BANDS IN THE SPECTRUM OF A PERIODIC ELASTIC WAVEGUIDE

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Abstract: We study the spectral linear elasticity problem in an unbounded periodic waveguide, which consists of a sequence of identical bounded cells connected by thin ligaments of diameter of order $h > 0$. The essential spectrum of the problem is known to have band-gap structure. We derive asymptotic formulas for the position of the spectral bands and gaps, as $h \to 0$.

1. INTRODUCTION

We study the essential spectrum of the linearized elasticity system with traction–free boundary conditions in unbounded periodic waveguides denoted by Π_h. The waveguide, see Fig. 1.1, consists of infinitely many identical, translated bounded cells connected with small cylindrical ligaments, the length and radius of cross-section of which are both proportional to a small parameter $h > 0$ so that the volume of the ligament is $O(h^3)$. A number of papers (e.g [3], [33], [34]) has been devoted to geometrically similar waveguides consisting of arrays of macroscopic cells connected with thin structures, and, using rigorous perturbation arguments, the existence of gaps in the essential spectra has been detected. This has been done for elliptic boundary problems in elasticity, linear water-wave theory, piezo-electricity etc.

In this geometric setting the emerging of gaps is explained by that for small h, the problem can be seen as a perturbation of a "limit spectral problem" $(h = 0)$ on a bounded domain, which consists of a single cell ϖ_0. The spectrum of such a problem is in general a sequence $(\lambda_k)_{k=1}^\infty$ of eigenvalues , and the spectral bands of the original problem are situated "close" to the eigenvalues λ_k. To analyse this closeness and its dependence on h becomes a mathematical challenge: if that can be done accurately enough, one finds that disjoint eigenvalues correspond to spectral bands with a gap in between. In fact, this scheme can work only for a finite number of the lowest eigenvalues in the sense that for each fixed small h, at most a finite number of gaps can be found.

The purpose of this work is to refine the existing results by proving more accurate estimates than before for the end-points of spectral bands and gaps. For example, in [33] it was shown that for small h and also k, the kth spectral band is situated within a distance Ch from λ_k, the kth eigenvalue of the limit problem, where $C > 0$ is a constant depending on the shape of the cells and on the physical constants of the elastic material, but not on the size of the small ligaments. In comparison, we shall find here an asymptotic formula for the spectral bands Υ_k. Namely, by the Floquet-Bloch-theory of periodic problems, the bands are formed by eigenvalues

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$\Lambda_k^h(\eta)$ of the "model problem" depending on the parameter $\eta \in [0, 2\pi)$. It is in fact well known that the essential spectrum of the original problem (later (2.5)–(2.6)) equals

$$(1.1) \qquad \sigma_{\mathrm{ess}} = \bigcup_{k=1}^{\infty} \Upsilon_k^h \ , \quad \Upsilon_k^h = \{\Lambda_k^h(\eta) : \eta \in [0, 2\pi)\}.$$

In Theorem 4.1, see (4.1), (4.2), we determine the first order (with respect to the small parameter h) correction term for the difference of $\Lambda_k^h(\eta)$ and λ_k. The result includes the following claim:

For all k we have the estimate

$$(1.2) \qquad \left|\Lambda_k^h(\eta) - (\lambda_k + h\Lambda_k'(\eta))\right| \leq C_k h^{3/2} \quad \forall 0 < h < 1/10 \ , \ \eta \in [0, 2\pi),$$

where

$$(1.3) \qquad \Lambda_k'(\eta) = 2a\lambda_k + \left(A_k + e^{i\eta}B_k\right)^{\top}\mathbf{M}^+\left(A_k + e^{-i\eta}B_k\right),$$

the column vectors A_k, $B_k \in \mathbb{R}^3$ and the positive definite matrix $\mathbf{M}^+ \in \mathbb{R}^{3\times 3}$ do not depend on h or η, and the number a equals 0 or 1 (according to the domain, see Section 2.1).

The quantities A_k, B_k, $\mathbf{M}^+$ will be determined in Sections 3.2, 3.3. Anyway, the coefficient $\Lambda_k'(\eta)$ of the correction term depends only on the geometry of the limit problem. (It might thus be desirable to provide numerical experiments on the effect of this coefficient and the geometry of the limit domain to the appearance of spectral gaps, but we do not provide such information here.) However, information on the position and length of Υ_k obtained here is much more precise than before, since until now there has not even existed a criterion to distinguish, if the band has positive length or it consists of only a single point, which is an eigenvalue of infinite multiplicity. In many cases the vector B_k is nonzero, and it follows from (1.3) that length of the band Υ_k^h is positive.

In addition, we find an asymptotic representation for the corresponding eigen-functions U_k^h of the model problem, which includes the leading terms both near the junctions of the periodicity cells and at a distance of these points. Since it is difficult to describe this without a number of definitions, we refer to Theorem 4.1 for details.

In literature spectral gaps in essential spectra for scalar equations and Maxwell's system in infinite periodic media have been considered in many papers, see for example [7], [8], [9], [12], [13], [36]. For results on gaps in (periodic, unbounded) waveguides we mention the papers [2], [6], [10], [26]; for an approach based on parameter–dependent Korn–type inequalities, see [5], [27], [28], [30], [29]. A comparison of the present work with the paper [33] was already presented above. We just mention that the method of [33] is based on the max-min principle for eigenvalues. We finally also mention the paper [3], which contains an analysis of spectral bands much like in the present work, but in the more simple setting of the linear water wave equation.

As for the structure of this paper, we present in Section 2 the geometry of the waveguide Π_h, the formulation of the linear spectral elasticity problem and its variational formulation. The parameter dependent problem arising from the FBG-transform is presented in Section 2.2, together with the variational formulation in a special Sobolev-type Hilbert space $\mathcal{H}^{h,\eta}$. The limit problem is studied in Section 2.3, and additional technical devices are introduced in Section 2.4. Section 3 contains the formal asymptotic analysis, in particular the construction of the asymptotic ansätze
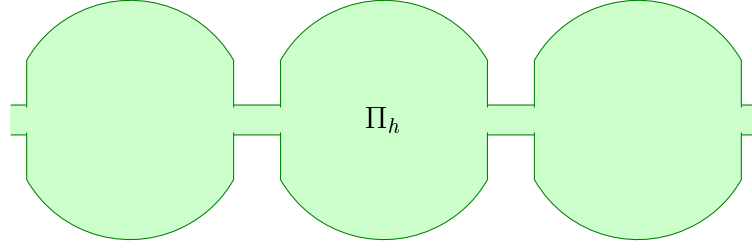
FIGURE 1.1. The waveguide Π_h

for the eigenfunctions of the model problem. Section 4.1 contains the main result, and proof is given in Sections 4.2–4.3. One point of the proof needs the existence of spectral gaps, which can be obtained by an adaptation of the method [33]; this is presented in the appendix, Section 5.

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2. Problem formulation, band-gap spectrum and limit problem

2.1. **Spectral elasticity problem in the waveguide.** The problem domain is a periodic waveguide Π_h, Fig. 1.1, depending on a small geometric parameter $h \in (0, \frac{1}{10}]$ and consisting of infinitely many disjoint translated copies of a bounded cell connected by thin cylinder. Let us describe the exact definition. We denote by ϖ_0 a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary $\partial \varpi_0$ and assume that the points $P^\pm = (0, 0, \pm 1/2)$ belong to $\partial \varpi_0$. We also assume that

$$\varpi_0 \subset \{x = (x_1, x_2, x_3) = (y, z) \colon y \in \mathbb{R}^2, |z| < 1/2\}$$

and in a neighbourhood of the points $P^\pm$ the boundary $\partial \varpi_0$ is "flat": it consists of an open subset of the plane $\{(y_1, y_2, \pm 1/2)\}$.

We also denote by $\theta \subset \mathbb{R}^2$ a bounded domain which has Lipschitz boundary $\partial \theta$ and which is star-shaped with respect to the point $(0, 0) \in \theta$. (Any point of θ can be joined with $(0, 0)$ by a line segment running in θ.) We also denote for $h > 0$

$$\theta_h = h\theta = \{hy \in \mathbb{R}^2 : y \in \theta\}.$$

By a thin cylinder we mean the set $\Theta_h = \theta_h \times \mathbb{R}$.

We fix a number a to be either 0 or 1 (see below for a discussion) and denote by a_h the number $(1 - ah)^{-1} \in \mathbb{R}$ and also the multiplier $a_h : x \mapsto (1 - ah)^{-1}x$ in any space $\mathbb{R}^n \ni x$, $n \in \mathbb{N}$. The meaning will be clear from the context. Let us define a scaled, translated family of bodies
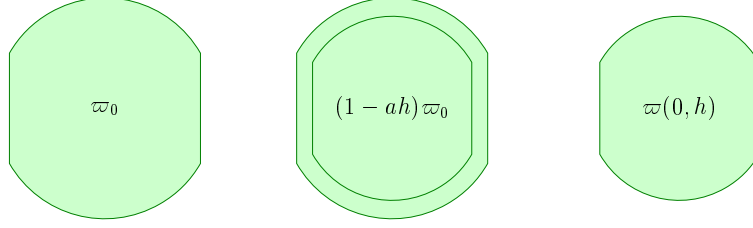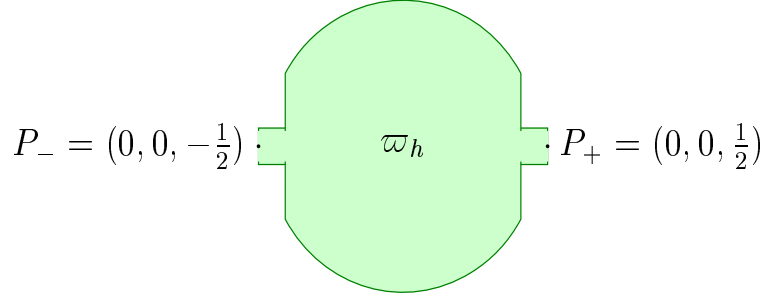
$$(2.1) \qquad \varpi(j, h) = \{x : a_h(y, z - j) \in \varpi_0\} \ \text{ with } \varpi(h) := \varpi(0, h) = a_h^{-1}(\varpi_0),$$

see Fig. 2.1. The periodicity cell $\varpi_h \supset \varpi(h)$, Fig. 2.2, is the set

$$(2.2) \qquad \varpi_h = \varpi(h) \cup \left(\theta_h \times \left(-\frac{1}{2}, \frac{1}{2} \right) \right),$$

in other words, ϖ_h is the disjoint union of $\varpi(h)$ and the small sets

$$(2.3) \qquad \Theta_h^- = \theta_h \times (-1/2, -1/2 + ah/2], \quad \Theta_h^+ = \theta_h \times (1/2 - ah/2, 1/2].$$

FIGURE 2.1. Transformation of the set ϖ_0

FIGURE 2.2. Periodicity cell ϖ_h

Finally, the waveguide Π_h, Fig. 1.1, is defined as

(2.4)
$$\Pi_h = \Theta_h \cup \bigcup_{j \in \mathbb{Z}} \varpi(j, h).$$

Note that when $h = 0$, the waveguide Π_h becomes a union of disconnected sets.

If $a = 0$ above, then of course $a_h = 1$ is independent of h, and the cells $\varpi(j, h)$ contact each other: the cylinder Θ_h loses its role and the cells are connected by holes, "apertures" of diameter h in their boundaries. In this case also the volume of the periodicity cells is independent of h. The subsequent calculations are concentrated on treating the case $a = 1$. In many cases the calculations are unnecessarily complicated for $a = 0$, but that case can be considered as a by-product; some details will be omitted, and the reader is asked to keep it also in mind. If $a = 1$, the diameter of the connecting ligaments is of order h and the volume is proportional h^3. This means a certain difference to the related paper [33].

We consider the spectral elasticity problem in Π_h written in matrix form, cf. [21], [31]. This is formulated for an unknown $\mathbb{R}^3$-valued vector function $u(x)$, which describes the displacement vector of the given material. The system contains a first order differential operator matrix $D(\nabla_x)$, where ∇_x denotes the gradient with respect to the variable x and

$$D(x) = \begin{pmatrix} x_1 & 0 & 2^{-1/2}x_2 & 2^{-1/2}x_3 & 0 & 0 \\ 0 & x_2 & 2^{-1/2}x_1 & 0 & 2^{-1/2}x_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}x_1 & 2^{-1/2}x_2 & x_3 \end{pmatrix}^\top , \quad x = (x_1, x_2, x_3)^\top,$$

and a matrix A of dimension 6×6, describing elastic moduli. The matrix A is assumed to be positive definite and, for technical simplicity, constant. The problem can be written as

(2.5)
$$L_x u(x) = \lambda u(x) , \quad x \in \Pi_h,$$

(2.6) $$N_x^h u(x) = 0 \text{ for a.e. } x \in \partial\Pi_h \,,$$

where

(2.7) $$L_x = D(-\nabla_x)^\top A D(\nabla_x) \,, \quad N_x^h = D(\nu(x))^\top A D(\nabla_x) \,,$$

ν is the outward normal vector defined for almost all points of the Lipshitz surface $\partial\Pi_h$ and λ is a spectral parameter. The boundary operator N_x^h depends on h via the domain Π_h. (Later on, ν will always denote the outward normal of a boundary, which will be clear from the context.) Also, L_x is just a constant coefficient partial differential operator containing only second order terms. In general, λ could be multiplied in (2.5) by a fixed function describing the material density, but we assume this function to be equal to 1 for simplicity.

The variational formulation of the spectral problem (2.5)-(2.6) reads as

(2.8) $$a(u, v; \Pi_h) = \lambda(u, v)_{\Pi_h}, \quad v \in H^1(\Pi_h)^3.$$

Here, given a regular enough domain $\Omega \subset \mathbb{R}^3$, we use the notation

(2.9) $$a(f, g; \Omega) = (AD(\nabla_x)f, D(\nabla_x)g)_\Omega \,, \text{ and } (f, g)_\Omega = \sum_{j=1}^k \int_\Omega f\overline{g}dx$$

for the usual (complex valued) inner product in $L^2(\Omega)^k$, $k = 1, 2, 3$; the latter notation will also be used for domains in $\mathbb{R}^2$. We denote the standard Sobolev space of first order on Ω by $H^1(\Omega)$. The bilinear form (2.9) is positive and closed in the Sobolev space $H^1(\Pi_h)^3$ and consequently (see [4]) our problem can be rewritten as an abstract operator equation $\mathcal{T}^h u = \lambda u$, where $\mathcal{T}^h$ is an unbounded, self-adjoint densely defined operator in the Hilbert space $L^2(\Pi_h)^3$ and thus the spectrum $\sigma(\mathcal{T}^h) \ni \lambda$ is a subset of $\overline{\mathbb{R}_+} = [0, +\infty)$. The embedding $H^1(\Pi_h) \subset L^2(\Pi_h)$ is not compact due to the unboundedness of the domain Π_h, hence, the essential spectrum $\sigma_{\text{ess}}(\mathcal{T}^h)$ is not empty (see [4], Th.10.15). Finally, spectral concepts of the problem (2.5)–(2.6) are defined with the help of the operator $\mathcal{T}^h$. In particular, $\sigma_{\text{ess}} := \sigma_{\text{ess}}(\mathcal{T}^h)$.

