^{2^n-1} (j+1)^{-\varepsilon}\,(2^n-j)^{-\beta} \\
&\leq C\,(t-s)^{-\beta-\varepsilon}\, 2^{n(\beta+\varepsilon)} \int_0^{2^n} q^{-\varepsilon}(2^n-q)^{-\beta}dq \\
&= C\,(t-s)^{-\beta-\varepsilon}\, 2^n B(1-\varepsilon, 1-\beta).
\end{aligned}
$$

Here, $B(\cdot,\cdot)$ stands for the Euler Beta function.

Consequently, this results in

$$
\left|\widetilde{N}^n_{ts}-\widetilde{N}^{n+1}_{ts}\right|_{D_\varepsilon} \leq Cc_1(t-s)^{\alpha'}2^{-n\alpha'} + Cc_2\,(t-s)^{\nu-\varepsilon-\beta}\,2^{-n(\nu-1)}. \tag{3.36}
$$

Since the right hand side is again summable, we obtain that $\widetilde{N}^n_{ts}\to\widetilde{N}_{ts}$ in D_ε as $n\to\infty$.

In order to obtain (3.34) we only have to show that $\widetilde{N}\equiv N$. We have

$$
\left|N_{ts}-\widetilde{N}_{ts}\right| = \lim_{n\to\infty}\left|N^n_{ts}-\widetilde{N}^n_{ts}\right| = \lim_{n\to\infty}\left|\Xi_{t\overline{v}_n}+S(t-\overline{u}_n)\Xi_{\overline{u}_n s}\right| \overset{(3.19)}{\leq} Cc_1\lim_{n\to\infty}\frac{(t-s)^\alpha}{2^{n\alpha}} = 0.
$$

Therefore, we conclude that $\overline{N}\equiv N$, which proves (3.34).
Now by (2.1) and (3.19) we see that for $\varepsilon<\alpha$

$$
|S(v-u)\Xi_{vu}|_{D_\varepsilon} \leq Cc_1(v-u)^{\alpha-\varepsilon}.
$$

Regarding that $\alpha\leq\nu-\beta$, (3.34) leads to (3.35). □

Now we can apply these results to estimate $\delta\mathcal{I}\Xi$.

Corollary 3.12. *Given the assumptions of Theorem 3.6. Then for all $\gamma < \alpha$ it holds*

$$|(\delta\mathcal{I}\Xi)_{ts}| \leq C(c_1 + c_2)\,(t-s)^{\gamma}\,T^{\alpha-\gamma}. \tag{3.37}$$

Proof. By applying (3.22) and (3.35) with $\varepsilon = \gamma$ we get

$$\begin{aligned}|(\delta\mathcal{I}\Xi)_{ts}| \leq & \left|\left(\hat{\delta}\mathcal{I}\Xi\right)_{ts}\right| + \left|(S(t-s) - \mathrm{Id})\,(\hat{\delta}\mathcal{I}\Xi)_{s0}\right| \\ \leq & C(c_1 + c_2)\,(t-s)^{\alpha} + C\,(t-s)^{\gamma}\left|(\hat{\delta}\mathcal{I}\Xi)_{s0}\right|_{D_\gamma} \\ \leq & C(c_1+c_2)\,(t-s)^{\alpha} + C(c_1+c_2)\,(t-s)^{\gamma}\,s^{\alpha-\gamma} \\ \leq & C(c_1+c_2)\,(t-s)^{\gamma}\,T^{\alpha-\gamma}.\end{aligned}$$

□

This immediately implies the next result.

Corollary 3.13. *Given the assumptions of Theorem 3.6. Then for all $\gamma < \alpha$ it holds*

$$\begin{aligned}\|\mathcal{I}\Xi\|_{\gamma} &\leq C(c_1+c_2)T^{\alpha-\gamma}, \\ \|\mathcal{I}\Xi\|_{\gamma,\gamma} &\leq C(c_1+c_2)T^{\alpha}.\end{aligned}$$

Remark 3.14. *Note that by construction $\mathcal{I}$ is a linear mapping. More precisely, according to [20, Section 4] or [36, Section 3.3] one can introduce the space $\hat{C}^{\alpha,\nu,\beta}(\Delta_T, W)$ of all elements Ξ satisfying assumptions* (3.19) *and* (3.20)*.*
Then one can show that the mapping $\mathcal{I}\colon \hat{C}^{\alpha,\nu,\beta}(\Delta_T, W) \to C^{\gamma,\gamma}([0,T], W)$, for $\gamma < \alpha$, is linear.
In particular, considering Ξ^1, Ξ^2 with

$$\begin{aligned}\left|\Xi^1_{vu} - \Xi^2_{vu}\right| &\leq \widetilde{c}_1\,(v-u)^{\alpha}, \\ \left|\left(\hat{\delta}_2\Xi^1\right)_{vmu} - \left(\hat{\delta}_2\Xi^2\right)_{vmu}\right| &\leq \widetilde{c}_2 u^{-\beta}\,(v-u)^{\nu}, \quad \text{for all } 0 < u \leq m \leq v \leq T,\end{aligned}$$

yields

$$\left\|\mathcal{I}\Xi^1 - \mathcal{I}\Xi^2\right\|_{\gamma,\gamma} = \left\|\mathcal{I}(\Xi^1 - \Xi^2)\right\|_{\gamma,\gamma} \leq C(\widetilde{c}_1 + \widetilde{c}_2)T^{\alpha}.$$

Finally, as announced in Section 3.1 we are able now to state further estimates for (1.2) in the Young integral case $\alpha > \frac{1}{2}$. The calculations done in the proof of Lemma 3.2 directly imply by Corollary 3.13

Corollary 3.15. *Given the assumptions of Lemma 3.2 we have the estimates*

$$\begin{aligned}\left\|\int_0^{\cdot} S(\cdot - r)G(y_r)d\omega_r\right\|_{\beta} &\leq C\,\|\omega\|_{\alpha}\,(1 + \|y\|_{\beta,\beta})T^{\alpha-\beta}, \\ \left\|\int_0^{\cdot} S(\cdot - r)G(y_r)d\omega_r\right\|_{\beta,\beta} &\leq C\,\|\omega\|_{\alpha}\,(1 + \|y\|_{\beta,\beta})T^{\alpha-\beta}.\end{aligned}$$

3.4 Construction of the Supporting Processes

We recall that $\boldsymbol{\omega} = (\omega, \omega^{(2)})$ is an α-Hölder rough path with $\alpha \in (\frac{1}{3}, \frac{1}{2})$, see Section 2.2.
Let $0 \leq s \leq \tau \leq t \leq T$ be fixed. As argued in Section 3.2, in order to introduce an infinite-dimensional rough integral we first need to define the following processes and investigate their algebraic and analytic properties. Recall that throughout this section K and E should be interpreted

as placeholder which stand for G, respectively DG. Keeping Section 3.2 in mind, we begin analyzing a, c and ω^S. More precisely,

$$\omega^S_{ts}\colon \mathcal{L}(V;W) \to W, \qquad \omega^S_{ts}(K) := \int_s^t S(t-r)K d\omega_r. \tag{3.38}$$

$$a_{ts}\colon \mathcal{L}(W\otimes V;W) \times W \to W, \qquad a_{ts}(E,x) := \int_s^t S(t-r)ES(r-s)x d\omega_r. \tag{3.39}$$

$$c_{ts}\colon \mathcal{L}(W\otimes V;W) \times \mathcal{L}(V;W) \to W, \qquad c_{ts}(E,K) := \int_s^t S(t-r)EK(\delta\omega)_{rs} d\omega_r. \tag{3.40}$$

Remark 3.16. *Note that some of the processes above exist even if ω is not smooth, as shown in the following deliberations. However, at the very first sight, it is not at all clear why for instance* (3.40) *is well-defined.*

Similar to [27] we consider a smooth approximating sequence $\left(\omega^n, \omega^{(2),n}\right) \to \left(\omega, \omega^{(2)}\right)$ in $C^\alpha\left([0,T], V\right) \times C^{2\alpha}\left(\Delta_T, V\otimes V\right)$, prove that the previous processes exist for this approximation terms and finally pass to the limit. Therefore, we analyze

$$\omega^{S,n}_{ts}(K) := \int_s^t S(t-r)K d\omega^n_r \tag{3.41}$$

$$a^n_{ts}(E,x) := \int_s^t S(t-r)ES(r-s)x d\omega^n_r \tag{3.42}$$

$$c^n_{ts}(E,K) := \int_s^t S(t-r)EK\left(\delta\omega^n\right)_{rs} d\omega^n_r. \tag{3.43}$$

In the following we establish algebraic and analytic properties which will be employed further on. We begin with the algebraic structure.

Lemma 3.17. *The properties*

$$(\hat{\delta}_2\omega^{S,n})_{t\tau s}(K) = 0 \tag{3.44}$$

$$(\hat{\delta}_2 a^n)_{t\tau s}(E,x) = a^n_{t\tau}(E,(S(\tau-s)-Id)x) \tag{3.45}$$

$$(\hat{\delta}_2 c^n)_{t\tau s}(E,K) = \omega^{S,n}_{t\tau}(EK(\delta\omega)_{\tau s}) \tag{3.46}$$

are satisfied.

Proof. One can easily verify that

$$(\hat{\delta}_2\omega^{S,n})_{t\tau s}(K) = \int_s^t S(t-r)K d\omega^n_r - \int_\tau^t S(t-r)K d\omega^n_r - \int_s^\tau S(t-r)K d\omega^n_r = 0.$$

Furthermore,

$$
\begin{aligned}
(\hat{\delta}_2 a^n)_{t\tau s}(E,x) &= \int_s^t S(t-r)ES(r-s)x d\omega_r^n - \int_\tau^t S(t-r)ES(r-\tau)x d\omega_r^n \\
&\quad - \int_s^\tau S(t-r)ES(r-s)x d\omega_r^n \\
&= \int_\tau^t S(t-r)E\left(S(r-s) - S(r-\tau)\right) x d\omega_r^n \\
&= \int_\tau^t S(t-r)ES(r-\tau)\left(S(\tau-s) - id\right) x d\omega_r^n \\
&= a_{t\tau}^n(E, (S(\tau-s) - \mathrm{Id})\, x).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
(\hat{\delta}_2 c^n)_{t\tau s}(E,K) &= \int_s^t S(t-r)EK(\delta\omega^n)_{rs} d\omega_r^n - \int_\tau^t S(t-r)EK(\delta\omega^n)_{r\tau} d\omega_r^n \\
&\quad - \int_s^\tau S(t-r)EK(\delta\omega^n)_{rs} d\omega_r^n \\
&= \int_\tau^t S(t-r)EK(\delta\omega^n)_{\tau s} d\omega_r^n \\
&= \omega_{t\tau}^{S,n}(EK(\delta\omega^n)_{\tau s}).
\end{aligned}
$$

□

The analytic estimates are contained in the next result. Throughout this section c_S stands for a constant which exclusively depends on the semigroup.

Lemma 3.18. *For the processes $\omega_{ts}^{S,n}$, a_{ts}^n and c_{ts}^n the following estimates hold true:*

$$\left|\omega_{ts}^{S,n}(K)\right| \leq C \left\|\omega^n\right\|_\alpha |K| (t-s)^\alpha \tag{3.47}$$

$$|a_{ts}^n(E,x)| \leq C \left\|\omega^n\right\|_\alpha |E| \, |x|_W (t-s)^\alpha, \quad \text{for } x \in W \tag{3.48}$$

$$\left|a_{ts}^n(E,x) - \omega_{ts}^{S,n}(Ex)\right| \leq C \left\|\omega^n\right\|_\alpha |E| \, |x|_{D_\beta} (t-s)^{\alpha+\beta}, \quad \text{for } x \in D_\beta \tag{3.49}$$

$$|c_{ts}^n(E,K)| \leq C \left(\left\|\omega^n\right\|_\alpha + \left\|\omega^{(2),n}\right\|_{2\alpha}\right) |E| \, |K| (t-s)^{2\alpha}. \tag{3.50}$$

Proof. Using the integration by parts formula, see Theorem 3.5 in [55], leads to

$$\omega_{ts}^{S,n}(K) = \int_s^t S(t-r)K d\omega_r^n = S(t-s)K(\delta\omega^n)_{ts} - A\int_s^t S(t-r)K(\delta\omega^n)_{tr} dr,$$

$$
\begin{aligned}
a_{ts}^n(E,x) &= \int_s^t S(t-r)ES(r-s)x d\omega_r^n = -\int_s^t \partial_r \omega_{tr}^{S,n}\left(ES(r-s)x\right) dr \\
&= \omega_{ts}^{S,n}(Ex) + \int_s^t \omega_{tr}^{S,n}\left(EAS(r-s)x\right) dr,
\end{aligned}
$$

and

$$c^n_{ts}(E,K) = \int_s^t S(t-r)EK\,(\delta\omega^n)_{rs} d\omega^n_r = \int_s^t S(t-r)EK(\delta\omega^n)_{ts} d\omega^n_r - \int_s^t S(t-r)EK(\delta\omega^n)_{tr} d\omega^n_r$$
$$= \omega^{S,n}_{ts}(EK(\delta\omega^n)_{ts}) - \int_s^t S(t-r)EK\; d\omega^{(2),n}_{tr}$$
$$= \omega^{S,n}_{ts}(EK\,(\delta\omega^n)_{ts}) - S(t-s)EK\,\omega^{(2),n}_{ts} - \int_s^t AS(t-r)EK\,\omega^{(2),n}_{tr} dr.$$

For a similar construction, see [10, Section 6.1]. Based on these identities we easily derive the analytic estimates as follows.

A standard computation immediately entails

$$\left|\omega^{S,n}_{ts}(K)\right| \le |S(t-s)K(\delta\omega^n)_{ts}| + \left|\int_s^t AS(t-r)K(\delta\omega^n)_{tr} dr\right|$$
$$\le c_S\,|K|\, \|\omega^n\|_\alpha\,(t-s)^\alpha\,.$$

Recalling (3.44) we infer that

$$|a^n_{ts}(E,x)| = \left|\omega^{S,n}_{ts}(Ex) + \int_s^t \omega^{S,n}_{tr}\,(EAS(r-s)x)dr\right|$$
$$= \left|\omega^{S,n}_{ts}(Ex) + \int_s^t \omega^{S,n}_{ts}\,(EAS(r-s)x)dr - \int_s^t S(t-r)\omega^{S,n}_{rs}\,(EAS(r-s)x)dr\right|$$
$$\le \left|\omega^{S,n}_{ts}(ES(t-s)x)\right| + \left|\int_s^t S(t-r)\omega^{S,n}_{rs}\,(EAS(r-s)x)dr\right|$$
$$\le c_S\,|E|\,|x|_W\, \|\omega^n\|_\alpha\,(t-s)^\alpha + c_S\,|E|\,|x|_W\, \|\omega^n\|_\alpha \int_s^t (r-s)^{\alpha-1}dr$$
$$\le c_S\,|E|\,|x|_W\, \|\omega^n\|_\alpha\,(t-s)^\alpha\,.$$

For our aims it is also necessary to derive estimates for $x \in D_\beta$ with $0 < \beta \le 1$. In this situation we have

$$\left|a^n_{ts}(E,x) - \omega^{S,n}_{ts}(Ex)\right| = \left|\int_s^t \omega^{S,n}_{tr}\,(EAS(r-s)x)dr\right|$$
$$\le c_S\,|E|\,|x|_{D_\beta}\, \|\omega^n\|_\alpha \int_s^t (t-r)^\alpha\,(r-s)^{\beta-1}dr$$
$$= c_S\,|E|\,|x|_{D_\beta}\, \|\omega^n\|_\alpha\,(t-s)^{\alpha+\beta}\,.$$

Furthermore, we obtain

$$\begin{aligned}|c^n_{ts}(E,K)| \leq& \left|\omega^{S,n}_{ts}(EK\,(\delta\omega^n)_{ts})\right| + \left|S(t-s)EK\,\omega^{(2),n}_{ts}\right| + \left|\int_s^t AS(t-r)EK\,\omega^{(2),n}_{tr}dr\right| \\ \leq& c_S\,|E|\,|K|\,\|\omega^n\|^2_\alpha\,(t-s)^{2\alpha} + c_S\,|E|\,|K|\left\|\omega^{(2),n}\right\|_{2\alpha}(t-s)^{2\alpha} \\ &+ c_S\,|E|\,|K|\left\|\omega^{(2),n}\right\|_{2\alpha}\int_s^t (t-r)^{2\alpha-1}dr \\ \leq& c_S\,|E|\,|K|\left(\|\omega^n\|^2_\alpha + \left\|\omega^{(2),n}\right\|_{2\alpha}\right)(t-s)^{2\alpha}.\end{aligned}$$

□

Consequently, keeping Lemma 3.18 in mind we are justified to define the supporting processes via

$$\omega^S_{ts}(K) := S(t-s)K(\delta\omega)_{ts} - A\int_s^t S(t-r)K(\delta\omega)_{tr}dr \tag{3.51}$$

$$a_{ts}(E,x) := \omega^S_{ts}(Ex) + \int_s^t \omega^S_{tr}\,(EAS(r-s)x)dr \tag{3.52}$$

$$c_{ts}(E,K) := \omega^S_{ts}(EK\,(\delta\omega)_{ts}) - S(t-s)EK\,\omega^{(2)}_{ts} - \int_s^t AS(t-r)EK\,\omega^{(2)}_{tr}dr. \tag{3.53}$$

Lemma 3.19. *We have that*

$$\begin{aligned}&\omega^{S,n} \to \omega^S \text{ in } C^\alpha([0,T],\mathcal{L}(\mathcal{L}(V;W),W)) \\ &a^n \to a \text{ in } C^\alpha([0,T],\mathcal{L}(\mathcal{L}(W\otimes V;W)\times W,W)) \\ &c^n \to c \text{ in } C^{2\alpha}([0,T],\mathcal{L}(\mathcal{L}(W\otimes V;W)\times\mathcal{L}(V;W),W)).\end{aligned}$$

Proof. Similarly to the proof of Lemma 3.18 we obtain

$$\begin{aligned}\left|\left(\omega^S-\omega^{S,n}\right)_{ts}(K)\right| =& \left|S(t-s)K(\delta(\omega-\omega^n))_{ts} - A\int_s^t S(t-r)K(\delta(\omega-\omega^n))_{tr}dr\right| \\ \leq& c_S\,\|\omega-\omega^n\|_\alpha\,|K|\,(t-s)^\alpha,\end{aligned}$$

which shows that $\omega^{S,n}\to\omega^S$ in $C^\alpha([0,T],\mathcal{L}(\mathcal{L}(V;W),W))$.

The same deliberations as in the proof of (3.49) lead to

$$\begin{aligned}|a_{ts}(E,x)-a^n_{ts}(E,x)| \leq& \left|\left(\omega^S-\omega^{S,n}\right)_{ts}(S(t-s)Ex)\right| \\ &+\left|\int_s^t S(t-r)\left(\omega^S-\omega^{S,n}\right)_{rs}(EAS(r-s)x)dr\right| \\ \leq& c_S\,|E|\,|x|\,\|\omega-\omega^n\|_\alpha\,(t-s)^\alpha.\end{aligned}$$

The last term yields

$$\begin{aligned}|c_{ts}(E,K)-c^n_{ts}(E,K)| \leq& \left|\omega^S_{ts}(EK\,(\delta\omega)_{ts}) - \omega^{S,n}_{ts}(EK\,(\delta\omega^n)_{ts})\right| \\ &+\left|S(v-u)EK\,\omega^{(2)}_{ts} - S(v-u)EK\,\omega^{(2),n}_{ts}\right| \\ &+\left|\int_s^t AS(t-r)EK\,\omega^{(2)}_{tr}dr - \int_s^t AS(t-r)EK\,\omega^{(2),n}_{tr}dr\right| \\ \leq& c_S\left(\|\omega\|_\alpha\,\|\omega-\omega^n\|_\alpha + \left\|\omega^{(2)}-\omega^{(2),n}\right\|_{2\alpha}\right)|E|\,|K|\,(t-s)^{2\alpha}.\end{aligned}$$

□

Remark 3.20. *Note that the algebraic and analytic properties proved in Lemmas 3.17 and 3.18 remain valid.*

The next statement gives an extension of (3.47) if one considers a regularizing placeholder K. The proof follows the same lines as the proof of Lemma 3.18.

Lemma 3.21. *Let $K \in \mathcal{L}(V; D_\gamma)$ for $0 \leq \gamma \leq 1$. Then*

$$\left|\omega^S_{ts}(K)\right|_{D_\gamma} \leq C \left\| \omega \right\|_\alpha |K|_{\mathcal{L}(V;D_\gamma)} (t-s)^\alpha$$

Proof. The proof can immediately be derived using (3.51). □

Furthermore, we observe.

Corollary 3.22. *For an arbitrary partition $\mathcal{P} = \mathcal{P}(s,t)$ the following identity holds true*

$$a_{ts}(E,x) = \sum_{[u,v]\in\mathcal{P}} S(t-v)a_{vu}(E, S(u-s)x).$$

Proof. Using (3.45) and the bilinearity of a we notice that

$$\begin{aligned} a_{ts}(E,x) &= a_{t\tau}(E,x) + S(t-\tau)a_{\tau s}(E,x) + a_{t\tau}(E,(S(\tau-s)-\mathrm{Id})x) \\ &= a_{t\tau}(E, S(\tau-s)x) + S(t-\tau)a_{\tau s}(E,x). \end{aligned}$$

Iterating this identity for any given partition $\mathcal{P}(s,t)$ proves the claim. □

Remark 3.23. *Alternatively, these processes can also be defined using Theorem 3.6. For a better comprehension we illustrate this technique for a and emphasize the fact that both approaches are equivalent.*

Heuristically, similar to Section 3.2, we notice that for a smooth function ω we can approximate a_{ts} as follows:

$$\begin{aligned} a_{ts}(E,x) := \int_s^t S(t-r)ES(r-s)x d\omega_r &= \sum_{[u,v]\in\mathcal{P}} S(t-v) \int_u^v S(v-r)ES(r-s)x d\omega_r \\ &\approx \sum_{[u,v]\in\mathcal{P}} S(t-v) \int_u^v S(v-r)ES(u-s)x d\omega_r \\ &= \sum_{[u,v]\in\mathcal{P}} S(t-v)\, \omega^S_{vu}(ES(u-s)x). \end{aligned}$$

Keeping this in mind, the deliberations made in Section 3.3 lead to the following result.

Lemma 3.24. *Let $0 \leq s \leq T$. For all $s \leq \tau \leq t \leq T$ we define*

$$\Xi^{(a),s}_{t\tau}(E,x) := \omega^S_{t\tau}(ES(\tau-s)x). \tag{3.54}$$

Then we have

$$a_{ts} = \left(\hat{\delta}\mathcal{I}\Xi^{(a),s}\right)_{ts}. \tag{3.55}$$

Proof. In order to apply Theorem 3.6 we have to analyze the term $\Xi^{(a),s}_{vu}$. Therefore, we estimate

$$\left|\Xi^{(a),s}_{vu}(E,x)\right| = \left|\omega^S_{vu}(ES(u-s)x)\right| \leq c_S |E|\, |x| \left\|\omega\right\|_\alpha (v-u)^\alpha$$

and

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(a),s})_{vmu}\right| &= \left|\omega^S_{vu}(ES(u-s)x) - \omega^S_{vm}(ES(m-s)x) - S(v-m)\omega^S_{mu}(ES(u-s)x)\right| \\ &= \left|\omega^S_{vm}(E\left(S(u-s)-S(m-s)\right)x)\right| \\ &\leq c_S\,|E|\,|x|_{D_\beta}\,\|\omega\|_\alpha\,(u-s)^{-\beta}\,(v-u)^{\alpha+\beta}\,.\end{aligned}$$

Hence, Theorem 3.6 yields the existence of $\mathcal{I}\Xi^{(a),s}$ and by Corollary 3.8

$$(\hat{\delta}\mathcal{I}\Xi^{(a),s})_{t\tau} = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}(\tau,t)} S(t-v)\Xi^{(a),s}_{vu},$$

which further implies

$$(\hat{\delta}\mathcal{I}\Xi^{(a),s})_{ts}(E,x) = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}(s,t)} S(t-v)\omega^S_{vu}(ES(u-s)x).$$

We define

$$\widetilde{a}_{ts} := (\hat{\delta}\mathcal{I}\Xi^{(a),s})_{ts}$$

and show that $a=\widetilde{a}$.
By Corollary 3.22 we know that

$$a_{ts}(E,x) = \sum_{[u,v]\in\mathcal{P}(s,t)} S(t-v)a_{vu}(E,S(u-s)x).$$

Particularly, this also holds for the limit $|\mathcal{P}|\to 0$.

Regarding this, in order to prove the statement, i.e. that $a=\widetilde{a}$, we have to estimate the difference between a_{vu} and ω^S_{vu}. To this aim, we consider now a dyadic partition $\mathcal{P}_n$ and have that

$$\begin{aligned}&\left|\sum_{[u,v]\in\mathcal{P}_n}\left(a_{vu}(E,S(u-s)x) - \omega^S_{vu}(ES(u-s)x)\right)\right| \\ &\leq \sum_{[u,v]\in\mathcal{P}_n}\left|a_{vu}(E,S(u-s)x) - \omega^S_{vu}(ES(u-s)x)\right|.\end{aligned}$$

We apply (3.49) for the first term with $\beta=0$ and for the other terms with $1-\alpha<\beta<1$, and obtain

$$\begin{aligned}&\sum_{[u,v]\in\mathcal{P}_n}\left|a_{vu}(E,S(u-s)x) - \omega^S_{vu}(ES(u-s)x)\right| \\ &\leq c_s\,|E|\,\|\omega\|_\alpha\,|x|\,\frac{(t-s)^\alpha}{2^{n\alpha}} + \sum_{\substack{[u,v]\in\mathcal{P}_n\\ u\neq s}} c_s\,|E|\,\|\omega\|_\alpha\,|x|\,(u-s)^{-\beta}\,\frac{(t-s)^{\alpha+\beta}}{2^{n(\alpha+\beta)}} \\ &\leq c_s\,|E|\,\|\omega\|_\alpha\,|x|\left(\frac{(t-s)^\alpha}{2^{n\alpha}} + \frac{(t-s)^{\alpha+\beta-1}}{2^{n(\alpha+\beta-1)}}\int_s^t (q-s)^{-\beta}\,dq\right) \\ &\leq c_s\,|E|\,\|\omega\|_\alpha\,|x|\,(t-s)^\alpha\left(2^{-n\alpha}+2^{-n(\alpha+\beta-1)}\right)\overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

This proves the statement. □

In order to complete the construction of the supporting processes, recall Section 3.2 we focus now on

$$
\begin{aligned}
&b_{ts}\colon \mathcal{L}(W\otimes V;W)\times\mathcal{L}(V;W)\to W,\\
&b_{ts}(E,K):=\int\limits_s^t S(t-r)E\int\limits_s^r S(r-q)K d\omega_q d\omega_r.
\end{aligned}
\tag{3.56}
$$

Remark 3.25. *To our best knowledge it is not possible to define this process via integration by parts, see Remark 4.3 in [10]. We use Theorem 3.6 to show that at least under some additional regularity assumption on K (specified in Lemma 3.26) it is possible to define $b(E,K)$.*

Inspired by the definition of a we follow the heuristic intuition given in Section 3.2. We saw in (3.16), that for a smooth ω we have

$$
b_{ts}(E,K)\approx\sum_{[u,v]\in\mathcal{P}} S(t-v)\left[\omega^S_{vu}(E\omega^S_{us}(K))+c_{vu}(E,K)\right].
$$

At the first sight the approximation above appears quite arbitrarily but we will rigorously show that this gives us the right approach to define b.

As previously argued, we consider again a smooth approximating sequence $\left(\omega^n,\,\omega^{(2),n}\right)$ of $\left(\omega,\,\omega^{(2)}\right)$ and define

$$
b^n_{ts}(E,K):=\int\limits_s^t S(t-r)E\int\limits_s^r S(r-q)K d\omega^n_q d\omega^n_r. \tag{3.57}
$$

Furthermore, for all $0\le s\le\tau\le t\le T$ we introduce

$$
\Xi^{(b),s}_{t\tau}(E,K):=\omega^S_{t\tau}(E\omega^S_{\tau s}(K))+c_{t\tau}(E,K).
$$

Here the additional regularity assumption on K plays a crucial role. This translates into the restriction on the diffusion coefficient G, recall assumption **(G)** in Section 2.1.

