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On Integral Operators and Equations Generated by Cosine and Sine Fourier Transforms

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Abstract. In this paper, we study some properties of a class of integral operators that depends on the cosine and sine Fourier transforms. In particular, we will exhibit properties related with their invertibility, the spectrum, Parseval type identities and involutions. Moreover, a new convolution will be proposed and consequent integral equations will be also studied in detail. Namely, we will characterize the solvability of two integral equations which are associated with the integral operator under study. Moreover, under appropriate conditions, the unique solutions of those two equations are also obtained in a constructive way.

INTRODUCTION AND BASIC PROPERTIES

Integral operators have a great importance in applications due to their potential use in modeling a huge variety of applied problems. Most of the time this is done by constructing equations or systems of equations characterized by such operators. Among the integral operators we may classify a great variety of different types of operators, cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. In some cases, there are convolutions somehow associated with the integral operators and allow the consideration of consequent convolution type equations.

In the present paper we will be concerned with a class of integral operators which is generated by the Fourier sine and cosine transforms (as well as the identity operator). Namely, we will study the operator

$$T = aI + bT_c + cT_s, \tag{1}$$

with $b \neq \pm c$ and $bc \neq 0$, in the framework of $L^2(\mathbb{R}^n)$. In order to interpret in a complete way (1), let us recall the cosine and sine Fourier transforms T_c and T_s, defined by

$$(T_c f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cos(xy) f(y) dy$$

and

$$(T_s f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sin(xy) f(y) dy.$$

We shall also recall the concept of algebraic operators. For this matter, let X be a linear space over the complex field $\mathbb{C}$, and let $L(X)$ be the set of all linear operators with domain and range in X.

Definition 1 (see [11, 12]) *An operator $K \in L(X)$ is said to be algebraic if there exists a normed (non-zero) polynomial*

$$P_K(t) = t^m + \alpha_1 t^{m-1} + \cdots + \alpha_{m-1} t + \alpha_m,$$

$\alpha_j \in \mathbb{C},\ j = 1, 2, \ldots, m$ *such that* $P_K(K) = 0$ *on* X.

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We say that an algebraic operator $K \in L(X)$ is of order m if there does not exist a normed polynomial $Q(t)$ of degree $k < m$ such that $Q(K) = 0$ on X. In this case, $P_K(t)$ is called the *characteristic polynomial* of K, and the roots of this polynomial are called the *characteristic roots* of K. Algebraic operators with a characteristic polynomial $t^m - 1$ or $t^m + 1$ ($m \geq 2$) are called *involutions* or *anti-involutions of order m*, respectively. An involution (or anti-involution) of order 2 is called, in brief, *involution* (or *anti-involution*).

We know that T_c and T_s are algebraic operators in $L^2(\mathbb{R}^n)$ and $T_c^3 = T_c$ and $T_s^3 = T_s$, so

$$P_{T_c}(t) = P_{T_s}(t) = t^3 - t,$$

which roots are $0, -1, 1$ (see [11, 12]). In this way, we have three projectors for each one of those operators. Therefore, based on that information, we are able to consider three projectors corresponding to T_c:

$$\begin{cases} P_1 = I - T_c^2, \\ P_2 = \frac{T_c^2 - T_c}{2}, \\ P_3 = \frac{T_c^2 + T_c}{2}, \end{cases} \tag{2}$$

which satisfy the following identities:

$$\begin{cases} P_j P_k = \delta_{jk} P_k, \qquad \text{for} \quad j, k = 1, 2, 3, \\ P_1 + P_2 + P_3 = I, \\ T_c = -P_2 + P_3, \end{cases} \tag{3}$$

and three projectors corresponding to T_s:

$$\begin{cases} Q_1 = I - T_s^2, \\ Q_2 = \frac{T_s^2 - T_s}{2}, \\ Q_3 = \frac{T_s^2 + T_s}{2}, \end{cases} \tag{4}$$

which satisfy the following identities:

$$\begin{cases} Q_j Q_k = \delta_{jk} P_k, \qquad \text{for} \quad j, k = 1, 2, 3, \\ Q_1 + Q_2 + Q_3 = I, \\ T_s = -Q_2 + Q_3 \end{cases} \tag{5}$$