2.2. **Floquet-Bloch-Gelfand transform and essential spectrum.** To analyse the band-gap structure spectrum of the problem (2.5)–(2.6) we use the FBG-transform

$$v(y, z) \mapsto V(y, z; \eta) = \frac{1}{\sqrt{2\pi}} \sum_{j \in \mathbb{Z}} \exp(-i\eta j)v(y, z + j),$$

where $(y, z) \in \Pi_h$ on the left, while $\eta \in [0, 2\pi)$ and $(y, z) \in \varpi_h$ on the right. As well known, this operator establishes an isometric isomorphism between Lebesgue spaces $L^2(\Pi_h)^3$ and $L^2(0, 2\pi; L^2(\varpi_h)^3)$, where $L^2(0, 2\pi; B)$ is the Lebesgue space of functions with values in the Banach space B, endowed with the norm $\|V; L^2(0, 2\pi; B)\|^2 = \int_0^{2\pi} \|V(\eta); B\|^2 d\eta$. The FBG-transform is also an isomorphism from the Sobolev space $H^1(\Pi_h)^3$ onto $L^2(0, 2\pi; H^1_\eta(\varpi_h)^3)$ and from $H^2(\Pi_h)^3$ onto $L^2(0, 2\pi; H^2_\eta(\varpi_h)^3)$. Here, for a fixed h and η, the space $H^2_\eta(\varpi_h)$ is the space of Sobolev functions f on ϖ_h, which satisfy quasiperiodicity conditions

(2.10) $$f(y, 1/2) = e^{i\eta}f(y, -1/2) \,, \quad \partial_z f(y, 1/2) = e^{i\eta}\partial_z f(y, -1/2) \,, \quad y \in \theta_h.$$

Similarly, the space $H^1_\eta(\varpi_h)$ consists of H^1-functions satisfying the condition (2.10) only. In the following we denote by $\mathcal{H}^{h,\eta}$ the space $H^1_\eta(\varpi_h)^3$ endowed with the norm

$\|f; \mathcal{H}^{h,\eta}\|$ coming from the inner product

$$(2.11) \qquad \langle f, g \rangle_{\mathcal{H}^{h,\eta}} = (f, g)_{\varpi_h} + \big(AD(\nabla_x)f, D(\nabla_x)g\big)_{\varpi_h};$$

notice that the last form on the right is Hermitian and positive on $H_\eta^1(\varpi_h)^3$. Obviously, $\|f; \mathcal{H}^{h,\eta}\| \le C\|f; H^1(\varpi_h)\|$ for some constant independent of h and η, since the matrix A was assumed to be constant.

Lemma 2.1. *The norm $\|\cdot; \mathcal{H}^{h,\eta}\|$ is equivalent to $\|\cdot; H^1(\varpi_h)\|$ with constants independent of $h > 0$ and $\eta \in [0, 2\pi)$.*

Proof. It is enough to show that the Korn inequality

$$(2.12) \qquad \|f; H^1(\varpi_h)\|^2 \le C\big(\|f; L^2(\varpi_h)\|^2 + \big(AD(\nabla_x)f, D(\nabla_x)f\big)_{\varpi_h}$$

holds for Sobolev-functions f in such a way that the constant $C > 0$ can be chosen independently of h, η. It is quite obvious that replacing ϖ_h by $\varpi(h)$ (see (2.1)) in (2.12), the corresponding inequality would hold with constant independent of h, η. But the same is true also, when ϖ_h is replaced by the two sets $\Theta_h^\pm$, see (2.3). This follows from the result [17], Th. 1, Th. 2. For example the set Θ_h^+ has diameter $c_1 h$ and it is in the terminology of the citation starshaped with respect to a ball with center in $(0, 0, 1/2 - ah/4)$ and radius $c_2 h$, where c_1, c_3 do not depend on h (see the assumptions on θ in Section 2.1). Then, (2.12) follows by combining these facts. $\square$

Using the FBG-transform the problem (2.5)-(2.6) turns into a model problem on the periodicity cell for the unknown $U = U(x; \eta)$,

$$(2.13) \qquad \overline{D(-\nabla_x)}^\top AD(\nabla_x)U = \Lambda U \quad \text{in } \varpi_h,$$

$$(2.14) \qquad \overline{D(\nu(x))}^\top AD(\nabla_x)U = 0 \quad \text{a.e. in } \partial\varpi_h \setminus (\theta_h^- \cup \theta_h^+),$$

$$(2.15) \qquad U(y, 1/2; \eta) = e^{i\eta}U(y, -1/2; \eta), \quad y \in \theta_h,$$

$$(2.16) \qquad \partial_z U(y, 1/2; \eta) = e^{i\eta}\partial_z U(y, -1/2; \eta), \quad y \in \theta_h,$$

where the overline denotes complex conjugation and Λ is a spectral parameter. In the weak form this amounts to finding $0 \ne U \in \mathcal{H}^{h,\eta}$ and $\Lambda \in \mathbb{C}$ with

$$(2.17) \qquad a(U, V; \varpi_h) = \Lambda(U, V)_{\varpi_h}$$

for all $V \in \mathcal{H}^{h,\eta}$. The problem (2.13)-(2.16) can be associated with a self-adjoint, positive and compact operator $\mathcal{B}^{h,\eta} : \mathcal{H}^{h,\eta} \to \mathcal{H}^{h,\eta}$. We define $\mathcal{B}^{h,\eta}$ in a standard way by requiring that the identity

$$(2.18) \qquad \langle \mathcal{B}^{h,\eta}U, V \rangle_{\mathcal{H}^{h,\eta}} = (U, V)_{\varpi_h}$$

holds for all $U, V \in \mathcal{H}^{h,\eta}$. The problem (2.17) is then equivalent to the spectral problem $\mathcal{B}^{h,\eta}U = \mu U$ with a new spectral parameter $\mu = (1 + \Lambda)^{-1}$. The spectrum of $\mathcal{B}^{h,\eta}$ consists of a decreasing sequence $\{\mu_k^h(\eta)\}_{k \ge 1}$ of eigenvalues and the point 0 of the essential spectrum. As a consequence, the spectrum of the problem (2.13)-(2.16) can be presented as the eigenvalue sequence (counting multiplicities)

$$(2.19) \qquad 0 \le \Lambda_1^h(\eta) \le \Lambda_2^h(\eta) \le \ldots \le \Lambda_p^h(\eta) \le \ldots \to +\infty.$$

We denote the corresponding eigenfunctions by $U_p^h \in H_\eta^1(\varpi_h)^3$, where the dependence on η will usually not be shown. We require the orthogonality property

$$(2.20) \qquad (U_p^h, U_q^h)_{\varpi_h} = \delta_{p,q}, \quad p, q = 1, 2, \ldots.$$

By [11], [18], [23], [25], [32], Theorem 3.4.6, [33], Theorem 2.1), for example, a number Λ belongs to the resolvent set or the discrete spectrum of $\mathcal{T}^h$ (end of Section 2.1), if and only if it does not coincide with $\Lambda_p^h(\eta)$ for any $\eta \in [0, 2\pi]$ and p. Hence, the essential spectrum of $\mathcal{T}^h$ and thus also of the original problem (2.5)–(2.6) have band-gap structure (1.1),

$$(2.21) \qquad \sigma_{\text{ess}}(\mathcal{T}^h) = \bigcup_{p=1}^{\infty} \Upsilon_p^h, = \bigcup_{p=1}^{\infty} \{\Lambda_p^h(\eta) : \eta \in [0, 2\pi)\};$$

where the spectral bands Υ_p^h are closed intervals (possibly single points).

2.3. **Spectrum of the limit model problem.** When $h \to 0$, the cylinder Θ_h turns into a negligible set and the quasi-periodicity conditions (2.15)-(2.16) lose their meaning. The model problem (2.13)-(2.16) turns into the so called limit problem on the isolated elastic body ϖ_0,

$$(2.22) \qquad D(-\nabla_x)^\top AD(\nabla_x)u = \Lambda^0 u \quad \text{in } \varpi_0,$$

$$(2.23) \qquad D(\nu(x))^\top AD(\nabla_x)u = 0 \quad \text{a.e. in } \partial\varpi_0,$$

where u is the unknown function on ϖ_0 and Λ^0 is a spectral parameter. The problem has for any $\eta \in (0, 2\pi]$ the same eigenvalues as the case $\eta = 0$, so we can restrict to this single limit case, see [33] for some more details. We again denote by $a(u, v; \varpi_0)$ the sesquilinear form corresponding to the problem (2.22)-(2.23). It is positive and closed on $H^1(\varpi_0)^3$. We also define the space $\mathcal{H}^0$ (with no quasiperiodicity conditions) and the self-adjoint, positive, compact operator $\mathcal{B}^0 : \mathcal{H}^0 \to \mathcal{H}^0$, associated to the problem (2.22)–(2.23), analogously to $\mathcal{H}^{h,\eta}$ and $\mathcal{B}^{h,\eta}$ of the previous section. Since ϖ_0 is a bounded Lipschitz domain, the limit problem has an eigenvalue sequence

$$(2.24) \qquad 0 = \lambda_1 = \ldots = \lambda_6 < \lambda_7 \le \lambda_8 \le \ldots \le \lambda_p \le \ldots \to +\infty.$$

Here the $\lambda_1, \ldots, \lambda_6$ are the eigenvalues of the rigid motions including three translations and rotations. For every p, let u_p be the eigenfunction corresponding to λ_p, normalized in $L^2(\varpi_0)^3$ by

$$(2.25) \qquad (u_p, u_q)_{\varpi_0} = \delta_{p,q}, \quad p, q = 1, 2, \ldots$$

We also state a scaled version of the problem (2.22)–(2.23) (see (2.1)):

$$(2.26) \qquad D(-\nabla_x)^\top AD(\nabla_x)u = \Lambda^0 u \quad \text{in } \varpi(h),$$

$$(2.27) \qquad D(\nu(x))^\top AD(\nabla_x)u = 0 \quad \text{a.e. in } \partial\varpi(h).$$

It is a trivial consequence of the above remarks that the eigenvalues and corresponding eigenvectors of (2.26)–(2.27) are now given by

$$(2.28) \qquad \lambda_j^h := a_h^2 \lambda_j = \lambda_j + O(h), \quad u_j^h(x) := a_h^{3/2} u_j(a_h x),$$

and the relation $(u_j^h, u_k^h)_{\varpi(h)} = \delta_{j,k}$ still holds.

Lemma 2.2. *There exist $b_1 > 0$ such that for all $k \in \mathbb{N}$, the bounds $|u_k(x)| \le C_k$, $|\nabla u_k(x)| \le C_k$ hold for $x \in \varpi_0 \subset \mathbb{R}^3$ with $|x - P^\pm| < b_1$.*

The proof of this fact follows from standard local elliptic estimates and our smoothness assumption of the boundary $\partial\varpi_0$ near the points $P^\pm$. The lemma is proven in detail in [33], Lemma 3.1. As a corollary, we also have

$$(2.29) \qquad |u_k^h(x)| \le C_k, \quad |\nabla u_k^h(x)| \le C_k$$
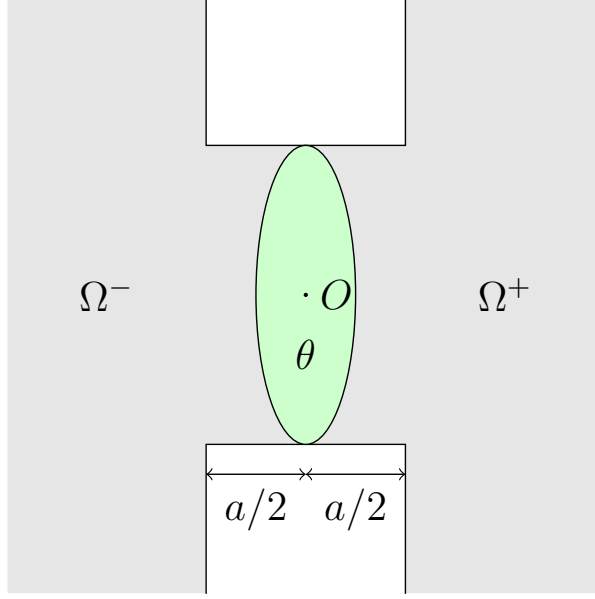
FIGURE 2.3. The set Ω.

for all small enough $h > 0$, for all $x \in \varpi(h)$ with $|x - P^\pm| < b_1$.

2.4. **Additional notation.** We finish this section by fixing some more notation. We write in any dimension n, $B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\} \subset \mathbb{R}^n$ and $A(s, r) = \{x \in \mathbb{R}^n : s < |x| < r\} \subset \mathbb{R}^n$ for $0 < s < r$, and $e^{(j)} \in \mathbb{R}^n$, $j = 1, \ldots, n$, for the canonical basis vectors.

We define some smooth cut-off functions to be used at several points later. First, let $\chi_\theta : \mathbb{R}^3 \to [0, 1]$ be a C^∞-smooth function equal to 1 on the set $\theta \times [-1, 1]$ and equal to 0 outside some neighbourhood of this set. More precisely, we assume that χ_θ is chosen such that

$$(2.30) \quad \varpi_h \cap \{x : \mathcal{X}_+^h(x) < 1\} \subset \varpi(h) \quad \text{and} \quad \varpi_h \cap \{x : \mathcal{X}_-^h(x) < 1\} \subset \varpi(h).$$

for all $h \in (0, \frac{1}{10}]$, where $\mathcal{X}_\pm^h(x) = \chi_\theta(h^{-1}(x - P^\pm))$. We also set for $x \in \mathbb{R}^3$

$$(2.31) \qquad\qquad \mathcal{X}^h(x) = 1 - \mathcal{X}_-^h(x) - \mathcal{X}_+^h(x),$$

As a consequence,

$$(2.32) \qquad\qquad \varpi_h \cap \operatorname{supp} \mathcal{X}^h \subset \varpi(h) \quad \text{and} \quad |\nabla \mathcal{X}^h| \leq C h^{-1},$$

and $\mathcal{X}^h$ equals 0 in both balls $B(P^\pm, ch)$ and 1 outside the balls $B(P^\pm, c_1 h)$ for some constants $0 < c < c_1$. Also, notice that at least for small enough h, the boundary $\partial \varpi(h)$ is infinitely smooth (planar) and (2.29) holds inside the domain $\{x \in \mathbb{R}^3 : \mathcal{X}^h(x) < 1\}$.