Lemma 3.26. *Let $K\in\mathcal{L}(V;D_\beta)$ with $\alpha+2\beta>1$ and $\alpha>\beta$. Then there exists*

$$
b_{ts}:=\left(\hat{\delta}\mathcal{I}\Xi^{(b),s}\right)_{ts}.
$$

Moreover, the following statements are valid

(i) analytic property:

$$
|b_{ts}(E,K)|\le c_S\,|E|\,|K|_{D_\beta}\left(\|\omega\|^2_\alpha+\left\|\omega^{(2)}\right\|_{2\alpha}\right)(t-s)^{2\alpha}. \tag{3.58}
$$

(ii) continuous dependence on the paths of the noise:

$$
b^n\to b \text{ in } C^{2\alpha}\left([0,T],\mathcal{L}(\mathcal{L}(W\otimes V;W)\times\mathcal{L}(V;D_\beta),W)\right). \tag{3.59}
$$

(iii) algebraic property:

$$
(\hat{\delta}_2 b)_{t\tau s}(E,K)=a_{t\tau}(E,\omega^S_{\tau s}(K)). \tag{3.60}
$$

Proof. As seen before, in order to apply Theorem 3.6, we have to analyze $\Xi^{(b),s}_{t\tau}$. Obviously,

$$
\left|\Xi^{(b),s}_{t\tau}(E,K)\right|\le\left|\omega^S_{t\tau}(E\omega^S_{\tau s}(K))\right|+|c_{t\tau}(E,K)|.
$$

Applying (3.47) and (3.50) we have

$$\begin{aligned}&\left|\omega^S_{t\tau}(E\omega^S_{\tau s}(K))\right| + |c_{t\tau}(E,K)|\\&\leq c_S \left\|\omega\right\|_\alpha \left|E\omega^S_{\tau s}(K)\right|(t-\tau)^\alpha + c_S\left(\left\|\omega\right\|_\alpha + \left\|\omega^{(2)}\right\|_{2\alpha}\right)|E|\,|K|\,(t-\tau)^{2\alpha}\\&\leq c_S \left\|\omega\right\|^2_\alpha |E|\,|K|\,(\tau-s)^\alpha (t-\tau)^\alpha + c_S\left(\left\|\omega\right\|_\alpha + \left\|\omega^{(2)}\right\|_{2\alpha}\right)|E|\,|K|\,(t-\tau)^{2\alpha}.\end{aligned}$$

Furthermore, we compute

$$\hat{\delta}_2\Xi^{(b),s}_{vmu}(E,K) = (\hat{\delta}_2\omega^S)_{vmu}(E\omega^S_{us}(K)) + \omega^S_{vm}(E\left(\omega^S_{us}(K)-\omega^S_{ms}(K)\right)) + (\hat{\delta}_2 c)_{vmu}(E,K).$$

By applying (3.44), (3.46) and (3.47) we obtain

$$\begin{aligned}\hat{\delta}_2\Xi^{(b),s}_{vmu}(E,K) &= \omega^S_{vm}(E\left(\omega^S_{us}(K)-\omega^S_{ms}(K)\right)) + \omega^S_{vm}(EK(\delta\omega)_{mu})\\&= \omega^S_{vm}(E\left(K(\delta\omega)_{mu}-\omega^S_{ms}(K)+\omega^S_{us}(K)\right)).\end{aligned}$$

This further entails

$$\left|\hat{\delta}_2\Xi^{(b),s}_{vmu}(E,K)\right| \leq |E|\left\|\omega\right\|_\alpha (v-m)^\alpha \left|K(\delta\omega)_{mu}-\omega^S_{ms}(K)+\omega^S_{us}(K)\right|. \tag{3.61}$$

Consequently, we need appropriate estimates for the last term. We apply (3.44) and infer that

$$\begin{aligned}\left|K(\delta\omega)_{mu}-\omega^S_{ms}(K)+\omega^S_{us}(K)\right| &= \left|K(\delta\omega)_{mu}-\omega^S_{mu}(K)-(S(m-u)-\mathrm{Id})\,\omega^S_{us}(K)\right|\\&\leq \left|K(\delta\omega)_{mu}-\omega^S_{mu}(K)\right| + \left|(S(m-u)-\mathrm{Id})\,\omega^S_{us}(K)\right|.\end{aligned}$$

For the next steps the additional assumption $K : V \to D_\beta$ is required. Keeping this in mind and using (3.51) we estimate the first term of the previous inequality as follows:

$$\begin{aligned}\left|K(\delta\omega)_{mu}-\omega^S_{mu}(K)\right| &\leq |(S(m-u)-\mathrm{Id})\,K(\delta\omega)_{mu}| + \left|\int_u^m AS(m-r)K(\delta\omega)_{mr}dr\right|\\&\leq c_S\,|K|_{D_\beta}\left\|\omega\right\|_\alpha (m-u)^{\alpha+\beta}.\end{aligned}$$

On the other hand, we have

$$\left|(S(m-u)-\mathrm{Id})\,\omega^S_{us}(K)\right| \leq c_S\left|\omega^S_{us}(K)\right|_{D_{2\beta}}(m-u)^{2\beta}.$$

Applying again (3.51) we derive

$$\begin{aligned}\left|\omega^S_{us}(K)\right|_{D_{2\beta}} &\leq |S(u-s)K(\delta\omega)_{us}|_{D_{2\beta}} + \left|A\int_s^u S(u-r)K(\delta\omega)_{ur}dr\right|_{D_{2\beta}}\\&\leq c_S\,|K|_{D_\beta}\left\|\omega\right\|_\alpha (u-s)^{\alpha-\beta} + c_S\,|K|_{D_\beta}\left\|\omega\right\|_\alpha \int_s^u (u-r)^{\alpha-\beta-1}\,dr\\&\leq c_S\,|K|_{D_\beta}\left\|\omega\right\|_\alpha (u-s)^{\alpha-\beta}.\end{aligned}$$

This finally leads to

$$\left|(S(m-u)-\mathrm{Id})\,\omega^S_{us}(K)\right| \leq c_S\,|K|_{D_\beta}\left\|\omega\right\|_\alpha (u-s)^{\alpha-\beta}(m-u)^{2\beta}.$$

Putting all these together we get

$$\left|K(\delta\omega)_{mu}-\omega^S_{ms}(K)+\omega^S_{us}(K)\right| \leq c_S T^{\alpha-\beta}\,|K|_{D_\beta}\left\|\omega\right\|_\alpha (m-u)^{2\beta}.$$

Consequently, regarding (5.26), we obtain

$$\left|\hat{\delta}_2\Xi^{(b),s}_{vmu}(E,K)\right| \leq c_S T^{\alpha-\beta}\,|E|\,|K|_{D_\beta}\,\|\omega\|^2_\alpha\,(v-u)^{\alpha+2\beta}\,.$$

Theorem 3.6 ensures the existence of $\mathcal{I}\Xi^{(b),s}$ such that for all $s \leq \tau \leq t \leq T$ we have

$$(\hat{\delta}\mathcal{I}\Xi^{(b),s})_{t\tau}(E,K) = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(\tau,t)} S(t-v)\left[\omega^S_{vu}(E\omega^S_{us}(K)) + c_{vu}(E,K)\right]$$

and

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(b),s})_{t\tau}(E,K)\right| \leq c_S\,|E|\,|K|_{D_\beta}\left(\|\omega\|^2_\alpha + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(t-\tau)^\alpha\left((\tau-s)^\alpha + (t-\tau)^\alpha\right).$$

In particular setting $\tau = s$ we can define $b_{ts} := (\hat{\delta}\mathcal{I}\Xi^{(b),s})_{ts}$ and infer from the previous estimate that

$$|b_{ts}(E,K)| \leq c_S\,|E|\,|K|_{D_\beta}\left(\|\omega\|^2_\alpha + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(t-s)^{2\alpha}\,, \tag{3.62}$$

which precisely gives us (i).

In order to prove (iii) we compute as before

$$\begin{aligned}
(\hat{\delta}_2 b)_{t\tau s}(E,K) &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(s,t)} S(t-v)\left[\omega^S_{vu}(E\omega^S_{us}(K)) + c_{vu}(E,K)\right] \\
&\quad - \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(\tau,t)} S(t-v)\left[\omega^S_{vu}(E\omega^S_{u\tau}(K)) + c_{vu}(E,K)\right] \\
&\quad - \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(s,\tau)} S(t-v)\left[\omega^S_{vu}(E\omega^S_{us}(K)) + c_{vu}(E,K)\right] \\
&= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(\tau,t)} S(t-v)\omega^S_{vu}(E\left(\omega^S_{us} - \omega^S_{u\tau}\right)(K)) \\
&= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}(\tau,t)} S(t-v)\omega^S_{vu}(E\omega^S_{\tau s}(K)) \\
&= a_{t\tau}(E,\omega^S_{\tau s}(K)).
\end{aligned}$$

It only remains to show that assertion (ii) holds true. Regarding that ω^n and $\omega^{(2),n}$ are smooth approximation terms, we are allowed to choose $\alpha > 1/2$.

We define

$$\Xi^{(b),n,s}_{t\tau}(E,K) := \omega^{S,n}_{t\tau}(E\omega^{S,n}_{\tau s}(K)) + c^n_{t\tau}(E,K).$$

Similar computations entail the existence of

$$\widetilde{b}^n_{ts}(E,K) := (\hat{\delta}\mathcal{I}\Xi^{(b),n,s})_{ts}(E,K).$$

Moreover, using the same deliberations as above we obtain the analytic estimate

$$\left|\widetilde{b}^n_{ts}(E,K)\right| \leq c_S\,|E|\,|K|_{D_\beta}\left(\|\omega^n\|^2_\alpha + \left\|\omega^{(2),n}\right\|_{2\alpha}\right)(t-s)^{2\alpha}$$

together with the algebraic structure

$$(\hat{\delta}_2\widetilde{b}^n)_{t\tau s}(E,K) = a^n_{t\tau}(E,\omega^{S,n}_{\tau s}(K)).$$

A straightforward computation for b^n gives us

$$
\begin{aligned}
(\hat{\delta}_2 b^n)_{t\tau s}(E,K) = & \int_s^t S(t-r)E \int_s^r S(r-q)K d\omega_q^n d\omega_r^n \\
& - \int_\tau^t S(t-r)E \int_\tau^r S(r-q)K d\omega_q^n d\omega_r^n \\
& - \int_s^\tau S(t-r)E \int_s^r S(r-q)K d\omega_q^n d\omega_r^n \\
= & \int_\tau^t S(t-r)E \left(\int_s^r S(r-q)K d\omega_q^n - \int_\tau^r S(r-q)K d\omega_q^n \right) d\omega_r^n \\
= & \int_\tau^t S(t-r)ES(r-\tau) \int_s^\tau S(\tau-q)K d\omega_q^n d\omega_r^n \\
= & a_{t\tau}^n(E, \omega_{\tau s}^{S,n}(K)).
\end{aligned}
$$

Consequently, we obtain that $\hat{\delta}_2(b^n - \widetilde{b}^n) \equiv 0$. Hence, for all E, K by Lemma 2.4 there exists $\kappa \in C([0,T], W)$ such that

$$
\kappa_0 = 0 \quad \text{and} \quad (\hat{\delta}\kappa)_{ts} = (b^n - \widetilde{b}^n)_{ts}(E,K).
$$

We have

$$
\left|(\hat{\delta}\kappa)_{ts}\right| \leq |b_{ts}^n(E,K)| + \left|\widetilde{b}_{ts}^n(E,K)\right| \leq C(S, \omega^n, \omega^{(2),n}, E, K)\,(t-s)^{2\alpha}.
$$

Since we assumed $\alpha > \frac{1}{2}$ Lemma A.1 yields $\kappa \equiv 0$ which implies $b^n = \widetilde{b}^n$. We know by Remark 3.14 that $\widetilde{b}^n$ converges to b which proves the assertion. □

Remark 3.27. *All supporting processes defined in this section depend on the underlying rough path $\boldsymbol{\omega} = (\omega, \omega^{(2)})$. More precisely, ω^S and the process a are independent of $\omega^{(2)}$, so only are induced by the path component ω.*

Recall (2.11) *and* (2.13) *and Lemma 2.8. For our latter purpose it is important to consider integrals with respect to an appropriate time-shifted rough path $(\theta_\tau\omega, \overline{\theta}_\tau\, \omega^{(2)})$ for $\tau \in \mathbb{R}$.*

Note that the supporting terms $\overline{\theta}_\tau\omega^S$ and $\overline{\theta}_\tau a$ are induced by the shifted path component $\theta_\tau\omega$. Moreover, $\overline{\theta}_\tau b$ and $\overline{\theta}_\tau c$ are induced by the shifted rough path $(\theta_\tau\omega, \overline{\theta}_\tau\, \omega^{(2)})$.

3.5 Local Solutions for Rough Evolution Equations

Throughout this section we impose $\frac{1}{3} < \beta < \alpha \leq \frac{1}{2}$ such that $\alpha + 2\beta > 1$. Recall that all necessary assumptions on the coefficients and on the noise were stated in Chapter 2.

We now derive the existence of a solution for (1.1) which is given by a pair (y,z), as argued in Section 3.2. Here $(y_t)_{t\in[0,T]}$ stands for a W-valued path and $(z_{ts})_{(t,s)\in\Delta_T}$, $z_{ts} \in \mathcal{L}(\mathcal{L}(W\otimes V; W); W)$ denotes the area term.

Therefore, we are justified to introduce the Banach space

$$
\begin{aligned}
X_{\omega,T} := \Big\{(y,z):\ & y \in C^{\beta,\beta}([0,T],W),\\
& z \in C^{\alpha}\left(\Delta_T, \mathcal{L}(\mathcal{L}(W\otimes V;W);W)\right) \cap C^{\alpha+\beta,\beta}\left(\Delta_T, \mathcal{L}(\mathcal{L}(W\otimes V;W);W)\right),\\
& (\hat{\delta}_2 z)_{t\tau s} = \omega^S_{t\tau}(\cdot(\delta y)_{\tau s})\Big\},
\end{aligned}
$$

endowed with norm

$$
\|(y,z)\|_X := \|y\|_\infty + |\!|\!|y|\!|\!|_{\beta,\beta} + \|z\|_\alpha + \|z\|_{\alpha+\beta,\beta}\,. \tag{3.63}
$$

Remark 3.28. *Note that the norm given above is equivalent to*

$$
\|y\|_\infty + |\!|\!|y|\!|\!|_{\beta,\beta} + \sup_{0\le s<t\le T}\frac{|z_{ts}|}{(t-s)^\alpha} + \sup_{0<s<t\le T} s^\beta \frac{|z_{ts}|}{(t-s)^{\alpha+\beta}},
$$

which essentially simplifies the computation. By a slight abuse of notation we use the same symbols.

Using the same notations as in Section 3.2, we consider the map

$$
\mathcal{M}_T\colon X_{\omega,T}\to X_{\omega,T} \qquad \mathcal{M}_T(y,z) = (\widetilde{y},\widetilde{z}),
$$

where

$$
\begin{aligned}
\widetilde{y}_t &= S(t)\xi + \mathcal{I}\Xi^{(y)}(y,z)_t\\
\overline{y}_t &= \widetilde{y}_t - S(t)\xi = \mathcal{I}\Xi^{(y)}(y,z)_t.
\end{aligned}
$$

Furthermore, for $E\in\mathcal{L}(W\otimes V;W)$ the second component of the solution is constituted by

$$
\begin{aligned}
\widetilde{z}_{ts}(E) &= \left(\hat{\delta}\mathcal{I}\Xi^{(z)}(y,\widetilde{y})\right)_{ts}(E) - \omega^S_{ts}(E\widetilde{y}_s),\\
&= \left(\hat{\delta}\mathcal{I}\Xi^{(z)}(y,\overline{y})\right)_{ts}(E) - \omega^S_{ts}(E\overline{y}_s) + a_{ts}(E,S(s)\xi) - \omega^S_{ts}(ES(s)\xi),\\
\overline{z}_{ts}(E) &= \left(\hat{\delta}\mathcal{I}\Xi^{(z)}(y,\overline{y})\right)_{ts}(E) - \omega^S_{ts}(E\overline{y}_s).
\end{aligned}
$$

Regarding this we define for $(u,v)\in\Delta_T$

$$
\begin{aligned}
\Xi^{(y)}_{vu} &= \Xi^{(y)}(y,z)_{vu} = \omega^S_{vu}(G(y_u)) + z_{vu}(DG(y_u)),\\
\Xi^{(z)}_{vu}(E) &= \Xi^{(z)}(y,\overline{y})_{vu}(E) = b_{vu}(E,G(y_u)) + a_{vu}(E,\overline{y}_u).
\end{aligned}
$$

In order to show that $\mathcal{M}_T$ maps $X_{\omega,T}$ into itself and is a contraction we have to derive suitable a-priori estimates. We proceed step by step and split these results into several Lemmas.

Remark 3.29. *Note that the universal constant C occurring in the estimates below depends on $|\!|\!|\omega|\!|\!|_\alpha$, $\left\|\omega^{(2)}\right\|_{2\alpha}$, α, β, $S(\cdot)$, G uniformly with respect to T. We stress that this is independent of ξ.*

Lemma 3.30 (Estimates of the y-integral)**.** *For a pair $(y,z)\in X_{\omega,T}$ the following estimates are valid:*

$$
\left|(\hat{\delta}\overline{y})_{ts}\right| \le C\left(1+\|(y,z)\|_X^2\right)(t-s)^\alpha\,, \tag{3.64}
$$

$$
|\overline{y}_s|_{D_\beta} \le C\left(1+\|(y,z)\|_X^2\right)s^{\alpha-\beta}, \tag{3.65}
$$

$$
|\!|\!|\overline{y}|\!|\!|_{\beta,\beta} \le C\left(1+\|(y,z)\|_X^2\right)T^\alpha, \tag{3.66}
$$

$$
\left|(\hat{\delta}\overline{y})_{ts} - \omega^S_{ts}(G(y_s))\right| \le C\left(1+\|(y,z)\|_X^2\right)s^{-\beta}(t-s)^{\alpha+\beta}\,. \tag{3.67}
$$

Proof. Regarding the definition of $\Xi^{(y)}_{vu}$, the α-Hölder continuity of ω, the regularity of G and the definition of the norm in $X_{\omega,T}$ we infer

$$\begin{aligned}\left|\Xi^{(y)}_{vu}\right| &\leq \left|\omega^S_{vu}(G(y_u))\right| + |z_{vu}(DG(y_u))| \\ &\leq C \left\|\omega\right\|_\alpha |G(y_u)| (v-u)^\alpha + C \left\|z\right\|_\alpha |DG(y_u)| (v-u)^\alpha \\ &\leq C \left(\left\|\omega\right\|_\alpha (1 + \|y\|_\infty) + \|z\|_\alpha\right) (v-u)^\alpha \\ &\leq C(1 + \|(y,z)\|_X) (v-u)^\alpha .\end{aligned}$$

Recalling (3.13)

$$(\hat{\delta}_2\Xi^{(y)})_{vmu} = \omega^S_{vm}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}) + z_{vm}(DG(y_u) - DG(y_m)),$$

together with the regularity assumptions on y and z, further results in

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| &\leq \left|\omega^S_{vm}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu})\right| + |z_{vm}(DG(y_u) - DG(y_m))| \\ &\leq C \left\|\omega\right\|_\alpha |G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}| (v-m)^\alpha \\ &+ C \|z\|_{\alpha+\beta,\beta} |DG(y_u) - DG(y_m)| m^{-\beta} (v-m)^{\alpha+\beta} \\ &\leq C \left\|\omega\right\|_\alpha |y_u - y_m|^2 (v-m)^\alpha + C \|z\|_{\alpha+\beta,\beta} |y_u - y_m| m^{-\beta} (v-m)^{\alpha+\beta} .\end{aligned}$$

We observe that we have two different possibilities to estimate $|y_u - y_m|$. Obviously,

$$|y_u - y_m| \leq 2 \|y\|_\infty \quad \text{and} \quad |y_u - y_m| \leq \left\|y\right\|_{\beta,\beta} u^{-\beta} (m-u)^\beta .$$

Therefore, on the one hand we get

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| &\leq C \left\|\omega\right\|_\alpha \left\|y\right\|^2_{\beta,\beta} u^{-2\beta} (m-u)^{2\beta} (v-m)^\alpha \\ &+ C \|z\|_{\alpha+\beta,\beta} \left\|y\right\|_{\beta,\beta} u^{-\beta} (m-u)^\beta m^{-\beta} (v-m)^{\alpha+\beta} \\ &\leq C \left(\left\|\omega\right\|_\alpha \left\|y\right\|^2_{\beta,\beta} + \|z\|_{\alpha+\beta,\beta} \left\|y\right\|_{\beta,\beta}\right) u^{-2\beta} (v-u)^{\alpha+2\beta} \\ &\leq C \left(1 + \|(y,z)\|^2_X\right) u^{-2\beta} (v-u)^{\alpha+2\beta} .\end{aligned}$$

By applying Theorem 3.6 we can show the existence of $\mathcal{I}\Xi^{(y)} = \mathcal{I}\Xi^{(y)}(y,z)$ and obtain the estimate

$$\left|(\hat{\delta}\overline{y})_{ts}\right| = \left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts}\right| \leq C \left(1 + \|(y,z)\|^2_X\right) (t-s)^\alpha .$$

Corollary 3.11 entails

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts}\right|_{D_\beta} \leq C \left(1 + \|(y,z)\|^2_X\right) (t-s)^{\alpha-\beta} ,$$

which implies that

$$|\overline{y}_s|_{D_\beta} \leq C \left(1 + \|(y,z)\|^2_X\right) s^{\alpha-\beta}.$$

Furthermore, by Corollary 3.13 we obtain

$$\|\overline{y}\|_{\beta,\beta} \leq C \left(1 + \|(y,z)\|^2_X\right) T^\alpha. \tag{3.68}$$

On the other hand we also have

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| &\leq C \left\|\omega\right\|_\alpha \|y\|_\infty \left\|y\right\|_{\beta,\beta} u^{-\beta} (m-u)^\beta (v-m)^\alpha \\ &+ C \|z\|_{\alpha+\beta,\beta} \|y\|_\infty m^{-\beta} (v-m)^{\alpha+\beta} \\ &\leq C \left(\left\|\omega\right\|_\alpha \|y\|_\infty \left\|y\right\|_{\beta,\beta} + \|z\|_{\alpha+\beta,\beta} \|y\|_\infty\right) u^{-\beta} (v-u)^{\alpha+\beta} \\ &\leq C \left(1 + \|(y,z)\|^2_X\right) u^{-\beta} (v-u)^{\alpha+\beta} .\end{aligned}$$

Hence, we can apply Corollary 3.7 and obtain

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} - \Xi^{(y)}_{ts}\right| \leq C\left(1+\|(y,z)\|_X^2\right) s^{-\beta}\,(t-s)^{\alpha+\beta},$$

which leads to

$$\begin{aligned}\left|(\hat{\delta}\overline{y})_{ts} - \omega^S_{ts}(G(y_s))\right| &\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} - \Xi^{(y)}_{ts}\right| + |z_{ts}(DG(y_s))| \\ &\leq C\left(1+\|(y,z)\|_X^2\right) s^{-\beta}\,(t-s)^{\alpha+\beta}.\end{aligned}$$

□

We now focus in deriving suitable estimates for $\overline{z}$.

Lemma 3.31 (Estimates of the z-integral)**.** *Let $(y,z)\in X_{\omega,T}$. The following estimates are valid:*

$$|\overline{z}_{ts}(E)| \leq C|E|\left(1+\|(y,z)\|_X^2\right)\left[(t-s)^{2\alpha} + s^{\alpha-\beta}\,(t-s)^{\alpha+\beta}\right], \tag{3.69}$$

$$\|\overline{z}\|_{\alpha+\beta} \leq C\left(1+\|(y,z)\|_X^2\right) T^{\alpha-\beta}. \tag{3.70}$$

Proof. Applying Theorem 3.6 we get

$$\left|\Xi^{(z)}_{vu}(E)\right| \leq |b_{vu}(E,G(y_u))| + |a_{vu}(E,\overline{y}_u)|.$$

Furthermore, due to (3.48) and (3.58) together with the Lipschitz continuity of the mapping $G : W \to \mathcal{L}(V, D_\beta)$, we infer that

$$\begin{aligned}\left|\Xi^{(z)}_{vu}(E)\right| \leq& C\,|E|\,(1+\|y\|_\infty)\left(\|\omega\|_\alpha^2 + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(v-u)^{2\alpha} + C\,|E|\,\|\overline{y}\|_\infty\,\|\omega\|_\alpha\,(v-u)^\alpha \\ \leq& C\,|E|\left[(1+\|y\|_\infty)\left(\|\omega\|_\alpha^2 + \left\|\omega^{(2)}\right\|_{2\alpha}\right) + \|\overline{y}\|_\infty\,\|\omega\|_\alpha\right](v-u)^\alpha.\end{aligned}$$

Using (3.66) we obtain

$$\left|\Xi^{(z)}_{vu}(E)\right| \leq C\,|E|\left(1+\|(y,z)\|_X^2\right)(v-u)^\alpha.$$

Furthermore, we have by (3.45) and (3.60) that

$$\begin{aligned}(\hat{\delta}_2\Xi^{(z),s})_{vmu}(E) =& (\hat{\delta}_2 b)_{vmu}(E,G(y_u)) + b_{vm}(E,G(y_u)-G(y_m)) \\ &+ (\hat{\delta}_2 a)_{vmu}(E,\overline{y}_u) + a_{vm}(E,\overline{y}_u-\overline{y}_m) + \\ =& a_{vm}(E,\omega^S_{mu}(G(y_u))) + b_{vm}(E,G(y_u)-G(y_m)) \\ &+ a_{vm}(E,(S(m-u)-\mathrm{Id})\overline{y}_u) + a_{vm}(E,\overline{y}_u-\overline{y}_m) \\ =& a_{vm}(E,\omega^S_{mu}(G(y_u)) - (\hat{\delta}\overline{y})_{mu}) + b_{vm}(E,G(y_u)-G(y_m)).\end{aligned}$$

This leads to

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(z),s})_{vmu}(E)\right| \leq& \left|a_{vm}(E,\omega^S_{mu}(G(y_u)) - (\hat{\delta}\overline{y})_{mu})\right| + |b_{vm}(E,G(y_u)-G(y_m))| \\ \leq& C\,\|\omega\|_\alpha\,|E|\left|\omega^S_{mu}(G(y_u)) - (\hat{\delta}\overline{y})_{mu}\right|(v-m)^\alpha \\ &+ C\left(\|\omega\|_\alpha^2 + \left\|\omega^{(2)}\right\|_{2\alpha}\right)|E|\,|G(y_u)-G(y_m)|\,(v-m)^{2\alpha}.\end{aligned}$$

Applying (3.67) entails

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(z),s})_{vmu}(E)\right| \leq& C\,|E|\left(1+\|(y,z)\|_X^2\right)u^{-\beta}\,(m-u)^{\alpha+\beta}\,(v-m)^\alpha \\ &+ C\,|E|\,\|y\|_{\beta,\beta}\,u^{-\beta}\,(m-u)^\beta\,(v-m)^{2\alpha} \\ \leq& C\,|E|\left(1+\|(y,z)\|_X^2\right)u^{-\beta}\,(v-u)^{2\alpha+\beta}.\end{aligned}$$

Hence, we can apply again Theorem 3.6 however, this does not give us the appropriate estimates. By a slightly different computation we obtain

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(z),s})_{vmu}(E)\right| &\leq C \left\|\!\left|\omega\right|\!\right\|_\alpha |E| \left|\left(\omega^S_{mu}(G(y_u))\right| + \left|(\delta\overline{y})_{mu}\right|\right)(v-m)^\alpha \\ &\quad + C\left(\left\|\!\left|\omega\right|\!\right\|_\alpha^2 + \left\|\omega^{(2)}\right\|_{2\alpha}\right)|E|\,|G(y_u)-G(y_m)|\,(v-m)^{2\alpha}.\end{aligned}$$

Now (3.64) entails

$$\left|(\hat{\delta}_2\Xi^{(z),s})_{vmu}(E)\right| \leq C\,|E|\left(1+\|(y,z)\|_X^2\right)(v-u)^{2\alpha}.$$

Hence, by Corollary 3.7 we derive

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| \leq C\,|E|\left(1+\|(y,z)\|_X^2\right)(t-s)^{2\alpha}.$$

Furthermore, let us consider

$$\left|\Xi^{(z)}_{ts}(E) - \omega^S_{ts}(E\overline{y}_s)\right| \leq |b_{ts}(E,G(y_s))| + \left|a_{ts}(E,\overline{y}_s) - \omega^S_{ts}(E\overline{y}_s)\right|.$$

Applying (3.49) and (3.58) entails

$$\begin{aligned}\left|\Xi^{(z)}_{ts}(E) - \omega^S_{ts}(E\overline{y}_s)\right| &\leq C\,|E|\,(1+\|y\|_\infty)\left(\left\|\!\left|\omega\right|\!\right\|_\alpha^2 + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(t-s)^{2\alpha} \\ &\quad + C\,|E|\left\|\!\left|\omega\right|\!\right\|_\alpha |\overline{y}_s|_{D_\beta}(t-s)^{\alpha+\beta}.\end{aligned}$$

By using (3.65) we obtain

$$\left|\Xi^{(z)}_{ts}(E) - \omega^S_{ts}(E\overline{y}_s)\right| \leq C\,|E|\left(1+\|(y,z)\|_X^2\right)\left[s^{\alpha-\beta}(t-s)^{\alpha+\beta} + (t-s)^{2\alpha}\right].$$

Summarizing, we conclude

$$\begin{aligned}|\overline{z}_{ts}(E)| &= \left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \omega^S_{ts}(E\overline{y}_s)\right| \\ &\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| + \left|\Xi^{(z)}_{ts}(E) - \omega^S_{ts}(E\overline{y}_s)\right| \\ &\leq C|E|\left(1+\|(y,z)\|_X^2\right)\left[(t-s)^{2\alpha} + s^{\alpha-\beta}(t-s)^{\alpha+\beta}\right].\end{aligned}$$

Consequently, we get

$$\|\overline{z}\|_{\alpha+\beta} \leq C\left(1+\|(y,z)\|_X^2\right)T^{\alpha-\beta}.$$ □

After establishing suitable analytic properties we focus now on the algebraic setting.