(see [11, 12]). So, we can write our operator in terms of these projectors as follows:

$$\begin{aligned} T &= (a - b)\, P_2 + (a + b)\, P_3 + (a - c)\, Q_2 + (a + c)\, Q_3 \\ &=: [0; a - b; a + b; 0; a - c; a + c]\,. \end{aligned} \tag{6}$$

Proposition 2 *The characteristic polynomial associated with this operator is*

$$P_T(t) = t^4 - 4at^3 + (6a^2 - b^2 - c^2)t^2 - 2(2a^3 - ab^2 - ac^2)t + (a^2 - c^2)(a^2 - b^2), \tag{7}$$

whose characteristic roots are $t_1 = a - b$, $t_2 = a + b$, $t_3 = a - c$ and $t_4 = a + c$.

Proof. By (6) and some computations, we have that

$$\begin{aligned} & T^4 - 4aT^3 + (6a^2 - b^2 - c^2)T^2 - 2(2a^3 - ab^2 - ac^2)T + (a^2 - c^2)(a^2 - b^2)I \\ &= \left[0; (a - b)^4; (a + b)^4; 0; (a - c)^4; (a + c)^4\right] \\ &\quad -4a\left[0; (a - b)^3; (a + b)^3; 0; (a - c)^3; (a + c)^3\right] \\ &\quad +(6a^2 - b^2 - c^2)\left[0; (a - b)^2; (a + b)^2; 0; (a - c)^2; (a + c)^2\right] \\ &\quad -2(2a^3 - ab^2 - ac^2)\,[0; a - b; a + b; 0; a - c; a + c] \\ &\quad +(a^2 - c^2)(a^2 - b^2)\,[0; 1; 1; 0; 1; 1] \\ &= [0; 0; 0; 0; 0; 0] = 0. \end{aligned}$$

Now, we will prove that there does not exist any polynomial Q, with $\deg(Q) < 4$, such that $Q(T) = 0$. For that, suppose that there is a such polynomial

$$Q(t) = t^3 + mt^2 + nt + p.$$

This is equivalent to

$$\begin{cases} (a-b)^3 + (a-b)^2 m + (a-b)n + p = 0 \\ (a+b)^3 + (a+b)^2 m + (a+b)n + p = 0 \\ (a-c)^3 + (a-c)^2 m + (a-c)n + p = 0 \\ (a+c)^3 + (a+c)^2 m + (a+c)n + p = 0 \end{cases}$$

whose solutions are $b = c = 0$ or $b = \pm c$ and $c \neq 0$ or $b = 0$ and $c \neq 0$ or $b = 0$ and $c \neq 0$. Any solution is not under the conditions $b \neq \pm c$ and $bc \neq 0$. So, such polynomial Q does not exist.

A NEW CONVOLUTION AND STRUCTURAL PROPERTIES OF THE OPERATOR T

In this section we will consider invertibility properties of T as well as other structural properties. This is the case of a Parseval-type identity and an involution property. Moreover, a new convolution operation will be introduced which will allow a factorization property with T.

Proposition 3 *The spectrum of the operator T is given by $\{a-b, a+b, a-c, a+c\}$.*

Proof. Let φ_k denote the multi-dimensional Hermite functions (see [13]). Namely, we have (see [14, 15])

$$(T_c\varphi_k)(x) = \begin{cases} \varphi_k(x), & \text{if } |k| \equiv 0 \pmod 4 \\ 0, & \text{if } |k| \equiv 1,3 \pmod 4 \\ -\varphi_k(x), & \text{if } |k| \equiv 2 \pmod 4 \end{cases} \tag{8}$$

and

$$(T_s\varphi_k)(x) = \begin{cases} 0, & \text{if } |k| \equiv 0,2 \pmod 4 \\ \varphi_k(x), & \text{if } |k| \equiv 1 \pmod 4 \\ -\varphi_k(x), & \text{if } |k| \equiv 3 \pmod 4. \end{cases} \tag{9}$$