We also denote $\chi_\pm : \mathbb{R}^3 \to [0, 1]$ some C^∞-cut-off functions such that

$$(2.33) \quad \chi_\pm(x) = 1 \text{ for } x \in B(P^\pm, S) \quad \text{and} \quad \chi_\pm(x) = 0 \text{ for } x \notin B(P^\pm, R);$$

for some $0 < S < R < 1/2$. Obviously, $\chi_+(y, z) = 0$ for $z \leq 0$ and $\chi_-(y, z) = 0$ for $z \geq 0$ and also for small enough h,

$$(2.34) \qquad\qquad\qquad \chi_\pm \mathcal{X}_\pm^h = \mathcal{X}_\pm^h$$

We define for $a \in \{0, 1\}$ the domains

$$\Omega := \{\xi \in \mathbb{R}^3 \,:\, |\xi_3| > a/2\} \cup \big(\theta \times [-a/2, a/2]\big) \quad \text{and}$$
(2.35) $$\qquad \Omega^\pm = \{\xi \in \Omega \,:\, \pm\xi_3 > 0\},$$

see Fig. 2.3. The reader should keep in mind that if $x \in \varpi_h \cap \text{supp}\,\chi_\pm$, then $\xi = h^{-1}(x - P^\pm) \in \Omega^\mp$. We define the coordinate change mappings $\tau_\pm^h(x) = (x - P^\pm)/h$ and denote $\tau_\pm := \tau_\pm^1$.

3. Formal asymptotic procedure

3.1. **The ansätze, the first step.** In this section we present asymptotic formulas as $h \to 0$, for the eigenvalues $\Lambda_k^h(\eta)$ and for the eigenfunctions U_k^h, see (2.19), (2.20) and (3.1), (3.2). We start with the case that an eigenvalue λ_k of the limit problem is simple, see (2.24). Let us fix such a k.

As for the corresponding eigenvalue $\Lambda_k^h(\eta)$ of the model problem (2.13)–(2.16), we expect it to be a perturbation of λ_k, and the asymptotic formula or ansatz is written as

(3.1) $$\Lambda_k^h(\eta) = \lambda_k + h\Lambda_k'(\eta) + \widetilde{\Lambda}_k^h(\eta),$$

where $\Lambda_k'(\eta)$ is the main correction term, to be calculated in Section 3.3, and $\widetilde{\Lambda}_k^h(\eta)$ is a remainder, which will be shown to be of small order $O(h^{3/2})$ in Section 4. Next, we introduce the following asymptotic formula for the eigenfunction U_k^h,

(3.2) $$U_k^h(x) = \mathcal{X}^h(x)\Big(u_k(a_h x) + hu_k'(a_h x))\Big)$$
$$+ \sum_\pm \chi_\pm(x)V_k^\mp(h^{-1}(x - P^\pm)) - \sum_\pm \chi_\pm(x)\mathcal{X}^h(x)u_k(P^\pm)$$
$$+ \widetilde{U}_k^h(x) \,, \quad x \in \varpi_h, \ \eta \in [0, 2\pi),$$

where u_k is as in (2.25) and the notation of Section 2.4 is used, in particular, the cut-off functions are given in (2.31) and (2.33), and the function $\mathcal{X}^h u_k \circ a_h$ is extended from $\varpi(h)$ to ϖ_h as 0 by using (2.30). The first and second rows of the right hand side of (3.2) are called the outer and inner expansions, respectively. They describe the behaviour of U_k^h in ϖ_h (at a distance of $P^\pm$) and close to $P^\pm$, respectively; see the definitions of the cut-off functions.

To motivate the ansatz (3.2) we follow the standard asymptotic scheme described in [14] and the concordance method. As the *first step* in the construction (3.2) we observe that according to (2.1), (2.2), at some distance of the points $P^\pm$ the domain ϖ_h is just a scaling of ϖ, hence, the leading term $u_k \circ a_h$ is expected to describe the behaviour of U_k^h inside ϖ_h.

The other terms of (3.2) will be considered in subsequent sections, in particular the main correction term $u_k' = u_k'(x, \eta)$ of the outer expansion will be defined in Section 3.3. The behaviour of U_k^h near the point $P^\pm$ is quite subtle, and the main task of Section 3.2 will be to motivate the inner expansion, which describes this. In particular we shall determine in Section 3.2 the functions $V_k^\pm$ and thus the boundary layer terms of the second row of (3.2). The term $\widetilde{U}_k^h$ is again a small remainder to be evaluated in Section 4.

3.2. **Second step: construction of the inner expansion.** We next derive the
expression for the inner expansion in the ansatz (3.2),

$$(3.3) \qquad \sum_{\pm} \chi_{\pm}(x) V_k^{\mp}(h^{-1}(x - P^{\pm})) - \sum_{\pm} \chi_{\pm}(x) \mathcal{X}^h(x) u_k(P^{\pm}).$$

In other words, we aim to determine the functions $V_k^{\pm}$ and boundary layer type terms
near the points $P^{\pm}$, a task which is a priori quite unclear. The starting point is that
the above fixed first term $u_k \circ a_h$ does not satisfy the quasiperiodicity conditions,
and we thus require that the principal terms containing $V_k^{\pm}$ should compensate the
discrepancy caused by that fact. We next show how this condition will fix the
functions $V_k^{\pm}$, see (3.28) below.

By Lemma 2.2 and the mean value theorem, $u_k(a_h x) = u_k(P^{\pm}) + O(h)$, if x belongs
to some small neighbourhood $B(P^{\pm}, ch)$ of $P^{\pm}$. Denoting $\xi = \tau_{\pm}^h(x) = h^{-1}(x - P^{\pm})$,
we have $\xi \in \Omega^{\mp}$ for $x \in \varpi_h \cap \operatorname{supp} \chi_{\pm}$ (see (2.35)), and the scaling $L_x = h^{-2} L_{\xi} = h^{-2} D(-\nabla_{\xi})^{\top} A D(\nabla_{\xi})$ holds for the operator (2.7). So, to compensate the above
discrepancy, we expect the functions $V_k^{\pm}$ to be the solutions of the problems

$$(3.4) \qquad\qquad L_{\xi} V_k^{\pm}(\xi) = 0, \quad \xi \in \Omega^{\pm},$$

$$(3.5) \qquad\qquad N_{\xi}^{(\pm)} V_k^{\pm}(\xi) = 0, \quad \xi \in \partial\Omega^{\pm} \setminus (\theta \times \{0\}),$$

$$(3.6) \qquad\qquad V_k^{\pm}(\xi) \to u_k(P^{\mp}), \quad |\xi_3| \to \infty,$$

with additional linking boundary conditions

$$(3.7) \qquad V_k^{-}(\xi) = e^{i\eta} V_k^{+}(\xi), \quad -N_{\xi}^{(-)} V_k^{-}(\xi) = e^{i\eta} N_{\xi}^{(+)} V_k^{+}(\xi), \quad \xi \in \theta \times \{0\}.$$

Here we have denoted by $N_{\xi}^{(+)} = D(\nu(\xi))^{\top} A D(\nabla_{\xi})$ the boundary operator associ-
ated with the domain Ω^{+}, and similarly for the sign "$-$". The conditions (3.7) are
related to the quasiperiodicity conditions for U_k^h.

The rest of this section is devoted to looking for a solution to (3.4)–(3.7) and to
stating some of its properties. First, note that if a function V_k is a solution of

$$(3.8) \qquad\qquad L_{\xi} V_k(\xi) = 0, \quad \xi \in \Omega,$$

$$(3.9) \qquad\qquad N_{\xi} V_k(\xi) = 0, \quad \xi \in \partial\Omega,$$

$$(3.10) \qquad\qquad V_k(\xi) \to u_k(P^{+}), \quad \xi_3 < 0, \quad |\xi| \to \infty,$$

$$(3.11) \qquad\qquad V_k(\xi) \to e^{i\eta} u_k(P^{-}), \quad \xi_3 > 0, \quad |\xi| \to \infty,$$

where $N_{\xi} = D(\nu(\xi))^{\top} A D(\nabla_{\xi})$ on the boundary of Ω, then the functions $V_k^{\pm}$ can be
found as the restrictions

$$(3.12) \qquad\qquad V_k^{-} = V_k|_{\Omega^{-}}, \quad V_k^{+} = e^{-i\eta} V_k|_{\Omega^{+}}.$$

Definition 3.1. For $j = 1, 2, 3$, we denote by $X_j : \Omega \to \mathbb{R}^3$ the three functions
which are the solutions of the problems

$$(3.13) \qquad\qquad L_{\xi} X_j(\xi) = 0, \quad \xi \in \Omega,$$

$$(3.14) \qquad\qquad N_{\xi} X_j(\xi) = 0, \quad \xi \in \partial\Omega,$$

with asymptotics $X_j(\xi) \to \pm e^{(j)}$ when $\pm \xi_3 > 0$ and $|\xi| \to \infty$.

We shall soon prove the existence of such functions X_j. Taking that for granted we define the function V_k by

$$(3.15) \qquad V_k(\xi) = \sum_{j=1}^{3} \left(a_j^{(k)}(\eta)e^{(j)} + b_j^{(k)}(\eta)X_j(\xi) \right)$$

where the coefficients are to be chosen such that (3.8)–(3.11) are satisfied. Since the functions X_j do not depend on η or k, it is plain that V_k depends on η and k only via the coefficients, which are determined by the six equations

$$(3.16) \qquad a^{(k)}(\eta) - b^{(k)}(\eta) = u_k(P^+), \quad a^{(k)}(\eta) + b^{(k)}(\eta) = e^{i\eta}u_k(P^-)$$

with $a^{(k)} = (a_1^{(k)}, a_2^{(k)}, a_3^{(k)})^\top$ and $b^{(k)} = (b_1^{(k)}, b_2^{(k)}, b_3^{(k)})^\top$. Solving the system (3.16) yields

$$(3.17) \quad a^{(k)}(\eta) = \frac{1}{2}(u_k(P^+) + e^{i\eta}u_k(P^-)) \, , \, b^{(k)}(\eta) = \frac{1}{2}(-u_k(P^+) + e^{i\eta}u_k(P^-)).$$

We now turn to the existence and some properties of the functions X_j.

Lemma 3.2. *For every $j = 1, 2, 3$, there exists a unique solution to the problem (3.13)–(3.14), and it can be written in the form*

$$(3.18) \qquad X_j = \pm(1 - \chi_\theta)e^{(j)} + (1 - \chi_\theta)\sum_{l=1}^{3} M_{lj}^\pm T_l^\pm \circ \tau_\pm + \widetilde{T}_j \quad \text{in } \Omega^\pm,$$

where $\tau_\pm(\xi) = \xi - P^\pm$ and $M_{lj}^\pm \in \mathbb{R}$ are unique coefficients and $T_l^\pm$ denote the Poisson kernels for the operator L_ξ in the domains $\{\xi \in \mathbb{R}^3 : \pm\xi_3 > 0\}$, respectively (see (3.20)–(3.21)). The remainders $\widetilde{T}_j$ satisfy for $\xi \in \mathbb{R}^3$ the estimates

$$(3.19) \qquad |\widetilde{T}_j(\xi)| \leq \frac{C}{1 + |\xi|^2}, \quad |\nabla_\xi \widetilde{T}_j(\xi)| \leq \frac{C}{1 + |\xi|^3}.$$

Proof. We define the Poisson kernels $T_l^\pm$ as solutions of the half-space problems

$$(3.20) \qquad\qquad L_\xi T_l^\pm(\xi) = 0, \quad \xi \in \mathbb{R}^3, \ \pm\xi_3 > 0,$$
$$(3.21) \qquad\qquad N_\xi T_l^\pm(\xi) = \delta_{(\xi_1, \xi_2)}e^{(l)}, \quad \pm\xi_3 = 0.$$

Here, the differential operators L_ξ and N_ξ are understood as distributional derivatives (notice the direction of the normal vector in (3.21), e.g. it is $-e^{(3)}$ in the case of the sign "$+$") and $\delta_{(\xi_1, \xi_2)}$ stands for the 2-dimensional Dirac measure of the point $(0, 0)$ in the plane $\{\xi \in \mathbb{R}^3 : \xi_3 = 0\}$. The solutions of the problem (3.20) are known to be homogeneous functions of order -1 (see e.g. [20], Chapter 8), in other words, the function $T_l^\pm$ has a singularity at 0, and for all $\xi \in \mathbb{R}^3$, $\pm\xi_3 > 0$, $0 < r \in \mathbb{R}$, we have

$$(3.22) \qquad T_l^\pm(r\xi) = \frac{1}{r}T_l^\pm(\xi) \, , \, |T_l^\pm(\xi)| \leq \frac{C}{|\xi|} \, , \, |\nabla_\xi T_l^\pm(\xi)| \leq \frac{C}{|\xi|^2}.$$

Let us define for a moment a smooth function in Ω by $\Theta^{(j)}(\xi) = \pm(1 - \chi_\theta(\xi))e^{(j)}$, where $\xi \in \Omega^\pm$ and χ_θ is as in (2.30). We now look for a function $X_j : \Omega \to \mathbb{R}$ in the form $X_j = \Theta^{(j)} + \widetilde{X}_j$, where the function $\widetilde{X}_j$ satisfies the problem

$$L_\xi \widetilde{X}_j(\xi) = -L_\xi \Theta^{(j)}(\xi), \quad \xi \in \Omega,$$
$$N_\xi \widetilde{X}_j(\xi) = -N_\xi \Theta^{(j)}(\xi), \quad \xi \in \partial\Omega,$$

and the functions on the right are smooth and have compact supports. According to the general results from [15], Chapter 1, such a function $\widetilde{X}_j$ exists and is unique in the function space

$$(3.23) \qquad \left\{ v \in H^1_{\mathrm{loc}}(\Omega) \ : \ \|\nabla_\xi v; L^2(\Omega)\| + \|(1 + |\xi|)^{-1} v; L^2(\Omega)\| < \infty \right\}.$$