Lemma 3.32. *The following algebraic property*

$$(\hat{\delta}_2\overline{z})_{vmu}(E) = \omega^S_{vm}(E(\delta\overline{y})_{mu})$$

holds true.

Proof. By (3.44) and Lemma 2.4 ((**i**)) we have

$$\begin{aligned}(\hat{\delta}_2\overline{z})_{vmu}(E) &= (\hat{\delta}_2\hat{\delta}\mathcal{I}\Xi^{(z)})_{vmu}(E) - (\hat{\delta}_2\omega^S)_{vmu}(E\overline{y}_u) + \omega^S_{vm}(E(\delta\overline{y})_{mu}) \\ &= \omega^S_{vm}(E(\delta\overline{y})_{mu}).\end{aligned}$$ □

We now have all the necessary ingredients to analyze the mapping $\mathcal{M}_T$ and proceed towards our fixed-point argument.

Theorem 3.33. *The mapping $\mathcal{M}_T$ maps $X_{\omega,T}$ into itself. Moreover, the estimate*

$$\|\mathcal{M}_T(y,z)\|_X \leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right) \tag{3.71}$$

holds true.

Proof. Recall

$$\widetilde{y}_t = S(t)\xi + \overline{y}_t.$$

Hence, applying (3.66) we derive

$$\begin{aligned}\|\widetilde{y}\|_\infty &\leq C\left(|\xi| + \|\overline{y}\|_\infty\right)\\ &\leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right)\end{aligned}$$

and

$$\begin{aligned}\|\!|\widetilde{y}|\!\|_{\beta,\beta} &\leq \|\!|S(\cdot)\xi|\!\|_{\beta,\beta} + \|\!|\overline{y}|\!\|_{\beta,\beta}\\ &\leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right).\end{aligned}$$

Moreover, we have

$$\widetilde{z}_{ts}(E) = \overline{z}_{ts}(E) + a_{ts}(E, S(s)\xi) - \omega_{ts}^S(ES(s)\xi).$$

On the one hand by (3.70), (3.48) and (3.47) we get

$$\begin{aligned}\|\widetilde{z}\|_\alpha &\leq \|\overline{z}\|_{\alpha+\beta}T^\beta + C\|S(\cdot)\xi\|_\infty\\ &\leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right).\end{aligned}$$

On the other hand we apply (3.70) and (3.49) and infer that

$$\begin{aligned}\|\widetilde{z}\|_{\alpha+\beta,\beta} &\leq \|\overline{z}\|_{\alpha+\beta}T^\beta + C\sup_{0<s<T} s^\beta |S(s)\xi|_{D_\beta}\\ &\leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right).\end{aligned}$$

Summarizing we obtain the required estimate

$$\|\mathcal{M}(y,z)\|_X \leq C\left(|\xi| + \left(1 + \|(y,z)\|_X^2\right)T^\alpha\right).$$

The last step is to prove the corresponding algebraic relation. Due to Lemma 3.32 and of the algebraic relations (3.44) and (3.45) we compute

$$\begin{aligned}(\hat{\delta}_2\widetilde{z})_{t\tau s}(E) =& (\hat{\delta}_2\overline{z})_{t\tau s}(E) + (\hat{\delta}_2 a)_{t\tau s}(E, S(s)\xi) + a_{t\tau}(E, S(s)\xi - S(\tau)\xi)\\ &+ (\hat{\delta}_2\omega^S)_{t\tau s}(ES(s)\xi) + \omega_{t\tau}^S(E(S(s)\xi - S(\tau)\xi))\\ =& \omega_{t\tau}^S(E(\overline{y}_\tau - \overline{y}_s)) + a_{t\tau}(E, S(\tau)\xi - S(s)\xi) + a_{t\tau}(E, S(s)\xi - S(\tau)\xi)\\ &+ \omega_{t\tau}^S(E(S(s)\xi - S(\tau)\xi))\\ =& \omega_{t\tau}^S(E(\widetilde{y}_\tau - \widetilde{y}_s)).\end{aligned}$$

□

In order to show the existence of a unique local mild solution for (1.1) by means of Banach's Fixed-Point Theorem we verify that $\mathcal{M}$ is a contraction. To this aim, analogously to Lemmas 3.30 and 3.31 we derive the necessary estimates.

Lemma 3.34 (estimate for δy-integral)**.** *Let (y^1,z^1), and $(y^2,z^2)\in X_{\omega,T}$. Then we have*

$$\left|(\hat\delta(\overline{y}^1-\overline{y}^2))_{ts}\right| \le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X (t-s)^\alpha, \tag{3.72}$$

$$\left|\overline{y}^1_s-\overline{y}^2_s\right|_{D_\beta} \le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X s^{\alpha-\beta} \tag{3.73}$$

$$\left\|\overline{y}^1-\overline{y}^2\right\|_{\beta,\beta} \le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X T^\alpha \tag{3.74}$$

as well as

$$\begin{aligned}&\left|(\hat\delta(\overline{y}^1-\overline{y}^2))_{ts}-\omega^S_{ts}(G(y^1_s)-G(y^2_s))\right|\\ &\le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X s^{-\beta}(t-s)^{\alpha+\beta}.\end{aligned} \tag{3.75}$$

Proof. We have

$$\overline{y}^1_t-\overline{y}^2_t=\mathcal{I}\Xi^{(y)}(y^1,z^1)_t-\mathcal{I}\Xi^{(y)}(y^2,z^2)_t=\mathcal{I}\left(\Xi^{(y)}(y^1,z^1)-\Xi^{(y)}(y^2,z^2)\right)_t.$$

We make the same deliberations as in Lemma 3.30 and use the assumptions on G.

$$\begin{aligned}&\left|\Xi^{(y)}(y^1,z^1)_{vu}-\Xi^{(y)}(y^2,z^2)_{vu}\right|\\
&\le\left|\omega^S_{vu}(G(y^1_u)-G(y^2_u))\right|+\left|z^1_{vu}(DG(y^1_u))-z^2_{vu}(DG(y^2_u))\right|\\
&\le\left|\omega^S_{vu}(G(y^1_u)-G(y^2_u))\right|+\left|z^1_{vu}\left(DG(y^1_u)-DG(y^2_u)\right)\right|+\left|\left(z^1-z^2\right)_{vu}(DG(y^2_u))\right|\\
&\le C\left\|\omega\right\|_\alpha\left\|y^1-y^2\right\|_\infty(v-u)^\alpha+C\left\|z^1\right\|_\alpha\left\|y^1-y^2\right\|_\infty(v-u)^\alpha+C\left\|z^1-z^2\right\|_\alpha(v-u)^\alpha\\
&\le C\left(1+\left\|(y^1,z^1)\right\|_X\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\left|(\hat\delta_2\Xi^{(y)}(y^1,z^1))_{vmu}-(\hat\delta_2\Xi^{(y)}(y^2,z^2))_{vmu}\right|\\
&\le\left|\omega^S_{vm}\left(G(y^1_u)-G(y^1_m)+DG(y^1_u)(\delta y^1)_{mu}-G(y^2_u)+G(y^2_m)-DG(y^2_u)(\delta y^2)_{mu}\right)\right|\\
&+\left|z^1_{vm}(DG(y^1_u)-DG(y^1_m))-z^2_{vm}(DG(y^2_u)-DG(y^2_m))\right|\\
&\le C\left\|\omega\right\|_\alpha\left|G(y^1_u)-G(y^1_m)+DG(y^1_u)(\delta y^1)_{mu}-G(y^2_u)+G(y^2_m)-DG(y^2_u)(\delta y^2)_{mu}\right|(v-u)^\alpha\\
&+\left|z^1_{vm}(DG(y^1_u)-DG(y^1_m)-DG(y^2_u)+DG(y^2_m))\right|+\left|\left(z^1-z^2\right)_{vm}(DG(y^2_u)-DG(y^2_m))\right|.\end{aligned}$$

As in Lemma 3.30 we have two possibilities to estimate these terms.

By (A.3) and (A.4) we infer

$$\begin{aligned}&\left|(\hat\delta_2\Xi^{(y)}(y^1,z^1))_{vmu}-(\hat\delta_2\Xi^{(y)}(y^2,z^2))_{vmu}\right|\\
&\le C\left\|\omega\right\|_\alpha\left[\left(\left|\!\left|\!\left|y^1\right|\!\right|\!\right|_{\beta,\beta}+\left|\!\left|\!\left|y^2\right|\!\right|\!\right|_{\beta,\beta}\right)\left|\!\left|\!\left|y^1-y^2\right|\!\right|\!\right|_{\beta,\beta}+\left|\!\left|\!\left|y^2\right|\!\right|\!\right|^2_{\beta,\beta}\left\|y^1-y^2\right\|_\infty\right]u^{-2\beta}(v-u)^{\alpha+2\beta}\\
&+C\left\|z^1\right\|_{\alpha+\beta,\beta}\left[\left|\!\left|\!\left|y^1-y^2\right|\!\right|\!\right|_{\beta,\beta}+\left|\!\left|\!\left|y^2\right|\!\right|\!\right|_{\beta,\beta}\left\|y^1-y^2\right\|_\infty\right]u^{-2\beta}(v-u)^{\alpha+2\beta}\\
&+C\left\|z^1-z^2\right\|_{\alpha+\beta,\beta}\left|\!\left|\!\left|y^2\right|\!\right|\!\right|_{\beta,\beta}u^{-2\beta}(v-u)^{\alpha+2\beta}\\
&\le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X u^{-2\beta}(v-u)^{\alpha+2\beta}.\end{aligned}$$

Again, applying Theorem 3.6 entails

$$\left|(\hat\delta(\overline{y}^1-\overline{y}^2))_{ts}\right|\le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X(t-s)^\alpha.$$

By Corollary 3.11 we obtain

$$\left|\overline{y}^1_s-\overline{y}^2_s\right|_{D_\beta}\le C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X s^{\alpha-\beta},$$

and with Corollary 3.13 we have

$$\left\|\overline{y}^1-\overline{y}^2\right\|_{\beta,\beta}\leq C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X T^\alpha.$$

On the other hand using (A.4) and (A.7) we get

$$\begin{aligned}
&\left|(\hat{\delta}_2\Xi^{(y)}(y^1,z^1))_{vmu}-(\hat{\delta}_2\Xi^{(y)}(y^2,z^2))_{vmu}\right|\\
&\leq C\,\|\omega\|_\alpha\left(\|y^1\|_{\beta,\beta}+\|y^2\|_{\beta,\beta}+\|y^2\|_{\beta,\beta}\left\|y^2\right\|_\infty\right)\left\|y^1-y^2\right\|_\infty u^{-\beta}\,(v-u)^{\alpha+\beta}\\
&+C\left\|z^1\right\|_{\alpha+\beta,\beta}\left(\left\|y^1-y^2\right\|_\infty+\left\|y^2\right\|_\infty\left\|y^1-y^2\right\|_\infty\right)u^{-\beta}\,(v-u)^{\alpha+\beta}\\
&+C\left\|z^1-z^2\right\|_{\alpha+\beta,\beta}\left\|y^2\right\|_\infty u^{-\beta}\,(v-u)^{\alpha+\beta}\\
&\leq C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X u^{-\beta}\,(v-u)^{\alpha+\beta}.
\end{aligned}$$

Hence, we can apply Corollary 3.7 and obtain

$$\begin{aligned}
&\left|(\hat{\delta}(\overline{y}^1-\overline{y}^2))_{ts}-\omega^S_{ts}(G(y^1_s)-G(y^2_s))\right|\\
&\leq C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X s^{-\beta}\,(t-s)^{\alpha+\beta}.
\end{aligned}$$

□

Lemma 3.35 (estimate for δz-integral)**.** *Let (y^1,z^1), and $(y^2,z^2)\in X_{\omega,T}$. Then the following estimates are valid*

$$\begin{aligned}
\left|(\overline{z}^1_{ts}-\overline{z}^2_{ts})(E)\right|\leq C|E|&\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X\\
&\left[(t-s)^{2\alpha}+s^{\alpha-\beta}\,(t-s)^{\alpha+\beta}\right],
\end{aligned}\tag{3.76}$$

$$\left\|\overline{z}^1-\overline{z}^2\right\|_{\alpha+\beta}\leq C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X T^{\alpha-\beta}.\tag{3.77}$$

Proof. Recall that

$$\begin{aligned}
(\overline{z}^1_{ts}-\overline{z}^2_{ts})(E)&=(\hat{\delta}\mathcal{I}\Xi^{(y)}(y^1,\overline{y}^1))_{ts}(E)-\omega^S_{ts}(E\overline{y}^1_s)-(\hat{\delta}\mathcal{I}\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)+\omega^S_{ts}(E\overline{y}^2_s)\\
&=(\hat{\delta}\mathcal{I}[\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2)])_{ts}(E)-\left(\omega^S_{ts}(E\overline{y}^1_s)-\omega^S_{ts}(E\overline{y}^2_s)\right).
\end{aligned}$$

Building the difference of $\Xi^{(z)}$ for $(y^1,\overline{y}^1)$ and $(y^2,\overline{y}^2)$ entails

$$\begin{aligned}
&\left|\Xi^{(z)}(y^1,\overline{y}^1)_{vu}(E)-\Xi^{(z)}(y^2,\overline{y}^2)_{vu}(E)\right|\\
&\leq\left|b_{vu}(E,G(y^1_u)-G(y^2_u))\right|+\left|a_{vu}(E,\overline{y}^1_u-\overline{y}^2_u)\right|\\
&\leq C\,|E|\left(\|\omega\|_\alpha^2+\left\|\omega^{(2)}\right\|_{2\alpha}\right)\left\|y^1-y^2\right\|_\infty(v-u)^{2\alpha}+C\,|E|\,\|\omega\|_\alpha\left\|\overline{y}^1-\overline{y}^2\right\|_\infty(v-u)^\alpha.
\end{aligned}$$

By (3.74) we get

$$\begin{aligned}
&\left|\Xi^{(z)}(y^1,\overline{y}^1)_{vu}(E)-\Xi^{(z)}(y^2,\overline{y}^2)_{vu}(E)\right|\\
&\leq C\,|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X(v-u)^\alpha.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\left|(\hat{\delta}_2\Xi^{(z)}(y^1,\overline{y}^1))_{vmu}(E)-(\hat{\delta}_2\Xi^{(z)}(y^2,\overline{y}^2))_{vmu}(E)\right|\\
&\leq\left|a_{vm}(E,\omega^S_{mu}(G(y^1_u)-G(y^2_u))-(\hat{\delta}\overline{y}^1)_{mu}+(\hat{\delta}\overline{y}^2)_{mu})\right|\\
&+\left|b_{vm}(E,G(y^1_u)-G(y^1_m)-G(y^2_u)+G(y^2_m))\right|\\
&\leq C\,\|\omega\|_\alpha\,|E|\left|\omega^S_{mu}(G(y^1_u)-G(y^2_u))-(\hat{\delta}(\overline{y}^1-\overline{y}^2))_{mu}\right|(v-u)^\alpha\\
&+C\left(\|\omega\|_\alpha^2+\left\|\omega^{(2)}\right\|_{2\alpha}\right)|E|\left|G(y^1_u)-G(y^1_m)-G(y^2_u)+G(y^2_m)\right|(v-u)^{2\alpha}.
\end{aligned}$$

By applying (3.75) and Lemma A.3 we derive

$$\begin{aligned}&\left|(\hat{\delta}_2\Xi^{(z)}(y^1,\overline{y}^1))_{vmu}(E)-(\hat{\delta}_2\Xi^{(z)}(y^2,\overline{y}^2))_{vmu}(E)\right|\\&\leq C\,|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X u^{-\beta}\,(v-u)^{2\alpha+\beta}\,.\end{aligned}$$

On the other hand we can estimate

$$\begin{aligned}&\left|(\hat{\delta}_2\Xi^{(z)}(y^1,\overline{y}^1))_{vmu}(E)-(\hat{\delta}_2\Xi^{(z)}(y^2,\overline{y}^2))_{vmu}(E)\right|\\&\leq C\,|E|\left(\left|\omega^S_{mu}(G(y^1_u)-G(y^2_u))\right|+\left|(\hat{\delta}(\overline{y}^1-\overline{y}^2))_{mu}\right|\right)(v-u)^{\alpha}\\&+C\,|E|\left(\left|G(y^1_u)-G(y^2_u)\right|+\left|G(y^1_m)-G(y^2_m)\right|\right)(v-u)^{2\alpha}\,.\end{aligned}$$

By applying (3.72) we obtain

$$\begin{aligned}&\left|(\hat{\delta}_2\Xi^{(z)}(y^1,\overline{y}^1))_{vmu}(E)-(\hat{\delta}_2\Xi^{(z)}(y^2,\overline{y}^2))_{vmu}(E)\right|\\\leq&C\,|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X(v-u)^{2\alpha}\,.\end{aligned}$$

Again with Corollary 3.7 we conclude

$$\begin{aligned}&\left|(\hat{\delta}\mathcal{I}[\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2)])_{ts}(E)-(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)\right|\\&\leq C\,|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X(t-s)^{2\alpha}\,.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}&\left|(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)-\omega^S_{ts}(E(\overline{y}^1_s-\overline{y}^2_s))\right|\\&\leq\left|b_{ts}(E,G(y^1_s)-G(y^2_s))\right|+\left|a_{ts}(E,\overline{y}^1_s-\overline{y}^2_s)-\omega^S_{ts}(E(\overline{y}^1_s-\overline{y}^2_s))\right|\,.\end{aligned}$$

Then (3.49) yields

$$\begin{aligned}&\left|(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)-\omega^S_{ts}(E(\overline{y}^1_s-\overline{y}^2_s))\right|\\&\leq C\,|E|\left\|y^1-y^2\right\|_\infty\left(\|\omega\|_\alpha^2+\left\|\omega^{(2)}\right\|_{2\alpha}\right)(t-s)^{2\alpha}\\&+C\,|E|\,\|\omega\|_\alpha\left|\overline{y}^1_s-\overline{y}^2_s\right|_{D_\beta}(t-s)^{\alpha+\beta}\,.\end{aligned}$$

By applying (3.73) we see

$$\begin{aligned}&\left|(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)-\omega^S_{ts}(E(\overline{y}^1_s-\overline{y}^2_s))\right|\\&\leq C\,|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X\left[s^{\alpha-\beta}(t-s)^{\alpha+\beta}+(t-s)^{2\alpha}\right].\end{aligned}$$

Finally, we derive

$$\begin{aligned}\left|(\overline{z}^1_{ts}-\overline{z}^2_{ts})(E)\right|&\leq\left|(\hat{\delta}\mathcal{I}[\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2)])_{ts}(E)-(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)\right|\\&+\left|(\Xi^{(z)}(y^1,\overline{y}^1)-\Xi^{(z)}(y^2,\overline{y}^2))_{ts}(E)-\omega^S_{ts}(E(\overline{y}^1_s-\overline{y}^2_s))\right|\\&\leq C|E|\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X\\&\quad\left[(t-s)^{2\alpha}+s^{\alpha-\beta}(t-s)^{\alpha+\beta}\right].\end{aligned}$$

Consequently, we get

$$\left\|\overline{z}^1-\overline{z}^2\right\|_{\alpha+\beta}\leq C\left(1+\left\|(y^1,z^1)\right\|_X^2+\left\|(y^2,z^2)\right\|_X^2\right)\left\|(y^1-y^2,z^1-z^2)\right\|_X T^{\alpha-\beta}.$$

□

Now, putting all these results together, we can state the main theorem of this section.

Theorem 3.36. *Let $r > 0$ with $|\xi| \leq r$. Then there exist $\varrho = \varrho(r, \omega)$ and $T = T(\omega, \varrho) > 0$ such that the mapping $\mathcal{M}_{T,\varrho} := \mathcal{M}_T|_{B_X(0,\varrho)} \colon B_X(0, \varrho) \to B_X(0, \varrho)$ is a contraction and possesses a unique fixed point.*

Proof. By Theorem 3.33 we know that $\mathcal{M}_T$ maps $X_{\omega,T}$ into itself and

$$\|\mathcal{M}_T(y, z)\|_X \leq C \left(|\xi| + \left(1 + \|(y, z)\|_X^2\right) T^\alpha\right).$$

Setting $\varrho := 2Cr$, we have

$$\|\mathcal{M}_{T,\varrho}(y, z)\|_X \leq \frac{\varrho}{2} + C \left(1 + \varrho^2\right) T^\alpha.$$

Hence, we can choose T small enough and obtain

$$\|\mathcal{M}_{T,\varrho}(y, z)\|_X \leq \varrho,$$

which means that $\mathcal{M}_{T,\varrho}$ maps $B_X(0, \varrho)$ into itself.

Since $\widetilde{y}^1 - \widetilde{y}^2 = \overline{y}^1 - \overline{y}^2$ and $\widetilde{z}^1 - \widetilde{z}^2 = \overline{z}^1 - \overline{z}^2$, applying Lemmas 3.34 and 3.35 we derive

$$\left\|\mathcal{M}_T(y^1, z^1) - \mathcal{M}_T(y^2, z^2)\right\|_X \leq C \left(1 + \left\|(y^1, z^1)\right\|_X^2 + \left\|(y^2, z^2)\right\|_X^2\right) \left\|(y^1 - y^2, z^1 - z^2)\right\|_X T^\alpha.$$

Hence,

$$\left\|\mathcal{M}_{T,\varrho}(y^1, z^1) - \mathcal{M}_{T,\varrho}(y^2, z^2)\right\|_X \leq C \left(1 + 2\varrho^2\right) \left\|(y^1 - y^2, z^1 - z^2)\right\|_X T^\alpha.$$

Again, we can choose T small enough such that

$$\left\|\mathcal{M}_{T,\varrho}(y^1, z^1) - \mathcal{M}_{T,\varrho}(y^2, z^2)\right\|_X \leq \frac{1}{2} \left\|(y^1 - y^2, z^1 - z^2)\right\|_X,$$

which proves the contraction property of $\mathcal{M}_{T,\varrho}$. Consequently, Banach's fixed-point Theorem entails that $\mathcal{M}_{T,\varrho}$ has a unique fixed point in $B_X(0, \varrho)$. □

We showed the existence of a unique local solution in an appropriate ball. This means that another local mild solution for (1.1) could exist outside this ball. The following theorem excludes this case. In order to prove this statement we need some additional results.

For the existence proof we considered a fixed $\boldsymbol{\omega} \in \mathscr{C}^\alpha$ and a fixed initial condition $\xi \in W$. From now on we want to investigate the dependence of the solution on these parameters. Therefore, we emphasize the $\boldsymbol{\omega}$-dependence in the approximating terms $\Xi_\omega^{(y)}$ and $\Xi_\omega^{(z)}$ and introduce the following notation

$$\begin{aligned} \mathcal{M}_{T,\omega,\xi} &\colon X_{\omega,T} \to X_{\omega,T} \\ \mathcal{M}_{T,\omega,\xi}(y, z)_t^{(1)} &= S(t)\xi + \mathcal{I}\Xi_\omega^{(y)}(y, z)_t \\ \mathcal{M}_{T,\omega,\xi}(y, z)_{ts}^{(2)}(E) &= \left(\hat{\delta}_1 \mathcal{I}\Xi_\omega^{(z)}(y, y)\right)_{ts}(E) - \omega_{ts}^S(Ey_s). \end{aligned}$$

Remark 3.37. *We are aware that the given notations are a little bit sloppy. In fact an accurate notation which, for instance emphasizes the $\boldsymbol{\omega}$-dependence of $\Xi_\omega^{(y)}$ could have the form $\Xi_{\boldsymbol{\omega}}^{(y)}$ or $\Xi_{(\omega,\omega^{(2)})}^{(y)}$. However, the first expression is hard to spot while the second one is a too extensive notation. Hence, for notational simplicity we chose the notations given above.*
Furthermore, we are especially interested in considering time shifted noise, see (2.11). Hence, thanks to Lemma 2.8 the second order process will be clear from the context.

The next Lemma is a direct consequence of Remark 3.27

Lemma 3.38. *Let $\tau > 0$. Then the following identities hold true*

$$\begin{aligned}\overline{\theta}_\tau \Xi^{(y)}_\omega(y,z) &= \Xi^{(y)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau z),\\ \overline{\theta}_\tau \Xi^{(z)}_\omega(y,y) &= \Xi^{(z)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau y).\end{aligned}$$

Proof. We directly have that

$$\begin{aligned}\overline{\theta}_\tau \Xi^{(y)}_\omega(y,z)_{vu} &= \Xi^{(y)}_\omega(y,z)_{v+\tau,u+\tau} = \omega^S_{v+\tau,u+\tau}(G(y_{u+\tau})) + z_{v+\tau,u+\tau}(DG(y_{u+\tau}))\\ &= \overline{\theta}_\tau \omega^S_{vu}(G(\overline{\theta}_\tau y_u)) + \overline{\theta}_\tau z_{vu}(DG(\overline{\theta}_\tau y_u)) = \Xi^{(y)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau z)_{vu},\end{aligned}$$

as well as

$$\begin{aligned}\overline{\theta}_\tau \Xi^{(z)}_\omega(y,y)_{vu}(E) &= \Xi^{(z)}_\omega(y,y)_{v+\tau,u+\tau}(E) = b_{v+\tau,u+\tau}(E, G(y_{u+\tau})) + a_{v+\tau,u+\tau}(E, y_{u+\tau})\\ &= \overline{\theta}_\tau b_{vu}(E, G(\overline{\theta}_\tau y_u)) + \overline{\theta}_\tau a_{vu}(E, \overline{\theta}_\tau y_u) = \Xi^{(z)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau y)_{vu}(E).\end{aligned}$$ □

The first step in establishing the uniqueness of the local solution is contained in the next result. Note that this is referred to as the *cocycle property* in the theory of random dynamical systems, see [2]. This will be dealt in Section 4.1.