By (8)–(9), we obtain

$$(T\varphi_k)(x) = \begin{cases} (a+b)\,\varphi_k(x), & \text{if } \quad |k| \equiv 0 \pmod 4 \\ (a+c)\,\varphi_k(x), & \text{if } \quad |k| \equiv 1 \pmod 4 \\ (a-b)\,\varphi_k(x), & \text{if } \quad |k| \equiv 2 \pmod 4 \\ (a-c)\,\varphi_k(x), & \text{if } \quad |k| \equiv 3 \pmod 4 \end{cases} \tag{10}$$

Therefore, the Hermite functions are eigenfunctions of T with the eigenvalues $a \pm b$ and $a \pm c$. By using (10), we deduce that $\{a-b, a+b, a-c, a+c\} \subset \sigma(T)$. On the other hand, for any $\lambda \in \mathbb{C}$, we have

$$\begin{aligned} t^4 - 4at^3 + (6a^2 - b^2 - c^2)t^2 \; - \; & 2(2a^3 - ab^2 - ac^2)t + (a^2 - c^2)(a^2 - b^2) \\ = \;\; & (t-\lambda)\Big[t^3 + (\lambda - 4a)\,t^2 + \Big(\lambda^2 - 4a\lambda + 6a^2 - b^2 - c^2\Big)\,t \\ & + \Big(\lambda^3 - 4a\lambda^2 + (6a^2 - b^2 - c^2)\lambda - 2(2a^3 - ab^2 - ac^2)\Big)\Big] + P_T(\lambda). \end{aligned}$$

Suppose that $\lambda \notin \{a-b, a+b, a-c, a+c\}$. This implies that

$$P_T(\lambda) = \lambda^4 - 4a\lambda^3 + (6a^2 - b^2 - c^2)\lambda^2 - 2(2a^3 - ab^2 - ac^2)\lambda + (a^2 - c^2)(a^2 - b^2) \neq 0. \tag{11}$$

Then, the operator $T - \lambda I$ is invertible, and the inverse operator is defined by

$$\begin{aligned} (T - \lambda I)^{-1} \;\; = \;\; & -\frac{1}{P_T(\lambda)}\Big[T^3 + (\lambda - 4a)\,T^2 + \Big(\lambda^2 - 4a\lambda + 6a^2 - b^2 - c^2\Big)\,T \\ & + \Big(\lambda^3 - 4a\lambda^2 + (6a^2 - b^2 - c^2)\lambda + (a^2 - c^2)(a^2 - b^2)\Big)\,I\Big]. \end{aligned}$$

Thus, we have $\sigma(T) = \{a-b+d, a+b+d, a-c, a+c\}$.

Theorem 4 (Inversion theorem) *The operator T is an invertible operator if and only if*

$$a \neq \pm b, \quad \text{and} \quad a \neq \pm c. \tag{12}$$

In case of (12) holds, the inverse operator is defined by

$$T^{-1} \quad := \quad \frac{1}{(c^2-a^2)(a^2-b^2)}\left[a\left(-a^2+b^2+c^2\right)I + b^2\,(a-c)\,T_c + c\left(a^2-b^2\right)T_s - ab^2T_c^2 - ac^2T_s^2\right]. \tag{13}$$

Proof. If T is invertible, then it is injective. Taking into account the Hermite functions φ_k, we have already observed that (8)–(9) holds true. Indeed, by (10), we see that the Hermite functions are eigenfunctions of T with eigenvalues $a \pm b$ and $a \pm c$. So, we deduce that

$$(c^2-a^2)(a^2-b^2) \neq 0,$$

which is equivalent to (12).

Conversely, suppose that $(c^2-a^2)(a^2-b^2) \neq 0$. Hence, it is possible to consider the operator defined in (13) and, by a straightforward computation, verify that this is the inverse of T.

We will denote by $\Re\{x\}$ the real part of the complex number x.