Using Theorem 5.8 of [24] and the results of Section 3 in [22], the solution $\widetilde{X}_j$ is, say in a bounded ball $B(0, 2)$, at least C^2-smooth bounded function, and outside $B(0, 2)$, a linear combination of the Poisson kernels plus a perturbation which is small at the infinity. More precisely, we can write for some coefficients $M^\pm_{lj}$,

$$(3.24) \qquad \widetilde{X}_j(\xi) = (1 - \chi_\theta(\xi)) \sum_{l=1}^{3} M^\pm_{lj} T^\pm_l(\xi - P^\pm) + \widetilde{T}_j(\xi) \ , \ \xi \in \Omega^\pm,$$

where the functions $(1 - \chi_\theta) T^\pm_l(\xi - P^\pm)$ are well defined in $\Omega^\pm$ and the perturbation $\widetilde{T}_j$ satisfies (3.19); moreover, $\widetilde{T}_j|_{B(0,r)} \in L^2(\Omega^\pm)$ for any fixed radius $r > 0$. More information on the coefficients $M^\pm_{lj}$ will be obtained in Lemma 3.3 below. $\qquad \square$

We now define V_k by putting (3.18) into (3.15) and using (3.17). We also denote, keeping in mind that $b^{(k)}$ and thus the consequent functions depend also on η,

$$(3.25) \qquad W^\pm_k(\xi) = \left(1 - \chi_\theta \circ \tau_\mp(\xi)\right) \sum_{j,l=1}^{3} b^{(k)}_j M^\pm_{lj} T^\pm_l(\xi) \ , \ \ \tau_\mp(\xi) \in \Omega^\pm,$$

$$(3.26) \qquad \widetilde{W}^\pm_k(\xi) = \sum_{j=1}^{3} b^{(k)}_j \left(\mp \chi_\theta(\xi) e^{(j)} + \widetilde{T}_j(\xi) \right), \ \ \xi \in \Omega^\pm,$$

where $\tau_\pm(\xi) = \xi - P^\pm$. We remark that due to (3.22), (3.25), we have $W^\pm_k \in L^2(\Omega^\pm)$. By the remarks on $\widetilde{T}_j$ in the proof of Lemma 5.2, $\widetilde{W}^\pm_k|_{B(0,r)} \in L^2(\Omega^\pm)$ for any constant radius $r > 0$, and moreover

$$(3.27) \qquad \|W^\pm_k; L^2(\Omega^\pm)\| \leq C_k \ , \ \ \|\widetilde{W}^\pm_k|_{B(0,r)}; L^2(\Omega^\pm)\| \leq C_k.$$

We finally find the inner expansion by taking the restrictions (3.12), which leads to

$$(3.28) \qquad V^\pm_k(\xi) = u_k(P^\mp) + W^\pm_k \circ \tau_\pm(\xi) + \widetilde{W}^\pm_k(\xi), \ \ \ \xi \in \Omega^\pm.$$

From (3.25), (3.22) we see that there exists some constant $M > 2$, which we fix now, such that for all $\xi \in \mathbb{R}^3$ with $|\xi| \geq M/2$

$$(3.29) \qquad W^\pm_k(r\xi) = \frac{1}{r} W^\pm_k(\xi) \ ,$$

Also, (3.22), (3.19) imply for all $\xi \in \mathbb{R}^3$

$$(3.30) \qquad |W^\pm_k(\xi)| \leq \frac{C_k}{1 + |\xi|} \ , \ \ |\nabla_\xi W^\pm_k(\xi)| \leq \frac{C_k}{1 + |\xi|^2} \ ,$$

$$(3.31) \qquad |\widetilde{W}^\pm_k(\xi)| \leq \frac{C_k}{1 + |\xi|^2} \ , \ \ |\nabla_\xi \widetilde{W}^\pm_k(\xi)| \leq \frac{C_k}{1 + |\xi|^3}.$$

We complete the present study of the inner expansion by the following remarks.

Lemma 3.3. *The matrices* $\mathbf{M}^{\pm} = (M_{jl}^{\pm})_{j,l=1}^{3}$ *defined by the coefficients in* (3.18), (3.24) *(which do not depend on η or k) have the properties*

$$(3.32) \qquad\qquad \mathbf{M}^{+} = -\mathbf{M}^{-} \quad , \quad \mathbf{M}^{\pm} = \left(\mathbf{M}^{\pm}\right)^{\top},$$

and, moreover, $\mathbf{M}^{+}$ *is positive definite.*

Proof. Let us use the (generalized) Green formula (see [19]) for X_l and $e^{(j)}$ in the domain $B(0, \varrho) \cap \Omega$, where $\varrho > 0$ large enough, in particular such that $\chi_\theta(\xi) = 0$ for $|\xi| \geq \varrho - 1/2$. So, by (3.13), (3.14) and Fig. 2.3.,

$$0 = (L_\xi X_l, e^{(j)})_{B(0,\varrho)\cap\Omega} = (N_\xi X_l, e^{(j)})_{S_\varrho^+} + (N_\xi X_l, e^{(j)})_{S_\varrho^-},$$

where $S_\varrho^{\pm} = \{\xi : |\xi| = \varrho, \frac{a}{2} < \pm\xi_3\}$. Taking into account the representation (3.18), we get

$$0 = (N_\xi X_l, e^{(j)})_{S_\varrho^+} + (N_\xi X_l, e^{(j)})_{S_\varrho^-} = \sum_{\pm} \sum_{m=1}^{3} M_{ml}^{\pm}(N_\xi T_m^{\pm} \circ \tau_\pm, e^{(j)})_{S_\varrho^{\pm}}$$

$$= -\sum_{\pm} \sum_{m=1}^{3} M_{ml}^{\pm}(N_\xi T_m^{\pm} \circ \tau_\pm, e^{(j)})_{\{\pm\xi_3=a/2\}}$$

$$= -\sum_{\pm} \sum_{m=1}^{3} M_{ml}^{\pm}(N_\xi T_m^{\pm}, e^{(j)})_{\{\pm\xi_3=0\}} = -M_{jl}^+ - M_{jl}^-,$$

where we also used (3.21) and it was necessary to change the sign due to the direction of the normal vector. Notice that the contribution of the term $\widetilde{T}_m$ must vanish as a consequence of the second inequality (3.19), since $(N_\xi \widetilde{T}_m, e^{(j)})_{S_\varrho^{\pm}} = O(\varrho^{-1})$ for large ϱ, whereas the terms $(N_\xi T_m^{\pm}, e^{(j)})_{S_\varrho^{\pm}}$ are constant with respect to ϱ. We have proven the first identity in (3.32).

To prove the second identity of (3.32) we use the Green formula for the functions X_j and X_l:

$$\begin{aligned}0 &= (L_\xi X_j, X_l)_{B(0,\varrho)\cap\Omega} = (N_\xi X_j, X_l)_{S_\varrho^+} \\ (3.33) \qquad &+ (N_\xi X_j, X_l)_{S_\varrho^-} - (X_j, N_\xi X_l)_{S_\varrho^+} - (X_j, N_\xi X_l)_{S_\varrho^-}.\end{aligned}$$

Since $N_\xi T_m^{\pm}(\xi) = O(\varrho^{-2})$ and $T_m^{\pm}(\xi) = O(\varrho^{-1})$ for $\xi \in S_\varrho$, the scalar products $(N_\xi T_m^{\pm}, T_l^{\pm})_{S_\varrho^{\pm}}$ tend to zero as $\varrho \to +\infty$. Thus, by (3.33) and the same observations as in the first case,

$$\begin{aligned}0 &= \lim_{\varrho \to +\infty} \sum_{m=1}^{3} \Big(M_{mj}^{+}(N_\xi T_m^{+}, e^{(l)})_{S_\varrho^+} + M_{mj}^{-}(N_\xi T_m^{-}, -e^{(l)})_{S_\varrho^-} \\ (3.34) \qquad &- M_{ml}^{+}(e^{(j)}, N_\xi T_m^{+})_{S_\varrho^+} - M_{ml}^{-}(-e^{(j)}, N_\xi T_m^{-})_{S_\varrho^-} \Big) = 2M_{lj}^{-} - 2M_{jl}^{+}.\end{aligned}$$

Finally, for the positive definiteness of $\mathbf{M}^+$ we consider a vector $0 \neq v = (v_1, v_2, v_3)^{\top} \in \mathbb{R}^3$ and the function $w = \sum_{j=1}^{3} v_j X_j$. Obviously, w is not a rigid motion. Then, by the positivity of the operator L_ξ,

$$0 < (L_\xi w, w)_{\Omega^+} = (N_\xi w, w)_{\partial\Omega^+} = -(N_\xi w, w)_{S_\varrho^+}$$

$$(3.35) \qquad = -\sum_{j,l=1}^{3} v_j \overline{v_l} \int_{S_\varrho^+} X_l(\xi) N_\xi X_j(\xi) \, ds_\xi.$$

It remains to note as above that

$$\lim_{\varrho \to +\infty} \int_{S_\varrho^+} X_l(\xi) N_\xi X_j(\xi) = -M_{jl}^+ ,$$

hence, $v^\top \mathbf{M}^+ \overline{v} = (L_\xi w, w)_{\Omega^+}$. The claim follows from (3.35). $\quad\square$

3.3. Third step: construction of the outer expansion and the leading correction term for the eigenvalue.

We next look for the correction term u_k' of the outer expansion. Let us write for a moment $E(x, h) := u_k(a_h x) + h\tilde{u}_k(a_h x, \eta)$ for some $\tilde{u}_k \in H^1(\varpi_0)$. Since $L_x u_k(x) = \lambda_k u_k(x)$, we can calculate

$$L_x E(x, h) = \lambda_k u_k(a_h x) + 2ah\lambda_k u_k(a_h x) + hL_x \tilde{u}_k(a_h x, \eta) + O(h^2)$$

and on the other hand, assuming (3.1),

$$\Lambda_k^h(\eta) E(x, h) = \lambda_k u_k(a_h x) + h\Lambda_k'(\eta) u_k(a_h x) + \lambda_k h\tilde{u}_k(a_h x) + O(h^{3/2})$$

Hence, the assumption that $L_x E(x, h) = \Lambda_k^h E(x, h)$ is correct up to terms of order $h^{3/2}$, leads to the following problem to find $\tilde{u}_k$ in ϖ_0:

$$(3.36) \qquad (L_x - \lambda_k)\tilde{u}_k(x, \eta) = (-2a\lambda_k + \Lambda_k'(\eta))u_k(x), \quad x \in \varpi_0,$$

$$(3.37) \qquad N_x \tilde{u}_k(x, \eta) = 0, \quad x \in \partial\varpi_0,$$

$$(3.38) \qquad \tilde{u}_k(x, \eta) \sim W_k^\pm(x - P^\mp), \quad x \to P^\mp.$$

The asymptotic condition (3.38) comes by looking at the inner expansion (3.3) near the points $P^\pm$. We write $\tilde{u}_k$ as

$$(3.39) \qquad \tilde{u}_k(x, \eta) = \sum_\pm \chi_\pm(x) W_k^\mp(x - P^\pm) + u_k'(x, \eta),$$

where $\chi_\pm$ are as in (2.33), and state the following problem for u_k':

$$(3.40) \qquad (L_x - \lambda_k)u_k'(x, \eta) = F(x), \quad x \in \varpi_0,$$

$$(3.41) \qquad N_x u_k'(x, \eta) = G(x), \quad x \in \partial\varpi_0.$$

Here we denote (cf. (2.33))

$$(3.42) \qquad F = \big(-2a\lambda_k + \Lambda_k'(\eta)\big)u_k - (L_x - \lambda_k)\sum_\pm \chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm \quad \text{in } \varpi_0,$$

$$(3.43) \qquad G(x) = \begin{cases} -\sum_\pm N_x(\chi_\pm(x)\mathcal{W}_k^\mp(x - P^\pm) , & x \notin \partial\varpi_0 \cap B(P^\pm, S/2) \\ 0 , & x \in \partial\varpi_0 \cap B(P^\pm, S/2), \end{cases}$$

where $\mathcal{W}_k^\mp$ are defined as $W_k^\mp$ in (3.25) but without the cut-off function χ_θ:

$$(3.44) \qquad \mathcal{W}_k^-(\xi) = \sum_{j,l=1}^3 b_j^{(k)} M_{lj}^- T_l^-(\xi) , \quad \xi \in \Omega^-,$$

and similarly for W_k^+. Note that $\chi_\pm(x) = 1$ for x close to $P^\pm$. Hence, (3.20), (3.22), (3.25) imply that the functions $L_x(\chi_\pm W_k^\mp \circ \tau_\pm)$ are equal to 0 near the points $P^\pm$, and thus $F \in L^q(\varpi_0)$ for every $q < 3$. The function G is smooth and thus bounded, by similar arguments and (3.21); in fact, G equals the expression on the first row of (3.43) everywhere except at $P^\pm$, where the latter has the Dirac measure singularity.

Lemma 3.4. *If*

$$(3.45) \qquad \Lambda'_k(\eta) = 2a\lambda_k + 2b^{(k)}(\eta)^\top \mathbf{M}^+ \overline{b^{(k)}(\eta)}$$

then the problem (3.40)-(3.41) *has a unique solution in* $H^1(\varpi_0)^3$.