Lemma 3.39. *Let $T > 0$ and $(y,z) \in X_{\omega,T}$ be a fixed-point of $\mathcal{M}_{T,\omega,\xi}$. Then for any $\tau \in [0,T)$ there exists a fixed-point of $\mathcal{M}_{T-\tau,\theta_\tau\omega,y_\tau}$ given by $(\overline{\theta}_\tau y, \overline{\theta}_\tau z)$.*

Proof. This is a direct consequence of Corollary 3.10 and Lemma 3.38. By standard computations we get

$$\begin{aligned}\overline{\theta}_\tau y_t = y_{t+\tau} &= S(t+\tau)\xi + \mathcal{I}\Xi^{(y)}_\omega(y,z)_{t+\tau}\\ &= S(t)y_\tau + (\hat{\delta}\mathcal{I}\Xi^{(y)}_\omega(y,z))_{t+\tau,\tau}\\ &= S(t)y_\tau + \mathcal{I}\Xi^{(y)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau z)_t.\end{aligned}$$

Furthermore,

$$\begin{aligned}\overline{\theta}_\tau z_{ts}(E) = z_{t+\tau,s+\tau}(E) &= (\hat{\delta}\mathcal{I}\Xi^{(z)}_\omega(y,y))_{t+\tau,s+\tau}(E) - \omega^S_{t+\tau,s+\tau}(Ey_{s+\tau})\\ &= (\hat{\delta}\mathcal{I}\Xi^{(z)}_{\theta_\tau\omega}(\overline{\theta}_\tau y, \overline{\theta}_\tau y))_{ts}(E) - \overline{\theta}_\tau \omega^S_{ts}(E\overline{\theta}_\tau y_s),\end{aligned}$$

where we used Remark 3.27 in the last step. □

Remark 3.40. *If (y,z) is a fixed point of $\mathcal{M}_{T,\omega,\xi}$ than for any $\tilde{T} < T$ the restriction of (y,z) on $[0,\tilde{T}] \times \Delta_{\tilde{T}}$ is a fixed point of $\mathcal{M}_{\tilde{T},\omega,\xi}$.*

Now we can state the uniqueness result of the local solution.

Theorem 3.41. *Let (y^i, z^i), $i = 1,2$ be two fixed-points of $\mathcal{M}_{T,\omega,\xi}$. Then it must hold that $(y^1,z^1) = (y^2,z^2)$.*

Proof. We set $\overline{T} := \sup\left\{\tilde{T} > 0 \colon (y^1,z^1)\mid_{[0,\tilde{T}]} = (y^2,z^2)\mid_{[0,\tilde{T}]}\right\}$ and assume that $(y^1,z^1) \neq (y^2,z^2)$. Then $\overline{T} < T$. Using the continuity of the solution we have that $y^1_{\overline{T}} = y^2_{\overline{T}}$. By Lemma 3.39 we know that $(\theta_{\overline{T}} y^1, \theta_{\overline{T}} z^1)$ and $(\theta_{\overline{T}} y^2, \theta_{\overline{T}} z^2)$ are fixed points of $\mathcal{M}_{T-\overline{T},\theta_{\overline{T}}\omega, y^1_{\overline{T}}}$. According to Remark 3.40 we can choose a small $T^* \in [0, T-\overline{T}]$, apply (3.74) and (3.77). This leads to

$$\begin{aligned}&\left\|(\overline{\theta}_{\overline{T}} y^1, \overline{\theta}_{\overline{T}} z^1) - (\overline{\theta}_{\overline{T}} y^2, \overline{\theta}_{\overline{T}} z^2)\right\|_{X,T^*}\\ &= \left\|\mathcal{M}_{T-\overline{T},\theta_{\overline{T}}\omega,y^1_{\overline{T}}}((\overline{\theta}_{\overline{T}} y^1, \overline{\theta}_{\overline{T}} z^1)) - \mathcal{M}_{T-\overline{T},\theta_{\overline{T}}\omega,y^1_{\overline{T}}}((\overline{\theta}_{\overline{T}} y^2, \overline{\theta}_{\overline{T}} z^2))\right\|_{X,T^*}\\ &\leq C\left(1 + \left\|(\overline{\theta}_{\overline{T}} y^1, \overline{\theta}_{\overline{T}} z^1)\right\|^2_X + \left\|(\overline{\theta}_{\overline{T}} y^2, \overline{\theta}_{\overline{T}} z^2)\right\|^2_X\right)\left\|(\overline{\theta}_{\overline{T}} y^1 - \overline{\theta}_{\overline{T}} y^2, \overline{\theta}_{\overline{T}} z^1 - \overline{\theta}_{\overline{T}} z^2)\right\|_X (T^*)^\alpha.\end{aligned}$$

If T^* is sufficiently small we see that $(\overline{\theta}_{\overline{T}} y^1, \overline{\theta}_{\overline{T}} z^1) = (\overline{\theta}_{\overline{T}} y^2, \overline{\theta}_{\overline{T}} z^2)$ on $[0,T^*]$ which yields $(y^1,z^1) = (y^2,z^2)$ on $[0, \overline{T}+T^*]$. Therefore, we obviously reached a contradiction with the definition of $\overline{T}$. □

We conclude this section collecting three important results which immediately follow from the previous deliberations. We first indicate why taking more regular initial data leads to simpler arguments.

Corollary 3.42. *If $\xi \in D_\beta$ and (y,z) is the unique fixed-point of $\mathcal{M}_{T,\omega,\xi}$ we have that $y \in C^\beta$ and $z \in C^{\alpha+\beta}$.*

Proof. Since

$$(\delta y)_{ts} = (\delta \widetilde{y})_{ts} = (\hat{\delta}\overline{y})_{ts} + (S(t-s) - \mathrm{Id})\overline{y}_s + (S(t) - S(s))\xi.$$

Using (3.64) and (3.65) we conclude that

$$\|\!|y|\!\|_\beta \leq C|\xi|_{D_\beta} + C(1 + ||(y,z)||_X^2)T^{\alpha-\beta}.$$

Recall that

$$\widetilde{z}_{ts}(E) = \overline{z}_{ts}(E) + a_{ts}(E, S(s)\xi) - \omega^S_{ts}(ES(s)\xi).$$

Therefore, applying (3.70) and (3.49) proves the statement. □

Furthermore, similar to deterministic Evolution Equation we see a smoothing effect for the path component of the solution.

Corollary 3.43. *Let (y,z) be the unique fixed-point of $\mathcal{M}_{T,\omega,\xi}$ then we have for all $0 < t \leq T$.*

$$|y_t|_{D_\beta} \leq C\left(|\xi|\, t^{-\beta} + (1 + \|(y,z)\|_X^2)\, t^{\alpha-\beta}\right). \tag{3.78}$$

Proof. The proof ís a direct consequence of (2.1) and (3.65) since

$$|y_t|_{D_\beta} = |S(t)\,\xi + \overline{y}_t|_{D_\beta} \leq |S(t)\xi|_{D_\beta} + |\overline{y}_t|_{D_\beta}$$ □

Remark 3.44. *Keeping Lemma 3.19 and* (3.59) *in mind one can easily show that the solution continuously depends on the noisy input.*

3.6 Global Solutions for Rough Evolution Equations

As recalled in the previous section, working with (3.63) leads to quadratic estimates for the norm of (y,z) in $X_{\omega,T}$. From this approach it is not clear how/if one can extend the unique local solution on an arbitrary time horizon. Therefore, we need different arguments for the global-in-time existence. To this aim, similar to the finite-dimensional case, see [20, Section 8.5], it is convenient to work with the norm of certain remainder terms, which is common in the rough paths theory.

Definition 3.45. Let $(y,z) \in X_{\omega,T}$. Then we define the remainders

$$\begin{aligned} R^y_{ts} &:= (\hat{\delta}y)_{ts} - \omega^S_{ts}(G(y_s)),\\ R^z_{ts}(E) &:= z_{ts}(E) - b_{ts}(E, G(y_s)). \end{aligned}$$

Remark 3.46. *If $S = Id$ and (y,z) is a fixed-point of $\mathcal{M}$, then the previous terms read as*

$$R^y_{ts} = (\delta y)_{ts} - G(y_s)(\delta\omega)_{ts},$$

respectively

$$R^z_{ts}(E) = E\int_s^t R^y_{rs} d\omega_r.$$

The expression for the remainder R^y is the same as the one in the finite-dimensional case, compare [20, Section 8.5]. In contrast to the finite-dimensional setting, R^z is required here to estimate the quadratic terms appearing in (3.71).

Definition 3.47. Let $(y, z) \in X_{\omega,T}$ such that

$$\Phi_T(y, z) := \|y\|_{\infty,D_{2\beta},T} + \|R^y\|_{2\beta,T} + \|R^z\|_{\alpha+2\beta,T} < \infty, \tag{3.79}$$

where

$$\|R^y\|_{2\beta,T} := \sup_{0\leq s<t\leq T} \frac{|R^y_{ts}|}{(t-s)^{2\beta}},$$
$$\|R^z\|_{\alpha+2\beta,T} := \sup_{0\leq s<t\leq T} \sup_{|E|\leq 1} \frac{|R^z_{ts}(E)|}{(t-s)^{\alpha+2\beta}}.$$

The space of all pairs (y, z) satisfying (3.79) is denoted by $\widetilde{X}_{\omega,T}$.

This leads to the next result.

Lemma 3.48. *Let $(y, z) \in \widetilde{X}_{\omega,T}$. Then we obtain the following estimates*

$$|||y|||_{\beta,T} \leq C\left(T^{\alpha-\beta} + T^{\beta}\Phi_T(y, z)\right), \tag{3.80}$$
$$\|z\|_{\alpha+\beta,T} \leq C\left(T^{\alpha-\beta} + T^{\beta}\Phi_T(y, z)\right). \tag{3.81}$$

Proof. Regarding the definition of R^y, and applying (3.47) yields

$$\begin{aligned}
|||y|||_{\beta,T} &= \sup_{0\leq s<t\leq T} \frac{|(\delta y)_{ts}|}{(t-s)^{\beta}} \\
&\leq \sup_{0\leq s<t\leq T} \frac{|R^y_{ts}|}{(t-s)^{\beta}} + \sup_{0\leq s<t\leq T} \frac{|(S(t-s)-\mathrm{Id})y_s|}{(t-s)^{\beta}} + \sup_{0\leq s<t\leq T} \frac{\left|\omega^S_{ts}(G(y_s))\right|}{(t-s)^{\beta}} \\
&\leq \|R^y\|_{2\beta,T}\, T^{\beta} + C\, \|y\|_{\infty,D_{2\beta},T}\, T^{\beta} + C\, |||\omega|||_{\alpha,T}\, T^{\alpha-\beta} \\
&\leq C\left(T^{\alpha-\beta} + T^{\beta}\Phi_T(y, z)\right),
\end{aligned}$$

which proves the first statement.

Furthermore, due to (3.58), the estimates for z result in

$$\begin{aligned}
\|z\|_{\alpha+\beta,T} &= \sup_{0\leq s<t\leq T} \sup_{|E|\leq 1} \frac{|z_{ts}(E)|}{(t-s)^{\alpha+\beta}} \\
&\leq \sup_{0\leq s<t\leq T} \sup_{|E|\leq 1} \frac{|R^z_{ts}(E)|}{(t-s)^{\alpha+\beta}} + \sup_{0\leq s<t\leq T} \sup_{|E|\leq 1} \frac{|b_{ts}(E, G(y_s))|}{(t-s)^{\alpha+\beta}} \\
&\leq \|R^z\|_{\alpha+2\beta,T}\, T^{\beta} + C\left(|||\omega|||^2_{\alpha} + \left\|\omega^{(2)}\right\|_{2\alpha}\right) T^{\alpha-\beta} \\
&\leq C\left(T^{\alpha-\beta} + T^{\beta}\Phi_T(y, z)\right).
\end{aligned}$$

□

The next result indicates the connection between the space-regularity of y and of the initial data ξ.

Lemma 3.49. *Let $\xi \in D_{\beta}$ and (y, z) be a fixed-point of $\mathcal{M}_{T,\omega,\xi}$. Then for $t \in (0, T]$ we have that $y_t \in D_{2\beta}$.*

Proof. By Corollary 3.42 we know that $y \in C^{\beta}$ and $z \in C^{\alpha+\beta}$. In order to apply Corollary 3.11 we have to estimate

$$(\hat{\delta}_2\Xi^{(y)})_{vmu} = \omega^S_{vm}(G(y_u) - G(y_m)) + (\hat{\delta}_2 z)_{vmu}(DG(y_u)) + z_{vm}(DG(y_u) - DG(y_m)),$$

see (3.13). Therefore, we have

$$\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| \leq \left|\omega^S_{vm}(G(y_u)-G(y_m)+DG(y_u)(\delta y)_{mu})\right| + \left|z_{vm}(DG(y_u)-DG(y_m))\right|. \quad (3.82)$$

For the first term we have applying (3.47) that

$$\left|\omega^S_{vm}(G(y_u)-G(y_m)+DG(y_u)(\delta y)_{mu})\right| \leq C \left\|\omega\right\|_\alpha (v-m)^\alpha \left|G(y_u)-G(y_m)+DG(y_u)(\delta y)_{mu}\right|.$$

Furthermore,

$$\left|G(y_u)-G(y_m)+DG(y_u)(\delta y)_{mu}\right| \leq C \left\|y\right\|^2_{\beta,T} (m-u)^{2\beta},$$

and

$$\left|z_{vm}(DG(y_u)-DG(y_m))\right| \leq C \left\|z\right\|_{\alpha+\beta} \left\|y\right\|_{\beta,T} (v-u)^{\alpha+2\beta}.$$

Summarizing, we obtain

$$\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| \leq C\left(1+\left\|y\right\|^2_{\beta,T}+\left\|z\right\|^2_{\alpha+\beta,T}\right)(v-u)^{\alpha+2\beta}. \quad (3.83)$$

On the other hand

$$\begin{aligned}\left|S(v-u)\Xi^{(y)}_{vu}\right|_{D_{2\beta}} &\leq \left|S(v-u)\omega^S_{vu}(G(y_u))\right|_{D_{2\beta}} + \left|S(v-u)z_{vu}(DG(y_u))\right|_{D_{2\beta}}\\ &\leq C(v-u)^{-\beta}\left|\omega^S_{vu}(G(y_u))\right|_{D_\beta} + C(v-u)^{-2\beta}\left|z_{vu}(DG(y_u))\right|.\end{aligned}$$

Using Lemma 3.21 we get

$$\left|S(v-u)\Xi^{(y)}_{vu}\right|_{D_{2\beta}} \leq C(1+\left\|z\right\|_{\alpha+\beta,T})(v-u)^{\alpha-\beta}.$$

Hence, Corollary 3.11 entails

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts}\right|_{D_{2\beta}} = \left|(\hat{\delta}y)_{ts}\right|_{D_{2\beta}} \leq C\left(1+\left\|y\right\|^2_{\beta,T}+\left\|z\right\|^2_{\alpha+\beta,T}\right)(t-s)^{\alpha-\beta},$$

which simply yields

$$\begin{aligned}\left|y_t\right|_{D_{2\beta}} &\leq \left|S(t)\xi\right|_{D_{2\beta}} + \left|(\hat{\delta}y)_{t0}\right|_{D_{2\beta}}\\ &\leq Ct^{-\beta}\left|\xi\right|_{D_\beta} + C\left(1+\left\|y\right\|^2_{\beta,T}+\left\|z\right\|^2_{\alpha+\beta,T}\right)t^{\alpha-\beta}. \quad (3.84)\end{aligned}$$

This proves the statement. □

Lemma 3.50. *If $\xi \in D_{2\beta}$ and (y,z) is a fixed-point of $\mathcal{M}_{T,\omega,\xi}$ then $\Phi_T(y,z) < \infty$.*

Proof. First of all, note that if $\xi \in D_{2\beta}$, (3.84) immediately entails that $\|y\|_{\infty,2\beta,T} < \infty$. We now investigate R^y. To this aim, we verify (3.19) using (3.47). This obviously results in

$$\begin{aligned}\left|\Xi^{(y)}_{vu}\right| &\leq \left|\omega^S_{vu}(G(y_u))\right| + \left|z_{vu}(DG(y_u))\right|\\ &\leq C\|\omega\|_\alpha(v-u)^\alpha + C\|z\|_\alpha(v-u)^\alpha.\end{aligned}$$

Now (3.20) is verified by (3.83). Therefore, we obtain (3.23), namely

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} - \Xi^{(y)}_{ts}\right| \leq C\left(1+\left\|y\right\|^2_{\beta,T}+\left\|z\right\|^2_{\alpha+\beta,T}\right)(t-s)^{\alpha+\beta}.$$

This yields

$$\begin{aligned}|R^y_{ts}| = |(\hat{\delta}y)_{ts} - \omega^S_{ts}(G(y_s))| &\leq |(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} - \Xi^{(y)}_{ts}| + |z_{ts}(DG(y_s))| \\ &\leq C\left(1 + \|y\|^2_{\beta,T} + \|z\|^2_{\alpha+\beta,T} + \|z\|_{\alpha+\beta,T}\right)(t-s)^{\alpha+\beta}.\end{aligned}$$

We infer from the previous computation that $\|R^y\|_{2\beta,T} < \infty$.

We now prove that $\|R^z\|_{\alpha+2\beta,T} < \infty$. We make the same deliberations as for R^y. Estimates (3.48) and (3.58) entail

$$|\Xi^{(z)}_{vu}(E)| \leq |b_{vu}(E, G(y_u))| + |a_{vu}(E, y_u)| \leq C|E|(1 + \|y\|_{\infty,T})(v-u)^{\alpha}$$

and

$$\begin{aligned}\left|(\hat{\delta}_2\Xi^{(z)})_{vmu}(E)\right| &= |a_{vm}(E, \omega^S_{mu}(G(y_u)) - (\hat{\delta}y)_{mu}) + b_{vm}(E, G(y_u) - G(y_m))| \\ &\leq C\,\|\omega\|_{\alpha}(v-m)^{\alpha}\,|E|\,\|R^y\|_{2\beta,T}(m-u)^{2\beta} \\ &\quad + C\left(\|\omega\|^2_{\alpha} + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(v-m)^{2\alpha}\,|E|\,\|y\|_{\beta,T}(m-u)^{\beta} \\ &\leq C|E|(\|R^y\|_{2\beta,T} + \|y\|_{\beta,T})(v-u)^{\alpha+2\beta}.\end{aligned}$$

Thus, (3.23) implies

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| \leq C|E|(\|R^y\|_{2\beta,T} + \|y\|_{\beta,T})(t-s)^{\alpha+2\beta}. \tag{3.85}$$

Consequently

$$\begin{aligned}|R^z_{ts}(E)| &= |z_{ts}(E) - b_{ts}(E, G(y_s))| \\ &\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| + |a_{ts}(E, y_s) - \omega^S(Ey_s)|.\end{aligned}$$

Applying (3.49) yields

$$|R^z_{ts}(E)| \leq C|E|(\|R^y\|_{2\beta,T} + \|y\|_{\beta,T})(t-s)^{\alpha+2\beta} + C|E||y_s|_{D_{2\beta}}(t-s)^{\alpha+2\beta}.$$

This proves that $\|R^z\|_{\alpha+2\beta,T} < \infty$. □

We now derive the following a-priori estimate of the solution mapping of (1.1). The computations rely on similar arguments as in the previous Lemma.

Lemma 3.51. *Let $\xi \in D_{2\beta}$ and let (y,z) be a fixed-point of $\mathcal{M}_{T,\omega,\xi}$ with $0 < T \leq 1$. Then it holds*

$$\Phi_T(y,z) \leq C\left(|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)\right). \tag{3.86}$$

Proof. Recall that $\Phi_T(y,z) = \|y\|_{\infty,D_{2\beta},T} + \|R^y\|_{2\beta,T} + \|R^z\|_{\alpha+2\beta,T}$. We begin with $\|R^y\|_{2\beta,T}$. and further use that

$$\begin{aligned}|G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}| &= \left|\int_0^1 [DG(y_u + q(\delta y)_{mu}) - DG(y_u)]\,dq\;(\delta y)_{mu}\right| \\ &\leq \int_0^1 |DG(y_u + q(\delta y)_{mu}) - DG(y_u)|\,dq \\ &\qquad \cdot \left[|R^y_{mu}| + |(S(m-u) - \mathrm{Id})y_u| + \left|\omega^S_{mu}(G(y_u))\right|\right] \\ &\leq C\left[\|R^y\|_{2\beta,T}(m-u)^{2\beta} + \|y\|_{\infty,D_{2\beta},T}(m-u)^{2\beta}\right. \\ &\qquad \left. + \|y\|_{\beta,T}\,\|\omega\|_{\alpha}(m-u)^{\alpha+\beta}\right].\end{aligned}$$

Applying (3.80) results in

$$|G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu}| \leq C\left[\Phi_T(y,z)(m-u)^{2\beta} + (T^{\alpha-\beta} + T^{\beta}\Phi_T(y,z))(m-u)^{\alpha+\beta}\right].$$

All in all we obtain for the first term in (3.82)

$$\left|\omega^S_{vm}(G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu})\right| \leq C(1+\Phi_T(y,z))(v-u)^{\alpha+2\beta}.$$

For the second term in (3.82) we have

$$\begin{aligned}
&|z_{vm}(DG(y_u) - DG(y_m))| \\
&\leq |R^z_{vm}(DG(y_u) - DG(y_m))| + |b_{vm}(DG(y_u) - DG(y_m), G(y_m))| \\
&\leq C\,\|R^z\|_{\alpha+2\beta,T}\,(v-m)^{\alpha+2\beta} + C\left(\|\omega\|^2_{\alpha} + \left\|\omega^{(2)}\right\|_{2\alpha}\right)(v-m)^{2\alpha}\,\|y\|_{\beta,T}\,(m-u)^{\beta}.
\end{aligned}$$

Again, we apply (3.80) and derive

$$|z_{vm}(DG(y_u) - DG(y_m))| \leq C(1+\Phi_T(y,z))(v-u)^{\alpha+2\beta}.$$

Summarizing, we obtain

$$\left|(\hat{\delta}_2\Xi^{(y)})_{vmu}\right| \leq C(1+\Phi_T(y,z))\,(v-u)^{\alpha+2\beta}.$$

Then (3.23) yields

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts} - \Xi^{(y)}_{ts}\right| \leq C(1+\Phi_T(y,z))\,(t-s)^{\alpha+2\beta}.$$

Consequently,

$$\begin{aligned}
|R^y_{ts}| &= \left|(\hat{\delta}y)_{ts} - \omega^S_{ts}(G(y_s))\right| \\
&\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(y)}_{ts} - \Xi^{(y)}_{ts}\right| + |z_{ts}(DG(y_s))| \\
&\leq C(1+\Phi_T(y,z))\,(t-s)^{\alpha+2\beta} + \|z\|_{\alpha+\beta,T}\,(t-s)^{\alpha+\beta}.
\end{aligned}$$

Now (3.81) entails the first important estimate on the 2β-norm of R^y, namely

$$\|R^y\|_{2\beta,T} \leq C\left(T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)\right). \tag{3.87}$$

We now continue investigating $\|y\|_{\infty,D_{2\beta},T}$. In order to apply Corollary 3.11 we firstly consider

$$\begin{aligned}
\left|S(v-u)\Xi^{(y)}_{vu}\right|_{D_{2\beta}} &\leq \left|S(v-u)\omega^S_{vu}(G(y_u))\right|_{D_{2\beta}} + |S(v-u)z_{vu}(DG(y_u))|_{D_{2\beta}} \\
&\leq C\,(v-u)^{-\beta}\left|\omega^S_{vu}(G(y_u))\right|_{D_{\beta}} + C(v-u)^{-2\beta}\,|z_{vu}(DG(y_u))|.
\end{aligned}$$

Using Lemma 3.21 and (3.81) we get

$$\begin{aligned}
\left|S(v-u)\Xi^{(y)}_{vu}\right|_{D_{2\beta}} &\leq C(1+\|z\|_{\alpha+\beta,T})\,(v-u)^{\alpha-\beta} \\
&\leq C(1+T^{\beta}\Phi_T(y,z))(v-u)^{\alpha-\beta}.
\end{aligned}$$

Hence, by Corollary 3.11 we obtain

$$\begin{aligned}
\left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{ts}\right|_{D_{2\beta}} = \left|(\hat{\delta}y)_{ts}\right|_{D_{2\beta}} &\leq C(1+T^{\beta}\Phi_T(y,z))(t-s)^{\alpha-\beta} + C(1+\Phi_T(y,z))(t-s)^{\alpha} \\
&\leq C(T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)).
\end{aligned}$$

Regarding this we immediately obtain

$$\begin{aligned}|y_t|_{D_{2\beta}} &\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(y)})_{t0}\right|_{D_{2\beta}} + |S(t)\xi|_{D_{2\beta}}\\ &\leq C(|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)).\end{aligned}$$

This obviously implies the second important estimate, namely

$$\|y\|_{\infty,D_{2\beta},T} \leq C(|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)). \tag{3.88}$$

Finally, we only have to compute $\|R^z\|_{\alpha+2\beta,T}$ analogously to the proof of Lemma 3.50.

Applying (3.87) and (3.80) to (3.85) entails

$$\left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| \leq C(T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z))\,|E|\,(t-s)^{\alpha+2\beta}.$$

Consequently, (3.49) further leads to

$$\begin{aligned}|R^z_{ts}(E)| &\leq \left|(\hat{\delta}\mathcal{I}\Xi^{(z)})_{ts}(E) - \Xi^{(z)}_{ts}(E)\right| + \left|a_{ts}(E,y_s) - \omega^S_{ts}(Ey_s)\right|\\ &\leq C(T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z))\,|E|\,(t-s)^{\alpha+2\beta} + C\,\|\omega\|_{\alpha}\,\|y\|_{\infty,D_{2\beta},T}\,|E|\,(t-s)^{\alpha+2\beta}.\end{aligned}$$

Regarding this and plugging in (3.88), we derive the third and final important estimate for the terms defining Φ_T, namely

$$\|R^z\|_{\alpha+2\beta,T} \leq C\left(|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)\right). \tag{3.89}$$

This proves the statement, i.e.

$$\Phi_T(y,z) \leq C\left(|\xi|_{D_{2\beta}} + T^{\alpha-\beta} + T^{\alpha}\Phi_T(y,z)\right).$$

□

We now derive a crucial estimate which will be required for the concatenation procedure.

Lemma 3.52. *Let $T > 0$, $r \geq 1 \vee |\xi|_{D_{2\beta}}$ and let (y,z) be a fixed-point of $\mathcal{M}_{T,\omega,\xi}$. Then there exists a constant $M > 0$ independent of r, such that*

$$\|y\|_{\infty,D_{2\beta},T} \leq rMe^{MT}.$$

Proof. By Remark 3.40 we know that by restricting the solution on a smaller time interval $[0,\tilde{T}]$, with $\tilde{T} < T$, we obtain a fixed-point of $\mathcal{M}_{\tilde{T},\omega,\xi}$. According to Lemma 3.51 we have for all $0 < \tilde{T} \leq 1$ that

$$\Phi_{\tilde{T}}(y,z) \leq C\left(|\xi|_{D_{2\beta}} + \tilde{T}^{\alpha-\beta} + \tilde{T}^{\alpha}\Phi_{\tilde{T}}(y,z)\right).$$

We now choose $0 < T_0^* \leq \tilde{T}$ sufficiently small such that $C(T_0^*)^{\alpha} \leq \frac{1}{2}$. This yields for all $T_0 < T_0^*$ that

$$\Phi_{T_0}(y,z) \leq 2C\left(|\xi|_{D_{2\beta}} + 1\right) \leq 4Cr.$$

Consequently, this means that

$$\|y\|_{\infty,D_{2\beta},T_0} \leq 4Cr.$$

At this point it is important to note that the choice of T_0^* is independent of r and T.

If $T \leq T_0^*$ the statement follows choosing $M \geq 4C$. Otherwise we can find an $N \in \mathbb{N}$ (not necessarily unique), such that $\frac{T_0^*}{2} < \frac{T}{N} \leq T_0^*$. In this case we set $T_0 := \frac{T}{N}$.