Theorem 5 (Parseval-type identity) *For any $f, g \in L^2(\mathbb{R}^n)$, the identities*

$$\langle Tf, Tg\rangle_2 \quad = \quad |a|^2\langle f,g\rangle_2 + 2\Re\left\{a\bar{b}\right\}\langle f, T_c g\rangle_2 + 2\Re\left\{a\bar{c}\right\}\langle f, T_s g\rangle_2 + |b|^2\langle f, T_c^2 g\rangle_2 + |c|^2\langle f, T_s^2\rangle_2 \tag{14}$$

$$\begin{aligned}
\langle T^{-1}f, T^{-1}g\rangle_2 \quad = \quad & \left|\frac{1}{(c^2-a^2)(a^2-b^2)}\right|^2 \Big\{|a(-a^2+b^2+c^2)|^2\langle f,g\rangle_2 \\
& +2\left[\Re\left\{a(-a^2+b^2+c^2)\overline{b^2(a-c)}\right\} - \Re\left\{b^2(a-c)\overline{ab^2}\right\}\right]\langle f, T_c g\rangle_2 \\
& +2\left[\Re\left\{a(-a^2+b^2+c^2)\overline{c(a^2-b^2)}\right\}\right. \\
& \left.-\Re\left\{c(a^2-b^2)\overline{ac^2}\right\}\right]\langle f, T_s g\rangle_2 \\
& +\left[|b^2(a-c)|^2 + |-ab^2|^2 - 2\Re\left\{a(-a^2+b^2+c^2)\overline{ab^2}\right\}\right]\langle f, T_c^2 g\rangle_2 \\
& +\left[-2\Re\left\{a(-a^2+b^2+c^2)\overline{ac^2}\right\}\right. \\
& \left.+|c(a^2-b^2)|^2 + |-ac^2|^2\right]\langle f, T_s^2 g\rangle_2\Big\}
\end{aligned} \tag{15}$$

$$\begin{aligned}
\langle Tf, T^{-1}g\rangle_2 \quad = \quad & \overline{\frac{1}{(c^2-a^2)(a^2-b^2)}}\Big\{a\overline{a(-a^2+b^2+c^2)}\langle f,g\rangle_2 \\
& +\left[a\overline{(b^2(a-c))} + b\overline{a(-a^2+b^2+c^2)} - b\overline{ab^2}\right]\langle f, T_c g\rangle_2 \\
& +\left[a\overline{c(a^2-b^2)} + c\overline{a(-a^2+b^2+c^2)} - c\overline{ac^2}\right]\langle f, T_s g\rangle_2 \\
& +\left[-a\overline{ab^2} + b\overline{b^2(a-c)}\right]\langle f, T_c^2 g\rangle_2 \\
& +\left[-a\overline{ac^2} + c\overline{c(a^2-b^2)}\right]\langle f, T_s^2 g\rangle_2\Big\}
\end{aligned} \tag{16}$$

hold.

Proof. For any $f, g \in L^2(\mathbb{R}^n)$, a direct computation yields $\langle T_s f, g\rangle_2 = \langle f, T_s g\rangle_2$, $\langle T_c f, g\rangle_2 = \langle f, T_c g\rangle_2$. Thus, (14)–(16) appears from a straightforward computation when having in mind the definition of T and the just presented identities.

Theorem 6 (Unitary property) *T is not a unitary operator.*

Proof. Let T^* be the adjoint operator of T, this is, the operator satisfying

$$\langle Tf, g\rangle_2 = \langle f, T^* g\rangle_2,$$

for any $f, g \in L^2(\mathbb{R}^n)$. From the last identity, we obtain

$$T^* = \bar{a}I + \bar{b}T_c + \bar{c}T_s$$

and we know that T is a unitary operator if T is bijective and $T^* = T^{-1}$. In this way, taking into account (13), we have

$$\begin{cases} \bar{a} & = \frac{a(-a^2+b^2+c^2)}{(c^2-a^2)(a^2-b^2)} \\ \bar{b} & = \frac{b^2(a-c)}{(c^2-a^2)(a^2-b^2)} \\ \bar{c} & = \frac{c(a^2-c^2)}{(c^2-a^2)(a^2-b^2)} \\ 0 & = \frac{-ab^2}{(c^2-a^2)(a^2-b^2)} \\ 0 & = \frac{-ac^2}{(c^2-a^2)(a^2-b^2)}, \end{cases}$$

which implies that $b = 0$ or $c = 0$, which is not under the conditions of the definition of the operator T.