Proof. The problem (3.40)-(3.41) can be rewritten in the weak formulation as

$$(3.46) \quad (AD(\nabla_x)u'_k, D(\nabla_x)v)_{\varpi_0} - (G, v)_{\partial\varpi_0} - \lambda_k(u'_k, v)_{\varpi_0} = (F, v)_{\varpi_0}, \quad v \in H^1(\varpi_0)^3.$$

Equation (3.46) is equivalent to

$$\langle u'_k, v \rangle_{\mathcal{H}^0} - (1 + \lambda_k)\langle \mathcal{B}^0 u'_k, v \rangle_{\mathcal{H}^0} = (G, v)_{\partial\varpi_0} + (F, v)_{\varpi_0} , \quad v \in \mathcal{H}^0,$$

where $\mathcal{B}^0$ and $\mathcal{H}^0$ were defined below (2.23). We use the fact that functional defined by the formula $\mathcal{F}(v) = (G, v)_{\partial\varpi_0} + (F, v)_{\varpi_0}$ is linear and continuous on $H^1(\varpi)^3$. The problem (3.40)-(3.41) can thus be rewritten as the equation $\mathcal{B}^0 u'_k - \mu_k u'_k = -\mu_k \mathcal{F}$, where $\mu_k = (1 + \lambda_k)^{-1}$. According to the Fredholm alternative, this equation has a solution if and only if the right-hand side is orthogonal to u_k (solution of the homogeneous problem, Section 2.3). So we get the solvability condition

$$(3.47) \qquad 0 = (G, u_k)_{\partial\varpi_0} + (F, u_k)_{\varpi_0}.$$

Here we use the Green formula and take into account (2.22) and (2.23):

$$(F, u_k)_{\varpi_0} = -2a\lambda_k + \Lambda'_k(\eta) - \sum_\pm \int_{\varpi_0} (L_x - \lambda_k)(\chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm)\overline{u}_k dx$$

$$= -2a\lambda_k + \Lambda'_k(\eta) + \sum_\pm \int_{\varpi_0} \lambda_k \chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm \overline{u}_k dx - \sum_\pm \int_{\varpi_0} \chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm L_x \overline{u}_k dx$$

$$- \sum_\pm \int_{\partial\varpi_0} N_x^0(\chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm)\overline{u}_k ds_x + \sum_\pm \int_{\partial\varpi_0} \chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm N_x^0 \overline{u}_k ds_x$$

$$= -2a\lambda_k + \Lambda'_k(\eta) - \sum_\pm \int_{\partial\varpi_0} N_x^0(\chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm)\overline{u}_k ds_x.$$

By the remarks just before Lemma 3.4, the last integral equals $(G, u_k)_{\partial\varpi_0}$ except that $N_x^0(\chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm)$ contains Dirac measures at $P^\pm$ which G does not contain. This remark, (3.21), (3.25), and (3.47) yield

$$-2a\lambda_k + \Lambda'_k(\eta) = b(\eta)^\top \mathbf{M}^- u_k(P^+) + e^{-i\eta}b(\eta)^\top \mathbf{M}^+ u_k(P^-) = 2b(\eta)^\top \mathbf{M}^+ \overline{b(\eta)}. \ \ \Box$$

Remark 3.5. The facts that $F \in L^q(\varpi_0)$ for every $q < 3$, the function G is smooth, and the boundary of ϖ_0 is smooth near points $P^\pm$, imply that $u'_k \in W_q^2(\varpi_0)$, by standard elliptic estimates. Due to the Sobolev embedding theorem and (3.39) we get for all $p < \infty$ that $\nabla_x u'_k \in L^p(\varpi_0)$ and $u'_k \in L^\infty(\varpi_0)$ with the corresponding norm bounds independent of η.

3.4. **Comments on multiple eigenvalues.** Let us consider the case of a multiple eigenvalue λ_k, see (2.24). Assume that its multiplicity is $m \geq 2$ and let u_{k+j}, where $j = 0, \ldots, m-1$, be the corresponding eigenfunctions satisfying the orthogonality and normalization condition (2.25). Now the principal term of the asymptotic of U_{k+l}^h, where $l = 0, \ldots, m-1$, is a linear combination $\sum_{j=0}^{m-1} \alpha_{l,j} u_{k+j} \circ a_h =: u_{k,l} \circ a_h$. Repeating the procedure of the concordance method of asymptotic expansions in this case we again get the problem (3.40)-(3.41) for the main correction terms u'_{k+l}

and $\Lambda'_{k+l}(\eta)$. Now it turns out that we have m solvability conditions which are equivalent to a system of m linear equations

$$(-2a\lambda_k + \Lambda'_{k+l}(\eta))\alpha_{l,j} = 2\sum_{q=0}^{m-1} \alpha_{l,q} b^{(k+q)}(\eta)^{\top} \mathbf{M}^+ \overline{b^{(k+j)}(\eta)},$$

where $j = 0, \ldots, m-1$, while $\mathbf{M}^+$ is as in Section 3.2 and $b^{(k+q)}(\eta) = \frac{1}{2}(u_{k+q}(P^-) - e^{-i\eta} u_{k+q}(P^+))$ (see (3.17)). As a consequence, the expression $-2a\lambda_k + \Lambda'_{k+l}(\eta)$ is one of the m eigenvalues (multiplicities counted) of the $m \times m$ matrix

$$\mathbf{B}(\eta) = \left(b^{(k+q)}(\eta)^{\top} \mathbf{M}^+ \overline{b^{(k+j)}(\eta)} \right)_{q,j=1}^{m},$$

and the coefficient sequence $(\alpha_{l,j})_{j=1}^{m}$ is found as the eigenvector of $\mathbf{B}(\eta)$ corresponding to the eigenvalue $-2a\lambda_k + \Lambda'_{k+l}(\eta)$. It is clear that the rank of the matrix $\mathbf{B}$ does not exceed 3.

3.5. **The case $\lambda = 0$.** The sequence (2.24) begins with six eigenvalues equal to 0 corresponding to the rigid motions

$$u_j(x) = \beta_j e^{(j)} \text{ for } j = 1,2,3, \quad u_j(x) = \beta_j x \times e^{(j-3)} \text{ for } j = 4,5,6,$$

where β_j are the normalization multipliers, i.e. $\beta_1 = \beta_2 = \beta_3 = |\varpi_0|^{-1/2}$, $\beta_4 = (J_2 + J_3)^{-1/2}$, $\beta_5 = (J_1 + J_3)^{-1/2}$, $\beta_6 = (J_2 + J_1)^{-1/2}$. Here J_k is the moment of inertia of the body ϖ_0 around the axis $e^{(k)}$. Calculating the vectors $b^{(k)}(\eta)$ and suppressing the inessential index l of $\alpha_{l,j}$, we get

$$b^{(k)}(\eta) = 2^{-1}|\varpi_0|^{-1/2}(1 - e^{-i\eta})e^{(k)} =: \alpha_1 e^{(k)}, \quad k = 1,2,3;$$
$$b^{(4)}(\eta) = 0;$$
$$b^{(5)}(\eta) = -4^{-1}|J_1 + J_3|^{-1/2}(1 + e^{-i\eta})e^{(3)} =: \alpha_5 e^{(3)};$$
$$b^{(6)}(\eta) = 4^{-1}|J_1 + J_2|^{-1/2}(1 + e^{-i\eta})e^{(2)} =: \alpha_6 e^{(2)}.$$

Thus, using the symmetry of matrix $\mathbf{M}^+$,

$$\mathbf{B}(\eta) = \begin{pmatrix} |\alpha_1|^2 M_{11}^+ & |\alpha_1|^2 M_{12}^+ & |\alpha_1|^2 M_{13}^+ & 0 & \alpha_1\overline{\alpha_5}M_{13}^+ & \alpha_1\overline{\alpha_6}M_{12}^+ \\ |\alpha_1|^2 M_{12}^+ & |\alpha_1|^2 M_{22}^+ & |\alpha_1|^2 M_{23}^+ & 0 & \alpha_1\overline{\alpha_5}M_{23}^+ & \alpha_1\overline{\alpha_6}M_{22}^+ \\ |\alpha_1|^2 M_{13}^+ & |\alpha_1|^2 M_{23}^+ & |\alpha_1|^2 M_{33}^+ & 0 & \alpha_1\overline{\alpha_5}M_{33}^+ & \alpha_1\overline{\alpha_6}M_{23}^+ \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \overline{\alpha_1}\alpha_5 M_{13}^+ & \overline{\alpha_1}\alpha_5 M_{23}^+ & \overline{\alpha_1}\alpha_5 M_{33}^+ & 0 & |\alpha_5|^2 M_{33}^+ & \alpha_5\overline{\alpha_6}M_{23}^+ \\ \overline{\alpha_1}\alpha_6 M_{12}^+ & \overline{\alpha_1}\alpha_6 M_{22}^+ & \overline{\alpha_1}\alpha_6 M_{23}^+ & 0 & \alpha_6\overline{\alpha_5}M_{23}^+ & |\alpha_6|^2 M_{22}^+ \end{pmatrix}.$$

This can be rewritten in shorter form

$$\mathbf{B}(\eta) = \begin{pmatrix} & 0 & & & & \\ |\alpha_1|^2\mathbf{M}^+ & 0 & \alpha_1\overline{\alpha_5}\mathbf{M}_3^+ & \alpha_1\overline{\alpha_6}\mathbf{M}_2^+ \\ & 0 & & & & \\ 0 & 0 & 0\ 0 & 0 & 0 \\ \overline{\alpha_1}\alpha_5(\mathbf{M}_3^+)^{\top} & 0 & |\alpha_5|^2 M_{33}^+ & \alpha_5\overline{\alpha_6}M_{23}^+ \\ \overline{\alpha_1}\alpha_6(\mathbf{M}_2^+)^{\top} & 0 & \alpha_6\overline{\alpha_5}M_{32}^+ & |\alpha_6|^2 M_{22}^+ \end{pmatrix},$$

where $\mathbf{M}_k^+$ is the kth column of the matrix $\mathbf{M}^+$. If $e^{(j)}$, $1 \leq j \leq 6$, are the standard basis vectors in $\mathbb{R}^6$ it is easy to see that the vectors

$$e^{(4)}, \overline{\alpha_5}e^{(3)} - \overline{\alpha_1}e^{(5)}, \overline{\alpha_6}e^{(2)} - \overline{\alpha_1}e^{(6)}$$

are in the kernel of the matrix $\mathbf{B}(\eta)$. So, the corresponding linear combinations of the functions u_j form the ansatz of the eigenfunctions of the perturbed problem, but the asymptotic corrections of order h for the eigenvalues are 0. To find the other three corrections of order h we must find the nonzero eigenvalues of the matrix $\mathbf{B}(\eta)$. To do this we have to calculate the characteristic function $\phi(t) = \det(\mathbf{B}(\eta) - t\mathbf{E}_6)$, where $\mathbf{E}_n$ denotes the unit matrix of dimension $n \times n$. One can write

$$\phi(t) = \det \begin{pmatrix} & 0 & 0 & 0 \\ |\alpha_1|^2\mathbf{M}^+ - t\mathbf{E}_3 & 0 & 0 & \overline{\alpha_1^{-1}\alpha_6 t} \\ & 0 & \overline{\alpha_1^{-1}\alpha_5 t} & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ \overline{\alpha_1}\alpha_5(\mathbf{M}_3^+)^\top & 0 & -t & 0 \\ \overline{\alpha_1}\alpha_6(\mathbf{M}_2^+)^\top & 0 & 0 & -t \end{pmatrix}$$

$$= \det \begin{pmatrix} & 0 & 0 & 0 \\ \widetilde{\mathbf{M}} - t\mathbf{E}_3 & 0 & 0 & 0 \\ & 0 & 0 & 0 \\ 0 & 0 & 0 & -t & 0 & 0 \\ \overline{\alpha_1}\alpha_5(\mathbf{M}_3^+)^\top & 0 & -t & 0 \\ \overline{\alpha_1}\alpha_6(\mathbf{M}_2^+)^\top & 0 & 0 & -t \end{pmatrix} = -t^3 \det(\widetilde{\mathbf{M}} - t\mathbf{E}_3),$$

where

$$\widetilde{\mathbf{M}} = \begin{pmatrix} |\alpha_1|^2 M_{11}^+ & |\alpha_1|^2 M_{12}^+ & |\alpha_1|^2 M_{13}^+ \\ (|\alpha_1|^2 + |\alpha_6|^2) M_{12}^+ & (|\alpha_1|^2 + |\alpha_6|^2) M_{22}^+ & (|\alpha_1|^2 + |\alpha_6|^2) M_{23}^+ \\ (|\alpha_1|^2 + |\alpha_5|^2) M_{13}^+ & (|\alpha_1|^2 + |\alpha_5|^2) M_{23}^+ & (|\alpha_1|^2 + |\alpha_5|^2) M_{33}^+ \end{pmatrix}$$
$$= \mathrm{diag}(|\alpha_1|^2, |\alpha_1|^2 + |\alpha_6|^2, |\alpha_1|^2 + |\alpha_5|^2)\mathbf{M}^+$$

and $\mathrm{diag}(a, b, c)$ is a diagonal matrix with elements a, b and c.

So in the case $\lambda = 0$ there are six main asymptotic corrections $\Lambda_j'(\eta)$: for $j = 1, 2, 3$ they are the eigenvalues of the matrix $\widetilde{\mathbf{M}}$, and for $j = 4, 5, 6$ they are just 0.

4. Main result: position of spectral gaps.

4.1. **Main theorem.** We state our main result on the position of the spectral bands for the linear elasticity problem in the domain Π_h. Recall from (1.1) that the bands consist of the eigenvalues $\Lambda_k^h(\eta)$.

Theorem 4.1. *For every k there exists a constant $C_k > 0$ such that for all $0 < h < 1/10$, $\eta \in [0, 2\pi)$,*

$$(4.1) \qquad |\Lambda_k^h(\eta) - (\lambda_k + h\Lambda_k'(\eta))| \leq C_k h^{3/2} \quad , \quad \|U_k^h - \mathcal{U}_{h,k}; \mathcal{H}^{h,\eta}\| \leq C_k h^{3/2}.$$

Here $\lambda_k + h\Lambda_k'(\eta)$ is the approximate eigenvalue from (3.1), λ_k is the kth eigenvalue of the limit problem, see (2.24), and $\Lambda_k'(\eta)$ is determined in Lemma 3.4 of Section 3.3,

$$(4.2) \qquad \Lambda_k'(\eta) = 2a\lambda_k + \left(A_k + e^{i\eta}B_k\right)^\top \mathbf{M}^+ \left(A_k + e^{-i\eta}B_k\right),$$

where the column vectors A_k, $B_k \in \mathbb{R}^3$ and the positive definite matrix $\mathbf{M}^+ \in \mathbb{R}^{3\times 3}$ do not depend on h or η. Moreover,

$$(4.3) \qquad \mathcal{U}_{h,k} = \mathcal{X}^h\left(u_k \circ a_h + hu_k' \circ a_h\right) + \sum_\pm \chi_\pm V_k^\mp \circ \tau_\pm^h - \sum_\pm \chi_\pm \mathcal{X}^h u_k(P^\pm)$$

is the approximate eigenvector from (3.2); the functions $V_k^\pm$ and u_k' are determined in Sections 3.2 and 3.3, respectively, and the cut-off-functions and $\tau_\pm^h$ are defined in Section 2.4.

According to (3.45), (3.17), we have $A_k = -u_k(P^+)/2$, $B_k = u_k(P^-)/2$, and thus $u_k(P^-) \neq 0$ is a sufficient criterion that the corresponding, kth spectral band is an interval and not an eigenvalue of infinite multiplicity. The proof of Theorem 4.1 will be given in Sections 4.2–4.3, and it needs another proof for the existence of the gaps, which will be postponed to the appendix. The proof in the appendix does not use the machinery of Sections 3 and 4; it is an adaptation of the methods of [33].

In the following we shall also use the notation

$$(4.4) \qquad \mu_k = \left(1 + \lambda_k + h\Lambda_k'(\eta)\right)^{-1} , \quad \mathcal{U}_{h,k}^\natural = \|\mathcal{U}_{h,k}; \mathcal{H}^{h,\eta}\|^{-1}\mathcal{U}_{h,k}$$

4.2. **Lemma on near eigenvalues and eigenvectors.** We shall need in Section 4.3 the following operator theoretic result in the form given in [1]; see also [4] or [35] for more simple formulations corresponding to $n = 1$, $\gamma = 0$ and $t = \tau$.