Now, combining Lemma 3.51 and Lemma 3.39 we obtain for $1 \leq n \leq N-1$ that

$$\Phi_{T_0}(\widetilde{\theta}_{nT_0}y, \widetilde{\theta}_{nT_0}z) \leq C\left(|y_{nT_0}|_{D_{2\beta}} + T_0^{\alpha-\beta} + T_0^{\alpha}\Phi_{T_0}(\widetilde{\theta}_{nT_0}y, \widetilde{\theta}_{nT_0}z)\right),$$

Since $CT_0^\alpha \leq \frac{1}{2}$ and $y_{nT_0} = (\widetilde{\theta}_{(n-1)T_0}y)_{T_0}$, the previous estimate results in

$$\Phi_{T_0}(\widetilde{\theta}_{nT_0}y, \widetilde{\theta}_{nT_0}z) \leq 2C\Big(\left\|\widetilde{\theta}_{(n-1)T_0}y\right\|_{\infty,D_{2\beta},T_0} + 1\Big),$$

which yields

$$\left\|\widetilde{\theta}_{nT_0}y\right\|_{\infty,D_{2\beta},T_0} \leq \Phi_{T_0}(\widetilde{\theta}_{nT_0}y, \widetilde{\theta}_{nT_0}z) \leq 2C\Big(\left\|\widetilde{\theta}_{(n-1)T_0}y\right\|_{\infty,D_{2\beta},T_0} + 1\Big).$$

By induction we infer that

$$\left\|\widetilde{\theta}_{nT_0}y\right\|_{\infty,D_{2\beta},T_0} \leq (4C)^{n+1}r, \quad \text{for all } n = 0, \ldots, N-1.$$

From this we finally conclude

$$\|y\|_{\infty,D_{2\beta},T} = \max_{n=0,\ldots,N-1}\left\|\widetilde{\theta}_{nT_0}y\right\|_{\infty,D_{2\beta},T_0} \leq (4C)^N r = (4C)^{\frac{T}{T_0}} r \leq \left((4C)^{\frac{2}{T_0^*}}\right)^T r \leq Me^{MT}r.$$

for a sufficiently large M. □

Now we state the main step required in order to obtain a global solution. We show that by the concatenation of two local solutions we obtain a solution on a larger time interval. For similar arguments and techniques, see [27].

Lemma 3.53. *Let (y^1, z^1) be a fixed-point of $\mathcal{M}_{T_1,\omega,\xi}$ and (y^2, z^2) be a fixed-point of $\mathcal{M}_{T_2,\theta_{T_1}\omega,y^1_{T_1}}$. Then we obtain a fixed-point (y, z) of $\mathcal{M}_{T_1+T_2,\omega,\xi}$ via*

$$y_t := \begin{cases} y^1_t, & 0 \leq t \leq T_1 \\ y^2_{t-T_1}, & T_1 \leq t \leq T_1 + T_2, \end{cases}$$

and

$$z_{ts}(E) = \begin{cases} z^1_{ts}(E), & 0 \leq s \leq t \leq T_1 \\ \omega^S_{tT_1}(E(\delta y^1)_{T_1 s}) + z^2_{t-T_1,0}(E) + S(t-T_1)z^1_{T_1 s}(E), & 0 \leq s \leq T_1 \leq t \leq T_1 + T_2 \\ z^2_{t-T_1,s-T_1}(E), & T_1 \leq s \leq t \leq T_1 + T_2. \end{cases}$$

Proof. The statement follows by a standard computation. We only focus on certain cases, since the rest are straightforward. For the beginning we consider $T_1 \leq t \leq T_1 + T_2$. We recall that we use the notation $\Xi^{(y/z)}_\omega$ and $\Xi^{(y/z)}_{\theta.\omega}$ in order to indicate the appropriate shifts with respect to ω.

$$\begin{aligned} S(t)\xi + \mathcal{I}\Xi^{(y)}_\omega(y,z)_t &= S(t-T_1)S(T_1)\xi + S(t-T_1)\mathcal{I}\Xi^{(y)}_\omega(y,z)_{T_1} + (\hat{\delta}\mathcal{I}\Xi^{(y)}_\omega(y,z))_{tT_1} \\ &= S(t-T_1)\left(S(T_1)\xi + \mathcal{I}\Xi^{(y)}_\omega(y^1,z^1)_{T_1}\right) + (\hat{\delta}\mathcal{I}\Xi^{(y)}_\omega(y^2_{\cdot-T_1}, z_{\cdot-T_1,\cdot-T_1}))_{tT_1} \\ &= S(t-T_1)y^1_{T_1} + (\hat{\delta}\mathcal{I}\Xi^{(y)}_\omega(y^2_{\cdot-T_1}, z^2_{\cdot-T_1,\cdot-T_1}))_{tT_1}. \end{aligned}$$

Recall that

$$\Xi^{(y)}_\omega(y^2_{\cdot-T_1}, z^2_{\cdot-T_1,\cdot-T_1})_{vu} = \omega^S_{vu}(G(y^2_{u-T_1})) + z^2_{v-T_1,u-T_1}(DG(y^2_{u-T_1})),$$

which further leads to

$$\begin{aligned}\widetilde{\theta}_{T_1}\Xi^{(y)}_{\omega}(y^2_{\cdot-T_1},z^2_{\cdot-T_1,\cdot-T_1})vu &= \omega^S_{v+T_1,u+T_1}(G(y^2_u))+z^2_{vu}(DG(y^2_u))\\ &= \widetilde{\theta}_{T_1}\omega^S_{vu}(G(y^2_u))+z^2_{vu}(DG(y^2_u)).\end{aligned}$$

Now, Lemma 3.10 and Lemma 3.38 entail

$$(\hat{\delta}\mathcal{I}\Xi^{(y)}_{\omega}(y^2_{\cdot-T_1},z^2_{\cdot-T_1,\cdot-T_1}))_{tT_1}=(\hat{\delta}\mathcal{I}\Xi^{(y)}_{\theta_{T_1}\omega}(y^2,z^2))_{t-T_1,0}.$$

Consequently,

$$S(t)\xi+\mathcal{I}\Xi^{(y)}_{\omega}(y,z)_t=S(t-T_1)y^1_{T_1}+\mathcal{I}\Xi^{(y)}_{\theta_{T_1}\omega}(y^2,z^2)_{t-T_1}=y^2_{t-T_1}=y_t.$$

Now, let $0\le s\le T_1\le t\le T_1+T_2$. Then we have

$$\begin{aligned}&(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y,y))_{ts}(E)-\omega^S_{ts}(Ey_s)\\ =&(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y,y))_{tT_1}(E)+S(t-T_1)(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y,y))_{T_1s}(E)-\omega^S_{tT_1}(Ey_s)-S(t-T_1)\omega^S_{T_1s}(Ey_s)\\ =&(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y^2_{\cdot-T_1},y^2_{\cdot-T_1}))_{tT_1}(E)+S(t-T_1)(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y^1,y^1))_{T_1s}(E)-\omega^S_{tT_1}(Ey^1_s)-S(t-T_1)\omega^S_{T_1s}(Ey^1_s)\\ =&S(t-T_1)z^1_{T_1s}(E)+\omega^S_{tT_1}(E(\delta y^1)_{T_1s})+(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y^2_{\cdot-T_1},y^2_{\cdot-T_1}))_{tT_1}(E)-\omega^S_{tT_1}(Ey^2_0),\end{aligned}$$

where we use in the last step that $y^2_0=y^1_{T_1}$.

Hence, we infer using Lemma 3.10 and 3.38 that

$$\begin{aligned}&(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\omega}(y,y))_{ts}(E)-\omega^S_{ts}(Ey_s)\\ =&S(t-T_1)z^1_{T_1s}(E)+\omega^S_{tT_1}(E(\delta y^1)_{T_1s})+(\hat{\delta}\mathcal{I}\Xi^{(z)}_{\theta_{T_1}\omega}(y^2,y^2))_{t-T_1,0}(E)-\omega^S_{tT_1}(Ey^2_0)\\ =&S(t-T_1)z^1_{T_1s}(E)+\omega^S_{tT_1}(E(\delta y^1)_{T_1s})+z^2_{t-T_1,0}(E)=z_{ts}(E).\end{aligned}$$

□

Regarding all the previous deliberations we can now state the main results of this section.

Theorem 3.54. *Let ξ in $D_{2\beta}$. Then for any $T>0$ there exists a unique global solution, i.e. there exists a unique fixed-point of $\mathcal{M}_{T,\omega,\xi}$.*

Proof. Let $r=1\vee|\xi|_{D_{2\beta}}$. By Lemma 3.52 we know that every fixed-point of $\mathcal{M}_{T,\omega,\xi}$ must satisfy the estimate

$$\|y\|_{\infty,D_{2\beta},T}\le rMe^{MT}=:\widetilde{r}.$$

Particularly, this means that $|y_t|_{D_{2\beta}}\le\widetilde{r}$, for all $t\le T$. Applying Theorem 3.36 with $|\xi|\le|\xi|_{D_{2\beta}}\le\widetilde{r}$ entails the existence of a local solution on a time interval $[0,T^*]$, where $T^*=T^*(\widetilde{r})$, i.e. there is a fixed-point (y,z) of $\mathcal{M}_{T^*,\omega,\xi}$. For simplicity, since we can choose T^* arbitrary small, we set $N:=\frac{T}{T^*}\in\mathbb{N}$ for $N\ge 2$.

Note that $|y_{T^*}|\le|y_{T^*}|_{D_{2\beta}}\le\widetilde{r}$. Hence, we can derive by using again Theorem 3.36 the existence of a unique fixed-point of $\mathcal{M}_{T^*,\theta_{T^*}\omega,y_{T^*}}$. Furthermore, Lemma 3.53 shows that we can concatenate them and obtain a fixed-point (y,z) of $\mathcal{M}_{2T^*,\omega,\xi}$. Again we have $|y_{2T^*}|\le\widetilde{r}$.

Iterating this argument entails the existence of a unique fixed-point (y,z) of $\mathcal{M}_{T,\omega,\xi}$ for any $T>0$ which is unique by Theorem 3.41. □

Corollary 3.55. *Let ξ in W. Then for any $T>0$ there exists a unique fixed-point of $\mathcal{M}_{T,\omega,\xi}$.*

Proof. Theorem 3.36 guarantees the existence of a unique fixed-point (y,z) of $\mathcal{M}_{T_1,\omega,\xi}$, where $T_1=T_1(\omega,\xi)$. Furthermore, due to (3.78) we obtain $y_{T_1}\in D_\beta$. Then, by Theorem 3.36 we know that there exists a unique-fixed point of $\mathcal{M}_{T_2,\theta_{T_1}\omega,y_{T_1}}$, which according to Lemma 3.53 can be concatenated with the previous one to a fixed-point (y,z) of $\mathcal{M}_{T_1+T_2,\omega,\xi}$. Lemma 3.49 entails $y_{T_1+T_2}\in D_{2\beta}$. Hence, we are in the setting of Theorem 3.54 and obtain the existence of a global fixed-point of $\mathcal{M}_{T-T_1-T_2,\theta_{T_1+T_2}\omega,y_{T_1+T_2}}$. Again, this can be concatenated to a fixed-point of $\mathcal{M}_{T,\omega,\xi}$ due to Lemma 3.53. This procedure gives us the global-in-time solution. □

3.7 An Application

We indicate an example for the abstract theory proven above. For further applications consult [25, Section 5] and [27, Section 7].

Example 3.56. We consider an open bounded C^2-domain $\mathcal{O} \in \mathbb{R}^d$, for $d \geq 1$. Furthermore, let A stand for the Laplace operator or for a second order uniformly elliptic operator augmented by Dirichlet boundary conditions. Then we know that A generates an analytic C_0-semigroup on $W := L^2(\mathcal{O})$. Moreover, we can identify the domains of the fractional powers of A with Sobolev-Slobodetski spaces depending on the range of θ. We have according to Theorem 16.12 in [60] that

$$D((-A)^\theta) = \begin{cases} H^{2\theta}(\mathcal{O}), & 0 \leq \theta < 1/4 \\ H_D^{2\theta}(\mathcal{O}), & 1/4 < \theta \leq 1. \end{cases}$$

Here H_D stands for the Sobolev space that incorporates the boundary conditions, in particular $D(-A) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$.

Having stated the assumptions on the linear part we now focus on G. Therefore, we firstly set for simplicity $V := L^2(\mathcal{O})$. Let $g : \overline{\mathcal{O}} \times \mathbb{R} \to \mathbb{R}$ be a three times continuously differentiable function with bounded derivatives which is zero on $\{0, 1\} \times \mathbb{R}$. We interpret g as the kernel of the following integral operator

$$G(\varphi)(\psi)[x] := \int_{\mathcal{O}} g(x, \varphi(\widetilde{x}))\psi(\widetilde{x})\; d\widetilde{x}. \tag{3.90}$$

As in [43, Section XVII.3] one can show that G is three times continuously Frèchet-differentiable and compute the derivatives as follows

$$\begin{aligned} DG(\varphi)(\psi, h_1)[x] &= \int_{\mathcal{O}} D_2 g(x, \varphi(\widetilde{x}))\psi(\widetilde{x})h_1(\widetilde{x})\; d\widetilde{x}, \\ D^2G(\varphi)(\psi, h_1, h_2)[x] &= \int_{\mathcal{O}} D_2^2 g(x, \varphi(\widetilde{x}))\psi(\widetilde{x})h_1(\widetilde{x})h_2(\widetilde{x})\; d\widetilde{x}, \\ D^3G(\varphi)(\psi, h_1, h_2, h_3)[x] &= \int_{\mathcal{O}} D_2^3 g(x, \varphi(\widetilde{x}))\psi(\widetilde{x})h_1(\widetilde{x})h_2(\widetilde{x})h_3(\widetilde{x})\; d\widetilde{x}, \end{aligned}$$

for h_1, h_2 and h_3 belonging to W. Due to the assumptions on g, these expressions are obviously bounded.

It is left to show that $G : W \to \mathcal{L}(W, D_\beta)$ is Lipschitz continuous. Here $\beta \geq 1/3$ as assumed in **(G)**. To this aim let ψ^1 and $\psi^2 \in W$ and compute

$$\begin{aligned} |G(\varphi^1) - G(\varphi^2)|_{\mathcal{L}(W, D_\beta)} &= \sup_{|\psi|=1} |G(\varphi^1)(\psi) - G(\varphi^2)(\psi)|_{D_\beta} \\ &\leq C \sup_{|\psi|=1} |G(\varphi^1)(\psi) - G(\varphi^2)(\psi)|_{D(-A)} \\ &= C \sup_{|\psi|=1} \left| \int_{\mathcal{O}} \big(g(\cdot, \varphi^1(\widetilde{x})) - g(\cdot, \varphi^2(\widetilde{x}))\big)\, \psi(\widetilde{x})\; d\widetilde{x} \right|_{D(-A)}. \end{aligned}$$

Therefore, we estimate for $k = 0, 1, 2$:

$$\begin{aligned}
&\Big| \int_{\mathcal{O}} \Big(D_1^k g(\cdot, \varphi^1(\widetilde{x})) - D_1^k g(\cdot, \varphi^2(\widetilde{x})\Big) \psi(\widetilde{x})\ d\widetilde{x} \Big|^2_{L^2(\mathcal{O})} \\
&= \int_{\mathcal{O}} \Big| \int_{\mathcal{O}} \Big(D_1^k g(x, \varphi^1(\widetilde{x})) - D_1^k g(x, \varphi^2(\widetilde{x})\Big) \psi(\widetilde{x})\ d\widetilde{x} \Big|^2 dx \\
&\leq C \int_{\mathcal{O}} \Big| \int_{\mathcal{O}} |D_2 D_1^k g|_\infty\ |\varphi^1(\widetilde{x}) - \varphi^2(\widetilde{x})|\ |\psi(\widetilde{x})|\ d\widetilde{x} \Big|^2 dx \\
&\leq C|\mathcal{O}|\ |D_2 D_1^k g|^2_\infty\ |\varphi^1 - \varphi^2|^2_{L^2(\mathcal{O})}\ |\psi|^2_{L^2(\mathcal{O})}.
\end{aligned}$$

We finally obtain that

$$|G(\varphi^1) - G(\varphi^2)|_{\mathcal{L}(W, D_\beta)} \leq C|\varphi^1 - \varphi^2|_{L^2(\mathcal{O})},$$

where the constant C depends on $|\mathcal{O}|$ and g.

In conclusion our theory can be applied to parabolic SPDEs driven by multiplicative fractional noise as described in (3.90).

Remark 3.57. *Note that we do not make any additional assumptions on the eigenvalues of A. This is natural in the context of rough path theory, compare [10]. However, working with different techniques such as presenting the infinite-dimensional integral as a sum of one-dimensional integrals [50], ([27]) may lead to further assumptions on the asymptotic of the eigenvalues and implicitly to a restriction of the domains.*

3.8 Concluding Remarks on the Coefficients

As announced in Remark 2.2 we now have a closer look at the coefficients.

At first let us how to deal with $F \not\equiv 0$. Assuming F to be Lipschitz continuous, which is essential even in the deterministic case, we obtain the existence of the Bochner integral, see [55, Section 4.3]. This fulfills

$$\Big| \int_s^t S(t-r)F(y_r)dr \Big|_{D_\gamma} \leq C(1 + \|y\|_\infty)(t-s)^{1-\gamma}, \qquad \text{for all } 0 \leq \gamma \leq 1, \tag{3.91}$$

which yields its Lipschitz continuity. In fact, for all $0 \leq s \leq \tau \leq t \leq T$ we have

$$\Big| \int_s^t S(t-r)F(y_r)dr - \int_s^\tau S(\tau - r)F(y_r)dr \Big| \leq C(1 + \|y\|_\infty)(t - \tau). \tag{3.92}$$

Following the theory developed in this chapter, we see that if $F \not\equiv 0$ a solution (y, z) of (1.1) fulfills

$$y_t = S(t)\xi + \int_0^t S(t-r)F(y_r)dr + \mathcal{I}\Xi^{(y)}(y, z)_t \tag{3.93}$$

and for a placeholder $E \in \mathcal{L}(W \otimes V; W)$ we have

$$z_{ts}(E) = \int_s^t S(t-r)E \int_s^r S(r-q)F(y_q)\ dq\ d\omega_r + (\hat{\delta}\mathcal{I}\Xi^{(z)}(y, y))_{ts}(E) - \omega^S_{ts}(Ey_s), \tag{3.94}$$

By (3.91) and (3.92) we see that the additional term $\int_0^{\cdot} S(\cdot - r)F(y_r)dr$ is contained in the domain of A and Lipschitz continuous. Hence, since $2\beta < 1$ all deliberations done in Section 3.5 and 3.6 remain the same.

When considering (3.94) we have to give meaning to the additional term

$$\int_s^t S(t-r)E\int_s^r S(r-q)F(y_q)\ dq\ d\omega_r.$$

Thanks to the Lipschitz continuity of the inner integral, see (3.92), we can define

$$\int_s^t S(t-r)E\int_s^r S(r-q)F(y_q)\ dq\ d\omega_r = \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} S(t-v)\,\omega_{vu}^S\Big(E\int_s^u S(u-q)F(y_q)dq\Big) \quad (3.95)$$

in the Young sense. For the sake of completeness let us give the proof using the Sewing Lemma. Set

$$\Xi_{vu}^{(F),s} = \omega_{vu}^S\Big(E\int_s^u S(u-q)F(y_q)dq\Big).$$

(3.47) and (3.91) yield

$$\left|\Xi_{vu}^{(F),s}\right| \le C\,\|\omega\|_\alpha\,|E|(1+\|y\|_\infty)(u-s)^1(v-u)^\alpha. \quad (3.96)$$

By (3.44) we obtain

$$(\hat{\delta}_2\Xi^{(F),s})_{vmu} = \omega_{vm}^S\Big(E\Big[\int_s^u S(u-q)F(y_q)dq - \int_s^m S(m-q)F(y_q)dq\Big]\Big).$$

By (3.47) and (3.92) we see

$$\left|(\hat{\delta}_2\Xi^{(F),s})_{vmu}\right| \le C\,\|\omega\|_\alpha\,|E|\,(1+\|y\|_\infty)(v-u)^{1+\alpha}.$$

Hence, the Sewing Lemma, more precisely Corollary 3.8, yields (3.95).

Furthermore, we can derive another estimate. (3.96) yields $\left|\Xi_{ts}^{(F),s}\right| = 0$. Hence, by (3.23) we conclude that

$$\begin{aligned}\Big|\int_s^t S(t-r)E\int_s^r S(r-q)F(y_q)\ dq\ d\omega_r\Big| &= \left|(\hat{\delta}\mathcal{I}\Xi^{(F),s})_{ts}\right| \\ &\le \left|(\hat{\delta}\mathcal{I}\Xi^{(F),s})_{ts} - \Xi_{ts}^{(F),s}\right| + \left|\Xi_{ts}^{(F),s}\right| \\ &\le C\,\|\omega\|_\alpha\,|E|(1+\|y\|_\infty)(t-s)^{1+\alpha}.\end{aligned}$$

Since $1+\alpha > \alpha + 2\beta$ the deliberations in Section 3.5 and 3.6 remain the same.

Moreover, we shortly want to consider the case of an affine linear diffusion term $G(x) = G_1x + G_2$. In this case one can use a Doss-Sussmann type transformation in order to convert the stochastic equation into a random equation which can be solved pathwise, see [13] and [59] for the transformation and [58], [9] and [14] for applications to stochastic equations.

While we can derive the existence of a local solution because Theorem 3.33 does not require boundedness of G, Theorem 3.54 does not apply to affine linear G. To the best of our knowledge it

is not clear if one can drop the boundedness condition on G.
However, if you follow the calculations in Section 3.6 you see that in the case of an affine linear G we still obtain a global solution.

In Section 3.6 we used the boundedness of G in order to avoid quadratic terms. Now, for affine linear G we have that $DG \equiv G_1$ and $D^2G \equiv 0$ and thus

$$G(y_u) - G(y_m) + DG(y_u)(\delta y)_{mu} = G_1 y_u + G_2 - G_1 y_m - G_2 + G_1(\delta y)_{mu} = 0,$$

compare the proof of Lemma 3.51.
Furthermore, we only have to consider $z_{vu}(DG(y_u)) = z_{vu}(G_1)$ which is independent of y.

So, one can prove Lemma 3.51 with the same calculations as given in Section 3.6 under the assumption that G is affine linear. Since this is the crucial lemma of Section 3.6, all further results hold true.

We conclude this chapter with a final remark on the generic constant.

Remark 3.58. *In Chapter 2 and this chapter we have worked with a generic constant C which depends on the semigroup S the nonlinear coefficients F and G, the noisy input $\boldsymbol{\omega}$ and on further parameters α, β. For notational simplicity we have omitted these dependencies. However, it is not hard to see that C is of multiplicative structure in all computations done in this chapter, more precisely $C = C_S C_F C_G C(\omega)$ where $C_S = C_{S,\alpha,\beta}$ and $C(\omega)$ is a polynomial in $\|\omega\|_\alpha$ and $\left\| \omega^{(2)} \right\|_{2\alpha}$. For analyzing the dynamics of solutions of* (1.1) *in the next chapter it will be necessary to state these dependencies.*

4. Dynamics of Rough Evolution Equations

Referring to the monograph of Arnold [2], it is well-known that an Itô-type stochastic differential equation generates a random dynamical system under natural assumptions on the coefficients. This fact is based on the flow property, see [46, 57], which can be obtained by Kolmogorov's theorem about the existence of a (Hölder)-continuous random field with finite-dimensional parameter range, i.e. the parameters of this random field are the time and the non-random initial data.

The generation of a random dynamical system from an Itô-type SPDE has been a long-standing open problem, since Kolmogorov's theorem breaks down for random fields parametrized by infinite-dimensional Hilbert spaces, see [52]. As a consequence it is not trivial how to obtain a random dynamical system from an SPDE, since its solution is defined almost surely, which contradicts the cocycle property. Particularly, this means that there are exceptional sets which depend on the initial condition and it is not clear how to define a random dynamical system if more than countably many exceptional sets occur.
Thus, dynamical aspects for (1.1) such as asymptotic stability, Lyapunov exponents, multiplicative ergodic theorems, random attractors, random invariant manifolds have not been investigated in their full generality.

In this chapter we show that the solution of (1.1) generates a random dynamical system and in the following prove asymptotic stability of the trivial solution.

4.1 Random Dynamical Systems

Based on the results derived in the previous chapter we investigate random dynamical systems for (1.1). There are very few works that deal with random dynamical systems for SPDEs driven by nonlinear multiplicative rough noise, see for instance [19]. In the finite-dimensional setting this topic was considered in [3].

We start by introducing the next fundamental concept in the theory of random dynamical systems, which describes a model of the driving noise, see [2].

Definition 4.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ stand for a probability space and $\theta : \mathbb{R} \times \Omega \to \Omega$ be a family of $\mathbb{P}$-preserving transformations (i.e., $\theta_t \mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$) having the following properties:

(i) the mapping $(t, \omega) \mapsto \theta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$-measurable;

(ii) $\theta_0 = \mathrm{Id}_\Omega$;

(iii) $\theta_{t+s} = \theta_t \circ \theta_s$ for all $t, s, \in \mathbb{R}$.

Then the quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is called a metric dynamical system.

Motivated by this we precisely describe the random input driving (1.1). Therefore, our aim is introduce the (canonical) probability space associated to a Hilbert space-valued α-Hölder rough path. We recall that $\alpha \in (\frac{1}{3}, \frac{1}{2})$ was fixed at the beginning of this work. An example is constituted by a trace-class V-valued fractional Brownian motion with Hurst index $H \in (1/3, 1/2]$. In order to construct it, we recall that a *two-sided* real-valued fractional Brownian motion $\widetilde{\beta}^H(\cdot)$ with a Hurst index $H \in (0,1)$ is a centered Gaussian process with covariance function

$$\mathbb{E}(\widetilde{\beta}^H(t)\widetilde{\beta}^H(s)) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \quad \text{for } s, t \in \mathbb{R}.$$

In order to introduce a V-valued process, we let Q stand for a positive symmetric operator of *trace-class* on V, i.e. $\mathrm{tr}_V Q < \infty$. This has a discrete spectrum which will be denoted by $(\lambda_n^Q)_{n\in\mathbb{N}}$.

It is well-known that the eigenvectors $(e_n)_{n\in\mathbb{N}}$ build an orthonormal basis in V. Then a V-valued two-sided Q-fractional Brownian motion ω is represented by

$$\omega_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n^Q}\, \widetilde{\beta}_n^H(t) e_n, \quad t \in \mathbb{R}, \tag{4.1}$$

where $(\widetilde{\beta}_n^H(\cdot))_{n\in\mathbb{N}}$ is a sequence of one-dimensional independent standard two-sided fractional Brownian motions with the same Hurst parameter H and $\mathrm{tr}_V Q = \sum\limits_{n=1}^{\infty} \lambda_n^Q < \infty$. In the following sequel we further fix $H \in (\frac{1}{3}, \frac{1}{2}]$.
Keeping (4.1) in mind it is not hard to show that ω is locally Hölder continuous by using Kolmogorov's continuity criterion, see [46, Theorem 1.4.1].

Lemma 4.2. *Let ω be a fractional Brownian motion given by* (4.1). *Then, ω is locally α-Hölder continuous for all $\alpha < H$ almost surely.*
Furthermore, we obtain

$$\mathbb{E}\, \|\omega\|_{\alpha} \leq C_{\alpha,H} \sqrt{tr_V Q}. \tag{4.2}$$

Proof. It is well known that a one-dimensional fractional Brownian motion is locally α-Hölder regular for all $\alpha < H$, see[53, Section 1.4]. Keeping this in mind we obtain

$$\mathbb{E}\, \|\omega\|_{\alpha}^2 \leq \sum_{n=1}^{\infty} \lambda_n^Q \mathbb{E} \|\widetilde{\beta}_n^H\|_{\alpha}^2 = \|\widetilde{\beta}_1^H\|_{\alpha}^2 \mathrm{tr}_V Q.$$

Jensen's inequality proves (4.2) which yields the rest of the statement. □

Keeping Lemma 4.2 in mind we are justified to introduce the canonical probability space $(C_0(\mathbb{R}, V), \mathcal{B}(C_0(\mathbb{R}, V)), \mathbb{P}, \theta)$. Here $C_0(\mathbb{R}, V)$ denotes the set of all V-valued continuous functions which are zero in zero endowed with the compact open topology and $\mathbb{P}$ is the fractional Gauß-measure which is uniquely determined by Q and H.
As already introduced in Section 2.2, we take for θ the usual Wiener-shift, namely

$$\theta_\tau \omega_t = \omega_{t+\tau} - \omega_\tau, \quad \text{for } \omega \in C_0(\mathbb{R}, V).$$

Further, for our aims we restrict it to the set $\Omega := C_0^{\alpha'}(\mathbb{R}, V)$ of all α'-Hölder-continuous paths on any compact interval, where $\frac{1}{3} < \alpha < \alpha' < H \leq \frac{1}{2}$. We equip this set with the trace σ-algebra $\mathcal{F} := \Omega \cap \mathcal{B}(C_0(\mathbb{R}, V))$ and take the restriction of $\mathbb{P}$ as well. Then $\Omega \subset C_0(\mathbb{R}, V)$ has full measure and is θ-invariant. Moreover, the new quadrupel $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as introduced above forms again a metric dynamical system which we will further be restricted later on.