Remark 7 *For any $n \in \mathbb{N}$, we have*

$$\begin{aligned} T^n &= a^n I + \left(\sum_{j=0}^{n} C^n_{2j+1} a^{n-(2j+1)} b^{2j+1}\right) T_c + \left(\sum_{j=0}^{n} C^n_{2j+1} a^{n-(2j+1)} c^{2j+1}\right) T_s \\ &\quad + \left(\sum_{j=1}^{n} C^n_{2j} a^{n-2j} b^{2j}\right) T_c^2 + \left(\sum_{j=1}^{n} C^n_{2j} a^{n-2j} c^{2j}\right) T_s^2. \end{aligned} \tag{17}$$

Let us consider the operator written in terms of projectors (6) and

$$\begin{cases} a-b & = e^{i\frac{2k_1\pi}{n}}, \\ a+b & = e^{i\frac{2k_2\pi}{n}}, \\ a-c & = e^{i\frac{2k_3\pi}{n}}, \\ a+c & = e^{i\frac{2k_4\pi}{n}}, \end{cases} \tag{18}$$

with $k_1, k_2, k_3, k_4 = 0, 1, \ldots, n-1$.

Theorem 8 *The operator T is an involution if and only if*

(i) *n is odd and $k_1 \neq k_2$ and $k_3 \neq k_4$; or*
(ii) *n is even and $k_1 \neq k_2$, $k_3 \neq k_4$, $k_2 \neq \frac{n}{2} + k_1$ and $k_4 \neq \frac{n}{2} + k_3$*

(where the notation of (18) is here used).

Proof. Let us consider the operator T written as (6). So, we can write

$$T^n = (a-b)^n P_2 + (a+b)^n P_3 + (a-c)^n Q_2 + (a+c)^n Q_3. \tag{19}$$

The operator T is an involution if $T^n = I$, which is equivalent to

$$\begin{cases} (a-b)^n = 1 \\ (a+b)^n = 1 \\ (a-c)^n = 1 \\ (a+c)^n = 1. \end{cases}$$

So, we obtain (18) and, equivalently,

$$\begin{cases} a & = \frac{1}{2}\left(e^{i\frac{2k_1\pi}{n}} + e^{i\frac{2k_2\pi}{n}}\right) = \frac{1}{2}\left(e^{i\frac{2k_3\pi}{n}} + e^{i\frac{2k_4\pi}{n}}\right) \\ b & = \frac{1}{2}\left(e^{i\frac{2k_2\pi}{n}} - e^{i\frac{2k_1\pi}{n}}\right) \\ c & = \frac{1}{2}\left(e^{i\frac{2k_4\pi}{n}} - e^{i\frac{2k_3\pi}{n}}\right), \end{cases}$$

wich is equivalent to (i) or (ii).

Definition 9 *Considering T and (12), we define the new multiplication (convolution), for any two elements $f, g \in L^2(\mathbb{R}^n)$, as follows:*