Lemma 4.2. *Let $\mathcal{B}$ be a selfadjoint, positive, and compact operator in Hilbert space $\mathcal{H}$ with the inner product $(\cdot,\cdot)_\mathcal{H}$. If there are numbers $\mu > 0$, $n \in \mathbb{N}$, and $\gamma \in (0, 1/n)$, as well as elements $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \mathcal{H}$ such that $\left|(\mathcal{U}_i, \mathcal{U}_j)_\mathcal{H} - \delta_{i,j}\right| \leq \gamma$ and $\|\mathcal{B}\mathcal{U}_j - \mu\mathcal{U}_j; \mathcal{H}\| \leq t$ for some $t \in (0, \mu)$, then the interval $[\mu - \tau, \mu + \tau]$ contains at least n eigenvalues of $\mathcal{B}$, with multiplicities counted, where $\tau = tn^{1/2}(1 - n\gamma)^{-1/2}$.*

In this section we moreover prove the following η-independent lower bound for the L^2-norm of the function $\mathcal{U}_{h,k}$.

Lemma 4.3. *There exists a constant $C = C(k, \varpi, \theta) > 0$ such that*

$$(4.5) \qquad \left|\|\mathcal{U}_{h,k}; L^2(\varpi_h)\|^2 - 1\right| \leq Ch \quad \forall \eta \in [0, 2\pi].$$

Proof. First of all note that by (2.25), $\int_{\varpi(h)} |u_k \circ a_h|^2 dx = a_h^{-2}$, hence,

$$(4.6) \quad \left|\|\mathcal{X}^h u_k \circ a_h; L^2(\varpi_h)\|^2 - 1\right| \leq \int_{\varpi_h \setminus \varpi(h)} \left(|\mathcal{X}^h u_k \circ a_h|^2 - |u_k \circ a_h|^2\right)dx \leq Ch^3$$

where the last estimate uses the facts that u_k is a bounded function near $P^\pm$, by Lemma 2.2, and the volume of the set $\varpi_h \setminus \varpi(h) \ni P^\pm$ is $O(h^3)$.

Denoting $\widetilde{\mathcal{U}}_{h,k} = \mathcal{U}_{h,k} - \mathcal{X}^h u_k \circ a_h$, (4.6) and the Cauchy-Schwartz inequality yield

$$\left|\|\mathcal{U}_{h,k}; L^2(\varpi_h)\|^2 - 1)\right| \leq \left|2(\mathcal{X}^h u_k \circ a_h, \widetilde{\mathcal{U}}_{h,k})_{\varpi_h}\right| + \|\widetilde{\mathcal{U}}_{h,k}; L^2(\varpi_h)\|^2 + Ch$$

$$\leq C'\|\widetilde{\mathcal{U}}_{h,k}; L^2(\varpi_h)\|^2 + C'h.$$

Thus it is enough to prove that $\|\widetilde{\mathcal{U}}_{h,k}; L^2(\varpi_h)\| \leq Ch^{1/2}$. By (4.3), $\|\widetilde{\mathcal{U}}_{h,k}; L^2(\varpi_h)\|$ can be bounded from above by the sum of the norms (cf. (2.34))

$$(4.7) \qquad \|\mathcal{X}_\pm^h u_k(P^\pm); L^2(\varpi_h)\|,$$

$$(4.8) \qquad \|\chi_\pm\left(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm)\right); L^2(\varpi_h)\|,$$

$$(4.9) \qquad h\|\mathcal{X}^h u_k' \circ a_h; L^2(\varpi_h)\|.$$

We complete the proof by estimating these separately. The bound $Ch^{3/2}$ for (4.7) follows immediately by recalling that the supports of the functions $\mathcal{X}_\pm^h$ are contained in balls of radius $O(h)$: the L^2-integral is taken over a volume bounded by Ch^3.

To treat (4.8) we recall the relation (3.28) and use (3.30):

$$\left\| \chi_\pm W_k^\mp \circ \tau_\mp \circ \tau_\pm^h; L^2(\varpi_h) \right\|^2$$

$$(4.10) \qquad \leq \int\limits_{\varpi_h} \frac{1}{1 + h^{-2}|x - P^\pm - hP^\mp|^2} dx \leq h^3 \int\limits_{B(0,2dh^{-1})} \frac{1}{1 + |\xi|^2} d\xi \leq C'h^2,$$

where the coordinate change $h^{-1}(x - P^\pm - hP^\pm) \mapsto \xi$ was used and $d > 0$ denotes the diameter of the set ϖ_h. In the same way, using (3.31) and the change $h^{-1}(x - P^\pm) \mapsto \xi$, we also get $\|\chi_\pm \widetilde{W}_k^\mp \circ \tau_\pm^h; L^2(\varpi_h)\|^2 \leq Ch^3$. These estimates and (3.28) imply that (4.8) is bounded by Ch.

Finally, $u'_k \in L^\infty(\varpi_h)$ by Remark 3.5, which yields the bound Ch for (4.9). $\quad\square$

4.3. **Proof of Theorem 4.1.** To prove Theorem 4.1 we need to prove the main estimate (4.1), or, (1.2). For simplicity of presentation we assume here that the eigenvalue λ_k is simple. Multiple eigenvalues could be treated using the above formulation of Lemma 4.2, but we leave the details to the reader. So, let us take in Lemma 4.2 $n = 1$, $\gamma = 0$ and $t = \tau$; in addition, we take for $\mathcal{B}$ the operator $\mathcal{B}^{h,\eta} : \mathcal{H}^{h,\eta} \to \mathcal{H}^{h,\eta}$, and $\mu = \mu_k = (1 + \lambda_k + h\Lambda'_k(\eta))^{-1}$, and $\mathcal{U}_1 = \mathcal{U}^\natural_{h,k}$, see (2.18) and (4.4). We are going to prove that

$$(4.11) \qquad \tau = \|\mathcal{B}^{h,\eta}\mathcal{U}^\natural_{h,k} - \mu_k \mathcal{U}^\natural_{h,k}; \mathcal{H}^{h,\eta}\| \leq Ch^{3/2}.$$

Then the lemma implies that the operator $\mathcal{B}^{h,\eta}$ has an eigenvalue $\mu_k^h(\eta)$ such that

$$|\mu_k^h(\eta) - \mu_k| \leq Ch^{3/2} , \quad \text{or} \quad |\Lambda^h_{j(k)}(\eta) - (\lambda_k + h\Lambda'_k(\eta))| \leq C_k h^{3/2},$$

where $\Lambda^h_{j(k)} := (\mu_k^h)^{-1} - 1$ is some eigenvalue of the model problem. However, since λ_k is a simple eigenvalue of the limit problem, the result (5.1) of the Appendix implies that the only eigenvalue of the model problem, which is near λ_k, must be $\Lambda^h_k(\eta)$, i.e., $\Lambda^h_{j(k)}(\eta) = \Lambda^h_k(\eta)$. Thus, (4.1) follows.

By (2.18),

$$\|\mathcal{B}^{h,\eta}\mathcal{U}^\natural_{h,k} - \mu_k \mathcal{U}^\natural_{h,k}; \mathcal{H}^{h,\eta}\| = \sup_Z \langle \mathcal{B}^{h,\eta}\mathcal{U}^\natural_{h,k} - \mu_k \mathcal{U}^\natural_{h,k}, Z \rangle_{\mathcal{H}^{h,\eta}}$$

$$= \sup_Z \left((\mathcal{U}^\natural_{h,k}, Z)_{\varpi_h} - \mu_k(\mathcal{U}^\natural_{h,k}, Z)_{\varpi_h} - \mu_k(AD(\nabla_x)\mathcal{U}^\natural_{h,k}, D(\nabla_x)Z)_{\varpi_h} \right)$$

$$= \|\mathcal{U}_{h,k}; \mathcal{H}^{h,\eta}\|^{-1} \mu_k \sup_Z |\mathcal{A}(\mathcal{U}_{h,k}, Z)| ,$$

where supremum is taken over all $Z \in \mathcal{H}^{h,\eta}$ with $\|Z; \mathcal{H}^{h,\eta}\| = 1$ and

$$(4.12) \quad \mathcal{A}(\mathcal{U}_{h,k}, Z) = (\lambda_k + h\Lambda'_k(\eta))(\mathcal{U}_{h,k}, Z)_{\varpi_h} - \left(AD(\nabla_x)\mathcal{U}_{h,k}, D(\nabla_x)Z\right)_{\varpi_h}.$$

By Lemma 4.3, $\|\mathcal{U}_{h,k}; \mathcal{H}^{h,\eta}\| \geq C$, and moreover, $\mu_k \leq 1$. So we see that the proof of (4.11) and thus of Theorem 4.1 will be completed by showing that

$$(4.13) \qquad \sup_Z |\mathcal{A}(\mathcal{U}_{h,k}, Z)| \leq Ch^{3/2}.$$

Lemma 4.4. *We have*

$$\mathcal{A}\big(\mathcal{X}^h(u_k \circ a_h + hu'_k \circ a_h) + \mathcal{X}^h_\pm u_k(P^\pm), Z\big)$$

$$(4.14) \qquad = -h(F_1 \circ a_h, Z)_{\varpi(h)} - h(G \circ a_h, Z)_{\partial\varpi(h)} + O(h^{3/2})$$

for all $Z \in \mathcal{H}^{h,\eta}$, where F and G are defined in (3.40)-(3.41) and we denote

$$-F_1 = -F + \left(-2a\lambda_k + \Lambda'_k(\eta)\right)u_k = (L_x - \lambda_k)\sum_\pm \chi_\pm \mathcal{W}_k^\mp \circ \tau_\pm.$$

Proof. Due to the cut-off function $\mathcal{X}^h$ we can write the left hand side of (4.14) as

(4.15) $\qquad \widetilde{\mathcal{A}}(f, Z) := (\lambda_k + h\Lambda'_k(\eta))(f, Z)_{\varpi(h)} - \left(AD(\nabla_x)f, D(\nabla_x)Z\right)_{\varpi(h)}$

where $f = u_k \circ a_h + hu'_k \circ a_h$. We first prove that

(4.16) $\qquad \left|\widetilde{\mathcal{A}}(\mathcal{X}^h_\pm(u_k \circ a_h - u_k(P^\pm) + hu'_k \circ a_h), Z)\right| \leq Ch^{3/2}.$

Since the mean value theorem and Lemma 2.2 imply $|u_k \circ a_h(x) - u_k(P^\pm)| \leq Ch$ in the set $\operatorname{supp}\mathcal{X}^h_\pm$, we get

$$\left|(\mathcal{X}^h_\pm(u_k \circ a_h - u_k(P^\pm)), Z)_{\varpi(h)}\right|$$

$$\leq \left(\int_{\operatorname{supp}\mathcal{X}^h_\pm \cap \varpi(h)} |u_k \circ a_h - u_k(P^\pm)|^2 dx\right)^{1/2} \|Z; \mathcal{H}^{h,\eta}\|$$

$$\leq C\left(\int_{\operatorname{supp}\mathcal{X}^h_\pm} h^2 dx\right)^{1/2} \leq Ch^{5/2}.$$

Moreover, since $|D(\nabla_x)\mathcal{X}^h_\pm(x)| \leq |\nabla_x\mathcal{X}^h_\pm(x)| \leq C/h$ for all x,

$$\left|(AD(\nabla_x)\mathcal{X}^h_\pm(u_k(P^\pm) - u_k \circ a_h)), D(\nabla_x)Z)_{\varpi(h)}\right|$$

$$\leq c\|Z; \mathcal{H}^{h,\eta}\|\left(\left(\int_{\operatorname{supp}\mathcal{X}^h_\pm \cap \varpi(h)} \left|\nabla_x(u_k \circ a_h - u_k(P^\pm))\right|^2 dx\right)^{1/2}\right.$$

$$\left. + \left(\int_{\operatorname{supp}\nabla_x\mathcal{X}^h_\pm} \left|[D(\nabla_x), \mathcal{X}^h_\pm](u_k \circ a_h(x) - u_k(P^\pm))\right|^2 dx\right)^{1/2}\right)$$

$$\leq C\left((h^3)^{1/2} + (h^3(h^{-2}h^2))^{1/2}\right) \leq C'h^{3/2},$$

where in the commutator, the function $\mathcal{X}^h_\pm$ is understood as the multiplication operator with this function, hence, $[D(\nabla_x), \mathcal{X}^h_\pm] = D(\nabla_x)\mathcal{X}^h_\pm - \mathcal{X}^h_\pm D(\nabla_x)$ is just a multiplication by the smooth function $D(\nabla_x)\mathcal{X}^h_\pm$ with support contained in $\operatorname{supp}\nabla_x\mathcal{X}^h_\pm$.

To estimate the same expressions with $h\mathcal{X}^h_\pm(x)u'_k \circ a_h$ we take into account Remark 3.5. We get, by the boundedness of u'_k,

$$\left|(h\mathcal{X}^h_\pm u'_k \circ a_h, Z)_{\varpi_h}\right| \leq h\left(\int_{\operatorname{supp}\mathcal{X}^h_\pm \cap \varpi(h)} |u'_k \circ a_h|^2 dx\right)^{1/2} \leq Ch^{5/2}.$$

and by the L^3-estimate for $\nabla_x u'_k$,

$$\left|(AD(\nabla_x)(h\mathcal{X}^h_\pm u'_k \circ a_h), D(\nabla_x)Z)_{\varpi_h}\right|$$

$$\leq ch\left(\int_{\operatorname{supp}\mathcal{X}^h_\pm \cap \varpi(h)} |\nabla_x u'_k \circ a_h|^2 dx + h^{-2}\int_{\operatorname{supp}\mathcal{X}^h_\pm \cap \varpi(h)} |u'_k \circ a_h|^2 dx\right)^{1/2}$$

$$\leq ch\left(\left(\int_{\text{supp}\,\mathcal{X}_\pm^h\cap\varpi(h)}1^3dx\right)^{1/3}\left(\int_{\text{supp}\,\mathcal{X}_\pm^h\cap\varpi(h)}|\nabla_x u_k'\circ a_h(x)|^3dx\right)^{2/3}+c'h\right)^{1/2}\leq Chh^{1/2}.$$

These arguments prove (4.16). Using that and also (2.34), the left hand side of (4.14) equals

$$(4.17) \qquad \widetilde{\mathcal{A}}\big(u_k\circ a_h+hu_k'\circ a_h,Z\big)+O(h^{3/2}).$$

Next, since u_k is a solution of the problem (2.22)–(2.23) we get

$$(AD(\nabla_x)u_k\circ a_h,D(\nabla_x)Z)_{\varpi(h)}=\lambda_k a_h^2(u_k\circ a_h,Z)_{\varpi(h)}$$

and thus also the estimate

$$(4.18)\ \Big|(AD(\nabla_x)u_k\circ a_h,D(\nabla_x)Z)_{\varpi(h)}-\lambda_k(1+2ah)(u_k\circ a_h,Z)_{\varpi(h)}\Big|\leq Ch^2.$$

Combining this with the definition of $\widetilde{\mathcal{A}}$, (4.15), we get

$$\widetilde{\mathcal{A}}(u_k\circ a_h,Z)=h(\Lambda_k'(\eta)-2a\lambda_k)(u_k,Z)_{\varpi(h)}+O(h^2).$$

Since u_k' is a solution of the problem (3.40)-(3.41), we can write

$$h\widetilde{\mathcal{A}}(u_k'\circ a_h,Z)$$
$$=h\Big(\big(-AD(\nabla_x)(u_k'\circ a_h),D(\nabla_x)Z\big)_{\varpi(h)}+\lambda_k(u_k'\circ a_h,Z)_{\varpi(h)}\Big)+O(h^2)$$
$$(4.19)\ =-h(F\circ a_h,Z)_{\varpi(h)}-h(G\circ a_h,Z)_{\partial\varpi(h)}+O(h^2).$$

Notice that here we can commute differentiation and composition with the function a_h, for example $hD(\nabla_x)(u_k'\circ a_h)=h(D(\nabla_x)u_k')\circ a_h+O(h^2)$ and so on. Now (4.14) follows from (4.17)–(4.19). $\square$

In view of (4.3), (4.12), and Lemma 4.4, we can prove the bound (4.13) and thus also Theorem 4.1 by proving the following lemma, where $0<h<1/10$ and the above introduced notation is used.