We point out the following result regarding the existence/construction of the Lévy-area $\omega^{(2)}$ for an element $\omega \in \Omega$. We stress the fact that it is necessary to let ω be α'-Hölder continuous for $\frac{1}{3} < \alpha < \alpha' < H \leq \frac{1}{2}$. This is required in order to lift ω to an α-Hölder rough path $\boldsymbol{\omega} = (\omega, \omega^{(2)})$. To this aim we furthermore have to consider the restriction of ω on compact intervals. The precise setting is stated below.

Lemma 4.3. *Let $\frac{1}{3} < \alpha < \alpha' < H \leq \frac{1}{2}$ and $\omega \in \Omega$ be a Q-fractional Brownian motion with Hurst index H. Then there is a θ-invariant subset $\Omega' \subset \Omega$ of full measure such that for any $\omega \in \Omega'$ and for any compact interval $J \subset \mathbb{R}$ there exists a Lévy-area $\omega^{(2)} \in C^{2\alpha}(\Delta_J, V \otimes V)$ such that $\boldsymbol{\omega} = (\omega, \omega^{(2)})$ defines an α-Hölder rough path. This can further be approximated by a sequence $\boldsymbol{\omega}^n := ((\omega^n, \omega^{(2),n}))_{n\in\mathbb{N}}$ in the corresponding $d_{\alpha,J}$-metric. Here $(\omega^n)_{n\in\mathbb{N}}$ are piecewise dyadic linear functions and*

$$\omega_{ts}^{(2),n} = \int_s^t (\delta\omega^n)_{rs} \otimes d\omega_r^n.$$

Proof. Let $j, k \in \mathbb{N}$ and $T \in \mathbb{N}$ be such that $J \subseteq [-T, T]$. We introduce

$$\omega_{ts}^{(2)}(j,k) := \int_s^t (\widetilde{\beta}_j^H(r) - \widetilde{\beta}_j^H(s))\, d\widetilde{\beta}_k^H(r), \quad \text{for } -T \leq s \leq t \leq T. \tag{4.3}$$

This process exists almost surely according to Theorem 2 in [8], see also [20, Corollary 10.10]. Regarding (4.1) we can represent the infinite-dimensional Lévy-area $\omega_{ts}^{(2)} \in V \otimes V$ component-wise as

$$\omega_{ts}^{(2)} = \sum_{j,k=1}^{\infty} \sqrt{\lambda_j}\sqrt{\lambda_k}\ \omega_{ts}^{(2)}(j,k)\ e_j \otimes e_k. \tag{4.4}$$

This is well-defined almost surely due to the fact that $\mathrm{tr}_V Q < \infty$. Moreover, one has that $\omega^{(2),n} \to \omega^{(2)}$ in $C^{2\alpha}(\Delta_{[-T,T]}, V \otimes V)$ almost surely. The proof of these assertions relies on a standard Borel-Cantelli argument combined with the Garsia-Rodemich-Rumsey inequality and follows the lines of Lemma 2 in [24]. Since $J \subseteq [-T, T]$, one clearly concludes that $\boldsymbol{\omega}^n$ converges to $\boldsymbol{\omega}$ with respect to the $d_{\alpha,J}$-metric. This immediately yields that Ω' has full measure and is θ-invariant. □

From now on we work with the metric dynamical system $(\Omega', \mathcal{F}', \mathbb{P}', \theta)$ corresponding to Ω' constructed in Lemma 4.3. As above we set $\mathcal{F}' := \Omega' \cap \mathcal{F}$ and take $\mathbb{P}'$ as the restriction of $\mathbb{P}$.

Note that thanks to Lemma 4.3 there is a unique way to lift the path ω of a fractional Brownian motion to rough path $\boldsymbol{\omega} \in C_g^{0,\alpha}$. Whenever this is possible in a more general setting the following statements of this Section hold true.

For considering the dynamics of a given solution of (1.1) we are especially interested in the path component of this solution while the area component is less important. Roughly speaking the path component is the observed object while the area component has only supporting character. Hence, we define

$$\varphi \colon \mathbb{R}_+ \times \Omega' \times W \to W, \qquad \varphi(t, \omega, \xi) := y_t,$$

where (y, z) is the unique fixed point of $\mathcal{M}_{t,\omega,\xi}$. One can say $\varphi(\cdot, \omega, \xi)$ gives the path component of the solution of (1.1) on $\mathbb{R}_+$ for given noise $\omega \in \Omega'$ and initial condition $\xi \in W$.

In case we want to emphasize the dependence of the solution on the coefficients F and G, we write $\varphi(t, \omega, \xi, F, G)$.

The concept introduced next is the basis for the investigation of many dynamical aspects. For a general overview see [2].

Definition 4.4. A random dynamical system on W over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a mapping

$$\overline{\varphi} : \mathbb{R}_+ \times \Omega \times W \to W, \qquad (t, \omega, x) \mapsto \overline{\varphi}(t, \omega, x),$$

which is $(\mathcal{B}(\mathbb{R}_+) \times \mathcal{F} \times \mathcal{B}(W), \mathcal{B}(W))$-measurable and satisfies:

(i) $\overline{\varphi}(0, \omega, \cdot) = \mathrm{Id}_W$ for all $\omega \in \Omega$;

(ii) $\overline{\varphi}(t + \tau, \omega, x) = \overline{\varphi}(t, \theta_\tau\omega, \overline{\varphi}(\tau, \omega, x))$, for all $x \in W$, $t, \tau \in \mathbb{R}_+$, $\omega \in \Omega$.

If one additionally assumes that

(iii) $\overline{\varphi}(t, \omega, \cdot) : W \to W$ is continuous for all $t \in \mathbb{R}_+$ and all $\omega \in \Omega$,

then $\overline{\varphi}$ is called a *continuous random dynamical system.*

The second property in Definition 4.4 is referred to as the *cocycle property.* As already mentioned in Section 3.5 the fact that φ fulfills the cocycle property is a direct consequence of Lemma 3.39. However, since this property is fundamental, we provide all necessary deliberations in detail.

Lemma 4.5. *φ fulfills the cocycle property.*

Proof. By definition we have

$$\varphi(t + \tau, \omega, \xi) = y_{t+\tau},$$

where (y, z) is the unique fixed point of $\mathcal{M}_{t+\tau,\omega,\xi}$. Lemma 3.39 states that $(\overline{\theta}_\tau y, \overline{\theta}_\tau z)$ is a fixed point of $\mathcal{M}_{t,\theta_\tau\omega,y_\tau}$. This entails

$$y_{t\tau} = \overline{\theta}_\tau y_t = \varphi(t, \theta_\tau\omega, y_\tau).$$

By Remark 3.40 we know that $y_\tau = \varphi(\tau\omega, \xi)$ which yields the statement. $\square$

One can expect the solution operator of (1.1) to generate a random dynamical system. Indeed, a big advantage in working with a pathwise interpretation of the stochastic integral, is that no exceptional sets occur.

We can now state the main result of this section. Recall that Ω' was constructed in Lemma 4.3.

Theorem 4.6. *$\varphi : \mathbb{R}_+ \times \Omega' \times W \to W$ generates a random dynamical system.*

Proof. Due to Theorem 3.54 we know that we can define the solution (y, z) of (1.1) on any time-interval $[0, T]$ for $T > 0$. The cocycle property is given by Lemma 4.5. The continuity of φ with respect to time and initial condition is clear, we only have to show the measurability. Therefore, we consider a sequence of solutions $((y^n, z^n))_{n\in\mathbb{N}}$ corresponding to the smooth approximations $((\omega^n, \omega^{(2),n}))_{n\in\mathbb{N}}$, recall Lemma 4.3. Note that the mapping $\omega \mapsto (\omega^n, \omega^{(2),n})$ is measurable. Due to the fact that ω^n is smooth y^n is a classical solution of (1.1). Hence, the mapping

$$[0, T] \times \Omega' \times W \ni (t, \omega, \xi) \mapsto y^n_t \in W$$

is $(\mathcal{B}([0, T]) \otimes \mathcal{F}' \otimes \mathcal{B}(W), \mathcal{B}(W))$-measurable. Regarding Lemma 3.19 one can immediately infer that the solution (y, z) continuously depends on $(\omega^n, \omega^{(2),n})$. This leads to

$$\lim_{n\to\infty} y^n_t = y_t, \tag{4.5}$$

which gives us the measurability of y_t with respect to $\mathcal{F}' \otimes \mathcal{B}(W)$. Since y is continuous with respect to t, we obtain by Lemma 3 in [4] the jointly measurability, i.e. the $(\mathcal{B}([0, T]) \otimes \mathcal{F}' \otimes \mathcal{B}(W), \mathcal{B}(W))$ measurability of the mapping

$$[0, T] \times \Omega' \times W \ni (t, \omega, \xi) \mapsto y_t \in W. \tag{4.6}$$

Since (4.6) holds true for any $T > 0$, one obviously concludes that φ is $(\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}' \otimes \mathcal{B}(W), \mathcal{B}(W))$-measurable. $\square$

4.2 Local Exponential Stability

In this section and also in the following one we want to analyze the asymptotic behavior of the path component of the solution of (1.1). Now we are interested in establishing local exponential stability of the trivial solution.

The most important drawback we have to face is that our driving noise is in general not Markovian. Hence, we can not use techniques provided in [44]. However, new approaches have been developed in order to show stability for pathwise solutions. In [29] and [30] the authors show local stability in an ODE setting where the driving noise is given by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, respectively $H > \frac{1}{3}$. In [16] and [17] the authors consider driving Gaussian noise in an ODE setting. The goal of this section is to prove local exponential stability in the very general setting given in (1.1), with three additional assumptions. At first, we specify the assumptions on the given linear operator

(SL1) For a $\lambda > 0$ the operator $A + \lambda\,\mathrm{Id}$ is strictly negative. Thus, it generates an analytic exponentially stable semigroup S meaning that for all $t > 0$ we have

$$\|S(t)\| \leq e^{-\lambda t}.$$

Let us state one important estimate of the semigroup provided by assumption **(SL1)**. For more details see [55, Section 2.6].

Lemma 4.7. *Given assumption* ***(SL1)*** *for $\eta, \kappa \in \mathbb{R}$ and $t \geq 0$ we have*

$$\|S(t)\|_{\mathcal{L}(D_\kappa, D_\eta)} \leq C_S\, t^{\kappa-\eta} e^{-\lambda t} \qquad \text{for } \kappa \leq \eta. \tag{4.7}$$

Furthermore, we impose some additional properties on the coefficients.

(SL2) $F(0) = 0$, $DF(0) = 0$,

(SL3) $G(0) = 0$, $DG(0) = 0$,

Clearly under the assumptions **(SL2)** and **(SL3)** the solution of (1.1) is trivial if $\xi = 0$.

Let us give a formal definition for local exponential stability of this trivial solution, see [30, Definition 8].

Definition 4.8 (Local Exponential Stability)**.** The trivial solution of (1.1) is called locally exponentially stable with a rate $\lambda' > 0$ if there exists a random neighborhood of zero $U_0(\omega)$ and a random variable $M(\omega) > 0$ such that for almost all $\omega \in \Omega$ the path component fulfills

$$\sup_{\xi \in U_0(\omega)} |y_t| \leq M(\omega) e^{-\lambda' t} \qquad \text{for all } t \geq 0. \tag{4.8}$$

In preparation of the following deliberations for notational simplicity we introduce for $(y, z) \in X_{\omega,T}$

$$\mathrm{I}_{t,\omega}(y, z, F, G) = \int_0^t S(t-r) F(y_r) dr + \mathcal{I}\Xi^{(y)}_\omega(y, z)_t, \qquad 0 \leq t \leq T. \tag{4.9}$$

Moreover, we set $\mathrm{I}_\omega(y, z, F, G) := \big(\mathrm{I}_{t,\omega}(y, z, F, G)\big)_{t \in [0,T]}$.
We emphasize the dependence of the coefficients F and G because eventually it will be necessary to consider different coefficients. However, if we only have to consider one pair of coefficients F and G we will suppress this dependence and only write $\mathrm{I}_{t,\omega}(y, z)$ and $\mathrm{I}_\omega(y, z)$.

The following lemma gives the basis for both of our long time behavior analysis (local and global, see Section 4.3). We consider a solution of (1.1) on $\mathbb{R}_+$ and split it via the cocycle property into a sequence of solutions on compact intervals which depend on the (shifted) noise.

Lemma 4.9. *Let (y,z) be a solution of (1.1) on $\mathbb{R}_+$. Furthermore, consider a sequence of increasing random times $(T_n)_{n\in\mathbb{N}_0} = (T_n(\omega))_{n\in\mathbb{N}_0}$ with $T_0 \equiv 0$ For $n \in \mathbb{N}_0$ define the function*

$$\begin{aligned} y_t^n &:= y_{t+T_n}, && \text{for } 0 \le t \le T_{n+1} - T_n. \\ z_{ts}^n &:= z_{t+T_n,s+T_n}, && \text{for } 0 \le s \le t \le T_{n+1} - T_n. \end{aligned}$$

Then, we have for all $n \in \mathbb{N}_0$ and $t \in [T_n, T_{n+1}]$

$$y_{t-T_n}^n = S(t)\xi + \sum_{j=0}^{n-1} S(t - T_{j+1}) \mathrm{I}_{T_{j+1}-T_j,\theta_j\omega}(y^j, z^j) + \mathrm{I}_{t-T_n,\theta_n\omega}(y^n, z^n). \tag{4.10}$$

Proof. By definition we have $y^0 = \varphi(\cdot,\omega,\xi)$ on $[0,T_1]$. Furthermore, Lemma 4.5 implies that $y^n = \varphi(\cdot, \theta_{T_n}\omega, y^{n-1}_{T_n - T_{n-1}})$ on $[0, T_{n+1} - T_n]$ for all $n \ge 1$.

Consequently we obtain for $t \in [T_n, T_{n+1}]$

$$y_t = y_{t-T_n}^n = S(t - T_n) y^{n-1}_{T_n - T_{n-1}} + \mathrm{I}_{t-T_n,\theta_{T_n}\omega}(y^n, z^n).$$

Plugging in the same formula for $y^{n-1}_{T_n - T_{n-1}}$ yields

$$y_{t-T_n}^n = S(t - T_{n-1}) y^{n-2}_{T_{n-1}-T_{n-2}} + S(t - T_n) \mathrm{I}_{T_n - T_{n-1},\theta_{n-1}\omega}(y^{n-1}, z^{n-1}) + \mathrm{I}_{t-T_n,\theta_n\omega}(y^n, z^n).$$

Iterating this calculation yields the proposed statement. □

For the sake of completeness we have proven Lemma 4.9 in a very general setting for arbitrary random times $(T_n)_{n\in\mathbb{N}_0}$.

For showing local exponential stability it is sufficient to set $T_n = n$ for all $n \in \mathbb{N}_0$ whereas in Section 4.3 it will be necessary to consider a sequence of stopping times which will be specified later on.

Now, consider a solution (y,z) of (1.1) on $\mathbb{R}_+$. For $n \in \mathbb{N}_0$ define the functions

$$\begin{aligned} y_t^n &:= y_{t+n}, && \text{for } 0 \le t \le 1. \\ z_{ts}^n &:= z_{t+n,s+n}, && \text{for } 0 \le s \le t \le 1. \end{aligned}$$

Then, Lemma 4.9 shows for $t \in [n, n+1]$

$$y_{t-n}^n = S(t)\xi + \sum_{j=0}^{n-1} S(t - j - 1) \mathrm{I}_{1,\theta_j\omega}(y^j, z^j, F, G) + \mathrm{I}_{t-n,\theta_n\omega}(y^n, z^n, F, G). \tag{4.11}$$

In order to prove local stability we will work with piecewise truncated functions. First let us introduce the notion of such a cut-off function. Here, we use the same cut-off function as in [29, Section 5].

For an arbitrary Banach space W consider the cut-off function

$$\chi \colon W \to \bar{B}_W(0,1), \text{ with}$$
$$\chi(u) = \begin{cases} u & \text{if } |u| \le \frac{1}{2}, \\ 0 & \text{if } |u| \ge 1, \end{cases}$$

where χ is bounded by 1 and twice continuously differentiable with bounded derivatives $\|D\chi\|_\infty =: L_{D_\chi}$ and $\|D^2\chi\|_\infty =: L_{D^2_\chi}$.

For arbitrary $\varrho > 0$ define

$$\chi_\varrho \colon W \to B(0,\varrho),$$
$$\chi_\varrho(u) := \varrho\, \chi\left(\frac{u}{\varrho}\right).$$

Let W' be a further Banach space. For a function $\mathcal{T}: W \to W'$ we set

$$\mathcal{T}_\varrho: W \to W',$$
$$\mathcal{T}_\varrho(u) := \mathcal{T}(\chi_\varrho(u)).$$

The next lemma gives a fundamental property of truncated functions. You can find a finite-dimensional version in [29, Lemma 4]. In the infinite-dimensional case we have to additionally assume boundedness of the first derivative which is automatically given in the finite-dimensional case by the compactness of the unit ball.

Lemma 4.10. *Let W, W' be Banach spaces and $\mathcal{T}: B_W(0,1) \to W'$ a continuously differentiable function with bounded first derivative and $\mathcal{T}(0) = 0$. Then, there exists a measurable function $(0,1] \ni \bar{\varrho} \mapsto \varrho \in (0,1]$ such that*

$$|\mathcal{T}(x)| \leq \bar{\varrho} \qquad \text{for all } x \in B_W(0,\varrho).$$

and there exists $\kappa > 0$ such that for all $\bar{\varrho} \in (0,1]$ we have the estimate

$$\frac{\varrho(\bar{\varrho})}{\bar{\varrho}} \geq \kappa.$$

Proof. We have

$$|\mathcal{T}(x)| \leq \|D\mathcal{T}\|_\infty |x|.$$

Consequently, choosing $\varrho = \varrho(\bar{\varrho}) = \frac{\bar{\varrho}}{\|D\mathcal{T}\|_\infty}$ entails

$$|\mathcal{T}(x)| \leq \bar{\varrho} \qquad \text{for all } x \in B_W(0,\varrho)$$

and

$$\frac{\varrho(\bar{\varrho})}{\bar{\varrho}} = \frac{1}{\|D\mathcal{T}\|_\infty} =: \kappa. \qquad \square$$

Now, let us state important estimates for the truncated coefficients F_ϱ and G_ϱ. All follow directly by Lemma 4.10 with standard computations. Detailed calculations can be found in [30, Appendix].

Lemma 4.11. *Given assumption **(SL2)** and **(SL3)**, then for any $\bar{\varrho} > 0$ there exists a positive $\varrho \leq 1$ such that for all $x^1, x^2 \in W$ we have*

$$\left|F_\varrho(x^1)\right| \leq L_{D_\chi}\bar{\varrho}\left|x^1\right|, \tag{4.12}$$
$$\left|G_\varrho(x^1)\right| \leq L_{D_\chi}\bar{\varrho}\left|x^1\right|, \tag{4.13}$$
$$\left|G_\varrho(x^1)\right|_{\mathcal{L}(V;D_\beta)} \leq \bar{\varrho}, \tag{4.14}$$
$$\left|G_\varrho(x^1) - G_\varrho(x^2)\right|_{\mathcal{L}(V;D_\beta)} \leq L_{D_\chi}\bar{\varrho}\left|x^1 - x^2\right|, \tag{4.15}$$
$$\left|DG_\varrho(x^1)\right| \leq L_{D_\chi}\bar{\varrho}, \tag{4.16}$$
$$\left|DG_\varrho(x^1) - DG_\varrho(x^2)\right| \leq (L_{D^2_\chi + L^2_{D_\chi}})\left|x^1 - x^2\right|. \tag{4.17}$$

Remark 4.12. *Note that unlike in [30] we do not assume $D^2G(0) = 0$. Hence, the estimate (4.17) slightly differs. Furthermore, we work in an infinite-dimensional setting and we are considering some estimate in the fractional domain D_β. However, the required calculations are still the same as in [30].*

For a given truncation constant $\varrho > 0$ we want to consider the truncated equation

$$\begin{cases} dy_t = (Ay_t + F_\varrho(y_t))dt + G_\varrho(y_t)d\omega_t, & t \in [0,T] \\ y_0 = \xi. \end{cases} \tag{4.18}$$

By the theory developed in Chapter 3, its solution is given by

$$y_t = S(t)\xi + \int_0^t S(t-r)F_\varrho(y_r)dr + \mathcal{I}\Xi^{(y),\varrho}(y,z)_t, \qquad \text{and} \tag{4.19}$$

$$z_{ts}(E) = \int_s^t S(t-r)E\int_s^r S(r-q)F_\varrho(y_q)\, dq\, d\omega_r + (\hat{\delta}\mathcal{I}\Xi^{(z),\varrho}(y,y))_{ts}(E) - \omega^S_{ts}(Ey_s), \tag{4.20}$$

where

$$\Xi^{(y),\varrho}_{vu}(y,z) := \omega^S_{vu}(G_\varrho(y_u)) + z_{vu}(DG_\varrho(y_u)) \quad \text{and}$$
$$\Xi^{(z),\varrho}_{vu}(y,y)(E) := b_{vu}(E, G_\varrho(y_u)) + a_{vu}(E, y_u).$$

The following lemma states some fundamental estimates for the solution of the truncated equation (4.18). These are very similar to the estimates given in Section 3.6 but are specified because of the additional assumptions **(SL2)** and **(SL3)** and the truncation of the coefficients. More precisely,

We have a term Φ instead of $(1+\Phi)$ due to **(SL2)** and **(SL3)**. (4.21)

We are not interested in the dependence on the time horizont T.
Instead, we exploit the effect of the truncation, see Lemmas 4.10 and 4.13. (4.22)

Consequently, the proofs are also very similar, the only difference is the necessity of applying Lemma 4.11. Hence, we will omit the proof.

Lemma 4.13. *Given assumptions **(SL2)** and **(SL3)** and let* $0 < T \le 1$. *Then, for all* $\bar{\varrho} > 0$ *there exists* $0 < \varrho \le 1$ *such that the solution* $(y,z) = (y^\varrho, z^\varrho)$ *of* (4.18) *on a time interval* $[0,T]$*, fulfills*

$$\|y\|_\beta \le C_S\,(1 + \|\omega\|_\alpha\,\bar{\varrho})\ \Phi, \tag{4.23}$$

$$\|z\|_{\alpha+\beta} \le C_S\,\big(1 + \big(\,\|\omega\|^2_\alpha + \big\|\,\omega^{(2)}\big\|_{2\alpha}\big)\bar{\varrho}\big)\ \Phi, \tag{4.24}$$

$$\Big|\int_s^t S(t-r)F_\varrho(y_r)dr\Big|_{D_\gamma} \le C_S\,\bar{\varrho}\,\|y\|_\infty\,(t-s)^{1-\gamma}, \qquad \text{for all } 0 \le \gamma \le 1, \tag{4.25}$$

$$|\mathcal{I}\Xi^{y,\varrho}(y,z)_t|_{D_\gamma} \le C_S\ \bar{\varrho}\ \Phi(t-s)^{\alpha+\beta-\gamma}, \qquad \text{for all } \beta \le \gamma < \alpha+\beta, \tag{4.26}$$

$$\|R^y\|_{2\beta} \le C_S C(\omega)\bar{\varrho}\,\Phi, \tag{4.27}$$

$$\|R^z\|_{\alpha+2\beta} \le C_S C(\omega)\bar{\varrho}\,\Phi + C_S\,\|\omega\|_\alpha\,\|y\|_{\infty,D_{2\beta}}\,. \tag{4.28}$$

Here $C(\omega)$ *is polynomial containing* $\|\omega\|_\alpha$ *and* $\big\|\,\omega^{(2)}\big\|_{2\alpha}$ *and the mapping* $\bar{\varrho} \mapsto \varrho$ *is measurable.*

In order to proof local exponential stability we will consider a sequence of solutions of properly truncated equations and show that their norms tend to zero, while under suitable assumptions on the initial condition this sequence coincides with the original solution.

Therefore, let $\varrho\colon \Omega' \to (0,1)$ be a random variable, which will be specified later on. We define a sequence of local path components for the truncated equation (4.19), namely

$$y^{0,\varrho} = \varphi(\omega, \xi, F_{\varrho(\omega)}, G_{\varrho(\omega)}),$$
$$y^{n,\varrho} = \varphi(\theta_n\omega, y_1^{n-1,\varrho}, F_{\varrho(\theta_n\omega)}, G_{\varrho(\theta_n\omega)}), \qquad \text{for } n \ge 1,$$

and $(z^{n,\varrho})_{n\in\mathbb{N}_0}$ is the sequence of corresponding area components.

Recall the functional Φ as given in (3.79). Note that it depends on the path component, the area component and the driving noise ω. Since we have to deal with a whole sequence of solutions and consequently a sequence of functionals, we simplify the notation. For all $n \in \mathbb{N}$ we introduce

$$\Phi^n := \|y^{n,\varrho}\|_{\infty,D_{2\beta},1} + \|R^{y,n}\|_{2\beta,1} + \|R^{z,n}\|_{\alpha+2\beta,1}\,,$$

with

$$\begin{aligned} R^{y,n}_{ts} &:= (\hat{\delta} y^{n,\varrho})_{ts} - \theta_n\omega^S_{ts}(G_{\varrho(\theta_n\omega)}(y^{n,\varrho})), \qquad \text{and} \\ R^{z,n}_{ts} &:= z^{n,\varrho}_{ts} - \theta_n b_{ts}(\cdot, G_{\varrho(\theta_n\omega)}(y^{n,\varrho})). \end{aligned}$$

In order to estimate Φ^n we will use a discrete version of the Gronwall Lemma. The proof can be found in [29, Lemma 7].

Lemma 4.14 (discrete Gronwall lemma). *Let* $(\mathrm{a}_n)_{n\in\mathbb{N}}$ *and* $(\mathrm{b}_n)_{n\in\mathbb{N}}$ *non negative sequences and* $\mathrm{c} > 0$ *such that*

$$\mathrm{a}_n \leq \mathrm{c} + \sum_{j=0}^{n-1} \mathrm{b}_j \mathrm{a}_j, \quad n = 0, 1, \ldots$$

then

$$\mathrm{a}_n \leq \mathrm{c} \prod_{j=0}^{n-1} (1 + \mathrm{b}_j), \quad n = 0, 1, \ldots$$

Indeed for a sequence of functionals Φ^n we can derive an estimate which allows us to apply the discrete Gronwall Lemma.

Lemma 4.15. *Let assumptions **(SL1)**–**(SL3)** be fulfilled and let* $\bar{\varrho}\colon \Omega' \to (0,\infty)$ *be measurable function. Then, there exists a measurable function* $\varrho\colon \Omega' \to (0,\infty)$ *such that for all* $n \in \mathbb{N}_0$ *we have*

$$\Phi^n \leq C_S C(\theta_n\omega)\Big[e^{-\lambda n}\,|\xi|_{D_{2\beta}} + \sum_{j=0}^{n-1} e^{-\lambda(n-j-1)} C(\theta_j\omega)\ \bar{\varrho}(\theta_j\omega)\ \Phi^j + C(\theta_n\omega)\ \bar{\varrho}(\theta_n\omega)\ \Phi^n\Big],$$

where $C(\omega)$ *is a polynomial containing* $\|\omega\|_\alpha$ *and* $\left\|\omega^{(2)}\right\|_{2\alpha}$.

Proof. For each $\bar{\varrho}(\omega)$ on can choose $\varrho(\omega)$ according to Lemma 4.13.