$$
\begin{aligned}
(f * g)(x) \quad = \quad & \frac{1}{(c^2-a^2)(a^2-b^2)} \Big\{ a^3(-a^2+b^2+c^2) f(x)g(x) \\
& + \frac{a^2b(-a^2+b^2+c^2)}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cos(xu) \left[f(x)g(u) + f(u)g(x) \right] du \\
& + \frac{a^2c(-a^2+b^2+c^2)}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sin(xu) \left[f(x)g(u) + f(u)g(x) \right] du \\
& + \frac{a^2b^2(a-c)}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cos(xu) f(u)g(u)\, du + \frac{a^2c(a^2-b^2)}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sin(xu) f(u)g(u)\, du \\
& + \frac{ab^2(-a^2+b^2+c^2)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(xu) \left(f(u+v) + f(u-v) \right) g(v)\, dudv \\
& + \frac{ac^2(-a^2+b^2+c^2)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(xu) \left(f(u+v) - f(u-v) \right) g(v)\, dudv \\
& + \frac{abc(-a^2+b^2+c^2)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \sin(xu) \left[\left(f(v-u) + f(u-v) \right) g(v) + f(v) \left(g(v-u) + g(u-v) \right) \right] dudv \\
& + \frac{ab^3(a-c)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(vu) \left[f(v) \left(g(x-u) + g(u-x) \right) + \left(f(x-u) + f(u-x) \right) g(v) \right] dudv \\
& + \frac{ab^2c(a-c)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \sin(vu) \left[f(v) \left(g(u+x) + g(u-x) \right) + \left(f(u+x) + f(u-x) \right) g(v) \right] dudv \\
& + \frac{abc(a^2-b^2)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \sin(vu) \left[f(v) \left(g(x-u) + g(u-x) \right) + \left(f(x-u) + f(u-x) \right) g(v) \right] dudv \\
& + \frac{ac^2(a^2-b^2)}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(vu) \left[f(v) \left(g(x-u) - g(u-x) \right) + \left(f(x-u) - f(u-x) \right) g(v) \right] dudv \\
& - \frac{a^3b^2}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(vu) \left(f(x-u)g(x-u) + f(u-x)g(u-x) \right) dudv \\
& - \frac{a^3c^2}{2(2\pi)^n} \int_{\mathbb{R}^{2n}} \cos(vu) \left(f(x-u)g(x-u) - f(u-x)g(u-x) \right) dudv \\
& + \frac{b^4(a-c)}{2(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \cos(wu) \left[f(x-u+v) + f(x-u-v) + f(u-x+v) + f(u-x-v) \right] g(v)\, dudvdw \\
& + \frac{b^3c(a-c)}{4(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \sin(wu) \left[\left(f(v-u-x) + f(u+x-v) + f(v-u+x) + f(u-x-v) \right) g(v) \right. \\
& \qquad \left. + f(v) \left(g(v-u-x) + g(u+x-v) + g(v-u+x) + g(u-x-v) \right) \right] dudvdw \\
& + \frac{b^2c^2(a-c)}{4(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \cos(wu) \left[f(x-u+v) - f(x-u-v) + f(u-x+v) - f(u-x-v) \right] g(v)\, dudvdw \\
& + \frac{b^2c(a^2-b^2)}{4(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \sin(wu) \left[f(x-u+v) + f(x-u-v) + f(u-x+v) + f(u-x-v) \right] g(v)\, dudvdw \\
& + \frac{bc^2(a^2-b^2)}{4(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \cos(wu) \left[\left(f(v-x+u) + f(x-u-v) - f(v-u+x) - f(u-x-v) \right) g(v) \right. \\
& \qquad \left. + f(v) \left(g(v-x+u) + g(x-u-v) - g(v-u+x) - g(u-x-v) \right) \right] dudvdw \\
& + \frac{c^3(a^2-b^2)}{4(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \sin(wu) \left[f(x-u+v) - f(x-u-v) + f(u-x+v) - f(u-x-v) \right] g(v)\, dudvdw \\
& - \frac{a^2b^3}{2(2\pi)^{\frac{3n}{2}}} \int_{\mathbb{R}^{3n}} \cos(xw)\cos(vu) \left[f(v) \left(g(w-u) + g(u-w) \right) + \left(f(w-u) + f(u-w) \right) g(v) \right] dudvdw
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{a^2b^2c}{2(2\pi)^{\frac{3n}{2}}}\int_{\mathbb{R}^{3n}}\cos(xw)\sin(vu)\left[f(v)\left(g(u+w)+g(u-w)\right)+\left(f(u+w)+f(u-w)\right)g(v)\right]dudvdw\\
&-\frac{a^2bc^2}{2(2\pi)^{\frac{3n}{2}}}\int_{\mathbb{R}^{3n}}\sin(xw)\sin(vu)\left[f(v)\left(g(w-u)+g(u-w)\right)+\left(f(w-u)+f(u-w)\right)g(v)\right]dudvdw\\
&-\frac{a^2c^3}{2(2\pi)^{\frac{3n}{2}}}\int_{\mathbb{R}^{3n}}\sin(xw)\cos(vu)\left[f(v)\left(g(w-u)-g(u-w)\right)+\left(f(w-u)-f(u-w)\right)g(v)\right]dudvdw\\
&-\frac{ab^4}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\cos(xt)\cos(wu)\left[f(t-u+v)+f(t-u-v)+f(u-t+v)+f(u-t-v)\right]g(v)\,dudvdwdt\\
&-\frac{ab^3c}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\cos(xt)\sin(wu)\left[\left(f(v-u-t)+f(u+t-v)+f(v-u+t)+f(u-t-v)\right)g(v)\right.\\
&\qquad\left.+f(v)\left(g(v-u-t)+g(u+t-v)+g(v-u+t)+g(u-t-v)\right)\right]dudvdwdt\\
&-\frac{ab^2c^2}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\cos(xt)\cos(wu)\left[f(t-u+v)-f(t-u-v)+f(u-t+v)-f(u-t-v)\right]g(v)\,dudvdwdt\\
&-\frac{ab^2c^2}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\sin(xt)\sin(wu)\left[f(t-u+v)+f(t-u-v)+f(u-t+v)+f(u-t-v)\right]g(v)\,dudvdwdt\\
&-\frac{abc^3}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\sin(xt)\cos(wu)\left[\left(f(v-t+u)+f(t-u-v)-f(v-u+t)-f(u-t-v)\right)g(v)\right.\\
&\qquad\left.+f(v)\left(g(v-t+u)+g(t-u-v)-g(v-u+t)-g(u-t-v)\right)\right]dudvdwdt\\
&\left.-\frac{ac^4}{4(2\pi)^{2n}}\int_{\mathbb{R}^{4n}}\sin(xt)\sin(wu)\left[f(t-u+v)-f(t-u-v)+f(u-t+v)-f(u-t-v)\right]g(v)\,dudvdwdt\right\}.
\end{aligned}
\tag{20}
$$