Lemma 4.5. *We have the estimate*

$$\sup_Z\Big|\mathcal{A}\Big(\sum_\pm\chi_\pm\big(V_k^\mp\circ\tau_\pm^h-u_k(P^\pm)\big),Z\Big)$$

$$(4.20)\qquad -h(F_1\circ a_h,Z)_{\varpi(h)}-h(G\circ a_h,Z)_{\varpi(h)}\Big|\leq Ch^{3/2},$$

Proof. We divide the proof into two parts: in the first one we rewrite the expression inside the moduli in (4.20) in a different form, and in the second one we perform the final estimation.

Part I. We use the Green formula for the term (cf. the second term on the right of (4.12))

$$-\Big(AD(\nabla_x)\chi_\pm\big(V_k^\mp\circ\tau_\pm^h-u_k(P^\pm)\big),D(\nabla_x)Z\Big)_{\varpi(h)}$$

and write the term inside the moduli in (4.20) as

$$\sum_\pm\lambda_k\Big(\chi_\pm\big(V_k^\mp\circ\tau_\pm^h-u_k(P^\pm)\big),Z\Big)_{\varpi_h}$$
$$+\sum_\pm h\Lambda_k'(\eta)\Big(\chi_\pm\big(V_k^\mp\circ\tau_\pm^h-u_k(P^\pm)\big),Z\Big)_{\varpi_h}$$

$$- \sum_{\pm} \Big(L_x \chi_{\pm} (V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm})), Z \Big)_{\varpi_h}$$

$$- \sum_{\pm} \Big(N_x^h \chi_{\pm} (V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm})), Z \Big)_{\partial \varpi_h}$$

$$+ h \Big(\sum_{\pm} (L_x - \lambda_k)(\chi_{\pm} \mathcal{W}_k^{\mp} \circ \tau_{\pm}) \circ a_h, Z \Big)_{\varpi(h)}$$

$$+ h \sum_{\pm} \big(N_x^h (\chi_{\pm} \mathcal{W}_k^{\mp} \circ \tau_{\pm}) \circ a_h, Z \big)_{\partial \varpi(h) \setminus B_h}$$

$$(4.21) \qquad = T^{(I)} + T^{(II)} + T^{(III)} + T^{(IV)} + T^{(V)} + T^{(VI)},$$

where $B_h := a_h^{-1} B(P^{\pm}, S/2)$, cf. (3.43). We take into account that $h \chi_{\pm} \circ a_h = h \chi_{\pm} + O(h^2)$ for $T^{(V)}$ and $T^{(VI)}$, and combine

$$S^{(I)} := T^{(I)} + T^{(III)} + T^{(V)}$$

$$= \sum_{\pm} \Big(- L_x + \lambda_k) \chi_{\pm} \big(V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm}) - h \mathcal{W}_k^{\mp} \circ \tau_{\pm} \circ a_h \big), Z \Big)_{\varpi(h)}$$

$$(4.22) \qquad + \sum_{\pm} \Big((-L_x + \lambda_k) \chi_{\pm} \big(V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm}) \big), Z \Big)_{\varpi_h \setminus \varpi(h)} + O(h^2)$$

(see the remark after (4.19) for commuting the differentiation and a_h). The function $N_x^h(\chi_{\pm} \mathcal{W}_k^{\mp} \circ \tau_{\pm}) \circ a_h$ vanishes in the set $\partial \varpi(h) \cap B_h$, by (3.21), (3.25), except at the point $(a_h)^{-1} P^{\pm}$. We pick up a constant $\tilde{b} > 2M$ (M as in (3.29)) such that $\varpi_h \setminus \cup_{\pm} B(P^{\pm}, \tilde{b}h) \subset \varpi(h)$ for all h. Noticing that $(a_h)^{-1} P^{\pm} \in B(P^{\pm}, \tilde{b}h) \subset B_h$ for small h, we can write

$$T^{(IV)} + T^{(VI)} = \sum_{\pm} \Big(N_x^h \chi_{\pm} \big(V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm}) - h \mathcal{W}_k^{\mp} \circ \tau_{\pm} \circ a_h \big), Z \Big)_{\partial \varpi(h) \setminus B(P^{\pm}, \tilde{b}h)}$$

$$+ \sum_{\pm} \Big(N_x^h \chi_{\pm} \big(V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm}) \big), Z \Big)_{\partial \varpi_h \cap B(P^{\pm}, \tilde{b}h)} + O(h^2) =: S^{(II)}.$$

Thus, (4.21) equals $O(h^2)$ plus

$$(4.23) \qquad \sum_{\pm} \big(S^{(I)} + T^{(II)} + S^{(II)} \big).$$

Part II. The second and final part of the proof consists of showing that the modulus of the expression (4.23) is bounded by $Ch^{3/2}$ for $0 < h < 1/10$. We treat the terms separately.

$1°.a)$ We treat the first inner product of $S^{(I)}$, (4.22). We remark that $V_{\mp} \circ \tau_{\pm}^h$ are at least C^2-smooth in the sets $B(P^{\pm}, Mh)$, and the suprema of these functions and their derivatives up to the order 2 are bounded by constants independent of h. Hence, by the same argument as below (4.16),

$$(4.24) \qquad \big| \lambda_k \big(\chi_{\pm} (V_k^{\mp} \circ \tau_{\pm}^h - u_k(P^{\pm})), Z \big)_{B(P^{\pm}, Mh)} \big| \leq Ch^{3/2}$$

and the same estimates hold even if λ_k is replaced by L_x. Moreover, the term $L_x \mathcal{W}_k^{\pm}$ vanishes due to (3.44) and (3.20), and (3.22) implies

$$(4.25) \quad h \big| \lambda_k \big(\chi_{\pm} \mathcal{W}_k^{\mp} \circ \tau_{\pm}, Z \big)_{B(P^{\pm}, Mh)} \big| \leq h \Big(\int_{B(P^{\pm}, Mh)} \frac{C}{|x - P^{\pm}|^2} dx \Big)^{1/2} \leq Ch^{3/2}$$

As a consequence of the estimates (4.24) and (4.25), the first inner product in (4.22) can only be taken over the set $\varpi(h) \setminus \cup_\pm B(P^\pm, Mh)$.

The relations (3.22), (3.44) imply $(a = 1)$,

$$h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h(x) = h\mathcal{W}_k^\mp \big((1 - h)^{-1}x - P^\pm\big)$$
$$= (1 - h)h\mathcal{W}_k^\mp \big(x - P^\pm + hP^\pm\big)$$
$$(4.26) \qquad = (1 - h)\mathcal{W}_k^\mp \big(h^{-1}(x - P^\pm) - P^\mp\big) = (1 - h)\mathcal{W}_k^\mp \circ \tau_\mp \circ \tau_\pm^h(x).$$

Moreover, if $|x - P^\pm| \geq Mh$, then $|\tau_\mp \circ \tau_\pm^h(x)| \geq M/2$, and by (3.44), (3.25), and the choice of M in (3.29) we have $\mathcal{W}_k^\pm \circ \tau_\mp \circ \tau_\pm^h(x) = W_k^\pm \circ \tau_\mp \circ \tau_\pm^h(x)$. Hence, by (3.28),

$$\big\|\chi_\pm\big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h\big); L^2(\varpi(h) \setminus \cup_\pm B(P^\pm, Mh))\big\|^2$$
$$\leq \big\|\chi_\pm\big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - (1 - h)W_k^\mp \circ \tau_\mp \circ \tau_\pm^h\big); L^2(\varpi(h))\big\|^2$$
$$(4.27)\precsim \big\|\chi_\pm\widetilde{W}_k^\mp \circ \tau_\pm^h; L^2(\varpi(h))\big\|^2 + h\big\|\chi_\pm W_k^\mp \circ \tau_\mp \circ \tau_\pm^h; L^2(\varpi(h))\big\|^2$$

Here, the argument in (4.10) and below it yields

$$(4.28) \qquad \big\|\chi_\pm\widetilde{W}_k^\mp \circ \tau_\pm^h; L^2(\varpi(h))\big\|^2 \leq \big\|\chi_\pm\widetilde{W}_k^\mp \circ \tau_\pm^h; L^2(\varpi_h)\big\|^2 \leq Ch^3.$$

Also, (4.10) implies that the second term on the right of (4.27) is bounded by Ch^3.

There remain the terms with L_x, to be evaluated on $\varpi \setminus B(P^\pm, Mh)$. We first observe that due to (3.8), (3.12), (3.20), (3.44),

$$L_x\chi_\pm\big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h\big)$$
$$(4.29) \qquad = \widetilde{L}_x^\pm\big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h\big),$$

where $\widetilde{L}_x^\pm$ is a first order differential operator having as coefficients bounded, non-constant functions with supports contained in $B(P^\pm, R) \setminus \bar{B}(P^\pm, S)$ for the constants $0 < S < R < 1/2$, see (2.33); notice that $L_x\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h = 0$ in spite of the scaling a_h, since L_x only contains second order terms. Moreover, again by (4.26)–(4.27),

$$V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h = hW_k^\mp \circ \tau_\mp \circ \tau_\pm^h + \widetilde{W}_k^\mp \circ \tau_\pm^h$$

so that using (3.30), (3.31) including the gradient estimates, the argument similar to (4.10) again yields

$$(4.30) \qquad \big\|L_x\chi_\pm\big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h\big); L^2(\varpi(h))\big\|^2 \leq C''h^3.$$

$1°.b)$ We estimate the inner product over $\varpi_h \setminus \varpi(h)$ in $S^{(I)}$, (4.22). First, the term with $-L_x$ vanishes, since V satisfies (3.8) and the supports of the coefficients of the operator $\widetilde{L}_x^\pm$ do not intersect the set $\varpi_h \setminus \varpi(h)$, see the remarks after (4.29). Second, the estimate for the term with λ_k is already contained in (4.24). From this, (4.22) and (4.26)–(4.30) we obtain

$$(4.31) \qquad\qquad\qquad |S^{(I)}| \leq Ch^{3/2}.$$

$2°.$ The estimate for $T^{(II)}$ can be done as the estimate for (4.8) in (4.10), and it yields that $T^{(II)} \leq Ch^3$.

$3°.$ Finally, to estimate the term $S^{(II)}$ we write it as the sum over $\pm$ of the terms

$$\big(\chi_\pm N_x^h(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h), Z\big)_{\partial\varpi(h) \setminus B(P^\pm, \tilde{b}h)}$$

$$+ \big(\widetilde{N}_x^h(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h\mathcal{W}_k^\mp \circ \tau_\pm \circ a_h), Z\big)_{\partial\varpi(h) \setminus B(P^\pm, \tilde{b}h)}\big)$$

$$(4.32) \qquad + \Big(\widetilde{N}_x^h \chi_\pm \big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm), Z \big) \Big)_{\partial \varpi_h \cap B(P^\pm, \tilde{b}h)}$$

where $\widetilde{N}_x^h$ is a bounded, smooth multiplier with support in $A^\pm$.

$3°.a)$ To estimate the first term in (4.32) we observe that the functions $V_k^\pm$ satisfy (3.5) on the boundary $\partial \Omega^\pm$ with $\theta \times \{0\}$ excluded. Moreover, by (3.21), (3.44), $N_x^h \mathcal{W}_k^\mp \circ \tau_\pm \circ a_h$ vanishes in the set $\partial \varpi(h) \setminus B(P^\pm, \tilde{b}h) = \partial \varpi(h) \setminus B(P^\pm, \tilde{b}h)$, except possibly outside a set $B(P^\pm, \hat{b})$ for some constant $\hat{b} > 0$ (where the function $\tau_\pm \circ a_h$ does not map $\partial \varpi(h)$ into the plane $\{\xi_3 = 0\}$). Combining these observations we deduce that the inner product can be taken only over the set $\partial \varpi(h) \setminus B(P^\pm, \hat{b})$. In this set we have, by the argument (4.26) and the remark after it,

$$h \mathcal{W}_k^\mp \circ \tau_\pm \circ a_h = (1-h)\mathcal{W}_k^\mp \circ \tau_\mp \circ \tau_\pm^h = (1-h)W_k^\mp \circ \tau_\mp \circ \tau_\pm^h,$$

hence, proceeding as in (4.22) we obtain

$$\Big| \int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \chi_\pm N_x^h \big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) - h \mathcal{W}_k^\mp \circ \tau_\pm \circ a_h \big) Z ds_x \Big|$$

$$(4.33) \quad \leq C \Big(\int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \chi_\pm \big| N_x^h \big(\widetilde{W}_k^\mp \circ \tau_\pm^h + h W_k^\mp \circ \tau_\mp \circ \tau_\pm^h \big) \big|^2 ds_x \Big)^{1/2} \Big(\int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} |Z|^2 ds_x \Big)^{1/2}.$$

Here, we use (3.30), (3.31), and the same argument as in (4.10) to estimate the first factor on the right by the square root of a constant times

$$\int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \Big| (\nabla_x \widetilde{W}_k^\mp) \Big(\frac{x - P^\pm}{h} \Big) \Big|^2 ds_x + h^2 \int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \Big| W_k^\mp \Big(\frac{x - P^\pm - h P^\mp}{h} \Big) \Big|^2 ds_x$$

$$\leq \int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \frac{C}{1 + h^{-6} |x - P^\pm|^6} ds_x + h^2 \int_{\partial \varpi(h) \setminus B(P^\pm, \hat{b})} \frac{C}{1 + h^{-4} |x - P^\pm - h P^\mp|^4} ds_x \leq ch^3.$$

The second factor on the right hand side of (4.33) is bounded by $C\|Z; \mathcal{H}^{h,\eta}\|$, by the Sobolev embedding theorem. We get the bound $Ch^{3/2}$ for the first term in (4.32).