In order to estimate Φ^n we have to check each summand. Let us start with $\|y^n\|_{\infty,D_{2\beta},1}$. By (4.11) and (4.7) we see

$$\begin{aligned} \|y^{n,\varrho}\|_{\infty,D_{2\beta},1} \leq C_S e^{-\lambda n}\,|\xi|_{D_{2\beta}} &+ \sum_{j=0}^{n-1} C_S e^{-\lambda(n-j-1)} \left\|\mathrm{I}_{\theta_j\omega}(y^{j,\varrho}, z^{j,\varrho}, F_{\varrho(\theta_j\omega)}, G_{\varrho(\theta_j\omega)})\right\|_{\infty,D_{2\beta},1} \\ &+ \left\|\mathrm{I}_{\theta_n\omega}(y^{n,\varrho}, z^{n,\varrho}, F_{\varrho(\theta_n\omega)}, G_{\varrho(\theta_n\omega)})\right\|_{\infty,D_{2\beta},1}. \end{aligned}$$

Applying (4.25) and (4.26) yields

$$\begin{aligned} \|y^{n,\varrho}\|_{\infty,D_{2\beta},1} \leq C_S e^{-\lambda n}\,|\xi|_{D_{2\beta}} &+ \sum_{j=0}^{n-1} C_S e^{-\lambda(n-j-1)} C(\theta_j\omega)\bar{\varrho}(\theta_j\omega)\Phi^j \\ &+ C_S C(\theta_n\omega)\bar{\varrho}(\theta_n\omega)\Phi^n. \end{aligned} \tag{4.29}$$

Furthermore, considering $R^{y,n}$ we know by (4.27)

$$\|R^{y,n}\|_{2\beta,1} \leq C_S\, C(\theta_n\omega)\,\bar{\varrho}(\theta_n\omega)\,\Phi^n. \tag{4.30}$$

Finally, we have to consider $R^{z,n}$. (4.28) states

$$\|R^{z,n}\|_{\alpha+2\beta,1} \leq C_S\, C(\theta_n\omega)\, \bar{\varrho}(\theta_n\omega)\, \Phi^n + C_S\, \|\theta_n\omega\|_\alpha\, \|y^{n,\varrho}\|_{\infty,D_{2\beta}}\,.$$

Hence, (4.29) yields

$$\begin{aligned}\|R^{z,n}\|_{\alpha+2\beta} \leq C_S C(\theta_n\omega)\Big[e^{-\lambda n}\,|\xi|_{D_{2\beta}} + \sum_{j=0}^{n-1} e^{-\lambda(n-j-1)} C(\theta_j\omega)\,\bar{\varrho}(\theta_j\omega)\,\Phi^j \\ + C(\theta_n\omega)\,\bar{\varrho}(\theta_n\omega)\,\Phi^n\Big].\end{aligned} \tag{4.31}$$

Combining (4.29), (4.30) and (4.31) we obtain the statement. □

Corollary 4.16. *For any $\varepsilon > 0$ there exists $\bar{\varrho} = \bar{\varrho}_\varepsilon \colon \Omega' \to (0,1)$ such that for all $n \in \mathbb{N}_0$ we have*

$$\Phi^n \leq C_S C(\theta_n\omega)\,|\xi|_{D_{2\beta}} \left(e^{-\lambda} + \varepsilon\right)^n. \tag{4.32}$$

Proof. Choose

$$\bar{\varrho}(\omega) := \frac{\varepsilon}{2C_S C(\omega)^2}$$

Lemma 4.15 guarantees the existence of $\varrho\colon \Omega' \to (0,1)$ such that

$$e^{\lambda n}\frac{\Phi^n}{C(\theta_n\omega)} \leq C_S\,|\xi|_{D_{2\beta}} + \varepsilon\,\sum_{j=0}^{n-1} e^{-\lambda} e^{\lambda j}\frac{\Phi^j}{C(\theta_j\omega)}.$$

So, applying Lemma 4.14 with $\mathrm{a}_n := e^{\lambda n}\frac{\Phi^n}{C(\theta_n\omega)}$, $\mathrm{b}_n := \varepsilon e^{-\lambda}$ and $\mathrm{c} := C_S\,|\xi|_{D_{2\beta}}$ yields

$$e^{\lambda n}\frac{\Phi^n}{C(\theta_n\omega)} \leq C_S\,|\xi|_{D_{2\beta}} \left(1 + \varepsilon e^{-\lambda}\right)^n.$$

Finally, we conclude

$$\Phi^n \leq C_S C(\theta_n\omega)\,|\xi|_{D_{2\beta}} \left(e^{-\lambda} + \varepsilon\right)^n. \qquad \square$$

We can choose $\varepsilon > 0$ small enough such that we obtain an exponential decay with rate $(e^{-\lambda} + \varepsilon)^n$. However, $C(\theta_n\omega)$ depends on n as well. So we have to make sure that this is dominated by an exponential term. This property for random variables is called temperedness.

Definition 4.17. A positive random variable X is called *tempered from above*, see [2] if

$$\limsup_{t\to\infty} \frac{\log^+ X(\theta_t\omega)}{t} = 0 \qquad \text{for almost all } \omega \in \Omega. \tag{4.33}$$

X is called *tempered from bellow* if $\frac{1}{X}$ is tempered from above.

Remark 4.18. *If $t \mapsto X(\theta_t\omega)$ is continuous, an equivalent formulation to* (4.33) *is:*
For all $\delta > 0$ there exists a random constant $M_\delta(\omega)$ such that

$$X(\theta_t\omega) \leq M_\delta(\omega)e^{\delta t} \qquad \textit{for almost all } \omega \in \Omega \textit{ and for all } t \geq 0.$$

Lemma 4.19. *The random variable $C(\omega)$ is tempered from above and $\bar{\varrho}(\omega)$ is tempered from below.*

Proof. It is known that $\|\omega\|_\alpha$ and $\left\|\,\omega^{(2)}\right\|_{2\alpha}$ are tempered from above, see [30, Lemma 20]. Clearly, each polynomial of tempered random variables is tempered, too. Hence, $C(\omega)$ is tempered from a above and consequently $\bar{\varrho}$ is tempered from below. □

Theorem 4.20. *Let $\omega \in \Omega'$ fulfill* (4.33) *then for all $0 < \lambda' < \lambda$ there exists $B_{\lambda'}(\omega) > 0$ such that, if $|\xi|_{D_{2\beta}} \leq B_{\lambda'}(\omega)$, then for all $t \geq 0$ the path component of* (1.1) *fulfills*

$$|y_t|_{D_{2\beta}} \leq e^{-\lambda' t}.$$

Proof. Since $C(\omega)$ is tempered from above there exists $0 < \delta < \lambda - \lambda'$, and $M_\delta(\omega) > 0$ such that $C(\theta_t\omega) < M_\delta(\omega)e^{\delta t}$.

Choose $\varepsilon > 0$ small enough such that $\delta + \log(e^{-\lambda} + \varepsilon) < -\lambda'$ then (4.32) yields for all $n \in \mathbb{N}$

$$\|y^{n,\varrho}\|_{\infty,D_{2\beta}} \leq \Phi^n \leq C_S M_\delta(\omega) B_{\lambda'}(\omega) e^{-\lambda' n}. \tag{4.34}$$

Furthermore, $\bar{\varrho}(\omega) := \frac{\varepsilon}{2C_S C(\omega)^2}$ is tempered from below, i.e. together with Lemma 4.10 we have

$$\varrho(\theta_t\omega) \geq \kappa\bar{\varrho}(\theta_t\omega) \geq \overline{M}_{\lambda'}(\omega)e^{-\lambda' t}.$$

Hence, if

$$B_{\lambda'}(\omega) \leq \frac{\overline{M}_{\lambda'}(\omega)}{2C_S M_\delta(\omega)}$$

we see for all $n \in \mathbb{N}$

$$\|y^{n,\varrho}\|_{\infty,D_{2\beta}} \leq \frac{\varrho(\theta_n\omega)}{2}.$$

But on this domain the truncated functions and the original coefficients coincide.
So, the solutions coincide too, i.e. $y^{n,\varrho} = y^n$ for all $n \in \mathbb{N}_0$.

Finally, if furthermore $B_{\lambda'}(\omega) \leq \frac{e^{-\lambda'}}{C_S M_\delta(\omega)}$ we obtain for all $n \in \mathbb{N}_0$ and $t \in [n, n+1]$

$$|y_t|_{D_{2\beta}} = \left|y^n_{t-n}\right|_{D_{2\beta}} \leq \|y^n\|_{\infty,D_{2\beta}} \leq \|y^{n,\varrho}\|_{\infty,D_{2\beta}} \leq e^{-\lambda'(n+1)} \leq e^{-\lambda' t}.$$ □

Remark 4.21. *Following the lines of the proof of Theorem 4.20 one further sees that the mapping $t \mapsto B_{\lambda'}(\theta_t\omega)$ is continuous for all $\omega \in \Omega'$ and $0 < \lambda' < \lambda$.*

We conclude this section by showing the result of Theorem 4.20 for an arbitrary initial condition $\xi \in W$ with sufficient small norm. The proof is very similar to the one of Corollary 3.55.

Corollary 4.22. *Let $\omega \in \Omega'$ fulfill* (4.33). *Then for all $0 < \lambda' < \lambda$ there exists $\widehat{B}_{\lambda'}(\omega) > 0$ and $M_{\lambda'}(\omega) > 0$ such that, if $|\xi| \leq \widehat{B}_{\lambda'}(\omega)$, then for all $t \geq 0$ the path component of* (1.1) *fulfills*

$$|y_t| \leq M_{\lambda'}(\omega)e^{-\lambda' t}.$$

Proof. We split the proof into two steps. At first consider $\widetilde{\xi} \in D_\beta$. For all $0 < \lambda' < \lambda$ Theorem 4.20 guarantees the existence of $B_{\lambda'}(\omega)$.

Let $\widetilde{y} = \varphi(\cdot, \omega, \widetilde{\xi})$ and $\widetilde{z}$ the corresponding area term. Considering (3.84) and keeping Remark 4.21 in mind we can choose $0 < \widetilde{T} = \widetilde{T}(\omega) \leq 1$ such that

$$\begin{aligned}
&\left|\widetilde{y}_{\widetilde{T}}\right|_{D_{2\beta}} \leq C_S C(\omega)\left|\widetilde{\xi}\right|_{D_\beta}\widetilde{T}^{-\beta} + \frac{1}{3}B_{\lambda'}(\omega),\\
&\frac{2}{3}B_{\lambda'}(\omega) \leq B_{\lambda'}(\theta_{\widetilde{T}}\omega).
\end{aligned}$$

We see that one can choose $\left|\widetilde{\xi}\right|_{D_\beta} \leq \widetilde{B}_{\lambda'}(\omega)$ such that

$$\left|\widetilde{y}_{\widetilde{T}}\right|_{D_{2\beta}} \leq \frac{2}{3}B_{\lambda'}(\omega) \leq B_{\lambda'}(\theta_{\widetilde{T}}\omega).$$

Thus, Lemma 4.5 and Theorem 4.20 show that for all $t \geq \widetilde{T}$

$$|y_t|_{D_{2\beta}} \leq e^{-\lambda'(t-\widetilde{T})} = e^{\lambda'\widetilde{T}} e^{-\lambda' t} \tag{4.35}$$

holds true.
In the second step let $\xi \in W$. The deliberations are very similar to the one of the first step. Let $y = \varphi(\cdot, \omega, \xi)$ and z the corresponding area term. Now, consider (3.78) and note that analogously to Remark 4.21 the mapping $t \mapsto \widetilde{B}_{\lambda'}(\theta_t \omega)$ is continuous, too. So, consider $0 < T = T(\omega) \leq 1$ such that

$$|y_T|_{D_\beta} \leq C_S C(\omega) \, |\xi|_W \, T^{-\beta} + \frac{1}{3} \widetilde{B}_{\lambda'}(\omega),$$
$$\frac{2}{3} \widetilde{B}_{\lambda'}(\omega) \leq \widetilde{B}_{\lambda'}(\theta_T \omega).$$

Hence, there is $\widehat{B}_{\lambda'}(\omega) > 0$ such that if $|\xi| \leq \widehat{B}_{\lambda'}(\omega)$ we have

$$|y_T|_{D_\beta} \leq \frac{2}{3} \widetilde{B}_{\lambda'}(\omega) \leq \widetilde{B}_{\lambda'}(\theta_T \omega).$$

Consequently, by Lemma 4.5 and (4.35) we see that for all $t \geq T(\omega) + \widetilde{T}(\theta_{T(\omega)}\omega)$ the estimate

$$|y_t| \leq |y_t|_{D_{2\beta}} \leq e^{\lambda'(T + \widetilde{T}(\theta_T \cdot))} e^{-\lambda' t}$$

holds true. Finally, for $t \leq T + \widetilde{T}(\theta_T \cdot) \leq 2$ we know by Lemma 3.30 that y remains bounded which concludes the proof. □

4.3 Global Exponential Stability

Unlike in the Section 4.2 now we are interested in proving global exponential stability of the trivial solution. The drawback of an in general not Markovian random input is still present, compare Section 4.2. In [18] and [17] the authors show global exponential stability for the trivial solution in the Young respectively in the rough case but always for small noise, meaning that influence of the random noise is chosen sufficiently small. Here, we want omit this restriction and analyze a fractional Brownian motion with arbitrary but fixed trace class operator Q.

Definition 4.23 (Global Exponential Stability)**.** The trivial solution of (1.1) is called globally exponentially stable with a rate $\lambda' > 0$ if there exists a random variable $M(\omega) > 0$ such that for almost all $\omega \in \Omega$ the path component fulfills

$$|y_t| \leq M(\omega) e^{-\lambda' t} \qquad \text{for all } t \geq 0. \tag{4.36}$$

Clearly, global exponential stability with rate $\lambda' > 0$ implies local exponential stability with rate λ'. Furthermore, we only assume

(SG1) For a $\lambda > 0$ the operator $A + \lambda \mathrm{Id}$ is strictly negative. Thus, it generates an analytic exponentially stable semigroup S.

(SG2) $F(0) = 0$,

(SG3) $G(0) = 0$.

Note that **(SG1)** coincides with **(SL1)** while **(SG2)**–**(SG3)** are less restrictive than **(SL2)**–**(SL3)**. Hence, we can in general not expect to get as good convergence rates as in Section 4.2.

Moreover, during this section we will focus on the case of a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Recall that the fractional Brownian motion is not Markovian for $H \neq \frac{1}{2}$, as

mentioned in Section 4.2. However, it still has stationary increments which is equivalent to the fact that for all $t \in \mathbb{R}$ the shift $\theta_t\omega$ is a fractional Brownian motion, too. Since in this section we will work with a sequence of stopping times we want to show this property for a non-deterministic time shift by a stopping time. For this purpose, we further use an integral representation w.r.t. to a Brownian motion. Hence, we cannot consider $\Omega = C_0^{\alpha'}(\mathbb{R}, V)$ with $\frac{1}{2} < \alpha < \alpha' < H$ as in Section 4.1 as the Brownian motion cannot be embedded into $C_0^{\alpha'}(\mathbb{R}, V)$.
We consider a two-sided Brownian motion $B = (B_t)_{t\in\mathbb{R}}$ on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, by [53, Proposition 2.3] or [49] we have specific integral representation for a fractional Brownian motion, namely for all $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ there is a constant $c_H > 0$ such that

$$B_t^H := c_H\Big(\int_{-\infty}^{0} \Big[(t-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]\, dB_r + \int_0^t (t-r)^{H-\frac{1}{2}} dB_r\Big), \qquad t \geq 0, \tag{4.37}$$

is a one-sided fractional Brownian motion with Hurst parameter H.

With the help of this representation we can show the next lemma.

Lemma 4.24. *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ and $T = T(\omega) \geq 0$ be a stopping time w.r.t. to the augmented Brownian filtration $(\mathcal{F}_t^B)_{t\in\mathbb{R}}$. Then, $\theta_T B^H$ is again a fractional Brownian motion with Hurst parameter H.*
Consequently, for each stopping time $T = T(\omega) > 0$ w.r.t the augmented fractional Brownian filtration $(\mathcal{F}_t^{B^H})_{t\geq 0}$ the randomly shifted process $\theta_T B^H$ is again a fractional Brownian motion with Hurst parameter H.

Proof. By (4.37) we have

$$\begin{aligned}
(\theta_T(\omega) B^H)_t &= B^H_{t+T(\omega)} - B^H_{T(\omega)} \\
&= c_H\Big(\int_{-\infty}^{0} \Big[(t+T(\omega)-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]\, dB_r + \int_0^{t+T(\omega)} (t+T(\omega)-r)^{H-\frac{1}{2}} dB_r\Big) \\
&\quad - c_H\Big(\int_{-\infty}^{0} \Big[(T(\omega)-r)^{H-\frac{1}{2}} - (-r)^{H-\frac{1}{2}}\Big]\, dB_r + \int_0^{T(\omega)} (T(\omega)-r)^{H-\frac{1}{2}} dB_r\Big) \\
&= c_H\Big(\int_{-\infty}^{t} (t-r)^{H-\frac{1}{2}} d\theta_{T(\omega)}B_r - \int_{-\infty}^{0} (-r)^{H-\frac{1}{2}} d\theta_{T(\omega)}B_r\Big).
\end{aligned}$$

By the strong Markov property we know that $\theta_{T(\omega)}B$ is a Brownian motion. Hence, by Representation (4.37) we know that $\theta_{T(\omega)}B^H$ is a fractional Brownian motion with Hurst parameter H.
Furthermore, Representation (4.37) yields that for each $t \geq 0$ we have $\mathcal{F}_t^{B^H} \subset \mathcal{F}_t^B$. Thus, each stopping time $T \geq 0$ w.r.t $(\mathcal{F}_t^{B^H})_{t\geq 0}$ is a stopping time w.r.t. $(\mathcal{F}_t^B)_{t\in\mathbb{R}}$ which finishes the proof. □

Note that for the rest of this section we will work with the one-sided fractional Brownian motion given by Representation (4.37). For notational simplicity we set $\omega = B^H$. However, in contrast to all previous sections ω is locally α-Hölder continuous with $\alpha < H$ just almost surely.
Here it is important to mention that the set $C_0^\alpha(\mathbb{R}_+, V)$ is θ-invariant. Hence, we can follow the pathwise deliberations given in Chapter 3 and obtain the same results but just for those paths with $\omega \in C_0^\alpha(\mathbb{R}_+, V)$.

Let us recall important estimates given in Corollary 3.15 and (3.92). Keeping in mind Remark 3.58 and (4.21) for $\omega \in C^{\alpha}$ and $y \in C^{\beta,\beta}$ with $\alpha+\beta>1$ we have for $0<s\leq t\leq T$

$$\left|\int_s^t S(t-r)F(y_r)dr\right| \leq C_S C_F \left\|y\right\|_{\beta,\beta,T}(t-s) \qquad \text{and}$$
$$\left|\int_s^t S(t-r)G(y_r)d\omega_r\right| \leq C_S C_G \left\|\omega\right\|_{\alpha,T}\left\|y\right\|_{\beta,\beta,T}(t-s)^{\beta-\alpha}.$$

Hence, by combining these results we obtain

$$\left\|\mathrm{I}_\omega(y,F,G)\right\|_{\beta,\beta,T} \leq \left(C_S C_F T + C_S C_G T^{\alpha}\left\|\omega\right\|_{\alpha,T}\right)\left\|y\right\|_{\beta,\beta,T} \tag{4.38}$$

Consider a solution y of (1.1) on $\mathbb{R}_+$ for $H>\frac{1}{2}$. In Section 4.2 we used a truncation technique in order to prove local exponential stability. Now, we will exploit the possibility to choose the time parameter in estimate (4.38) small.

Therefore, consider a fixed $0<\mu\leq\frac{1}{2}$, and similar to [29], define a sequence of increasing stopping times $(T_n)_{n\in\mathbb{N}_0}=(T_n(\omega,\mu))_{n\in\mathbb{N}_0}$ with $T_0\equiv 0$ and

$$C_S(C_F(T_{n+1}-T_n)+C_G(T_{n+1}-T_n)^{\alpha}\left\|\theta_{T_n}\omega\right\|_{\alpha,T_{n+1}-T_n}) = \mu \qquad \text{for all } n\in\mathbb{N}_0, \tag{4.39}$$

if $\omega\in C_0^{\alpha}(\mathbb{R}_+,V)$ and $T_n=0$ for all $n\in\mathbb{N}_0$ else.

Let us summarize the important properties of this sequence of stopping times in the next lemma.

Lemma 4.25. *For all $\omega\in\Omega$ and $\mu\in(0,\frac{1}{2}]$ it holds*

(i) The sequence $(T_n)_{n\in\mathbb{N}_0}$ is well defined.

(ii) We can iteratively calculate the stopping times meaning that

$$T_{n+1}(\omega)=T_n(\omega)+T_1(\theta_{T_n(\omega)}\omega). \tag{4.40}$$

(iii) For all $0\leq s<t$ we have

$$s+T_1(\theta_s\omega)<t+T_1(\theta_t\omega). \tag{4.41}$$

(iv) For all $n\in\mathbb{N}_0$ we have the estimates

$$T_{n+1}-T_n\leq\frac{\mu}{C_S C_F} \tag{4.42}$$

and

$$T_{n+1}-T_n\geq\frac{\mu^{\frac{1}{\alpha}}}{(\mu^{1-\alpha}C_S^{\alpha}C_F^{\alpha}+C_S C_G\left\|\theta_{T_n}\omega\right\|_{\alpha})^{\frac{1}{\alpha}}}. \tag{4.43}$$

(v) T_n tends to infinity for $n\to\infty$ and

$$\lim_{n\to\infty}\frac{T_n}{n}=\mathbb{E}T_1 \qquad \textit{almost surely.}$$

Proof. For defining the sequence according to (4.39) the first time T_1 is crucial. It is implicitly given by

$$C_S(C_F T_1 + C_G T_1^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,T_1}) = \mu.$$

Consider the function

$$\begin{aligned} &\mathcal{T}_\omega\colon \mathbb{R}_+ \to \mathbb{R}_+, \\ &\mathcal{T}_\omega(q) = C_S(C_F\, q + C_G\, q^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,q}). \end{aligned}$$

This function is monotonically increasing and continuous and fulfills $\mathcal{T}_\omega(0) = 0$ and $\lim\limits_{q\to\infty} \mathcal{T}_\omega(q) = \infty$. Hence, there exists T_1 such that $\mathcal{T}_{\mu,\omega}(T_1) = \mu$.

The same deliberations yield (4.40) and so the existence of the whole sequence.

In order to prove (4.41) we assume the contrary. This yields

$$\begin{aligned} \mu &= C_S(C_F T_1(\theta_s\omega) + C_G T_1(\theta_s\omega)^\alpha \left\|\!\left|\theta_s\omega\right|\!\right\|_{\alpha,T_1(\theta_s\omega)}) \\ &= C_S(C_F T_1(\theta_s\omega) + C_G T_1(\theta_s\omega)^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,s,s+T_1(\theta_s\omega)}) \end{aligned}$$

We have $s < t \le t + T_1(\theta_t\omega) \le s + T_1(\theta_s\omega)$ which yields $T_1(\theta_s\omega) > T_1(\theta_t\omega)$. Hence,

$$\begin{aligned} \mu &= C_S(C_F T_1(\theta_s\omega) + C_G T_1(\theta_s\omega)^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,[s,s+T_1(\theta_s\omega)]}) \\ &> C_S(C_F T_1(\theta_t\omega) + C_G T_1(\theta_t\omega)^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,[t,t+T_1(\theta_t\omega)]}) \\ &= C_S(C_F T_1(\theta_t\omega) + C_G T_1(\theta_t\omega)^\alpha \left\|\!\left|\theta_t\omega\right|\!\right\|_{\alpha,T_1(\theta_t\omega)}) = \mu. \end{aligned}$$

Clearly, this is not possible.

Estimate (4.42) follows directly by (4.39). By using this estimate we derive

$$\mu = C_S C_F T_1 + C_S C_G T_1^\alpha \left\|\!\left|\omega\right|\!\right\|_{\alpha,T_1} \le \left(C_S C_F \frac{\mu^{1-\alpha}}{C_S^{1-\alpha} C_F^{1-\alpha}} + C_S C_G \left\|\!\left|\omega\right|\!\right\|_{\alpha,T_1} \right) T_1^\alpha.$$

This yields

$$T_1 \ge \frac{\mu^{\frac{1}{\alpha}}}{\left(C_S^\alpha C_F^\alpha \mu^{1-\alpha} + C_S C_G \left\|\!\left|\omega\right|\!\right\|_{\alpha,T_1} \right)^{\frac{1}{\alpha}}}.$$

So, (4.40) implies (4.43). Finally, (4.40) entails

$$\frac{T_n}{n} = \frac{1}{n} \sum_{j=0}^{n-1} T_{j+1} - T_j = \frac{1}{n} \sum_{j=0}^{n-1} T_1(\theta_{T_j}\omega).$$

Since $(\theta_{T_n})_{n\in\mathbb{N}_0}$ is measure preserving, see Lemma 4.24, Birkhoff's Ergodic Theorem guarantees the existence of the limit for all ω on a $(\theta_{T_n})_{n\in\mathbb{N}_0}$-invariant set with full measure. Define

$$T^*(\omega) := \begin{cases} \lim\limits_{n\to\infty} \frac{T_n(\omega)}{n}, & \text{if the limit exists,} \\ 0, & \text{else.} \end{cases}$$

Then, it holds $\mathbb{E}T^* = \mathbb{E}T_1$.

However, to our best knowledge it is not clear if this shift $(\theta_{T_n})_{n\in\mathbb{N}_0}$ is ergodic. Hence, we have to manually prove that the limit T^* is constant almost surely.

We will prove that for all $t > 0$ and all $n \in \mathbb{N}_0$ we have

$$T_n(\omega) < t + T_n(\theta_t\omega). \tag{4.44}$$

The case $n = 0$ is trivial and $n = 1$ is given by (4.41). Furthermore, by (4.40) we have

$$\begin{aligned} T_{n+1}(\omega) &= T_n(\omega) + T_1(\theta_{T_n(\omega)}\omega) \qquad \text{and} \\ t + T_{n+1}(\theta_t\omega) &= t + T_n(\theta_t\omega) + T_1(\theta_{T_n(\theta_t\omega)}\theta_t\omega) \\ & t + T_n(\theta_t\omega) + T_1(\theta_{T_n(\theta_t\omega)+t}\omega). \end{aligned}$$

By induction assumption we have $T_n(\omega) < t + T_n(\theta_t\omega)$. So, (4.41) yields

$$\begin{aligned} T_{n+1}(\omega) &= T_n(\omega) + T_1(\theta_{T_n(\omega)}\omega) \\ &< t + T_n(\theta_t\omega) + T_1(\theta_{T_n(\theta_t\omega)+t}\omega) \\ &= t + T_{n+1}(\theta_t\omega). \end{aligned}$$

Consequently, for all $t > 0$ there exists $k = k(t, \omega) \in \mathbb{N}_0$ such that

$$T_k(\omega) < t \leq T_{k+1}(\omega).$$

Then, (4.44) yields for all $j \in \mathbb{N}$.

$$T_{k+j}(\omega) < t + T_j(\theta_t\omega) \leq T_{k+1+j}(\omega).$$

This implies

$$T^*(\omega) = \lim_{n\to\infty} \frac{T_n(\omega)}{n} = \lim_{n\to\infty} \frac{T_n(\theta_t\omega)}{n} = T^*(\theta_t\omega)$$

for all $t > 0$. Since $(\theta_t)_{t\geq 0}$ is ergodic, see [31] it follows that the limit is constant almost surely. □

Now, we have all tools ready for proving global exponential stability.