Theorem 10 *For the operator T and $f, g \in L^2(\mathbb{R}^n)$, we have the following T–factorization:*

$$T(f * g)(x) = (Tf)(Tg). \tag{21}$$

Proof. Using the definition of T and a straightforward computation, we obtain the equivalence between (20) and

$$
\begin{aligned}
f*g \;=\; &\frac{1}{(c^2-a^2)(a^2-b^2)}\left[a\left(-a^2+b^2+c^2\right)I+b^2(a-c)T_c+c\left(a^2-b^2\right)T_s-ab^2T_c^2-ac^2T_s^2\right]\\
&[(Tf)(Tg)].
\end{aligned}
\tag{22}
$$

Thus, having in mind (13), which provides the formula for the inverse of T, we identify the last identity with

$$f * g = T^{-1}\left[(Tf)(Tg)\right],$$

which is equivalent to (21), as desired.

SOLVABILITY OF INTEGRAL EQUATIONS GENERATED BY THE OPERATOR T

Within the framework of $L^2(\mathbb{R}^n)$, we will consider integral equations of the type

$$\alpha\varphi + \beta T\varphi + \gamma T^2\varphi = g, \tag{23}$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, with $|\alpha| + |\beta| + |\gamma| \neq 0$, and $g \in L^2(\mathbb{R}^n)$ are the given data, and the operator T is defined, in $L^2(\mathbb{R}^n)$, by (1).

The polynomial (7) has the single roots $t_1 := a - b$, $t_2 := a + b$, $t_3 := a - c$ and $t_4 = a + c$. Having this in mind, we are able to built projections, induced by T, in the sense of the Lagrange interpolation formula. Namely:

$$R_1 \;:=\; \frac{(T-t_2I)(T-t_3I)(T-t_4I)}{(t_1-t_2)(t_1-t_3)(t_1-t4)};$$

$$
\begin{aligned}
R_2 &:= \frac{(T-t_1I)(T-t_3I)(T-t_4I)}{(t_2-t_1)(t_2-t_3)(t_2-t_4)};\\
R_3 &:= \frac{(T-t_1I)(T-t_2I)(T-t_4I)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)};\\
R_4 &:= \frac{(T-t_1I)(T-t_2I)(T-t_3I)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)};
\end{aligned}
$$

Then, we have $R_jR_k = \delta_{jk}R_k$, and

$$T^\ell = t_1^\ell R_1 + t_2^\ell R_2 + t_3^\ell R_3 + t_4^\ell R_4,$$

for any $j, k = 1, 2, 3, 4$ and $\ell = 0, 1, 2$.

By using these projectors, we are in conditions to rewrite (23) in the following equivalent way:

$$m_1R_1\varphi + m_2R_2\varphi + m_3R_3\varphi + m_4R_4\varphi = g, \quad \text{with} \quad m_j = \alpha + \beta t_j + \gamma t_j^2, \;\; j = 1, 2, 3, 4. \tag{24}$$

Theorem 11 (i) *The integral equation (24) (or (23)) has a unique solution if and only if* $m_1m_2m_3m_4 \neq 0$.