$3°.b)$ As for the second term in (4.32), the integrand is again supported outside the sets $B(P^\pm, S)$. By the same argument as in $a)$, this term is bounded by $Ch^{3/2}$.

$3°.c)$ To evaluate the third term in (4.22) we first observe that the support of the function $\widetilde{N}_x^h$ does not intersect the integration domain $\partial \varpi_h \cap B(P^\pm, \tilde{b}h)$. Thus, on the integration domain,

$$N_x^h \chi_\pm \big(V_k^\mp \circ \tau_\pm^h - u_k(P^\pm) \big) = N_x^h (V_k^\mp \circ \tau_\pm^h).$$

Moreover, the function on the right vanishes everywhere else in $\partial \varpi_h \cap B(P^\pm, \tilde{b}h)$ except for the sets $\theta_h \times \{P^\pm\}$, cf. (3.4)–(3.12) and (2.2). However, due to the quasiperiodicity conditions

$$\sum_\pm \int_{P^\pm \times \theta_h} N_x^h \big(V_k^\mp \circ \tau_\pm^h \big) \bar{Z} ds_x = 0,$$

hence, the third term in (4.32) vanishes.

We have thus shown that $|S^{(II)}| \leq Ch^{3/2}$. By this, (4.31), and $2°$, the moduli of the expressions (4.23) and (4.21) have the bound $Ch^{3/2}$. This finishes the proof of (4.20). $\square$

As remarked before Lemma 4.5, we have now completed the proofs of (4.13) and Theorem 4.1, modulo the results in the Appendix.

5. Appendix: existence of spectral gaps.

5.1. Upper estimate for the bands.
In this section we complete the paper by giving a proof, independent of the considerations in Sections 3–4, for the existence of spectral gaps in the essential spectrum σ_{ess} of the original problem (2.5), (2.6). More precisely, we show that σ_{ess}, corresponding to the parameter value h, has a gap between Υ_j^h and Υ_{j+1}^h, if j is such that $\lambda_j \neq \lambda_{j+1}$ and if h is small enough.

This immediately follows from the following result: we shall show that for all $j \in \mathbb{N}$ there exist numbers $h_j > 0$ and $C_j > 0$ (depending also on ϖ, A, and ϱ), such that

$$(5.1) \qquad \lambda_j - C_j h \leq \Lambda_j^h(\eta) \leq \lambda_j + C_j h$$

for all $h \leq h_j$ and $\eta \in [0, 2\pi)$. Since this holds for all elements $\Lambda_j^h(\eta)$ of a band Υ_j^h, (5.1) yields a rough estimate for the position of the band and thus also the desired result on gaps.

To this end we follow and modify the argument of [33] to prove (5.1). We start with an upper estimate for $\Lambda_j^h(\eta) \in \Upsilon_j^h$ in terms of λ_j.

Lemma 5.1. *For all $j \in \mathbb{N}$ there exist numbers $h_j > 0$ and $C_j > 0$, which depend also on ϖ, A, and ϱ, and which satisfy*

$$(5.2) \qquad \Lambda_j^h(\eta) \leq \lambda_j + C_j h$$

for all $h \leq h_j$ and $\eta \in [0, 2\pi)$.

Proof. For all $j \in \mathbb{N}$, $x \in \varpi(h)$, we set $\mathcal{V}_j^h = (1 - \mathcal{X}^h)u_j^h$, where u_j^h is as in (2.28), and extend the product as 0 for $x \in \varpi_h \setminus \varpi(h)$ (see (2.1), (2.2), (2.28), (2.31)). The definition guarantees that the extensions become smooth and that the support of $\mathcal{V}_j^h - u_j^h$ is contained in a set υ_h with volume estimate $|\upsilon_h| \leq Ch^3$. Moreover, the bounds (2.29) hold true in υ_h. Hence, we get

$$\|u_j^h - \mathcal{V}_j^h; L^2(\varpi_h)\|^2 \leq \int_{\upsilon_h} C_j dx \leq C_j' h^3,$$

$$\|\nabla_x u_j^h - \nabla_x \mathcal{V}_j^h; L^2(\varpi_h)\|^2$$

$$\leq \int_{\upsilon_h} \left(|\nabla_x u_j^h| + |\nabla_x \mathcal{X}^h||u_j^h|\right)^2 dx \leq C_j(h^3 + h) \leq c_j' h,$$

$$(5.3) \qquad \|u_j^h; L^2(\upsilon_h)\|^2 \leq C_j h^3 \ , \quad \|\nabla_x u_j^h; L^2(\upsilon_h)\|^2 \leq C_j h^3.$$

For every $j \in \mathbb{N}$ we now pick an h_j, $0 < h_j < 1$, such that $h_j < h_{j-1}$ and, say, $(1+\tilde{C}_j)h_j \leq 2^{-j-3}$, where $\tilde{C}_j$ is the largest of the constants C_j appearing in (5.3). Let us fix the index j for the rest of the proof, and consider numbers h satisfying $h < h_j$. First, by using the choice of the numbers h_j, classical arguments and the fact that the functions $u_1^h, \ldots, u_j^h$ form an orthonormal set in $L^2(\varpi(h))$ (see below (2.28)), we deduce that also the small perturbations $\mathcal{V}_1^h, \ldots, \mathcal{V}_j^h$ are linearly independent in $L^2(\varpi(h))^3$ and also in $L^2(\varpi_h)^3$.

Let now $(b_p)_{p=1}^j$ be any sequence of numbers normalized so that $\sum_{p=1}^j |b_p|^2 = 1$, and let $\mathcal{W}_j^h = \sum_{p=1}^j b_p \mathcal{V}_p^h$, $W_j := \sum_{p=1}^j b_p u_p^h$, hence, $\|W_j; L^2(\varpi(h))\| = 1$. We

evaluate $a(\mathcal{W}_j^h, \mathcal{W}_j^h; \varpi(h))$. First, by (2.24), (2.25), (2.28), $|a(W_j, W_j; \varpi(h))| \leq \lambda_j$. Moreover,

$$
a(\mathcal{W}_j^h, \mathcal{W}_j^h; \varpi(h)) = a(W_j, W_j; \varpi(h)) + \sum_{p,q=1}^{j} b_p \bar{b}_q \Big(a(\mathcal{V}_p^h - u_p^h, u_q^h; \varpi(h))
$$

$$
(5.4) \qquad\qquad + a(u_p^h, \mathcal{V}_q^h - u_q^h; \varpi(h)) + a(\mathcal{V}_p^h - u_p^h, \mathcal{V}_q^h - u_q^h; \varpi(h)) \Big).
$$

The expression for $a(\cdot, \cdot, \varpi(h))$, (2.9), the Cauchy-Schwartz inequality, the volume bound for υ_h, and (5.3) imply

$$
|a(\mathcal{V}_p^h - u_p^h, u_q^h; \varpi(h))| \leq C_{p,q} \|\mathcal{V}_p^h - u_p^h; H^1(\upsilon_h)\| \|u_q^h; H^1(\upsilon_h)\| \leq C_j h^2.
$$

The third term on the right hand side of (5.4) has the same bound, but the fourth one only the bound Ch. From these estimates and $\sum_{p,q=1}^{j} b_p \bar{b}_q \leq 2j$ we thus obtain

$$
(5.5) \qquad\qquad |a(\mathcal{W}_j^h, \mathcal{W}_j^h; \varpi(h))| \leq \lambda_j + C_j h .
$$

In the same vain one can estimate using (5.3)

$$
(\mathcal{W}_j^h, \mathcal{W}_j^h)_{\varpi(h)} = (W_j, W_j)_{\varpi(h)} + \sum_{p,q=1}^{j} b_p \bar{b}_q \Big((\mathcal{V}_p^h - u_p^h, u_q^h)_{\varpi(h)}
$$

$$
(5.6) \qquad + (u_p^h, \mathcal{V}_q^h - u_q^h)_{\varpi(h)} + (\mathcal{V}_p^h - u_p^h, \mathcal{V}_q^h - u_q^h)_{\varpi(h)} \Big) \geq 1 - C_j h.
$$

To apply these estimates we use the max–min principle [4, Th.10.2.2],

$$
\Lambda_j^h(\eta) = \max_{\mathcal{H}_j} \inf_{\mathcal{V} \in \mathcal{H}_j \setminus \{0\}} \frac{a(\mathcal{V}, \mathcal{V}; \varpi_h)}{\|\mathcal{V}; L^2(\varpi_h)\|^2},
$$

where $\mathcal{H}_j$ stands for any subspace in $H^1_{\text{per}}(\varpi_h)$ of codimension $j - 1$. Since the sequence $(\mathcal{V}_p)_{p=1}^{j}$ is linearly independent, we can find from any $\mathcal{H}_j$ an element $\mathbf{W}_j^h$ of the form $\mathbf{W}_j^h(x, \eta) = e^{-i\eta z} \mathcal{W}_j^h(x)$. Moreover, by the definition of the form a, (2.9), (2.17), and the definitions above, we have

$$
a(\mathbf{W}_j^h, \mathbf{W}_j^h; \varpi_h) = a(\mathcal{W}_j^h, \mathcal{W}_j^h; \varpi(h)), \quad \|\mathbf{W}_j^h, L^2(\varpi_h)\| = \|\mathcal{W}_j^h, L^2(\varpi(h))\|.
$$

These, together with (5.5), (5.6), imply

$$
\Lambda_j^h(\eta) \leq \frac{a(\mathbf{W}_j^h, \mathbf{W}_j^h; \varpi_h)}{\|\mathbf{W}_j^h; L^2(\varpi_h)\|^2} = \frac{a(\mathcal{W}_j^h, \mathcal{W}_j^h; \varpi(h))}{\|\mathcal{W}_j^h; L^2(\varpi(h))\|^2} \leq \frac{\lambda_j + C_j h}{1 - C_j h} \leq \lambda_j + C' h. \;\square
$$

5.2. **Lower estimate for the bands.** We finally prove the lower estimate for the numbers $\Lambda_j^h(\eta)$, see (5.1). We use the following Korn inequality, which was proven in [33]: for all $f \in \mathcal{H}^{h,\eta}$, there holds

$$
(5.7) \qquad \|f; L^2(\varpi_h \setminus \varpi(h))\|^2 \leq Ch^2 \big(a(f, f; \varpi_h) + \|f; L^2(\varpi_h)\|^2 \big).
$$

Lemma 5.2. *For every j there exist a constant $C_j = C_j(\varpi, A, \varrho) > 0$ and a number $\tilde{h}_j > 0$ such that $\Lambda_j^h(\eta) \geq \lambda_j - C_j h$ for all $0 < h < \tilde{h}_j$.*

Proof. We consider the eigenvectors $U_j^h \in \mathcal{H}^{h,\eta}$ see (2.20): we have $a(U_j^h, U_j^h; \varpi_h) = \Lambda_j^h(\eta)$. By (5.2), if $h \leq h_j$, every $\Lambda_j^h(\eta)$ can be bounded by a positive number depending only on j, so (5.7) implies

$$
\|(U_j^h; L^2(\varpi(h))\| \geq \|(U_j^h; L^2(\varpi_h)\| - C_j h ,
$$

$$(5.8) \qquad |(U_j^h, U_k^h)_{\varpi(h)} - (U_j^h, U_k^h)_{\varpi_h}| \le C_j h$$

Thus, for small enough $\tilde{h}_j > 0$ and $h \le \tilde{h}_j$, the sequence $U_1^h, \dots, U_j^h$ remains linearly independent in $L^2(\varpi(h))^3$.

We fix j and η, and assume $h \le \tilde{h}_j$. The sequence $(e^{i\eta z} U_p^h)_{p=1}^j$ is still linearly independent in $L^2(\varpi(h))^3$, hence, any subspace $\mathcal{H}_j \subset L^2(\varpi(h))^3$ of codimension $j-1$ contains a linear combination $\sum_{p=1}^j b_p e^{i\eta z} U_p^h$ such that $\sum_{p=1}^j |b_p|^2 = 1$. We denote $\mathcal{Y}_j^h = \sum\limits_{p=1}^j b_p U_p^h$; we have $\|\mathcal{Y}_j^h; L^2(\varpi_h)\| = 1$.

The eigenvector property of U_p^h and (2.19) imply $a(\mathcal{Y}_j^h, \mathcal{Y}_j^h; \varpi_h) \le \Lambda_j^h(\eta)$. On the other hand $|a(e^{i\eta z}\mathcal{Y}_j^h, e^{i\eta z}\mathcal{Y}_j^h; \varpi_h)| = |a(\mathcal{Y}_j^h, \mathcal{Y}_j^h; \varpi_h)|$, and the estimates (5.8) also yield

$$\|e^{i\eta z}\mathcal{Y}_j^h; L^2(\varpi(h))\|^2 = \|\mathcal{Y}_j^h; L^2(\varpi(h))\|^2 \ge 1 - C_j h.$$

Hence, we obtain

$$\lambda_j \le \frac{a(e^{i\eta z}\mathcal{Y}_j^h, e^{i\eta z}\mathcal{Y}_j^h; \varpi(h))}{\|e^{i\eta z}\mathcal{Y}_j^h; L^2(\varpi(h))\|^2} \le \frac{a(e^{i\eta z}\mathcal{Y}_j^h, e^{i\eta z}\mathcal{Y}_j^h; \varpi_h)}{\|e^{i\eta z}\mathcal{Y}_j^h; L^2(\varpi(h))\|^2} = \frac{a(\mathcal{Y}_j^h, \mathcal{Y}_j^h; \varpi_h)}{\|\mathcal{Y}_j^h; L^2(\varpi(h))\|^2}$$
$$\le a_\eta(\mathcal{Y}_j^h, \mathcal{Y}_j^h; \varpi_h)(1 + C_j h) \le \Lambda_j^h(\eta)(1 + C_j h). \quad \square$$

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