Theorem 4.26. *Given the coefficients F and G fulfilling **(SG2)** and **(SG3)**, a trace class fractional Brownian motion with $tr_V Q < \infty$ and a linear operator A that generates an analytic semigroup and satisfying **(SG1)** with a sufficiently large $\lambda > 0$, i.e. $\lambda > \lambda_0(C_S, C_F, C_G, \alpha, tr_V Q)$, then the trivial solution of* (1.1) *for $H > \frac{1}{2}$ is globally exponentially stable.*

Proof. For $n \in \mathbb{N}_0$ define the function

$$y^n_t := y_{t+T_n}, \qquad \text{for } 0 \leq t \leq T_{n+1} - T_n.$$

Lemma 4.9 yields that for $t \in [T_n, T_{n+1}]$

$$y^n_{t-T_n} = S(t)\xi + \sum_{j=0}^{n-1} S(t - T_{j+1})\mathrm{I}_{T_{j+1}-T_j,\theta_j\omega}(y^j, F, G) + \mathrm{I}_{t-T_n,\theta_n\omega}(y^n, F, G).$$

Hence, by (4.7) and (4.38) it follows

$$\begin{aligned} &\|y^n\|_{\beta,\beta,T_{n+1}-T_n} \\ &\leq C_S e^{-\lambda T_n} |\xi| \\ &+ C_S \sum_{j=0}^{n-1} e^{-\lambda(T_n-T_{j+1})}(C_F(T_{j+1} - T_j) + C_G(T_{j+1} - T_j)^\alpha \left\|\theta_{T_j}\omega\right\|_{\alpha,T_{j+1}-T_j}) \left\|y^j\right\|_{\beta,\beta,T_{j+1}-T_j} \\ &+ C_S(C_F(T_{n+1} - T_n) + C_G(T_{n+1} - T_n)^\alpha \|\theta_{T_n}\omega\|_{\alpha,T_{n+1}-T_n}) \|y^n\|_{\beta,\beta,T_{n+1}-T_n}. \end{aligned}$$

The definition of the stopping times yields

$$e^{\lambda T_n} \|y^n\|_{\beta,\beta,T_{n+1}-T_n} \leq \frac{C_S}{1-\mu} |\xi| + \frac{\mu}{1-\mu} \sum_{j=0}^{n-1} e^{\lambda(T_{j+1}-T_j)} e^{\lambda T_j} \left\|y^j\right\|_{\beta,\beta,T_{j+1}-T_j}.$$

We apply the Gronwall Lemma (Lemma 4.14) with $\mathrm{a}_n = e^{\lambda T_n}\left\|y^n\right\|_{\beta,\beta,T_{n+1}-T_n}$, $\mathrm{b}_n = \frac{\mu}{1-\mu}e^{\lambda(T_{n+1}-T_n)}$ and $\mathrm{c} = C_S\left|\xi\right|$. This leads to

$$
\begin{aligned}
\left\|y^n\right\|_{\beta,\beta,T_{n+1}-T_n} &\leq \frac{C_S}{1-\mu}\left|\xi\right| e^{-\lambda T_n}\prod_{j=0}^{n-1}(1+\frac{\mu}{1-\mu}e^{\lambda(T_{j+1}-T_j)}) \\
&= 2C_S\left|\xi\right|\prod_{j=0}^{n-1}(e^{-\lambda(T_{j+1}-T_j)}+\frac{\mu}{1-\mu}) \\
&= 2C_S\left|\xi\right|\exp\Big(\sum_{j=0}^{n-1}\log(e^{-\lambda(T_{j+1}-T_j)}+\frac{\mu}{1-\mu})\Big).
\end{aligned}
$$

We know by (4.42) that

$$\lambda(T_{n+1}-T_n) \leq \frac{\lambda\mu}{C_S C_F}, \qquad \text{for all } n \in \mathbb{N}_0.$$

If $\lambda > 2C_SC_F$ we can set $\mu = \frac{C_SC_F}{\lambda} \leq \frac{1}{2}$ which yields $\lambda(T_{n+1}-T_n) \leq 1$.
Since the function $q \mapsto \log(e^{-q}+\frac{\mu}{1-\mu})$ is convex, we have for all $0 \leq q \leq 1$ the estimate

$$\log(e^{-q}+\frac{\mu}{1-\mu}) \leq \log(1+\frac{\mu}{1-\mu}) - \Big(\log(1+\frac{\mu}{1-\mu}) - \log(e^{-1}+\frac{\mu}{1-\mu})\Big)\, q. \tag{4.45}$$

The coefficient of q is monotonically decreasing in μ, it attains it minimum for $\mu = \frac{1}{2}$, namely $\log(2) - \log(1+e^{-1}) > e^{-1}$.
Consequently, we obtain

$$\sum_{j=0}^{n-1}\log(e^{-\lambda(T_{j+1}-T_j)}+\frac{\mu}{1-\mu}) \leq n\log(1+\frac{\mu}{1-\mu}) - \frac{\lambda}{e}\,T_n.$$

Hence, by dividing this inequality by $(n\mu)$ and passing to the limit we see

$$\limsup_{n\to\infty}\frac{1}{n\mu}\sum_{j=0}^{n-1}\log(e^{-\lambda(T_{j+1}-T_j)}+\frac{\mu}{1-\mu}) \leq \frac{1}{\mu}\log(1+\frac{\mu}{1-\mu}) - \frac{\lambda}{e\,\mu}\,\limsup_{n\to\infty}\frac{T_n}{n}.$$

We have by Lemma 4.25 (v) and (4.43)

$$\lim_{n\to\infty}\frac{T_n}{n} = \mathbb{E}T_1 \geq \mathbb{E}\Big[\frac{\mu^{\frac{1}{\alpha}}}{(\mu^{1-\alpha}C_S^\alpha C_F^\alpha + C_SC_G\left\|\omega\right\|_{\alpha,T_1})^{\frac{1}{\alpha}}}\Big]$$

Jensen's inequality and Lemma 4.2 show

$$\mathbb{E}T_1 \geq \frac{\mu^{\frac{1}{\alpha}}}{(\mu^{1-\alpha}C_S^\alpha C_F^\alpha + C_SC_G\mathbb{E}\left\|\omega\right\|_{\alpha})^{\frac{1}{\alpha}}} \geq \frac{\mu^{\frac{1}{\alpha}}}{(\mu^{1-\alpha}C_S^\alpha C_F^\alpha + C_SC_Gc_Q)^{\frac{1}{\alpha}}},$$

where $c_Q = c_{\alpha,H}\sqrt{\mathrm{tr}_V Q}$. So, finally we have

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n\mu}\sum_{j=0}^{n-1}\log(e^{-\lambda(T_{j+1}-T_j)}+\frac{\mu}{1-\mu}) \\
&\leq \frac{1}{\mu}\log(1+\frac{\mu}{1-\mu}) - \frac{\lambda}{e\,\mu}\,\frac{\mu^{\frac{1}{\alpha}}}{(\mu^{1-\alpha}C_S^\alpha C_F^\alpha + C_SC_Gc_Q)^{\frac{1}{\alpha}}} \\
&\leq \frac{1}{1-\mu} - \frac{1}{e}\,\frac{\lambda^{2-\frac{1}{\alpha}}(C_SC_F)^{\frac{1}{\alpha}-1}}{(\mu^{1-\alpha}C_S^\alpha C_F^\alpha + C_SC_Gc_Q)^{\frac{1}{\alpha}}} \\
&\leq 2 - \frac{1}{e}\,\frac{\lambda^{2-\frac{1}{\alpha}}(C_SC_F)^{\frac{1}{\alpha}-1}}{(C_S^\alpha C_F^\alpha + C_SC_Gc_Q)^{\frac{1}{\alpha}}} =: -\mathrm{K}(\lambda, C_S, C_F, C_G, \alpha, \mathrm{tr}_V Q).
\end{aligned}
$$

Hence, we conclude that if $\lambda > 0$ is large enough then $\mathrm{K} = \mathrm{K}(\lambda, C_S, C_F, C_G, \alpha, \mathrm{tr}_V Q) > 0$ and there exists a constant $M = M(\omega) > 0$ such that

$$\|y^n\|_{\beta,\beta,T_{n+1}-T_n} \leq MC_S\,|\xi|\,e^{-n\mu\mathrm{K}} \qquad \text{for all } n \in \mathbb{N}_0.$$

This entails for $t \in [T_n, T_{n+1}]$

$$|y_t| \leq \|y^n\|_{\beta,\beta,T_{n+1}-T_n} \leq MC_S\,|\xi|\,e^{-n\mu\mathrm{K}} \leq MC_S\,|\xi|\,e^{-\frac{n}{n+1}C_SC_FT_{n+1}\mathrm{K}} \leq MC_S\,|\xi|\,e^{-\frac{n}{n+1}C_SC_F\mathrm{K}t},$$

which grants global exponential stability for all $\lambda' < C_SC_F\mathrm{K}$. □

We point out a concluding remark.

Remark 4.27. *During this section we assumed $H > \frac{1}{2}$ in order to obtain $\alpha > \frac{1}{2} \Leftrightarrow 2 - \frac{1}{\alpha} > 0$. This guarantees* K *to be positive for sufficiently large λ. However, the general approach does not need this assumption. At the moment we are not able to show ergodicity of $(\theta_{T_n})_{n\in\mathbb{N}_0}$. If this would be true we could benefit from much better estimates than* (4.45)*. Thus, we could derive a much better stability rate and prove global exponential stability also for $H \in (\frac{1}{3}, \frac{1}{2})$.*

5. Conclusion and Outlook

We summarize the main results of this thesis and indicate possible extensions.

This work can be structured into two main parts. Firstly, we developed a solution theory for rough evolution equations and afterwards we analyzed the dynamics of the corresponding solution.

One general drawback is given by Assumption **(G)** as we demand a smoothing property of G. This assumption was necessary in order to rigorously define the supporting process b, see Lemma 3.26, as well as for showing the necessary spatial regularity for the path component of the solution, see (3.84). Combined with the assumption that the diffusion coefficient is three times Frechét differentiable it is hard to think of more general examples than the one presented in Section 3.7. A promising approach to relax **(G)** is to consider estimates in a negative fractional domain of the linear operator, see [32]. We strongly believe that by similar deliberations as in Chapter 3 working with negative fractional domains leads to the same results for diffusion coefficient satisfying

(G)' $G\colon D_{-\beta} \to D_{-\beta}$ is bounded and three times Frechét differentiable with bounded derivatives. Furthermore, the restriction $G\colon W \to W$ is Lipschitz continuous.

This assumption is quite similar to the ones stated in [32], which are the most general as far as we know. However, the authors in [32] are working with finite-dimensional noise.

The algebraic framework developed in Chapter 3 is influenced by the rough paths theory for Evolution Equations, see [37], [10] or [32] but also uses ideas from fractional calculus, see [50], [25] or [27]. We are aware that the algebraic framework respectively the solution space, see (3.63), we are working with is more complicated than the Gubinelli space used in the rough paths theory. The setting introduced in this thesis is somehow more natural due to the time-singularity of the semigroup in zero and can be hopefully extended to discontinuous p-variation processes, as in the original works of [61] and [48]. The Hölder (semi)norms used in (3.63) can be transformed to variation (semi)norms.

We conjecture that the ideas of Section 3.3 can be adapted to the discontinuous p-variation case, compare [61], [48], [21]. One of the main obstacles is then to give meaning to the supporting processes, introduced in Section 3.4, in order to give meaning to the rough integral (1.2).

In Chapter 4 we considered a trace class fractional Brownian motion as noisy input and analyzed the stability of the solution of (1.1). Clearly, one can consider more general classes of stochastic processes, like e.g. Gaussian processes with stationary increments, see [20], [22] or Volterra processes investigated in [6], [7] which are in general not Gaussian nor have stationary increments and thus, do not define a metric dynamical system. In case we are able to establish the solution theory for discontinuous noise analyzing Lévy processes, see [1], would be of interest.

Furthermore, one natural extension is to generalize the results of Section 4.3 for rough noise, e.g. a fractional Brownian motion with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. We believe that the techniques used in the Sections 4.2 and 4.3 with slight modifications can lead to global stability results. In case one can prove ergodicity of the random shift family, see Remark 4.27, one could even derive better estimates in the Young case as well as in the rough case.

Moreover, one could analyze further dynamical aspects, for instance random fixed points, random attractors or invariant manifolds, see [26], [23], [5], [41], [45], [15] or [34]. Finally, one could be interested in generalizing the linear operator, e.g. to consider non-autonomous rough semilinear SPDEs, as investigated in [33]. Since the estimates for the evolution family stated in [55, Chapter 5] are similar to the one for the semigroup, see Section 2.1, we believe that one can extend the results to the non-autonomous case by similar deliberations.

A. Appendix

A.1 Preliminary Results for the Sewing Lemma

The following facts will be employed in Section 3.3.

Lemma A.1. *Let $\varrho > 1$ and $\kappa \in C([0,T], W)$ such that*

$$\kappa_0 = 0 \qquad \text{and} \qquad \left|(\hat{\delta}\kappa)_{ts}\right| \leq c\,(t-s)^{\varrho}\,.$$

Then $\kappa \equiv 0$.

Proof. For any partition $\mathcal{P}$ of $[0,t]$ we have

$$\kappa_t = (\hat{\delta}\kappa)_{t0} = \sum_{[u,v]\in\mathcal{P}} S(t-v)(\hat{\delta}\kappa)_{vu}.$$

Hence, we derive

$$|\kappa_t| \leq c_S c\,|\mathcal{P}|^{\varrho-1}\,t, \quad \text{which tends to } 0 \text{ as } |\mathcal{P}| \to 0.$$

Consequently $\kappa \equiv 0$. □

Lemma A.2. *Let $0 < \alpha,\ \beta < 1$, $\varrho > 1$ and $\kappa \in C([0,T], W)$ such that*

$$\begin{aligned}
&\kappa_0 = 0,\\
&\left|(\hat{\delta}\kappa)_{ts}\right| \leq C\,(t-s)^{\alpha}\,,\\
&\left|(\hat{\delta}\kappa)_{ts}\right| \leq C s^{-\beta}\,(t-s)^{\varrho}\,, \quad \text{for } s \neq 0.
\end{aligned}$$

Then $\kappa \equiv 0$.

Proof. Let $t \in [0,T]$ be arbitrary but fixed. Consider $\mathcal{P}_n$ a dyadic partition of $[0,t]$. We have

$$\kappa_t = (\hat{\delta}\kappa)_{t0} = \sum_{[u,v]\in\mathcal{P}_n} S(t-v)(\hat{\delta}\kappa)_{vu}.$$

Consequently,

$$\begin{aligned}
|\kappa_t| &\leq C \sum_{[u,v]\in\mathcal{P}_n} \left|(\hat{\delta}\kappa)_{vu}\right|\\
&\leq C\frac{t^{\alpha}}{2^{n\alpha}} + C \sum_{\substack{[u,v]\in\mathcal{P}_n\\ u\neq 0}} u^{-\beta}\frac{t^{\varrho}}{2^{n\varrho}}\\
&= C\frac{t^{\alpha}}{2^{n\alpha}} + C\frac{t^{\varrho-1}}{2^{n-1}} \sum_{\substack{[u,v]\in\mathcal{P}_n\\ u\neq 0}} u^{-\beta}\frac{t}{2^{n}}\\
&\leq C\frac{t^{\alpha}}{2^{n\alpha}} + C\frac{t^{\varrho-1}}{2^{n-1}} \int_0^t q^{-\beta}dq\\
&\leq C\frac{t^{\alpha}}{2^{n\alpha}} + C\frac{t^{\varrho-\beta}}{2^{n-1}}.
\end{aligned}$$

Since, this statement has to be valid for all $n \in \mathbb{N}$ we must have $\kappa \equiv 0$. □

Lemma A.3. *Given* $0 < \gamma,\ \varepsilon < 1$ *then there is a constant* $C = C(\gamma, \varepsilon)$ *such that for all* $n \in \mathbb{N}$ *the estimate*

$$\sum_{k=1}^{n-1} k^{-\gamma}(n-k)^{-\varepsilon} \leq C \sum_{k=0}^{n-1} (k+1)^{-\gamma}(n-k)^{-\varepsilon} \tag{A.1}$$

holds true.

Proof. For the right sum we see

$$\sum_{k=0}^{n-1} (k+1)^{-\gamma}(n-k)^{-\varepsilon} = n^{-\varepsilon} + \sum_{k=1}^{n-1} (k+1)^{-\gamma}(n-k)^{-\varepsilon}$$

Hence, (A.1) is fulfilled if

$$\sum_{k=1}^{n-1} \left[k^{-\gamma} - (k+1)^{-\gamma}\right](n-k)^{-\varepsilon} \leq Cn^{-\varepsilon}$$

Furthermore, we estimate

$$\begin{aligned}\sum_{k=1}^{n-1} \left[k^{-\gamma} - (k+1)^{-\gamma}\right](n-k)^{-\varepsilon} &= \sum_{k=1}^{n-1} \int_k^{k+1} \gamma x^{-\gamma-1} dx\, (n-k)^{-\varepsilon} \\ \leq \sum_{k=1}^{n-1} \int_k^{k+1} \gamma x^{-\gamma-1}(n-x)^{-\varepsilon} dx &= \int_1^n \gamma x^{-\gamma-1}(n-x)^{-\varepsilon} dx.\end{aligned}$$

Therefore, it is sufficient to show that for all $u \geq 1$ it holds

$$g(u,\gamma,\varepsilon) := \int_1^u \gamma x^{-\gamma-1}(u-x)^{-\varepsilon} dx \leq Cu^{-\varepsilon}.$$

To prove this we define

$$\begin{aligned}h(u,\gamma,\varepsilon) :=& u^{\gamma+\varepsilon}\, g(u,\gamma,\varepsilon) \\ =& u^{\gamma+\varepsilon} \int_1^u \gamma x^{-\gamma-1}(u-x)^{-\varepsilon} dx \\ =& \int_1^u \gamma \left(\frac{u}{x}\right)^{\gamma-1} \left(1 - \frac{x}{u}\right)^{-\varepsilon} \frac{u}{x^2} dx \\ =& -\int_1^u \gamma \left(\frac{u}{x}\right)^{\gamma-1} \left(1 - \frac{x}{u}\right)^{-\varepsilon} d\left(\frac{u}{x}\right) \\ =& \int_1^u \gamma x^{\gamma-1} \left(1 - \frac{1}{x}\right)^{-\varepsilon} dx.\end{aligned}$$

If $\gamma + \varepsilon \leq 1$ consider

$$h(u,\gamma,1-\gamma) = \int_1^u \gamma\,(x-1)^{\gamma-1} dx = (u-1)^{\gamma} \leq u^{\gamma}.$$

Since $0 < 1 - \frac{1}{x} < 1$ for all $x \geq 1$ we see that h is monotonously increasing in ε. Hence, we obtain for $0 \leq \varepsilon \leq 1-\gamma$

$$h(u,\gamma,\varepsilon) \leq h(u,\gamma,1-\gamma) \leq u^{\gamma},$$

consequently

$$g(u,\gamma,\varepsilon) \leq u^{-\varepsilon}.$$

If $\gamma + \varepsilon > 1$ we estimate

$$\begin{aligned} h(u,\gamma,\varepsilon) &= \int_1^u \gamma x^{\gamma-1}\left(1-\frac{1}{x}\right)^{-\varepsilon} dx \\ &= \int_1^u \gamma x^{\gamma+\varepsilon-1}(x-1)^{-\varepsilon} dx \\ &\leq \gamma u^{\gamma+\varepsilon-1}\int_1^u (x-1)^{-\varepsilon} dx \\ &= \frac{\gamma}{1-\varepsilon} u^{\gamma+\varepsilon-1}(u-1)^{1-\varepsilon} \leq \frac{\gamma}{1-\varepsilon} u^{\gamma}, \end{aligned}$$

which directly yields

$$g(u,\gamma,\varepsilon) \leq \frac{\gamma}{1-\varepsilon} u^{-\varepsilon}.$$

□

A.2 Fundamental Estimates for the Fixed-Point Argument

In the next deliberations we illustrate a technique which is required in Section 3.5. This is based on the division property for smooth functions, see p. 109 in [20]. This is also used for rough SDEs, in order to estimate the difference of the norm of two controlled rough paths, consult [20, Chapter 8], especially the proof of Theorem 8.4 in [20].

In the following C_G stands for a universal constant which exclusively depends on G and its derivatives. The next result can immediately be obtained applying the mean value theorem.

Lemma A.4. *Let $\hat{W}$ be a separable Banach space, $G \in C_b^2(W,\hat{W})$ and $x_1, x_2, x_3, x_4 \in W$. The following estimate*

$$\begin{aligned} &|G(x_2) - G(x_1) - G(x_4) + G(x_3)| \\ &\leq C_G\,|x_2 - x_1 - x_4 + x_3| + C_G\,|x_4 - x_3|\,(|x_3 - x_1| + |x_4 - x_2|) \end{aligned} \tag{A.2}$$

holds true.

Keeping this in mind we derive the following result.

Corollary A.5. *Let $y^i \in C^{\beta,\beta}([0,T];W)$ for $i = 1,2$ and $G \in C_b^2(W,\hat{W})$. Then, for all $0 < s < t \leq T$ we have*

$$\begin{aligned} &\left|G(y_t^1) - G(y_s^1) - G(y_t^2) + G(y_s^2)\right| \\ &\leq C_G\left(\left\|\left|y^1 - y^2\right|\right\|_{\beta,\beta} + \left\|\left|y^2\right|\right\|_{\beta,\beta}\left\|y^1 - y^2\right\|_{\infty}\right) s^{-\beta}(t-s)^{\beta}, \end{aligned} \tag{A.3}$$

as well as

$$\begin{aligned} &\left|G(y_t^1) - G(y_s^1) - G(y_t^2) + G(y_s^2)\right| \\ &\leq C_G\left(\left\|y^1 - y^2\right\|_{\infty} + \left\|y^2\right\|_{\infty}\left\|y^1 - y^2\right\|_{\infty}\right). \end{aligned} \tag{A.4}$$

Proof. Applying Lemma A.4, see also Lemma 7.1 in [54], with $x_1 = y_s^1$, $x_2 = y_t^1$, $x_3 = y_s^2$ and $x_4 = y_t^2$ we infer

$$\begin{aligned}&\left|G(y_t^1) - G(y_s^1) - G(y_t^2) + G(y_s^2)\right|\\ &\leq C_G \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right| \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right)\end{aligned}$$

We have two possibilities to estimate this expression. On the one hand we have

$$\begin{aligned}&\left|G(y_t^1) - G(y_s^1) - G(y_t^2) + G(y_s^2)\right|\\ &\leq C_G \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right| \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right)\\ &\leq C_G \left(\left\|\left|y^1 - y^2\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta} \left\|y^1 - y^2\right\|_\infty\right) s^{-\beta} (t-s)^\beta .\end{aligned}$$

On the other hand we obtain

$$\begin{aligned}&\left|G(y_t^1) - G(y_s^1) - G(y_t^2) + G(y_s^2)\right|\\ &\leq C_G \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right| \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right)\\ &\leq C_G \left(\left\|y^1 - y^2\right\|_\infty + \left\|y^2\right\|_\infty \left\|y^1 - y^2\right\|_\infty\right) .\end{aligned}$$

This proves the statement. □

Lemma A.6. *Let $G \in C_b^3(W, \hat{W})$ and $x_1, x_2, x_3, x_4 \in W$. Then*

$$\begin{aligned}&|G(x_2) - G(x_1) - DG(x_1)(x_2 - x_1) - G(x_4) + G(x_3) + DG(x_3)(x_4 - x_3)|\\ &\leq C_G \left(|x_2 - x_1| + |x_4 - x_3|\right) |x_2 - x_1 - x_4 + x_3|\\ &+ C_G |x_4 - x_3|^2 \left(|x_3 - x_1| + |x_4 - x_2|\right) .\end{aligned} \tag{A.5}$$

For a complete proof, see p. 2716 in [42].

This helps us further obtain an essential estimate for our fixed-point argument.

Corollary A.7. *Given $y^i \in C^{\beta,\beta}([0,T]; W)$ for $i = 1, 2$ and $G \in C_b^3(W, \hat{W})$. Then the following estimates are valid for all $0 < s < t \leq T$:*

$$\begin{aligned}&\left|G(y_s^1) - G(y_t^1) + DG(y_s^1)(y_t^1 - y_s^1) - \left(G(y_s^2) - G(y_t^2) + DG(y_s^2)(y_t^2 - y_s^2)\right)\right|\\ &\leq C_G \left[\left(\left\|\left|y^1\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta}\right) \left\|\left|y^1 - y^2\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta}^2 \left\|y^1 - y^2\right\|_\infty\right] s^{-2\beta} (t-s)^{2\beta} ,\end{aligned} \tag{A.6}$$

as well as

$$\begin{aligned}&\left|G(y_s^1) - G(y_t^1) + DG(y_s^1)(y_t^1 - y_s^1) - \left(G(y_s^2) - G(y_t^2) + DG(y_s^2)(y_t^2 - y_s^2)\right)\right|\\ &\leq C_G \left(\left\|\left|y^1\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta} \left\|y^2\right\|_\infty\right) \left\|y^1 - y^2\right\|_\infty s^{-\beta} (t-s)^\beta .\end{aligned} \tag{A.7}$$

Proof. As previously argued, we apply Lemma A.6 with $x_1 = y_s^1$, $x_2 = y_t^1$, $x_3 = y_s^2$ and $x_4 = y_t^2$. This results in

$$\begin{aligned}&\left|G(y_s^1) - G(y_t^1) + DG(y_s^1)(y_t^1 - y_s^1) - \left(G(y_s^2) - G(y_t^2) + DG(y_s^2)(y_t^2 - y_s^2)\right)\right|\\ &\leq C_G \left(\left|y_t^1 - y_s^1\right| + \left|y_t^2 - y_s^2\right|\right) \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right|^2 \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right) .\end{aligned}$$

Again, we use have two possibilities to obtain the following inequalities. First of all we infer that

$$\begin{aligned}&\left|G(y_s^1) - G(y_t^1) + DG(y_s^1)(y_t^1 - y_s^1) - \left(G(y_s^2) - G(y_t^2) + DG(y_s^2)(y_t^2 - y_s^2)\right)\right|\\ &\leq C_G \left(\left|y_t^1 - y_s^1\right| + \left|y_t^2 - y_s^2\right|\right) \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right|^2 \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right)\\ &\leq C_G \left(\left\|\left|y^1\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta}\right) \left\|\left|y^1 - y^2\right\|\right|_{\beta,\beta} s^{-2\beta} (t-s)^{2\beta} + C_G \left\|\left|y^2\right\|\right|_{\beta,\beta}^2 \left\|y^1 - y^2\right\|_\infty s^{-2\beta} (t-s)^{2\beta}\\ &\leq C_G \left[\left(\left\|\left|y^1\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta}\right) \left\|\left|y^1 - y^2\right\|\right|_{\beta,\beta} + \left\|\left|y^2\right\|\right|_{\beta,\beta}^2 \left\|y^1 - y^2\right\|_\infty\right] s^{-2\beta} (t-s)^{2\beta} .\end{aligned}$$

On the other hand we finally get

$$
\begin{aligned}
&\left|G(y_s^1) - G(y_t^1) + DG(y_s^1)(y_t^1 - y_s^1) - \left(G(y_s^2) - G(y_t^2) + DG(y_s^2)(y_t^2 - y_s^2)\right)\right| \\
&\leq C_G \left(\left|y_t^1 - y_s^1\right| + \left|y_t^2 - y_s^2\right|\right) \left|y_t^1 - y_s^1 - y_t^2 + y_s^2\right| + C_G \left|y_t^2 - y_s^2\right|^2 \left(\left|y_s^1 - y_s^2\right| + \left|y_t^1 - y_t^2\right|\right) \\
&\leq C_G \left(\left\|\left|y^1\right|\right\|_{\beta,\beta} + \left\|\left|y^2\right|\right\|_{\beta,\beta}\right) \left\|y^1 - y^2\right\|_\infty s^{-\beta} (t-s)^\beta + C_G \left\|\left|y^2\right|\right\|_{\beta,\beta} \left\|y^2\right\|_\infty \left\|y^1 - y^2\right\|_\infty s^{-\beta} (t-s)^\beta \\
&\leq C_G \left(\left\|\left|y^1\right|\right\|_{\beta,\beta} + \left\|\left|y^2\right|\right\|_{\beta,\beta} + \left\|\left|y^2\right|\right\|_{\beta,\beta} \left\|y^2\right\|_\infty\right) \left\|y^1 - y^2\right\|_\infty s^{-\beta} (t-s)^\beta . \qquad \square
\end{aligned}
$$

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Jena, den 30.08.2019 Robert Hesse