(ii) *If* $m_1m_2m_3m_4 \neq 0$*, then the unique solution of (24) is given by*

$$\varphi = m_1^{-1}R_1g + m_2^{-1}R_2g + m_3^{-1}R_3g + m_4^{-1}R_4g. \tag{25}$$

(iii) *If* $m_j = 0$*, for some* $j = 1, 2, 3, 4$*, then the equation (24) has solution if and only if* $R_jg = 0$.

(iv) If $R_jg = 0$, for some $j = 1, 2, 3, 4$, then the equation (24) has an infinite number of solutions given by

$$\varphi = \sum_{\substack{j\le 4\\ m_j\neq 0}} m_j^{-1}R_jg + z, \quad \textit{where} \quad z \in \ker\left(\sum_{\substack{j\le 4\\ m_j\neq 0}} R_j\right). \tag{26}$$

Proof. Suppose that the equation (23) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying P_j to both sides of the equation (24), we obtain a system of four equations

$$m_jP_j\varphi = P_jg,$$

$j = 1, 2, 3, 4$. In this way, if $m_1m_2m_3m_4 \neq 0$, then we have the following equivalent system of equations

$$\begin{cases} R_1\varphi = m_1^{-1}R_1g\\ R_2\varphi = m_2^{-1}R_2g\\ R_3\varphi = m_3^{-1}R_3g\\ R_4\varphi = m_4^{-1}R_4g. \end{cases}$$

Using the identity

$$R_1 + R_2 + R_3 + R_4 = I,$$

we obtain (25). Conversely, we can directly verify, by substitution, that φ given by (25) fulfills (24).

If $m_1m_2m_3m_4 = 0$, then $m_j = 0$, for some $j \in \{1, 2, 3, 4\}$. Therefore, it follows that $R_jg = 0$. Then, we have

$$\sum_{\substack{j\le 4\\ m_j\neq 0}} R_j\varphi = \sum_{\substack{j\le 4\\ m_j\neq 0}} m_j^{-1}R_jg.$$

Using the fact that $R_jR_k = \delta_{jk}R_k$, we get

$$\left(\sum_{\substack{j\le 4\\ m_j\neq 0}} R_j\right)\varphi = \left(\sum_{\substack{j\le 4\\ m_j\neq 0}} R_j\right)\left[\sum_{\substack{j\le 4\\ m_j\neq 0}} m_j^{-1}R_jg\right]$$

or, equivalently,

$$\left(\sum_{\substack{j\le 4\\ m_j\neq 0}} R_j\right)\left[\varphi - \sum_{\substack{j\le 4\\ m_j\neq 0}} m_j^{-1} R_j g\right] = 0.$$

By this, we obtain the solution (26).

Conversely, we can directly verify that all the elements φ, with the form of (26), fulfill (24). As the Hermite functions are eigenfunctions of T (cf. (8)–(9)), we conclude that the cardinality of all elements φ in (26) is infinite.

Now, let us consider the convolution type equation

$$\lambda\varphi(x) + (k * \varphi)(x) = h(x), \tag{27}$$

where $0 \neq \lambda \in \mathbb{C}$, $k, h \in L^2(\mathbb{R}^n)$ are given functions and $\varphi \in L^2(\mathbb{R}^n)$ is unknown.

Theorem 12 *The equation (27) as a unique solution if and only if*

$$\lambda + (Tk)(x) \neq 0,$$

for all $x \in \mathbb{R}^n$. The solution is given by

$$\varphi(x) = T^{-1}\left[\frac{(Th)(x)}{\lambda + (Tk)(x)}\right] \tag{28}$$

Proof. Let us consider that the equation (27) has a solution $\varphi \in L^2(\mathbb{R}^n)$. Applying T to both sides of the equation, we obtain

$$(T\varphi)(x) = \frac{(Th)(x)}{\lambda + (Tk)(x)}. \tag{29}$$

By Theorem 4, we obtain (28).

Conversely, by substitution, we can verify that φ given by (28) fulfills (27).

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